ISTANBUL BILGI UNIVERSTIY

INSTITUTE OF SOCIAL SCIENCES

ECONOMICS PH.D. PROGRAM

THREE ESSAYS ON COALITION FORMATION GAMES

SEÇKİN ÖZBİLEN

111808001

PROF. DR. İPEK ÖZKAL SANVER

ISTANBUL

MARCH 2019

Three Essays on Coalition Formation Games

Koalisyon Oluşum Oyunları Alanında Üç Makale

Seçkin Özbilen

111808001

Tez Danışmanı: **Prof. Dr. İpek Özkal Sanver** ....
İstanbul Bilgi Üniversitesi

Jüri Üyeleri: **Prof. Dr. Jean Lainé** ....
Le Conservatoire National des Arts et Métiers

Prof. Dr. Remzi Sanver ....
CNRS, LAMSADE, Université Paris Dauphine

Doç. Dr. Ayça Ebru Giritligil ....
İstanbul Bilgi Üniversitesi

Dr. Öğr. Üyesi Hayrullah Dindar ....
İstanbul Bilgi Üniversitesi

Tezin Onaylandığı Tarih: 21.03.2019

Toplam Sayfa Sayısı: 109

Anahtar Kelimeler (Türkçe)	**Anahtar Kelimeler (İngilizce)**
1) Koalisyon Oluşum Oyunları	1) Coalition Formation Games
2) Kararlılık	2) Stability
3) Çekirdek	3) Core
4) Kısıtlı Tanım Kümesi	4) Domain Restriction
5) Değişim Kararlılık	5) Exchange Stability

ACKNOWLEDGEMENTS

Undertaking this Ph.D. has been a truly life-changing experience for me and it would not have been possible to do without the support and guidance that I received from many people.

I would like to express my gratitude to my supervisor Prof. İpek Özkal Sanver for all the support and encouragement she gave me during the completion of this thesis. I am extremely grateful for her patience and immense knowledge.

Besides my supervisor, I would like to say a very big thank you to Assoc. Prof. Ayça Ebru Giritligil and Prof. Jean Augustin Lainé for making themselves available whenever I needed their guidance and support. Ayça and Jean always spared time whenever I needed to talk and consult, no matter how busy their schedules were. I will be forever indepted to them.

I am very grateful to Prof. Remzi Sanver and Prof. Walter Trockel for elaborate discussions, helpful comments, and elucidative research talks. Furthermore, I would like to thank Assist. Prof. Hayrullah Dindar for his insightful comments, his encouragement, and his instructive friendship during my Ph.D. studies.

I would like to thank Murat Sertel Center for Advanced Economic Studies and İstanbul Bilgi University for their financial support during my Ph.D. studies which enabled me to meet wonderful people in academia and to present earlier versions of this work in several prestigious conferences and get feedback. I would like to express my thanks to The Scientific and Technological Research Council of Turkey (TÜBİTAK) for supporting the third chapter of this thesis and financing my visit to Maastricht University under the project **1059B1415000065**. Furthermore, Dr. Burak Can deserves special thanks for his generous effort, brilliant comments, and his friendship during my visit at Maastricht University.

I am also very grateful to all the professors at the Department of Economics, especially Prof. Burhan Şenatalar, Prof. Benan Orbay, and Assist. Prof. Umut Keskin for providing very supportive and friendly environment for the time I worked as a Research Assistant. I also need to thank Neslihan Sözeri, Tuğçe Erdem, İlknur Toygar Değer, Demet Gürkan, and Ebru Serdar Karaduman for always being very helpful and polite with administrative matters.

I would like to thank my graduate friends and office mates, especially to Orkun Doğan, Fatma Aslan, Nazlı Şahanoğulları, Rabia Zeynep Yurdakul, and Erdal Terkivatan for the stimulating discussions, and for all their friendship.

I would also like to say a heartfelt thank you to my grandmother Fatma, to my parents Hatem and Ramazan, and to my sister Seçil for always believing me and encouraging me to follow my dreams. My grandmother and my parents have sacrificed their lives for my sister and myself, and provided unconditional love and care. I would not have made it this far without them. Special thanks to my wife's family, Ayten and Mehmet, who all have been supportive and caring, especially to her mother for all her help whenever we needed.

And finally to my wife, Pınar, who has been by my side throughout this Ph.D., living every single minute of it, without whom, I would not have had the courage to embark on this journey in the first place. Last but not least, I want to thank our little baby Adel Bahar for the motivation she provided and for making it possible for me to complete what I started.

This thesis is dedicated to women of my family and to my father.

I always felt their support in every moment of my life.

To my grandmother Fatma Özbilen,

to my mother Hatem Özbilen,

to my father Ramazan Özbilen,

to my wife Pınar Özbilen,

to my sister Seçil Özbilen Erol,

to my mother-in-law Ayten Yozgatlı,

and to our little daughter Adel Bahar Özbilen.

ABSTRACT

This dissertation is about hedonic coalition formation games. A hedonic coalition formation game consists of a finite set of individuals and a preference profile of individuals. Every individual's preference depends only on the members of her coalition. An outcome of a hedonic game is partitioning of the individual set into disjoint subsets. An outcome is called as coalition structure. The quality and desirability of a coalition structure is analyzed using stability concepts.

In this dissertation, we study three stability concepts, namely core stability, strong coalitional stability, and strong exchange stability. The main scope of the dissertation is exploring sufficient domain conditions for the existence of coalition structures in the context of aforementioned stability concepts. In the Introduction chapter, we firstly introduce hedonic coalition formation games and present the literature. Then, we introduce the formal model of hedonic coalition formation games. Afterwards, we present some stability concepts which are excessively studied in the literature. The second chapter is the first essay of this dissertation. In that chapter, we focus on core stability. Three new domain restrictions namely $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity are introduced. All three domain restrictions are individually sufficient for the existence of core stable coalition structures. The third chapter is the second essay of this dissertation. The main focus of that chapter is a new stability concept called strong coalitional stability. We show that if $\mathcal{A}$-

responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity are intensified with mutuality, mutuality, and symmetry, respectively, they become sufficient for the existence of strongly coalitionally stable coalition structures. Afterwards, we present the relation between strong coalitional stability and other stability concepts. The fourth chapter is the last essay of this dissertation. We introduce a new stability concept called strong exchange stability. We present the relation between strong exchange stability and the domain restrictions $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity. Additionally, we show that strongly exchange stable coalition structures exist in some domains which are previously defined. The last chapter is dedicated to final remarks open questions, and conclusion.

ÖZET

Bu tez, hedonik koalisyon oluşum oyunlarıyla ilgilidir. Bir hedonik koalisyon oluşum oyununda sonlu sayıda birey yer almaktadır. Her bireyin sadece kendisinin içinde yer aldığı altkümeler (koalisyonlar) üzerine yansıyan, karşılaştırılabilir ve geçişken tercih bağıntısı vardır. Tüm bireylerin tercih bağıntıları düşünüldüğünde bir hedonik koalisyon oluşumu oyununun çıktısı birey kümesinin partisyonlara (koalisyon yapısı) ayrılmasıdır. Koalisyon yapılarının sınıflandırılması ve analizi çeşitli kararlılık kavramları kullanılarak yapılmaktadır. Bir koalisyon yapısının kararlı olması bireysel ya da grup halinde herhangi bir tür yer değiştirme hareketine imkan sağlamaması ile alakalıdır. Literatürde üzerinde çalışma yapılan çok sayıda kararlılık kavramı mevcuttur. Genel olarak yapılan analizler, kararlılık özelliğini sağlayan koalisyon yapılarının hangi tanım kümeleri üzerinde var olduğu ve tek olduğu üzerinedir. Neredeyse tüm kararlılık kavramları (birkaç zayıf kararlılık kavramı dışında) en geniş tanım kümesinde (tüm hedonik koaliyon oluşum oyunları kümesi) bile var olmayabilmektedir. Dolayısıyla kararlılık özelliği gösteren koalisyon yapılarının varlığı veya tekliğini araştırmak için daha kısıtlı tanım kümeleri üzerinde çalışılmaktadır.

Bu tezde çekirdek kararlılık, kuvvetli koalisyonel kararlılık ve kuvvetli değişim kararlılık kavramları ele alınmaktadır. Tezin ana kapsamı bu kararlılık kavramlarını sağlayan koalisyon yapılarının var olduğu tanım kümelerini araştırmaktır. Giriş bölümünde ilk önce hedonik koalisyon oluşum oyunları tanıtılmakta ve literatürden

bahsedilmektedir. Ardından hedonik koalisyon oluşum oyunlarının formel tanımı yapılmakta ve tezde kullanılan gösterimler ile kavramlar tanıtılmaktadır. Ardından literatürde sıkça çalışılan kararlılık kavramları tanımlanmaktadır. İkinci bölüm bu tezin ilk makalesidir. Bu bölümde çekirdek kararlılık kavramı üzerine odaklanılmaktadır. $\mathcal{A}$-duyarlılık, $\mathcal{B}$-duyarlılık ve $\mathcal{G}$-tekillik isminde üç yeni tanım kümesi tanıtılmaktadır. Her üç tanım kümesi de çekirdek kararlı koalisyon yapılarının varlığı için yeter koşul oluşturmaktadır. Üçüncü bölüm bu tezin ikinci makalesidir. Bu bölümde kuvvetli koalisyonel kararlılık kavramı tanıtılmakta ve bu kararlılık kavramını sağlayan koalisyon yapılarının varlığı araştırılmaktadır. $\mathcal{A}$-duyarlılık, $\mathcal{B}$-duyarlılık ve $\mathcal{G}$-tekillik koşulları sırasıyla üst karşılıklılık, üst karşılıklılık ve üst simetri koşulları ile kuvvetlendirildiğinde, her bir tanım kümesinde kuvvetli koalisyonel kararlı koalisyon yapılarının var olduğu ispatlanmaktadır. Daha sonra kuvvetli koalisyonel kararlılık kavramı ile diğer kararlılık kavramları karşılaştırılmaktadır. Dördüncü bölüm bu tezin son makalesidir. Bu bölümde kuvvetli değişim kararlılık kavramı tanıtılmakta ve analiz edilmektedir. Kuvvetli değişim kararlı koalisyon yapılarının var olduğu tanım kümeleri araştırılmakta, kuvvetli değişim kararlılık ile $\mathcal{A}$-duyarlılık, $\mathcal{B}$-duyarlılık ve $\mathcal{G}$-tekillik kümeleri arasındaki ilişki anlatılmaktadır. Son bölüm sonuçlara, cevaplanmamış sorulara ve son yorumlara ayrılmıştır.

Table of Contents

List of Figures

1. INTRODUCTION

Humans are social entities. They display several collaborative behaviors during their lives. In order to enhance collaboration, they prefer to act in groups (coalitions) rather than staying alone. Hobby groups comprised by friends, research groups of academics, project groups of business people, homework groups of students, political parties, and trade unions are simple examples of coalitions from everyone's daily lives in which people aim to increase their acquisitions via cooperation. Other important examples of coalitions are various international agreements between countries such a the European Union, the Kyoto Protocol, the Shangai Cooperation Organization (Shangai Five), or the North Atlantic Treaty Organization (NATO).

In Game Theory, these and many other cooperative situations could be modelled by coalition formation games. Von Neumann and Morgenstern pointed out the the importance of coalition formation, and they devoted the significant part of their seminal work, Theory of Games and Economic Behavior (1944) to formal analysis of coalition formation. Von Neumann and Morgenstern studied coalition formation games within the framework of characteristic function (coalitional function) games. A characteristic function game consists of a finite set of players (individuals) and a real valued function (which maps every coalition of players to a real number). The real valued function is called characteristic function (or sometimes coalitional function) and it specifies for each coalition of players the total amount of payoff that its members can jointly guarantee themselves and that payoff can be transferred without loss between them. These games are called *N-person Transferable Utility Games* (TU games, for short), or *games with side payments*.

Definition 1.0.1. *A **TU game** is a pair (N, v), where $N = \{1, 2, ..., n\}$ denotes the set of players and $v : 2^N \rightarrow \mathbb{R}$ is **the characteristic function** that assigns to every coalition $S \subseteq N$ a value $v(S)$, representing the total payoff to this group of players when they cooperate. By convention $v(\emptyset) = 0$.*

The expression "transferable utility" means that there is some medium of exchange between the players, e.g., money, and that the players' utilities are linear in money. Players can compare their utility and transfer some utility without any loss.

Early studies in TU games supposed that the game is superadditive, i.e., if two disjoint coalitions act together, they can get at least as much as they can when they act separately. In such situations, it is natural to expect the formation of the grand coalition $\{N\}$. Thus, these early studies concentrated on describing the reasonable ways of apportioning the payoff available to the grand coalition to individuals. Several solution concepts have been defined for superadditive games such as the core, the nucleolus, the kernel, the Shapley value, von Neumann-Morgenstern solutions, various bargaining sets, and others.

Although early literature focused on superadditive TU games, many situations are not superadditive. For example, political parties form coalitions in order to get more votes however, forming large coalitions obligates to compromise on a neutral candidate who finally do not satisfy any voter. Individuals come together in communities in order to share/decrease the costs of the production of local public goods. On the other hand, if public good is used by many consumers, some individuals may not receive it or there may be a congestion. Aumann and Drèze (1974, p. 233) also points out the existence of non-superadditive situations: "acting together may be difficult, costly, or illegal, or the players may for various "personal reasons" not wish to do so.".

As a result, it can be summarized that individuals who have similar tastes, similar objectives, or close locations, form subcoalitions. Then, given an n-person TU game, two fundamental questions show up which need to be answered:

1) *Which coalitions can be expected to form* ?

2) *How individuals/players in coalitions apportion their joint profit/payoff* ?

Shenoy (1979) stated that answers of these two questions intertwine with each other, i.e., the coalition structure influence the payoffs and vice versa. On one hand, the final distribution of payoffs to the players depends on the coalitions that finally form and, on the other hand, coalitions that finally show up depend on the payoffs available to each player in each of these coalitions.

Most of the research in the literature focused on forecasting player's payoffs while assuming that coalition structure is given exogenously (see Aumann and Maschler (1964) and Aumann and Drèze (1974)). Hart and Kurz (1983) used a model which considers endogenous coalition structures. Their model combines two kinds of concepts: value and stability. They first evaluate player's payoffs in various coalition structures and then, based on these values they try to find which ones are stable. Similar approaches can also be found in Yi (1997), Ray and Vohra (1999), and Belleflamme (2000).

Bennet and Zame (1988) defined *bargaining aspiration outcomes*, Zhou (1994) defined *bargaining set*, and Gerber (2000) defined *C-solution* concepts. These are some of the studies which simultaneously provide answer to the question of payoff distribution as well as to the question of coalition formation.

All previously mentioned studies supposed that players have a common scale to measure the worth of a coalition and the utility is freely transferable among players. In general, such a scale may not exist and side payments may not be possible for various reasons. In such situations, it is better to represent each coalition's possibilities by a set of payoff vectors. This model is called *N-person Nontransferable Utility Games* (NTU games, for short), or *games without side payments*.

Definition 1.0.2. *An **NTU game** is a pair (N,V), where $N=\{1,2,...,n\}$ denotes the set of players and V is a map assigning to every coalition $S\subseteq N$ a subset $V(S)$ of $\mathbb{R}^S$ such that $V(\emptyset)=\emptyset$ and for all $S\subseteq N, S\neq\emptyset$:*

(i) $V(S)$ is a nonempty, closed and convex subset of $\mathbb{R}^S$,
(ii) $V(S)$ is comprehensive, i.e., if $x\in V(S)$ and $y\leq x$, then $y\in V(S)$,
(iii) $V(S)\cap\mathbb{R}^S_+$ is bounded.

The interpretation of an NTU game (N,V) is that $V(S)$ is the set of feasible payoff (utility) vectors for coalition S if that coalition forms. A solution then predicts or prescribes a final payoff vector or a set of payoff vectors. NTU games generalize TU games, i.e., every TU game (N,v) can be reformulated as an NTU game by defining $V(S)=\{x\in\mathbb{R}^S|\sum_{i\in S}x_i\leq v(S)\}\ \forall S\subseteq N$ such that $S\neq\emptyset$.

Compared to TU games, the literature on NTU games is very narrow. Early studies on NTU games focused on the analysis of exchange or production economies and markets (Arrow and Debreu (1954)). Following studies focused on existence and uniqueness of solutions or axiomatic characterizations for various solution concepts such as the core, the nucleolus, the kernel, the Shapley value, and bargaining set. A detailed analysis of the literature of NTU games can be found in Peters (2009).

In some situations (e.g., people working in project groups, students in homework groups, sport clubs, ...) it is not possible to specify an exact value for each coalition or a set of feasible payoffs for each player in each coalition she could belong to. The only thing that matters for a player in these situations becomes her membership in a coalition itself. The game is then given by a finite set of players (individuals) and their personal preferences for membership in specific coalitions. A feasible allocation in such a game becomes partitioning of players. We speak about hedonic coalition formation games.

In a Hedonic Coalition Formation Game (simply hedonic game)[1] there exists finite number of individuals. Every non-empty subset of the set of individuals is called as coalition. Every individual only cares about which individuals are in her coalition, but does not care how other individuals are grouped. An individual's preference is complete, and transitive over the coalitions of which she is a member. The duple, finite set of individuals and profile of preferences of individuals, is called as hedonic coalition formation game.

Definition 1.0.3. *A **hedonic coalition formation game** is a pair $(N, \succsim)$ where $N = \{1, 2, ..., n\}$ is the set of individuals, and $\succsim = \{\succsim_1, \succsim_2, ..., \succsim_n\}$ denotes the profile of preferences, specifying for each individual $i \in N$ her preference relation, i.e., a complete and transitive binary relation on $\mathcal{C}_i^N = \{S \subseteq N | i \in S\}$.*

Every hedonic game $(N, \succsim)$ can be reformulated as an NTU game $(N, V, \succsim)$ by defining a unique outcome x_S and defining $V(S) = \{x_S\}$ for each coalition $S \subseteq N$.

Several hedonic game examples can be given from social life. Allocation of sophomore, junior, and senior university students into dormitory rooms, designation of homework groups in a class, government formation process of political parties after elections, allocation of deputies into parliamentary committees, and Central (Axis)

[1] The hedonic aspect of preferences was introduced by Dréze and Greenberg (1980) in a context of local public goods where individual's preferences rely on their consumption of the public good as well as the coalition they belong to.

Powers and Allied Powers during World War 1 (World War 2) are all hedonic games. In each of the examples, an individual (player) only cares about which individuals (player) are in her group. She does not care how other individuals (players) are grouped. Moreover, each individual (player) can be placed only in one coalition which is also meaningful and consistent with real life. For example, a student can only live in one room in a dormitory, or a political party either takes part in coalitional government or in opposition, but not in each side. Similarly, a country cannot ally with central and axis powers at the same time, which is against the grain of the war.

An outcome of a hedonic game is a partitioning of the set of individuals into mutually disjoint coalitions. An outcome is called as coalition structure. The quality and desirability of an outcome is analyzed using equilibrium concepts. In the literature of hedonic games, an equilibrium concept is called as **stability concept**[2].

When considering and analyzing stability concepts and stable coalition structures, two fundamental issues must be considered. First one is *who deviates from a given coalition structure for what purpose* and the second one is *when a move (deviation) from a coalition structure is subjected to the approval of the individuals whom it affects*. To put these issues more explicitly, an individual may deviate from a given coalition structure with an intention to either stand alone or join an existing coalition. In addition to that, a group of individuals may deviate from a given coalition structure with an intention to either construct a new coalition, or join an existing coalition, or display a more complex behavior. However, these intentional moves may be subject to the consent of a collection of individuals who are not directly involved in these moves but affected by these moves. For example, when incumbents perceive that they will be worse off if entrants join them, they will not welcome

[2] There are several stability concepts in the literature. See Sung and Dimitrov (2007) for classification of all stability concepts which are studied in the literature.

the entrants. When some individuals plan to migrate, if remaining people may get worse off after migration, they may not approve that people's migration.

In the literature of hedonic coalition formation games, there are several stability concepts. They are all defined and analyzed by taking the first issue into consideration. Regrettably, the existence and the importance of the second issue is ruled out in majority of the studies. The second issue is introduced and analyzed in depth in Sertel (1982 and 1992) by introducing four different *membership property right concepts*, namely *free exit*, *approved exit*, *free entry*, and *approved entry*. When a move (deviation) is intended, these concepts initiate four different *membership property right codes*, namely *free exit-free entry* (FX-FE), *free exit-approved entry* (FX-AE), *approved exit-free entry* (AX-FE), and *approved exit-approved entry* (AX-AE) [3] [4]. A membership property right code defines the set of individuals whose approval is needed (for this move to take place) when a group of individuals plan to deviate. Under *free exit-free entry* (FX-FE) membership property right code, deviating individuals have a right to move among coalitions of a given coalition structure without receiving permissions of the coalitions that they leave or join. For example, a family can move their house from one city to another without permission of residents of the two cities. Under *free exit-approved entry* (FX-AE) membership property right code, deviating individuals can move from their current coalitions without any permission of their current coalition, but they can join a new coalition if every member of that coalition welcomes new arrivals. For example, if a researcher in a co-authorship team receives an attractive offer from a new co-authorship team

[3] It must be kept in mind that when defining a stability (equilibrium) concept, membership property right concepts have to be taken into consideration. Defining the set of individuals whose approval is needed for a move to take place means that membership property right codes define the state of the world. These codes mark off the scope of the deviations.

[4] The membership property rights concept has been formulated in an abstract setting by Eren (1993), applied to the production of pure and impure (respectively) public good production problem by Asan and Sanver (2003 and 2015), applied to matching problems by Ozkal-Sanver (2005) and Nizamogullari and Ozkal-Sanver (2011).

and if her move is beneficial both for herself and for the welcoming researchers, she leaves her old team and joins to new one. Under *approved exit-free entry* (AX-FE) membership property right code, deviating individuals can leave their current coalition if every member of that coalition approves their leaving, while their joining to a new coalition does not necessitate a permission of anyone. For example, communities who carry on a business on volunteer basis accept new members without any permission of anybody. However, an individual can leave that kind of communities only if every member of that community permits (e.g. voluntary military service). Lastly, under *approved exit-approved entry* (AX-AE) membership property right code, deviating individuals need to get permissions of the coalitions they leave and they join. For example, let us consider employee-employer couple as a coalition. In order to change a job, an unsatisfied employee firstly has to break her contract with her current coalition (employer) and then has to sign a new contract with a new coalition (employer).

Sertel (2001 and 2003) introduced the notion of *Rechtsstaat*, which is fitted with code of rights, and studied designing its elements. *Rechtsstaat* can be esteemed as a comprehensive tool for economic design. Let N be any set of individuals. Let $\dot{N} = 2^N$ denote the set of all subsets of N, $\ddot{N} = 2^{\dot{N}}$ denote the space of all families of subsets of N, and $\dddot{N} = 2^{\ddot{N}}$ denote the space of families of families. Let $\mathcal{S}$ be the *state space* (such as $r, s, t \in \mathcal{S}$) that the individuals of N can be confronted with. Given $N, \dot{N}, \ddot{N}, \dddot{N}$, and $\mathcal{S}$, a *Rechtsstaat* is any ordered triplet $< \alpha, \beta, \gamma >$ where, α, β, and γ are certain functions called *ability*, *benefit*, and *a code*, respectively, which are defined on $\mathcal{S} \times \mathcal{S}$, *the space of alterations*.

For any alteration $(s, t) \in \mathcal{S} \times \mathcal{S}$, $\alpha : \mathcal{S} \times \mathcal{S} \to \ddot{N}$ gives us the family $\alpha(s, t) \in \ddot{N}$ (or $\alpha(s, t) \subset \dot{N}$) of subsets which are able/capable of altering a state of world s into a state of world t. At any alteration $(s, t) \in \mathcal{S} \times \mathcal{S}$, $\beta : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^{\dot{N}}$ indicates the benefit $\beta^C(s,t) \in \mathbb{R}$ that falls to each subset C of N. When $\beta^C(s,t) > 0$, we say that C benefits from the alteration of s to t, or C is *willing* to alter s to t. When $\beta^C(s,t) = 0$, we say that C *approves* the alteration of s to t, and disapproves the alteration of s to t if $\beta^C(s,t) < 0$. By *a code of rights*, $\gamma : \mathcal{S} \times \mathcal{S} \rightarrow \ddot{N}$ specifies for each alteration $(s,t) \in \mathcal{S} \times \mathcal{S}$, the family $\gamma(s,t) \subset \ddot{N}$ of families of subsets who are given the right to interfere in that alteration. If there is no code family $\mathfrak{C} \in \gamma(s,t)$, each of whose member subsets $C \in \mathfrak{C}$ approves, i.e., every code family owns a disapproving subset, then the alteration (s,t) can not be enacted, even if there is an able subset $H \in \alpha(s,t)$ which is willing to alter s to t (i.e., $\beta^H(s,t) > 0$).

Considering all, Sertel's concept of equilibrium is based on ability, willingness, and approval. The state $s \in \mathcal{S}$ is an *equilibrium* of a *Rechtsstaat* $s =< \alpha, \beta, \gamma >$ iff, for each alteration $(s,t) \in \mathcal{S} \times \mathcal{S}$ of s to a state t where there is a *willing able* subset $H \in \alpha(s,t)$, every code family $\mathfrak{C} \in \gamma(s,t)$ incorporates some subset $C \in \mathfrak{C}$ which is disapproving the alteration, i.e., $\beta^H(s,t) < 0$.

Sertel's work constitutes a roof for several studies in microeconomic theory. All the membership property rights concepts mentioned in this dissertation are derived and customised from the notion of *Rechtsstaat*.

In hedonic coalition formation games, the state space $\mathcal{S}$ is the set of all coalition structures $\prod^N$, i.e., each coalition structure $\pi \in \prod^N$ symbolizes a state of the world. The functions α, β, and γ are defined on the space $\prod^N \times \prod^N$.

Consider any alteration $(\pi^1, \pi^2) \in \prod^N \times \prod^N$. $\alpha(\pi^1, \pi^2)$ are the coalitions which are able/capable of altering coalition structure π^1 into π^2 via some certain move and for each coalition $H \subseteq N$, $\beta^H(\pi^1, \pi^2)$ is interpreted ordinally. Then, willing able coalition $H \in \alpha(\pi^1, \pi^2)$ corresponds to the notion of blocking π^1 via some certain move. When H blocks π^1, $\beta^H(\pi^1, \pi^2) > 0$. When alteration (π^1, π^2) is enacted, coalition T such that $\beta^T(\pi^1, \pi^2) = 0$ does not dissolve, and continue to exist in π^2.

The alteration (π^1, π^2) is subjected to the approval of individuals whom it affects, i.e., $\gamma(\pi^1, \pi^2)$ is the family of families of coalitions who are given the right to interfere in that alteration. Following Sertel (1982 and 1992), we studied the notions *free exit* (FX), *approved exit* (AX), *free entry* (FE), and *approved entry* (AE). Then hte family of families of coalitions $\gamma(\pi^1, \pi^2)$ changes when we assume that the code is *free exit-free entry* (FX-FE), *free exit-approved entry* (FX-AE), *approved exit-free entry* (AX-FE), or *approved exit-approved entry* (AX-AE).

Two stability concepts, namely core (coalitional) stability and Nash stability[5], are predominantly studied in the literature. A coalition structure is Nash stable if no individual can benefit by moving from her current coalition to another existing (possibly empty) coalition. A coalition structure is core stable if there exists no coalition of individuals whose members strictly prefer that coalition to their current coalition. Unfortunately, neither core stable coalition structures nor Nash stable coalition structures exist everywhere (in the domain of all hedonic games).

In this dissertation, we study three stability concepts under different membership property rights codes. We firstly study core stability under FX-FE membership property right codes. We introduce three mutually independent domain conditions and prove that all of the three domain conditions are individually sufficient for the existence of core stable coalition structures. Then, we introduce a new stability concept called *strong coalitional stability* which is a particular refinement of core

[5] The equilibrium notions called *stability* and *free-mobility equilibrium* (or *individual stability*) in the literature of local public good production games (see Conley and Konishi (2002) and Asan and Sanver (2015)) are reflected into the literature of hedonic games as *core stability* and *Nash stability*, respectively. Stability concepts considering individual deviations are analyzed in Bogomolnaia and Jackson (2002) and in Sung and Dimitrov (2007). Note that individual stability under FX-FE membership property right code (respectively, FX-AE, AX-AE) is called Nash stability (respectively, *individual stability*, *contractual individual stability*) by Bogomolnaia and Jackson (2002). Individual stability under AX-FE membership property right code is called *contractual Nash stability* by Sung and Dimitrov (2007). Note that Nash stability implies individual stability and contractual Nash stability. Both individual stability and contractual Nash stability imply contractual individual stability.

stability and Nash stability. We show that under FX-FE membership property right codes, strongly coalitionally stable coalition structures always exist in particular domains. Then, we discuss the relation between strong coalitional stability and other stability concepts under different membership property right codes. Lastly, we study *strong exchange stability* under FX-FE membership property right codes. We discuss the existence of strongly exchange stable coalition structures and associate strong exchange stability with other stability concepts.

In the next section, we elaborate on the literature of hedonic games.

1.1 Literature

The literature of hedonic coalition formation games mainly splits up into two constituent parts. The first part of the literature consists of the theoretical studies which focuses on the analysis of stability concepts and coalition formation rules. The second part incorporates the computational complexity analysis of hedonic coalition formation problems in various domains of hedonic games.

The first part constitutes the vast majority of the literature. Studies involving the first part mainly focuses on revealing and finding out the sufficient and necessary domain conditions mostly for the existence and sometimes for the uniqueness of stable coalition structures for various notions of stability. In addition, there exists some studies which analyze and characterize coalition formation rules in various domains of hedonic games.

There are several stability concepts in the literature. They are formulated considering individual moves or collective moves of group of individuals. We cannot speak of the existence of coalition structures for most of the stability concepts in the full domain (in the set of all hedonic games). Only contractually core stable

(Sung and Dimitrov (2007)), contractually individually stable (Sung and Dimitrov (2007)), and inner stable (Ozbilen (2018)) coalition structures exist in the full domain. In order to find coalition structures which have the characteristics of one of the remaining stabilities, we need to restrict the domain of all hedonic games. Domain conditions/restrictions are imposed either on the individual preferences or on the profile of preferences.

Core stability is one of the stability concepts which is predominantly studied in the literature. A coalition structure is called core stable if there does not show up a coalition S whose members strictly prefer being in S to their current coalition. Unfortunately, core stable coalition structures may not exist in the full domain. Most of the studies in the literature related with core stability seek the domains in which a core stable coalition structure exists or is unique. Although there exist several domain conditions which are sufficient for the existence of core stability, there exist only two domain conditions which are necessary (see Pápai (2004) and Iehlé (2007)).

Banerjee et al. (2001) showed that for various restrictions over profile of preferences, the existence of core stable coalition structures is not guaranteed. Then, they introduced a condition called *weak top coalition property*, which is sufficient for the existence of core stable coalition structures. They also showed that when preferences are strict, if a hedonic game satisfies the stronger version of the weak top coalition property, namely the *top coalition property*, then there exists a unique core stable coalition structure. Bogomolnaia and Jackson (2002) introduced two domain conditions called *ordinal balancedness* and *weak consecutiveness* both of which guarantee the existence of core stable coalition structures. Burani and Zwicker (2003) introduced interesting results related to core stability and *separable preferences*. They showed that the utility functions representing symmetric and additively separable preferences can be decomposed into two components, namely the cardinal component and the ordinal component. They proved that if preferences are represented only by purely cardinal component, then there always exist core stable coalition structures. If preferences are represented only by ordinal component, then there may not exist core stable coalition structures. They also introduced *descending separable preferences* which are weaker than preferences represented by cardinal component and showed that under descending separable preferences there always exist core and Nash stable coalition structures. Cechlárová and Romero-Medina (2001) introduced two preference relations called β*-preferences* and w-*preferences*. They assumed that every individual ranks all individuals (except herself) from the best individual to the worst individual. An individuals β*-preferences* and w*-preferences* are derived from the comparison of the best and the worst individuals of that coalitions, respectively. They showed that when individuals' preferences over individuals are strict, for any hedonic game with β-preferences, there always exist strict core and core stable coalition structures. They also get parallel results when individuals' preferences over individuals are strict and when a hedonic game is with w*-preferences*. Cechlárová and Hadjuková (2004b) studied βw-*preferences* and $w\beta$*-preferences*. They showed that when individuals' preferences over individuals are strict, there always exist core stable coalition structures for hedonic games with βw-preferences and $w\beta$-preferences. Alcalde and Revilla (2004) brought in a natural preference restriction called *top responsiveness* under which core stable coalition structures always exist. Dimitrov and Sung (2007) showed that top responsiveness is sufficient for the existence of strictly core stable coalition structures, as well as core stable coalition structures. Pápai (2004) studied *permissible sets* and that model generalizes the model of hedonic coalition formation games. Pápai investigated the uniqueness of core stable coalition structures in hedonic coalition formation models with *permissible coalitions*. She showed that only the *single lapping hedonic models* have a unique coalition structure for each preference profile, i.e., the single lapping property is both necessary and sufficient condition for the existence of a unique core stable coalition structures in hedonic games with permissible coalitions. Alcalde and Romero-Medina (2006) introduced four mutually independent natural preference restrictions called *union responsiveness*, *intersection responsiveness*, *singularity*, and *essentiality*. They showed that under each condition, core stable coalition structures always exist. Dimitrov et al. (2006) introduced conditions called *appreciation of friends* and *aversion to enemies*, and showed that they are sufficient for existence of strictly core stable and core stable coalition structures, respectively[6]. Sung and Dimitrov (2007) studied the taxonomy of stability concepts. They scrutinized the nativity of several stability concepts. They introduced contractual strict core stability and showed that contractually strictly core stable coalition structures[7] always exist in the full domain. Iehlé (2007) provided the domain condition called *pivotal balancedness* and showed that pivotal balancedness is **necessary** and **sufficient** for the existence of core stable coalition structures. Suzuki and Sung (2010) introduced a domain condition called *bottom refusedness* which is the counterpart of the *top responsiveness* condition of Alcalde and Revilla (2004). Suzuki and Sung showed that hedonic games satisfying bottom refusedness always have core stable coalition structures.

Stability concepts which are explained via individual moves (Nash stability, individual stability, contractual individual stability) are also populously studied in the literature. A coalition structure is called Nash stable if no individual can benefit by moving from her current coalition to another existing (possibly empty) coalition. A coalition structure is called individually stable if it is immune to individual move-

[6] Preferences satisfying appreciation of friends and aversion to enemies are two subclasses of additively separable preferences (see Burani and Zwicker (2003)).

[7] Contractual strict core stability is equivalent to the strict core stability under AX-AE membership property right codes.

ments which benefit the moving player and do not hurt any member of the coalition she joins. A coalition structure is called contractually individually stable if it is immune to individual movements which benefit the moving player and do not hurt any member of the coalition she joins as well as any remaining member of the coalition she left. Bogomolnaia and Jackson (2002) studied Nash stability and individual stability. They showed that *additively separable* and *symmetric* preferences are sufficient for the existence of a Nash stable coalition structure. They proved that if symmetry condition is weakened to *mutuality*, not only Nash stable coalition structures but also individually stable ones may fail to exist even if an additional strong condition called *single-peakedness on a tree* is imposed. Then, they introduced *ordered characteristics* and *consistency* conditions and showed that individually stable coalition structures always exist in the domain of hedonic games satisfying ordered characteristics and consistency. They also proved that contractually individually stable coalition structures always exist in the full domain. Dimitrov and Sung (2006) proved that Nash stable coalition structures always exist when preferences satisfy top responsiveness and mutuality. Sung and Dimitrov (2007) studied contractual Nash stability. A coalition structure is called contractually Nash stable if it is immune to individual movements which benefit the moving player and do not hurt any remaining member of the coalition she left. They proved that contractually Nash stable coalition structures always exist in the domain of hedonic games which satisfy *separability* and *weak mutuality* simultaneously.

There exist some studies in the literature which focus on different stability concepts other than core stability and individual stabilities. Diamantoudi and Xue (2003) studied hedonic games with farsighted and conservative individuals. In their model, individuals have ability to look many steps ahead, and an individual or group of individuals deviate only if each possible ultimate outcome makes the deviating individuals strictly better off. They showed that core stable, Nash stable, and individually stable coalition structures always exist when individuals are endowed with foresight and they are conservative. Karakaya (2011) introduced a stability concept called strong Nash stability. It is one of the strongest stability concepts in the literature. Karakaya showed that *top choice property* is sufficient condition for the existence of strongly Nash stable coalition structures. Then, he analyzed the relation between strong Nash stability and other stability concepts under different membership property right codes.

The domain conditions introduced by Banerjee et al. (2001), Bogomolnaia and Jackson (2002), Iehlé (2007), Pápai (2004), and Karakaya (2011) impose restrictions on the preference profile. For this reason, it is very difficult to identify whether individuals' preferences satisfy these conditions or not.

The domain conditions introduced by Cechlárová and Romero-Medina (2001), Burani and Zwicker (2003), Cechlárová and Hadjuková (2003, 2004a, and 2004b), Alcalde and Revilla (2004), Alcalde and Romero-Medina (2006), Dimitrov et al. (2006), Suzuki and Sung (2010), and the new domain conditions introduced in this thesis ($\mathcal{A}$-*responsiveness*, $\mathcal{B}$-*responsiveness*, and $\mathcal{G}$-*singularity*) impose restrictions on each individuals' preferences, i.e., preference profiles can be expressed as a Cartesian product of individuals' preferences. This fact produces two interesting advantages. Firstly, it is very easy to identify whether or not individuals' preferences satisfy our conditions. Hence, it becomes possible to design efficient algorithms which select stable coalition structures for various stability concepts. Secondly, when conditions are stated on individuals' preferences, it becomes very easy to study whether introducing a new individual into the problem introduces instabilities.

There are also some studies which analyze coalition formation rules. A coalition formation rule is a correspondence from a particular domain of hedonic games to the set of all coalition structures which associate each hedonic coalition formation game with a non-empty set of coalition structures. Pápai (2004) showed that given an initial set of coalitions that satisfy the single-lapping property, the associated single-lapping rule is the unique coalition formation rule which satisfies strategy-proofness, individual rationality, and constrained efficiency when individuals are restricted to prefer any coalition in the initial set to any other coalition. Alcalde and Revilla (2004) studied the existence of strategy-proof stable mechanisms. They proved that when individuals preferences satisfy top-responsiveness, the mechanism induced by the *top-covering algorithm* is the only strategy-proof mechanism that always selects core stable coalition structures. Rodriguez-Alvarez (2009) focused on strategy-proof coalition formation rules on the domain of additively separable hedonic games. He characterized the family of single-lapping rules that satisfy strategy-proofness, individual rationality, and constrained efficiency in the domain of additively separable hedonic games. Takayima (2013) studied hedonic coalition formation problems from the viewpoint of mechanism design under the model of permissible coalitions. Takayima proved that if the mechanism is strategy-proof and respects coalitional unanimity, then for each preference profile that mechanism selects the unique strictly core stable coalition structure whenever it is available. Takayima also proved that if the requirement of strategy-proofness is strengthened to coalition strategy-proofness in the above result, then for every preference profile there exists only one strictly core stable coalition structure and the mechanism chooses that coalition structure. Karakaya and Klaus (2017) showed that on the domain of solvable hedonic games, core is characterized by coalitional unanimity and Maskin monotonicity. They also showed that core can be characterized by unanimity, Maskin monotonicity, and either competition sensitivity or resource sensitivity in the domain of solvable hedonic games. Moreover, they proved that there exists a coalition formation rule which is not equal to core and can be characterized by unanimity, consistency, competition sensitivity, and resource sensitivity.

The second part of the literature is about computational complexity analysis of hedonic coalition formation problems in various domains. Studies on computational complexity mainly focus on reaching stable coalition structures or confirming stability of coalition structures in various domains of hedonic games for various stability notions. Ballester (2004) studied computational complexity of hedonic games with arbitrary preferences in the domain of all hedonic games. He showed that the problems of deciding the existence of a core stable, a Nash stable, or an individually stable coalition structure are *NP-hard*. He also showed that these decision problems remain *NP-hard* even if individuals' preferences are assumed to be *strictly anonymous*. Dimitrov et al. (2006) studied computational complexity of hedonic games with two special classes of separable preferences. They proved that under friends appreciation, a strictly core stable coalition structure can be reached in *polynomial time*, whereas finding a core stable coalition structure under aversion to enemies is *NP-hard*. Sung and Dimitrov (2010) proved that checking whether a Nash stable, an individually stable, or a core stable coalition structure exists in an additively separable hedonic game is *NP-hard*. Cechlárova and Romero-Medina (2001) proved that when individuals' preferences over individuals are strict, for any hedonic game with β-preferences, a strictly core stable coalition structure can be found by a polynomial time algorithm called *B-stable*. Cechlárová and Hajuková (2003) proved that finding a core stable or a strictly core stable coalition structure for hedonic games with β-preferences become *NP-complete* when indifferences among individuals are allowed. Cechlárová and Hajuková (2004a) showed that when individuals' preferences over individuals are strict, for any hedonic game with w-preferences, there exists a polynomial time algorithm for finding strictly core stable or core stable coalition structures. Detailed analysis of computational complexities of various hedonic coalition formation problems can be found in Aziz and Savani (2016).

In addition, there exist some studies in the literature which can be formulated as hedonic games. Darmann et al. (2012) introduced *group activity selection problem* which is a generalization of anonymous hedonic games. In their model, each individual participates in at most one activity, and individuals preferences over activities depend on the number of participants in the activity. The outcome of a group activity selection problem is partitioning the set of individuals into activities with respect to their preferences over activity and group size pair. Spradling et al. (2013) introduced a new variant of hedonic games called *roles and teams hedonic games*. In that model, individuals have two levels of preference on their own coalitions, namely preference on the set of roles that makes up the coalition and preference on their own role with in the coalition. Lee and Shoahm (2015) introduced *stable invitations problems* which can be formulated as a hedonic game as follows. Consider an organizer who is trying to convene an event and needs to choose whom should be invited out of a given set of individuals. Each individual has preferences over how many attendees should be invited and who the attendees are. The organizer aims to find an invitation of maximum size with respect to preferences of individuals.

In this dissertation, we have introduced two new equilibrium (stability) concepts, namely, strong coalitional stability and strong exchange stability. Strong coalitional stability is a particular refinement of the core stability and the Nash stability. Thus, it corrects the deficiencies of them. Strong exchange stability is the minimal refinement of the core stability and the exchange stability. In the same manner, strong exchange stability corrects the deficiencies of the core stability and the exchange stability. To our knowledge, strong coalitional stability, exchange stability, and strong exchange stability are not defined and studied before. Moreover, they generalize the coalitional (core) stability, individual stability, and exchange stability concepts in one sided matching games, two sided one-to-one, one-to-many and many-to-many matching games.

We have introduced three new and independent domain restrictions, namely, $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity. $\mathcal{A}$-responsiveness and $\mathcal{B}$-responsiveness conditions are born for the first time in this study. They are particular monotonicity conditions. $\mathcal{G}$-singularity condition generalizes the singularity definition of Alcalde and Romero-Medina (2006). All three domain conditions are imposed on individuals' preferences. Using this nice property, we proved the possibility of extending the domain of existence when we combine individuals with these three preference restrictions together. Moreover, this property allowed us to define five efficient and finite time algorithms which run on individual preferences and look for stable coalition structures.

Furthermore, obeying the advises of Sertel (1982 and 1992) we incorporated membership property rights explicitly into the definition of stability concepts. We studied strong coalitional stability and strong exchange stability under different membership property rights.

Considering all of these, besides its significant contribution to the literature of hedonic games, this study lighted the fuse of several new research questions. For example, computational complexity analysis of finding stable coalition structures for various stability concepts in $\mathcal{A}$-responsive, $\mathcal{B}$-responsive, or $\mathcal{G}$-singular domains is still an intact question. Moreover, efficiency analysis of $\mathcal{A}1$, $\mathcal{B}1$, $\mathcal{B}2$, $\mathcal{G}1$, and $\mathcal{G}2$ algorithms are reserved for future study. Moreover, the characterization of coalition formation rules and strategy-proofness property of coalition formation rules which always bring core stable (strongly coalitionally stable, strongly exchange stable) coalition structures in $\mathcal{A}$-responsive, $\mathcal{B}$-responsive, or $\mathcal{G}$-singular domains are still intact questions.

1.2 Definitions and Notation

Let $\mathbb{Z}^+$ be the set of potential individuals (players). Let $\mathcal{N}$ be the set of all non-empty finite subsets of $\mathbb{Z}^+$, i.e., $\mathcal{N} = \{N \subset \mathbb{Z}^+ | 0 < |N| < \infty\}$[8]. We will work with finite set of individuals, i.e., we will work with any $N \in \mathcal{N}$.

For any $N = \{1, 2, ..., n\} \in \mathcal{N}$:

- cardinality of N will be represented by $|N| = n$.
- a non-empty subset S of N is called a *coalition* of N.
- lower-case letters i, j, k will symbolize representative individuals from N, i.e., $i, j, k \in N$.
- upper-case letters $H, S, T, U, V, ...$ will symbolize coalitions of N.
- $\mathcal{C} = 2^N \backslash \{\emptyset\}$ will denote the set of all non-empty coalitions of N.
- for any $V \in \mathcal{C}$, $\mathcal{C}|_V = 2^V \backslash \{\emptyset\}$ will denote the set of all non-empty coalitions of V.

Consider any $N \in \mathcal{N}$. For any individual $i \in N$, $\mathcal{C}_i^N = \{S \subseteq N | i \in S\}$ will denote the set of all coalitions of N containing the individual i. We assume that every individual $i \in N$ is endowed with a preference $\succsim_i$ over $\mathcal{C}_i^N$, i.e., a binary relation over $\mathcal{C}_i^N$ which is complete and transitive[9]. For any individual $i \in N = \{1, 2, ..., n\}$, $R(\mathcal{C}_i^N)$ will denote the set of all weak orders over $\mathcal{C}_i^N$. The vector $\succsim = \{\succsim_1, \succsim_2, ..., \succsim_n\} \in \mathcal{R}^N$ is called *preference profile* where $\mathcal{R}^N = R(\mathcal{C}_1^N) \times R(\mathcal{C}_2^N) \times ... \times R(\mathcal{C}_n^N)$ is the set of all preference profiles. For any $i \in N$ and $\succsim_i \in R(\mathcal{C}_i^N)$ and for

[8] $A \subset B$ means that *A is a proper subset of B* and $A \subseteq B$ means that *A is a subset of B such that A can be equal to B*.

[9] A binary relation $\succsim_i$ is *complete* if and only if $\forall S, T \in \mathcal{C}_i^N$, $S \succsim_i T$ or $T \succsim_i S$ and *transitive* if and only if $\forall S, T, U \in \mathcal{C}_i^N$, if $S \succsim_i T$ and $T \succsim_i U$ then $S \succsim_i U$.

any $S, T \in \mathcal{C}_i^N$, $S \succ_i T$ denotes that i *strictly prefers* S *to* T, $S \succsim_i T$ denotes that i *weakly prefers* S *to* T, and $S \sim_i T$ denotes that i *is indifferent between* S *and* T. Let $S \in \mathcal{C}_i^N$. If $S \succsim_i \{i\}$, we say that individual i finds coalition S *acceptable* and if $\{i\} \succ_i S$, we say that individual i finds coalition S *unacceptable*.

Consider any $N \in \mathcal{N}$. For any $S \in \mathcal{C}$, $i \in S$, and $\succsim_i \in R(\mathcal{C}_i^N)$, $CH(i, S) = \{T \in \mathcal{C}_i^S | \forall U \in \mathcal{C}_i^S, T \succsim_i U\}$ denotes the set of all maximal sets of individual i from the coalition S under $\succsim_i$. When $|CH(i, S)| = 1$, $ch(i, S)$ will denote the unique maximal set of individual i from S under $\succsim_i$. Notice that, if individual i has strict preferences, $CH(i, S)$ is always singleton.

A **hedonic coalition formation game**, or simply a **hedonic game**, is a pair $(N, \succsim)$ which consists of a preference profile $\succsim \in \mathcal{R}^N$ and a set of individuals $N \in \mathcal{N}$. The outcome of a hedonic coalition formation game is called as **coalition structure**. A coalition structure π of a hedonic game $(N, \succsim)$ is a set $\pi = \{T_1, T_2, ..., T_K\}$ ($K \leq |N|$ is a positive integer) such that π partitions the individuals' set N [10]. For any $i \in N$ and any $S \in \mathcal{C}$, $\pi(i)$ denotes the unique coalition in π which includes individual i and by $\pi(S)$, we denote the collection of coalitions $\pi(i)$, i.e., $\pi(S) = \{\pi(i)\}_{i \in S}$.

For any $N \in \mathcal{N}$, we let $\prod^N$ stand for all possible partitions of N. $\aleph = \{N\}$ will denote the **grand coalition** and $\daleth = \{\{1\}, \{2\}, ..., \{n\}\}$ will denote the coalition structure which consists of singletons. Given any coalition structure π, we will represent a coalition $T \in \pi$ via round brackets, i.e., if T consists of individuals i, j, and k, we will write $T = (ijk)$ or sometimes $T = (i, j, k)$.

[10] $\pi = \{T_1, T_2, ..., T_K\}$ partitions N if and only if (i) $\forall l \in \{1, 2, ..., K\}$ $T_l \neq \emptyset$, (ii) $\forall l, m \in \{1, 2, ..., K\}$ with $l \neq m$, $T_l \cap T_m = \emptyset$, and (iii) $\bigcup_{l=1}^{K} T_l = N$.

1.3 Stability Concepts

Definition 1.3.1. *Let* $(N, \succsim)$ *be a hedonic coalition formation game. Let* $\pi \in \prod^N$ *be a coalition structure.*

- *Individual* $i \in N$ ***blocks*** *the coalition structure* π *if* $\exists S \in (\pi \cup \{\emptyset\})$ *such that* $S \cup \{i\} \succ_i \pi(i)$.
- *A coalition* $S \subseteq N$ ***blocks*** *the coalition structure* π *if every individual* $i \in S$ *strictly prefers* S *to his current coalition* $\pi(i)$, *i.e.,* $\forall i \in S$: $S \succ_i \pi(i)$.
- *A coalition* $S \subseteq N$ ***weakly blocks*** *the coalition structure* π *if every individual* $i \in S$ *weakly prefers* S *to* $\pi(i)$ *and there exists at least one individual* $j \in S$ *who strictly prefers* S *to his current coalition* $\pi(j)$, *i.e.,* $\forall i \in S$: $S \succsim_i \pi(i)$ *and* $\exists j \in S$: $S \succ_j \pi(j)$.

Definition 1.3.2.

- *A coalition structure* π *is called* ***Nash stable*** (NS) *if it is not blocked by any individual* $i \in N$.
- *A coalition structure* π *which admits no blocking coalition is said to be* ***core stable*** (CS).
- *A coalition structure* π *which admits no weakly blocking coalition is said to be* ***strictly core stable*** (SCS).
- *A coalition structure* π *is called* ***individually rational*** (IR) *if no individual has an incentive to become alone, i.e.,* $\forall i \in N, \pi(i) \succsim_i \{i\}$.
- *A coalition structure* π *is called* ***Pareto optimal*** (PO) *if there exists no coalition structure* $\pi' \in \prod^N$ *such that* $\pi'(j) \succsim_j \pi(j)$ *for all* $j \in N$ *and* $\pi'(i) \succ_i \pi(i)$ *for some* $i \in N$.

- *A coalition structure π is called **weakly Pareto optimal** (wPO) if there exists no coalition structure $\pi' \in \prod^N$ such that $\pi'(i) \succ_i \pi(i)$ for all $i \in N$.*

We observe that, strict core stability implies core stability and Pareto optimality, and core stability implies individual rationality. However, core stability and Pareto optimality do not follow from each other. Moreover, core stability and Nash stability do not have any implication relation inter se. The following example reveals the relation between above concepts.

Example 1.3.1. *Consider the hedonic game $(N, \succsim)$ with $N = \{1, 2, 3\}$ and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$
12	*12 $\sim$ 123*	*123*
13	*23*	*13*
123	*2*	*23*
1		*3*

We observe that the coalition structure $\pi^1 = \{(12), (3)\}$ is strictly core stable, however it is not Nash stable. Individual 3 by joining the existing coalition (12) blocks π^1. The coalition structure $\pi^2 = \{(123)\}$ is both Nash stable and core stable, though it is not strictly core stable. The coalition (12) weakly blocks π^2. The coalition structure $\pi^3 = \{(13), (2)\}$ is neither core stable nor Nash stable, but it is Pareto optimal. The coalition (12) by coming together as well as the individual 2 by joining (13) block π^3. $\pi^4 = \{(1), (23)\}$ is purely and simply individually rational. Individual 1 might block it by joining the existing coalition (23), coalition (12) might block it by coming together, or the coalition structure $\pi^2 = \{(123)\}$ Pareto dominates it.

2. CORE STABILITY: SOME DOMAIN RESTRICTIONS

2.1 Core Stability

In this chapter, we study core stability. Core stability is one of the stability concepts which is predominantly studied in the literature. A coalition structure is called core stable if there does not show up a coalition S whose members strictly prefer being in S to their current coalition. Unfortunately, core stable coalition structures may not exist in the full domain. Most of the studies in the literature related with core stability seek the domains in which a core stable coalition structure exists or is unique. Although there exist several domain conditions which are sufficient for the existence of core stability, there exist only two domain conditions which are necessary (see Pápai (2004) and Iehlé (2007)).

Banerjee et al. (2001) showed that for various restrictions over profile of preferences, the existence of core stable coalition structures is not guaranteed. Then, they introduced a condition called *weak top coalition property*, which is sufficient for the existence of core stable coalition structures. They also showed that when preferences are strict, if a hedonic game satisfies the stronger version of the weak top coalition property, namely the *top coalition property*, then there exists a unique core stable coalition structure. Bogomolnaia and Jackson (2002) introduced two domain conditions called *ordinal balancedness* and *weak consecutiveness* both of which guarantee the existence of core stable coalition structures. Burani and Zwicker

(2003) introduced interesting results related to core stability and *separable preferences*. They showed that the utility functions representing symmetric and additively separable preferences can be decomposed into two components, namely the cardinal component and the ordinal component. They proved that if preferences are represented only by purely cardinal component, then there always exist core stable coalition structures. If preferences are represented only by ordinal component, then there may not exist core stable coalition structures. They also introduced *descending separable preferences* which are weaker than preferences represented by cardinal component and showed that under descending separable preferences there always exist core and Nash stable coalition structures. Cechlárová and Romero-Medina (2001) introduced two preference relations called β*-preferences* and w*-preferences*. They assumed that every individual ranks all individuals (except herself) from the best individual to the worst individual. An individuals β*-preferences* and w*-preferences* are derived from the comparison of the best and the worst individuals of that coalitions, respectively. They showed that when individuals' preferences over individuals are strict, for any hedonic game with β-preferences, there always exist strict core and core stable coalition structures. They also get parallel results when individuals' preferences over individuals are strict and when a hedonic game is with w*-preferences*. Cechlárová and Hadjuková (2004) studied βw*-preferences* and $w\beta$*-preferences*. They showed that when individuals' preferences over individuals are strict, there always exist core stable coalition structures for hedonic games with βw-preferences and $w\beta$-preferences. Alcalde and Revilla (2004) brought in a natural preference restriction called *top responsiveness* under which core stable coalition structures always exist. Dimitrov and Sung (2007) showed that top responsiveness is sufficient for the existence of strictly core stable coalition structures, as well as core stable coalition structures. Pápai (2004) studied *permissible sets* and that model generalizes the model of hedonic coalition formation games. Pápai investigated the uniqueness of core stable coalition structures in hedonic coalition formation models with *permissible coalitions*. She showed that only the *single lapping hedonic models* have a unique coalition structure for each preference profile, i.e., the single lapping property is both necessary and sufficient condition for the existence of a unique core stable coalition structures in hedonic games with permissible coalitions. Alcalde and Romero-Medina (2006) introduced four mutually independent natural preference restrictions called *union responsiveness*, *intersection responsiveness*, *singularity*, and *essentiality*. They showed that under each condition, core stable coalition structures always exist. Dimitrov et al. (2006) introduced conditions called *appreciation of friends* and *aversion to enemies*, and showed that they are sufficient for existence of strictly core stable and core stable coalition structures, respectively. Sung and Dimitrov (2007) studied the taxonomy of stability concepts. They scrutinized the nativity of several stability concepts. They introduced contractual strict core stability and showed that contractually strictly core stable coalition structures always exist in the full domain. Iehlé (2007) provided the domain condition called *pivotal balancedness* and showed that pivotal balancedness is **necessary** and **sufficient** for the existence of core stable coalition structures. Suzuki and Sung (2010) introduced a domain condition called *bottom refusedness* which is the counterpart of the *top responsiveness* condition of Alcalde and Revilla (2004). Suzuki and Sung showed that hedonic games satisfying bottom refusedness always have core stable coalition structures.

In this chapter, we introduce three new and independent domain conditions, namely $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity. We prove that all the three domain conditions are sufficient for the existence of core stable coalition structures. Moreover, we prove that it is possible to extend the domain of existence via combining individuals with $\mathcal{A}$-responsive, $\mathcal{B}$-responsive, and $\mathcal{G}$-singular preferences together, i.e., if a set of individuals is partitioned into three subsets such that every individual in the first subset has $\mathcal{A}$-responsive preferences, every individual in the second subset has $\mathcal{B}$-responsive preferences, and every individual in the last subset has $\mathcal{G}$-singular preferences, there always exist a core stable coalition structure. All the domain conditions that we introduce in this paper are directly imposed on preferences of individuals. This nice property of preferences enables us to design four nice algorithms which work on individuals' preferences and look for core stable coalition structures.

We start by remembering definitions of core stability and strict core stability. Then, we introduce the domain conditions and the algorithms that work on these domains. Afterwards, we prove our propositions. The last chapter is dedicated to final remarks and open questions.

2.2 Definitions and Notation

Definition 2.2.1. *Let $(N, \succsim)$ be a hedonic coalition formation game. Let $\pi \in \prod^N$ be a coalition structure.*

- *A coalition $S \subseteq N$ **blocks** the coalition structure π if every individual $i \in S$ strictly prefers S to his current coalition $\pi(i)$, i.e., $\forall i \in S$: $S \succ_i \pi(i)$.*
- *A coalition $S \subseteq N$ **weakly blocks** the coalition structure π if every individual $i \in S$ weakly prefers S to $\pi(i)$ and there exists at least one individual $j \in S$ who strictly prefers S to his current coalition $\pi(j)$, i.e., $\forall i \in S$: $S \succsim_i \pi(i)$ and $\exists j \in S$: $S \succ_j \pi(j)$.*

Definition 2.2.2.
- *A coalition structure π which admits no blocking coalition is said to be **core stable** (CS).*

- *A coalition structure π which admits no weakly blocking coalition is said to be* ***strictly core stable*** *(SCS).*

2.3 Main result

Definition 2.3.1. *Let $(N, \succsim)$ be a hedonic game and $i \in N$ be any individual. $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{A}$-**responsive** iff*

- $|CH(i, N)| = 1$,
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \subseteq S \subset T$ we have $S \succ_i T$,*
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \subseteq S$, $ch(i, N) \subseteq T$, and $\mid S \mid = \mid T \mid$ we have $S \sim_i T$,*
- *For all $S \in \mathcal{C}_i^N$ such that $ch(i, N) \backslash S \neq \emptyset$, we have $N \succsim_i S$*

*We say that $\succsim \in \mathcal{R}^N$ satisfies $\mathcal{A}$-responsiveness iff $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{A}$-responsive $\forall i \in N$. A hedonic game $(N, \succsim)$ with $\mathcal{A}$-responsive preference profile $\succsim \in \mathcal{R}^N$ is said to satisfy $\mathcal{A}$-**responsiveness**.*

$\mathcal{A}$-responsiveness is a kind of monotonicity condition which regards top preferences. If an individual i has $\mathcal{A}$-responsive preferences, he puts $ch(i, N)$ to top and places coalitions including $ch(i, N)$ under $ch(i, N)$ with respect to increasing cardinality. All the remaining coalitions are placed weakly below the grand coalition. If two coalitions which contain $ch(i, N)$ have the same cardinality, then individual i is indifferent between them. If $ch(i, N) = \{i\}$, then preferences of individual i just becomes size monotonic in increasing order. Hedonic games satisfying *$\mathcal{A}$-responsiveness* always have core stable coalition structures.

Example 2.3.1. *Consider the hedonic game $(N, \succsim)$ with $N = \{1, 2, 3, 4\}$ and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
123	24	13	124
1234	124 $\sim$ 234	123 $\sim$ 134	1234 $\sim$ 4
12 $\sim$ 13 $\sim$ 14	1234	1234	24
134 $\sim$ 124	23 $\sim$ 123	23	134
1	2 $\sim$ 12	234	14 $\sim$ 34
		3	234
		34	

The above hedonic game $(N, \succsim)$ satisfies $\mathcal{A}$-responsiveness.

Now, we are going to prove the theorem "**If a hedonic game $(N, \succsim)$ satisfies $\mathcal{A}$-responsiveness, then it has a core stable coalition structure.**". Before we prove this theorem, we firstly define an algorithm called $\mathcal{A}1$ **Algorithm**. In the domain of $\mathcal{A}$-responsive hedonic games, $\mathcal{A}1$ **Algorithm** always brings core stable coalition structures.

We start by defining the following recursive function which comprises a basis for the algorithm.

Let $t \in \mathbb{Z}^+$ be a positive integer. For every $i \in N$ and $S \in \mathcal{C}_i^N$, let define the function $\mathcal{X}^t : N \times \mathcal{C} \to \mathcal{C}$ as follows:

$\mathcal{X}^1(i, S) = ch(i, S)$.

$\mathcal{X}^{t+1}(i, S) = \bigcup_{j \in \mathcal{X}^t(i,S)} ch(j, S)$ for each positive integer t.

By definition, $\mathcal{X}^t(i, S) \subseteq \mathcal{X}^{t+1}(i, S) \subseteq S$ and $\mathcal{X}^{|N|+1}(i, S) = \mathcal{X}^{|N|}(i, S)$. By $\mathcal{X}\mathcal{X}(i, S)$ we denote $\mathcal{X}^{|N|}(i, S)$.

$\mathcal{A}1$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{A}$-responsiveness.

Step 1: Set $\mathcal{P}^1 := N$ and $\pi^0 := \emptyset$.

Step 2: For $m = 1$ to $|N|$:

Step 2.1: Select $i \in \mathcal{P}^m$ satisfying $|\mathcal{XX}(i, N)| \leq |\mathcal{XX}(j, N)|$ for each $j \in N$.

Set $S^m := \mathcal{XX}(i, N)$.

Step 2.2: If $S^m \subseteq \mathcal{P}^m$, set $\pi^m := \pi^{m-1} \cup \{S^m\}$ and $\mathcal{P}^{m+1} := \mathcal{P}^m \backslash S^m$.

Step 2.3: If $S^m \nsubseteq \mathcal{P}^m$, set $\pi^m := \pi^{m-1} \cup \{(i)\}_{i \in \mathcal{P}^m}$ and $\mathcal{P}^{m+1} = \mathcal{P}^m$.

Go to Step 3.

Step 3: Select $H \subseteq \mathcal{P}^{m+1}$ such that $\forall i \in H : H \succsim_i \{i\}$ and $H \succ_i H^{'}$ $\forall i \in H \cap H^{'}, \forall H^{'} \in \mathcal{P}^{m+1}$.

Set $S^{m+1} := H$ and $\pi^{m+1} := \pi^{m-1} \cup \{S^{m+1}\} \cup \{(i)\}_{i \in \mathcal{P}^{m+1} \backslash S^{m+1}}$.

Return π^{m+1} as outcome.

Step 4: If $\nexists H \subseteq \mathcal{P}^{m+1}$ such that for every $i \in H$: $H \succsim_i \{i\}$, return π^m as outcome.

We have finite number of individuals, thus $\mathcal{A}1$ algorithm terminates in finite steps. There exists $K \in \{1, ..., |N|\}$ such that π^K is the outcome of the algorithm. π^K incorporates (if exists) mutually best coalitions, *least common supersets*[1] of maximal sets, coalitions which are *mutually acceptable*[2], and of course singletons. $\mathcal{A}1$ algorithm firstly detects and separates out the individuals whose maximal sets are singletons, i.e., $ch(i, N) = \{i\}$. Then, if exists, it detects mutually best coalitions and continues to separate them out. Subsequently, it brings out least common supersets. In step 3, the algorithm detects mutually acceptable sets above $\{i\}$ and

[1] Let A and B be arbitrary sets. Set C is the **least common superset** of A and B iff $C = A \cup B$. Set D is the **greatest common subset** of A and B iff $D = A \cap B$.

[2] A coalition $S \in \mathcal{C}$ is a **mutually acceptable** coalition iff $S \succ_i \{i\}$ for every $i \in S$.

separates them out. Eventually, remnant individuals stand alone. All these steps are guaranteed by hierarchy in preferences which is given away by $\mathcal{A}$-responsiveness.

Remark 2.3.1. *If we run $\mathcal{A}1$ Algorithm more than ones, we might get different coalition structures which are still core stable. This opportunity is provided by the weak preferences and the freedom of choice in step 2.1 of the algorithm. We might have more than one coalition which satisfies the criterion "$|\mathcal{X}\mathcal{X}(i,N)| \leq |\mathcal{X}\mathcal{X}(j,N)|$ for each $j \in N$". Selection of different coalitions in step 2.1 initiate algorithm to bring out different coalition structures.*

Let us explain the $\mathcal{A}1$ Algorithm and the above recursive function in the following example.

Example 2.3.2. *Consider the Example 2.2.1 .*

We observe that:

$\mathcal{X}^1(1,N) = ch(1,N) = (123)$,
$\mathcal{X}^2(1,N) = \cup_{i \in ch(1,N)} ch(i,N) = ch(1,N) \cup ch(2,N) \cup ch(3,N) = (1234)$,
$\mathcal{X}^2(1,N) = \mathcal{X}^3(1,N) = \mathcal{X}^4(1,N)$. *Thus,* $\mathcal{X}\mathcal{X}(1,N) = (1234)$.

$\mathcal{X}^1(2,N) = ch(2,N) = (24)$,
$\mathcal{X}^2(2,N) = \cup_{i \in ch(2,N)} ch(i,N) = ch(2,N) \cup ch(4,N) = (24)$,
$\mathcal{X}^2(2,N) = \mathcal{X}^3(2,N) = \mathcal{X}^4(2,N)$. *Thus,* $\mathcal{X}\mathcal{X}(2,N) = (24)$.

$\mathcal{X}^1(3,N) = ch(3,N) = (13)$,
$\mathcal{X}^2(3,N) = \cup_{i \in ch(3,N)} ch(i,N) = ch(1,N) \cup ch(3,N) = (123)$,
$\mathcal{X}^3(3,N) = \cup_{i \in \mathcal{X}^2(3,N)} ch(i,N) = ch(1,N) \cup ch(2,N) \cup ch(3,N) = (1234)$,
$\mathcal{X}^3(3,N) = \mathcal{X}^4(3,N)$. *Thus,* $\mathcal{X}\mathcal{X}(3,N) = (1234)$.

$\mathcal{X}^1(4,N) = ch(4,N) = (24)$,
$\mathcal{X}^2(4,N) = \cup_{i \in ch(4,N)} ch(i,N) = ch(2,N) \cup ch(4,N) = (24)$,

$\mathcal{X}^2(4,N) = \mathcal{X}^3(4,N) = \mathcal{X}^4(4,N)$. Thus, $\mathcal{XX}(4,N) = (24)$.

If we run $\mathcal{A}1$ Algorithm:

$\mathcal{P}^1 := N = \{1,2,3,4\}$ and $\pi^0 := \emptyset$.
For $m = 1$, for individuals 2 and 4, we observe that $|\mathcal{XX}(2,N)| \leq |\mathcal{XX}(j,N)|$ for each $j \in N$ and $|\mathcal{XX}(4,N)| \leq |\mathcal{XX}(j,N)|$ for each $j \in N$.
Let choose $\mathcal{XX}(2,N)$ and set $S^1 = \mathcal{XX}(2,N)$.
$S^1 = \mathcal{XX}(2,N) \subseteq \mathcal{P}^1$, so that set $\pi^1 = \{S^1\} = \{(24)\}$.
Now, $m = 2$.
$\mathcal{P}^2 = \mathcal{P}^1 \backslash S^1 = \{1,3\}$. But $\mathcal{XX}(1,N)$ and $\mathcal{XX}(3,N)$ do not belong to $\mathcal{P}^2$.
Now, $m = 3$.
Then, set $\mathcal{P}^3 = \mathcal{P}^2$ and $\pi^3 = \pi^1 \cup \{(1)\} \cup \{(3)\}$.
Consider all $H \subseteq \mathcal{P}^3 = \{1,3\}$ such that $\forall i \in H : H \succsim_i \{i\}$ and $H \succ_i H'$ $\forall i \in H \cap H', \forall H' \in \mathcal{P}^3$.
(13) obeys the criterion. Set $S^3 = (13)$. Then, $\pi^3 = \{(24),(13)\}$.
Now, $m = 4$. $\mathcal{P}^4 = \emptyset$.

Therefore, $\pi^3 = \{(24),(13)\}$ is the final outcome of the $\mathcal{A}1$ Algorithm. π^3 is core stable.

Theorem 2.3.1. *If a hedonic game $(N, \succsim)$ satisfies $\mathcal{A}$-responsiveness, then it has a core stable coalition structure.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{A}$-responsiveness. Let π^K be the outcome of the $\mathcal{A}1$ algorithm. At the very early iterations of the algorithm, individuals whose maximal sets are singletons are separated out. For every such individual i, for every $H \in \mathcal{C}_i^N$, we have $\{i\} \succ_i H$. Then, algorithm selects mutually best coalitions. Thus, if S^m is a mutually best coalition which is selected in the m-th iteration, we know that $S^m = \pi^K(i) \succ_i H$ for every $i \in S^m$ and for every $H \in \mathcal{C}_i^N$.

In the subsequent iterations, least common supersets are extracted. Observe that, every least common superset in the m-th iteration are mutually independent. The opposite case is not probable, because it contradicts with the first condition of $\mathcal{A}$-responsiveness. If S^m is the selected least common superset, then for every $i \in S^m$, $S^m \succ_i T$ for all $T \supset S^m$ and for every T which is weakly ranked below N (guaranteed by the *2nd* and *3rd* conditions of $\mathcal{A}$-responsiveness). Therewithal, there exists no $H \subset S^m$ such that $H \succ_j S^m$ for every $j \in H \cap S^m$. Otherwise, it contradicts with S^m being one of the least common supersets which is selected in the m-th iteration. Then, mutually acceptable coalitions, which are above singletons are extracted. Ultimately, individuals who are not settled by the algorithm stand alone. We observe that, there exists no $H \in \mathcal{C}$ such that $H \succ_i \pi^K(i)$ for every $i \in H$. Hence, π^K is core stable.

□

Definition 2.3.2. *Let $(N, \succsim)$ be a hedonic game and $i \in N$ be any individual. $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{B}$-**responsive** iff*

- $|CH(i, N)| = 1$,
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \supseteq S \supset T$ we have $S \succ_i T$,*
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \supseteq S$, $ch(i, N) \supseteq T$, and $| S |=| T |$ we have $S \sim_i T$,*
- *For all $S \in \mathcal{C}_i^N$ such that $S \backslash ch(i, N) \neq \emptyset$, we have $\{i\} \succsim_i S$*

*We say that $\succsim \in \mathcal{R}^N$ satisfies $\mathcal{B}$-responsiveness iff $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{B}$-responsive $\forall i \in N$. A hedonic game $(N, \succsim)$ with $\mathcal{B}$-responsive preference profile $\succsim \in \mathcal{R}^N$ is said to satisfy $\mathcal{B}$-**responsiveness**.*

$\mathcal{B}$-responsiveness is a kind of monotonicity condition regarding top preferences. If an individual i has $\mathcal{B}$-responsive preferences, he puts $ch(i, N)$ to top and places coalitions which are subsets of $ch(i, N)$ under $ch(i, N)$ with respect to decreasing cardinality. All the remaining coalitions are placed weakly below the coalition $\{i\}$. If two proper subsets of $ch(i, N)$ have the same cardinality, then individual i is indifferent between them. If $ch(i, N) = \{N\}$, then preferences of individual i just becomes size monotonic in decreasing order. Hedonic games satisfying *$\mathcal{B}$-responsiveness* always have core stable coalition structures.

Example 2.3.3. *Consider the hedonic game $(N, \succsim)$ with $N = \{1, 2, 3, 4\}$ and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
123	*24*	*134*	*1234*
12 $\sim$ 13	*2*	*13 $\sim$ 34*	*124 $\sim$ 134 $\sim$ 234*
1 $\sim$ 1234	*12 $\sim$ 234*	*3*	*14 $\sim$ 24 $\sim$ 34*
14 $\sim$ 124	*23 $\sim$ 123*	*23*	*4*
134	*1234*	*234*	
		1234	
		123	

The above hedonic game $(N, \succsim)$ satisfies $\mathcal{B}$-responsiveness.

Now, we are going to prove the theorem "**If a hedonic game $(N, \succsim)$ satisfies $\mathcal{B}$-responsiveness, then it has a core stable coalition structure.**". Before we prove this theorem, we firstly define an algorithm called $\mathcal{B}1$ **Algorithm**. In the domain of $\mathcal{B}$-responsive hedonic games, $\mathcal{B}1$ **Algorithm** always brings core stable coalition structures.

We start by defining a linear order on $\prod^N$ in order to impose ordering on coalition structures.

Let $N = \{1, ..., n\}$ be a set of individuals. Let $\pi^1, \pi^2 \in \prod^N$ be two coalition structures such that $\pi^1 = \{S_1, ..., S_m\}$ with $|S_t| \geq |S_{t+1}|$ for every $t \in \{1, ..., m-1\}$ and $\pi^2 = \{T_1, ..., T_l\}$ with $|T_r| \geq |T_{r+1}|$ for every $r \in \{1, ..., l-1\}$.
Define a linear order $\geq^*$ on $\prod^N$ such that:
- $\pi^1 >^* \pi^2 \Leftrightarrow \exists t \leq min\{m, l\} : |S_t| > |T_t|$ and $|S_r| = |T_r|\ \forall r < t$, and
- $\pi^1 =^* \pi^2 \Leftrightarrow m = l$ and $\forall t \leq m : |S_t| = |T_t|$.

$\mathcal{B}1$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{B}$-responsiveness.

Step 1: Set $\mathcal{P}^1 := N$ and $\pi^0 := \{(i)\}_{i \in \mathcal{P}^1}$.

Step 2: $\forall m \in \{1, ..., |N|\}$, construct the set

$\mathcal{M}^m := \{S \subseteq \mathcal{P}^m | \forall i \in S : S \succ_i \pi^{m-1}(i)\}$.

Step 3: Choose $S \in \mathcal{M}^m$ such that $\forall T \in \mathcal{M}^m : |S| \geq |T|$.

Step 4: Set $S^m := S$, $\pi^m := \{S^m\} \cup \{(i)\}_{i \in \mathcal{P}^m \setminus S^m}$, and $\mathcal{P}^{m+1} := \mathcal{P}^m \setminus S^m$.

Step 5: If $\mathcal{M}^{m+1} = \emptyset$, stop. Return π^m as outcome.

$\mathcal{B}$-responsiveness ensures that for all $i \in N$, every coalition $S \subseteq ch(i, N)$ is ranked below $ch(i, N)$ with respect to decreasing cardinality. This property allows for one of the sets with maximum size to be selected in each iteration at *Step 3*. On top of that, the coalitions selected in the each next iteration can not have greater cardinality than the previously selected coalitions. Thus, after each iteration, we get $\pi^m \geq^* \pi^{m-1}$ for all $m \in \{1, ..., |N|\}$. We have finite number of individuals, thus $\mathcal{B}1$ algorithm terminates in finite steps. There exists $K \in \{1, ..., |N|\}$ such that π^K is the outcome of the algorithm and $\pi^K >^* \pi^m$ for all $m \in \{0, 1, ..., K-1\}$. π^K incorporates (if exist) mutually best coalitions, *greatest common subsets* of maximal sets, and singletons. $\mathcal{B}1$ algorithm firstly detects and separates out the mutually best coalitions. Mutually best coalitions are incorporated into π^K incrementally and with respect to decreasing cardinality. Then, in the same manner, algorithm brings out the greatest common subsets of maximal sets. Eventually, remnant individuals stand alone. Obviously, π^K is individually rational, i.e., for all $i \in N$: $\pi^K(i) \succsim_i \{i\}$. All of these are guaranteed by hierarchy in preferences which is provided by $\mathcal{B}$-responsiveness.

Remark 2.3.2. *If we run $\mathcal{B}1$ Algorithm more than ones, we might get different coalition structures which are still core stable. This opportunity is provided by the weak preferences and the freedom of choice in step 3 of the algorithm. We might have more than one coalition which satisfies the criterion "$\forall T \in \mathcal{M}^m : |S| \geq |T|$". Selection of different coalitions in step 3 initiate algorithm to bring out different coalition structures.*

Let us explain the $\mathcal{B}1$ Algorithm in the following example.

Example 2.3.4. *Consider the Example 2.2.3 . If we run the $\mathcal{B}1$ Algorithm, we observe that:*

$\mathcal{P}^1 := N = \{1, 2, 3, 4\}$ *and* $\pi^0 := \{(1), (2), (3), (4)\}$.
For $m = 1$, $\mathcal{M}^1 = \{S \subseteq \mathcal{P}^1 | \forall i \in S : S \succ_i \pi^0\} = \{(13), (24)\}$.
Choose (13) *and set* $S^1 = (13)$, $\pi^1 = \{(13)\} \cup \{(2)\} \cup \{(4)\}$.
Now $m = 2$, $\mathcal{P}^2 = \mathcal{P}^1 \backslash S^1 = \{2, 4\}$.
$\mathcal{M}^2 = \{S \subseteq \mathcal{P}^2 | \forall i \in S : S \succ_i \pi^1\} = \{(24)\}$. *and* $\pi^2 = \{(13), (24)\}$.
Now $m = 3$, $\mathcal{P}^3 = \mathcal{P}^2 \backslash S^2 = \emptyset$. *Thus,* $\mathcal{M}^3 = \emptyset$.

Therefore, $\pi^2 = \{(13), (24)\}$ *is the final outcome of the* $\mathcal{B}1$ *algorithm.* π^2 *is core stable.*

Theorem 2.3.2. *If a hedonic game $(N, \succsim)$ satisfies $\mathcal{B}$-responsiveness, then it has a core stable coalition structure.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{B}$-responsiveness. Let π^K be the outcome of the $\mathcal{B}1$ Algorithm. Assume that, π^K is not core stable. Then, there exists $H \in \mathcal{C}$ such that $H \succ_i \pi^K(i)$ for every $i \in H$. Because π^K is the $>^*$-maximal and $\mathcal{B}1$ algorithm terminated, H can not be in $\mathcal{M}^{K+1}$. So, $\exists l$ $(0 < l < K)$ such that H should have been selected in l-th or $l+1$-th iteration. But, this contradicts with the fact that S^l and S^{l+1} are maximal coalitions. Therefore, π^K has no blocking coalition $S \in \mathcal{C}$, i.e., it is core stable.

□

Definition 2.3.3. *Let $(N, \succsim)$ be a hedonic game and $i \in N$ be any individual. $\succsim_i \in R(\mathcal{C}_i^N)$ satisfies $\mathcal{G}$-**singularity** iff for all $S \in \mathcal{C}_i^N$, $S \succ_i \{i\} \Rightarrow S \in CH(i, N)$.*

*We say that $\succsim \in \mathcal{R}^N$ satisfies $\mathcal{G}$-singularity iff $\succsim_i \in R(\mathcal{C}_i^N)$ satisfies $\mathcal{G}$-singularity $\forall i \in N$. A hedonic game $(N, \succsim)$ is said to satisfy $\mathcal{G}$-**singularity** iff $\succsim \in \mathcal{R}^N$ satisfies $\mathcal{G}$-singularity.*

If an individual's preferences satisfy $\mathcal{G}$-singularity, he puts every alternative in $CH(i, N)$ to top and then he prefers to become single. All other coalitions are aligned weakly below $\{i\}$. Roughly speaking, if an individual's preferences satisfy $\mathcal{G}$-singularity, then we say that this individual never cooperates unless he obtains what he wants. Hedonic games satisfying $\mathcal{G}$-singularity always have core stable coalition structures.

$\mathcal{G}$-singularity condition generalizes the *singularity* condition of Alcalde and Romero-Medina (2006). In their definition, every individual i has unique maximal set of N under preferences $\succsim_i$, i.e., $\forall i \in N$, $|CH(i, N)| = 1$. Every other coalition is aligned weakly below (i). In our $\mathcal{G}$-singularity condition, we assume that $\forall i \in N$, $|CH(i, N)| \geq 1$. Thus, $\mathcal{G}$-singularity enriches the mobility of individuals.

Example 2.3.5. *Consider the hedonic game $(N, \succsim)$ with $N = \{1, 2, 3, 4\}$ and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
12 $\sim$ 13	*12 $\sim$ 23*	*23 $\sim$ 34*	*1234 $\sim$ 14*
1	*2*	*3 $\sim$ 13*	*4 $\sim$ 34*
rest	*rest*	*rest*	*rest*

The above hedonic game $(N, \succsim)$ satisfies $\mathcal{G}$-singularity.

Theorem 2.3.3. *If a hedonic game $(N, \succsim)$ satisfies $\mathcal{G}$-singularity, then it has core stable coalition structures.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{G}$-singularity. Consider the following algorithm:

$\mathcal{G}1$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{G}$-singularity.

Step 1: Set $V_0 := N$, $S^0 := \emptyset$ and $\pi^0 := \{(i)\}_{i \in V_0}$.

Step 2: $\forall m \in \{0, 1, ..., |N| - 1\}$, search for $T \in \mathcal{C}|_{V_m}$ such that $|T| \geq 2$ and T is mutually best for all $i \in T$.

Step 2.1: If there exists such T, set $S^{m+1} := T$.

Set $V_{m+1} := V_m \backslash S^{m+1}$ and $\pi^{m+1} := \{S^{m+1}\} \cup (\pi^m \backslash \{(i)\}_{i \in S^{m+1}})$.

Step 2.2: If there does not exist such T, stop. Define $\pi^{m+1} = \pi^m$.

Go to Step 3.

Step 3: Return π^m as outcome.

We have finite number of individuals, thus the above algorithm terminates for some $K \in \{0, 1, ..., |N|\}$. The coalition structure π^K incorporates mutually best coalitions and singletons. There exists no $S \in \mathcal{C}$ such that $S \succ_i \pi^K(i)$ for every $i \in S$. Therefore, $\pi^{\mathcal{G}}$ is core stable.

$\square$

Remark 2.3.3. *If we run $\mathcal{G}1$ algorithm more than ones, we might get different coalition structures which are still core stable. This opportunity is provided by the weak preferences and the freedom of choice in step 2 of the algorithm. Some individuals might have more than one maximal set in $CH(i, N)$ and algorithm selects any mutually best coalition among them. Notice that, only the $>^*$-maximal coalition structure is strictly core stable. All the remaining core stable coalition structures are weakly blocked.*

Example 2.3.6. *Consider the Example 2.2.5 . If we run the $\mathcal{G}1$ Algorithm, we observe that:*

$V_0 = N = \{1, 2, 3, 4\}$, $S^0 = \emptyset$, *and* $\pi^0 = \{(1), (2), (3), (4)\}$.
For $m = 0$, $\mathcal{C}_{|V_0} = \mathcal{C}$, *and the corresponding "collection of mutually best coalitions" with coalitions which has cardinality* ≥ 2 *is* $\{(12), (23)\}$.
Choose $S^1 = (12)$. *Then* $V_1 = V_0 \backslash S^1 = \{3, 4\}$ *and we set* $\pi^1 = \{(12), (3), (4)\}$.
Now, $m = 1$. $\mathcal{C}_{|V_1} = \{(34), (3), (4)\}$. *But, there does not exist any mutually best coalition in* $\mathcal{C}_{|V_1}$ *with coalition size* ≥ 2. *Thus, the algorithm stops.*
Therefore, $\pi^2 = \pi^1 = \{(12), (3), (4)\}$ *is the outcome of the algorithm.* π^2 *is core stable.*

If (23) *have been chosen in the first iteration, we would get* $\pi^2 = \{(23), (1), (4)\}$ *as the final outcome of the algorithm which is also core stable.*

Theorem 2.3.4. *$\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity conditions are independent.*

Proof. Let $(N, \succsim)$ be a hedonic game with $N = \{1, 2, 3, 4, 5\}$. Consider the below preference profiles of individual 1:

$\succsim_1^a$: $(123) \succ_1^a (1234) \sim_1^a (1235) \succ_1^a (12345) \succ_1^a (12) \succ_1^a (1) \succ_1^a \ldots$

$\succsim_1^b$: $(123) \succ_1^b (12) \sim_1^b (13) \succ_1^b (1) \succ_1^b (14) \succ_1^b (145) \succ_1^b ...$

$\succsim_1^c$: $(123) \succ_1^c (1) \succ_1^c (1235) \succ_1^c (1234) \sim_1^c (12) \succ_1^c (134) \succ_1^c ...$

We observe that $\succsim_1^a$ satisfies $\mathcal{A}$-responsiveness, but it does not satisfy $\mathcal{B}$-responsiveness and $\mathcal{G}$-singularity. $\succsim_1^b$ satisfies $\mathcal{B}$-responsiveness, but it does not satisfy $\mathcal{A}$-responsiveness and $\mathcal{G}$-singularity. Lastly, $\succsim_1^c$ satisfies $\mathcal{G}$-singularity, but it does not satisfy $\mathcal{A}$-responsiveness and $\mathcal{B}$-responsiveness.

□

Remark 2.3.4. *If $|CH(i,N)| = 1$ and $ch(i,N) = (i)$, then the preference of individual i given by*
$\succsim_i$: $(i) \succ_i (ij) \succ_i (ijk) \succ_i ...$ *satisfies $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity.*

The domain conditions $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity are all independent. This situation brings to mind such a question "*is it possible to extend the domain of existence via modifying or disassembling and assembling these three domain conditions separately or indiscrete ?*". The answer of this question is partially affirmative. Designing a new preference restriction from old ones via modifying or disassembling and assembling is not probable. On the other hand, if an individual set is partitioned into three subsets such that the first subset consists of individuals with $\mathcal{A}$-responsive preferences, the second subset consists of individuals with $\mathcal{B}$-responsive preferences, and the last subset consists of individuals with $\mathcal{G}$-singular preferences, we can find a core stable coalition structure in such a hedonic game. This attempt is not a preference restriction, but it still extends the domain of existence.

Theorem 2.3.5. *Let* $(N, \succsim)$ *be a hedonic game such that* $N = N^{\mathcal{A}} \cup N^{\mathcal{B}} \cup N^{\mathcal{G}}$ *and every individual* $i \in N^{\mathcal{A}}$ *has* $\mathcal{A}$*-responsive preferences, every individual* $j \in N^{\mathcal{B}}$ *has* $\mathcal{B}$*-responsive preferences, and every individual* $k \in N^{\mathcal{G}}$ *has* $\mathcal{G}$*-singular preferences. Then, there always exists a core stable coalition structure.*

Before we prove this theorem, we firstly define an algorithm called $\mathcal{B}2$ **Algorithm** which is a modified version of $\mathcal{B}1$ Algorithm.

$\mathcal{B}2$ Algorithm

Given: $(N, \succsim)$ hedonic game such that $N = N^{\mathcal{A}} \cup N^{\mathcal{B}} \cup N^{\mathcal{G}}$.

Step 1: Set $\mathcal{P}^1 := N$ and $\pi^0 := \{(i)\}_{i \in \mathcal{P}^1}$.

Step 2: $\forall m \in \{1, ..., |N|\}$, construct the set
$\mathcal{M}^m := \{S \subseteq \mathcal{P}^m | \forall i \in S : S \succ_i \pi^{m-1}(i)\}$.

Step 3.1: Sort and rename all the coalitions in $\mathcal{M}^m$ with respect to descending cardinality so that $\mathcal{M}^m = \{S_m^1, ..., S_m^{l_m}\}$ and $\forall t \in \{1, ..., l_m - 1\} : |S_m^t| \geq |S_m^{t+1}|$.

Step 3.2: Find $S_m^t \in \mathcal{M}^m$ with the greatest cardinality such that
$\forall X \in \mathcal{C}$ with $\forall i \in S_m^t \cap X$: $S_m^t \succsim_i X$ and $\forall j \in S_m^t \backslash X$: $S_m^t \succsim_j [X, \pi](i)$.

Step 3.3: Set $T^m := S_m^t$, $\pi^m := \{T^m\} \cup \{(i)\}_{i \in \mathcal{P}^m \backslash T^m}$, and $\mathcal{P}^{m+1} := \mathcal{P}^m \backslash T^m$.

Step 4: If $\exists\ \mathcal{M}^{K+1}$ such that $\mathcal{M}^K = \emptyset$, stop. Return π^K as outcome.

We have finite number of individuals, thus $\mathcal{B}2$ algorithm terminates in finite steps. There exists $K \in \{1, 2, ..., n\}$ such that π^K is the outcome of the algorithm. π^K incorporates (if exists) mutually best coalitions, least common supersets of maximal sets, greatest common subsets of maximal sets, coalitions which are mutually acceptable, and of course singletons.

$\mathcal{B}2$ algorithm detects the coalitions with maximum cardinality in each iteration. If it is impossible to tear up these coalitions via some blocking, they are separated out in order to be incorporated in the final coalition structure. At the end of the last iteration, remnant individuals stand alone.

Proof. (*Theorem 2.2.5*) Let $(N, \succsim)$ be a hedonic game such that $N = N^{\mathcal{A}} \cup N^{\mathcal{B}} \cup N^{\mathcal{G}}$ and every individual $i \in N^{\mathcal{A}}$ has $\mathcal{A}$-responsive preferences, every individual $j \in N^{\mathcal{B}}$ has $\mathcal{B}$-responsive preferences, and every individual $k \in N^{\mathcal{G}}$ has $\mathcal{G}$-singular preferences. Let π^K be the outcome of the $\mathcal{B}2$ Algorithm.

Assume that, π^K is not core stable. Then, there exists $H \in \mathcal{C}$ such that $\forall i \in H$: $H \succ_i \pi^K(i)$.

$\forall m \in \{1, ..., |N|\}$, $\forall t \in \{1, ..., l_m\}$, $H \nsubseteq S^t_m$. Otherwise, $T_m \neq S^t_m$ and the algorithm should have selected the next coalition $S^{t^*}_m$ which satisfies the property $\forall X \in \mathcal{C}$ with $\forall i \in (S^{t^*}_m \cap X)$: $S^{t^*}_m \succsim_i X$ and $\forall j \in (S^{t^*}_m \backslash X)$: $S^{t^*}_m \succsim_j [X, \pi](i)$.

Assume that $H \subseteq \{T_{x_1}, ..., T_{x_1}\}$. Because $H \cap T_{x_1} \neq \emptyset$, $\forall i \in (H \cap T_{x_1})$ we have $H \succ_i T_{x_1}$. But then, $T_{x_1} = S^{l_p}_{x_1}$ should not have been selected in the x_1-th iteration.

Therefore, π^K is core stable. □

3. STRONG COALITIONAL STABILITY IN HEDONIC GAMES

3.1 Strong Coalitional Stability

In this chapter, we study a new stability concept, called strong coalitional stability, under different membership property right codes. Strong coalitional stability is a particular refinement of core stability and Nash stability.

Core stability and Nash stability[1], are predominantly studied in the literature. A coalition structure is Nash stable if no individual can benefit by moving from her current coalition to another existing (possibly empty) coalition. A coalition structure is core stable if there exists no coalition of individuals whose members strictly prefer that coalition to their current coalition. Unfortunately, neither core stable coalition structures nor Nash stable coalition structures exist everywhere. Conditions under which core stable or Nash stable coalition structures exist have been studied by several authors.

When allocating incoming freshman students into dormitory rooms, when designating homework groups in a classroom, or when assigning projects to groups of employees, dormitory administration, teachers, or managers, respectively desire to partition individuals into disjoint coalitions such that no objection, no complaint, and no displeasure will show up. Core stability and Nash stability do not cure the

[1] The equilibrium notions called *stability* and *free-mobility equilibrium* (or *individual stability*) in the literature of local public good production games (see Conley and Konishi (2002) and Asan and Sanver (2015)) are reflected into the literature of hedonic games as *core stability* and *Nash stability*, respectively.

need stated above because some core stable coalition structures are not immune to individual deviations and some Nash stable coalition structures are not immune to coalitional deviations. Although new coalitions can not show up when a coalition structure is core stable, there is no impediment for some individuals to leaving their current coalitions and then joining in another existing coalition. By the same token, while there is no possibility of individual moves in a Nash stable coalition structure, some individuals may form a new coalition and get better off. Moreover, when considering the core stability of a coalition structure, the fact that the deviating coalition may have a desire to join an existing coalition is ignored. Taking all of these into account, there is a need for a new stability concept which possesses immunity against coalitional and individual deviations.

This chapter has two aims. The primary one is to define a comprehensive stability concept which cures the deficiencies of the core stability and the Nash stability. The second aim is to implement membership property right concepts to hedonic games properly. Thus, we introduce **strong coalitional stability**[2] under different membership property right codes. A coalition structure is called **strongly coalitionally stable under membership property right code** $RC \in \{FX-FE, FX-AE, AX-FE, AX-AE\}$ if and only if when a collection of individuals block the coalition structure either by constructing new coalitions or joining in existing coalitions, there exists at least one individual (among the individuals who must be consulted for that move to take place) who is worse off in the new coalition structure (the one that is

[2] Strong coalitional stability is stronger than the *Nash stable core* definition mentioned in Hasan et al. (2014). Hasan et al.'s stability definition is not immune to coalitional blockings such as the deviating coalitions may have a desire to join an existing coalition. Strong coalitional stability is a particular refinement of Nash stability and core stability which is immune to individual and coalitional blockings.
Strong coalitional stability is weaker than *coalitional Nash stability* and *strong Nash stability* concepts mentioned in Karakaya (2011). Coalitional move which is related with Karakaya's coalitional Nash block not only contains v-*move* of coalition S but also the possibility of S's joining in desolated coalitions. Reachability definition related with Karakaya's strong Nash stability is one of the broadest coalitional moves which incorporates complex moves such as exchanges and shuffles of individuals and coalitions as well as $v - move$ of a coalition.

wanted to be formed). We begin by defining coalitional move and coalitional blocking concepts. Then, we give the definitions of membership property right codes and strong coalitional stability based on these definitions. Given a coalition structure, by a coalitional move, we mean that a group of individuals leave their current coalitions, they partition with respect to some rule, and then each of these new coalitions in the partition join in an existing (possibly empty) coalition in the given coalition structure. We say that a coalition structure is coalitionally blocked if there exist a group individuals who all together get strictly better of when they carry out a coalitional move. Subsequently, we introduce three mutually independent preference restrictions, namely *$\mathcal{A}$-responsiveness*, *$\mathcal{B}$-responsiveness*, and *$\mathcal{G}$-singularity*. Each of the three preference restrictions guarantees the existence of core stable coalition structures. We show that with assistance of an additional condition (top-mutuality or top-symmetry), they guarantee the existence of strongly coalitionally stable coalition structures under FX-FE membership property rights. If a hedonic game satisfies (*i*) *$\mathcal{A}$-responsiveness* and *top-mutuality*, or (*ii*) *$\mathcal{B}$-responsiveness* and *top-mutuality*, or (*iii*) *$\mathcal{G}$-singularity* and *top-symmetry*, then there always exist strongly coalitionally stable coalition structures under FX-FE membership property rights. *$\mathcal{A}$-responsiveness* and *$\mathcal{B}$-responsiveness* are particular monotonicity conditions. If a hedonic game satisfies $\mathcal{A}$-responsiveness, every individual aligns the coalitions which are supersets of the unique maximal set to below the maximal set with respect to increasing size order till the grand coalition (the whole set of individuals). Remaining coalitions are released free weekly below the grand coalition. If a hedonic game satisfies $\mathcal{B}$-responsiveness, every individual aligns the coalitions which are subsets of the unique maximal set to below the maximal set with respect to decreasing size order till the singleton coalition (staying alone). Remaining coalitions are released free weekly below the singleton coalition. *$\mathcal{G}$-singularity* condition generalizes the *singularity* condition of Alcalde and Romero-Medina (2006)[3]. If an individual's preference satisfy $\mathcal{G}$-singularity, then we say that this individual never cooperates unless he obtains what he wants. Thus, $\mathcal{G}$-singularity condition characterizes the individuals who have restricted collaborative behavior. Symmetry and mutuality conditions are used to describe identical and similar behaviors, respectively. Top-symmetry condition tells of the individuals who has identical maximal sets. Top-mutuality condition describes the individuals who has common elements in their maximal sets. Followingly, we prove the existence of strongly coalitionally stable coalition structures in the stated domains. Afterwards, we compare stability concepts in the literature with strong coalitional stability and examine which stability concept follows from the other. We verify that under FX-FE membership property rights (as well as under FX-AE, AX-FE, and AX-AE), (strict) core stability and Nash stability follow from (strict) strong coalitional stability.

In the following section, we define coalitional move concept and membership property right codes. Then, we introduce the definition of strong coalitional stability under different membership property rights.

3.2 Definitions and Notation

3.2.1 Coalitional Moves

Let $N \in \mathcal{N}$ be arbitrary set of individuals. Let $\pi \in \prod^N$ be arbitrary coalition structure such that $\pi = \{T_1, ..., T_K\}$ ($K \leq |N|$ is a positive integer). For all $S \in \mathcal{C}$ and for all $\upsilon \in \prod^S$ such that $\upsilon = \{S_1, ..., S_m\}$ ($m \leq |S|$ is a positive integer),

[3] In Alcalde and Romero-Medina (2006), if a hedonic game satisfies singularity, then every individual has a maximal set and aligns the singleton coalition below the maximal set. Remaining coalitions are aligned weakly below the singleton coalition. If a hedonic game satisfies $\mathcal{G}$-singularity, individuals may have several maximal sets and singleton coalitions are aligned below maximal sets. Remaining coalitions are aligned weakly below singleton coalitions.

υ-**move of** S is a coalition structure

$$F_{S,\upsilon} = \{T \backslash S \in \mathcal{C} | T \in \pi\} \cup \{S_l \cup T_{\sigma(l)}\}_{l \in \{1,\ldots,m\}} \tag{3.1}$$

where

- $T_{\sigma(1)}, T_{\sigma(2)}, ..., T_{\sigma(m)} \in (\pi \backslash \pi(S)) \cup \{\emptyset\}$
- $\sigma : \{1, ...n, n+1, ..., m\} \to \{r_0, r_1, ..., r_n\}$ is an *onto function* such that
  $\sigma(l) = \begin{cases} r_l, & \text{for all } l \in \{1,..,n\} \\ r_0, & \text{for all } l \in \{n+1,..m\} \end{cases}$
- and $\sigma(l) = r_0 \Rightarrow T_{r_0} = \emptyset$.

υ-move of S means that, all members of S leave their current coalitions in π **coordinately** and **simultaneously**. They partition according to υ and each $S_l \in \upsilon = \{S_1, ..., S_m\}$ joins in (possibly empty) coalition $T_{\sigma(l)} \in (\pi \backslash \pi(S)) \cup \{\emptyset\}$ with respect to rule σ. For notational consistency, $F_{S,\upsilon}(i)$ denotes the unique coalition in the new coalition structure $F_{S,\upsilon}$ which includes individual i. When individuals of S leave their current coalitions in π, remaining individuals do not dissolve, i.e., $\forall j, k \in N \backslash S$ with $j \neq k$, $\pi(j) = \pi(k) \Leftrightarrow F_{S,\upsilon}(j) = F_{S,\upsilon}(k)$.

Example 3.2.1. *Let* $(N, \succsim)$ *be a hedonic game such that* $N = \{1, ..., 15\}$. *Let* $\pi \in \prod^N$ *be a coalition structure such that*
$\pi = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\} = \{(1,2,3), (4,7), (5,6,8,9), (10,12), (11,13), (14), (15)\}$.

Suppose that individuals $\{1, 8, 9, 13, 14\}$ *leave their current coalitions in* π. *So we observe that,* $S = \{1, 8, 9, 13, 14\}$. *Then, they partition such that* $\upsilon = \{S_1, S_2, S_3, S_4\}$ *and* $S_1 = (8, 9)$, $S_2 = (13)$, $S_3 = (14)$, $S_4 = (1)$.

Each S_l, $l \in \{1,2,3,4\}$ joins in a coalition $T_{\sigma(l)} \in (\pi \backslash \pi(S)) \cup \{\emptyset\}$ with respect to rule (onto function) $\sigma : \{1,2,3,4\} \to \{r_0, r_1, r_2, r_3\}$ such that $\sigma(1) = r_1 = 2$, $\sigma(2) = r_2 = 7$, $\sigma(3) = r_3 = 4$, and $\sigma(4) = r_0$.

Then, we observe that:

$(\pi \backslash \pi(S)) \cup \{\emptyset\} = (\pi \backslash \{(1,2,3),(5,6,8,9),(11,13),(14)\}) \cup \{\emptyset\} = \{(4,7),(10,12),(15),\emptyset\}$.

S_1 joins $T_{\sigma(1)} = T_2 = (4,7)$, S_2 joins $T_{\sigma(2)} = T_7 = (15)$, S_3 joins $T_{\sigma(3)} = T_4 = (10,12)$, and S_4 joins $T_{\sigma(4)} = T_{r_0} = \emptyset$.

After υ-move of the coalition S, we get the coalition structure
$F_{S,\upsilon} = \{(2,3),(5,6),(11)\} \cup \{(1),(4,7,8,9),(10,12,14),(13,15)\}$.

The definition *υ-move of coalition S* is less comprehensive than the *reachability* definition in Karakaya (2011). However, it is more comprehensive than *coalitional move* definitions of Conley and Konishi (2002) and Asan and Sanver (2015). In the definitions of Conley and Konishi (2002) and Asan and Sanver (2015), when coalition S deviates, they partition with respect to a certain rule and then each of the new coalitions only joins in empty set. In υ-move, new coalitions not only join in empty set, but also in coalitions in $\pi \backslash \pi(S)$. Reachability definition of Karakaya (2011) is one of the most comprehensive definitions. It allows members of S to conduct any kind of move (exchange, joining in an existing coalition, joining in empty set, and more than this). Here is to point that, if we augment the degrees of freedom of S, existence of the stability concept (related with the move) gets harder and harder.

In the following subsection, we define *membership property right codes*, which specifies the list of individuals who must be consulted for an υ-move to take place.

3.2.2 Membership Property Rights

Sertel (1982 and 1992) introduced four different *membership property right concepts*, namely *free exit*, *approved exit*, *free entry*, and *approved entry*. When a move (deviation) is intended, these concepts initiate four different *membership property right codes*, namely *free exit-free entry* (FX-FE), *free exit-approved entry* (FX-AE), *approved exit-free entry* (AX-FE), and *approved exit-approved entry* (AX-AE). A membership property right code defines the set of individuals whose approval is needed (for this move to take place) when a group of individuals plan to deviate.

Let $N \in \mathcal{N}$ be an arbitrary set of individuals. Given any $A \in \mathcal{C}$, any $i \in A$, and any $B \in (2^N \backslash \{A\})$, let $\mathcal{K}_{i,A,B} \subseteq N$ denote the set of individuals who must be consulted when individual i wants to leave coalition A and join the (possibly empty) coalition B. Then, the collection $\mathcal{K} = \{\mathcal{K}_{i,A,B}\}$ is the membership property right code of the set of individuals.

Definition 3.2.1. *For any $A \in \mathcal{C}$, any $i \in A$, and any $B \in (2^N \backslash \{A\})$, membership property rights are:*

- *FX-FE if and only if $\mathcal{K}_i = \{i\}$*
- *FX-AE if and only if $\mathcal{K}_i = \{i\} \cup B$*
- *AX-FE if and only if $\mathcal{K}_i = A$*
- *AX-AE if and only if $\mathcal{K}_i = A \cup B$*

Given any $N \in \mathcal{N}$, any $\pi \in \prod^N$, and any $S \in \mathcal{C}$, if υ-move of S takes place, then in the above definition we have $A = \pi(i)$ and $B = F_{S,\upsilon}(i)$. Unless indicated otherwise, we will use $\mathcal{K}_i$ in stead of $\mathcal{K}_{i,\pi(i),F_{S,\upsilon}(i)}$ for all $i \in N$ in the remaining of the paper. Moreover, we will use *membership property right code* and *membership property rights* interchangeably in the remaining of the paper.

In the following section, we introduce **coalitional blocking** and **strong coalitional stability** definitions.

3.2.3 Strong Coalitional Stabilities

Definition 3.2.2. *Let $(N, \succsim)$ be a hedonic coalition formation game. Let $\pi \in \prod^N$ be a coalition structure.*

- *A coalition $S \in \mathcal{C}$ **blocks** π **via** υ-**move** iff $F_{S,\upsilon}(i) \succ_i \pi(i)$ for all $i \in S$.*
- *A coalition $S \in \mathcal{C}$ **weakly blocks** π **via** υ-**move** iff $F_{S,\upsilon}(i) \succsim_i \pi(i)$ for all $i \in S$ and $F_{S,\upsilon}(j) \succ_j \pi(j)$ for some $j \in S$.*
- *π is **strongly coalitionally stable** under membership property right code $\mathcal{K}$ iff for all $S \in \mathcal{C}$ which blocks π via some υ-move, there exists $i \in S$ such that $\pi(j) \succ_j F_{S,\upsilon}(j)$ for some $j \in \mathcal{K}_i$.*
- *π is **strictly strongly coalitionally stable**[4] under membership property right code $\mathcal{K}$ iff for all $S \in \mathcal{C}$ which weakly blocks π via some υ-move, there exists $i \in S$ such that $\pi(j) \succ_j F_{S,\upsilon}(j)$ for some $j \in \mathcal{K}_i$.*

Remark 3.2.1. *Note that, as the list of individuals represented by the membership property right code gets tight, the set of strongly coalitionally stable coalition structures shrinks. In other words, if $\mathcal{K}_i^1 \subseteq \mathcal{K}_i^2$, then the strong coalitional stability of π under $\mathcal{K}^1$ implies the strong coalitional stability of π under $\mathcal{K}^2$.*

[4] Strict strong Nash stability which is introduced in Karakaya (2011) is one of the strongest stability concepts in the literature. Strict strong coalitional stability follows from strict strong Nash stability.

Let us facilitate the understanding of strong coalitional stability by degrading υ-move into one coalition.

- Let $N \in \mathcal{N}$ be arbitrary set of individuals, $\pi \in \prod^N$, and $S \in \mathcal{C}$.
- Let $\upsilon \in \prod^S$ be such that $\upsilon = \{S\}$, i.e., every individual in S leaves their current coalition in π, and they gather in S.
- Let $F_{S,\upsilon} = \{T \backslash S \in \mathcal{C} | T \in \pi\} \cup \{S \cup H\}$, where $H \in (\pi \backslash \pi(S))$.
- For all $i \in S$, $\mathcal{K}_i = \{i\}$, $\mathcal{K}_i = \{i\} \cup H$, $\mathcal{K}_i = \pi(i)$, and $\mathcal{K}_i = \pi(i) \cup H$ under FX-FE, FX-AE, AX-FE, and AX-AE membership property right codes, respectively.

Remark 3.2.2. *When membership property rights are assumed to be FX-FE, strong coalitional stability definition under FX-FE simplifies to:*

π is strongly coalitionally stable under FX-FE membership property right code iff no coalition $S \in \mathcal{C}$ blocks π via some υ-move.

For simplicity, we will abridge strong coalitional stability under FX-FE membership property rights as *strong coalitional stability* ($SCoS$) and strict strong coalitional stability under FX-FE membership property rights as *strict strong coalitional stability* ($SSCoS$). Under FX-AE, AX-FE, and AX-AE membership property rights, we will point out strong coalition stability parenthetically.

3.3 Existence of Strong Coalitional Stability for Specific Classes of Games

In this section, we assume that the membership property rights are FX-FE and discuss the existence of strongly coalitionally stable coalition structures. Because strong coalitional stabilities under FX-AE, AX-FE, and AX-AE membership property rights follow from strong coalitional stability under FX-FE membership property right, the results in this section straightforwardly hold when membership property rights are assumed to be FX-AE, AX-FE, and AX-AE. Remember that strong coalitional stability under FX-FE membership property rights simply restated as: π is *strongly coalitionally stable under FX-FE membership property rights (SCoS) iff no coalition* $S \in \mathcal{C}$ *blocks* π *via some* υ*-move*. We introduce a number of sufficient domain conditions in the previous chapter. $\mathcal{A}$*-responsiveness* and $\mathcal{B}$*-responsiveness* conditions are sufficient for the existence of core stability. When we intensify them with *top-mutuality*, they become sufficient for the existence of strong coalitional stability. $\mathcal{G}$*-singularity* is also sufficient for the existence of core stability. We assemble $\mathcal{G}$*-singularity* with *top-symmetry* and bring out that it is sufficient for the existence of strongly coalitionally stable ($SCoS$) coalition structures. Symmetry and mutuality conditions here are assistive domain conditions which are context-dependent. They predicate on the idea of "equality or similarity of both sides", respectively.

We start by introducing definitions of top-symmetry and top-mutuality. For simplicity, we firstly write definitions of top-symmetry and top-mutuality when $\forall i \in N$, $|CH(i,N)| = 1$. Followingly, we modify the definitions of top-symmetry and top-mutuality for the case $\forall i \in N$, $|CH(i,N)| \geq 1$. Then, we provide the definitions of $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity. Following this, we show that strongly coalitionally stable coalition structures exist when we intensify $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity either with top-mutuality or top-symmetry. All the proofs are presented constructively hinging upon algorithms.

Definition 3.3.1. *Let* $(N, \succsim)$ *be a hedonic game. Let* $i, j \in N$ *be arbitrary individuals. Assume that* $\forall i \in N$, $|CH(i,N)| = 1$.

- $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are* ***top-symmetric*** *iff* $i \in ch(j,N) \Rightarrow ch(i,N) = ch(j,N)$.

- $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are* ***top-mutual*** *iff* $i \in ch(j,N) \Rightarrow j \in ch(i,N)$.

- $\succsim \in \mathcal{R}^N$ *satisfies top-symmetry*[5] *iff* $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are top-symmetric* $\forall i,j \in N$ *such that* $j \in ch(i,N)$.

- $\succsim \in \mathcal{R}^N$ *satisfies top-mutuality iff* $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are top-mutual* $\forall i,j \in N$ *such that* $j \in ch(i,N)$.

We say that a hedonic game $(N,\succsim)$ *satisfies* ***top-symmetry*** *and* ***top-mutuality*** *iff* $\succsim \in \mathcal{R}^N$ *satisfies top-symmetry and top-mutuality, respectively.*

Definition 3.3.2. *Let* $(N,\succsim)$ *be a hedonic game. Let* $i,j \in N$ *be arbitrary individuals. Assume that* $\forall i \in N$*,* $|CH(i,N)| \geq 1$.

- $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are* ***top-symmetric*** *iff* $\exists X \in CH(i,N)$ *such that* $j \in X \Rightarrow X \in CH(j,N)$.

- $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are* ***top-mutual*** *iff* $\exists X \in CH(i,N)$ *such that* $j \in X \Rightarrow \exists Y \in CH(j,N)$ *such that* $i \in Y$.

- $\succsim \in \mathcal{R}^N$ *satisfies top-symmetry iff* $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are top-symmetric* $\forall i,j \in N$*,* $\forall X \in CH(i,N)$ *such that* $j \in X$.

[5] We say that a coalition $S \in \mathcal{C}$ is *mutually best* for every $i \in S$ iff every individual $i,j \in S$ has *top-symmetric* preferences, and vice versa. But, keep in mind that being *mutually best* is a property of a coalition, and being *top-symmetric* is a property of preferences of individuals. The presence of some mutually best coalitions does not necessitate the whole preference profile to be top-symmetric. On the contrary, when we have a top-symmetric preference profile, the top preferences incorporate mutually best coalitions.

- *$\succsim \in \mathcal{R}^N$ satisfies top-mutuality iff $\succsim_i \in R(\mathcal{C}_i^N)$ and $\succsim_j \in R(\mathcal{C}_j^N)$ are top-mutual $\forall i, j \in N$, $\forall X \in CH(i, N)$ such that $j \in X$, and $\forall Y \in CH(j, N)$ such that $i \in Y$.*

*We say that a hedonic game $(N, \succsim)$ satisfies **top-symmetry** and **top-mutuality** iff $\succsim \in \mathcal{R}^N$ satisfies top-symmetry and top-mutuality, respectively.*

Remark 3.3.1. *When preferences are strict, top-symmetry is sufficient for existence of a strongly coalitionally stable, as well as perfectly stable[6] coalition structure. If we allow weak preferences, it is possible to find hedonic games with no strongly coalitionally stable coalition structures.*

Example 3.3.1. *Consider the hedonic game $(N, \succsim)$ with $N = \{1, 2, 3\}$ and the preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$
12 $\sim$ 13	12 $\sim$ 23	13 $\sim$ 23
123	123	123
1	2	3

The above hedonic game satisfies top-symmetry, however non of the coalition structures are strongly coalitionally stable. The coalition structure $\{(123)\}$ is blocked by any duple and any coalition structure in $\{\{(ij), (k)\} | i, j, k \in N\}$ is blocked by the singleton k.

We are going to prove the theorem "**If a hedonic game $(N, \succsim)$ satisfies $\mathcal{A}$-responsiveness and top-mutuality, then it has a strongly coalitionally stable coalition structure.**". Before we prove this theorem, let us remember the definition of $\mathcal{A}$-responsiveness (*definition 2.2.1*) and the $\mathcal{A}$1 Algorithm.

[6] A coalition structure π is called **perfectly stable** iff $\pi(i)$ is the most preferred coalition for all $i \in N$. Perfect stability is the strongest stability concept ever.

Definition 3.3.3. *Let $(N, \succsim)$ be a hedonic game and $i \in N$ be any individual. $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{A}$-**responsive** iff*

- $|CH(i, N)| = 1$,
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \subseteq S \subset T$ we have $S \succ_i T$,*
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \subseteq S$, $ch(i, N) \subseteq T$, and $\mid S \mid = \mid T \mid$ we have $S \sim_i T$,*
- *For all $S \in \mathcal{C}_i^N$ such that $ch(i, N) \backslash S \neq \emptyset$, we have $N \succsim_i S$*

*We say that $\succsim \in \mathcal{R}^N$ satisfies $\mathcal{A}$-responsiveness iff $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{A}$-responsive $\forall i \in N$. A hedonic game $(N, \succsim)$ with $\mathcal{A}$-responsive preference profile $\succsim \in \mathcal{R}^N$ is said to satisfy $\mathcal{A}$-**responsiveness**.*

Before we call back the $\mathcal{A}1$ Algorithm to mind, let us define the following recursive function that comprises a basis for the algorithm.

Let $t \in \mathbb{Z}^+$ be a positive integer. For every $i \in N$ and $S \in \mathcal{C}_i^N$, let define the function $\mathcal{X}^t : N \times \mathcal{C} \to \mathcal{C}$ as follows:

$\mathcal{X}^1(i, S) = ch(i, S)$.

$\mathcal{X}^{t+1}(i, S) = \bigcup_{j \in \mathcal{X}^t(i,S)} ch(j, S)$ for each positive integer t.

By definition, $\mathcal{X}^t(i, S) \subseteq \mathcal{X}^{t+1}(i, S) \subseteq S$ and $\mathcal{X}^{|N|+1}(i, S) = \mathcal{X}^{|N|}(i, S)$. By $\mathcal{X}\mathcal{X}(i, S)$ we denote $\mathcal{X}^{|N|}(i, S)$.

$\mathcal{A}1$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{A}$-responsiveness.

Step 1: Set $\mathcal{P}^1 := N$ and $\pi^0 := \emptyset$.

Step 2: For $m = 1$ to $|N|$:

Step 2.1: Select $i \in \mathcal{P}^m$ satisfying $|\mathcal{X}\mathcal{X}(i, N)| \leq |\mathcal{X}\mathcal{X}(j, N)|$ for each $j \in N$.

Set $S^m := \mathcal{X}\mathcal{X}(i, N)$.

Step 2.2: If $S^m \subseteq \mathcal{P}^m$, set $\pi^m := \pi^{m-1} \cup \{S^m\}$ and $\mathcal{P}^{m+1} := \mathcal{P}^m \backslash S^m$.

Step 2.3: If $S^m \nsubseteq \mathcal{P}^m$, set $\pi^m := \pi^{m-1} \cup \{(i)\}_{i \in \mathcal{P}^m}$ and $\mathcal{P}^{m+1} = \mathcal{P}^m$.

Go to Step 3.

Step 3: Select $H \subseteq \mathcal{P}^{m+1}$ such that $\forall i \in H : N \succsim_i H \succsim_i \{i\}$ and $H \succ_i H'$ $\forall i \in H \cap H', \forall H' \in \mathcal{P}^{m+1}$.

Set $S^{m+1} := H$ and $\pi^{m+1} := \pi^{m-1} \cup \{S^{m+1}\} \cup \{(i)\}_{i \in \mathcal{P}^{m+1} \backslash S^{m+1}}$.

Return π^{m+1} as outcome.

Step 4: If $\nexists H \subseteq \mathcal{P}^{m+1}$ such that for every $i \in H$: $N \succsim_i H \succsim_i \{i\}$, return π^m as outcome.

Theorem 3.3.1. *If a hedonic game $(N, \succsim)$ satisfies $\mathcal{A}$-responsiveness and top-mutuality, then it has a strongly coalitionally stable coalition structure.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{A}$-responsiveness and top-mutuality. Let π^K be the outcome of the $\mathcal{A}1$ algorithm. Hierarchy provided by $\mathcal{A}$-responsiveness and reciprocal agreement provided by top-mutuality ensure that π^K incorporates only mutually best coalitions, least common supersets of maximal sets, and the individuals whose maximal sets are singletons. No coalition which is weakly ranked blow N is included in π^K. As well, no individual other than whose maximal set is standing alone, stands alone. Otherwise, we get a contradiction with top-mutuality.

Suppose that, π^K is not strongly coalitionally stable. Then, there exists $S \in \mathcal{C}$ which blocks π^K via some υ-move. υ-move of S can not be in the form of joining in empty set because π^K is straightforwardly core stable. Thus, S partitions with respect to $\upsilon = \{H_1, ..., H_K\}$ and each H_m joins in $T_{l_m} \in (\pi^K \backslash \pi^K(S))$. Moreover, $H_m \cup T_{l_m} \succ_i \pi^K(i)$ for every $i \in H_m$ and for every $H_m \in \upsilon$. Then, there exists two possibilities; for all $i \in H_m$ either (i) $|H_m \cup T_{l_m}| < |\pi^K(i)|$ or (ii) $|H_m \cup T_{l_m}| \geq |\pi^K(i)|$. In case of eventuation, both cases generate a contradiction with top-mutuality and $\mathcal{A}$-responsiveness, respectively. If (i) is the case, although $j \in ch(i, H)$ for every $j \in T_{l_m}$ and for every $i \in H_m$, $i \notin ch(j, N)$ for every $i \in H_m$ and for every $j \in T_{l_m}$. Case (ii) is not even a matter of discussion, because it is against the grain of $\mathcal{A}$-responsiveness. Thus, no coalition $S \in \mathcal{C}$ has an incentive to conduct an υ-move and block π^K. Therefore, π^K is strongly coalitionally stable. □

Remark 3.3.2. *In the above theorem, top-mutuality condition is crucial. If we replace it with* ***weak top-mutuality***[7]*, the existence of strong coalitional stability is no more guaranteed. When we run* $\mathcal{A}1$ *algorithm, it goes beyond coalition* N *in order to identify mutually acceptable coalitions. However, this rises a vulnerability such that an individual or a coalition join in an existing coalition. Consider the following example:*

Example 3.3.2. *Consider the game* $(N, \succsim)$ *with* $N = \{1, 2, 3, 4\}$ *and the following preference profile:*

[7] Let $i, j \in N$ be arbitrary individuals. $\succsim_i \in R(\mathcal{C}_i^N)$ and $\succsim_j \in R(\mathcal{C}_j^N)$ are **weakly top-mutual** iff $j \in ch(i, N) \Rightarrow \exists k \in N$ such that $i \in ch(k, N)$. $\succsim \in \mathcal{R}^N$ satisfies *weak top-mutuality* iff $\succsim_i \in R(\mathcal{C}_i^N)$ and $\succsim_j \in R(\mathcal{C}_j^N)$ are weakly top-mutual $\forall i, j \in N$.

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
14	*123*	*1234*	*14*
124 $\sim$ *134*	*1234*	*123* $\sim$ *134* $\sim$ *234*	*124* $\sim$ *134*
1234	*12* $\sim$ *23*	*13* $\sim$ *23* $\sim$ *34*	*1234*
1	*124* $\sim$ *234*	*3*	*4*
12 $\sim$ *13*	*2*		*24* $\sim$ *34*
123	*24*		*234*

This game satisfies $\mathcal{A}$-responsiveness and weak top-mutuality, but not top-mutuality. Non of the coalition structures are strongly coalitionally stable. $\mathcal{A}1$ algorithm returns the coalition structure $\pi = \{(14), (23)\}$. It is core stable, but not strongly coalitionally stable. Individual 3 and coalition (23) may block it by joining in existing coalitions.

Our second result is the theorem "**If a hedonic game $(N, \succsim)$ satisfies $\mathcal{B}$-responsiveness and top-mutuality, then it has a strongly coalitionally stable coalition structure.**". Before we prove this theorem, let us remember the definition of $\mathcal{B}$-responsiveness (*definition 2.2.2*) and the $\mathcal{B}1$ Algorithm.

Definition 3.3.4. *Let $(N, \succsim)$ be a hedonic game and $i \in N$ be any individual. $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{B}$-**responsive** iff*

- $|CH(i, N)| = 1$,
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \supseteq S \supset T$ we have $S \succ_i T$,*
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \supseteq S$, $ch(i, N) \supseteq T$, and $\mid S \mid = \mid T \mid$ we have $S \sim_i T$,*
- *For all $S \in \mathcal{C}_i^N$ such that $S \backslash ch(i, N) \neq \emptyset$, we have $\{i\} \succsim_i S$*

We say that $\succsim \in \mathcal{R}^N$ *satisfies* $\mathcal{B}$*-responsiveness iff* $\succsim_i \in R(\mathcal{C}_i^N)$ *is* $\mathcal{B}$*-responsive* $\forall i \in N$. *A hedonic game* $(N, \succsim)$ *with* $\mathcal{B}$*-responsive preference profile* $\succsim \in \mathcal{R}^N$ *is said to satisfy* $\mathcal{B}$***-responsiveness***.

$\mathcal{B}1$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{B}$-responsiveness.

Step 1: Set $\mathcal{P}^1 := N$ and $\pi^0 := \{(i)\}_{i \in \mathcal{P}^1}$.

Step 2: $\forall m \in \{1, ..., |N|\}$, construct the set

$\mathcal{M}^m := \{S \subseteq \mathcal{P}^m | \forall i \in S : S \succ_i \pi^{m-1}(i)\}$.

Step 3: Choose $S \in \mathcal{M}^m$ such that $\forall T \in \mathcal{M}^m : |S| \geq |T|$.

Step 4: Set $S^m := S$, $\pi^m := \{S^m\} \cup \{(i)\}_{i \in \mathcal{P}^m \setminus S^m}$, and $\mathcal{P}^{m+1} := \mathcal{P}^m \setminus S^m$.

Step 5: If $\mathcal{M}^{m+1} = \emptyset$, stop. Return π^m as outcome.

Theorem 3.3.2. *If a hedonic game* $(N, \succsim)$ *satisfies* $\mathcal{B}$*-responsiveness and top-mutuality, then it has a strongly coalitionally stable coalition structure.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{B}$-responsiveness and top-mutuality. Let π^K be the outcome of the $\mathcal{B}1$ Algorithm.

Suppose that, it is not strongly coalitionally stable. Then, there exists $S \in \mathcal{C}$ which blocks π^K via some υ-move. υ-move of S can not be in the form of joining in empty set because π^K is straightforwardly core stable. Thus, S partitions with respect to $\upsilon = \{H_1, ..., H_K\}$ and each H_m joins in $T_{l_m} \in (\pi^K \setminus \pi^K(S))$. Moreover, $H_m \cup T_{l_m} \succ_i \pi^K(i)$ for every $i \in H_m$ and for every $H_m \in \upsilon$. But then, top-mutuality and $\mathcal{B}$-responsiveness imply that $H_m \cup T_{l_m} \succ_j \pi^K(j)$ for every $j \in T_{l_m}$ and for every $T_{l_m} \in (\pi^K \setminus \pi^K(S))$. Following this, we observe that for every individual $i \in H_m \cup T_{l_m}$, $|H_m \cup T_{l_m}| \geq |\pi^K(i)|$. But, this contradicts with the fact that $\pi^K(i)$ is one of the maximum coalitions for every $i \in N$ in the iteration it is selected. Thus, π^K is strongly coalitionally stable.

□

Remark 3.3.3. *In the above theorem, top-mutuality condition is crucial. If we replace it with **weak top-mutuality**, the existence of strong coalitional stability is no more guaranteed.*

Example 3.3.3. *Consider the game* $(N, \succsim)$ *with* $N = \{1, 2, 3, 4\}$ *and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
14	123	1234	14
1	$12 \sim 23$	$123 \sim 134 \sim 234$	4
1234	2	$13 \sim 23 \sim 34$	1234
$124 \sim 134$	$124 \sim 234$	3	$124 \sim 134$
$12 \sim 13$	1234		$24 \sim 34$
123	24		234

This game satisfies $\mathcal{B}$*-responsiveness and weak top-mutuality, but not top-mutuality. Non of the coalition structures are strongly coalitionally stable.* $\mathcal{B}1$ *algorithm returns the coalition structure* $\pi = \{(14), (23)\}$*. Individual* 3 *blocks* π *by joining in the existing coalition* (14).

Now, we are going to prove the theorem "**If a hedonic game** $(N, \succsim)$ **satisfies** $\mathcal{G}$**-singularity and top-symmetry, then it has a strongly coalitionally stable coalition structure.**". Before we prove this theorem, let us remember the definition of $\mathcal{G}$-singularity (*definition 2.2.3*).

Definition 3.3.5. *Let* $(N, \succsim)$ *be a hedonic game and* $i \in N$ *be any individual.* $\succsim_i \in R(\mathcal{C}_i^N)$ *satisfies* $\mathcal{G}$*-**singularity** iff for all* $S \in \mathcal{C}_i^N$*,* $S \succ_i \{i\} \Rightarrow S \in CH(i, N)$.

We say that $\succsim \in \mathcal{R}^N$ *satisfies* $\mathcal{G}$*-singularity iff* $\succsim_i \in R(\mathcal{C}_i^N)$ *satisfies* $\mathcal{G}$*-singularity* $\forall i \in N$*. A hedonic game* $(N, \succsim)$ *is said to satisfy* $\mathcal{G}$***-singularity*** *iff* $\succsim \in \mathcal{R}^N$ *satisfies* $\mathcal{G}$*-singularity.*

Theorem 3.3.3. *If a hedonic game* $(N, \succsim)$ *satisfies* $\mathcal{G}$*-singularity and top-symmetry, then it has a strongly coalitionally stable coalition structure.*

In order to prove the above theorem, we define another $\mathcal{G}2$ algorithm. The difference between the new algorithm and the previous one ($\mathcal{G}1$ Algorithm) is that the freedom of choice in *step 2* is abolished. The new algorithm selects the mutually best coalitions with maximum cardinality in *step 2*.

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{G}$-singularity and top-symmetry. Consider the following algorithm:

$\mathcal{G}2$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{G}$-singularity and top-symmetry.

Step 1: Set $V_0 := N$, $S^0 := \emptyset$ and $\pi^0 := \{(i)\}_{i \in V_0}$.

Step 2: $\forall m \in \{0, 1, ..., |N| - 1\}$, for all $i \in V_m$, construct the set

$\mathcal{P}_i^m := \{S \in CH(i, N) | \; \forall T \in CH(i, N), |S| \geq |T|\}$.

Select one of the coalitions $H \in \bigcup_{i \in V_m} \mathcal{P}_i^m$ which has maximum cardinality.

Step 2.1: If there exists such H, set $S^{m+1} := H$. Set $V_{m+1} := V_m \backslash S^{m+1}$ and

$\pi^{m+1} := \{S^{m+1}\} \cup (\pi^m \backslash \{(i)\}_{i \in S^{m+1}})$.

Step 2.2: If there does not exist such H, stop. Define $\pi^{m+1} = \pi^m$.

Go to Step 3.

Step 3: Return π^m as outcome.

We have finite number of individuals, thus the above algorithm terminates for some

$K \in \{0, 1, ..., |N|\}$. The coalition structure π^K incorporates mutually best coalitions and singletons.

Suppose that π^K is not strongly coalitionally stable. Then, there exists $S \in \mathcal{C}$ which blocks π^K via some υ-move. υ-move of S can not be in the form of joining in empty set because π^K is straightforwardly core stable. Thus, S partitions with respect to $\upsilon = \{H_1, ..., H_K\}$ and each H_m joins in $T_{l_m} \in (\pi^K \backslash \pi^K(S))$. Moreover, $H_m \cup T_{l_m} \succ_i \pi^K(i)$ for every $i \in H_m$ and for every $H_m \in \upsilon$. But then, by top-symmetry we must have $H_m \cup T_{l_m} \succsim_j \pi^K(j)$ for every $j \in T_{l_m}$ and for every $T_{l_m} \in (\pi^K \backslash \pi^K(S))$. Following this, we observe that for every individual $i \in H_m \cup T_{l_m}$, $|H_m \cup T_{l_m}| \geq |\pi^K(i)|$. But, this contradicts with the fact that $\pi^K(i)$ is one of the maximum coalitions in the iteration it is selected, for every $i \in N$. Thus, π^K is strongly coalitionally stable. ■ □

Remark 3.3.4. *If we run $\mathcal{G}2$ algorithm more than ones, we might get different coalition structures which are still strongly coalitionally stable. As well, there might exist strongly coalitionally stable coalition structures which are never delivered by the $\mathcal{G}2$ algorithm in $\mathcal{G}$-singular and top-symmetric hedonic games.*

$\mathcal{G}$-singularity and top-symmetry together do not guarantee the existence of strict strong coalitional stability. In the following example, no coalition structure is strictly strongly coalitionally stable although the game satisfies $\mathcal{G}$-singularity and top-symmetry.

Example 3.3.4. *Consider the game $(N, \succsim)$ with $N = \{1, 2, 3\}$ and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$
12 $\sim$ *13*	*12*	*13*
1	*2*	*3*
123	*123* $\sim$ *23*	*123* $\sim$ *23*

In this game, the $\mathcal{G}2$ *algorithm brings coalition structures* $\pi^1 = \{(12), (3)\}$ *and* $\pi^2 = \{(13), (2)\}$. *They are strongly coalitionally stable, but not strictly strongly coalitionally stable. Coalition* (13) *weakly blocks* π^1, *and coalition* (12) *weakly blocks* π^2.

Remark 3.3.5. *Top-symmetry in Theorem 3.3.3 is a crucial condition. If top-symmetry is replaced with top-mutuality, existence of strong coalitional stability is imperiled.*

Example 3.3.5. *Consider the game* $(N, \succsim)$ *with* $N = \{1, 2, 3\}$ *and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$
12	*123*	*13*
1	*2*	*3*
123	*12*	*123*
13	*23*	*23*

In this game, individuals' preferences satisfy $\mathcal{G}$*-singularity and top-mutuality but not top-symmetry. We observe that* $\pi = \{(1), (2), (3)\}$ *is core stable but it is not strongly coalitionally stable. Individual* 1 *by joining in existing singletons* $\{2\}$ *and individual* 3 *by joining in existing singleton* $\{1\}$, *respectively, might block* π.

Remark 3.3.6. *Theorems 3.3.1, 3.3.2, and 3.3.3 bring to mind such a question. When a hedonic game satisfies top-mutuality and has multiple core stable coalition* *structures, may one of them be strongly coalitionally stable ? The answer is no. Consider the following example.*

Example 3.3.6. *Consider the game* $(N, \succsim)$ *with* $N = \{1, 2, 3\}$ *and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
14	24	34	1234
134	23	134	14
12 $\sim$ 13	12	23	34
1234	234	3	4
1	...	...	...
...			

This game satisfies top-mutuality. Coalition structures $\pi^1 = \{(14), (23)\}$ *and* $\pi^2 = \{(12), (34)\}$ *are core stable. However, non of them are strongly coalitionally stable. Individual* 3 *by joining in the existing coalition* (14) *blocks* π^1 *and individual* 1 *by joining in the existing coalition* (34) *blocks* π^2.

Remark 3.3.7. *Remember the Theorem 2.2.5. "Let* $(N, \succsim)$ *be a hedonic game such that* $N = N^{\mathcal{A}} \cup N^{\mathcal{B}} \cup N^{\mathcal{G}}$ *and every individual* $i \in N^{\mathcal{A}}$ *has* $\mathcal{A}$*-responsive preferences, every individual* $j \in N^{\mathcal{B}}$ *has* $\mathcal{B}$*-responsive preferences, and every individual* $k \in N^{\mathcal{G}}$ *has* $\mathcal{G}$*-singular preferences. Then, there always exists a core stable coalition structure."*

If we intensify individual preferences with top-symmetry, then we can always find (as an outcome of $\mathcal{B}2$ *Algorithm) coalition structure which satisfies strong coalitional stability. On the other hand, when we impose top-mutuality to individual preferences,* $\mathcal{B}2$ *Algorithm may not bring strongly coalitionally stable coalition structures. Consider the following example.*

Example 3.3.7. *Consider the hedonic game* $(N, \succsim)$ *with* $N = \{1, 2, 3, 4, 5, 6\}$ *and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$	$\succsim_6$
123	*12*	*13*	*45*	*456*	*56*
12 $\sim$ *13*	***	***	*4*	*5* $\sim$ *25*	*6* $\sim$ *136*
1 $\sim$ *16*	****	****	*Rest*	*Rest*	*Rest*
Rest	*****	*****			
	N	N			
	Rest	*Rest*			

In above hedonic game $(N, \succsim)$*, individual* 1*'s preference satisfies* $\mathcal{B}$*-responsiveness. Preferences of individuals* 2 *and* 3 *satisfy* $\mathcal{A}$*-responsiveness. Preferences of individuals* 4*,* 5*, and* 6 *satisfy* $\mathcal{G}$*-singularity. Moreover, the game* $(N, \succsim)$ *satisfies top-mutuality.*

*Arrays of *'s symbolize the coalitions which has cardinality equal to the number of *'s and which are superset of* $ch(i, N)$*. For example,* $ch(2, N) = (12)$*. Thus, (***) symbolizes the coalitions* $\{(123), (124), (125), (126)\}$*.*

If we run the $\mathcal{B}2$ *Algorithm, we get* $\pi = \{(123), (4), (5), (6)\}$*.* π *is a core stable coalition structure. However, it is not strongly coalitionally stable, because individual* 4 *could join individual* 5 *and gets better off.*

Theorem 3.3.4. *Let* $(N, \succsim)$ *be a hedonic game such that* $N = N^{\mathcal{A}} \cup N^{\mathcal{B}} \cup N^{\mathcal{G}}$ *and every individual* $i \in N^{\mathcal{A}}$ *has* $\mathcal{A}$*-responsive preferences, every individual* $j \in N^{\mathcal{B}}$ *has* $\mathcal{B}$*-responsive preferences, and every individual* $k \in N^{\mathcal{G}}$ *has* $\mathcal{G}$*-singular preferences. Moreover, suppose that the game* $(N, \succsim)$ *is top-symmetric. Then, there always exists a strongly coalitionally stable coalition structure.*

Proof. Skipped. It very similar to the proof of the *Theorem 3.3.3*. □

3.4 Relation Between Stability Concepts

In this section, we summarize the relation between strong coalitional stabilities and the stability concepts mentioned in section 1.3 using figures 3.1, 3.2, 3.3, and 3.4. In each figure, the implication arrow means that *the lower stability concept follows from the upper one*. A bi-directional implication arrow means that *the lower stability concept is equivalent to the upper one*. Figure 1 explains the relations between stability concepts under FX-FE membership property rights. In figures 3.2, 3.3, and 3.4, we exhibit the relations between stability concepts under the membership property right codes which are stated in parenthesis. The dashed implication arrows hold only when preferences are assumed to be **strict**. If preferences are **weak**, only **bi-directional dashed arrows** (iv and v) break down and the lower stability concept follows from the upper one. We prove only the labelled relations. Remaining relations can be proven using the labelled ones.

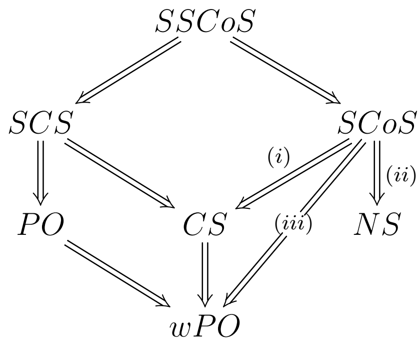

Fig. 3.1: Relation between stability concepts under FX-FE membership property rights.

Proposition 3.4.1. *Let $(N, \succsim)$ be a hedonic game. If a coalition structure $\pi \in \prod^N$ is strongly coalitionally stable, then it is (i) core stable, (ii) Nash stable, and (iii) weakly Pareto optimal.*

Proof. Let $(N, \succsim)$ be a hedonic game. Suppose that $\pi \in \prod^N$ is strongly coalitionally stable. Then, there exists no coalition $S \in \mathcal{C}$ which blocks π via some υ-move. In other words, for all $S \in \mathcal{C}$ and for all possible υ-move of S, we have $\pi(i) \succsim_i F_{S,\upsilon}(i)$ for every $i \in S$.

(i) Assume that π is not core stable. Then there exists $S \in \mathcal{C}$ such that $S \succ_i \pi(i)$ for all $i \in S$. When individuals in S leaves their current coalition and gather in S under FX-FE membership property rights, the coalition structure $\pi^{'} = \{S\} \cup \{T \backslash S | T \in \pi\}$ is formed. If we define $\upsilon = \{S\}$, then we completely get $\pi^{'} = F_{S,\upsilon}$. But then, π is blocked via some υ-move. This contradicts with the assumption that π is strongly coalitionally stable. Hence, π is core stable.

(ii) Assume that π is not Nash stable. Then, there exists $i \in S$ and $H \in (\pi \backslash \{\pi(i)\}) \cup \{\emptyset\}$ such that $H \cup \{i\} \succ_i \pi(i)$. When individual i leaves his current coalition $\pi(i)$ and joins in existing (possibly empty) coalition H under FX-FE membership property rights, the coalition structure $\pi^{'} = \pi \backslash \{\pi(i) \cup H\} \cup \{\pi(i) \backslash \{i\}\} \cup \{H \cup \{i\}\}$ is formed. If we define $\upsilon = \{i\}$, then we completely get $\pi^{'} = F_{S,\upsilon}$. But then, π is blocked via some υ-move. This contradicts with the assumption that π is strongly coalitionally stable. Hence, π is Nash stable.

(iii) $\pi \in \prod^N$ is strongly coalitionally stable, but not weakly Pareto optimal. Then, there exists a coalition structure $\pi^{'} \in (\prod^N \backslash \{\pi\})$ such that $\pi^{'}(i) \succ_i \pi(i)$ for all $i \in N$. If we insert $S = N$ in the definition of υ-move, $\pi^{'} \in (\prod^N \backslash \{\pi\})$ can be attained via some υ-move of N. In these premises, we get $\pi(i) \succ_i F_{N,\upsilon}(i) = \pi^{'}(i)$ for all $i \in N$, which contradicts with the fact that π is strongly coalitionally stable. Thus, π is weakly Pareto optimal. □

The converse of the proposition is not true. There exists coalition structures which are core stable, Nash stable, and weakly Pareto optimal, however they are not strongly coalitionally stable.

Example 3.4.1. *Consider the hedonic game* $(N, \succsim)$ *with* $N = \{1, 2, 3, 4\}$ *and the preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
14	*24*	*1234*	*34*
12	*12*	*13*	*124*
1234	*234*	*34*	*24*
13	*2*	*3*	*4*
1	...	...	...
...			

We observe that, coalition structure $\pi^1 = \{(13), (24)\}$ *is core stable, Nash stable, and weakly Pareto optimal. However, it is not strongly coalitionally stable, because individuals* 1 *and* 3 *by joining in the existing coalition* (24) *block* π^1 *and get better off. In this game, only the coalition structure* $\pi^2 = \{(12), (34)\}$ *is strongly coalitionally stable.*

Figure 3.2 exhibits the relation between strong coalitional stabilities and other stability concepts when membership property rights are assumed to be FX-AE. The bi-directional dashed implication arrow (*iv*) holds only when preferences are assumed to be strict. When there exists no preference restriction, it breaks down and the lower stability concept follows from the upper one. Other implication relations (one-sided and bi-directional ones) always hold.

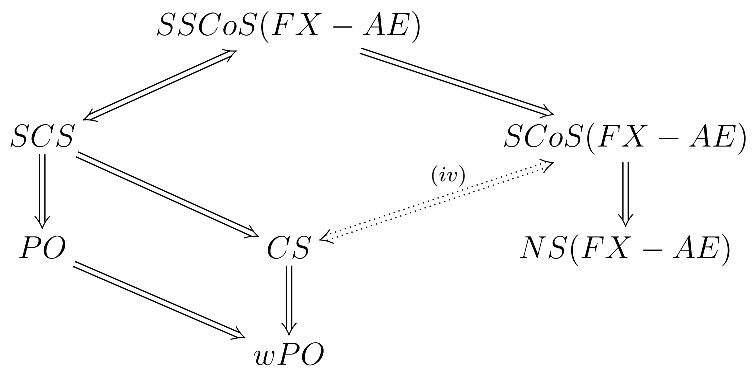

Fig. 3.2: Relation between stability concepts under FX-AE membership property rights.

Proposition 3.4.2. *Let $(N, \succ)$ be a hedonic game in which individuals have strict preferences. Under FX-AE membership property rights, a coalition structure $\pi \in \prod^N$ is strongly coalitionally stable iff it is core stable*[8]*(iv).*

Proof. Let $(N, \succ)$ be a hedonic game in which individuals have strict preferences. Let $\pi \in \prod^N$ be strongly coalitionally stable under FX-AE membership property rights. Suppose that, π is not core stable. Then, there exists at least one coalition $S \in \mathcal{C}$ which blocks π, i.e., $S \succ_i \pi(i)$ for all $i \in S$. Consider the coalition structure $\pi^{'} = \{S\} \cup \{T \backslash S | T \in \pi\}$. This coalition structure can be expressed as $\pi^{'} = F_{S,\upsilon}$ such that $\upsilon = \{S\} \in \prod^S$. It can be written that $F_{S,\upsilon}(i) = \pi^{'}(i) \succ_i \pi(i)$ for all $i \in S$ and there exists no $i \in S$ such that $\pi(j) \succ_j F_{S,\upsilon}(j)$ for some $j \in \mathcal{K}_i = \{i\} \cup \emptyset$. In other words, $S \in \mathcal{C}$ blocks π via υ-move and no individual $j \in \mathcal{K}_i$ oppose. This contradicts with the fact that π is strongly coalitionally stable under FX-AE membership property rights. Thus, π is core stable.

Now, let π be core stable. To the contrary, suppose that π is not strongly coalition-

[8] A coalition structure is called *core stable* if no group of individuals has an incentive to leave their current coalitions and form a new coalition on their own. Gathering together and constructing a new coalition is synonymous with joining in an empty set. Thus, *free entry* (FE) and *approved entry* (AE) concepts count for nothing when we consider (*weak*) *blocking*. Thus (strict) core stability incorporates FX-FE and FX-AE membership property rights together.

ally stable under FX-AE membership property rights. Then, there exists $S \in \mathcal{C}$ such that S blocks π via some υ-move and because preferences are strict, for all $i \in S$, for all $j \in \mathcal{K}_i$, $F_{S,\upsilon}(j) \succ_j \pi(j)$. π is core stable, thus υ-move can not incorporate moves such as joining in empty set. Then, the only way for υ-move to take place is, S partitions such that $\upsilon = \{S_1, ..., S_K\}$ and each $S_l \in \upsilon$ joins in an existing coalition $H_l \in (\pi \backslash \pi(S))$. By definition, for all $i \in S_l$ (for all $l \in \{1, ..., K\}$), $S_l \cup H_l \succ_i \pi(i)$ and for all $j \in H_l = \mathcal{K}_i$, $S_l \cup H_l \succ_j H_l$. If we define $S_l \cup H_l = T_l$ for all $l \in \{1, ..., K\}$, we observe that for all $T_l \in \{T_1, ..., T_K\}$, for all $i \in T_l$, $T_l \succ_i \pi(i)$. But then, each coalition T_l in the collection $\{T_1, ..., T_K\}$ blocks π. This contradicts with π being core stable. Thus, π is strongly coalitionally stable under FX-AE membership property rights.

□

Remark 3.4.1. *When there exists no preference restriction, the above proposition does not hold anymore. Under FX-AE membership property rights, core stability follows from strong coalitional stability, however the converse is not true.*

Example 3.4.2. *Consider the hedonic game $(N, \succsim)$ with $N = \{1, 2\}$ and the preference profile:*

$\succsim_1$	$\succsim_2$
12	*12* $\sim$ *2*
1	

When membership property rights are assumed to be FX-AE, we observe that the coalition structure $\pi = \{(1), (2)\}$ is core stable. It is not strongly coalitionally stable, because individual 1 blocks π by joining in existing coalition (2) and individual 2 does not get worse off. Thus, under weak preferences and FX-AE membership property rights, core stability does not imply strong coalitional stability.

Corollary 3.4.1. *Let $(N, \succsim)$ be a hedonic game. Under FX-AE membership property rights, if a coalition structure $\pi \in \prod^N$ is strongly coalitionally stable then it is core stable. The converse is not true.*

Remark 3.4.2. *When there exists no preference restriction, under FX-AE membership property rights, strict strong coalitional stability is equivalent to strict core stability. This fact can be proven in the same manner (see proposition 4.3).*

Corollary 3.4.2. *Let $(N, \succsim)$ be a hedonic game. Under FX-AE membership property rights, a coalition structure $\pi \in \prod^N$ is strictly strongly coalitionally stable iff it is strictly core stable.*

Figure 3.3 exhibits the relation between strong coalitional stabilities and other stability concepts when membership property rights are assumed to be AX-FE and when there exists no preference restrictions.

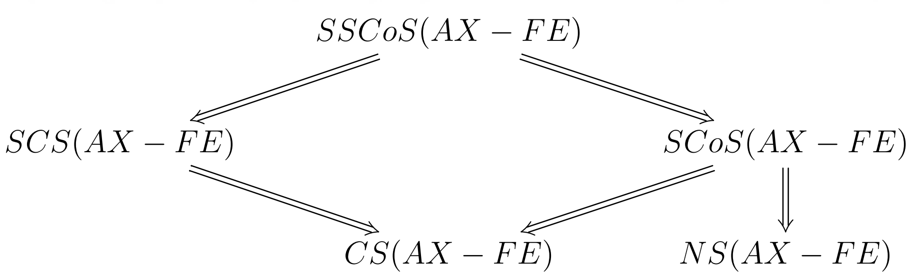

Fig. 3.3: Relation between stability concepts under AX-FE membership property rights.

Figure 3.4 exhibits the relation between strong coalitional stabilities and other stability concepts when membership property rights are assumed to be AX-AE. The bi-directional dashed implication arrow (*v*) holds only when preferences are assumed to be strict. When there exists no preference restriction, it breaks down and the lower stability concept follows from the upper one. Other implication relations (one-sided and bi-directional ones) always hold.

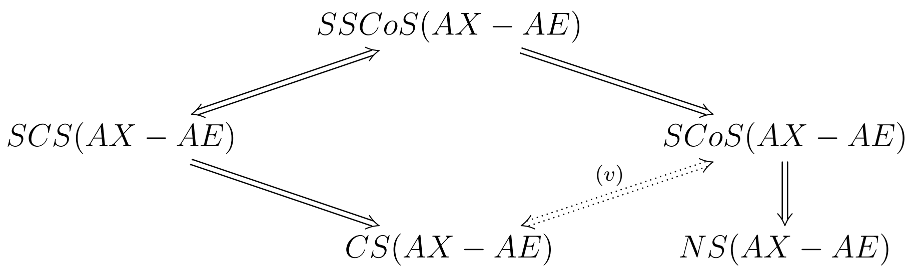

Fig. 3.4: Relation between stability concepts under AX-AE membership property rights.

Proposition 3.4.3. *Let $(N, \succ)$ be a hedonic game in which individuals have strict preferences. Under AX-AE membership property rights, a coalition structure $\pi \in \prod^N$ is strongly coalitionally stable iff it is core stable*[9] *(υ).*

Proof. Let $(N, \succ)$ be a hedonic game in which individuals have strict preferences. Let $\pi \in \prod^N$ be strongly coalitionally stable under AX-AE membership property rights. Suppose that, π is not core stable under AX-AE membership property rights. Then, there exists at least one coalition $S \in \mathcal{C}$ which blocks π, i.e., for all $i \in S$: $S \succ_i \pi(i)$ and for all $i \in S$, for all $j \in \mathcal{K}_i = \pi(i)$, $(\pi(i)\backslash S) \succ_j \pi(i)$. Consider the coalition structure $\pi^{'} = \{S\} \cup \{T\backslash S | T \in \pi\}$. This coalition structure can be expressed as $\pi^{'} = F_{S,\upsilon}$ such that $\upsilon = \{S\} \in \prod^S$. Moreover, we can write that $F_{S,\upsilon}(i) = \pi^{'}(i) \succ_i \pi(i)$ for all $i \in S$ and there exists no $i \in S$ such that $\pi(j) \succ_j F_{S,\upsilon}(j)$ for some $j \in \mathcal{K}_i = \pi(i)$. In other words, $S \in \mathcal{C}$ blocks π via υ-move and no individual $j \in \mathcal{K}_i$ oppose. On the top of it, they are all strictly better off in the new coalition structure. This contradicts with the fact that π is strongly coalitionally stable under AX-AE membership property rights. Thus, π is core stable under AX-AE membership property rights.

[9] (Strict) core stability under AX-AE membership property right code is called contractual (strict) core stability in Sung and Dimitrov (2007). Sung and Dimitrov (2007) proved that contractually strictly core stable coalition structures always exist. Thus, *Proposition 3.4.3* also implies that (strictly) strongly coalitionally stable coalition structures under AX-AE membership property right code always exist.

Now, let π be core stable under AX-AE membership property rights. To the contrary, suppose that π is not strongly coalitionally stable under AX-AE membership property rights. Then, there exists $S \in \mathcal{C}$ such that S blocks π via some υ-move and because preferences are strict, for all $i \in S$, for all $j \in \mathcal{K}_i$, $F_{S,\upsilon}(j) \succ_j \pi(j)$. π is core stable, thus υ-move can not incorporate moves such as joining in empty set. Then, the only way for υ-move to take place is, S partitions such that $\upsilon = \{S_1, ..., S_K\}$ and each $S_l \in \upsilon$ joins in an existing coalition $H_l \in (\pi \backslash \pi(S))$. Because membership property rights are AX-AE and preferences are strict, we have $S_l \cup H_l \succ_i \pi(i)$ for all $i \in S_l$ (for all $l \in \{1, ..., K\}$), $S_l \cup H_l \succ_j H_l$ for all $j \in H_l$, and $(\pi(i)\backslash S) \succ_j \pi(i)$ for all $j \in (\pi(i)\backslash S)$ where $\mathcal{K}_i = H_l \cup (\pi(i)\backslash S)$. If we define $S_l \cup H_l = T_l$ for all $l \in \{1, ..., K\}$, we observe that for all $T_l \in \{T_1, ..., T_K\}$, for all $i \in T_l$, $T_l \succ_i \pi(i)$. Moreover, $(\pi(i)\backslash S) \succ_j \pi(i)$ for all $j \in (\pi(i)\backslash S)$. But then, each coalition T_l in the collection $\{T_1, ..., T_K\}$ blocks π. As well, every individual $j \in (\pi(i)\backslash S)$ becomes strictly better off. This contradicts with π being core stable under AX-AE membership property rights. Thus, π is strongly coalitionally stable under AX-AE membership property rights.

$\square$

Remark 3.4.3. *When there exists no preference restriction, the above proposition does not hold anymore. Under AX-AE membership property rights, core stability follows from strong coalitional stability, however the converse is not true.*

Consider the Example 3.4.2. When membership property rights are assumed to be AX-AE, we observe that the coalition structure $\pi = \{(1), (2)\}$ *is core stable. It is not strongly coalitionally stable, because individual* 1 *blocks* π *by joining in existing coalition* (2) *and individual* 2 *does not get worse off. Thus, when preferences are weak and membership property rights are assumed to be AX-AE, core stability does not imply strong coalitional stability.*

Corollary 3.4.3. *Let $(N, \succsim)$ be a hedonic game. Under AX-AE membership property rights, if a coalition structure $\pi \in \prod^N$ is strongly coalitionally stable then it is core stable. The converse is not true.*

Remark 3.4.4. *When there exists no preference restriction, under AX-AE membership property rights, strict strong coalitional stability is equivalent to strict core stability. This fact can be proven in the same manner (see proposition 3.4.3).*

Corollary 3.4.4. *Let $(N, \succsim)$ be a hedonic game. Under AX-AE membership property rights, a coalition structure $\pi \in \prod^N$ is strictly strongly coalitionally stable iff it is strictly core stable.*

Proposition 3.4.4. *Let $(N, \succsim)$ be a hedonic game. If a coalition structure is Pareto optimal, then it is strictly strongly coalitionally stable under AX-AE membership property rights.*

Proof. Let $(N, \succsim)$ be a hedonic game. Let $\pi \in \prod^N$ be Pareto optimal. To the contrary, suppose that under AX-AE membership property rights it is not strictly strongly coalitionally stable. Then, there exists $S \in \mathcal{C}$ which weakly blocks π via some υ-move. Moreover, $\forall i \in S$, $\forall j \in \mathcal{K}_i = \pi(i) \cup F_{S,\upsilon}(i)$, $F_{S,\upsilon}(j) \succsim_j \pi(j)$. As well, every individual k who is not affected by υ-move is indifferent between $\pi(k)$ and $F_{S,\upsilon}(k)$, because $\pi(k) = F_{S,\upsilon}(k)$. Thus, for all $j \in N$, $F_{S,\upsilon}(j) \succsim_j \pi(j)$ and for some $i \in N$ (actually, for some $i \in S$) $F_{S,\upsilon}(i) \succ_i \pi(i)$. But then, $F_{S,\upsilon}$ Pareto dominates π. This contradicts with the assumption that π is Pareto optimal. Therefore, under AX-AE membership property rights, every Pareto optimal coalition structure is strictly strongly coalitionally stable. □

Remark 3.4.5. *The converse of the proposition does not hold. There exists hedonic games such that strictly strongly coalitionally stable coalition structures might be Pareto dominated. Consider the following example:*

Example 3.4.3. *Consider the hedonic game $(N, \succsim)$ with $N = \{1,2,3,4\}$ and the preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
12 $\sim$ *14*	*12* $\sim$ *23*	*23* $\sim$ *34*	*14* $\sim$ *24*
123	*123*	*123*	*34*
13	*24*	*13*	*4*
1	*2*	*3*	...
...	...	...	

When membership property rights are assumed to be AX-AE, we observe that the coalition structure $\pi^1 = \{(12),(34)\}$ is strictly strongly coalitionally stable. However, it is not Pareto optimal, because the coalition structure $\pi^2 = \{(14),(23)\}$ Pareto dominates it. Thus, sufficiency of strict strong coalitional stability for Pareto optimality is beside the point.

Acknowledgments

The preliminary version of this chapter was written when I was in Maastricht University as a visiting researcher. I would like to express my thanks to The Scientific and Technological Research Council of Turkey (TÜBİTAK) for supporting this project 1059B1415000065. I am very grateful to Burak Can, İpek Özkal-Sanver, Jean Lainé, Remzi Sanver, and Walter Trockel for elaborate discussions and helpful comments. Moreover, I am very grateful to two anonymous referees who helped to enrich the chapter with their suggestions and warnings. I also thank to the participants of the ASSET 2016 Annual Meeting in Thessaloniki, SSCW 2016 Meeting in Lund, and All-Istanbul Economics Workshop 2016 in Istanbul for helpful com-

ments. This chapter was earlier circulated with the title "Coalitional Nash stability in hedonic coalition formation games".

4. EXCHANGE STABILITY IN HEDONIC GAMES

4.1 Exchange Stability

In this chapter, we study a new stability concept called strong exchange stability. Strong exchange stability is the minimal refinement of core stability and exchange stability, i.e., a coalition structure is strongly exchange stable if and only if it is core stable and exchange stable. A coalition structure is called core stable if no group of individuals has an incentive to leave their current coalitions and form a new coalition on their own. A coalition structure is called *exchange stable* if there does not exist a collection of independent subcoalitions such that when subcoalitions consecutively exchange their places in a full cyclic order, individuals in all subcoalitions get better off. The main reasons that we focus on strong exchange stability rather than just focusing on exchange stability are threefold. Firstly, exchange stability is a complementary stability concept, i.e., some core stable coalition structures are not immune to blocking via exchanges. Secondly, exchange block, hence exchange stability are discernable and applicable only for coalition structures which has at least two coalitions with cardinality greater than or equal to 2. For example, exchange block is not applicable for the grand coalition $\aleph$ and for the coalition structure $\beth$ which consists of singletons. They become exchange stable by their nature. Lastly, there exist exchange stable coalition structures which are not core stable which cause conceptual gaps. Considering all these three reasons, it is more sensible and fruitful to study strong exchange stability.

In this study, we seek the domains of hedonic games in which strongly exchange stable coalition structures always exist. Because core stable coalition structures do not exist in the domain of all hedonic games, so does the strongly exchange stable coalition structures. Thus, we focus on some particular domain restrictions in which core stable coalition structures always exist and verify that core stable coalition structures are also exchange stable. Banerjee et al. (2001) introduced a domain condition called *weak top coalition property* which is sufficient for existence of core stable coalition structures. We prove that weak top coalition property is sufficient for the existence of exchange stability, as well as core stability. Followingly, we explore the existence of strongly exchange stable coalition structures in the domains of hedonic games satisfying $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity, separately. We find out that $\mathcal{B}$-responsiveness and $\mathcal{G}$-singularity conditions are sufficient conditions for exchange stability, however $\mathcal{A}$-responsiveness condition remains incapable and requires additional restrictive conditions.

A weaker version of exchange stability which is called *exchange stability* is widely studied in roommate market games and marriage market games. A matching is called *exchange stable* if no two agents can be made better off by exchanging their current matching position. Alcalde (1995) studied *Gale-Shapley stability* and *ξ-stability* (exchange stability). He showed that exchange stable matchings may not exist everywhere. He proved that if a roommate market game satisfies *α-reducibility* condition, then there exists a unique Gale-Shapley stable matching which is also exchange stable. He extensively emphasized the importance of *membership property rights* and dwelled on the distinction between ex-ante and ex-post matchings. Cechlárová (2002) studied exchange stability in roommate market games with the property that for every individual there exists both acceptable and unacceptable individuals and there exists possibility of inconsistent preference lists. Cechlárová and Manlove (2005) studied complexity of finding exchange stable matchings in marriage market games and roommate market games. They also studied weaker forms of exchange stability. Irving (2008) introduced *man-exchange stable* marriage problem and showed that finding an exchange stable matching in that world as well as its generalization to Hospital/Residents problem is NP-complete. Bodine-Baron et al. (2011) focused on many-to-one matchings with *peer effects* and *complementarities*. They showed that two-sided exchange stable matchings always exist when peer effects are derived from an underlying social network and socially optimum matchings are always stable. They also presented some algorithms for which convergence to a two-sided exchange stable matching is guaranteed and they studied complexity properties of that algorithms. Lazarova et al. (2015) studied two variants of exchange stability in many-to-one matching markets. They characterized Pareto optimal matchings by means of *contractual exchange stability* and matchings of maximum total reward by means of *compensational exchange stability*. Abizada (2019) studied exchange stability in roommate problems when preferences of individuals are assumed to be strict and number of rooms are limited. He defined new domains which generalizes Alcalde (1995)'s α-*reducibility* condition and sufficient for the existence of exchange stable matchings. Aziz and Goldwaser (2017) considered several notions of exchange stability for marriage and roommate problems, and they discussed the relation between these notions. To our knowledge, our paper is the first paper which studies exchange stability in hedonic games. Moreover, our paper generalizes all exchange concepts in one-sided and two-sided matching games.

In the first part, we define exchange stability and strong exchange stability under FX-FE membership property rights. Then, we prove the existence of strong exchange stability in various domains of hedonic games when membership property rights are assumed to be FX-FE. Lastly, we analyze exchange stability when membership property rights are FX-AE. The last chapter is dedicated to final remarks and open questions.

4.2 Definitions and Notation

Exchange is physically defined and meaningful for coalition structures in

$\prod_{\chi}^{N} = \{\pi \in \prod^{N} \mid \exists\, T_l \neq T_m$ such that $|T_l| \geq 2$ and $|T_m| \geq 2\}$.

If a coalition structure does not belong to $\prod_{\chi}^{N}$, then there does not exist at least two individuals or subcoalitions which exchange their places. For exchange to be applicable, we assume that there exist at least one survivor in each of the coalitions at which departure took place when at least two individuals or subcoalitions agree on exchanging their places. Those coalition structures in $\prod^{N} \setminus \prod_{\chi}^{N}$ are exchange stable by their nature because exchange block is not defined there.

$\prod_{\chi}^{N} = \emptyset$ when $|N| < 4$.

$|N| = 4 \Rightarrow |\prod_{\chi}^{N}| = 3$ and $\prod_{\chi}^{N} = \{\{(12),(34)\},\{(13),(24)\},\{(14),(23)\}\}$.

$|N| = 5 \Rightarrow |\prod_{\chi}^{N}| = 25$.

Thus $\prod_{\chi}^{N}$ is not a narrow set.

Throughout the paper, we assume that $|N| \geq 4$.

4.2.1 Blocking, Stability, and Exchange

Definition 4.2.1. *Let $(N, \succsim)$ be a hedonic coalition formation game. Let $\pi \in \prod^{N}$ be a coalition structure.*

- *A coalition $S \subseteq N$* ***blocks*** *the coalition structure π if every individual $i \in S$ strictly prefers S to his current coalition $\pi(i)$, i.e., $\forall i \in S$: $S \succ_i \pi(i)$.*

- *A coalition $S \subseteq N$* ***weakly blocks*** *the coalition structure π if every individual $i \in S$ weakly prefers S to $\pi(i)$ and there exists at least one individual $j \in S$ who strictly prefers S to his current coalition $\pi(j)$, i.e., $\forall i \in S$: $S \succsim_i \pi(i)$ and $\exists j \in S$: $S \succ_j \pi(j)$.*

- *Coalition structure π is said to be* ***core stable*** *(CS) if it is not blocked by any coalition.*

- *Coalition structure π is said to be* ***strictly core stable*** *(SCS) if it is not weakly blocked by any coalition.*

Definition 4.2.2. *Let $(N, \succsim)$ be a hedonic game. Let $\pi \in \prod_{\mathcal{X}}^{N}$ such that $\pi = \{T_1, ..., T_K\}$ ($K \leq |N|$ positive integer). $\zeta_l \subset T_l$ and $\zeta_m \subset T_m$ are called independent subcoalitions if $l \neq m$.*

Proper subsets of different coalitions are independent subcoalitions.

Definition 4.2.3. *The collection $\mathcal{Z}(\pi) = \{\zeta_{l_1}, ..., \zeta_{l_L}\}$ ($L \leq K$) is called collection of independent subcoalitions given $\pi \in \prod_{\mathcal{X}}^{N}$ if $\forall l_m \in \{l_1, ..., l_L\}$, we have $\zeta_{l_m} \subset T_{l_m}$ and $\nexists\, \zeta_{l_{m'}} \subset T_{l_m}$ such that $\zeta_{l_m} \neq \zeta_{l_{m'}}$.*

Definition 4.2.4. *Let $(N, \succsim)$ be a hedonic game. Let $\pi \in \prod^{N}$ such that $\pi = \{T_1, ..., T_K\}$ ($K \leq |N|$ positive integer). $\mathcal{Z}(\pi) = \{\zeta_{l_1}, ..., \zeta_{l_L}\}$* ***exchange block*** *$\pi$ if and only if there exists full cyclic permutation $\sigma : \{l_1, ..., l_L\} \to \{\sigma(l_1), ..., \sigma(l_L)\}$ such that $\forall l_m \in \{l_1, ..., l_L\}$, $\forall i \in \zeta_{\sigma(l_m)}$ we have $(T_{\sigma(l_{m+1})} \backslash \zeta_{\sigma(l_{m+1})}) \cup \zeta_{\sigma(l_m)} \succ_i T_{\sigma(l_m)}$ (mod l_L).*

Notice that the full cyclic permutation σ finds and aligns independent subcoalitions of π so that each former subcoalition take over the place of next subcoalition and every individual in subcoalitions gets better off after exchange block.

Example 4.2.1. *Let* $\pi = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$. *Assume that* $\zeta_{l_1} \subset T_2, \zeta_{l_2} \subset T_4$, *and* $\zeta_{l_3} \subset T_7$ *and* $\mathcal{Z}(\pi) = \{\zeta_{l_1}, \zeta_{l_2}, \zeta_{l_3}\}$ *exchange blocks* π *such that* $\sigma : \{l_1, l_2, l_3\} \rightarrow \{x_1, x_2, x_3\}$ *is a bijection where* $\sigma(l_1) = x_1, \sigma(l_3) = x_2$, *and* $\sigma(l_2) = x_3$.

Then, $\zeta_{x_1}(\zeta_{l_1})$ *takes over the place of* $\zeta_{x_2}(\zeta_{l_3})$, $\zeta_{x_2}(\zeta_{l_3})$ *takes over the place of* $\zeta_{x_3}(\zeta_{l_2})$, *and* $\zeta_{x_3}(\zeta_{l_2})$ *takes over the place of* $\zeta_{x_1}(\zeta_{l_1})$.

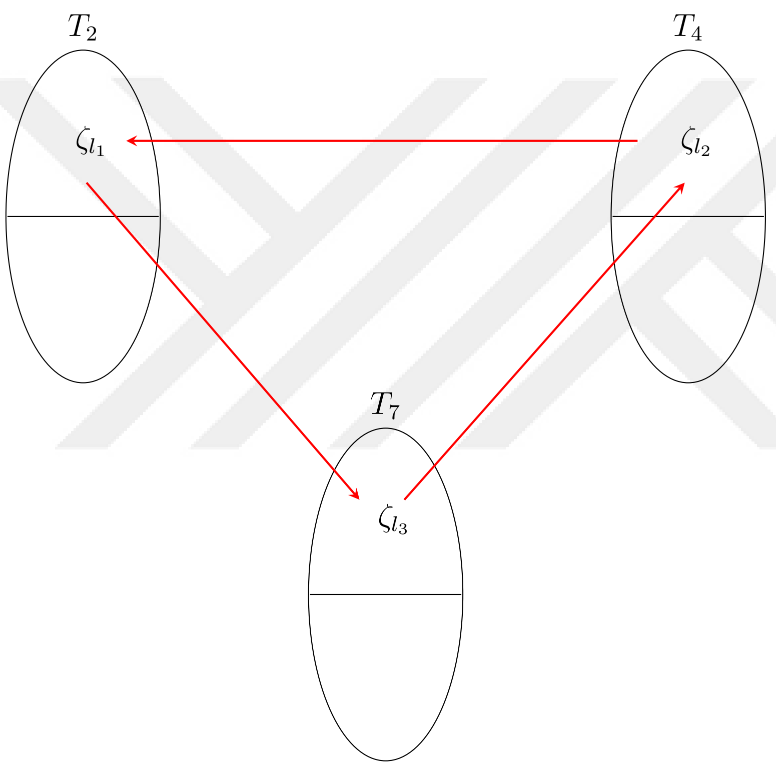

Fig. 4.1: Exchange block behaviour conducted by independent subcoalitions ζ_{l_1}, ζ_{l_1}, and ζ_{l_1}.

4.2.2 Membership Property Rights

Sertel (1982 and 1992) introduced four different *membership property right concepts*, namely *free exit*, *approved exit*, *free entry*, and *approved entry*. When a move (deviation) is intended, these concepts initiate four different *membership property* *right codes*, namely *free exit-free entry* (FX-FE), *free exit-approved entry* (FX-AE), *approved exit-free entry* (AX-FE), and *approved exit-approved entry* (AX-AE). A membership property right code defines the set of individuals whose approval is needed (for this move to take place) when a group of individuals plan to deviate.

Let $N \in \mathcal{N}$ be an arbitrary set of individuals. Given any $A \in \mathcal{C}$, any $i \in A$, and any $B \in (2^N \backslash \{A\})$, let $\mathcal{K}_{i,A,B} \subseteq N$ denote the set of individuals who must be consulted when individual i wants to leave coalition A and join the (possibly empty) coalition B. Then, the collection $\mathcal{K} = \{\mathcal{K}_{i,A,B}\}$ is the membership property right code of the set of individuals.

Definition 4.2.5. *For any $A \in \mathcal{C}$, any $i \in A$, and any $B \in (2^N \backslash \{A\})$, membership property rights are:*

- *FX-FE if and only if $\mathcal{K}_i = \{i\}$*
- *FX-AE if and only if $\mathcal{K}_i = \{i\} \cup B$*
- *AX-FE if and only if $\mathcal{K}_i = A$*
- *AX-AE if and only if $\mathcal{K}_i = A \cup B$*

Given any $N \in \mathcal{N}$, any $\pi = \{T_1, ..., T_K\} \in \prod_{\mathcal{X}}^N$, assume that subcoalitions $\zeta_A \subset T_A \in \pi$ and $\zeta_B \subset T_B \in \pi$ exchange their places. Then, if $i \in \zeta_A$, under FX-FE membership property rights $\mathcal{K}_i = \{i\}$ and under FX-AE membership property rights $\mathcal{K}_i = (T_B \backslash \zeta_B) \cup \{i\}$.

AX-FE and AX-AE membership property rights are not applicable when we consider exchange because AX concept makes our model dynamic. However, our exchange definition model is static.

In the following section, we introduce **exchange stability** and **strong exchange stability** definitions under FX-FE and FX-AE membership property rights.

4.2.3 Exchange Stability

Definition 4.2.6. *Let* $(N, \succsim)$ *be a hedonic game such that* $|N| \geq 4$.

- $\pi \in \prod^N$ *is called* ***exchange stable*** *under FX-FE membership property rights if and only if it is not exchange blocked by any collection* $\mathcal{Z}(\pi)$.
- $\pi \in \prod^N$ *is called* ***exchange stable*** *under FX-AE membership property rights if there exists a collection* $\mathcal{Z}(\pi)$ *which exchange block* π*, then* $\exists \zeta_{l_m} \in \mathcal{Z}(\pi)$ *and* $i \in \zeta_{l_m}$ *such that there exist at least one* $j \in \mathcal{K}_i$ *which is worse off.*

Example 4.2.2. *Consider the game* $(N, \succsim)$ *with* $N = \{1, 2, 3, 4\}$ *and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
14	24	13	124
134	234	34	34
12	12	123	24
13	2	3	4
1	...	...	...
...			

We observe that $\pi^1 = \{(13), (24)\}$ *and* $\pi^2 = \{(12), (34)\}$ *are core stable. However, individuals* 1 *and* 4 *by exchanging their places block* π^1 *and they get better off. Similarly, individuals* 2 *and* 3 *by exchanging their places block* π^2 *and they get better off. Thus, core stable coalition structures are not immune to blockings via exchanges.*

Definition 4.2.7. *Let* $(N, \succsim)$ *be a hedonic game such that* $|N| \geq 4$.

- *Assume that membership property rights are FX-FE. $\pi \in \prod^N$ is called* ***strongly exchange stable*** *if and only if it is exchange stable and core stable.*

- *Assume that membership property rights are FX-AE. $\pi \in \prod^N$ is called* ***strongly exchange stable*** *if and only if it is exchange stable and core stable.*

We are going to call exchange stability under FX-FE membership property rights for short exchange stability, and so strong exchange stability, during this study.

4.3 Main Result

In this section, we discuss the existence of strong exchange stability under FX-FE membership property rights. When membership property rights are assumed to be FX-FE, a coalition structure $\pi \in \prod^N$ is called **exchange stable** if and only if it is not exchange blocked by any collection $\mathcal{Z}(\pi)$, and it is called strongly exchange stable if and only if it is core stable and exchange stable.

The following definitions of top coalition property and weak top coalition property are due to Banerjee et al. (2001). We are going to prove that weak top coaltion property is sufficient for the existence of strong exchange stability as well as core stability.

Definition 4.3.1. *Let $(N, \succsim)$ be a hedonic game. Let $V \subseteq N$. $S \subseteq V$ is called as a* ***top coalition*** *of V if and only if $\forall i \in S$, $\forall T \subseteq V$ such that $i \in T$, we have $S \succsim_i T$.*

If S is a top coalition of V, we write $S = \mathcal{TC}(V)$.

A hedonic game $(N, \succsim)$ satisfies top coalition property if and only if for any non-empty set of players $V \subseteq N$, there exists a top coalition of V.

Definition 4.3.2. *Let $(N, \succsim)$ be a hedonic game. Let $V \subseteq N$. $S \subseteq V$ is called as a* ***weak top coalition*** *of V if and only if S has an ordered partition $\{S^1, ..., S^l\}$ such that*

(i) $\forall i \in S^1$, $\forall T \subseteq V$ with $i \in T$, we have $S \succsim_i T$

(ii) $\forall k > 1$, $\forall i \in S^k$, and $\forall T \subseteq V$ with $i \in T$, we have $S \succsim_i T$, we have $T \succ_i S \Rightarrow T \cap (\bigcup_{m<k} S_m) \neq \emptyset$.

A hedonic game $(N, \succsim)$ satisfies weak top coalition property if and only if for any non-empty set of players $V \subseteq N$, there exists a weak top coalition of V.

Theorem 4.3.1. *Let $(N, \succsim)$ be a hedonic game which satisfies weak top coalition property. Then, there always exists a strongly exchange stable coalition structure.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies weak top coalition property. Let $V_1 = N$, $S_1 \subseteq V_1$ be a weak top coalition of V_1, and $\{S_1^1, ..., S_1^{t_1}\}$ be its partition. Let $V_2 = V_1 \backslash S_1$, S_2 be a weak top coalition of V_2, and $\{S_2^1, ..., S_2^{t_2}\}$ be its partition. For each k, let define the triple $(V_k, S_k, \{S_k^1, ..., S_k^{t_k}\})$. Since N is finite, this procedure terminates in finite steps (i.e., there exists an integer K such that $V_K = \emptyset$ and $V_{K+1} = \emptyset$).

Let $\pi^* = \{S_1, ..., S_K\}$. Banerjee et al. (2001) proved that π^* is core stable. Assume that π^* is not exchange stable. Then, there exists a collection of independent subcoalitions $\mathcal{Z}(\pi^*) = \{\zeta_{l_1}, ..., \zeta_{l_L}\}$ that exchange blocks π^*. Then, there exists a full cyclic permutation $\sigma : \{l_1, ..., l_L\} \rightarrow \{\sigma(l_1), ..., \sigma(l_L)\}$ such that $\forall l_m \in \{l_1, ..., l_L\}$, $\forall i \in \zeta_{\sigma(l_m)}$ we have $(S_{\sigma(l_{m+1})} \backslash \zeta_{\sigma(l_{m+1})}) \cup \zeta_{\sigma(l_m)} \succ_i S_{\sigma(l_m)}$ (mod l_L).

By the above procedure, $\forall l_m \in \{l_1, ..., l_L\}$, $S_{\sigma(l_m)}$ is weak top coalition of $V_{\sigma(l_m)}$. $\mathcal{Z}(\pi^*)$ exchange blocks π^*, then consider the independent subcoalitions $\zeta_{\sigma(l_m)}$ and $\zeta_{\sigma(l_{m+1})}$ and coalitions $S_{\sigma(l_t)}$ and $S_{\sigma(l_{t+1})}$. $S_{\sigma(l_t)}$ has a partition $\{S^1_{\sigma(l_t)}, ..., S^{l_{\sigma(l_m)}}_{\sigma(l_m)}\} = \{T^1, ..., T^{l_{x_1}}\}$ and $S_{\sigma(l_{m+1})}$ has a partition $\{S^1_{\sigma(l_{m+1})}, ..., S^{l_{\sigma(l_{t+1})}}_{\sigma(l_{m+1})}\} = \{U^1, ..., U^{l_{x_2}}\}$.

$\forall i \in \zeta_{\sigma(l_m)}$ we have $(S_{\sigma(l_{m+1})} \backslash \zeta_{\sigma(l_{m+1})}) \cup \zeta_{\sigma(l_m)} \succ_i S_{\sigma(l_m)}$. WLOG, assume that $\sigma(l_m) < \sigma(l_{m+1})$, i.e., $S_{\sigma(l_m)}$ is incorporated into π^* before $S_{\sigma(l_{m+1})}$.

Thus, $\forall i \in T^1$, $\forall T \subseteq V_{\sigma(l_m)}$ with $i \in T$, we have $S_{\sigma(l_m)} \succsim_i T$. Then, $\zeta_{\sigma(l_m)}$ can not be a subset of T^1, i.e., $\zeta_{\sigma(l_m)} \nsubseteq T^1$. As well, none of the individuals in $\zeta_{\sigma(l_m)}$ can be a member of T^1, i.e., $\zeta_{\sigma(l_t)} \cap T^1 = \emptyset$. At best, $\zeta_{\sigma(l_t)} \subseteq T^2$. Let $\zeta_{\sigma(l_t)} \subseteq T^2$. Weak top coalition property implies that and $\forall k > 1$, $\forall i \in T^k$, and $\forall T \subseteq V_{\sigma(l_m)}$ with $i \in T$, we have $T \succ_i S_{\sigma(l_m)} \Rightarrow T \cap (\bigcup_{m<k} T^m) \neq \emptyset$. $S_{\sigma(l_m)}$ is incorporated into π^* before $S_{\sigma(l_{m+1})}$, thus $V_{\sigma(l_{m+1})} \subseteq V_{\sigma(l_m)}$ and $((S_{\sigma(l_{m+1})} \backslash \zeta_{\sigma(l_{m+1})}) \cup \zeta_{\sigma(l_m)}) = T \subseteq V_{\sigma(l_m)}$ and $((S_{\sigma(l_{m+1})} \backslash \zeta_{\sigma(l_{m+1})}) \cup \zeta_{\sigma(l_m)}) \cap T^1 = \emptyset$. Then, $\zeta_{\sigma(l_m)}$ can not be a subset of T^2, i.e., $\zeta_{\sigma(l_m)} \nsubseteq T^2$. As well, none of the individuals in $\zeta_{\sigma(l_m)}$ can be a member of T^2, i.e., $\zeta_{\sigma(l_t)} \cap T^2 = \emptyset$. At best, $\zeta_{\sigma(l_t)} \subseteq T^3$.... $S_{\sigma(l_m)}$ is weak top coalition of $V_{\sigma(l_m)}$ and this implies $\forall k > 1, \forall i \in T^k$, and $\forall T \subseteq V_{\sigma(l_m)}$ with $i \in T$, we have $T \cap \bigcap_{m<k} T^m = \emptyset$. Thus, $\forall T^k \in \{T^1, ..., T^{l_{x_1}}\}$, we have $\zeta_{\sigma(l_m)} \nsubseteq T^k$ and such an exchange contradicts with weak top coalition property. Then $\forall i \in S_{\sigma(l_m)}$, $S_{\sigma(l_m)} \succsim_i T$ which implies π^* is exchange stable. Therefore, π^* is strongly exchange stable.

□

Now, we are going to prove that $\mathcal{B}$-responsiveness condition is sufficient for the existence of strong exchange stability. Before we prove our result, let us remember $\mathcal{B}$-responsiveness condition and $\mathcal{B}1$ Algorithm.

Definition 4.3.3. *Let $(N, \succsim)$ be a hedonic game and $i \in N$ be any individual. $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{B}$-**responsive** iff*

- $|CH(i, N)| = 1$,
- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i, N) \supseteq S \supset T$ we have $S \succ_i T$,*

- *For all $S, T \in \mathcal{C}_i^N$ such that $ch(i,N) \supseteq S$, $ch(i,N) \supseteq T$, and $\mid S \mid = \mid T \mid$ we have $S \sim_i T$,*

- *For all $S \in \mathcal{C}_i^N$ such that $S \backslash ch(i,N) \neq \emptyset$, we have $\{i\} \succsim_i S$*

*We say that $\succsim \in \mathcal{R}^N$ satisfies $\mathcal{B}$-responsiveness iff $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{B}$-responsive $\forall i \in N$. A hedonic game $(N, \succsim)$ with $\mathcal{B}$-responsive preference profile $\succsim \in \mathcal{R}^N$ is said to satisfy $\mathcal{B}$-**responsiveness**.*

Now, let's define $\mathcal{B}1$ Algorithm.

$\mathcal{B}1$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{B}$-responsiveness.

Step 1: Set $\mathcal{P}^1 := N$ and $\pi^0 := \{(i)\}_{i \in \mathcal{P}^1}$.

Step 2: $\forall m \in \{1, ..., |N|\}$, construct the set

$\mathcal{M}^m := \{S \subseteq \mathcal{P}^m | \forall i \in S : S \succ_i \pi^{m-1}(i)\}$.

Step 3: Choose $S \in \mathcal{M}^m$ such that $\forall T \in \mathcal{M}^m : |S| \geq |T|$.

Step 4: Set $S^m := S$, $\pi^m := \{S^m\} \cup \{(i)\}_{i \in \mathcal{P}^m \backslash S^m}$, and $\mathcal{P}^{m+1} := \mathcal{P}^m \backslash S^m$.

Step 5: If $\mathcal{M}^{m+1} = \emptyset$, stop. Return π^m as outcome.

Theorem 4.3.2. *Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{B}$-responsiveness. Then there always exists a strongly exchange stable coalition structure.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{B}$-responsiveness. Assume that $\mathcal{B}1$ Algorithm yields the coalition structure $\pi = \{T_1, ..., T_K\}$. *Theorem 2.2.2* implies that π is core stable.

Thus, assume that π is not exchange stable. Then, $\exists \mathcal{Z}(\pi) = \{\zeta_{l_1}, ..., \zeta_{l_L}\}$ collection of independent subcoalitions which exchange block π. Then, $\exists \sigma$ (bijection) full cyclic permutation such that $\sigma : \{l_1, ..., l_L\} \rightarrow \{x_1, ..., x_L\}$ and $\forall x_l \in \{x_1, ..., x_L\}$, $\forall i \in \zeta_{x_l}$ we have $(T_{x_{l+1}} \backslash \zeta_{x_{l+1}}) \cup \zeta_{x_l} \succ_i T_{x_l}$ (mod L).

$l = 1 \Rightarrow \forall i \in \zeta_{x_1}$ we have $(T_{x_2} \backslash \zeta_{x_2}) \cup \zeta_{x_1} \succ_i T_{x_1}$

$l = 2 \Rightarrow \forall i \in \zeta_{x_2}$ we have $(T_{x_3} \backslash \zeta_{x_3}) \cup \zeta_{x_2} \succ_i T_{x_2}$

...

$l = L-1 \Rightarrow \forall i \in \zeta_{x_{L-1}}$ we have $(T_{x_L} \backslash \zeta_{x_L}) \cup \zeta_{x_{L-1}} \succ_i T_{x_{L-1}}$

$l = L \Rightarrow \forall i \in \zeta_{x_L}$ we have $(T_{x_1} \backslash \zeta_{x_1}) \cup \zeta_{x_L} \succ_i T_{x_L}$

All of the lines above come true only when $\forall x_l \in \{x_1, ..., x_L\}$ $|(T_{x_{l+1}} \backslash \zeta_{x_{l+1}}) \cup \zeta_{x_l}| > |T_{x_l}|$ because of $\mathcal{B}$-responsiveness. If we add all of the inequalities side by side, we get $\sum_{l=1}^{L} |(T_{x_{l+1}} \backslash \zeta_{x_{l+1}}) \cup \zeta_{x_l}| > \sum_{l=1}^{L} |T_{x_l}|$ (mod L). But the summation on both sides equal to $|N|$ and we can not get $|N| > |N|$. Thus, $|N| = |N|$. This implies that individuals in the last independent subcoalition get better off when they leave their coalition and take the place of the consecutive latter independent subcoalition which has smaller cardinality. $\forall i \in \zeta_{x_L}$ we have $(T_{x_1} \backslash \zeta_{x_1}) \cup \zeta_{x_L} \succ_i T_{x_L}$, but $|(T_{x_1} \backslash \zeta_{x_1}) \cup \zeta_{x_L}| < |T_{x_L}|$. This is a contradiction elicited by $\mathcal{B}$-responsiveness of preferences.

Therefore, π is strongly exchange stable. $\square$

Now, we are going to prove that $\mathcal{G}$-singularity condition is sufficient for the existence of strong exchange stability. Before we prove our result, let us remember $\mathcal{G}$-singularity condition and $\mathcal{G}1$ Algorithm.

Definition 4.3.4. *Let $(N, \succsim)$ be a hedonic game and $i \in N$ be any individual. $\succsim_i \in R(\mathcal{C}_i^N)$ satisfies $\mathcal{G}$-**singularity** iff for all $S \in \mathcal{C}_i^N$, $S \succ_i \{i\} \Rightarrow S \in CH(i, N)$.*

*We say that $\succsim \in \mathcal{R}^N$ satisfies $\mathcal{G}$-singularity iff $\succsim_i \in R(\mathcal{C}_i^N)$ satisfies $\mathcal{G}$-singularity $\forall i \in N$. A hedonic game $(N, \succsim)$ is said to satisfy $\mathcal{G}$-**singularity** iff $\succsim \in \mathcal{R}^N$ satisfies $\mathcal{G}$-singularity.*

Theorem 4.3.3. *Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{G}$-singularity. Then there always exists a strongly exchange stable coalition structure.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{G}$-singularity. Consider the $\mathcal{G}1$ Algorithm.

$\mathcal{G}1$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{G}$-singularity.

Step 1: Set $V_0 := N$, $S^0 := \emptyset$ and $\pi^0 := \{(i)\}_{i \in V_0}$.

Step 2: $\forall m \in \{0, 1, ..., |N| - 1\}$, search for $T \in \mathcal{C}|_{V_m}$ such that $|T| \geq 2$ and T is mutually best for all $i \in T$.

Step 2.1: If there exists such T, set $S^{m+1} := T$.

Set $V_{m+1} := V_m \backslash S^{m+1}$ and $\pi^{m+1} := \{S^{m+1}\} \cup (\pi^m \backslash \{(i)\}_{i \in S^{m+1}})$.

Step 2.2: If there does not exist such T, stop. Define $\pi^{m+1} = \pi^m$.

Go to Step 3.

Step 3: Return π^m as outcome.

Let π^m be the outcome of the $\mathcal{G}1$ Algorithm. π^m is core stable (*see Theorem 2.2.3*). The $\mathcal{G}1$ Algorithm always brings out mutually best coalitions and singletons. Thus, by definition, there does not exist discontented individuals who come together and form a collection of independent subcoalitions which exchange block π^m. Therefore, π^m is strongly exchange stable. $\square$

$\mathcal{A}$-responsiveness on its own is not sufficient for the existence of strong exchange stability. If it is intensified with top-mutuality, there always exist strongly exchange stable coalition structures.

Let us continue with the definition $\mathcal{A}$-responsiveness and $\mathcal{A}1$ Algorithm.

Definition 4.3.5. *Let $(N, \succsim)$ be a hedonic game and $i \in N$ be any individual. $\succsim_i \in R(\mathcal{C}_i^N)$ is $\mathcal{A}$-**responsive** iff*

- $|CH(i, N)| = 1$,

- *For all* $S, T \in \mathcal{C}_i^N$ *such that* $ch(i,N) \subseteq S \subset T$ *we have* $S \succ_i T$*,*
- *For all* $S, T \in \mathcal{C}_i^N$ *such that* $ch(i,N) \subseteq S$*,* $ch(i,N) \subseteq T$*, and* $\mid S \mid = \mid T \mid$ *we have* $S \sim_i T$*,*
- *For all* $S \in \mathcal{C}_i^N$ *such that* $ch(i,N) \backslash S \neq \emptyset$*, we have* $N \succsim_i S$

We say that $\succsim \in \mathcal{R}^N$ *satisfies* $\mathcal{A}$*-responsiveness iff* $\succsim_i \in R(\mathcal{C}_i^N)$ *is* $\mathcal{A}$*-responsive* $\forall i \in N$*. A hedonic game* $(N, \succsim)$ *with* $\mathcal{A}$*-responsive preference profile* $\succsim \in \mathcal{R}^N$ *is said to satisfy* $\mathcal{A}$***-responsiveness****.*

In order to define the $\mathcal{A}1$ Algorithm, we firstly introduce the following recursive function.

Let $t \in \mathbb{Z}^+$ be a positive integer. For every $i \in N$ and $S \in \mathcal{C}_i^N$, let define the function $\mathcal{X}^t : N \times \mathcal{C} \to \mathcal{C}$ as follows:

$\mathcal{X}^1(i,S) = ch(i,S)$.

$\mathcal{X}^{t+1}(i,S) = \bigcup_{j \in \mathcal{X}^t(i,S)} ch(j,S)$ for each positive integer t.

By definition, $\mathcal{X}^t(i,S) \subseteq \mathcal{X}^{t+1}(i,S) \subseteq S$ and $\mathcal{X}^{|N|+1}(i,S) = \mathcal{X}^{|N|}(i,S)$. By $\mathcal{X}\mathcal{X}(i,S)$ we denote $\mathcal{X}^{|N|}(i,S)$.

$\mathcal{A}1$ Algorithm

Given: $(N, \succsim)$ hedonic game which satisfies $\mathcal{A}$-responsiveness.

Step 1: Set $\mathcal{P}^1 := N$ and $\pi^0 := \emptyset$.

Step 2: For $m = 1$ to $|N|$:

Step 2.1: Select $i \in \mathcal{P}^m$ satisfying $|\mathcal{X}\mathcal{X}(i, N)| \leq |\mathcal{X}\mathcal{X}(j, N)|$ for each $j \in N$.

Set $S^m := \mathcal{X}\mathcal{X}(i, N)$.

Step 2.2: If $S^m \subseteq \mathcal{P}^m$, set $\pi^m := \pi^{m-1} \cup \{S^m\}$ and $\mathcal{P}^{m+1} := \mathcal{P}^m \backslash S^m$.

Step 2.3: If $S^m \nsubseteq \mathcal{P}^m$, set $\pi^m := \pi^{m-1} \cup \{(i)\}_{i \in \mathcal{P}^m}$ and $\mathcal{P}^{m+1} = \mathcal{P}^m$.

Go to Step 3.

Step 3: Select $H \subseteq \mathcal{P}^{m+1}$ such that $\forall i \in H : H \succsim_i \{i\}$ and $H \succ_i H'$ $\forall i \in H \cap H', \forall H' \in \mathcal{P}^{m+1}$.

Set $S^{m+1} := H$ and $\pi^{m+1} := \pi^{m-1} \cup \{S^{m+1}\} \cup \{(i)\}_{i \in \mathcal{P}^{m+1} \backslash S^{m+1}}$.

Return π^{m+1} as outcome.

Step 4: If $\nexists H \subseteq \mathcal{P}^{m+1}$ such that for every $i \in H$: $H \succsim_i \{i\}$, return π^m as outcome.

Remark 4.3.1. *$\mathcal{A}$-responsiveness is not sufficient for strong exchange stability. Consider the following example.*

Example 4.3.1. *Consider the hedonic game $(N, \succsim)$ with $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the following preference profile:*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$	$\succsim_6$	$\succsim_7$	$\succsim_8$
1234	*123*	*234*	*134*	*15*	*26*	*37*	*48*
*****	****	****	****	***	***	***	***
******	*****	*****	*****	****	****	****	****
*******	******	******	******	*****	*****	*****	*****
N	*******	*******	*******	******	******	******	******
⋮	*N*	*N*	*N*	*******	*******	*******	*******
⋮	⋮	⋮	⋮	*N*	*N*	*N*	*N*
⋮	⋮	⋮	⋮	*56 ∼ 57*	*67 ∼ 68*	*78 ∼ 789*	*58*
⋮	⋮	⋮	⋮	*58*	*56*	*67*	*78*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

The above hedonic game $(N, \succsim)$ *satisfies* $\mathcal{A}$*-responsiveness. Arrays of *'s symbolize the coalitions which has cardinality equal to the number of *'s and which are superset of* $ch(i, N)$*. For example,* $ch(1, N) = (1234)$*. Thus, (*****) symbolizes the coalitions* $\{(12345), (12346), (12347), (12348)\}$*.*

If we run the $\mathcal{A}1$ *Algorithm, we get* $\pi = \{(1234), (56), (78)\}$*.* π *is the only core stable coalition structure of the game. It is not strongly exchange stable, because individuals* 6 *and* 8 *exchange block* π*.*

Now, let us remember the definition of top-mutuality.

Definition 4.3.6. *Let $(N, \succsim)$ be a hedonic game. Let $i, j \in N$ be arbitrary individuals. Assume that $\forall i \in N$, $|CH(i, N)| = 1$.*

- $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are* ***top-mutual*** *iff* $i \in ch(j, N) \Rightarrow j \in ch(i, N)$.
- $\succsim \in \mathcal{R}^N$ *satisfies top-mutuality iff* $\succsim_i \in R(\mathcal{C}_i^N)$ *and* $\succsim_j \in R(\mathcal{C}_j^N)$ *are top-mutual* $\forall i, j \in N$ *such that* $j \in ch(i, N)$.

We say that a hedonic game $(N, \succsim)$ satisfies ***top-mutuality*** *iff* $\succsim \in \mathcal{R}^N$ *satisfies top-mutuality.*

Theorem 4.3.4. *Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{A}$-responsiveness and top-mutuality. Then there always exists a strongly exchange stable coalition structure.*

Proof. Let $(N, \succsim)$ be a hedonic game which satisfies $\mathcal{A}$-responsiveness and top-mutuality. Let $\pi^K = \{T_1, ..., T_K\}$ be the outcome of the $\mathcal{A}1$ algorithm. Hierarchy provided by $\mathcal{A}$-responsiveness and reciprocal agreement provided by top-mutuality ensure that π^K incorporates only mutually best coalitions, least common supersets of maximal sets, and the individuals whose maximal sets are singletons. No coalition which is weakly ranked blow N is included in π^K. As well, no individual other than whose maximal set is standing alone, stands alone. Otherwise, we get a contradiction with top-mutuality.

π^K is core stable (*see Theorem 2.2.1*). Suppose that, π^K is not strongly coalitionally stable. Then, $\exists \mathcal{Z}(\pi) = \{\zeta_{l_1}, ..., \zeta_{l_L}\}$ collection of independent subcoalitions which exchange block π^K. Then, $\exists \sigma$ (bijection) full cyclic permutation such that σ :

$\{l_1, ..., l_L\} \rightarrow \{x_1, ..., x_L\}$ and $\forall x_l \in \{x_1, ..., x_L\}$, $\forall i \in \zeta_{x_l}$ we have $(T_{x_{l+1}} \backslash \zeta_{x_{l+1}}) \cup \zeta_{x_l} \succ_i T_{x_l}$ (mod L).

$l = 1 \Rightarrow \forall i \in \zeta_{x_1}$ we have $(T_{x_2} \backslash \zeta_{x_2}) \cup \zeta_{x_1} \succ_i T_{x_1}$

$l = 2 \Rightarrow \forall i \in \zeta_{x_2}$ we have $(T_{x_3} \backslash \zeta_{x_3}) \cup \zeta_{x_2} \succ_i T_{x_2}$

...

$l = L-1 \Rightarrow \forall i \in \zeta_{x_{L-1}}$ we have $(T_{x_L} \backslash \zeta_{x_L}) \cup \zeta_{x_{L-1}} \succ_i T_{x_{L-1}}$

$l = L \Rightarrow \forall i \in \zeta_{x_L}$ we have $(T_{x_1} \backslash \zeta_{x_1}) \cup \zeta_{x_L} \succ_i T_{x_L}$

All of the lines above come true only when $\forall x_l \in \{x_1, ..., x_L\}$ $|(T_{x_{l+1}} \backslash \zeta_{x_{l+1}}) \cup \zeta_{x_l}| < |T_{x_l}|$ because of $\mathcal{A}$-responsiveness of preferences. If we add all of the inequalities side by side, we get $\sum_{l=1}^{L} |(T_{x_{l+1}} \backslash \zeta_{x_{l+1}}) \cup \zeta_{x_l}| < \sum_{l=1}^{L} |T_{x_l}|$ (mod L). But the summation on both sides equal to $|N|$ and we can not get $|N| < |N|$. Thus, $|N| = |N|$. This implies that individuals in the last independent subcoalition get better off when they leave their coalition and take the place of the consecutive latter independent subcoalition which has greater cardinality. $\forall i \in \zeta_{x_L}$ we have $(T_{x_1} \backslash \zeta_{x_1}) \cup \zeta_{x_L} \succ_i T_{x_L}$, but $|(T_{x_1} \backslash \zeta_{x_1}) \cup \zeta_{x_L}| > |T_{x_L}|$. This is a contradiction elicited by $\mathcal{A}$-responsiveness of preferences.

Therefore, π^K is strongly exchange stable. □

In the next proposition, we reveal the relation between echange stability and core stability under FX-AE membership property rights. If preferences of individuals are assumed to be strict, then exchange stability and core stability coincides under FX-AE membership property rights.

Proposition 4.3.1. *Let $(N, \succ)$ be a hedonic game in which individuals have strict preferences. Under FX-AE membership property rights, a coalition structure $\pi \in \prod^N$ is exchange stable if it is core stable.*

Proof. Let $(N, \succ)$ be a hedonic game in which individuals have strict preferences. Let $\pi = \{S_1, ..., S_K\} \in \prod^N$ be core stable. Moreover, suppose that membership property rights are FX-AE.

To the contrary, suppose that π is not exchange stable. Then, there exists a collection of independent subcoalitions $\mathcal{Z}(\pi) = \{\zeta_{l_1}, ..., \zeta_{l_L}\}$ that exchange blocks π. Then, there exists a full cyclic permutation $\sigma : \{l_1, ..., l_L\} \to \{\sigma(l_1), ..., \sigma(l_L)\}$ such that $\forall l_m \in \{l_1, ..., l_L\}$, $\forall i \in \zeta_{\sigma(l_m)}$ we have $(S_{\sigma(l_{m+1})} \backslash \zeta_{\sigma(l_{m+1})}) \cup \zeta_{\sigma(l_m)} \succ_i S_{\sigma(l_m)}$ (mod l_L). Moreover, $\forall l_m \in \{l_1, ..., l_L\}$, $\forall j \in (S_{\sigma(l_{m+1})} \backslash \zeta_{\sigma(l_{m+1})})$, we have $(S_{\sigma(l_{m+1})} \backslash \zeta_{\sigma(l_{m+1})}) \cup \zeta_{\sigma(l_m)} \succ_j S_{\sigma(l_{m+1})}$ (mod l_L).

But then, we say that the coalition $H = \bigcup_{l_m \in \{l_1, ..., l_L\}} S_{\sigma(l_m)}$ blocks π. This contradicts with the fact that π is core stable. Therefore, π is also exchange stable.

□

5. CONCLUSION

In this dissertation, we studied hedonic coalition formation games. We focused on three equilibrium concepts, namely core stability, strong coalitional stability, and strong exchange stability. Our main scope has been exploring the domains of hedonic games in which aforementioned stability concepts always exist. Moreover, we studied strong coalitional stability and strong exchange stability under different membership property rights (a la Sertel (1982 and 1992)) and compare it with well-known stability concepts with respect to implication relation.

In the second chapter, we studied core stability. We introduced three new domain restrictions for hedonic games, namely $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity. We proved that in all the three domains, core stable coalition structures always exist. These domain restrictions are directly imposed on individuals' preferences and they are independent. This property enabled us to prove an interesting result which can be regarded as a new domain extension property. We proved that if we have a hedonic game such that the set of individuals is partitioned into three subsets and the first subset consists of individuals with $\mathcal{A}$-responsive preferences, the second subset consists of individuals with $\mathcal{B}$-responsive preferences, and the last subset consists of individuals with $\mathcal{G}$-singular preferences, then we can always find a core stable coalition structure in such a hedonic game.

In order to prove the propositions we have claimed in the second chapter, we introduced four nice algorithms which run over individuals' preferences. The computational complexity and the efficiency analysis of these algorithms is not conducted in that chapter. This topic is all by itself a new research project which is still an interesting open question. Another interesting open question for future research is about coalition formation rules (algorithms). The questions "*when individuals preferences satisfy $\mathcal{A}$-responsiveness ($\mathcal{B}$-responsiveness, or $\mathcal{G}$-singularity) is the mechanism induced by the $\mathcal{A}1$ Algorithm ($\mathcal{B}1$ Algorithm, or $\mathcal{G}1$ Algorithm, respectively) is (the only) strategy-proof mechanism that always selects core stable coalition structures ?*" still wait an answer.

In the literature, several domain conditions are introduced. Although the great majority of the domain conditions are sufficient for the existence of core stability, there exist only two domain conditions which are necessary (see Pápai (2004) and Iehlé (2007)). The *single lapping property* of Pápai and the *pivotal balancedness* property of Iehlé are domain conditions which are imposed on the profile of preferences. However, there is still no attempt to find a necessary domain condition which is imposed on individuals' preferences. This is another open question for future research.

In the third chapter, we studied strong coalitional stability which is a particular refinement of core stability and Nash stability. We have shown that, strong coalitional stability cures the deficiencies of coalition structures which can not be healed by the core stability and the Nash stability. Firstly, we introduced some domain restrictions which are sufficient for the existence of strong coalitional stability under FX-FE membership property rights. $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity conditions are natural preference restrictions which are directly imposed on preferences of individuals. They are independent conditions and they all guarantee the existence of core stability. When they ally with top-mutuality, top-mutuality, and top-symmetry, respectively, they also guarantee the existence of strong coalitional stability under FX-FE membership property rights. Moreover, under FX-FE membership property rights, we proved that we can extend the domain of existence. We showed that strongly coalitionally stable coalition structures always exist when membership property rights are assumed to be AX-AE with the assistance of *Proposition 3.4.3* and the result of Sung and Dimitrov (2007). However, existence of strong coalitional stability under FX-AE and AX-FE membership property rights is still an open question. The question "*How $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, $\mathcal{G}$-singularity, and top-mutuality conditions should be weaked so that we always get strong coalitional stability under FX-AE and AX-FE membership property rights?*" waits an answer. Followingly, under different membership property rights, we revealed the relation between strong coalitional stability and other stability concepts. We received parallel results to Karakaya (2011).

$\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity conditions are directly imposed on preferences of individuals. By furnishing them with either top-mutuality or top-symmetry, we proved that they constitute a sufficient domain for the existence of strongly coalitionally stable coalition structures. In the literature, there are several domain conditions which are imposed on the whole preference profile such as *weak top coalition property* of Banerjee et al. (2001) and *ordinal balancedness* and *weak consecutiveness* properties of Bogomolnaia and Jackson (2002). The interesting question here comes to mind is: *how weak top coalition property*, *ordinal balancedness and weak consecutiveness can be strengthened so that they become sufficient for strong coalitional stability* ? Another interesting research direction is about necessary conditions. *How pivotal balancedness condition of* Iehlé (2007) *can be strengthened so that it becomes both sufficient and necessary for strong coalitional stability* ?

In the fourth chapter, we studied a new stability concept called strong exchange stability. Strong exchange stability is the minimal refinement of core stability and exchange stability, i.e., a coalition structure is strongly exchange stable if and only if it is core stable and exchange stable. The main reasons that we focused on strong exchange stability rather than just focusing on exchange stability are threefold. Firstly, exchange stability is a complementary stability concept, i.e., some core stable coalition structures are not immune to blocking via exchanges. Secondly, exchange block, hence exchange stability are discernable and applicable only for coalition structures which has at least two coalitions with cardinality greater than or equal to 2. For example, exchange block is not applicable for the grand coalition $\aleph$ and for the coalition structure $\gimel$ which consists of singletons. They become exchange stable by their nature. Lastly, there exist exchange stable coalition structures which are not core stable which cause conceptual gaps. Considering all these three reasons, it is more sensible and fruitful to study strong exchange stability.

In the fourth chapter, we firstly defined exchange stability and strong exchange stability under FX-FE membership property rights. Because core stable coalition structures do not exist in the domain of all hedonic games, so does the strongly exchange stable coalition structures. Thus, we focused on some particular domain restrictions in which core stable coalition structures always exist and verify that core stable coalition structures are also exchange stable. Banerjee et al. (2001) introduced a domain condition called *weak top coalition property* which is sufficient for existence of core stable coalition structures. We proved that weak top coalition property is sufficient for the existence of exchange stability, as well as core stability. Example 4.2.2 reveals that there exists some domains in which core stable coalition structures always exist, but core stable coalition structures are not exchange stable. In that example, the preference profile satisfies *pivotal balancedness* (Iehlé (2007)) which is both sufficient and necessary condition for the existence of core stability. Followingly, we explored the existence of strongly exchange stable coalition structures in the domains of hedonic games satisfying $\mathcal{A}$-responsiveness, $\mathcal{B}$-

responsiveness, and $\mathcal{G}$-singularity, separately. We found out that $\mathcal{B}$-responsiveness and $\mathcal{G}$-singularity conditions are sufficient conditions for exchange stability, however $\mathcal{A}$-responsiveness condition remains incapable and requires additional restrictive conditions. A good research topic then becomes *the generalization of the domains of hedonic games in which core stable coalition structures are also exchange stable*. Lastly, we studied exchange stability under FX-AE membership property rights and explained its relation with core stability.

To our knowledge, strong coalitional stability, exchange stability, and strong exchange stability are not defined and studied before. Moreover, they generalize the coalitional (core) stability, individual stability, and exchange stability concepts in one sided matching games, two sided one-to-one, one-to-many and many-to-many matching games. We have introduced three new and independent domain restrictions, namely, $\mathcal{A}$-responsiveness, $\mathcal{B}$-responsiveness, and $\mathcal{G}$-singularity. $\mathcal{A}$-responsiveness and $\mathcal{B}$-responsiveness conditions are born for the first time in this study. They are particular monotonicity conditions. All three domain conditions are imposed on individuals' preferences. Using this nice property, we proved the possibility of extending the domain of existence when we combine individuals with these three preference restrictions together. Moreover, this property allowed us to define new five efficient and finite time algorithms which run on individual preferences and look for stable coalition structures.

Considering all of these, besides its significant contribution to the literature of hedonic games, this dissertation lighted the fuse of several new stimulating research questions which are mentioned above.

REFERENCES

Abizada, A. (2019) “Exchange-stability in roommate problems", *Review of Economic Design*, *forthcoming*.

Alcalde, J. (1995) “Exchange-proofness or divorce-proofness? Stability in one-sided matching markets", *Economic Design*, 1: 275 - 287.

Alcalde, J. & P. Revilla (2004) “Researching with whom ? Stability and Manipulation", *Journal of Mathematical Economics*, 40: 869 - 887.

Alcalde, J. & A. Romero-Medina (2006) “Coalition formation and stability", *Social Choice and Welfare*, 27: 365 - 375.

Arrow, K. J. & G. Debreu (1954), “Existence of an equilibrium for a competitive economy", *Econometrica*, 22: 265 - 290.

Asan, G. & M. R. Sanver (2003) “Coalitional structural games and the voluntary provision of public goods", In M.R. Sertel & S. Koray (Eds.), Advances in economic design, Berlin: Springer-Verlag, 251-259.

Asan, G. & M. R. Sanver (2015) “The Tiebout hypothesis under membership property rights", *Theory and Decision*, 78: 457 - 469.

Aumann, R. J. & J. H. Drèze (1974) “Cooperative games with coalition structures", *International Journal of Game Theory*, 3(4): 217 - 237.

Aumann, R. J. & M. Maschler (1964) “The bargaining set for cooperative games", In Dresher, Shapley, and Trucker, editors, Advances in Game Theory, Annals of Mathematics Studies No. 52, Princeton: Princeton University Press: 443 - 476.

Aziz, H. & A. Goldwaser (2017) “Coalitional exchange stable matchings in marriage and roommate markets", In: AAMAS.

Aziz, H. & R. Savani (2016) “Hedonic Games" In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. Procaccia, editors, Handbook of Computational Social Choice, Cambridge University Press, In press.

Ballester, C. (2004) “NP-completeness in hedonic games", *Games and Economic Behavior*, 49(1): 1 - 30.

Banerjee, S., H. Konishi, & T. Sönmez (2001) “Core in a simple coalition formation game", *Social Choice and Welfare*, 18: 135 - 153.

Belleflamme, P. (2000) “Stable coalition structures with open membership and asymmetric firms", *Games and Economic Behavior*, 30: 1 - 21.

Bennett, E. & W. R. Zame (1988) “Bargaining in cooperative games", *International Journal of Game Theory*, 17(4): 279 - 300.

Bodine-Baron, E., C. Lee, A. Chong, B. Hassibi & A. Wierman (2011) “Peer effects and stability in matching markets", *Algorithmic Game Theory, Lecture Notes in Computer Science*, 6982: 117 - 129.

Bogomolnaia, A. & M. O. Jackson (2002) “The stability of hedonic coalition structures", *Games and Economic Behavior*, 38 (2): 201 - 230.

Burani, N. & W. S. Zwicker (2003) “Coalition formation games with separable preferences", *Mathematical Social Sciences*, 45: 27 - 52.

Cechlárová, K. (2002) “On the complexity of exchange-stable roommates", *Discrete Applied Mathematics*, 116: 279-287.

Cechlárová, K. & J. Hajduková (2003) “Computational complexity of stable partitions with B-preferences", *International Journal of Game Theory*, 31(3): 353 - 364.

Cechlárová, K. & J. Hajduková (2004a) “Stable partitions with W -preferences", *Discrete Applied Mathematics* 138(3): 333 - 347.

Cechlárová, K. & J. Hajduková (2004b) “Stability of partitions under WB-preferences And BW-preferences", *International Journal of Information Technology and Decision Making*, 3(4): 605 - 618.

Cechlárová, K. & D. Manlove (2005) “Exchange-stable marriage problem", *Discrete Applied Mathematics*, 152: 109-122.

Cechlárová, K. & A. Romero-Medina (2001) “Stability in coalition formation games", *International Journal of Game Theory*, 29(4): 487 - 494.

Conley, J.P., & H. Konishi (2002) “Migration-proof Tiebout equilibrium: existence and asymptotic efficiency", *Journal of Public Economics*, 86: 243-262.

Darmann, D., E. Elkind, S. Kurz, J. Lang, J. Schauer, & G. Woeginger (2012) “Group activity selection problem", In *Proceedings of the 8th International Conference on Internet and Network Economics, WINE'12*: 156 - 169.

Diamantoudi, E. & L. Xue (2003) “ Farsighted stability in hedonic games", *Social Choice and Welfare*, 21(1): 39 - 61.

Dimitrov, D., P. Borm, R. Hendrickx, & S. C. Sung (2006) “Simple priorities and core stability in hedonic games", *Social Choice and Welfare*, 26 (2): 421 - 433.

Dimitrov, D. & S. C. Sung (2006) “Top responsiveness and Nash stability in coalition formation games", *Kybernetika*, 42(4): 453-560.

Dimitrov, D. & S. C. Sung (2007) “On top responsiveness and strict core stability", *Journal of Mathematical Economics*, 43(2): 130-134.

Drèze, J. & J. Greenberg (1980) “Hedonic coalitions: optimality and stability", *Econometrica*, 48: 987 - 1003.

Eren, N.I. (1993) “Coalition structural games and stability under membership property right axioms", Unpublished M.A. Thesis, Bogazici University, Istanbul.

Gale, D. & L. S. Shapley (1962) “Collage admissions and the stability of marriage", *American Mathematical Monthly*, 69: 9 - 15.

Gerber, A. (2000) “Coalition formation in general NTU games", *Review of Economic Design*, 5: 149 - 175.

Hajduková, J. (2006) “Coalition formation games: A survey", *International Game Theory Review*, 8(4): 613 - 641.

Hart, S. & M. Kurz (1983) “Endogenous formation of coalitions", *Econometrica* 51: 1047 - 1064.

Hasan, C., E. Altman, & J. Gorce (2014) “On the Nash stability in the hedonic coalition formation games", mimeo.

Iehlé, V. (2007) “The core-partition of a hedonic game", *Mathematical Social Sciences*, 54: 176-185.

Irwing, R. W. (2008) “Stable matching problems with exchange restrictions", *Journal of Combinatorial Optimization*, 16: 344 - 360.

Karakaya, M. (2011) “Hedonic coalition formation games: A new stability notion", *Mathematical Social Sciences*, 61(3): 157 - 165.

Karakaya, M. & B. Klaus (2017) “Hedonic coalition formation games with variable populations: Core characterizations and (im)possibilities", *International Journal of Game Theory*, 46: 435 - 455.

Lazarova, E., P. Borm, & A. Estévez-Fernández (2016) “Transfer and exchange-stability in two-sided matching problems", *Theory and Decision*, 81: 53 - 71.

Lee, H. & Y. Shoham (2015) “ Stable invitations", In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, AAAI'15*: 965 - 971.

von Neumann, J.& O. Morgenstern (1944) “Theory of Games and Economic Behaviour", *Princeton University Press*.

Nizamogullari, D. & I. Ozkal-Sanver (2011) “Coalitional stability and efficiency of partitions in matching problems", *Theory and Decision*, 71(3): 395 - 407.

Ozbilen, S. (2018) “Coalition formation games: Inner stability", Unpublished M.Sc. Thesis, Istanbul Technical University, Istanbul.

Ozkal-Sanver, I. (2005) “Stability and efficiency of partitions in matching problems", *Theory and Decision*, 59(3): 193 - 205.

Peters, H. (2009) “NTU-Games", In U. Derigs., editor, Optimization and Operations Research - Vol. III, Encyclopedia of Life Support Systems, Oxford: 181 - 215.

Ray, D. & R. Vohra (1999) “A theory of endogenous coalition structures", *Games and Economic Behavior*, 26: 286 - 336.

Rodríguez-Álvarez, C. (2009) “Strategy-proof coalition formation", *International Journal of Game Theory*, 38(3): 431 - 452.

Roth, A. & M. Sotomayor (1990) “Two Sided Matching: A Study in Game-Theoretic

Modelling and Analysis", Cambridge University Press, New York.

Sertel, M. R. (1982) “Workers and incentives", in Contributions to Economic Analysis, vol. 140, North-Holland Publishing Company, Amsterdam.

Sertel, M. R. (1992) “Membership property rights, efficiency and stability", Research Papers, Bogazici University, Istanbul.

Sertel, M. R. (2001) “Designing rights: Invisible hand theorems, covering and membership", mimeo , Bogazici University, Istanbul.

Sertel, M. R. (2003) “Designing rights", in Advances in Economic Design (eds. S. Koray and M.R. Sertel), Springer- Verlag, Heidelberg.

Shenoy, L. S. (1979) “On coalition formation: A game theoretical approach", *International Journal of Game Theory*, 8(3): 133 - 164.

Spradling, M., J. Goldsmith, X. Liu, C. Dadi & Z. Li (2013) “ Roles and teams hedonic game", In *International Conference on Algorithmic Decision Theory ADT'13*: 351 - 362.

Sung, S. C. & D. Dimitrov (2007) “On myopic stability concepts for hedonic games", *Theory and Decision*, 62: 31 - 45.

Suzuki, K. & S. C. Sung (2010) “Hedonic coalition formation in conservative societies", Technical Report 1700921, Social Science Research Network.

Takamiya, K. (2013) “Coalitional unanimity versus strategy-proofness in coalition formation problems", *International Journal of Game Theory*, 42: 115 - 130.

Yi, S. S. (1997) “Stable coalition structures with externalities", *Games and Economic Behavior*, 20: 201 - 237.

Zhou, L. (1994) “A new bargaining set of an N-person game and endogenous coalition formation", *Games and Economic Behavior*, 6(3): 512 - 526.