CLASSIFICATION OF IRREDUCIBLE CUSPIDAL REPRESENTATIONS OF THE AUTOMORPHISM GROUPS OF REGULAR TREES

Şermin Çam

İSTANBUL BİLGİ ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATİK YÜKSEK LİSANS PROGRAMI

Tez danışmanı: Doç. Dr. Selçuk Demir
2011

CLASSIFICATION OF IRREDUCIBLE CUSPIDAL REPRESENTATIONS OF THE AUTOMORPHISM GROUPS OF REGULAR TREES

Şermin Çam

Tez danışmanı: Doç. Dr. Selçuk Demir ....

Jüri Üyesi: Dr. Kürşat Aker ....

Jüri Üyesi: Yrd. Doç. Dr. Sonat Süer.................................

Tezin Onaylandığı Tarih : 14.07.2011

Toplam Sayfa Sayısı : 59

Anahtar Kelimeler	Key Words
1)Otomorfizma Grubu	1)Automorphism group
2)Homojen Ağaç	2)Homogeneous tree
3)İndirgenemez temsil	3)Irreducible representation
4)Cebirsel temsil	4)Algebraic representation
5)Küspidal temsil	5)Cuspidal representation

ABSTRACT

CLASSIFICATION OF IRREDUCIBLE CUSPIDAL REPRESENTATIONS OF THE AUTOMORPHISM GROUPS OF REGULAR TREES

In this thesis we take a regular tree X of regularity greater than or equal to three and we give a detailed proof of G. I. Ol'shanskii's result about the classification of irreducible cuspidal representations of the automorphism group $Aut(X)$ of the tree X. First, we define a topology on the automorphism group $Aut(X)$ which makes it into a locally compact, Hausdorff, separable and totally disconnected topological group. Later, we work on some specific representations of the automorphism group $Aut(X)$. Finally we prove that irreducible cuspidal representations of the automorphism group $Aut(X)$ are induced from some specific representations of some specific open compact subgroups of the automorphism group $Aut(X)$.

ÖZET

HOMOJEN AĞAÇLARIN OTOMORFİZMA GRUPLARININ İNDİRGENEMEZ KÜSPİDAL TEMSİLLERİNİN SINIFLANDIRMASI

Bu tezde her noktasının derecesi üçten büyük eşit olan homojen bir X ağacı alınmış ve bu ağacın Aut(X) simgesi ile gösterilen otomorfizma grubunun indirgenemez küspidal temsillerinin ilk olarak G. I. Ol'shanskii tarafından yapılan sınıflandırması detaylı olarak çalışılmıştır. Öncelikle Aut(X) otomorfizma grubu üzerinde bir topoloji tanımlanmış ve Aut(X) otomorfizma grubunun bu topoloji ile birlikte yerel kompakt, Hausdorff, ayrılabilir ve bağlantısız bir topolojik grup olduğu gösterilmiştir. Daha sonra Aut(X) otomorfizma grubunun bazı özel temsilleri incelenmiştir. Son olarak Aut(X) otomorfizma grubunun indirgenemez küspidal temsillerinin, bu grubun bazı açık kompakt alt gruplarının bir takım özel temsillerinden yükseltilmiş temsiller olduklarının ispatı verilmiştir.

ACKNOWLEDGEMENT

I owe my deepest gratitude to my teacher and my thesis supervisor Assoc. Prof. Selçuk Demir whose support, encouragement and guidance from the preliminary to the concluding level of this study enabled me to develop an understanding of the subject.

I would like to thank Prof. Ali Nesin, Prof. Oleg Belegradek and again Assoc. Prof. Selçuk Demir who have all given me mathematics education. I am very proud of being a student of such wonderful teachers.

I thank to Kürşat Aker and Sonat Süer for their participation to my thesis committee.

I am deeply grateful to my fiance Fatih Çelik. He always gave me strength and motivation during my study.

I am also indepted to my mother Nuran Çam for the love and support she has given me through all my life.

INTRODUCTION

Let X be a regular tree with regularity greater than or equal to 3. The automorphism or isometry group G of X is equipped in a natural way with a topology which makes it into a locally compact separable metrizable group. If the regularity of X is of the form $p+1$, where p is a prime number, then the tree under discussion can be considered as the symmetric space corresponding to the p-adic $SL(2)$. In that case such trees are special cases of the so-called the Bruhat-Tits buildings associated to reductive p-adic groups. Then $PGL(2, \mathbb{Q}_p)$ becomes a closed subgroup of G.

The tree is in any case a kind of discrete version of the Poincare disk. It will be seen that the group G also has many properties similar to the $SL(2, \mathbb{R})$. But the group G does not have any Lie group or p-adic group structure. Therefore it is interesting to understand its representation theory. The groups we considered first studied by G. Ol'shanskii in the mid-seventies. The aim of this thesis is to work out the classification of the irreducible cuspidal representations of G in detail and to write in a nearly self-contained form. This classification shows some interesting properties of these representations. In particular, they are obtained by inducing some special representations from some special compact open subgroups. This property is shared by irreducible cuspidal representations of p-adic groups as well. But the p-adic analogue of this result has been established only recently. Therefore it seems quite natural to understand these groups better. Because they may shed some more light on the representation theory of p-adic and real Lie groups.

Contents

1 Automorphism Groups of Regular Trees

1.1 The topological group $AutX$

A *tree* is a connected non-empty graph with no circuits. A tree is called *k-regular* if every vertex is adjacent to exactly k many vertices, i.e. the degree of each vertex is equal to k.

Let X be a *k-regular* tree for $k \geq 3$.

Given two vertices x and y of the tree X, we define the distance between x and y by the number of edges on the unique geodesic joining x and y and we denote it by $l(x, y)$. Given a subtree Δ of the tree X, when we write $x \in \Delta$ we mean that x is a vertex of the subtree Δ. Also given two subtrees Δ and Δ' of the tree X, if Δ is a subtree of Δ' we write $\Delta \subseteq \Delta'$. A subtree Δ of the tree X is said to be bounded if its diameter $diam(\Delta) = sup\{l(x, y) : x, y \in \Delta\}$ is finite. For $x \in X$, set $l(x, \Delta) = inf\{l(x, y) : y \in \Delta\}$ and given $n \in \mathbb{N}$, let $V_n(\Delta) = \{x \in X : l(x, \Delta) \leq n\}$. Using the natural distance l on the tree X, we can define $Aut(X)$ as the group of all bijective isometries of the tree X. From now on, denote the group $Aut(X)$ by G and fix a vertex x_0 of the tree X.

Let $B_n := B_n(x_0) = \{x \in X : l(x, x_0) \leq n\}$.

Let $S_n := \{x \in X : l(x, x_0) = n\}$.

The set B_n is called the ball with center x_0 and radius n and the set S_n is called the sphere around x_0 of radius n.

Let $K_n := Stab(B_n) = \{g \in G : g(x) = x \;\; \forall x \in B_n\}$. Clearly, each K_n is a subgroup of G.

Then the function $d : G \times G \to [0, \infty)$ defined by

$$d(g, h) = \begin{cases} 1 & \text{if } g^{-1}h \notin K_n \;\; \forall n, \\ inf\left\{\frac{1}{n} : g^{-1}h \in K_n\right\} & \text{otherwise} \end{cases}$$

gives a metric space structure to the group G.

In fact, d is an ultrametric on the group G. To see this, let g, h and k be pairwise distinct elements of the group G. If $d(g,h) = \frac{1}{n}$ and $d(h,k) = \frac{1}{m}$ where $n \geq m > 1$, i.e. if $g = h$ on B_n and $h = k$ on B_m, then we get $g = k$ on B_m as $B_m(x_0) \subseteq B_n(x_0)$. So $d(g,k) \leq \frac{1}{m} = max(\frac{1}{n}, \frac{1}{m}) = max(d(g,h), d(h,k))$. If $d(g,h) = 1$ or $d(h,k) = 1$, we get $d(g,k) \leq 1 = max(d(g,h), d(h,k))$ as well.

Remarks:

• For all $f \in G$ and for all $g, h \in G$, we have $d(fg, fh) = d(g,h)$. This follows from the fact that $fg = fh$ on B_n iff $g = h$ on B_n for all n.

• G is a topological group with respect to the metric topology:

First, let us show that multiplication map

$$\begin{aligned} m : G \times G &\longrightarrow G \\ (x,y) &\longmapsto x.y \end{aligned}$$

is continuous.

So take a convergent sequence $\{(g_n, h_n)\}_n$ in the space $G \times G$ with its limit (g, h). Then we get that the sequence $\{d(g_n h, gh)\}_n$ converges to 0. To see this, it is enough to show that for all $k > 0$, $(gh)^{-1}(g_n h) \in K_k$ for n large enough. So let $k > 0$ and choose $l > k$ with $h(B_k) \subseteq B_l$. As the sequence $(g_n)_n$ converges to the element g in the group G, there exists $N > 0$ such that for all $n > N$, $g^{-1}g_n \in K_l$. Consequently, for all $n > N$, $(gh)^{-1}(g_n h) = h^{-1}(g^{-1}g_n)h \in h^{-1}K_l h \subset K_k$ as desired.

Now since

$$0 \leq d(g_n h_n, gh) \leq d(g_n h_n, g_n h) + d(g_n h, gh) = d(h_n, h) + d(g_n h, gh)$$

and the sequence $(h_n)_n$ converges to the element h in the group G, by Sandwich Lemma we get that the sequence $\{d(g_n h_n, gh)\}_n$ converges to 0, i.e. the sequence $(g_n h_n)_n$ converges to the element gh in the group G.

Secondly, let us show that the inversion map

$$\begin{aligned} i : G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned}$$

is continuous.

So take a convergent sequence $(g_n)_n$ in the group G with its limit g. Let $k > 0$. Choose $l > k$ such that $g^{-1}(B_k) \subset B_l$ and choose $N \in \mathbb{N}$ such that for all $n > N$, $g^{-1}g_n \in K_l$, i.e. $g_n = g$ on B_l. Let $x \in B_k$. Then since $g^{-1}(x) \in B_l$, for all $n > N$, $g_n(g^{-1}(x)) = g(g^{-1}(x))$, i.e. $g_n g^{-1}(x) = x$. Thus, we get $g_n^{-1} = g^{-1}$ on B_k for all $n > N$. This means that the sequence $(g_n^{-1})_n$ converges to the element g^{-1} in the group G as desired.

• $K_1 = \{g \in G : d(g,e) < 1\}$ and $K_n = \left\{g \in G : d(g,e) < \frac{1}{n-1}\right\}$ for all $n \geq 2$. Hence for all $n \geq 1$, K_n is an open subgroup of G. Moreover for each $k, n \in \mathbb{N}$, K_{n+k} is a normal subgroup of the group K_n of finite index. In particular, $[K_0 : K_n] < \infty$ for all $n \in \mathbb{N}$. Hence the subgroup K_0 of the group G is also open.

• Given $r > 0$, $K_n \subseteq B(e,r)$ for all $n > 1$ satisfying $\frac{1}{n} < r$. So, the sequence of open subgroups $(K_n)_n$ form a local basis at identity.

Lemma 1.1.1. *K_0 is a compact subgroup of the group G.*

Proof. Let $(g_n)_n$ be a sequence in the subgroup K_0. It suffices to find a convergent subsequence of the sequence $(g_n)_n$.

Note that each element g_n of the subgroup K_0 setwise stabilizes all spheres around x_0 . Then since the sphere S_n contains $k.(k-1)^{n-1}$ many vertices of the tree X, there are at most $(k.(k-1)^{n-1})!$ different actions of the elements of the subgroup K_0 on the sphere S_n for each n.

Therefore,

there exists a subsequence $(g_{1,n})_n$ of the sequence $(g_n)_n$ such that $g_{1,n} = g_{1,m}$ on the sphere S_1 for all $n, m \in \mathbb{N}$,

there exists a subsequence $(g_{2,n})_n$ of the subsequence $(g_{1,n})_n$ such that $g_{2,n} = g_{2,m}$ on the sphere S_2 for all $n, m \in \mathbb{N}$ and

there exists a subsequence $(g_{3,n})_n$ of the subsequence $(g_{2,n})_n$ such that $g_{3,n} = g_{3,m}$ on the sphere S_3 for all $n, m \in \mathbb{N}$.

Continuing this way, for each $k \geq 1$ we find a subsequence $(g_{k+1,n})_n$

of the subsequence $(g_{k,n})_n$ such that $g_{k+1,n} = g_{k+1,m}$ on the sphere S_{k+1} for all $n, m \in \mathbb{N}$.

Now let $g \in K_0$ be defined by $g = g_{n,n}$ on the sphere S_n for each n. Note that such an element g exists in the subgroup K_0. This is because $g_{n,n} = g_{k,k}$ on the sphere S_k for all n, k with $n > k$. We will show that this element g of the subgroup K_0 is the limit of the subsequence $(g_{n,n})_n$ of the sequence $(g_n)_n$.

So let $n, k \in \mathbb{N}$ with $n \geq k$. Then, $g_{n,n} = g_{k,m}$ for some $m \in \mathbb{N}$ so that $g = g_{k,k} = g_{k,m} = g_{n,n}$ on the sphere S_k. To sum up, $g = g_{n,n}$ on the sphere S_k for all $k \leq n$, i.e. $g = g_{n,n}$ on the ball B_n. So for all $n \in \mathbb{N}$, $d(g, g_{n,n}) \leq \frac{1}{n}$ and we are done.

□

Given $g \in G$ and $x \in X$, let us denote the vertex $g(x)$ by gx.

Corollary 1.1.2. *For all possible bounded subtrees Δ of the tree X, the subgroups*

$$K(\Delta) = \{g \in G : gx = x \ \forall x \in \Delta\}$$

are open compact. In particular, each subgroup K_n is compact.

Proof. Let $x_1, ..., x_n$ be the vertices of the subtree Δ. For $i = 1, .., n$, choose $g_i \in G$ satisfying $g_i(x_0) = x_i$. Then $K(\Delta) = \bigcap_{i=1}^n K(x_i) = \bigcap_{i=1}^n g_i K(x_0) g_i^{-1}$ where each $g_i K(x_0) g_i^{-1}$ is an open compact subgroup of the topological group G. Hence $K(\Delta)$ is an open compact subgroup of the group G.

□

By the facts we proved up to here, we obtain that the sequence $(K_n)_{n \in \mathbb{N}}$ is a sequence of open compact subgroups of the group G which form a local basis of unity. This implies that the group G is totally disconnected and first countable. To sum up, now we have the following theorem.

Theorem 1.1.3. *The automorphism group G of the regular tree X whose regularity is greater than or equal to three is a locally compact, Hausdorff and totally disconnected topological group.*

Note that $\{gK_0\}_{g\in G}$ is an open cover of the group G which has no finite subcover. Hence the group G is not compact.

1.2 Haar measure on the automorphism group

Let H be an arbitrary locally compact, Hausdorff topological group. Let $\mathscr{B}$ be the σ-algebra generated by all compact subgroups of H. A measure μ on $\mathscr{B}$ is called a left Haar measure if

1. μ is outer regular for all Borel subsets of the group H,
2. μ is inner regular for all open subsets of the group H,
3. μ is finite on all compact subsets of the group H,
4. $\mu(g.E) = \mu(E)$ for every subset E of the group H where $g.E = \{g.e : e \in E\}$.

For locally compact Hausdorff topological groups we always have a left Haar measure which enables us to take integrals of complex valued functions defined on the group H. In particular, the automorphism group G has a left Haar measure. This is guaranteed by the following theorem which was first fully proven by Andre Weil.

Theorem 1.2.1. *Every locally compact, Hausdorff topological group possesses a left Haar measure which is unique up to multiplication by a positive constant.*

1.3 On complete subtrees of the regular tree X

The set of extremities $\partial(\Delta)$ of the subtree Δ is the the set of vertices of Δ whose degrees in Δ are exactly one. A subtree Δ of the tree X is said to be complete if $V_1(\{x\}) \subseteq \Delta$ for each vertex x of the subtree Δ which is not contained in $\partial(\Delta)$. A tree consisting of one vertex or one edge is assumed to be complete.

Proposition 1.3.1. (a) *The nonempty intersection of complete trees is a complete tree.*

(b) *If $T \subset X$ is an arbitrary subtree, then $V_m(T)$ is a complete tree for $m = 1, 2, ...$*

Proof. a) Let $(\Delta_i)_{i \in I}$ be a family of complete trees and $\Delta = \bigcap_{i \in I} \Delta_i$. Let $x \in \Delta$ such that $x \notin \partial(\Delta)$. Then $x \in \Delta_i$ for all i and we can choose two distinct vertices $y, z \in \Delta$ which are adjacent to the vertex x. Since $y, z \in \Delta_i$ for all i, we get $x \notin \partial(\Delta_i)$ for all i. Hence $V_1(\{x\}) \subseteq \Delta_i$ for all i so that $V_1(\{x\}) \subseteq \Delta$.

b) Let $x \in V_m(T)$ such that $x \notin \partial V_m(T)$. Then $l(x, T) < m$. Let $z \in X$ with $l(x, z) = 1$. Then $l(z, T) \leq l(x, T) + 1 \leq m$. Hence $z \in V_m(T)$.

□

Assume now that Δ is a complete finite subtree of the tree X of diameter ≥ 2. Set $g\Delta = \{gx : x \in \Delta\}$ and $\widetilde{K}(\Delta) = \{g \in G : g\Delta = \Delta\}$.

Then $\widetilde{K}(\Delta)$ is the normalizer of the subgroup $K(\Delta)$ in G, in particular $K(\Delta) \trianglelefteq \widetilde{K}(\Delta)$: Let $g \in K(\Delta)$ and $h \in \widetilde{K}(\Delta)$. Then for all $x \in \Delta$, we have

$$hx \in \Delta \Rightarrow ghx = hx \Rightarrow h^{-1}ghx = x \Rightarrow h^{-1}gh \in K(\Delta).$$

Hence $\widetilde{K}(\Delta)$ normalizes $K(\Delta)$. Conversely, let $h \in N_G(K(\Delta))$ and $x \in \Delta$. If $hx \notin \Delta$, $n = l(hx, \Delta) \geq 1$. Let $y \in \Delta$ with $l(hx, y) = n$ and $(y, y_1, ..., y_{n-1}, hx)$ be the geodesic between the vertices y and hx. Since our tree' s regularity ≥ 3, there exists a vertex $z \in X$ adjacent to the vertex y_{n-1} and different from the vertices hx and y_{n-2}. Then we can choose an isometry $g \in K(\Delta)$ such that $g(y_i) = y_i$ for all $i = 1, ..., n-1$

and $g(hx) = z$. But then we get $h^{-1}ghx = h^{-1}z \neq h^{-1}(hx) = x$, i.e $h^{-1}gh \notin K(\Delta)$, a contradiction. Hence $hx \in \Delta$ so that $h \in \widetilde{K}(\Delta)$.

The quotient group $\widetilde{K}(\Delta)/K(\Delta)$ is isomorphic to the finite group $Isom(\Delta)$ of all isometries of the finite subtree Δ. Indeed, the restriction map on the subtree Δ

$$\begin{aligned} \varphi : \widetilde{K}(\Delta)/K(\Delta) &\longrightarrow Isom(\Delta) \\ gK(\Delta) &\longmapsto g|_{\Delta} \end{aligned}$$

is a well-defined isomorphism.

Let Δ' be a maximal complete subtree of Δ. If $diam(\Delta) = 2$, then as being complete $\Delta = V_1(\{x\})$ for some vertex $x \in X$ and the maximal complete subtrees of Δ are exactly the k many edges of Δ. If $diam(\Delta) > 2$, the maximal complete subtrees of Δ correspond bijectively to the vertices of $\partial(\Delta^0)$ where $\Delta^0 = \{x \in \Delta : x \notin \partial\Delta\}$. Indeed, if $v \in \partial(\Delta^0)$, then the vertex v corresponds to the maximal complete subtree of Δ that we obtain by deleting the $k-1$ vertices in $\partial\Delta$ which are adjacent to the vertex v.

Note that if $g \in \widetilde{K}(\Delta)$, then $g^{-1}K(\Delta')g = K(g^{-1}\Delta')$ for every subtree Δ' of Δ: If $h \in K(\Delta')$, $g^{-1}hg(g^{-1}x) = g^{-1}h(x) = g^{-1}(x)$ for all $x \in \Delta'$. Conversely, if $h \in K(g^{-1}\Delta')$, $h = g^{-1}(ghg^{-1})g$ where $ghg^{-1}(x) = gg^{-1}(x) = x$ for all $x \in \Delta'$, i.e. $ghg^{-1} \in K(\Delta')$.

Moreover, if Δ' is a maximal proper complete subtree of Δ, then so is the subtree $g\Delta'$. This is because $g\Delta'$ is the subtree of Δ that we obtain by deleting the vertices in $\partial(\Delta)$ which are adjacent to the vertex $gv \in \partial(\Delta^0)$ from the subtree Δ where $v \in \partial(\Delta^0)$ and Δ' is the subtree of Δ that we obtain by deleting the vertices in $\partial(\Delta)$ which are adjacent to v from the subtree Δ. Hence the internal automorphisms in $\widetilde{K}(\Delta)$ permute the stabilizers of maximal complete proper subtrees of Δ.

1.4 A relation between Gelfand pairs and unimodularity

Definition 1.4.1. A normed vector space $(A, \|\cdot\|)$ over $\mathbb{C}$ is called a *normed algebra* if it is an algebra satisfying $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in A$. A normed algebra A is called *Banach algebra* if the normed space $(A, \|\cdot\|)$ is a Banach space.

Let $L^1(H)$ be the space of complex valued functions defined on a locally compact, Hausdorff topological group H which are integrable with respect to a chosen left Haar measure μ on H. Note that this definition does not depend on the choice of a left Haar measure on the group H since any two left Haar measures on the group H are scalar multiples of each other.

$(L^1(H), \|\cdot\|_1)$ is a Banach space where

$$\|f\|_1 = \int_H |f(x)| \, \mathrm{d}\mu(x)$$

for all $f \in L^1(H)$.

Given two functions $f, g \in L^1(H)$ we define their convolution product $f * g$ by

$$f * g(x) = \int_H f(y)g(y^{-1}x) \, \mathrm{d}\mu(y)$$

for all $x \in H$. Since $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, $f * g \in L^1(H)$ and $(L^1(H), *)$ is a Banach algebra.

Definition 1.4.2. A complex valued function f defined a group H is said to K-left-invariant for a given subset K of the group H if $f(kx) = f(x)$ for all $k \in K$ and for all $x \in G$. The function f is said to be K-right-invariant if $f(xk) = f(x)$ for all $k \in K$ and for all $x \in G$. If the function f is both K-left-invariant and K-right-invariant, then the function f is said to be K-bi-invariant.

Definition 1.4.3. A pair (H, K), where H is a locally compact group and K a compact subgroup, is called a *Gelfand pair* if the subspace

$L^1(K\backslash H/K)$ of $L^1(H)$ consisting of K-bi-invariant functions is a commutative subalgebra of $L^1(H)$ under convolution. Note that the space $C_c(K\backslash H/K)$ of compactly supported, continuous and K-bi-invariant functions on the group H is dense in $L^1(K\backslash H/K)$. Hence, by the following proposition it is equivalent to require that $C_c(K\backslash H/K)$ is commutative.

Proposition 1.4.4. *Let A be a Banach algebra and B be a commutative and dense subalgebra of A. Then A is also commutative.*

Proof. Consider the map $f : A \times A \to A$ defined by $f(a,b) = ab - ba$. Since A is a Banach algebra, the map f is continuos on $A \times A$. Now since B is commutative, $B \times B \subseteq f^{-1}(\{0\})$ so that $\overline{B \times B} \subseteq \overline{f^{-1}(\{0\})}$. But since f is continuous, $f^{-1}(\{0\})$ is closed and $B \times B$ is dense in $A \times A$. So we get $A \times A \subseteq f^{-1}(\{0\})$, i.e. A is commutative. □

Definition 1.4.5. A locally compact Hausdorff topological group H is called *unimodular* if each left Haar measure on H is a right Haar measure.

Definition 1.4.6. Let μ be a left Haar measure on a locally compact Hausdorff topological group H. The *modular function* $\Delta : H \to \mathbb{R}^{>0}$ of H is defined by

$$\mu(E.t) = \Delta(t)\mu(E)$$

for every Borel subset E of H.

Note that the modular function Δ exists by the uniqueness of Haar measure. Note also that Δ is a continuous group homomorphism into the multiplicative group of positive real numbers.

By the two definitions above we get that H is unimodular iff $\Delta = 1$.

Since our tree X is locally finite (i.e. every vertex is adjacent to finite number of vertices), B_n is a finite set for all $n \in \mathbb{N}$. Then as $X = \cup_{n\in\mathbb{N}} B_n$, the number of vertices of X is countable. So we can enumerate the vertices of the tree X as $\{x_0, x_1, x_2, x_3...\}$. Now for all n, let $H_n = \{g \in G : g(x_0) = x_n\}$ and choose an element $g_n \in H_n$. Note that $H_n = g_n H_0$ for all n. This is because for all $h \in H_n$, $h = g_n g_n^{-1} h \in$

$g_n H_0 \subset H_n$. We already know $H_0 = K_0$ is compact. This implies that $H_n = g_n H_0$ is compact, hence has finite measure for all n. Then together with $G = \cup_{n \in \mathbb{N}} H_n$ we get G is a σ-finite measure space. So, from now on, we have the right to use Fubini' s theorem when it is needed.

Proposition 1.4.7. *Let H be a locally compact, Hausdorff topological group and fix a left Haar measure μ on H. Let $C_c(H)$ be the space of continuous, compactly supported, complex valued functions on H. Then for all $f \in C_c(H)$,*

$$\int_H f(x)\,\mathrm{d}\mu(x) = \int_H f(x^{-1})\Delta(x^{-1})\,\mathrm{d}\mu(x).$$

Proof. See Theorem 3.3.7, Harmonic Analysis for commutative spaces, Joseph Albert Wolf. □

Lemma 1.4.8. *Let (H, K) be a Gelfand pair. Then H is unimodular.*

Proof. Let μ be a left Haar measure on G. Since the positive real numbers with multiplication has no nontrivial compact subgroup and since the images of compact subgroups of H under Δ are compact subgroups of $\mathbb{R}^{>0}$ by the continuity of the group homomorphism Δ, Δ is trivial on any compact subgroup of H.

Thus, $\mu(Ek) = \Delta(k)\mu(E) = \mu(E)$ for every Borel subset E and for every $k \in K$ so that μ is K-right-invariant. Consequently, given $f \in C_c(H)$, the projection

$${}^K f^K(x) = \frac{1}{\mu(K)^2} \int_K \int_K f(kxk^{'})\ \mathrm{d}\mu(k)\ \mathrm{d}\mu(k^{'})$$

is contained in $C_c(K \backslash H/K)$.

Now let $f \in C_c(K\backslash H/K)$ be any. By Urysohn' s Lemma for locally compact Hausdorff spaces, there is a function $g \in C_c(H)$ such that $g = 1$ on the compact set $supp(f) \cup (supp(f))^{-1}$. Then ${}^K g^K \in C_c(K\backslash H/K)$ with ${}^K g^K = 1$ on $supp(f) \cup (supp(f))^{-1}$. Let $h =^K g^K$. Since (H, K) is a Gelfand pair, $C_c(K\backslash H/K)$ is commutative and so we get

$$\int_H f(x)\ \mathrm{d}\mu(x) = f * h(e) = h * f(e) = \int_H f(x^{-1})\ \mathrm{d}\mu(x).$$

So for any $f \in C_c(H)$,

$$\int_H {}^K f^K(x)\ \mathrm{d}\mu(x) = \int_H {}^K f^K(x^{-1})\ \mathrm{d}\mu(x).$$

But since μ is K-bi-invariant,

$$\begin{aligned}
\int_H {}^K f^K(x)\ \mathrm{d}\mu(x) &= \frac{1}{\mu(K)^2}\int_H\int_K\int_K f(kxk^{'})\ \mathrm{d}\mu(k)\ \mathrm{d}\mu(k^{'})\ \mathrm{d}\mu(x)\\
&= \frac{1}{\mu(K)^2}\int_K\int_K\int_H f(kxk^{'})\ \mathrm{d}\mu(x)\ \mathrm{d}\mu(k)\ \mathrm{d}\mu(k^{'})\\
&= \frac{1}{\mu(K)^2}\int_K\int_K\int_H f(x)\ \mathrm{d}\mu(x)\ \mathrm{d}\mu(k)\ \mathrm{d}\mu(k^{'})\\
&= \int_H f(x)\ \mathrm{d}\mu(x).
\end{aligned}$$

Similarly,

$$\int_H {}^K f^K(x^{-1})\ \mathrm{d}\mu(x) = \int_H f(x^{-1})\ \mathrm{d}\mu(x).$$

Hence for any $f \in C_c(H)$, we get

$$\int_H f(x^{-1})\Delta(x^{-1})\ \mathrm{d}\mu(x) = \int_H f(x^{-1})\ \mathrm{d}\mu(x).$$

Since the subgroup K is both open and compact, the characteristic function χ_K of K is in $C_c(H)$ by Urysohn' s Lemma for LCH spaces. Then for all $h \in H$,

$$\int_H \chi_{hK}(x^{-1})\Delta(x^{-1})\ \mathrm{d}\mu(x) = \int_H \chi_{hK}(x^{-1})\ \mathrm{d}\mu(x).$$

But since Δ is K-bi-invariant,

$$\begin{aligned}\int_H \chi_{hK}(x^{-1})\Delta(x^{-1}) \, \mathrm{d}\mu(x) &= \int_H \chi_{Kh^{-1}}(x)\Delta(x^{-1}) \, \mathrm{d}\mu(x) \\ &= \int_{Kh^{-1}} \Delta(x^{-1}) \, \mathrm{d}\mu(x) \\ &= \Delta(h)\mu(Kh^{-1})\end{aligned}$$

and

$$\int_H \chi_{hK}(x^{-1}) \, \mathrm{d}\mu(x) = \int_H \chi_{Kh^{-1}}(x) \, \mathrm{d}\mu(x) = \mu(Kh^{-1}).$$

Thus, $\Delta(h)\mu(Kh^{-1}) = \mu(Kh^{-1})$ so that $\Delta(h) = 1$.

□

1.5 Unimodularity of the automorphism group

Lemma 1.5.1. *(G, K_0) is a Gelfand pair. Hence G is unimodular.*

Proof. Let $u, v \in L^1(K_0 \backslash G / K_0)$. Then,

$$\begin{aligned}u * v(k' gk) &= \int_G u(h)v(h^{-1}k' gk) \, \mathrm{d}\mu(h) \\ &= \int_G u(k' h)v((k' h)^{-1}k' gk) \, \mathrm{d}\mu(h) \\ &= \int_G u(k' h)v(h^{-1}gk) \, \mathrm{d}\mu(h) \\ &= \int_G u(h)v(h^{-1}g) \, \mathrm{d}\mu(h) \\ &= u * v(g).\end{aligned}$$

So, the space $L^1(K_0 \backslash G / K_0)$ is a subalgebra of the convolution algebra $L^1(G)$.

Now let $g \in G$. Since $l(x_0, gx_0) = l(g^{-1}x_0, x_0)$, there exists $k \in K_0$ with $kg^{-1}x_0 = gx_0$. So, $g^{-1}kg^{-1} \in K_0$ which implies $g^{-1} \in K_0 g K_0$. Then if u and v are K-bi-invariant, we get

$$
\begin{aligned}
u * v(g) &= \int_G u(h)v(h^{-1}g) \ \mathrm{d}\mu(h) \\
&= \int_G u(h^{-1})v(h^{-1}g) \ \mathrm{d}\mu(h) \\
&= \int_G u((gh)^{-1})v((gh)^{-1}g) \ \mathrm{d}\mu(h) \\
&= \int_G u(h^{-1}g^{-1})v(h^{-1}) \ \mathrm{d}\mu(h) \\
&= \int_G v(h)u(h^{-1}g^{-1}) \ \mathrm{d}\mu(h) \\
&= v * u(g^{-1}) \\
&= v * u(g).
\end{aligned}
$$

□

The groups of the form $K(\Delta)$, where Δ is an edge are called Iwahori subgroups of G. If Δ and Δ' are two edges, we can choose $g \in G$ such that $g\Delta = \Delta'$. Then $gK(\Delta)g^{-1} = K(\Delta')$. So the Iwahori subgroups are conjugate to each other.

2 Irreducible Representations of Aut(X)

2.1 Terminology on representations

Let H be an arbitrary group. A *representation* of H on a vector space V over the field of complex numbers is a homomorphism $\pi : H \to GL(V)$ of H to the group of automorphisms of V. We call V itself a representation of H and we use the notation (π, V) to denote the representation π. We say that the homomorphism π gives V an H-module structure. Also given $x \in H$ and $v \in V$, we often write xv or $x \cdot v$ for $\pi(x)(v)$.

Let V be the space of complex valued functions defined on H and define $\pi_L : H \to GL(V)$ and $\pi_R : H \to GL(V)$ so that for $f \in V$ and $x \in H$, $\pi_L(x)(f)(y) = f(x^{-1}y)$ and $\pi_R(x)(f)(y) = f(yx)$ for all $y \in H$. Then (π_L, V) and (π_R, V) are representations of H and they are called the left regular representation of H and the right regular representation of H respectively.

A linear map T between two representations (π, V) and (σ, W) of the group H is called an *intertwinning operator* if $T(\pi(x)(v)) = \sigma(x)(T(v))$ for every $x \in H$ and $v \in V$. If there is a bijective intertwinning operator between two representations V and W, we denote it by $V \sim W$.

A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under the action of H.

A representation (π, V) is called *unitarizable* if there exists a positive definite, invariant Hermitian form on V.

A representation π is said to be a *unitary representation of* H *on a Hilbert space* V if for all $x \in H$, $\pi(x)$ is a unitary operator.

A unitary representation of H on a Hilbert space V is said to be *topologically irreducible* if V has no proper nontrivial closed invariant subspace and an arbitrary representation V is said to be *algebraically irreducible* if V has no proper nontrivial invariant subspace. From now on, by an irreducible representation we will mean an algebraically irreducible representation unless otherwise stated.

For a subset K of the group H, we denote the set of vectors in V which are invariant under the map $\pi(x)$ for each $x \in K$ by V^K, i.e.

$$V^K = \{v \in V : \pi(x)(v) = v \ \ \forall x \in K\}.$$

Now let us consider the automorphism group G again and fix a Haar measure μ on G.

A representation (π, V) of G is called *algebraic* if $V = \cup V^{K(\Delta)}$ where Δ runs through the set of finite(bounded) subtrees of the tree X, and *admissible*, if moreover, $\dim V^{K(\Delta)} < \infty$ for every finite subtree Δ of the tree X.

An admissible, unitarizable representation of G is called *unitary.*

Let (π, V) be an algebraic representation of G and V^* be the dual space of V. Then (π^*, V^*) is also a representation of G where

$$\pi^*(g)(v^*)(v) = v^*(\pi(g^{-1})(v)) = \left\langle \pi(g^{-1})(v), v^* \right\rangle$$

for all $g \in G$, $v \in V$ and $v^* \in V^*$. Now let $\widetilde{V} = \bigcup (V^*)^{K(\Delta)}$ where Δ runs through the set of finite(bounded) subtrees of the tree X and $\widetilde{\pi} = \pi^*|_{\widetilde{V}}$. Then $(\widetilde{\pi}, \widetilde{V})$ is an algebraic representation of G and it is called the *contragradient representation* of (π, V). Also given $\widetilde{v} \in \widetilde{V}$ and $v \in V$, a map of the form $g \to \langle \pi(g^{-1})(v), \widetilde{v} \rangle$ is called a *matrix coefficient* of the representation (π, V).

A representation of an algebra A is an homomorphism π of A into the algebra $End(V)$ of linear maps on a vector space V.

For the definition of the *formal dimension* of an algebraic representation, see [1] Section 5.2.

2.2 Hecke algebra

A complex valued function f defined on G is said to be *locally constant* if for every $x \in G$, there exists an open compact subgroup K_x of G such that f is constant on the set xK. Denote by $C(G)$ the space of locally constant complex valued functions on G, and by $C_K(G)$, ${}_KC(G)$

and ${}_KC_K(G)$, where K is a compact subgroup of G, respectively, the subspaces of $C(G)$ whose elements are K-left-invariant, K-right-invariant and K-bi-invariant.

We denote by $H(G)$ the space of locally constant, compactly supported complex valued functions on G and by $H(G,K)$, where K is an open compact subgroup of G, the subspace of $H(G)$ whose elements are K-bi-invariant.

Note that $H(G) = \bigcup_K H(G,K)$ where the union runs over open compact subgroups of G. This can be seen as follows: Let $f \in H(G)$ and let $F = supp(f)$. Since f is locally constant, for all $x \in F$, there exists an open compact subgroup K_x such that $f|_{xK_x}$ is constant. As $F \subseteq \bigcup_{x\in F} xK_x$ and F is compact, there exists $x_1, ..., x_n \in F$ such that $F \subseteq \bigcup_{i=1}^n x_iK_{x_i}$. Then for $K_1 = \bigcap_{i=1}^n K_{x_i}$, f is K_1- right-invariant. Similarly, we can find an open compact subgroup K_2 of G such that f is K_2-left-invariant. Then for $K = K_1 \bigcap K_2$, we get $f \in H(G,K)$.

Note that $H(G)$ is an associative algebra under convolution and $(H(G),*)$ is called the *Hecke algebra* of G.

Now let (π, V) be an algebraic representation of G. For $f \in H(G)$ and $v \in V$, we can choose an compact subgroup K such that both $f \in H(G,K)$ and $v \in V^K$. Then we define a vector $\pi(f)(v) \in V$ by

$$\pi(f)(v) = \left[\sum_{x\in G/K} f(x).\pi(x)(v)\right].\mu(K) \quad .$$

Note that the sum is finite as f is compactly supported and this definition does not depend on the choice of K. Also $\pi(f) \in End(V)$ for all $f \in H(G)$. Hence V is a representation of the convolution algebra $H(G)$. Moreover, a subspace W is a subrepresentation of V if and only if $\pi(f)(W) \subseteq W$ for every $f \in H(G)$. In fact, if $\pi(f)(W) \subseteq W$ for every $f \in H(G)$ and $w \in W \cap V^K$, then for any $g \in G$, $\pi(g)(w) = \pi(e_{gK})(w) \in W$ where for a given open compact subgroup K of G, $e_K = \frac{\chi_K}{\mu(K)} \in H(G)$.

There is a strong relationship between irreducible representations of G and irreducible representations of the convolution algebra $H(G,K)$.

The following theorem shows how strongly related they are.

Theorem 2.2.1. ([1, Lemma 4.2.7]) *Let (π, V) be a representation of G. Then (π, V) is irreducible iff $(\pi|_{H(G,K)}, V^K)$ is either 0 or an irreducible representation of $H(G, K)$. Furthermore, every irreducible representation of $H(G, K)$ appears in this way for some irreducible π. Also this π is unique. In other words, for two irreducible representations (π_1, V_1) and (π_2, V_2) of G, $(\pi_1, V_1) \sim (\pi_2, V_2)$ iff there is some open compact K such that $(\pi_1|_{H(G,K)}, V_1^K) \sim (\pi_2|_{H(G,K)}, V_2^K)$.*

2.3 On the smooth part of a unitary representation of the automorphism group on a Hilbert space

Given (π, V) a unitary representation of G on a complex Hilbert space V, for any $f \in L^1(G)$ we define an endomorphism $\pi(f)$ of V as follows:

For $v \in V$, define $F_v : V \to \mathbb{C}$ by

$$F_v(w) = \int_G f(g) \langle \pi(g)(v), w \rangle \ \mathrm{d}\mu(g).$$

F_v is clearly linear.

Moreover, for all $w \in V$,

$$\begin{aligned} |F_v(w)| &\leq \int_G |f(g)| \, |\langle \pi(g)(v), w \rangle| \ \mathrm{d}\mu(g) \\ &\leq \int_G |f(g)| \, \|\pi(g)(v)\| \, \|w\| \ \mathrm{d}\mu(g) \\ &= \int_G |f(g)| \, \|v\| \, \|w\| \ \mathrm{d}\mu(g) \\ &\leq \|f\|_1 \, \|v\| \, \|w\| \,. \end{aligned}$$

Hence F_v is a bounded linear operator on V with $|F_v| \leq \|f\|_1 \, \|v\|$, i.e. $f \in V^*$.

Then by Riesz Representation Theorem, there is a unique element $f_v \in V$ such that $F_v(w) = \langle f_v, w \rangle$ for all $w \in V$.

Now we define $\pi(f) : V \to V$ by $\pi(f)(v) = f_v$. Then for any $v_1, v_2 \in$

V, we get

$$\begin{aligned}
\langle \pi(f)(\lambda v_1 + \beta v_2), w\rangle &= \langle f_{\lambda v_1+\beta v_2}, w\rangle \\
&= F_{\lambda v_1+\beta v_2}(w) \\
&= \int_G f(g)\, \langle \pi(g)(\lambda v_1 + \beta v_2), w\rangle \; \mathrm{d}\mu(g) \\
&= \lambda \int_G f(g)\, \langle \pi(g)(v_1), w\rangle \; \mathrm{d}\mu(g) \\
&+ \beta \int_G f(g)\, \langle \pi(g)(v_2), w\rangle \; \mathrm{d}\mu(g) \\
&= \lambda F_{v_1}(w) + \beta F_{v_2}(w) \\
&= \lambda \langle f_{v_1}, w\rangle + \beta \langle f_{v_2}, w\rangle \\
&= \langle \lambda f_{v_1} + \beta f_{v_2}, w\rangle \\
&= \langle \lambda \pi(f)(v_1) + \beta \pi(f)(v_2), w\rangle
\end{aligned}$$

for all $w \in V$ and $\lambda, \beta \in \mathbb{C}$ so that $\pi(f)(\lambda v_1 + \beta v_2) = \lambda\pi(f)(v_1) + \beta\pi(f)(v_2)$. Thus, $\pi(f) \in End(V)$.

Moreover, $\|\pi(f)(v)\| \le \|f\|_1 . \|v\|$ for all $v \in V$ so that $\pi(f)$ is bounded. For this fact and for a more general result see Appendix 3, Theorem A.3.3., A Course in Abstract Harmonic Analysis, Gerald. B. Folland.

If K is a compact subgroup of G and $e_K = \frac{\chi_K}{\mu(K)}$, then $\pi(e_K)(V) = V^K$. It is easy to see that if $v \in V^K$, then $\pi(e_K)(v) = v$. This follows by the equation

$$\begin{aligned}
\langle \pi(e_K)(v), w\rangle &= \frac{1}{\mu(K)} \int_G \chi_K(g)\, \langle \pi(g)(v), w\rangle \;\; \mathrm{d}\mu(g) \\
&= \frac{1}{\mu(K)} \int_K \langle \pi(k)(v), w\rangle \;\; \mathrm{d}\mu(k) \\
&= \frac{1}{\mu(K)} \int_K \langle v, w\rangle \;\; \mathrm{d}\mu(k) \\
&= \frac{1}{\mu(K)} . \mu(K) . \langle v, w\rangle \\
&= \langle v, w\rangle
\end{aligned}$$

for all $w \in V$. Conversely, given $v \in V$ and $k \in K$,

$$\begin{aligned}
\langle \pi(k)\pi(e_K)(v), w \rangle &= \left\langle \pi(e_K)(v), \pi(k^{-1})w \right\rangle \\
&= \int_G e_K(g) \left\langle \pi(g)(v), \pi(k^{-1})w \right\rangle \ \mathrm{d}\mu(g) \\
&= \int_G e_K(g) \left\langle \pi(k)\pi(g)(v), w \right\rangle \ \mathrm{d}\mu(g) \\
&= \int_G e_K(k^{-1}g) \left\langle \pi(k)\pi(k^{-1}g)(v), w \right\rangle \ \mathrm{d}\mu(g) \\
&= \int_G e_K(g) \left\langle \pi(g)(v), w \right\rangle \ \mathrm{d}\mu(g) \\
&= \langle \pi(e_K)(v), w \rangle .
\end{aligned}$$

i.e. $\pi(e_K)(v) \in V^K$.

Now let (π, V) be a unitary representation of G on a Hilbert space V. Let $V^\infty = \bigcup_\Delta V^{K(\Delta)}$ where Δ runs over the finite complete subtrees of the tree X. Note that $K(g\Delta) = gK(\Delta)g^{-1}$ and consequently $\pi(g)V^{K(\Delta)} = V^{K(g\Delta)}$. Hence V^∞ is an invariant subspace of V. V^∞ is also nontrivial. Indeed, given $0 \neq v \in V$, since the map $g \mapsto \langle \pi(g)v, v \rangle$ is continuous at e and $\langle \pi(e)v, v \rangle = \|v\|^2 > 0$, there exists a finite complete subtree Δ of the tree X such that $Re\left(\langle \pi(g)v, v \rangle\right)$ is strictly positive for all $g \in K(\Delta)$. Then,

$$\int_{K(\Delta)} \langle \pi(g)v, v \rangle \ \mathrm{d}\mu(g) \neq 0.$$

But,

$$\int_{K(\Delta)} \langle \pi(g)v, v \rangle \ \mathrm{d}\mu(g) = \mu(K(\Delta)). \left\langle \pi(e_{K(\Delta)})v, v \right\rangle .$$

Hence, $0 \neq \pi(e_{K(\Delta)})v \in V^{K(\Delta)}$.

Let M be a nontrivial invariant subspace of V. Now let $v \in M$ and Δ be a complete finite subtree of the tree X. Assume v is $K(\Delta')$-invariant. Let $\Delta_0 = \Delta \bigcup \Delta'$. v is clearly $K(\Delta_0)$-invariant. $K(\Delta_0)$ is a finite index subgroup of $K(\Delta)$. So we can choose $g_1, ..., g_n \in G$ such that $K(\Delta) = \bigcup_{i=1}^n g_i K(\Delta_0)$ where $g_i K(\Delta_0) \bigcap g_j K(\Delta_0) = \emptyset$ for distinct i and j. Then, for all $w \in V$ we get

$$\begin{aligned}\left\langle\pi(e_{K(\Delta)})v,w\right\rangle &= (\mu(K(\Delta)))^{-1}\int_{K(\Delta)}\langle\pi(k)v,w\rangle \;\; \mathrm{d}\mu(k)\\ &= (\mu(K(\Delta)))^{-1}\sum_{i=1}^{n}\int_{g_iK(\Delta_0)}\langle\pi(g)v,w\rangle \;\; \mathrm{d}\mu(g)\\ &= (\mu(K(\Delta)))^{-1}.\mu(K(\Delta_0))\sum_{i=1}^{n}\langle\pi(g_i)v,w\rangle\\ &= \left\langle\sum_{i=1}^{n}\lambda\pi(g_i)v,w\right\rangle\end{aligned}$$

where $\lambda = (\mu(K(\Delta)))^{-1}.\mu(K(\Delta_0))$. Hence, $\pi(e_{K(\Delta)})v = \sum_{i=1}^{n}\lambda\pi(g_i)v \in M$, i.e $\pi(e_{K(\Delta)})M \subset M$.

It follows that V^{∞} is dense in V. If M is a nontrivial invariant subspace of V and $0 \neq v \in M$, then $\pi(e_{K(\Delta)})v \in M \cap V^{K(\Delta)}$ so that $M \cap V^{\infty} \neq \{0\}$. We know that $(V^{\infty})^{\perp}$ is an invariant subspace of H. So, if $(V^{\infty})^{\perp}$ is also nontrivial, we get that $(V^{\infty})^{\perp} \cap V^{\infty} \neq \{0\}$, a contradiction. Hence, $(V^{\infty})^{\perp} = \{0\}$ which means V^{∞} is dense in V.

Lemma 2.3.1. *Let (π, V) be a topologically irreducible unitary representation of G in a Hilbert space V; let π^{∞} be the restriction of π to the dense invariant subspace V^{∞}. Then, π^{∞} is algebraic, admissible and algebraically irreducible.*

Proof. π^{∞} is algebraic by definition of V^{∞}. For the part π^{∞} is admissible, see the paper of Ol'shanskii, *Representations of groups of automorphism of trees*. To see π^{∞} is algebraically irreducible, let M be a nontrivial invariant subpace of V^{∞}. Then $M^{\perp}$ is a closed invariant proper subspace of V. But since V is topologically irreducible, we get $M^{\perp} = \{0\}$ and this implies that M is dense in V. Then $\pi(e_{K(\Delta)})M$ is dense in $\pi(e_{K(\Delta)})V = V^{K(\Delta)}$ for any finite subtree Δ of the tree X by the continuity of the map $\pi(e_{K(\Delta)})$. Since π^{∞} is admissible, $V^{K(\Delta)}$ hence $\pi(e_{K(\Delta)})M$ are finite dimensional subspaces of V. Therefore, $\pi(e_{K(\Delta)})M$ is closed so that $\pi(e_{K(\Delta)})M = V^{K(\Delta)}$. But since $\pi(e_{K(\Delta)})M \subset M$, we get $V^{K(\Delta)} \subset M$. Hence, $M = V^{\infty}$ as desired.

□

It is also known that any algebraic, algebraically irreducible representations of G is admissible (due to a more general result of Harish-Chandra). So it is natural for us to use the language of algebraic and admissible representations.

Henceforward, all representations are assumed to be algebraic.

2.4 Complete reducibility of unitarizable admissible modules

We call a topological group H an *ℓ-group* if there is a fundamental system of neighbourhoods of the unit element e consisting of open compact subgroups.

Note that the automorphism group G of the tree X is an ℓ-group.

Theorem 2.4.1. *Let H be an ℓ-group, V an admissible unitarizable H-module. Then every irreducible subrepresentation W of V is complemented.*

Proof. Let W be an irreducible nonzero submodule of V. For each compact open subgroup K of H, let $U(K)$ be the subspace of V^K such that $V^K = W^K \oplus U(K)$. Since V is admissible, each V^K is finite dimensional, hence complete. Therefore we have such subspaces. Now let $U = H\langle \cup_K U(K) \rangle$. Since $V = \cup_K V^K$, it is clear that $V = W + U$. Since W is irreducible, we have $W \cap U = \{0\}$. Otherwise we would have $W \subset U$. Therefore $V = W \oplus U$. □

3 The Representations $L(\Delta)$ and $H(\Delta)$

For a finite subtree $\Delta \subset X$, let $L(\Delta)$ be the subspace of $C_{K(\Delta)}(G)$ which consists of left invariant functions relative to some open compact subgroups of G.

If f is a measurable function on G and K is an open compact subgroup of G, we define the functions f^K and ${}^K f$ on G by

$$f^K(g) = \frac{1}{\mu(K)} \int_K f(gk) \ \mathrm{d}\mu(k)$$

and

$${}^K f(g) = \frac{1}{\mu(K)} \int_K f(k^{-1}g) \ \mathrm{d}\mu(k).$$

The functions f^K and ${}^K f$ are called the right averaging of f on K and the left averaging of f on K respectively.

Now for $\Delta \subset X$ complete, finite and *diam* ≥ 2, let $H(\Delta)$ be the subspace of functions f in $L(\Delta)$ such that for any complete proper subtree Δ_0 of Δ, $f^{K(\Delta_0)} \equiv 0$.

Note that the left regular representation π_L of G restricted to $L(\Delta)$ (hence to $H(\Delta)$) is algebraic, i.e. for any $f \in L(\Delta)$, there exists a finite complete subtree Δ_f of the tree X where $f \in L(\Delta)^{K(\Delta_f)}$: If $f \in L(\Delta)$ and K is an open compact subgroup of G with f is K-left-invariant, then $\pi_L(k)(f) = f$ for all $k \in K$. As K is a neighborhood of unity, there exists a bounded subtree Δ' of the tree X such that $K(\Delta') \subset K$. Then, $\Delta' \subset V_1(\Delta') \Rightarrow K(V_1(\Delta')) \subset K(\Delta') \Rightarrow K(V_1(\Delta')) \subset K$. By choosing $\Delta_f = V_1(\Delta')$, we are done.

Proposition 3.0.2. (a) *If f is measurable on G, f^K is K-right-invariant and ${}^K f$ is K-left-invariant. Moreover the maps $f \mapsto f^K$ and $f \mapsto {}^K f$ preserve continuity and map compactly supported functions onto compactly supported functions.*

(b) *The map $\alpha(\Delta, \Delta_0)$ defined by $f \mapsto f^{K(\Delta_0)}$ is an intertwinning operator from $L(\Delta)$ to $L(\Delta_0)$. With this notation*

$$H(\Delta) = \bigcap_{1 \leq i \leq n} Ker \ \alpha(\Delta, \Delta^i)$$

where $\Delta^1, \Delta^2, ..., \Delta^n$ are maximal complete subtrees of Δ.

Proof. (*a*)

- Let $g \in G$ and $k_0 \in K$. Then,

$$\begin{aligned} f^K(gk_0) &= \frac{1}{\mu(K)} \int_K f(gk_0k) \ \mathrm{d}\mu(k) \\ &= \frac{1}{\mu(K)} \int_{k_0K} f(gk) \ \mathrm{d}\mu(k) \\ &= \frac{1}{\mu(K)} \int_K f(gk) \ \mathrm{d}\mu(k) \end{aligned}$$

  which is equal to $f^K(g)$. Hence f^K is K-right-invariant.

  Now let $h(k) = f(k^{-1}k_0^{-1}g)$. Then,

$$\begin{aligned} {}^K f(k_0^{-1}g) &= \frac{1}{\mu(K)} \int_K f(k^{-1}k_0^{-1}g) \ \mathrm{d}\mu(k) \\ &= \frac{1}{\mu(K)} \int_K h(k) \ \mathrm{d}\mu(k) \\ &= \frac{1}{\mu(K)} \int_K h(k_0^{-1}k) \ \mathrm{d}\mu(k) \\ &= \frac{1}{\mu(K)} \int_K f(k^{-1}k_0k_0^{-1}g) \ \mathrm{d}\mu(k) \\ &= \frac{1}{\mu(K)} \int_K f(k^{-1}g) \ \mathrm{d}\mu(k) \\ &= \ {}^K f(g) \end{aligned}$$

  Hence ${}^K f$ is K-left-invariant.

- f^K is K-right-invariant implies that it is locally constant, hence continuous. Similarly, ${}^K f$ is also continuous.

- Let f be compactly supported and let $supp(f) = K'$. If $g \notin K'K$, i.e. $gk \notin K'$ for all $k \in K$, then $f(gk) = 0$ for all $k \in K$, i.e.$f^K(g) = 0$. So, $supp(f^K) \subseteq K'K$.

(*b*) Let $f \in L(\Delta)$. Let K be an open compact subgroup of G with f is K-left invariant. Then, for all $k' \in K$

$$f^{K(\Delta_0)}(k'g) = \int_{K(\Delta_0)} f(k'gk) \ \mathrm{d}\mu(k) = \int_{K(\Delta_0)} f(gk) \ \mathrm{d}\mu(k) = f^{K(\Delta_0)}(g).$$

So, $f^{K(\Delta_0)}$ is also K-left invariant. Together with part (a), we get $f^{K(\Delta_0)} \in L(\Delta_0)$ so that $\alpha(\Delta, \Delta_0) : L(\Delta) \to L(\Delta_0)$ is well defined.

Now, let $g \in G$. Then, for all $x \in G$

$$\begin{aligned}
\alpha(\Delta, \Delta_0)(\pi_L(g)f)(x) &= \int_{K(\Delta_0)} \pi_L(g)f(xk) \ \mathrm{d}\mu(k) \\
&= \int_{K(\Delta_0)} f(g^{-1}xk) \ \mathrm{d}\mu(k) \\
&= \alpha(\Delta, \Delta_0)(f)(g^{-1}x) \\
&= \pi_L(g)(\alpha(\Delta, \Delta_0)(f))(x).
\end{aligned}$$

Hence $\alpha(\Delta, \Delta_0)$ is an intertwinning operator from $L(\Delta)$ to $L(\Delta_0)$.

Finally let us show, $H(\Delta) = \bigcap_{1 \leq i \leq n} Ker\ \alpha(\Delta, \Delta^i)$.

$(\subseteq)$ follows from the definition of $H(\Delta)$. Conversely, let f be a function in the right hand side. Then, $f \in Ker\ \alpha(\Delta, \Delta^i)$ for all i. Let Δ_0 be a finite complete subtree of Δ. Then $\Delta_0 \subset \Delta^i$ for some i. Since the group $K(\Delta_0)/K(\Delta^i)$ is finite , we can write $K(\Delta_0) = \bigcup_{j=1}^{m} g_j K(\Delta^i)$ for some $g_1, ..., g_m \in G$. Then for all $g \in G$,

$$\begin{aligned}
\alpha(\Delta, \Delta_0)(f)(g) &= \int_{K(\Delta_0)} f(gk) \ \mathrm{d}\mu(k) \\
&= \sum_{j=1}^{m} \int_{g_j K(\Delta^i)} f(gk) \ \mathrm{d}\mu(k) \\
&= \sum_{j=1}^{m} \int_{K(\Delta^i)} f(gg_jk) \ \mathrm{d}\mu(k) \\
&= 0.
\end{aligned}$$

So $f \in H(\Delta)$.

□

3.1 $H(\Delta)$ is nontrivial

From now on, let Δ be a complete finite subtree of X with $diam \geq 2$. Also let $\Delta_1, ..., \Delta_n$ be maximal complete subtrees of Δ.

Definition 3.1.1. A representation of $\widetilde{K}(\Delta)$ is called *non-degenerate* if it has no non-zero $K(\Delta_i)$-stationary vectors for $i = 1, 2, ..., n$. Let $(\widetilde{K}(\Delta))'$

be the set of irreducible non-degenerate representations of $\widetilde{K}(\Delta)$ which are trivial on $K(\Delta)$.

We will now prove that $(\widetilde{K}(\Delta))' \neq \emptyset$. In particular, we will get that $H(\Delta) \neq \emptyset$. We first need a few lemmas.

Lemma 3.1.2. *Let H be a finite group and $H_1, ..., H_n$ be n subgroups of H. Then there exists an irreducible representation π of H such that π has no non-trivial H_i-invariant vectors for every $i = 1, ..., n$ if and only if there exists a function on H not identically zero such that $\sum_{h \in H_i} f(ght) = 0$ for every $g, t \in H$ and $i = 1, ..., n$.*

Proof. ($\Leftarrow$) Let

$$V = \left\{ f : H \to \mathbb{C} \ : \ \sum_{h \in H_i} f(ght) = 0 \ \ \forall g, t \in H \ \ \forall i = 1, ..., n \right\} \neq \{0\}$$

. V is clearly a subspace of $Func(G, \mathbb{C})$ which is the space of complex valued functions on H. If $H = \{h_1, ..., h_n\}$, the characteristic functions $\chi_{\{h_1\}}, ..., \chi_{\{h_n\}}$ generate the space $Func(H, \mathbb{C})$. So, $Func(H, \mathbb{C})$ and hence V are finite dimensional complex vector spaces.

If $f \in V$ and $h_0 \in H$, $\sum_{h \in H_i} \pi_L(h_0)(f)(ght) = \sum_{h \in H_i} f(h_0^{-1} ght) = 0$ for every $g, t \in H$ and $i = 1, ..., n$. Similarly, $\sum_{h \in H_i} \pi_R(h_0)(f)(ght) = \sum_{h \in H_i} f(ghth_0) = 0$ for every $g, t \in H$ and $i = 1, ..., n$. So, V is a bi-invariant H-module.

Then by Maschke' s Theorem, (π_L, V) is a direct sum of irreducible representations of H.

If $f \in V$ is H_i-invariant for some i,

$$\begin{aligned} f(t) &= \frac{1}{|H_i|} \sum_{h \in H_i} f(t) \\ &= \frac{1}{|H_i|} \sum_{h \in H_i} \pi_L(h)(f)(t) \\ &= \sum_{h \in H_i} f(h^{-1} g) = \sum_{h \in H_i} f(eht) \\ &= 0 \end{aligned}$$

for all $t \in H$, i.e. $f \equiv 0$.

Hence a non-zero vector in V is not H_i-invariant for all $i = 1, .., n$. In particular, any irreducible representation appearing in the decomposition of V does not contain a non-zero H_i-invariant vector for every $i = 1, .., n$.

($\Rightarrow$) Let V be an irreducible representation of H which has no non-trivial H_i-invariant vectors for every $i = 1, ..., n$. Since V is irreducible, it is equivalent to a subrepresentation N of $(\pi_L, Func(G, \mathbb{C}))$. The map $P_L(H_i) : Func(H, \mathbb{C}) \to Func(H, \mathbb{C})$ defined by

$$P_L(H_i)(f) = \frac{1}{|H_i|} \sum_{h \in H_i} \pi_L(h)(f)$$

is the projection on the space of H_i-left-invariant functions. So; if $f \in N$, $P_L(H_i)(f) \in N^{H_i} = \{0\}$, i.e. $P_L(H_i)(f) \equiv 0$. Then, since N is H-invariant, we get $P_L(H_i)\pi_L(g)(f)(t) = 0$ for all $g, t \in H$ and $f \in N$. But

$$\begin{aligned} P_L(H_i)\pi_L(g)(f)(t) &= \frac{1}{|H_i|} \sum_{h \in H_i} \pi_L(h)(\pi_L(g)(f))(t) \\ &= \frac{1}{|H_i|} \sum_{h \in H_i} \pi_L(g)(f)(h^{-1}t) \\ &= \frac{1}{|H_i|} \sum_{h \in H_i} f(g^{-1}h^{-1}t) \\ &= \frac{1}{|H_i|} \sum_{h \in H_i} f(g^{-1}ht) \end{aligned}$$

for all $g, t \in H$ and $f \in N$. It follows that for any $f \in N$, $\sum_{h \in H_i} f(ght) = 0$ for all $g, t \in H$ and $i = 1, ..., n$.

□

Lemma 3.1.3. *Let H be a finite group and $H' = H_1 \times ... \times H_n \subseteq H$ be the direct product of n nontrivial subgroups of H. Then if the inner automorphisms of H permute the subgroups $H_1, ..., H_n$; there exists an irreducible representation of H which has no nontrivial H_i-invariant vectors for all $1 \leq i \leq n$.*

Proof. We will prove the lemma in two steps.

In the first step, we will show that the lemma holds when $H^{'} = H$. By the previous lemma, it is enough to find a function f on H such that for all $i = 1,..,n$ and $g,t \in H$, $\sum_{x\in gH_it} f(x) = 0$. But since each H_i is normal in H, $gH_it = H_igt$ for all $g,t \in H$. So it is enough to find a function f on H satisfying $\sum_{x\in H_it} f(x) = 0$ for all $i = 1,..,n$ and $t \in H$.

Let $E_i = \{x_i, y_i\} \subseteq H_i$ with $x_i \neq y_i$ and $E = E_1 \times ... \times E_n$. For $e \in E$, let $N(e)$ be the number of x_i appearing in the coordinates of e. For instance, if $n = 4$ and $e = (x_1, x_2, y_3, x_4)$, then $N(e) = 3$. Now define f on H by

$$f(e) = \begin{cases} (-1)^{N(e)} & \text{if } e \in E, \\ 0 & \text{otherwise} \end{cases}$$

Let $h = (h_1, ..., h_n) \in H^{'} = H$. Given i with $1 \leq i \leq n$,

$$H_ih = \{(h_1, ..., h_{i-1}, h, h_{i+1}, ..., h_n) : h \in H_i\}.$$

Note that if $h_k \notin E_k$ for some $k \neq i$, $H_ih \cap E = \emptyset$. Hence $f \equiv 0$ on H_ih so that $\sum_{x\in H_ih} f(x) = 0$. If $h_k \in E_k$ for all $k \neq i$,

$$H_ih \cap E = \{(h_1, ..., h_{i-1}, x_i, h_{i+1}, ..., h_n), (h_1, ..., h_{i-1}, y_i, h_{i+1}, ..., h_n)\}.$$

Then since

$$N((h_1, ..., h_{i-1}, x_i, h_{i+1}, ..., h_n)) = N((h_1, ..., h_{i-1}, y_i, h_{i+1}, ..., h_n)) + 1$$

, we get

$$\begin{aligned}\sum_{x\in H_ih} f(x) &= \sum_{x\in H_ih\cap E} f(x) \\ &= f(h_1, ..., h_{i-1}, x_i, h_{i+1}, ..., h_n) + f(h_1, ..., h_{i-1}, y_i, h_{i+1}, ..., h_n) \\ &= 0.\end{aligned}$$

So in the case $H^{'} = H$, we are done.

As a second step, we will pass to the general case. Again by the previous lemma, it is enough to find a function F on H satisfying $\sum_{x\in gH_it} F(x) = 0$, $\forall g \in H$ and $\forall i = 1, ..., n$.

Since inner automorphisms of H permute $H_1, ..., H_n$, given $g \in H$ and i, there exists j such that $gH_ig^{-1} = H_j$. Then $\forall t \in H$, $gH_it = gH_ig^{-1}gt = H_jgt$. Hence, it is enough to find a function F on H satisfying $\sum_{x \in H_it} F(x) = 0$, $\forall t \in H$ and $\forall i = 1, ..., n$.

Just take $F = f\chi_{H'}$ where f is the map defined in the first step. Then for all $t \in H$ and for all i,

$$\sum_{x \in H_it} F(x) = \sum_{x \in H_it} f\chi_{H'}(x) = \sum_{x \in H_it \cap H'} f(x) = \sum_{x \in H_it \cap E} f(x) = 0$$

as desired.

□

Lemma 3.1.4. *Let H be a finite nonabelian group and $H^{'}$ is a subgroup of H such that every irreducible representation of H has a nontrivial $H^{'}$-invariant vector. Then,*

$$|H^*| < |H| \,/\, |H'| \,.$$

where by H^* we mean the dual object of H, i.e. the set of nonequivalent one dimensional representations of H.

Corollary 3.1.5. *Let $k+1 \geq 3$. There is an irreducible representation of $Sym(k+1)$ which has no nontrivial $Sym(k)$- invariant vectors. Here we consider $Sym(k)$ as the stability subgroup of the point $k+1$.*

Proof. It is enough to show that

$$|(Sym(k+1))^*| \geq |Sym(k+1)| \,/\, |Sym(k)| = (k+1)!/k! = k+1.$$

But since $Sym(k+1)$ is a finite group, $|(Sym(k+1))^*|$ equals to the number of conjugacy classes of $Sym(k+1)$. (See Corollary 2.7, Character Theory of Finite Groups, Irving Martin Isaacs). For each $i = 2, 3, ..., k+1$ choose an i-cycle C_i in $Sym(k+1)$. Then the conjugacy classes of Id, C_2,..., C_{k+1} are pairwise disjoint. So the result follows.

□

Theorem 3.1.6. *For every finite complete subtree* Δ *with* $diam(\Delta) \geq 2$, *the space* $(\widetilde{K}(\Delta))' \neq \emptyset$. *In particular,* $H(\Delta) \neq \{0\}$.

Proof. If $diam(\Delta) = 2$, then $\Delta = V_1(\{x\})$ for some $x \in X$ and the maximal complete subtrees of Δ are the k many edges of Δ. So, $\widetilde{K}(\Delta)/K(\Delta) \cong Sym(k+1)$ and $K(\Delta_i)/K(\Delta) \cong Sym(k)$ for all $i = 1, ..., k$ where $\Delta_1, ..., \Delta_k$ are maximal complete subtrees of Δ. Then by corollary 3.1.5, there is a unitary irreducible representation $(\widetilde{\pi}, V)$ of $\widetilde{K}(\Delta)/K(\Delta)$ which has no nonzero $K(\Delta_1)/K(\Delta)$ invariant vectors. Define $\pi : \widetilde{K}(\Delta) \to \mathrm{GL}(V)$ by $\pi(g) = \widetilde{\pi}(gK(\Delta))$. Let $gK(\Delta) = \widetilde{g}$ for all $g \in G$. Clearly, $\langle \pi(g)v, \pi(g)w \rangle = \langle \widetilde{\pi}(\widetilde{g})v, \widetilde{\pi}(\widetilde{g})w \rangle = \langle v, w \rangle$ for all $g \in G$, $v \in V$ and $\pi(k)v = \widetilde{\pi}(\widetilde{e})(v) = v$ for all $k \in K(\Delta)$, $v \in V$. So, (π, V) is a unitary irreducible representation of $\widetilde{K}(\Delta)$ which is trivial on $K(\Delta)$ and which has no nonzero $K(\Delta_1)$-stationary vectors. Assume (π, V) has a $K(\Delta_i)$-stationary vector v for some i. Let $g \in G$ be such that $gK(\Delta_i)g^{-1} = K(\Delta_1)$. Then for all $h \in K(\Delta_i)$, $\pi(ghg^{-1})(\pi(g)v) = \pi(gh)(v) = \pi(g)(v)$ so that $\pi(g)v$ is a $K(\Delta_1)$-stationary vector. Thus, (π, V) has no nonzero $K(\Delta_i)$-stationary vectors for all $i = 1, ..., k$.

Note that for all $i = 1, ..., n$, $\Delta^0 \subset \Delta_i$ and so $K(\Delta_i) \subset K(\Delta^0)$. Let $x_1, ..., x_n \in \partial(\Delta^0)$ and $\Delta_1, ..., \Delta_n$ be the corresponding maximal complete subtrees of Δ. Given $g \in K(\Delta^0)$, for each i, there exists an element g_i of $K(\Delta_i)$ with $g_i = g$ on the set $\{x : l(x, \Delta) = l(x, x_i)\}$ and $g_i = Id$ on the set $X - \{x : l(x, \Delta) = l(x, x_i)\}$. Then $g = \prod_{i=1}^n g_i$ and $\widetilde{g} = \prod_{i=1}^n \widetilde{g}_i$ in $K(\Delta^0)/K(\Delta)$. Since $K(\Delta_i) \cap K(\Delta_j) = K(\Delta)$, $K(\Delta_i)/K(\Delta)$ intersects trivially with $K(\Delta_j)/K(\Delta)$ for all distinct i, j. Hence $K(\Delta^0)/K(\Delta)$ is direct product of its subgroups $K(\Delta_1)/K(\Delta)$,..., $K(\Delta_n)/K(\Delta)$. Note that the inner automorphisms of $\widetilde{K}(\Delta)/K(\Delta)$ permute the groups $K(\Delta_1)/K(\Delta)$,
...,$K(\Delta_n)/K(\Delta)$. Then by lemma 3.1.3., $\widetilde{K}(\Delta)/K(\Delta)$ has an irreducible unitary representation which has no $K(\Delta_i)/K(\Delta)$ for all $i = 1, ..., n$ as desired.

Now let $(\pi, V) \in (\widetilde{K}(\Delta))'$ be unitary and let $v \in V$. Define a function

f from G to $\mathbb{C}$ by

$$f(g) = \begin{cases} \langle \pi(g)v, v \rangle & \text{if } g \in \widetilde{K}(\Delta) \\ 0 & \text{otherwise .} \end{cases}$$

Let $k, k^{'} \in K(\Delta)$. Clearly, $x \in \widetilde{K}(\Delta)$ iff $kxk^{'} \in \widetilde{K}(\Delta)$. Then since V is trivial on $K(\Delta)$, for $x \in \widetilde{K}(\Delta)$, $f(kxk^{'}) = \left\langle \pi(kxk^{'})(v), v \right\rangle = \langle \pi(x)(v), v \rangle = f(x)$ and for $x \notin \widetilde{K}(\Delta)$, $f(kxk^{'}) = 0 = f(x)$ so that f is $K(\Delta)$-bi-invariant. As $f(e) = \langle v, v \rangle = \|v\|^2 > 0$, f is nonzero. Now let Δ_i be a maximal complete subtree of the tree X. For any $x \notin \widetilde{K}(\Delta)$, $\int_{K(\Delta_i)} f(xk)\ dk = \int_{K(\Delta_i)} 0\ dk = 0$. Besides, for any $x \in \widetilde{K}(\Delta)$,

$$\begin{aligned}
\int_{K(\Delta_i)} f(xk)\ \mathrm{d}\mu(k) &= \int_{K(\Delta_i)} \langle \pi(xk)v, v \rangle\ \ \mathrm{d}\mu(k) \\
&= \mu(K(\Delta_i)) \int_G e_{K(\Delta_i)}(k) \left\langle \pi(k)v, \pi(x^{-1})v \right\rangle\ \ \mathrm{d}\mu(k) \\
&= \mu(K(\Delta_i)) \left\langle \pi(e_{K(\Delta_i)})v, \pi(x^{-1})v \right\rangle \\
&= 0
\end{aligned}$$

as $\pi(e_{K(\Delta_i)})v \in V^{K(\Delta_i)} = \{0\}$. Hence, $f \in H(\Delta)$.

□

3.2 Subrepresentations of $L(\Delta)$

Lemma 3.2.1. *Any nonzero G-invariant subspace $L^{'}$ of $L(\Delta)$ has a nonzero intersection with $L(\Delta)^{K(\Delta)}$. Conversely, if (T,H) is an irreducible representation of G and $H^{K(\Delta)} \neq \emptyset$, then T is equivalent to some subrepresentation of $L(\Delta)$.*

Proof. Let f be a nonzero element of $L^{'}$ with $f(g) \neq 0$. Then, $\pi_L(g^{-1})f(e) = f(g.e) = f(g) \neq 0$ and $\pi_L(g^{-1})f \in L^{'}$ as $L^{'}$ is G-invariant. So we can restart the proof by taking an element $f \in L^{'}$ with $f(e) \neq 0$.

Let $f^{\circ} = {}^{K(\Delta)}f$.

Firstly let us show that $f^{\circ} \in L^{'}$. Let $K^{'}$ be an open compact subgroup of G such that f is $K^{'}$-left-invariant. Set $M = K(\Delta) \cap K^{'}$.

Since $\{xM\}_{x\in K(\Delta)}$ is an open cover of $K(\Delta)$ and $K(\Delta)$ is compact, $K(\Delta) = \bigcup_{i=1}^{n} x_i M$ for some $x_i \in K(\Delta)$ where the sets $x_1M, ..., x_nM$ are pairwise disjoint.

Then,

$$\begin{aligned} f^{\circ}(g) &= \int_{K(\Delta)} f(k^{-1}g)\ \mathrm{d}\mu(k) \\ &= \sum_{i=1}^{n} \int_{x_iM} f(k^{-1}g)\ \mathrm{d}\mu(k) \\ &= \sum_{i=1}^{n} \mu(M) f(x_i^{-1}g) \\ &= \mu(M)\left[\sum_{i=1}^{n} \pi_L(x_i) f\right](g) \end{aligned}$$

for all $g \in G$. So, $f^{\circ} = \mu(M)\left[\sum_{i=1}^{n} \pi_L(x_i) f\right] \in L'$.

By proposition 3.0.2. part (a), f° is $K(\Delta)$-left-invariant.

Since f is $K(\Delta)$-right-invariant, $f \equiv f(e)$ on $K(\Delta)$. Then,

$$f^{\circ}(e) = \frac{1}{\mu(K(\Delta))} \int_{K(\Delta)} f(k^{-1})\ \mathrm{d}\mu(k)$$

$$= \frac{1}{\mu(K(\Delta))} \int_{K(\Delta)} f(e)\ \mu(k) = \frac{1}{\mu(K(\Delta))}.f(e).\mu(K(\Delta)) = f(e) \neq 0.$$

So, $f^{\circ} \not\equiv 0$.

Hence $f^{\circ} \in L(\Delta)^{K(\Delta)}$.

Now consider the mapping

$$\begin{aligned} \phi : H &\longrightarrow L(\Delta) \\ w &\longmapsto \phi(w) : G \longrightarrow \mathbb{C} \\ & \qquad\qquad g \longmapsto \langle T(g^{-1})w, \widetilde{v}\rangle \end{aligned}$$

where $\widetilde{v}$ is a nonzero element in $\widetilde{H}^{K(\Delta)}$. We can choose such an element as $H^{K(\Delta)}$ is nonempty.

For $w \in H$, let us denote $\phi(w)$ by ϕ_w. Now, ϕ_w is $K(\Delta)$-right-invariant. This is because for all $g \in G$ and $k \in K(\Delta)$, we have

$$\phi_w(gk) = \langle T(k^{-1}g^{-1})w, \widetilde{v}\rangle = \left\langle T(g^{-1})w, \widetilde{T}(k)\widetilde{v}\right\rangle = \langle T(g^{-1})w, \widetilde{v}\rangle = \phi_w(g).$$

If w is K-left invariant where K is an open compact subgroup of G,

$$\phi_w(kg) = \left\langle T(g^{-1})T(k^{-1})w, \widetilde{v}\right\rangle = \left\langle T(g^{-1})w, \widetilde{v}\right\rangle = \phi_w(g).$$

So ϕ_w is K-left-invariant, hence locally constant, hence continuous.

Thus the image of ϕ is in $L(\Delta)$ so that ϕ is well defined.

The map $\phi : H \to L(\Delta)$ is a nonzero intertwinning operator. This follows from the following equation:

$$\begin{aligned}
\phi(T(h)v)(g) &= \phi_{T(h)v}(g) \\
&= \left\langle T(g^{-1})T(h)v, \widetilde{v}\right\rangle \\
&= \left\langle T(g^{-1}h)v, \widetilde{v}\right\rangle \\
&= \phi_v(h^{-1}g) \\
&= \phi(v)(h^{-1}g) \\
&= \pi_L(h)\phi(v)(g)
\end{aligned}$$

for all $g, h \in G$ and $v \in H$.

Then $Ker\,(\phi) = \{0\}$ as it is a proper G-invariant subspace of the irreducible representation H.

Therefore ϕ is 1-1 and H is equivalent to the subrepresentation $Im\,(\phi)$ of $L(\Delta)$.

□

3.3 Finiteness of the functions in $H(\Delta)$

Lemma 3.3.1. *Let Δ be a finite complete subtree of the tree X with $diam(\Delta) \geq 2$. Let $\Delta^{'}$ be a complete subtree not containing Δ. Then there exists a proper complete subtree $\Delta_0 \subset \Delta$, $\Delta_0 \neq \Delta$, such that $K(\Delta_0) \subseteq K(\Delta^{'})K(\Delta)$.*

Proof. Without loss of generality we may assume that $\Delta \bigcap \Delta^{'}$ contains an edge. In fact, if their intersection is empty or consists of one vertex, there exists only one $x \in \Delta$ whose distance from $\Delta^{'}$ is minimal. Let r be the minimal distance between Δ and $\Delta^{'}$. If there exists $x, x^{'} \in \Delta$

and $y, y' \in \Delta'$ such that $x \neq x'$, $y \neq y'$ and $l(x,y) = l(x',y') = r$, then (x,y) and (x',y') cannot intersect with Δ or Δ'. So in the case $\Delta \bigcap \Delta' = \emptyset$, we get a circuit $[x,x'][x',y'][y',y][y,x]$ which leads to a contradiction. In the case $\Delta \bigcap \Delta'$ contains only one vertex, r=0 and consequently $\{x, x'\} \subseteq \Delta \bigcap \Delta'$, again a contradiction.

Now let $\Delta'' = \{x \in X : l(x, \Delta') \leq r+1\}$. As $\Delta'' = V_m(\Delta')$, it is complete. Let x be the vertex of Δ whose distance from Δ' is r. Clearly, $x \in \Delta''$. Since $diam(\Delta) \geq 2$, there exists $y \in \Delta$ with $l(x,y) = 1$ so that $l(y, \Delta'') = r+1$, i.e. $y \in \Delta''$. Then $\Delta \bigcap \Delta''$ contains the edge $[x,y]$. Moreover Δ'' does not contain Δ. Otherwise, every $y \in \Delta$ different from x is adjacent to x which contradicts either with Δ is complete or with $diam(\Delta) \geq 2$.

Since $\Delta' \subseteq \Delta''$, $K(\Delta'') \subseteq K(\Delta')$. Now our aim is to find a proper complete subtree $\Delta_0 \subset \Delta$, $\Delta_0 \neq \Delta$, such that $K(\Delta_0) \subseteq K(\Delta'')K(\Delta)$ because if we find such Δ_0 we get $K(\Delta_0) \subseteq K(\Delta'')K(\Delta) \subseteq K(\Delta')K(\Delta)$ as desired.

Set $\Delta_0 = \Delta'' \bigcap \Delta$. Since $\Delta'' \bigcap \Delta$ contains an edge, $diam(\Delta_0) \geq 1$. It is complete as it is the intersection of two complete trees and it is different from Δ as $\Delta \not\subseteq \Delta''$. For the boundary points $x_1, \ldots, x_n$ of Δ_0, let

$$D_i = \{x \in X : l(x, \Delta_0) = l(x, x_i)\} \bigcup \{x_i\}$$

and set

$$K_i = K\left((\cup_{j \neq i} D_j) \cup \Delta_0\right).$$

Clearly, $K_i \subseteq K(\Delta_0)$ for all i. Now, choose $g \in K(\Delta_0)$ and define $g_i \in G$ by $g_i = g$ on D_i and $g_i = Id$ on the complement of D_i. Since $D_i \bigcap (\bigcup_{j \neq i} D_j) = \emptyset$, we get $g_i = Id$ on $\bigcup_{j \neq i} D_j$ and since $D_i \bigcap (\Delta_0 \setminus \{x_i\}) = \emptyset$, we get $g_i = Id$ on $\Delta_0 \setminus \{x_i\}$. As these two force g_i to fix x_i, we finally get $g_i \in K_i$ for all i and that $g = \prod_{i=1}^{n} g_i$. Moreover, $K_i \bigcap K_j = \{Id\}$ for $i \neq j$. In fact, if $g \in K_i \bigcap K_j$, g fixes D_i's and Δ_0 by definition of K_i and K_j so that $g = Id$. Hence, $K(\Delta_0)$ is the direct product of its subgroups K_i, $1 \leq i \leq n$.

For a given i, if $\Delta \bigcap D_i = \{x_i\}$, $\Delta \subseteq (\bigcup_{j \neq i} D_j) \bigcup \Delta_0$, i.e. $K_i \subseteq K(\Delta)$. Similarly, if $\Delta^{''} \bigcap D_i = \{x_i\}$, $K_i \subseteq K(\Delta^{''})$. Therefore, if $K(\Delta_0) \nsubseteq K(\Delta^{''})K(\Delta)$, then for some i we have $K_i \nsubseteq K(\Delta)$ and $K_i \nsubseteq K(\Delta^{''})$, i.e. $\Delta \bigcap D_i \neq \{x_i\}$ and $\Delta^{''} \bigcap D_i \neq \{x_i\}$. Thus there exist $y_1, y_2 \in D_i$ such that $y_1 \in \Delta$, $y_2 \in \Delta^{''}$. Since Δ and $\Delta^{''}$ are complete trees, we can choose y_1 and y_2 so that $l(y_1, x_i) = l(y_2, x_i) = 1$. As $\Delta_0 \bigcap D_i = \{x_i\}$, $y_1 \neq y_2$. On the other hand, by the condition $diam(\Delta_0) \geq 1$, there exists $y \in \Delta_0$ with $l(y, x_i) = 1$. As $y_1, y_2 \notin \Delta_0$, we have $y \neq y_1, y_2$. So x_i is the extremity of the two edges in Δ and $\Delta^{''}$, i.e. x_i is not a boundary point of Δ and $\Delta^{''}$. Then since Δ and $\Delta^{''}$ are complete, all vertices adjacent to x_i are both in Δ and $\Delta^{''}$ which implies y_1 and y_2 are in Δ_0, i.e. x_i is not a boundary point of Δ_0, a contradiction.

□

Lemma 3.3.2. *Let Δ and $\Delta^{'}$ be complete trees with $diam(\Delta) \geq 2$. Let $f \in H(\Delta)^{K(\Delta^{'})}$. Then, $f(g) = 0$ if $g\Delta \nsubseteq \Delta^{'}$, i.e. $supp(f) \subseteq \{g \in G : g\Delta \subseteq \Delta^{'}\}$. In particular, if $f \in H(\Delta)^{K(\Delta)}$, f is supported on $\widetilde{K}(\Delta)$.*

Proof. First let us show that it sufficies to prove the theorem for $g = e$. For this, assume we are done with the case $g = e$. Let $g \in G$ be any and consider the function $\pi_R(g)f$. Observe that $\pi_R(g)f \in H(g\Delta)^{K(\Delta^{'})}$. Then since $diam(g\Delta) = diam(\Delta) \geq 2$, we get $\pi_R(g)f(e) = f(g) = 0$ if $g\Delta \nsubseteq \Delta^{'}$ and we are done.

Now we will see $f(e) = 0$ if $\Delta \nsubseteq \Delta^{'}$. Since f is left $K(\Delta^{'})$ and right $K(\Delta)$ invariant, $f|_{K(\Delta^{'}).K(\Delta)} \equiv f(e)$ so that $f|_{K(\Delta_0)} \equiv f(e)$ where Δ_0 is a proper complete subtree of Δ satisfying $K(\Delta_0) \subseteq K(\Delta^{'})K(\Delta)$. Then together with $f \in H(\Delta)$, we get

$$\begin{aligned} 0 &= \alpha(\Delta, \Delta_0)(f)(e) \\ &= \frac{1}{\mu(K(\Delta_0))} \int_{K(\Delta_0)} f(k)\ \mathrm{d}\mu(k) \\ &= \frac{1}{\mu(K(\Delta_0))} \int_{K(\Delta_0)} f(e)\ \mathrm{d}\mu(k) \\ &= \frac{1}{\mu(K(\Delta_0))}.f(e).\mu(K(\Delta_0)) \\ &= f(e) \end{aligned}$$

as desired.

□

Corollary 3.3.3. *If Δ is a finite complete subtree with diameter at least 2, then every element of $H(\Delta)$ is compactly supported.*

Proof. Let $f \in H(\Delta)$. Then there is a finite complete subtree Δ_f of X such that f is $K(\Delta_f)$-left invariant. This means by the previous lemma that $supp(f)$ is contained in the set $S = \{g \in G : g(\Delta) \subseteq \Delta_f\}$. Let $(g_n)_n$ be a convergent sequence in S with $\lim_{n\to\infty} g_n = g$. Choose $k \in \mathbb{N}$ such that $\Delta \subset B_k$ and choose $N \in \mathbb{N}$ such that $g = g_n$ on B_k for all $n \geq N$. Then $g\Delta = g_N\Delta \subset \Delta_f$, i.e $g \in S$. So, S is closed. Now let x be a vertex of Δ and $y_1, ..., y_m$ be distinct vertices of Δ_f. For each $j = 1, ..., m$, let $S_j = \{g \in G : g(x) = y_j\}$. Since each S_j is compact we get that S is a closed subset of the compact set $\bigcup_{j=1}^m S_j$. Then S, hence $supp(f)$ is compact.

□

Let us now state another corollary of lemma 3.3.2.

Corollary 3.3.4. *The operator of left averaging over $K(\Delta')$ in $H(\Delta)$ projects $H(\Delta)$ onto $H(\Delta)^{K(\Delta')}$ and coincides with multiplication by the characteristic function of the set $\{g \in G : g\Delta \subseteq \Delta'\}$.*

Proof. Let ϕ be the operator of left averaging over $K(\Delta^{'})$ and $f \in H(\Delta)$. Since f is $K(\Delta)$-right-invariant, for all $x \in G$ and $y \in K(\Delta)$,

$$\begin{aligned}
\phi(f)(xy) &= \frac{1}{\mu(K(\Delta'))} \int_{K(\Delta')} f(k^{-1}xy) \ \mathrm{d}\mu(k) \\
&= \frac{1}{\mu(K(\Delta'))} \int_{K(\Delta')} f(k^{-1}x) \ \mathrm{d}\mu(k) \\
&= \phi(f)(x).
\end{aligned}$$

Hence $\phi(f)$ is $K(\Delta)$-right-invariant. By proposition 3.0.2, $\phi(f)$ is continuous and $K(\Delta^{'})$-left-invariant. Also for all $i = 1, ..., n$ and $x \in G$,

$$\begin{aligned}
\phi(f)^{K(\Delta_i)}(x) &= \frac{1}{\mu(K(\Delta_i))} \int_{K(\Delta_i)} \phi(f)(xk) \ \mathrm{d}\mu(k) \\
&= \frac{1}{\mu(K(\Delta_i))} \int_{K(\Delta_i)} \frac{1}{\mu(K(\Delta'))} \int_{K(\Delta')} f(g^{-1}xk) \ \mathrm{d}\mu(g) \ \mathrm{d}\mu(k) \\
&= \frac{1}{\mu(K(\Delta'))} \int_{K(\Delta')} \frac{1}{\mu(K(\Delta_i))} \int_{K(\Delta_i)} f(g^{-1}xk) \ \mathrm{d}\mu(k) \ \mathrm{d}\mu(g) \\
&= \frac{1}{\mu(K(\Delta'))} \int_{K(\Delta')} f^{K(\Delta_i)}(g^{-1}x) \ \mathrm{d}\mu(g) \\
&= 0.
\end{aligned}$$

Thus, $\phi(f) \in H(\Delta)^{K(\Delta^{'})}$.

Now if $g\Delta \nsubseteq \Delta^{'}$, since $\phi(f) \in H(\Delta)^{K(\Delta^{'})}$, by lemma 3.3.2., we get $\phi(f)(g) = 0$. If $g\Delta \subseteq \Delta^{'}$, we get $K(\Delta^{'}) \subseteq K(g\Delta) = gK(\Delta)g^{-1}$. Then, for all $x \in K(\Delta^{'})$, there exists $y \in K(\Delta)$ with $x = gyg^{-1}$ so that $f(x^{-1}g) = f(gy^{-1}g^{-1}g) = f(gy^{-1}) = f(g)$. This implies that

$$\begin{aligned}
\phi(f)(g) &= (\mu(K(\Delta^{'})))^{-1} \int_{K(\Delta')} f(x^{-1}g) \ \mathrm{d}\mu(x) \\
&= (\mu(K(\Delta^{'})))^{-1} \int_{K(\Delta')} f(g) \ \mathrm{d}\mu(x) \\
&= f(g).
\end{aligned}$$

Hence $\phi(f) = f\chi_{\{g \in G : g\Delta \subseteq \Delta'\}}$ as desired.

□

Proposition 3.3.5. *Every function in $H(\Delta)$ is a finite sum of left translates of $K(\Delta)$-bi-invariant functions.*

Proof. Let $f \in H(\Delta)$. As f is compactly supported, there exist $g_1, ..., g_n \in G$ such that $supp(f) \subseteq \bigcup_{i=1}^{n} g_i\widetilde{K}(\Delta)$.

It is easy to check that $\pi_L(g^{-1})(f.\chi_{g\widetilde{K}(\Delta)}) = (\pi_L(g^{-1})(f)) .\chi_{\widetilde{K}(\Delta)}$ so that $f.\chi_{g\widetilde{K}(\Delta)} = \pi_L(g) \left[\pi_L(g^{-1})(f.\chi_{g\widetilde{K}(\Delta)})\right] = \pi_L(g) \left[(\pi_L(g^{-1})(f)) .\chi_{\widetilde{K}(\Delta)}\right]$ for all $g \in G$. Then,

$$\begin{aligned} f &= f.\chi_{\left(\bigcup_{i=1}^{n} g_i\widetilde{K}(\Delta)\right)} \\ &= f.\sum_{i=1}^{n} \chi_{\left(g_i\widetilde{K}(\Delta)\right)} = \sum_{i=1}^{n} f.\chi_{g_i\widetilde{K}(\Delta)} = \sum_{i=1}^{n} \pi_L(g_i) \left[\left(\pi_L(g_i^{-1})(f)\right) .\chi_{\widetilde{K}(\Delta)}\right] \end{aligned}$$

where $\left(\pi_L(g_i^{-1})(f)\right) .\chi_{\widetilde{K}(\Delta)} = {}^{K(\Delta)}\left(\pi_L(g_i^{-1})(f)\right)$ is a $K(\Delta)$-bi-invariant function for all $i = 1, ..., n$.

□

3.4 Admissibility of $H(\Delta)$

Let $T(\Delta)$ be the representation of G in $H(\Delta)$ by left translations.

Since any function $f \in H(\Delta)$ is finite (compactly supported) on G, $H(\Delta)$ is a subspace of the space of square integrable functions $L^2(G)$ so that we have an inner product on $H(\Delta)$ that is defined by

$$\langle f_1, f_2 \rangle = \int_G f_1\overline{f_2}(g) \ \mathrm{d}\mu(g)$$

for $f_1, f_2 \in H(\Delta)$. Note that this inner product is G-invariant, i.e.

$$\langle T(\Delta)(g)f_1, T(\Delta)(g)f_2 \rangle = \langle f_1, f_2 \rangle$$

for every $f_1, f_2 \in H(\Delta)$ and $g \in G$.

Moreover, for every complete subtree Δ' of the tree X, the space $H(\Delta)^{K(\Delta')}$ is finite dimensional so that $(T(\Delta), H(\Delta))$ is an admissible representation of G. Indeed, since for every $f \in H(\Delta)^{K(\Delta')}$, f is $K(\Delta)$-right-invariant and supported on the compact set $\left\{g \in G : g\Delta \subseteq \Delta'\right\}$, $H(\Delta)^{K(\Delta')}$ can be identified with the space V of functions on $G/K(\Delta)$ which take 0 on the cosets that do not intersect the set $\left\{g \in G : g\Delta \subseteq \Delta'\right\}$. As $\left\{g \in G : g\Delta \subseteq \Delta'\right\}$ is compact, only finitely many cosets, say

$g_1K(\Delta), ..., g_nK(\Delta)$ intersect with $\{g \in G : g\Delta \subseteq \Delta'\}$. Hence, V is generated by the functions $f_1, ..., f_n$ where $f_i = \chi_{g_iK(\Delta)}$ for all $i = 1, ..., n$ so that V is finite dimensional. Then, $H(\Delta)^{K(\Delta')}$ is also finite dimensional.

Consequently, the representation $(T(\Delta), H(\Delta))$ of G is unitary.

Note that as $(T(\Delta), H(\Delta))$ is algebraic, for all $f \in H(\Delta)$ and $f^* \in H(\Delta)^*$ the matrix coefficient $g \mapsto \langle T(\Delta)(g)f, f^*\rangle$ is locally constant, hence continuous.

3.5 A necessary and sufficient condition for $T(\Delta) \sim T(\Delta')$

Lemma 3.5.1. *Let Δ' be a finite complete tree and $diam(\Delta) \geq 2$. Then, $T(\Delta) \sim T(\Delta')$ iff $\Delta' = g\Delta$ for some $g \in G$.*

Proof. ($\Rightarrow$)Assume $\Delta' \neq g\Delta$ for all $g \in G$. Without loss of generality, assume $Card(ver\Delta) \geq Card(ver\Delta')$. Then $g\Delta \nsubseteq \Delta'$ for any $g \in G$. So if $f \in H(\Delta)^{K(\Delta')}$, by lemma 3.3.3. $f(g) = 0$ for all $g \in G$, i.e $f \equiv 0$. In other words, there are no non-zero $K(\Delta')$-stationary vectors in $H(\Delta)$. But by 3.2.1. $H(\Delta')$ contains a non-zero $K(\Delta')$-stationary vector. Thus, $T(\Delta)$ and $T(\Delta')$ are disjunct.

($\Leftarrow$) For the maximal complete subtrees $\Delta_1, ..., \Delta_n$ of Δ, the maximal complete subtrees of Δ' are exactly $g\Delta_1, ..., g\Delta_n$. Note that $K(g\Delta_i) = gK(\Delta_i)g^{-1}$ for all $1 \leq i \leq n$. Then, if $f \in H(\Delta)$, for all $x \in G$ and for all i we get

$$\begin{aligned}
[\pi_R(g)(f)]^{K(g\Delta_i)}(x) &= \frac{1}{\mu(K(g\Delta_i))} \int_{K(g\Delta_i)} \pi_R(g)(f)(xk) \ \mathrm{d}\mu(k) \\
&= \frac{1}{\mu(K(\Delta_i))} \int_{gK(\Delta_i)g^{-1}} f(xkg) \ \mathrm{d}\mu(k) \\
&= \frac{1}{\mu(K(\Delta_i))} \int_{K(\Delta_i)} f(xgk) \ \mathrm{d}\mu(k) \\
&= \frac{1}{\mu(K(\Delta_i))} f^{K(\Delta_i)}(xg) \\
&= 0
\end{aligned}$$

so that $\pi_R(g)(f) \in H(\Delta^{'})$.

Moreover, for all $f \in H(\Delta)$ and $x, h \in G$,

$$\begin{aligned}
[\pi_R(g) \circ T(\Delta)(h)]\,(f)(x) &= \pi_R(g)\,[T(\Delta)(h)f]\,(x) \\
&= [T(\Delta)(h)f]\,(xg) \\
&= f(h^{-1}xg) \\
&= [\pi_R(g)(f)]\,(h^{-1}x) \\
&= \left[T(\Delta^{'})(h) \circ \pi_R(g)\right](f)(x).
\end{aligned}$$

Thus $\pi_R(g)$ is an intertwinning operator from $H(\Delta)$ to $H(\Delta^{'})$. As $\pi_R(g)$ is also bijective, we get $T(\Delta) \sim T(\Delta^{'})$.

□

3.6 Irreducible subrepresentations of $H(\Delta)$

Let H be a unimodular, seperable, locally compact group and K be a closed subgroup of H. Let (π, W) be a unitary representation of K on a Hilbert space W and let V be the space of functions $f : H \to W$ satisfying the following conditions:

1. $f(hk) = \pi(k^{-1})f(h)$ for all $k \in K$ and $h \in H$.

2. there is an open compact subgroup K_f of H such that $f(kh) = f(h)$ for all $k \in K_f$ and $h \in H$.

3. f is compactly supported.

Then the representation $(Ind(\pi), V)$ acting according to the formula

$$Ind(\pi)(g_0)(f)(g) = f(g_0^{-1}g)$$

is said to be induced by (π, W). If $f \in V$, then $f \in V^{K_f}$ so that $(Ind(\pi), V)$ is algebraic.

Moreover we can define an invariant inner product on V by

$$\langle f, g \rangle = \int_H \langle f(x), g(x) \rangle \;\; \mathrm{d}\mu(x)$$

where μ is a left Haar measure on H.

Now let (π, V) be an irreducible nondegenerate representation of the compact group $\widetilde{K}(\Delta)$, which is trivial on $K(\Delta)$ where Δ is a complete finite subtree of diameter ≥ 2. Since the group $\widetilde{K}(\Delta)/K(\Delta)$ is finite and π is irreducible, π is finite dimensional. Then since π is a finite dimensional representation of the compact group $\widetilde{K}(\Delta)$, we can put an inner product on V that makes π a unitary representation on the Hilbert space V.

Let $(T(\Delta, \pi), V_\pi)$ be the representation of G induced by π.

Set $K = K(\Delta)$ and $\widetilde{K} = \widetilde{K}(\Delta)$.

Remark 3.6.1. Let $h \in V_\pi$. Then the map $\widetilde{h} : V_\pi \to \mathbb{C}$ defined by $\widetilde{h}(f) = \langle f, h \rangle$ is contained in $\widetilde{V_\pi}$.

Proposition 3.6.2. *Every matrix coefficient of π is a matrix coefficient of $T(\Delta, \pi)$ with support in $\widetilde{K}$.*

Proof. Given $v \in V$, define $f_v : G \to V$ by

$$f_v(x) = \begin{cases} \pi(x^{-1})v & \text{if} \quad x \in \widetilde{K}, \\ 0 & \text{otherwise} \end{cases}$$

Then, if $g, k \in \widetilde{K}$,

$$f_v(gk) = \pi(k^{-1}g^{-1})v = \pi(k^{-1})\pi(g^{-1})v = \pi(k^{-1})f_v(g)$$

and if $g \notin \widetilde{K}$ and $k \in \widetilde{K}$ again we get

$$f_v(gk) = 0 = \pi(k^{-1})f_v(g)$$

so that

$$f_v(gk) = \pi(k^{-1})f_v(g)$$

for all $g \in G$ and $k \in \widetilde{K}$.

Similarly if $g \in \widetilde{K}$ and $k \in K$,

$$f_v(kg) = \pi(g^{-1}k^{-1})v = \pi(g^{-1})\pi(k^{-1})v = \pi(g^{-1})v = f_v(g)$$

and if $g \notin \widetilde{K}$ and $k \in K$ again we get

$$f_v(kg) = 0 = f_v(g)$$

so that f_v is K-left-invariant.

Moreover, $supp(f_v) \subseteq \widetilde{K}$ so that f_v is compactly supported.

Thus, $f_v \in T(\Delta, \pi)$.

Now, for $g \in \widetilde{K}$ and $v, w \in V$,

$$\begin{aligned}
\langle T(\Delta,\pi)(g) f_v, f_w \rangle &= \int_G \langle T(\Delta,\pi)(g) f_v(x), f_w(x) \rangle \;\; \mathrm{d}\mu(x) \\
&= \int_G \left\langle f_v(g^{-1}x), f_w(x) \right\rangle \;\; \mathrm{d}\mu(x) \\
&= \int_{\widetilde{K}} \left\langle \pi(x^{-1}g)v, \pi(x^{-1})w \right\rangle \;\; \mathrm{d}\mu(x) \\
&= \int_{\widetilde{K}} \langle \pi(g)v, w \rangle \;\; \mathrm{d}\mu(x) \\
&= \mu(\widetilde{K}).\, \langle \pi(g)v, w \rangle \\
&= \left\langle \pi(g)v, \mu(\widetilde{K}).w \right\rangle
\end{aligned}$$

and for $g \notin \widetilde{K}$,

$$\langle T(\Delta,\pi)(g) f_v, f_w \rangle = 0.$$

Hence the matrix element $g \mapsto \langle T(\Delta,\pi)(g) f_v, f_w \rangle$ of $T(\Delta,\pi)$ is equal to the matrix element $g \mapsto \left\langle \pi(g)v, \mu(\widetilde{K})w \right\rangle$ of π on $\widetilde{K}$ and is equal to 0 elsewhere.

In other words, given a matrix element $g \mapsto \langle \pi(g)v, w \rangle$ of π, the map

$$g \mapsto \begin{cases} \langle \pi(g)v, w \rangle & \text{if} \quad g \in \widetilde{K}, \\ 0 & \text{otherwise} \end{cases}$$

is a matrix element of $T(\Delta,\pi)$.

□

Lemma 3.6.3. *Let Δ be a finite complete subtree of the tree X and $diam(\Delta) \geq 2$. Then,*

a) $T(\Delta)$ splits into the direct sum of representations $T(\Delta,\pi)$, where π runs through the irreducible nondegenerate representations of $\widetilde{K}(\Delta)$. The *representation $T(\Delta, \pi)$ is irreducible and the multiplicity of its occurrence in $T(\Delta)$ is equal to dim π.*

b)If $\pi_1 \nsim \pi_2$, then $T(\Delta, \pi_1) \nsim T(\Delta, \pi_2)$.

Proof. a)Since $H(\Delta)$ is unitary(i.e. admissible and unitarizable), by theorem 2.4.1. $H(\Delta)$ is a direct sum of irreducible representations, say $H(\Delta) = \oplus_{i\in I} H_i$. By lemma 3.2.1. each H_i contains a nonzero $K(\Delta)$-invariant vector. So if I is infinite, we get infinitely many linearly independent vectors in $H(\Delta)^{K(\Delta)}$. But this is impossible as $dim(H(\Delta)^{K(\Delta)}) < \infty$. Hence, $H(\Delta)$ is a direct sum of finitely many irreducible representations.

Now let H be an irreducible subrepresentation of $H(\Delta)$. Set $\overline{H} = H \bigcap H(\Delta)^K$. Then by theorem 2.2.1. each $\overline{H}$ is an irreducible representation of the convolution algebra $H(G, K)$. Now let us understand the action of the convolution algebra $H(G, K)$ on $\overline{H}$. So let $\varphi \in H(G, K)$ and $f \in \overline{H}$. Then,

$$\begin{aligned} T(\Delta)(\varphi)(f)(g) &= \mu(K) \left[\sum_{x \in G/K} \varphi(x) f(x^{-1}g) \right] \\ &= \int_G \varphi(x) f(x^{-1}g) \ \mathrm{d}\mu(x) \\ &= \varphi * f(g) \end{aligned}$$

for all $g \in G$. Hence, $T(\Delta)(\varphi)(f) = \varphi * f$. Since $\varphi * f \in H(\Delta)^K$, $supp(\varphi * f) \subseteq \widetilde{K}$. On the other hand, $supp(\varphi * f) \subseteq supp(\varphi).supp(f) \subseteq supp(\varphi).\widetilde{K}$. So, $supp(\varphi * f) \subseteq \widetilde{K} \bigcap supp(\varphi).\widetilde{K}$. Note that if $\widetilde{K} \bigcap supp(\varphi) = \emptyset$, then $\widetilde{K} \bigcap supp(\varphi).\widetilde{K} = \emptyset$ so that $\varphi * f \equiv 0$.

Note also that $\overline{H}$ is a representation of $\widetilde{K}$. Indeed, if $f \in \overline{H}$ and $x \in \widetilde{K}$, then given $k \in K$, $x^{-1}kx \in K$ as K is a normal subgroup of $\widetilde{K}$ and consequently

$$T(\Delta)(k)\left[T(\Delta)(x)(f)\right] = T(\Delta)(x)\left[T(\Delta)(x^{-1}kx)(f)\right] = T(\Delta)(x)(f).$$

Then $(\overline{H})^K = \overline{H}$ is also a representation of the convolution algebra $H(\widetilde{K}, K)$.

For $\varphi \in H(G,K)$, the restriction of φ on $\widetilde{K}$ remains locally constant as $\widetilde{K}$ is open.

Hence, $\varphi|_{\widetilde{K}} \in H(\widetilde{K},K)$. Then, we observe that for all $f \in \overline{H}$, if $g \in \widetilde{K} \bigcap supp(\varphi).\widetilde{K}$

$$\varphi * f(g) = \int_{\widetilde{K} \bigcap supp(\varphi)} \varphi(x) f(x^{-1}g) \quad \mathrm{d}\mu(x) = \varphi|_{\widetilde{K}} * f(g).$$

Since $supp(\varphi|_{\widetilde{K}} * f) \subseteq \widetilde{K} \bigcap supp(\varphi|_{\widetilde{K}}).\widetilde{K} \subseteq \widetilde{K} \bigcap (\widetilde{K} \bigcap supp(\varphi)).\widetilde{K} = \widetilde{K} \bigcap supp(\varphi).\widetilde{K}$, if $g \notin \widetilde{K} \bigcap supp(\varphi).\widetilde{K}$,

$$\varphi * f(g) = 0 = \varphi|_{\widetilde{K}} * f(g).$$

Thus we get $\varphi * f = \varphi|_{\widetilde{K}} * f$. Consequently, $\overline{H}$ is an irreducible representation of the algebra $H(\widetilde{K},K)$, hence of the compact group $\widetilde{K}$.

Let π be the representation of $\widetilde{K}$ on the space $\overline{H}$ by left translations. By definition of $H(\Delta)$, $H(\Delta)^K$ is a space of functions on $\widetilde{K}$, spanned over the matrix elements of all irreducible nondegenerate representations of $\widetilde{K}$ which are trivial on K. Hence π is nondegenerate and each nondegenerate representations of $\widetilde{K}$ trivial on K appears its dimension times.

Now define $\varphi : T(\Delta,\pi) \to \overline{H}$ as follows: Take $f \in T(\Delta,\pi)$. Since f is compactly supported, $supp(f) \subseteq \bigcup_{i=1}^{n} x_j\widetilde{K}$ for some $n \in \mathbb{N}$ such that $x_i\widetilde{K} \neq x_j\widetilde{K}$ for distinct i,j. Then let

$$\varphi(f) = \sum_{i=1}^{n} T(\Delta)(x_i) f(x_i).$$

Since

$$\begin{aligned}\varphi\left[T(\Delta,\pi)(g)(f)\right] &= \sum_{i=1}^{n} T(\Delta)(gx_i) T(\Delta,\pi)(g)(f)(gx_i)\\ &= \sum_{i=1}^{n} T(\Delta)(gx_i) f(x_i)\\ &= T(\Delta)(g) \sum_{i=1}^{n} T(\Delta)(x_i) f(x_i)\\ &= T(\Delta)(g)\varphi(f)\end{aligned}$$

φ is an intertwinning operator.

Note that for each i, $supp(T(\Delta)(x_i)f(x_i)) \subseteq x_i\widetilde{K}$. Hence, $supp(T(\Delta)(x_i)f(x_i))$ and $supp(T(\Delta)(x_j)f(x_j))$ do not intersect for distinct i, j so that $\varphi(f) = 0$ implies $f = 0$, i.e. φ is 1-1. Then since ϕ is a 1-1 intertwinning operator to an irreducible representation $\overline{H}$ of G, we get ϕ is onto.

Therefore, $T(\Delta, \pi) \sim \overline{H}$. Part (b) follows from proposition 3.6.2.

□

4 Classification of Cuspidal Representations of G

Since every irreducible algebraic representation of G is admissible, Ol'shanskii had worked the irreducible admissible representations of G to understand the irreducible algebraic ones. Also, he had worked the irreducible admissible representations of G by seperating them into three classes as follows.

Let Δ be a fixed edge of our tree X and x_1, x_2 its extremities.

An irreducible admissible representation (π, V) of G is called ***cuspidal*** if V has no nonzero $K(\Delta)$-stationary vectors.

An irreducible admissible representation (π, V) of G is called ***special*** if there is a nonzero $K(\Delta)$-stationary vector in V, but no nonzero vectors which are stationary with respect to $K(\{x_1\})$ and $K(\{x_2\})$.

An irreducible admissible representation (π, V) is called ***spherical*** if there is a nonzero K_0-stationary vector in V.

These definitions do not depend on the choice of Δ. For example to see that the definition of a special representation does not depend on the choice of Δ, let $\Delta^{'}$ be a different edge from Δ and x_3, x_4 be its extremities. Let v be a nonzero vector in V which is $K(\Delta)$-stationary. Let $h \in G$ be such that $h(x_1) = x_3$ and $h(x_2) = h(x_4)$. Then $hK(\Delta)h^{-1} = K(\Delta^{'})$ and consequently $\pi(h)(v)$ is a nonzero $K(\Delta^{'})$-stationary vector. Moreover, there is no $K(\{x_3\})$ and $K(\{x_4\})$ stationary vectors in V. Because, if v is a nonzero $K(\left\{x_{3(4)}\right\})$- stationary vector, then $\pi(h^{-1})v$ is a nonzero $K(\left\{x_{1(2)}\right\})$-stationary vector, a contradiction.

The following theorem is the main theorem of this work. It gives a classification for irreducible cuspidal representations of the automorphism group G.

4.1 The main theorem and its proof

Theorem 4.1.1. *1. Let (T, H) be an admissible representation of G. Then the following conditions are equivalent:*

a) T is irreducible and cuspidal;

b) $T \sim T(\Delta, \pi)$, for some Δ complete finite subtree of diameter ≥ 2 and for some π irreducible nondegenerate representation of $\widetilde{K}(\Delta)$, which is trivial on $K(\Delta)$.

c) T is irreducible and for all $v \in H$ and $\widetilde{v} \in \widetilde{H}$, the matrix element $g \mapsto \langle T(g)v, \widetilde{v}\rangle$ is compactly supported;

d) T is irreducible, unitary and its matrix coefficients lie in $L^1(G)$.

2. $T(\Delta, \pi) \sim T(\Delta^{'}, \pi^{'})$, where $(\Delta, \pi), (\Delta^{'}, \pi^{'})$ are in b, iff there exists $g \in G$ such that $g\Delta = \Delta^{'}$, and the representations $a \mapsto \pi(a)$ and $a \mapsto \pi^{'}(gag^{-1})$ of the group $\widetilde{K}(\Delta)$ are equivalent.

3. The formal dimension of the representation $T(\Delta, \pi)$ is equal to

$$(dim\pi)(vol(\widetilde{K}(\Delta)))^{-1}$$

where vol denotes the volume relative to the Haar measure on G.

Proof. 1. (a$\Rightarrow$b): Let (T, H) be an irreducible, cuspidal representation of G. Choose a complete finite subtree Δ of the tree X such that $H^{K(\Delta)} \neq \{0\}$ and for any complete finite subtree $\Delta^{'}$ whose diameter is less than the diameter of Δ, $H^{K(\Delta^{'})} = \{0\}$. Then by lemma 3.2.1. H is equivalent to a subrepresentation of $L(\Delta)$ and for any maximal complete subtree Δ_i of Δ, H is not equivalent to a subrepresentation of $L(\Delta_i)$. Let $\varphi : H \to L(\Delta)$ be a 1-1 intertwinning operator. Now fix i and consider the intertwinning operator $\varphi_i = \alpha(\Delta, \Delta_i) \circ \varphi$ from H to $L(\Delta_i)$. If $\varphi_i(H) \neq \{0\}$, again by lemma 3.2.1. $\varphi_i(H)$ hence H contains a nonzero $K(\Delta_i)$- invariant vector, a contradiction. Hence, for all i, $\varphi_i(H) = \alpha(\Delta, \Delta_i)\,(\varphi(H)) = \{0\}$

so that $\varphi(H) \subseteq H(\Delta)$. Hence we can consider H as a subrepresentation of $H(\Delta)$. Note that since H is cuspidal, $diam(\Delta) \geq 2$. Then by lemma 3.6.3. H is equivalent to $T(\Delta, \pi)$ for some π irreducible nondegenerate representation of $\widetilde{K}(\Delta)$, which is trivial on $K(\Delta)$.

(b⇒a): Let (T, H) be a representation of G given as in b. By lemma 3.6.3. T is equivalent to an irreducible subrepresentation of $H(\Delta)$. Now assume $H^{K(\Delta^{'})} \neq \{0\}$ for some edge $\Delta^{'}$ of the tree X. Then $H(\Delta)^{K(\Delta^{'})} \neq \{0\}$. But since $diam(\Delta^{'}) = 1$ and $diam(\Delta) \geq 2$ for any $g \in G$, $g\Delta \nsubseteq \Delta^{'}$. So by lemma 3.3.2. we get that if $f \in H(\Delta)^{K(\Delta^{'})}$, $f(g) = 0$ for all $g \in G$. So $H(\Delta)^{K(\Delta^{'})} = \{0\}$, a contradiction.

(b⇒c): It is enough to show that matrix coefficients of the representation $H(\Delta)$ are compactly supported. So let $f \in H(\Delta)$ and $\widetilde{f} \in \widetilde{H(\Delta)}$. Then there is a finite subtree $\Delta^{'}$ of X such that both $f \in H(\Delta)^{K(\Delta^{'})}$ and $\widetilde{f} \in \widetilde{H(\Delta)}^{K(\Delta^{'})}$. But $\widetilde{H(\Delta)}^{K(\Delta^{'})} = \widetilde{H(\Delta)^{K(\Delta^{'})}}$. Hence the matrix element $g \mapsto \left\langle T(\Delta)(g)f, \widetilde{f} \right\rangle$ of $H(\Delta)$ is also a matrix element of the Hilbert space $H(\Delta)^{K(\Delta^{'})}$. Note that $H(\Delta)^{K(\Delta^{'})}$ is a Hilbert space as it is a finite dimensional inner product space. Therefore it suffices to show that the matrix elements of $H(\Delta)^{K(\Delta^{'})}$ are compactly supported. Since $H(\Delta)^{K(\Delta^{'})}$ is a Hilbert space, if ϕ is a matrix element of $H(\Delta)^{K(\Delta^{'})}$,

$$\begin{aligned} \phi(g) &= \langle T(\Delta)(g)f, h \rangle \\ &= \int_G f(g^{-1}x)\overline{h}(x) \ \mathrm{d}\mu(x) \end{aligned}$$

for some $f, h \in H(\Delta)^{K(\Delta^{'})}$. Observe that if $g \notin supp(h).(supp(f))^{-1}$, then $\phi(g) = 0$. So, $supp(\phi) \subseteq supp(h).(supp(f))^{-1}$. As f and h are in $H(\Delta)$, both $supp(h)$ and $supp(f)$ are compact. Consequently, $supp(h).(supp(f))^{-1}$, hence $supp(\phi)$ is compact as desired.

(c⇒d): Matrix elements lie in $L^1(G)$ as they are compactly supported. Now fix a nonzero element F in $\widetilde{H}$. Since the maps f_v

and f_w which are respectively the maps $g \mapsto \langle T(g)(v), F\rangle$ and $g \mapsto \langle T(g)(w), F\rangle$ are finite on G, we are allowed to define $\langle v, w\rangle$ as the integral of $f_v.\overline{f_w}$ over G. As T is algebraic and irreducible, T is also admissible. Hence T is unitary.

(d⇒a): Assume $H^{K(\Delta)} \neq \{0\}$ for some edge Δ of X. Then T is either special or elementary. But in both cases, matrix elements does not lie in $L^1(G)$([3], Section 3 and Section 4), a contradiction.

2. (⇒) Assume $T(\Delta, \pi) \sim T(\Delta^{'}, \pi^{'})$. Then there exists $g \in G$ such that $g\Delta = \Delta^{'}$. Because, if $g\Delta \neq \Delta^{'}$ for all $g \in G$ and $Card(ver\Delta) \geq Card(ver\Delta^{'})$ as we have shown before $H(\Delta)$ has no nonzero $K(\Delta^{'})$-invariant vectors. Then since $T(\Delta, \pi)$ is equivalent to a subrepresentation of $H(\Delta)$, $T(\Delta, \pi)$ has no nonzero $K(\Delta^{'})$-invariant vectors. But $H(\Delta^{'})$ has a nonzero $K(\Delta^{'})$-invariant vector. Then since $T(\Delta^{'}, \pi^{'})$ is equivalent to a nonzero subrepresentation of $H(\Delta^{'})$, $T(\Delta^{'}, \pi^{'})$ has also a nonzero $K(\Delta^{'})$-invariant vector. To sum up, we get that $T(\Delta, \pi)^{K(\Delta^{'})} = \{0\}$ whereas $T(\Delta^{'}, \pi^{'})^{K(\Delta^{'})} \neq \{0\}$. Similarly, if $Card(ver\Delta^{'}) \geq Card(ver\Delta)$ we get $T(\Delta^{'}, \pi^{'})^{K(\Delta)} = \{0\}$ whereas $T(\Delta, \pi)^{K(\Delta)} \neq \{0\}$. So, both cases imply $T(\Delta, \pi) \nsim T(\Delta^{'}, \pi^{'})$, a contradiction.

   Now fix $g \in G$ with $g\Delta = \Delta^{'}$. Let σ be the representation $a \mapsto \pi^{'}(gag^{-1})$ of $\widetilde{K}(\Delta)$. Note that σ is irreducible, nondegenerate and trivial on $K(\Delta)$. We will show that the representations π and σ of the group $\widetilde{K}(\Delta)$ are equivalent.

   If $\pi \nsim \sigma$, then by lemma 3.6.3.b. $T(\Delta, \pi) \nsim T(\Delta, \sigma)$. But since $T(\Delta, \sigma) \sim T(\Delta^{'}, \pi^{'})$, we get $T(\Delta, \pi) \nsim T(\Delta^{'}, \pi^{'})$, a contradiction. To see $T(\Delta, \sigma) \sim T(\Delta^{'}, \pi^{'})$, define

$$
\begin{aligned}
\phi : T(\Delta, \sigma) &\longrightarrow T(\Delta^{'}, \pi^{'}) \\
f &\longmapsto \phi(f) : G \longrightarrow \pi^{'} \\
&\qquad\qquad h \longmapsto f(hg)
\end{aligned}
$$

Since $g\Delta = \Delta'$, $g\widetilde{K}(\Delta)g^{-1} = \widetilde{K}(\Delta')$. So given $k' \in \widetilde{K}(\Delta')$, $k' = gkg^{-1}$ for some $k \in \widetilde{K}(\Delta)$. Then for all $f \in T(\Delta, \sigma)$,

$$\begin{aligned}\phi(f)(hk') &= f(hk'g) \\ &= f(hgk) \\ &= \sigma(k^{-1})f(hg) \\ &= \pi'(gk^{-1}g^{-1})f(hg) \\ &= \pi'((k')^{-1})\phi(f)(h).\end{aligned}$$

Moreover, $supp(\phi(f)) \subseteq supp(f)g^{-1}$ so that $\phi(f)$ is compactly supported and $\phi(f)$ is K_f-left-invariant where f is K_f-left-invariant. Hence ϕ is well-defined. ϕ is clearly a 1-1 intertwinning operator. As $T(\Delta', \pi')$ is irreducible, ϕ is also onto as desired.

($\Leftarrow$) Assume there exists $g \in G$ such that $g\Delta = \Delta'$ and the representations π and σ(defined as above) of $\widetilde{K}(\Delta)$ are equivalent. Let T be a bijective intertwinning operator from π to σ. Now define a map ϕ from $T(\Delta, \pi)$ to $T(\Delta', \pi')$ as follows:

$$\begin{aligned}\phi : T(\Delta, \pi) &\longrightarrow T(\Delta', \pi') \\ f &\longmapsto \phi(f) : G \longrightarrow \pi' \\ &\qquad\qquad h \longmapsto T(f(hg))\end{aligned}$$

Let $k' \in \widetilde{K}(\Delta')$. Then $k' = gkg^{-1}$ for some $k \in \widetilde{K}(\Delta)$ and for all $f \in T(\Delta, \pi)$,

$$\begin{aligned}\phi(f)(hk') &= T(f(hk'g)) \\ &= T(f(hgk)) \\ &= T(\pi(k^{-1})f(hg)) \\ &= \pi'(gk^{-1}g^{-1})T(f(hg)) \\ &= \pi'((k')^{-1})\phi(f)(h)\end{aligned}$$

3. Since $T(\Delta, \pi)$ is unitarizable, its formal dimension $d_{Ind(\pi)}$ can be computed by the inverse of the L^2 norm of a given matrix coefficient

of $T(\Delta, \pi)$. Let $v \in V$ be such that $\|v\| = 1$. As we have seen before(remark 3.6.1. and prop 3.6.2.), the map $g \mapsto \langle \pi(g)v, v \rangle$ on $\widetilde{K}$ and which is supported on $\widetilde{K}$ is a matrix coefficient of $T(\Delta, \pi)$. Call this map ϕ. Therefore,

$$\begin{aligned} d_{Ind(\pi)} &= \left(\int_G |\phi(g)|^2 \ \mathrm{d}\mu(g) \right)^{-1} \\ &= \left(\int_{\widetilde{K}} |\langle \pi(g)v, v \rangle|^2 \ \mathrm{d}\mu(g) \right)^{-1} \\ &= \left[\mu(\widetilde{K}) \int_{\widetilde{K}} \frac{1}{\mu(\widetilde{K})} |\langle \pi(g)v, v \rangle|^2 \ \mathrm{d}\mu(g) \right]^{-1} \\ &= \frac{1}{\mu(\widetilde{K})} . \left[\frac{dim(\pi)}{\|v\|^2} \right] \\ &= dim(\pi).(\mu(\widetilde{K}))^{-1}. \end{aligned}$$

Note that the fourth equation follows from Schur's orthogonality relation on the representations of compact group which states that given a compact group H with its normalized Haar measure μ and an irreducible finite dimensional representation π of H,

$$\int_H |\langle \pi(h)(u), v \rangle|^2 \ \mathrm{d}\mu(h) = \frac{1}{dim(\pi)} \|u\|^2 \|v\|^2$$

for all $u, v \in \pi$.

□

REFERENCES

1. Bernstein, J., *Representation theory of p-adic groups*, Harvard Lectures, 1993.

2. Figa-Talamanca, A. and Nebbia, C., *Harmonic analysis and representation theory for groups acting on homogeneous trees*, London Math. Soc. Lecture Note Series, 162, Cambridge University Press 1991.

3. Ol'shanskii G. I., *Classification of irreducible representations of groups of automorphisms of Bruhat-Tits Trees*, Func. Anal. Appl., 11 (1976), pp: 26-34.