CLASSIFICATION OF LATTÈS MAPS

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MELİKE EFE

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APPROVED BY:

Asst. Prof. Sonat Süer ....
(Thesis Supervisor)

Asst. Prof. Ayberk Zeytin ....
(Thesis Co-Advisor)

Prof. Muhammed Uludağ ....

Asst. Prof. Pınar Uğurlu Kowalski ....

Assoc. Prof. Kemal Ilgar Eroğlu ....

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To my family...

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CLASSIFICATION OF LATTÈS MAPS

Melike Efe
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Thesis Supervisor: Asst. Prof. Sonat Süer
Thesis Co-Advisor: Asst. Prof. Ayberk Zeytin

The purpose of this thesis is to investigate Lattès maps on $\widehat{\mathbb{C}}$ which are holomorphically conjugate to an affine map on $\mathbb{C}/\Lambda$. In this work, we introduce some notions and facts from dynamical systems, algebraic topology and complex analysis in order to examine these maps deeply. A part of our work concerns the results of John Milnor related to Lattès maps. Specifically, we will see that the degree of a conjugating holomorphism is either $2, 3, 4$ or 6. Following this, we introduce the explicit form of a conjugating holomorphism of a given degree by using the aforementioned results in addition with the property that an elliptic function can be written as a rational function of Weierstrass' elliptic function and its derivative. Finally, we describe the ramification behaviour of Lattès maps.

Keywords: Lattès maps, ramification behaviour, Weierstrass' elliptic function,
Riemann-Hurwitz formula, dynamical systems

LATTÈS FONKSİYONLARI'NIN SINIFLANDIRMASI

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Tez Yardımcı Danışmanı: Asst. Prof. Ayberk Zeytin

Bu tezin amacı, holomorfik olarak $\mathbb{C}/\Lambda$ üzerinde tanımlı bir afin fonksiyona konjuge olan $\widehat{\mathbb{C}}$ üzerinde tanımlı Lattès fonksiyonlarını incelemektir. Bu fonksiyonları derinlemesine incelemek için, bu çalışmanın içinde dinamik sistemler, cebirsel topoloji ve kompleks analizden çeşitli kavramları inceliyoruz. Bu çalışmanın bir kısmı da John Milnor'un Lattès fonksiyonu ile alakalı sonuçları ile ilişkilidir. Spesifik olarak, konjugasyon holomorfizmasının derecesinin $2, 3, 4$ veya 6 olacağını göreceğiz. Bundan sonra, bu sonuçları ve bir eliptic fonksiyonun Weierstrass' eliptik fonsiyonu ve türevinin bir rasyonel fonksiyonu olarak yazılabildiği özelliğini kullanarak, derecesi bilinen konjugasyon holomorfizmasının formunu tanıtıyoruz. Son olarak Lattès fonksiyonlarının dallanma davranışlarını izah ediyoruz.

Anahtar Kelimeler: Lattès fonksiyonları, dallanma davranışı, Weierstrass' eliptik fonksiyon, Riemann-Hurwitz formül, dinamik sistemler

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P_f	Union of the orbits of the critical values of f
V_f	Set of the critical values of the map f
λ_z	Multiplier of z
$\mathcal{F}(f)$	Fatou set of the map f
$\mathcal{J}(f)$	Julia set of the map f
$\mathcal{O}_f(s)$	Orbit of s by the map f
ε_f	Exceptional set of the map f
$\widehat{\mathbb{C}}$	Extended complex plane
$r_f(y)$	Ramification index of y by the map f

1 INTRODUCTION

A Lattès map f is a rational map which is holomorphically conjugate to an affine map on $\mathbb{C}/\Lambda$ i.e we have the following commutative diagram;

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{L} & \mathbb{C}/\Lambda \\ \Theta\big\downarrow & & \big\downarrow\Theta \\ \widehat{\mathbb{C}}\setminus\epsilon_f & \xrightarrow{f} & \widehat{\mathbb{C}}\setminus\epsilon_f \end{array}$$

where L is an affine map and Θ is a finite-to-one holomorphic map and Λ is a rank 2 lattice in $\mathbb{C}$. Its name comes from the French mathematician Samuel Lattès who introduced these maps in [8] which constitute an extremely interesting class of maps from a dynamical viewpoint. In literature, one might come across the name *a finite quotient of an affine map* of which Lattès maps are a special case.

Finite quotients of an affine map are of interest to many mathematicians. In 1923, an American mathematician Ritt proved that if rational maps f and h whose degrees are bigger than 1 commute with each other and if no iterate of f is equal to any iterate of h (such maps are called *integrable* in literature), then they must be finite quotients of an affine map, [13]. Same theorem was proved by Eremenko using a different method in 1989, [3]. There are many conjectures about Lattès maps. One of these states that there is no rational map which admits an invariant line field on their Julia set except flexible Lattès maps which are Lattès maps where the degree of Θ is 2 and the derivative of L is an integer. Indeed, *a hyperbolic map* is a rational map such that the postcritical set and the Julia set are disjoint. We say that a rational map f has *an invariant line field on its Julia set* if there is a subset $E \subseteq \mathcal{J}(f)$ of positive Lebesgue measure with $f^{-1}(E) = E$ and a measurable family of tangent lines defined almost everywhere in E invariant by the tangent action of f. Mañé, Sad and Sullivan proved that if this conjucture is true, then hyperbolic maps are dense among all rational maps, see for instance [9] and [10].

The thesis is organized as follows: section 2 starts with basic notions of dynamical systems. Since finite quotients of affine maps are defined on the extended complex plane $\widehat{\mathbb{C}}$ except exceptional set, we first introduce and study exceptional sets of rational maps. We will show that if the degree of a rational map on $\widehat{\mathbb{C}}$ is bigger than 1, then exceptional set of this function contains at most 2 points. By using this fact, we can observe that if a finite quotient of an affine map is a Lattès map, then the exceptional set of this function can not contain any element, which we detail in Section 4.2. Consequently, Lattès maps are defined on all $\widehat{\mathbb{C}}$.

Holomorphic self maps of $\mathbb{C}/\Lambda$ where Λ is a discrete additive subgroup of $\mathbb{C}$ of rank 2 must be an affine map i.e. it must be equal $az+b$ for some complex numbers a and b. Furthermore, if a is not an integer, then a must be a root of a quadratic polynomial which we show in Section 3.3. Other important property of these maps which we use in the proof of Proposition 4.7 is that periodic points of these holomorphic maps are everywhere dense in $\mathbb{C}/\Lambda$.

Section 4 starts with definition of a finite quotient of an affine map. John Milnor showed that if we have a Lattès map, then Θ induces a canonical homeomorphism from the quotient space $\mathcal{T}/G_n$ to $\widehat{\mathbb{C}}$ where G_n is a finite cyclic group of rigid rotation of the torus about some base point. As a result of this theorem, we can show that n is necessarily either $2, 3, 4$ or 6 and L commutes with a generator of G_n. By using similar considerations, we can show that if the rank of Λ is 1, then finite quotients of affine maps must be holomorphically conjugate either to a power map or to a Chebyshev map.

Holomorphic functions from $\mathbb{C}/\Lambda$ to $\widehat{\mathbb{C}}$ where Λ is a lattice of rank 2 can be written as a rational function of Weierstrass' elliptic function and its derivative. Using this fact, the explicit form of Θ can be obtained. Finally, we determine ramification behaviour of Lattès maps in Section 5.2.

2 FUNDAMENTAL OBJECTS OF DYNAMICAL SYSTEMS

In this section with the aim of fixing notation, we will recall some standard constructions around dynamical systems. Throughout the section, $\widehat{\mathbb{C}}$ will denote the extended complex plane $\mathbb{C} \cup \{\infty\}$.

2.1 BASIC DEFINITIONS AND EXAMPLES

The theory of dynamical systems is a major mathematical discipline related to different areas of mathematics. We are interested in behaviour of orbits of maps.

Definition 2.1. A **dynamical system** is a pair (S, f) where:

- S is a nonempty set,
- f is a map from S to itself.

Definition 2.2. Let (S, f) be a dynamical system. The **(forward) orbit** of $s \in S$ is the set consisting of all the iterations of s under f i.e. it is equal to the set $\{f^n(s) : n \in \mathbb{N}\}$. It is denoted by $\mathcal{O}_f(s)$.

Example 2.3. Let us take $S = \widehat{\mathbb{C}}$ and $a \in \mathbb{Z}\backslash\{0\}$. Look at the map $f_a : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f_a(z) = z^a$ for all $z \in \mathbb{C}\backslash\{0\}$, if $a > 1$, then $f_a(\infty) = \infty$ and $f_a(0) = 0$ and if $a < 1$, then $f_a(\infty) = 0$ and $f_a(0) = \infty$. We can compute the orbit of $z \in \widehat{\mathbb{C}}$ for all z. If $a > 0$ then $\mathcal{O}_f(0) = \{0\}$ and $\mathcal{O}_f(\infty) = \{\infty\}$, if $a < 0$ then $\mathcal{O}_f(0) = \{\infty, 0\}$ and $\mathcal{O}_f(\infty) = \{0, \infty\}$.

Let $z \in \widehat{\mathbb{C}} \setminus \{0, \infty\}$ and $a > 1$. If z is either a^n-th or $(a^n - a^m)$-th root of unity for some natural numbers n and m, then $\mathcal{O}_f(z) = \{z^{a^k} : k \in \mathbb{N}\}$ is a finite set and has at most $n + 1$ many elements. Otherwise, the orbit of z is infinite set and we can find a bijection between $\mathcal{O}_f(z)$ and $\mathbb{N}$. Similar consideration holds if $a < 1$.

Let $z \in \widehat{\mathbb{C}} \setminus \{0, \infty\}$ and $a = 1$. Then f_a is identity map and $\mathcal{O}_f(z)$ contains only the element z. Finally, let $z \in \widehat{\mathbb{C}} \setminus \{0, \infty\}$ and $a = -1$. Then $\mathcal{O}_f(z) = \{z, \dfrac{1}{z}\}$.

Example 2.4. Let ϕ_r be a map from $\mathbb{R}/\mathbb{Z}$ to itself given by $\phi_r(x) = x + r - \lfloor x + r \rfloor$ where r is a real number and $\lfloor x + r \rfloor$ is the smallest integer less than or equal to $x + r$. Then n-th iteration of ϕ_r is equal to $x + nr - \lfloor x + nr \rfloor$. If r is rational number, then there exists $m \in \mathbb{N}$ such that $mr \in \mathbb{Z}$. So $\phi_r^m(x) = x + mr - \lfloor x + mr \rfloor = x$. As a result, if r is a rational number, then all orbits have finitely many elements. Conversely, suppose that $\phi_r^m(x) = \phi_r^n(x)$ for some $n, m \in \mathbb{N}$. Then $mr = nr + z$ for some integer z. So r must be a rational number.

We have shown that the orbit of x under ϕ_r has finitely many elements if and only if r is a rational number. Moreover, we can show that if r is irrational number, then the orbit of x under ϕ_r is dense. To see this, we will show that for all $\varepsilon > 0$ and for each irrational real number r, there exist integers n, m such that $0 < nr - m < \varepsilon$. Let $\varepsilon > 0$ and r be an irrational real number. Then there exist $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Divide the interval [0,1] into n many intervals and look at the elements $r, 2r, 3r, \ldots, nr, (n+1)r$. We have $n + 1$ many elements and n many intervals. So there exist i and j such that $ir - \lfloor ir \rfloor$ and $jr - \lfloor jr \rfloor$ are in the same interval. Suppose that $ir - \lfloor ir \rfloor$ is bigger than $jr - \lfloor jr \rfloor$. Then we have $0 < (i - j)r - (\lfloor ir \rfloor - \lfloor jr \rfloor) < \frac{1}{n} < \varepsilon$.

Now look at the orbit of 0 under ϕ_r where r is an irrational real number. It is equal to the set $\{kr - \lfloor kr \rfloor : k \in \mathbb{N}\}$. Let $y \in \mathbb{R}/\mathbb{Z}$. Then for all $n \in \mathbb{N}$ we can find some integers k and m such that $0 < kr - m < \frac{1}{n} + y$. So we have $0 < kr - \lfloor kr \rfloor - y < \frac{1}{n}$. Denote $r_k := kr - \lfloor kr \rfloor \in \{kr - \lfloor kr \rfloor : k \in \mathbb{N}\}$. This gives us a sequence $\{r_k\}_{k \in \mathbb{N}}$ in the orbit of 0 which converges to y. So the orbit of 0 is dense. Hence the orbit of each element under ϕ_r where r is an irrational real number in $\mathbb{R}/\mathbb{Z}$ are dense because $\phi_r^n(x) = x + \phi_r^n(0)$ for all $x \in \mathbb{R}/\mathbb{Z}$ and $n \in \mathbb{N}$.

Definition 2.5. Let (S, f) be a dynamical system. We say that s is a **periodic point** of f if there exist some positive integer n such that $f^n(s) = s$ i.e if the orbit of s has finitely many elements. The smallest n which satisfy the equality $f^n(s) = s$ is called the **period of** s under f. If $n = 1$, then we say that s is a **fixed point**.

Example 2.6. Let us consider the function in Example 2.3. Suppose that $f_a^n(z) = z$ for some positive integer n i.e. z is periodic point of f_a. By the definition of f_a, $z^{a^n} = z$. Then either z is zero or z is $(a^n - 1)$-th root of unity. This yields us periodic points of f_a are 0 and all of the $(a^n - 1)$-th root of unities. Conversely, we can observe that zero and all of the $(a - 1)$-th root of unities are fixed points.

Example 2.7. Consider the fuction ϕ_r in Example 2.4. We showed that if r is a rational number then all orbits have finitely many elements. This means that if r is a rational number then each element of $\mathbb{R}/\mathbb{Z}$ under ϕ_r is periodic. Moreover, periods of these elements are the same and equal to m where $r = \frac{n}{m}$ and greatest common divisor of n amd m is 1. On the other hand, we showed that if r is an irrational number then all orbits are dense in $\mathbb{R}/\mathbb{Z}$. As a result of this, we do not have any periodic point of ϕ_r if r is an irrational number.

Definition 2.8. Let (S, f) be a dynamical system. For an arbitrary but fixed $s \in S$, we define the set;

$$\mathcal{GO}_f(s) = \{z \in S : \quad \mathcal{O}_f(z) \cap \mathcal{O}_f(s) \neq \emptyset \ \}$$

It is called the **grand orbit** of s under f. The set of all elements $s \in S$ for which $|\mathcal{GO}_f(s)| < \infty$ is denoted by ε_f and called the **exceptional set**.

Example 2.9. Let us consider the function in Example 2.3. For all $k \in \mathbb{N}$, z^{a^k} can not be equal to 0 or ∞ if z is not equal to 0 or ∞. This means that, for all $z \in \widehat{\mathbb{C}} \setminus \{0, \infty\}$, $\mathcal{O}_f(z)$ is not contain 0 and ∞. So the intersection of $\mathcal{O}_f(z)$ and $\mathcal{O}_f(0)$ (and $\mathcal{O}_f(\infty)$) is empty set for all $z \in \widehat{\mathbb{C}} \setminus \{0, \infty\}$. Then grand orbits of 0 and ∞ are finite sets, because they contain only the elements 0 and ∞.

Let $z \in \mathbb{C} \setminus \{0, \infty\}$ and $a > 1$. Then z has exactly a many distinct inverses under f. Choose one of these and say z_1. Observe that $z_1 \in \mathcal{GO}_f(z)$. Similarly, z has exactly a^2 many distinct inverses under f^2. So, we can find an element $z_2 \in \widehat{\mathbb{C}}$ such that $z_2 \neq z_1$ and $z_2 \in \mathcal{GO}_f(z)$. By continuing inductively, for all $n \in \mathbb{N}$, we can find an element $z_n \in \widehat{\mathbb{C}}$ such that $z_n \notin \{z_1, \ldots, z_{n-1}\}$ and $z_n \in \mathcal{GO}_f(z)$. As a result, $\mathcal{GO}_f(z)$ has infinitely many elements. Similar considerations holds for $a < -1$.

Let $z \in \widehat{\mathbb{C}} \setminus \{0, \infty\}$. If $a = 1$ ($a = -1$), then grand orbit of z contains only the element z ($z, \dfrac{1}{z}$). So grand orbit of z is finite set for all z. As a result, exceptional set is $\{0, \infty\}$ if a is not equal to 1 or -1. Otherwise, it is equal to $\widehat{\mathbb{C}}$.

2.2 JULIA AND FATOU SETS

We have shown that if $|a| > 1$ then exceptional set of f_a consists of two points. More generally, we can show that exceptional set of any rational map whose degree is bigger than 2 contains at most two points. To prove this, we need a few lemmas about the Julia and Fatou sets. Let us start with the definition of equi-continuity.

Definition 2.10. Let (S_1, d_1) and (S_2, d_2) be metric spaces. Then a collection $\mathcal{F}$ containing maps from S_1 to S_2 is said to be **equi-continuous** at a point $s \in S_1$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_1(t, s) < \delta \Longrightarrow d_2(f(t), f(s)) < \varepsilon$$

for every $f \in \mathcal{F}$ and $t \in S_1$. We say that $\mathcal{F}$ is equi-continuous on a subset U of S_1 if it is equi-continuous at every element of U.

Let f be a map from S to itself. We say that f is **equi-continuous** at s if the set

$\{f^n : n \in \mathbb{N}\}$ is equi-continuous at s.

Example 2.11. Let us endow $\widehat{\mathbb{C}}$ with the metric $d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for $z, w \in \mathbb{C}$ and $d(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}$. This is indeed the *spherical metric* on the Riemann sphere. Consider the function $g_a : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by $g_a(z) = z + a$ where a is a nonzero integer and $g_a(\infty) = \infty$.

Let $z \in \mathbb{C}$ and $\varepsilon > 0$. Choose a real number r which is bigger than $|z|$. Then there exists $\delta_1 > 0$ such that if $d(z,w) < \delta_1$ then $|w| < r$. Take $\delta = min\{\frac{\varepsilon}{1+r^2}, \delta_1\}$ and suppose $d(z,w) < \delta$. Then we have the inequality $\frac{2|z-w|}{1+r^2} < d(z,w) < \delta$. This yields us $d(g_a^n(z), g_a^n(w)) < 2|g_a^n(z) - g_a^n(w)| = 2|z-w| < (1+r^2)\delta < \varepsilon$ for all $n \in \mathbb{N}$. Hence g_a is equi-continuous at z.

Let $z = \infty$ and choose $\varepsilon = \frac{2}{\sqrt{a^2+1}}$. Take $\delta > 0$ and suppose $d(w,\infty) < \delta$. Then we have $|w|^2 > \frac{4}{\delta^2} - 1$. So $d(\frac{2}{\delta}, \infty) < \delta$ and $d(\frac{-2}{\delta}, \infty) < \delta$. If a is a negative integer, then we can find positive integer n such that $-(n-1)a < \frac{2}{\delta} < -(n+1)a$. Therefore $|\frac{2}{\delta} + na|^2 < a^2$. This gives us $d(g_a^n(\frac{2}{\delta}), \infty) > \varepsilon$. Similarly, if a is positive integer, then we can find positive integer n such that $a(-n-1) < \frac{-2}{\delta} < a(-n+1)$. Therefore $|\frac{-2}{\delta} + na|^2 < a^2$. This gives us $d(g_a^n(\frac{-2}{\delta}), \infty) > \varepsilon$. Hence g_a is not equi-continuous at $z = \infty$ for each nonzero integer a. As a conclusion, g_a is equi-continuous on $\mathbb{C}$ for every fixed nonzero integer a.

Definition 2.12. Let f be a map from a metric space S to itself. The maximal open subset of S which f is equi-continuous on it is called as the **Fatou set** of f. It is denoted by $\mathcal{F}(f)$. The complement of the Fatou set is called as the **Julia set** of f and is denoted by $\mathcal{J}(f)$.

Example 2.13. Let $f_a : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the map defined by $f_a(z) = z^a$ where a is an integer with $|a| > 1$ and $f_a(\infty) = \infty$. We endow $\widehat{\mathbb{C}}$ with the spherical metric. We will find the Fatou set of the map f_a.

Let $z \in \widehat{\mathbb{C}}$ with $|z| < 1$ and $\varepsilon > 0$. Since f_a^n uniformly converges (with respect to Euclidean metric) to 0 on the closed neighborhood of z with radius $r < 1 - |z|$, there exist $N \in \mathbb{N}$ such that $|w^{a^n}| < \frac{\varepsilon}{4}$ for all $n \geq N$ and for all $|z - w| < r$. Now look at the maps f_a^n where $n \in \{1, \ldots, N-1\}$. Since f_a^n is continuous, there exists $\delta_n > 0$ such that $d(f_a^n(z), f_a^n(w)) < \varepsilon$ whenever $d(z, w) < \delta_n$. Take $\delta = min\{\delta_1, \ldots, \delta_2, \frac{r}{2}\}$ and suppose that $d(z, w) < \delta$. If $n < N$, then $d(f_a^n(z), \phi_a^n(w)) < \varepsilon$. On the other hand, we have that $|z - w| < d(z, w) < \delta < r$. This gives us if $n \geq N$, then $d(f_a^n(z), f_a^n(w)) = \frac{2|z^{a^n} - w^{a^n}|}{\sqrt{1+|z^{a^n}|^2}\sqrt{1+|w^{a^n}|^2}} < 2|z^{a^n} - w^{a^n}| < 2(|z^{a^n}| + |w^{a^n}|) < \varepsilon$. So, for all $n \in \mathbb{N}$, $d(f_a^n(z), f_a^n(w)) < \varepsilon$. This means that f_a is equi-continuous at any $z \in \widehat{\mathbb{C}}$ with $|z| < 1$.

Let $z \in \widehat{\mathbb{C}}$ with $|z| > 1$ and $\varepsilon > 0$. Since $|\frac{1}{z}| < 1$, there exist $\delta_1 > 0$ such that $d(f_a^n(\frac{1}{z}), f_a^n(\frac{1}{w})) < \frac{\varepsilon}{2}$ whenever $d(\frac{1}{z}, \frac{1}{w}) < \delta_1$ for all $n \in \mathbb{N}$. As $|z| > 1$, we can find δ_2 such that if $d(z, w) < \delta_2$, then $|w| > 1$. Take $\delta = min\{\frac{\delta_1}{2}, \delta_2\}$ and suppose $d(z, w) < \delta$. Then $d(\frac{1}{z}, \frac{1}{w}) = d(z, w) < \delta < \delta_1$. Thus $d(f_a^n(\frac{1}{z}), \phi_a^n(\frac{1}{w})) < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$. As a result, we have $d(f_a^n(z), f_a^n(w)) < \frac{2|z^{a^n} - w^{a^n}|}{|z^{a^n}||w^{a^n}|} < 2d(f_a^n(\frac{1}{z}), f_a^n(\frac{1}{w})) < \varepsilon$. Hence f_a is equi-continuous at $z \in \widehat{\mathbb{C}}$ whenever $|z| > 1$.

Let $z \in \widehat{\mathbb{C}}$ with $|z| = 1$ and choose $\varepsilon = \frac{1}{2}$. Take $\delta > 0$. We can find $w \in \widehat{\mathbb{C}}$ such that $|w| > 1$ and $d(z, w) < \delta$. Since $|w| > 1$, there exists some $n \in \mathbb{N}$ such that $|w^{a^n}| > 4$. Then $d(f_a^n(z) - f_a^n(w)) > \frac{|z^{a^n} - w^{a^n}|}{|w^{a^n}|} > \frac{|w^{a^n}| - 1}{|w^{a^n}|} > \frac{1}{2}$. So f_a is not equi-continuous at z. As a conclusion, the Julia set of f_a with respect to the spherical metric is equal to the unit circle.

Lemma 2.14. *Let (S, d) be a metric space and f be a holomorphic map from S to itself. Then, for any $n > 0$, the Julia sets of f and f^n are same.*

Proof. Let $z \in \mathcal{F}(f^n)$ for some positive integer n and $\varepsilon > 0$. We know that f^i is continuous function for all iterations of f. Then there exist $\delta_i > 0$ such that $d(f^i(z), f^i(w)) < \varepsilon$ whenever $d(z, w) < \delta_i$. Take $\varepsilon_n = min\{\delta_1, \ldots, \delta_{n-1}, \varepsilon\}$. Since

z is in the Fatou set of f^n, there exists $\gamma > 0$ such that $d(f^{nm}(z), f^{nm}(w)) < \varepsilon_n$ whenever $d(z, w) < \gamma$ for all $m \in \mathbb{N}$. Finally, take $\delta = min\{\varepsilon_n, \gamma\}$ and suppose $d(z, w) < \delta$. Then $d(z, w) < \gamma$. So $d(f^{nm}(z), f^{nm}(w)) < \varepsilon_k \leq \varepsilon$ for all $m \in \mathbb{N}$. As a result of this, $d(f^{nm+i}(z), f^{nm+i}(w)) = d(f^i(f^{nm}(z)), f^i(f^{nm}(w))) < \varepsilon$ for all $i \in \{1, \ldots, n-1\}$. Consequently, $d(f^k(z), f^k(w)) < \varepsilon$ for all $k \in \mathbb{N}$. This means that z is in the Fatou set of f. So $\mathcal{F}(f^n) \subset \mathcal{F}(f)$. Conversely, we can show that $\mathcal{F}(f) \subset \mathcal{F}(f^n)$ by using the definition of the Fatou set. Hence $\mathcal{F}(f) = \mathcal{F}(f^n)$ for all positive integers n. Since the Julia set is complement of the Fatou set, the Julia set of f and f^n is same for all $n \in \mathbb{N}$. □

After that, we will examine the Julia and Fatou sets of dynamical system whose domain is a Riemann surface. For this, let us recall:

Definition 2.15. We say that topological space X is a **Riemann surface** if it has the following properties:

(1) X is Hausdorff;
(2) There exist open sets U_α such that $X = \bigcup_\alpha U_\alpha$;
(3) For each α, we have a homeomorphism $\varphi_\alpha : U_\alpha \to V_\alpha$ where V_α is open set in $\mathbb{C}$ such that the map $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\beta \cap U_\alpha)$.

Example 2.16. Consider the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

(1) It is easy to see that $\widehat{\mathbb{C}}$ is Hausdorff.
(2) Let $U_1 = \widehat{\mathbb{C}} \backslash \{0\}$ and $U_2 = \widehat{\mathbb{C}} \backslash \{\infty\}$. U_1, U_2 are open subset of $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{C}} = U_1 \cup U_2$.
(3) Let $\varphi_1 : U_1 \to \mathbb{C}$ be a map given by $\varphi_1(z) = \frac{1}{z}$ if $z \neq \infty$ and $\varphi_1(\infty) = 0$ and $\varphi_2 : U_2 \to \mathbb{C}$ be a map given by $\varphi_2(z) = z$. We can observe that φ_1, φ_2 are homeomorphisms and $\varphi_1 \circ \varphi_2^{-1} : \mathbb{C} \backslash \{0\} \to \mathbb{C}$, $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C} \backslash \{0\} \to \mathbb{C}$ are holomorphic maps. Therefore, $\widehat{\mathbb{C}}$ is a Riemann surface.

Definition 2.17. Let X and Y be Riemann surfaces with $X = \bigcup_\alpha U_\alpha$ and $Y = \bigcup_\beta K_\beta$ where U_α and K_β are open sets for all α, β. We say that a map $f : X \to Y$ is **holomorphic** if the composite map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is holomorphic on its domain of definition for all α and β where ψ_α and φ_β are holomorphic maps as in Definition 2.15. We have the following diagram;

$$\begin{tikzcd}
X \supseteq U_\alpha \cap f^{-1}(K_\beta) \arrow[r, "f|_{U_\alpha \cap f^{-1}(K_\beta)}"] \arrow[d, "\varphi_\alpha"'] & K_\beta \subseteq Y \arrow[d, "\psi_\beta"] \\
\mathbb{C} \supseteq V_\alpha \cap \varphi_\alpha(f^{-1}(K_\beta)) \arrow[r, "\psi_\beta \circ f \circ \varphi_\alpha^{-1}"] & V_\beta \subseteq \mathbb{C}
\end{tikzcd}$$

Example 2.18. Consider the map f_a in Example 2.13 and φ_1 and φ_2 in Example 2.16. Then $\varphi_1 \circ f_a \circ \varphi_2, \theta_2 \circ f_a \circ \varphi_1 : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ are holomorphic maps such that $\varphi_1 \circ f_a \circ \varphi_2(z) = \varphi_2 \circ f_a \circ \varphi_1(z) = \frac{1}{z^a}$. Similarly, $\varphi_1 \circ f_a \circ \varphi_1, \varphi_2 \circ f_a \circ \varphi_2 : \mathbb{C} \to \mathbb{C}$ are holomorphic maps such that $\varphi_1 \circ f_a \circ \varphi_1(z) = \varphi_2 \circ f_a \circ \varphi_2(z) = z^a$. Therefore, f_a is holomorphic map on $\widehat{\mathbb{C}}$.

2.3 PERIODIC POINTS AND THEIR MULTIPLIERS

In this subsection, we will recall the multiplier of holomorphic maps. Let us start with the definition of multiplier.

Definition 2.19. Let z be a periodic point of dynamical system determined by a holomorphic map $f : S \to S$ where S is any Riemann surface and let m be the period of z. The **multiplier** of z is the first derivative of f^m at z. We usually denote the multiplier of $z \in S$ by λ_z.

Remark 2.20. If $z \in \mathbb{C}$ and the orbit z consists of the elements $z, z_1, \ldots, z_{m-1}$ then the multiplier $\lambda_z = (f^m(z))' = f'(z_1)f'(z_2)\ldots f'(z_{m-1})f'(z)$.

To define multiplier at ∞, we observe that if f is a rational map and z is a fixed point of f, then $f'(z) = (f^{\phi})'(w)$ where $\phi \in PGL_2(\mathbb{C}) = \{\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} : \phi(z) = \frac{az+b}{cz+d} \ ad-bc \neq 0\}$ is a change of coordinates, $z = \phi^{-1}(w)$ and $f^{\phi} = \phi^{-1} \circ f \circ \phi$ [15]. We first suppose that ∞ is a fixed point of f. Since $S = \widehat{\mathbb{C}}$ and f is holomorphic, then f is rational map. Let $\phi(z) = \frac{1}{z} \in PGL_2(\mathbb{C})$. Then $\infty = \phi^{-1}(0)$. This yields us $\lambda_\infty = f'(\infty) = \lim\limits_{z\to 0}(f^{\phi})'(z) = \lim\limits_{z\to 0} = \lim\limits_{z\to 0}(\frac{1}{f(\frac{1}{z})})' = \lim\limits_{z\to 0} \frac{z^{-2}f'(z^{-1})}{f(z^{-1})^2}$. Suppose that ∞ is periodic point of f with period m. Then ∞ is a fixed point of f^m. As a result, we can find the multiplier of ∞ by using the formula $\lambda_\infty = \lim\limits_{z\to 0} \frac{z^{-2}(f^m)'(z^{-1})}{(f^m(z^{-1}))^2}$.

Example 2.21. Let f be a holomorphic map given by $f(z) = az + b$ for some complex numbers a and b if $z \in \mathbb{C}$ and $f(\infty) = \infty$. If a is zero, then b is a fixed point of f with the multiplier $\lambda_b = f'(b) = 0$ and we do not have any periodic point other than b. If a is not zero, then we have that $f^n(z) = a^n z + b(a^{n-1} + \cdots + a + 1)$ for all $n \in \mathbb{N}\backslash\{0\}$. So we can observe the following;

- If a is not 1, then $\frac{b}{a-1}$ is a fixed point of f and its multiplier is $a = f'(b)$.
- If a is n-th root of unity for some $n \in \mathbb{N}$ other than 1, then each element $z \in \mathbb{C}\backslash\{\frac{b}{a-1}\}$ is periodic point of f and their period is n. So their multiplier is $\lambda_z = (f^n)'(z) = a^n$.
- $z = \infty$ is a fixed point of f and its multiplier is $\lambda_\infty = \lim\limits_{z\to 0} \frac{z^{-2}a}{(a\frac{1}{z}+b)^2} = \frac{1}{a}$ for all $a \neq 0$.
- If a is not n-th root of unity for all $n \in \mathbb{N}$, then we do not have any periodic point other than ∞ and $\frac{b}{a-1}$.
- If a is 1 and b is zero, then all elements are fixed point of f. If a is 1 and b is not zero, then ∞ is unique periodic point of f.

Definition 2.22. Let f be a holomorphic map on a Riemann surface S and λ_z be the multiplier of periodic point z. Then z is called;

- **repelling** if $|\lambda_z| > 1$,
- **neutral** if $|\lambda_z| = 1$,

- **attracting** if $|\lambda_z| < 1$,
- **superattracting** if $|\lambda_z| = 0$.

Example 2.23. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a map given by $f(z) = az$ for all $z \in \mathbb{C}$ and $f(\infty) = \infty$ where a is a positive integer. If a is 1, then f is identity map and each element in $\widehat{\mathbb{C}}$ is fixed point. We can observe that multiplier of each element is 1. So each element in $\widehat{\mathbb{C}}$ is neutral fixed point.

If $a > 1$, then 0 and ∞ is fixed points of f and f does not have any other periodic point. The multiplier of 0 is $a > 1$. So 0 is a repelling fixed point. The multiplier of ∞ is $\lambda_\infty = \lim\limits_{z \to 0} \frac{z^{-2}.a}{(\frac{a}{z})^2} = \frac{1}{a} < 1$. Then ∞ is an attracting fixed point.

Example 2.24. Consider the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $f(z) = z^2$ if $z \in \mathbb{C}$ and $f(\infty) = \infty$. Fixed points of f are $1, 0$ and ∞ and their multipliers are $\lambda_1 = 2$, $\lambda_0 = 0$ and $\lambda_\infty = \lim\limits_{z \to 0} \frac{z^{-2}.2z^{-1}}{(z^{-2})^2} = 0$. Then 0 and ∞ are superattracting fixed points of f and 1 is repelling fixed point of f.

Suppose that z is periodic point of f whose period $n > 1$. We showed that z is $(a^n - 1)$-th root of unity in Example 2.6. We can observe that $\mathcal{O}_f(z) = \{z, z^2, \ldots, z^{2^{n-1}}\}$. Then the multiplier $\lambda_z = f'(z)f'(z^2).\ldots.f'(z^{2^{n-1}}) = 2^n$ and $|2^n| > 1$. So z is repelling periodic point of f. As a result, each periodic point of f in $\mathbb{C}\backslash\{0\}$ is repelling periodic point.

2.4 NORMAL FAMILIES AND THE ARZELA-ASCOLI THEOREM

Studying on subsequences of a given family is sometimes more convenient than studying on all family. Fortunately, if we study a family of functions on a compact Riemann surface S, then we can use normality of the family of iterates $\{f^n : n \in \mathbb{N}\}$ for a holomorphic map f on S to find the Fatou set of f. We can observe this by using the

Arzela-Ascoli theorem. Let us start with the definition of normality.

Definition 2.25. Let U and V are open subsets of a Riemann surface S and consider a sequence of holomorphic maps $f_n : U \to V$. We say that;

- f_n **converges locally uniformly** if there exist a holomorphic map $f : U \to V$ such that for every compact subset $K \subseteq U$, the sequence f_n converges uniformly to f on K.
- f_n **diverges locally uniformly** if for every compact subset $K_1 \subseteq U$ and $K_2 \subseteq V$, there exists $N \in \mathbb{N}$ such that the intersection $K_2 \cap f_n(K_1) = \emptyset$ for all $n > N$. We can observe that if U and V are compact, then we can not find any sequences of maps which are locally uniformly divergent.

Let $\mathcal{F}$ be a collection of maps from a Riemann surface S_1 to a Riemann surface S_2. If each sequence in $\mathcal{F}$ contains either a locally uniformly convergent subsequence or locally uniformly divergent subsequence, then $\mathcal{F}$ is called a **normal** family.

Example 2.26. Consider the function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = z + a$ where a is a positive integer. We will show that $\{f^n\}_{n\in\mathbb{N}}$ is normal family. Let K_1 and K_2 are compact subsets of $\mathbb{C}$. Since K_1 and K_2 are compact, $d = sup\{|w_1 - w_2| : w_1, w_2 \in K_2\}$ and $d_z = sup\{|z - w| : w \in K_2\}$ exist for all $z \in K_1$. Then, for all $n > \frac{2d+d_z}{a}$, we have $|z + na - w| > |na|z - w|| = na - |z - w| > 2d + d_z - d_z = 2d$ for all $w \in K_2$. On the other hand, consider the open cover $\{B(z, \frac{d}{2})\}_{z\in K_1}$ of K_1 where $B(z, \frac{d}{2})$ is the open ball with center z and radius $\frac{d}{2}$. Then there exist finitely many elements $z_1, \ldots, z_k$ in $\mathbb{C}$ such that $K_1 \subseteq B(z_1, \frac{d}{2}) \cup \cdots \cup B(z_k, \frac{d}{2})$ since K_1 is compact.

Now, choose the smallest $N \in \mathbb{N}$ which satisfy $N > max\{\frac{2d+3d_{z_1}}{a}, \ldots, \frac{2d+3d_{z_k}}{a}\}$. Let $z \in K_1$. Then there exist $z_i \in \{z_1, \ldots, z_k\}$ such that $z \in B(z_i, \frac{d}{2})$ and we have $|z + na - w| > ||z_i + na - w| - |z - z_i|| > 2d - d = d$ for all $n > N$. Hence, for $n \geq N$, $f^n(z) \notin K_2$ for all $z \in K_1$ i.e. $K_2 \cap f^n(K_1) = \emptyset$. This means that, f^n diverges locally uniformly on $\mathbb{C}$. So $\{f^n\}_{n\in\mathbb{N}}$ is a normal family.

The following theorem due to Arzela-Ascoli gives necessary and sufficient conditions for a family $\mathcal{F}$ of functions defined on $\widehat{\mathbb{C}}$ to be a normal family.

Theorem 2.27 (Arzela-Ascoli). *Let U be a domain in $\widehat{\mathbb{C}}$ and $\mathcal{F}$ be a collection of continuous maps from U to a metric space S. Then $\mathcal{F}$ is normal family if and only if:*

- *for any $z \in U$, the values $f(z)$ are contained by a compact subset of S for all $f \in \mathcal{F}$,*
- *$\mathcal{F}$ is equi-continuous on every compact subset $K \subseteq U$.*

Corollary 2.28. *Let f be a holomorphic map on $\widehat{\mathbb{C}}$. Then $z \in \widehat{\mathbb{C}}$ has a neighborhood U such that the sequence of iterates $\{f^n|_U : n \in \mathbb{N}\}$ is normal family if and only if z is in the Fatou set of f.*

Remark 2.29 (Theorem A.2)**.** The topology of locally uniformly convergence on $\mathcal{F}$ where $\mathcal{F}$ is a family of holomorphic maps from S_1 to S_2 depends only on the topologies of S_1 and S_2 and not on the particular choice of metric for S_2.

Lemma 2.30 ([11])**.** *Let f be a holomorphic map on $\widehat{\mathbb{C}}$. If z is attracting periodic point then it is contained in the Fatou set of f.*

Proof. Let $z \in \widehat{\mathbb{C}}$ and suppose that z is attracting fixed point of f. Then we have $|\lambda_z| = |f'(z)| < 1$. By using the Taylor's theorem, we can find some constant r with $|\lambda_z| < r < 1$ such that $|f(w) - z| < r|w - z|$ in some neighborhood U of z. By using this inequality, we can find a neigborhood V of z such that $\{f^n|_V\}$ converges uniformly to z. So $\{f^n|_V\}$ is normal family. Then z is in the Fatou set of f.

We have shown that if z is attracting fixed point, then it is in the Fatou set of f.

Let $z \in \widehat{\mathbb{C}}$ and suppose that z is attracting periodic point. Then there exists some $m \in \mathbb{N}$ such that z is attracting fixed point of f^m. So $z \in \mathcal{F}(f^m)$. On the other hand, $\mathcal{F}(f^m) = \mathcal{F}(f)$ by the lemma 2.14. Hence z contained in the Fatou set of f.

$\square$

Lemma 2.31. *Let $\{f_i\}_{i\in\mathbb{N}}$ be a collection of holomorphic maps from $(\widehat{\mathbb{C}}, d)$ where d is the spherical metric to a Riemann surface (S, d_s) where d_s is a metric on S. If every point of $\widehat{\mathbb{C}}$ has a neighborhood U such that the collection $\{f_i|_U\}_{i\in\mathbb{N}}$ of restricted maps is a normal family, then the collection $\{f_i\}_{i\in\mathbb{N}}$ is a normal family.*

Proof. Let $z \in \widehat{\mathbb{C}}$. Then there exists a neighborhood U of z such that the collection $\{f_i|_U\}_{i\in\mathbb{N}}$ is a normal family. So, for all i, the values $f_i(z)$ lies in a compact subset of S by Theorem 2.27.

Let K be a compact subset of $\widehat{\mathbb{C}}$, $z_0 \in K$ and $\varepsilon > 0$. For each $z \in K$, there exists a neighborhood U_z of z such that $\{f_i|_{U_z}\}_{i\in\mathbb{N}}$ is a normal family. So $\{f_i|_{U_z}\}_{i\in\mathbb{N}}$ is equi-continuous on every compact subset of U_z by Theorem 2.27. Then there exist $\delta_z > 0$ such that $d_s(f_i(w), f_i(z)) < \frac{\varepsilon}{2}$ whenever $d(w, z) < \delta_z$ for all $i \in \mathbb{N}$. Consider the open cover $\{V_z\}_{z\in K}$ of K where V_z is an open ball in $\widehat{\mathbb{C}}$ with center z and radius $\frac{\delta_z}{2}$. Since K is a compact subset, there exist finitely many elements $z_1, \dots, z_k$ such that $K \subseteq V_{z_1} \cup \dots \cup V_{z_k}$. Choose $\delta = \frac{min\{\frac{\delta_{z_1}}{2}, \dots, \frac{\delta_{z_2}}{2}\}}{2}$ and suppose that $d(w, k) < \delta$. Since $z_0 \in K$, then $z_0 \in V_i$ for some $i \in \{1, \dots, k\}$. So $d(w, z_i) < d(w, z_0) + d(z_0, z_i) < \delta_{z_i}$. Hence, for all i, $d_s(f_i(w), f_i(k)) < d_s(f_i(w), f_i(z_i)) + d_s(f_i(z_i), f_i(k)) < \varepsilon$. Therefore, $\{f_i\}_{i\in\mathbb{N}}$ is equi-continuous at z_0. Since z_0 is arbitrary, then the collection $\{f_i\}_{i\in\mathbb{N}}$ is equi-continuous on K.

We have shown that, for any $z \in \widehat{\mathbb{C}}$, the values $f_i(z)$ lies in a compact subset of S for all $i \in \mathbb{N}$ and the collection $\{f_i\}_{i\in\mathbb{N}}$ is equi-continuous on every compact subset of $\widehat{\mathbb{C}}$. Then the collection $\{f_i\}_{i\in\mathbb{N}}$ is a normal family by Theorem 2.27.

$\square$

Lemma 2.32 ([1]). *Let $\{f_i\}_{i\in\mathbb{N}}$ be a uniformly convergent sequence of rational functions on the complex sphere. Then it converges to a rational function f and degree of f_i, denoted by d_i, and degree of f, denoted by d, are equal for sufficiently large i.*

Proof. Since $\{f_i\}_{i\in\mathbb{N}}$ converges uniformly to f, f is an analytic function. So f is rational function. Now we will show that $d_i = d$ for sufficiently large i.

Suppose that $f(\infty) \neq 0$. If $f(\infty)$ is zero, then we can look at the maps $\frac{1}{f}$ and $\{\frac{1}{f_i}\}_{i\in\mathbb{N}}$. Let $z_1, z_2, \ldots, z_k$ be distict zeros of f. They are contained by $\mathbb{C}$ as $f(\infty) \neq 0$. Take U_j be an open ball containing z_j and satisfy the following properties;

- intersection of U_j and U_s is empty set for all $j, s \in \{1, \ldots, k\}$,
- U_j does not contain any pole of f for each $i \in \{1, \ldots, k\}$.

Let $i \in \{1, \ldots, k\}$. Then we have that $\{f_i\}$ uniformly converges to f on each compact subset of U_j and f_i has no pole in U_j for sufficiently large i. However, $f \not\equiv 0$ in U_j and $f(z_j) = 0$. So f_i must have at least one zero in U_j for sufficiently large i by Hurwitz's theorem[1]. Since $\{f_i\}_{i\in\mathbb{N}}$ converges uniformly to f, $f_i(w) \neq 0$ whenever $f(w) \neq 0$ for sufficiently large i. Hence f_i and f have same number of zeros in each U_j for sufficiently large k.

Let K be the complement of the union $\bigcup_{j=1}^{k} U_j$. So K is compact subset. Since $\{f_i\}_{i\in\mathbb{N}}$ converges uniformly to f on the compact set K, f_i and f must have the same number of zeros in K for sufficiently large i. As a result, f_i and f have same number of zeros on the complex sphere for sufficiently large i. On the other hand, f_i and f have the same number of poles on the complex sphere for sufficiently large i since $\{f_i\}_{i\in\mathbb{N}}$ converges uniformly to f. Therefore, the degree of f_i and the degree of f are same for sufficiently large i.

[1] If $\{f_i\}_{i\in\mathbb{N}}$ is a non-vanishing analytic function in a region U and converges uniformly to f on every compact subset of U, then either $f \equiv 0$ in U or $f(z) \neq 0$ for all $z \in U$

$\square$

Proposition 2.33 ([11])**.** *Let f be a rational self map of $\widehat{\mathbb{C}}$ of degree $d \geq 2$. Then the Julia set $\mathcal{J}(f)$ is nonempty.*

Proof. Suppose that $\mathcal{J}(f)$ is empty. Then for each element $z \in \widehat{\mathbb{C}}$ there exists a neighborhood of z such that $\{f^n|_U\}$ is a normal family. So $\{f^n\}$ is a normal family by Lemma 2.31. Since $\widehat{\mathbb{C}}$ is compact, $\{f^n\}$ is uniformly convergent sequence on the complex sphere. Therefore it converges to a rational function g and the degree of f^n and the degree of g are equal for sufficiently large n by the Lemma 2.32. On the other hand, degree of f^n is d^n. So d must be 1 but $d \geq 2$. As a result, $\mathcal{J}(f)$ can not be equal to empty set. $\square$

Finally, let us recall the Montel's theorem:

Theorem 2.34 (Montel)**.** *Let S be a Riemann surface and $\mathcal{F}$ be a collection of holomorphic maps $f : S \to \widehat{\mathbb{C}}$. If $f(S) \subset \widehat{\mathbb{C}} \backslash \{a, b, c\}$ for some distinct elements $a, b, c \in \widehat{\mathbb{C}}$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

Now, we can prove the following theorem;

Theorem 2.35 ([11])**.** *If f is rational self map of $\widehat{\mathbb{C}}$ of degree $d \geq 2$, then ϵ_f contains at most 2 points.*

Proof. As f is a rational map on $\widehat{\mathbb{C}}$, it is onto. By using the definition of the grand orbit, we can show that f sends any grand orbit $\mathcal{GO}_f(z)$ to itself. If $\mathcal{GO}_f(z)$ is finite for some $z \in \widehat{\mathbb{C}}$, then $f(\mathcal{GO}_f(z)) = \mathcal{GO}_f(z)$. This means that, if the grand orbit of z is finite then each element of $\mathcal{GO}_f(z)$ is periodic point and has only one inverse under the map f. On the other hand, the degree of f is bigger than one. So the multiplicity of each element of $\mathcal{GO}_f(z)$ is bigger than one. Thus, the first derivative of each element of

$\mathcal{GO}_f(z)$ is zero. Hence they are superattracting periodic point of f and are contained in the Fatou set of f by the Lemma 2.30. As a result, if $\mathcal{GO}_f(z)$ is finite then it is a subset of the Fatou set of f.

Suppose that the exceptional set ε_f contains three distinct elements $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$. Then $\mathcal{GO}_f(z_i)$ is finite for each $i \in \{1, 2, 3\}$. Take U be a complement of the union of these finite grand orbits. So $f(U) = U$ and the complement of U is a subset of the Fatou set of f. Since $f(U) = U$, $f^n(U) \subset \widehat{\mathbb{C}} \backslash \{z_1, z_2, z_3\}$ for all $n \in \mathbb{N}$. Then $\{f^n|_U\}$ is a normal family by the Montel's theorem. So U is subset of the Fatou set of f. Hence the Julia set of f must be empty but this is impossible by the Proposition 2.33. Consequently, the exceptional set ε_f contains at most two points. $\square$

3 QUOTIENTS OF THE COMPLEX PLANE

In this section, we will recall basic definitions of covering spaces and universal covers. More precisely, we will examine covering space of quotients of the complex plane.

3.1 COVERING SPACES AND UNIVERSAL COVERS

Let X and Y be topological spaces and $p : X \to Y$ be a continuous map.

Definition 3.1. We say that p is a **covering map** if each point of $y \in Y$ has an open neighborhood U_y such that $p^{-1}(U_y)$ is a disjoint union of open sets V_α, each of which is mapped by p homeomorphically onto U_y. In this case, X is called a **covering space** of Y.

Example 3.2. Let $p : \mathbb{R} \to S^1$ be a map such that $p(t) = e^{2\pi it}$ where S^1 is the unit circle in $\mathbb{C}$. It can be shown that p is covering map. So $\mathbb{R}$ is covering space of S^1:

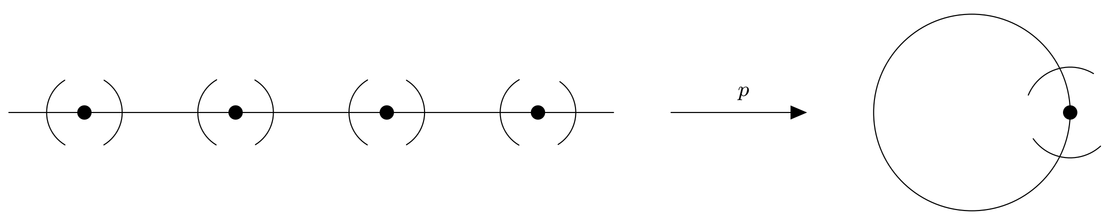

Figure 3.1.: The covering map of S^1

Let Y be a topological space and G be a group. An **action** of a group G on Y is a mapping $G \times Y \to Y$ (written as $(g, y) \mapsto g \cdot y$) satisfying the following properities:

(1) $g \cdot (h \cdot y) = (gh) \cdot y$ for all $g, h \in G$, $y \in Y$,

(2) $e \cdot y = y$ for all $y \in Y$ where e is the identity in G,

(3) for all $g \in G$, the map $f : Y \to Y$ given by $f(y) = g \cdot y$ is homeomorphism.

We say that G acts **evenly** if any point in Y has a neighborhood U such that $g \cdot U$ and $h \cdot U$ are disjoint for any distinct element g and h in G.

Example 3.3. Let $n \in \mathbb{N}_{>1}$ and G be a group consisting of all n-th root of unities. Clearly, G acts on $\mathbb{C}$ by multiplication: $w \cdot z = wz$. Let U be a neighborhood of 0 with $d = sup\{|z| : z \in U\} > 0$ and $\rho \in G$ be a generator of G. Then we can find an element $z_1 \in U$ such that $|z_1| < \frac{d}{|\rho|}$. This means that $\rho z_1 \in U$. Therefore, we have $\rho \cdot \rho z_1 = \rho^2 \cdot z_1 \in \rho \cdot U \cap \rho^2 \cdot U$. Hence G does not act evenly on $\mathbb{C}$.

Example 3.4. Let Λ be a discrete additive subgroup of $\mathbb{C}$ with one or two generators. Then Λ acts evenly on $\mathbb{C}$ by translation: $w \cdot z = w + z$. Let $z \in \mathbb{C}$ and $d = inf\{|w_1 - w_2| : w_1, w_2 \in \Lambda\}$. Since Λ is a discrete subgroup of $\mathbb{C}$, d is a positive real number. Take U be an open ball with center z and radius $\frac{d}{3}$. Let $w_1, w_2 \in \Lambda$ and suppose $w \in w_1 \cdot U \cap w_2 \cdot U$. Then there exist $z_1, z_2 \in U$ such that $w_1 + z_1 = w_2 + z_2$. This yields us $|w_1 - w_2| = |z_2 - z_1| < |z_2 - z| + |z - z_1| < \frac{2d}{3}$. However, this is impossible since $w_1, w_2 \in \Lambda$. Hence $w_1 \cdot U \cap w_2 \cdot U = \emptyset$. Since w_1 and w_2 are arbitrary, for each pair $w_1, w_2 \in \Lambda$, the intersection of $w_1 \cdot U$ and $w_2 \cdot U$ is empty set. This means that Λ acts evenly on $\mathbb{C}$.

Let Y be a topological space and G be a group. Suppose that G acts on Y. We say that two elements y_1 and y_2 in Y are equivalent if there is an element $g \in G$ such that $y_1 = g \cdot y_2$. This is an equivalence ralation. Let Y/G be the set consisting of equivalence classes. Then there is a *projection* $p : Y \to Y/G$ that sends an element $y \in Y$ to the equivalence class containing y. We endow Y/G with the quotient topology induced by p i.e. $U \subseteq Y/G$ open if and only if $p^{-1}(U)$ is open in Y. This way Y/G becomes a topological space whenever G acts on Y evenly.

Lemma 3.5 ([5], Lemma 11.17). *If a group G acts evenly on Y, then the projection $p : Y \to Y/G$ is a covering map.*

Definition 3.6. Let $p : Y \to X$ be a covering and $f : Z \to X$ be a continuous map. Then there exist a map $\widehat{f} : Z \to X$ such that $p \circ \widehat{f} = f$. Such a map is called a **lifting** of f.

Definition 3.7. Let X be a topological space. If covering space Y of X simply connected, then Y is called the **universal covering space** of X.

3.2 COMPLEX TORI

Let us investigate the case $Y = \mathbb{C}$ and G a discrete additive subgroup of $\mathbb{C}$ in detail.

Definition 3.8. A **lattice** is a discrete additive subgroup of $\mathbb{C}^n$ whose generating set is linearly independent over $\mathbb{R}$. It is denoted by Λ. The **rank** of Λ is the number of elements of maximal linearly independent set over $\mathbb{R}$ which spans Λ.

Example 3.9. Let Λ be a lattice of $\mathbb{C}$ with generator $1 + i$. Then all points of Λ lie on the Euclidean line passes through the points 0 and $1 + i$. More generally, if Λ is spaned by $w \in \mathbb{C}$ then all points lie on the Euclidean line passes through the points 0 and w.

Example 3.10. Let Λ be a lattice of $\mathbb{C}$ which is spaned by 1 and $w = 1 + i$. Then all points of Λ lie on the vertices of equivalent parallelograms which are known as a fundamental region. See Figure 2.2.

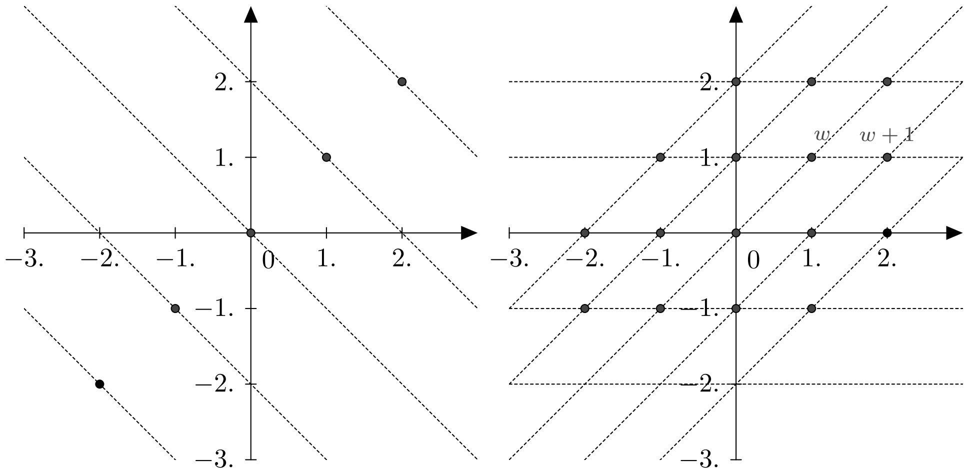

Figure 3.2.: $\Lambda = \langle 1+i \rangle$(at left), $\Lambda = \langle 1, 1+i \rangle$(at right)

Theorem 3.11 ([7], Theorem 3.1.3)**.** *Let Λ be a lattice in $\mathbb{C}$. Then Λ is isomorphic (as a group) to $\{0\}$ or $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.*

In our case the equivalence relation is given as:

$$z_1 \sim z_2 \quad if \ and \ only \ if \quad z_1 - z_2 \in \Lambda \tag{3.1}$$

Since Λ is a normal subgroup of the additive group $\mathbb{C}$, the quotient set $\mathbb{C}/\Lambda$ has the structure of a group. If we have Λ of rank 2, then we can identify $\mathbb{C}/\Lambda$ with the region $R = \{z = \lambda w_1 + \mu w_2 : \ \lambda, \mu \in [0,1] \ \}$, see Figure 3.3.

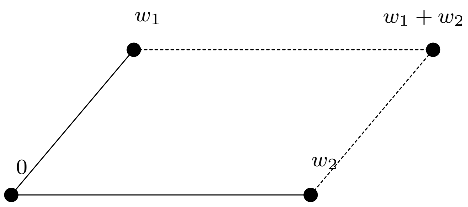

Figure 3.3.: The fundamental region of $\Lambda = \langle w_1, w_2 \rangle$

Since the Euclidean lines joining 0 to w_2 and w_1 to $w_1 + w_2$ are equivalent, then we can identify them:

Similarly, we can identify the Euclidean line joining 0 to w_1 with the Euclidean line

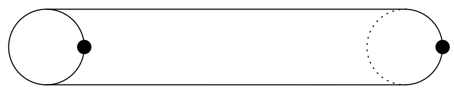

Figure 3.4.: The cylinder

joining w_2 to $w_1 + w_2$:

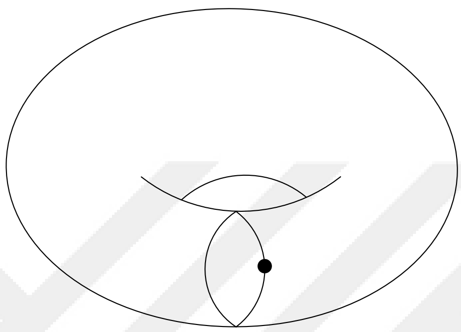

Figure 3.5.: The torus

Hence, we can topologically identify $\mathbb{C}/\Lambda$ with a resulting space which is called a torus. Similarly, if the rank of Λ is 1, then we can identify $\mathbb{C}/\Lambda$ with a cylinder.

As a conclusion of Section 3.1 and Section 3.2, we can observe that $\mathbb{C}$ is the universal covering space of $\mathbb{C}/\Lambda$ where Λ is a discrete additive subgroup of $\mathbb{C}$ of rank either 1 or 2 since $\mathbb{C}$ is simply connected.

3.3 EQUIVALENT LATTICES AND MAPS BETWEEN TORI

We say that two lattices Λ and Λ' of rank 2 are equivalent and write $\Lambda \sim \Lambda'$ if there is a non-zero complex number λ such that $\Lambda = \lambda\Lambda'$. Remark that this is an equivalence on the set of lattices. In particular, the lattice Λ generated by w_1 and w_2 is equivalent to the lattice generated by $\{1, \dfrac{w_1}{w_2}\}$ or $\{1, \dfrac{w_2}{w_1}\}$. As a result of this if we have a lattice Λ of rank 2, we can assume that generators of Λ are 1 and τ where $\tau \notin \mathbb{R}$. One may even choose $\tau = \frac{w_1}{w_2}$ as an element of the upper half plane. Given $\Lambda = \langle w_1, w_2 \rangle$, the map sending each element $z \in \Lambda$ to τz is an isomorphism between Λ and $\Lambda' = \langle 1, \tau \rangle$. We further have:

Theorem 3.12. *If $\Lambda \sim \Lambda'$, then $\mathbb{C}/\Lambda$ is biholomorphic to $\mathbb{C}/\Lambda'$.*

Proof. Firstly, recall that two subsets X and Y of $\mathbb{C}$ are *biholomorphic* if there exist a holomorphic bijective map whose inverse is also holomorphic between X and Y. Let $\Lambda = \langle w_1, w_2 \rangle$ and Λ' be lattices in $\mathbb{C}$ and suppose that $\Lambda \sim \Lambda'$. Then there is a non-zero complex number λ such that $\Lambda' = \lambda\langle w_1, w_2 \rangle = \langle \lambda w_1, \lambda w_2 \rangle$. On the other hand, each equivalent class $[z] \in \mathbb{C}/\Lambda$ can be represented uniquely by an element $aw_1 + bw_2 \in \mathbb{C}$ where $a, b \in [0, 1)$. Define the map $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ given by $f([aw_1 + bw_2]) = [a\lambda w_1 + b\lambda w_2] \in \mathbb{C}/\Lambda'$ for all $[aw_1 + bw_2] \in \mathbb{C}/\Lambda$ where $a, b \in [0, 1]$. f is a well-defined map because $[\lambda z] = \lambda[z] = \lambda[w] = [\lambda w]$ whenever $[z] = [w]$. We can observe that f is bijective.

Let $[z_1] \in \mathbb{C}/\Lambda$. Then $\lim\limits_{[z]\to[z_1]} \frac{f([z])-f([z_1])}{[z]-[z_1]} = \lim\limits_{[z]\to[z_1]} \frac{\lambda[z]-\lambda[z_1]}{[z]-[z_1]} = \lambda \in \mathbb{C}\backslash\{0\}$. So f is holomorphic. Similarly, we can observe that the inverse of f is holomorphic. As a result, $\mathbb{C}/\Lambda$ is biholomorphic to $\mathbb{C}/\Lambda'$.

□

Theorem 3.13 ([11]). *Let L be a holomorphic map from $\mathbb{C}/\Lambda$ to itself where Λ is a lattice of $\mathbb{C}$ with rank 2. Then there exist constants α and β in $\mathbb{C}$ such that $L(z) \equiv \alpha z + \beta$ (mod Λ).*

Lemma 3.14. *Let α and β be complex numbers and Λ of rank 1 or 2 be a lattice of $\mathbb{C}$. Then the map $L(z) = \alpha z + \beta$ defines a holomorphic self map of $\mathbb{C}/\Lambda$ if and only if $\alpha\Lambda \subset \Lambda$.*

Proof. Let $\Lambda = \langle 1, \tau \rangle$ where $\tau \in \mathbb{C}\backslash\mathbb{R}$. Then the map $L(z) = \alpha z + \beta$ defines a map $\mathbb{C}/\Lambda$ to itself if and only if $\alpha = L(z + 1) - L(z) \in \Lambda$ and $\alpha\tau = L(z + \tau) - L(z) \in \Lambda$ for all $z \in \mathbb{C}$. This is equivalent to $\alpha\Lambda \subset \Lambda$ as Λ is generated by 1 and τ. Similarly, we can show that if $\Lambda = \langle w \rangle$ where $w \in \mathbb{C}$, then $\alpha\Lambda \subset \Lambda$. $\square$

Observe that if $\alpha \in \mathbb{Z}$, then $L(z) = \alpha z + \beta$ is always holomorphic map from $\mathbb{C}/\Lambda$ to itsef with derivative α. Now we want to find other values of α. For this, we have the following lemma:

Lemma 3.15. *Let α be a complex number and Λ be a lattice of $\mathbb{C}$. Then:*

(1) If the rank of Λ is 2 and $\alpha\Lambda \subset \Lambda$ with $\alpha \notin \mathbb{Z}$ then there are integers q and d with $q^2 < 4d$ where $\alpha^2 + q\alpha + d = 0$ and $d = |\alpha|^2$.

(2) If the rank of Λ is 1 and $\alpha\Lambda \subset \Lambda$, then α is an integer.

Proof. (1) Let $\Lambda = \langle 1, \tau \rangle$ where $\tau \in \mathbb{C}\backslash\mathbb{R}$. Since $\alpha\Lambda \subset \Lambda$, $\alpha \in \Lambda$ and $\alpha\tau \in \Lambda$. So there exists integers a, b, c, e such that $\alpha = a + b\tau$ and $\alpha\tau = c + e\tau$. Then $\alpha^2 = (a + b\tau)(\frac{c}{\tau} + e)$. As a result of these, we have the equality,

$$\alpha^2 + (-a - e)\alpha + (ae - bc) = 0 \tag{3.2}$$

where $(-a - e)$ and $(ae - bc)$ are integers. Now we want to show that $|\alpha|^2 = ae - bc$. Since α and w are complex numbers, $\alpha = s + it$ and $\tau = x + iy$ for some real numbers s, t, x and y. Then $s + it = a + bx + biy$. This yields $s = a + bx$ and $t = by$. So

$$|\alpha|^2 = s^2 + t^2 = a(a + 2bx) + b(bx^2 + by^2) \tag{3.3}$$

On the other hand $(s + it)(x + iy) = c + ex + eiy$. Therefore we have

$$sx - ty = c + ex \tag{3.4}$$

$$sy + tx = ey \tag{3.5}$$

We can observe that $ay + 2bxy = ey$. So we have $a + 2bx = e$ because τ is not real number i.e. y is not zero. On the other hand, $bx^2 - by^2 = c + x(e - a) = c + 2bx^2$ by the

(3.4). Then we have $bx^2+by^2 = -c$. As a conclusion, $d = |\alpha|^2 = ae-bc$ by using (3.3). Now, let $q = -a-e$. To complete the proof, we need to show that $q^2 < 4d$. Since $a+2bx = e$, then $q = -a-e = -a-a-2bx = -2s$. Hence, $q^2 = 4s^2 < 4(s^2+t^2) = 4d$. As a conclusion, we have

$$\alpha^2 + q\alpha + d = 0 \tag{3.6}$$

where q and $d = |\alpha|^2$ are integers and $q^2 < 4d$. Furthermore, α can be equal to only two numbers $\dfrac{-q \pm \sqrt{q^2-4d}}{2}$.

(2) Let $\Lambda = \langle w \rangle$ where $w \in \mathbb{C}$. Since $\alpha\Lambda \subseteq \Lambda$, then $\alpha w \in \Lambda$. So there exists an integer k such that $\alpha w = kw$. This gives us $\alpha = k \in \mathbb{Z}$. $\square$

Now we want to show that periodic points of holomorphic self map of $\mathbb{C}/\Lambda$ are everywhere dense in $\mathbb{C}/\Lambda$. First of all, recall how to find the degree of L. The degree of L is the number of preimages of any element of $\mathbb{C}/\Lambda$. More generally, we have the following definition;

Definition 3.16. Let $L : X \to Y$ be a non-constant map between connected Riemann surfaces X and Y. Then the **number of preimages** counting multiplicities of any point $y \in Y$ is called the **degree** of L (This number is independent of the choice of $y \in Y$, see [2]). Whenever the multiplicity of a point is greater than 1, we call this point a **ramification point** of L.

Example 3.17. Let f be a self map of $\mathbb{C}/\Lambda$ where $\Lambda = \langle 1, \tau \rangle$ and $w \in \mathbb{C}/\mathbb{R}$ given by $f(z) = 2z$. We want to find degree of f. For this, we should count the preimages of any point of z. Since this number is independent of the choice of z, it is enough to look at the preimages of 0. We have that $f^{-1}(0) = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}\}$. So the degree of f is 4. Remark that the Euler characteristic of $\mathbb{C}/\Lambda$ is 0. Hence f has no ramification.

We can observe that these preimages divide the fundamental region 4 equal regions.

All of these regions have one preimage of 0. Even if we restrict f to one of these small regions, then f will be onto.

Remark 3.18. Let L be a holomorphic map from torus to itself with $L(z) = \alpha z + \beta$. We showed that $|\alpha|^2$ is an integer. We can observe that the degree of L is $|\alpha|^2$ since L carries a small region of area A to a region of area $|\alpha|^2 A$. On the other hand if we have lattice Λ of rank 2, then each fundamental region of this lattice have same area A. Therefore, if we take a region in $\mathbb{C}/\Lambda$ whose area is bigger than $\dfrac{A}{|\alpha|^2}$, then restriction of L to this region must be onto. Similarly, we can observe that the degree of L is $|\alpha|$ if L is a map from cylinder to itself given by $L(z) = \alpha z + \beta$.

Lemma 3.19. *Let L be a holomorphic map from $\mathbb{C}/\Lambda$ to itself where Λ is a lattice of $\mathbb{C}$ with rank 2 given by $L(z) = \alpha z + \beta$ and $|\alpha| \neq 0, 1$. Then periodic points of L are everywhere dense in $\mathbb{C}/\Lambda$.*

Proof. Observe that $L^n(z) = \alpha^n z + \beta(\alpha^{n-1} + \alpha^{n-2} + \cdots + 1)$. If $L^n(z) = z$ for some $n \in \mathbb{N}$ and $z \in \mathbb{C}/\Lambda$, then $(\alpha^n - 1)z + \gamma \equiv 0 \pmod{\Lambda}$ for some complex number γ. Now let U be an open subset of $\mathbb{C}/\Lambda$ with area A. We can find some $n \in \mathbb{N}$ such that $\dfrac{B}{|\alpha^n - 1|^2} < A$ where B is the area of a fundemantal region. Look at the map $f(z) = (\alpha^n - 1)z + \gamma$ from torus to itself. We want to examine $f^{-1}(0)$. We know that degree of f is $|\alpha^n - 1|^2$. This means that zero has $|\alpha^n - 1|^2$ many inverses. Moreover, U must contain one of these inverses by Remark 3.18. As a result of this, U must contain at least one periodic point of L. So periodic points of L are dense in $\mathbb{C}/\Lambda$. $\square$

By using similar arguments in Lemma 3.19, we can observe that if L is a holomorphic map from $\mathbb{C}/\Lambda$ where Λ is a lattice of $\mathbb{C}$ with rank 1 given by $L(z) = \alpha z + \beta$ and $|\alpha| \neq 0, 1$, then periodic points of L are everywhere dense in $\mathbb{C}/\Lambda$.

To summarize the above lemmas, we have the following theorem;

Theorem 3.20. *Let Λ be a lattice of $\mathbb{C}$ and $L : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ be a map given by $L(z) = \alpha z + \beta$ where $\alpha, \beta \in \mathbb{C}$. Then we have the followings;*

(1) L is a holomorphic map if and only if $\alpha\Lambda \subset \Lambda$.
(2) If the rank of Λ is 1, then $\alpha\Lambda \subset \Lambda$ if and only if α is an integer.
(3) If the rank of Λ is 2 and $\alpha\Lambda \subset \Lambda$, then α is either an integer or a quadratic integer.
(4) Periodic points of L are everywhere dense in $\mathbb{C}/\Lambda$ if $|\alpha| \neq 0, 1$.

Finally, we want to understand rotations of $\mathbb{C}/\Lambda$. We know that if we have a holomorphic self map f of $\mathbb{C}/\Lambda$ where Λ is a lattice of $\mathbb{C}$ with rank 2, then it must be equal $\alpha z + \beta$ for some complex numbers α and β. If β is zero and $|\alpha|$ is 1, then f is called as a **rotation** of torus. Since rotation maps are bijective linear maps, each rotation can be represented by an element of $SL_2(\mathbb{R}) = \{M = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) |\ det(M) = 1\ p, q, r, s \in \mathbb{R}\}$.

Proposition 3.21. *Let Λ be a lattice of $\mathbb{C}$ with generator 1 and τ where $\tau \notin \mathbb{R}$ and $M \in SL_2(\mathbb{R})$. Define $M \cdot \Lambda = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ \tau \end{smallmatrix}\right) = \langle p + q\tau, r + s\tau \rangle$. Then $M \cdot \Lambda = \Lambda$ if and only if $M \in SL_2(\mathbb{Z})$.*

Proof. Suppose that $M = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Since $p, q, r, s \in \mathbb{Z}$ then $\langle p + q\tau, r + s\tau \rangle$ is subset of $\langle 1, \tau \rangle$. We can observe the followings; $1 = s(p + q\tau) - q(r + s\tau)$ and $\tau = -r(p + q\tau) + p(r + s\tau)$. Then $\langle 1, \tau \rangle$ is subset of $\langle p + qw, r + s\tau \rangle$. Hence, $M \cdot \Lambda = \Lambda$.

For the other direction, suppose that $M \cdot \Lambda = \Lambda$ where $M = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Then $p + q\tau = a + b\tau$ and $r + s\tau = k + l\tau$ for some integers a, b, k and l. We can observe that if we have these two equalities, then we must necessarily have $p = a, q = b, r = k$ and $s = e$. Hence $M \in SL_2(\mathbb{Z})$. $\square$

As a result of this proposition, we conclude that if rotation sends lattice to itself, then rotation matrix must be in $SL_2(\mathbb{Z})$. Hence its trace must be an integer. Finally, we have the following proposition;

Proposition 3.22. *Let Λ be a lattice of $\mathbb{C}$ with generator w where $w = x + iy$ for $x, y \in \mathbb{R}$ and $M = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL_2(\mathbb{R})$. Define $M \cdot \Lambda = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \langle px + qy + i(rx + sy) \rangle$. Then $M \cdot \Lambda = \Lambda$ if and only if M is equal either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.*

Proof. Observe that if $\langle x + iy \rangle = \langle x_1 + iy_1 \rangle$, then $x_1 + iy_1$ is either equal $x + iy$ or $-x - iy$. On the other hand, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \langle x + iy \rangle$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \langle -x - iy \rangle$. Moreover, we know that if we have a matrix $M \in SL_2(\mathbb{R})$ such that $M \cdot \Lambda = \Lambda$, then M is unique. As a result of these, $M \cdot \Lambda = \Lambda$ if and only if M is equal either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. $\square$

4 LATTÈS MAPS ON $\widehat{\mathbb{C}}$

In this section, we will define a finite quotient of an affine map which is holomorphically conjugate to an affine map from $\mathbb{C}/\Lambda$ to itself and investigate these maps. If the rank of Λ is one, then it is holomorphically conjugate either to a power map or to a Chebyshev map whose details are given in Lemma 4.11. Firstly, we examine the results of John Milnor which made important observations about these maps [12]. After that, we will recall some properities of elliptic functions and compute the complete ramification data of Lattès maps.

4.1 FINITE QUOTIENTS OF AFFINE MAPS

Let $L(z) = az + b$ be an affine map from $\mathbb{C}/\Lambda$ to itself where a and b are complex numbers and Λ is a lattice of $\mathbb{C}$, Θ be an onto and finite-to-one holomorphic map from $\mathbb{C}/\Lambda$ to $\widehat{\mathbb{C}} \setminus \epsilon_f$, and f be a rational map whose degree is bigger than one. Look at the following diagram:

$$\begin{tikzcd}
\mathbb{C}/\Lambda \arrow[r, "L"] \arrow[d, "\Theta"'] & \mathbb{C}/\Lambda \arrow[d, "\Theta"] \\
\widehat{\mathbb{C}} \setminus \epsilon_f \arrow[r, "f"] & \widehat{\mathbb{C}} \setminus \epsilon_f
\end{tikzcd}$$

If this diagram commutes i.e. if we have semiconjugacy relation $f \circ \Theta = \Theta \circ L$, then f will be called a **finite quotient of an affine map**.

Lemma 4.1. *Let f, L and Θ as stated above. If $z \in \mathbb{C}/\Lambda$ is a periodic point of L, then $\Theta(z)$ is a periodic point of f. Conversely, if $w \in \widehat{\mathbb{C}} \setminus \epsilon_f$ is a periodic point of f, then $\Theta^{-1}(w)$ is a periodic point of L.*

Proof. Suppose that z is periodic point of L i.e. there exist some $n \in \mathbb{N}$ such that $L^n(z) = z$. Then $f^n(\Theta(z)) = \Theta(L^n(z)) = \Theta(z)$ since $f^n \circ \Theta = \Theta \circ L^n$. This means that $\Theta(z)$ is periodic point of f.

For the other direction, assume that $w \in \widehat{\mathbb{C}} \setminus \epsilon_f$ is periodic point of f i.e. there exist some $n \in \mathbb{N}$ such that $f^n(w) = w$. Since Θ is onto and finite-to-one map, there exist $z_1, \ldots, z_k$ such that $\Theta(z_1) = \cdots = \Theta(z_k) = w$ and if z is not equal to z_i for some $i \in \{1, \ldots, k\}$, then $\Theta(z) \neq w$. We have $w = f^n(w) = f^n(\Theta(z_i)) = \Theta(L^n(z_i))$ for all i. Thus, $\{L^n(z_1), \ldots, L^n(z_k)\} \subseteq \{z_1, \ldots, z_k\}$. We want to prove that these two sets are equal. Suppose that they are not equal. This means that there exist $z_s \in \{z_1, \ldots, z_k\}$ such that $L^n(z_i) \neq z_s$ for all $i \in \{1, \ldots, k\}$. Now, let m be a degree of L^n. Then z_s has m many inverses $l_1, \ldots, l_m$, which are not equal to z_i for all $i \in \{1, \ldots, k\}$, with counting multiplicity. Since $f^n(\Theta(l_j)) = \Theta(L^n(l_j)) = \Theta(z_s) = w$, then w and $l_1, \ldots, l_m$ are inverses of w under f^n. Then w has at least $m+1$ many inverses under f^n. On the other hand, degree of f^n is m since degree of L^n is m. Hence, we have a contradiction. Then $\{L^n(z_1), \ldots, L^n(z_k)\} = \{z_1, \ldots, z_k\}$ and we can regard $\{L^n(z_1), \ldots, L^n(z_k)\}$ as a symmetric group S_k. So it has a finite order k. Hence, each elements in this set is periodic point of L^n. As a result, they are periodic points of L. □

4.2 LATTÈS MAPS

Let f be a finite quotient of an affine map and Θ and L as stated in the definiton at the beginning of Section 3.1. If Λ has rank 2, then f is called as a Lattès map. In this case, we know that $\mathbb{C}/\Lambda$ is compact. Then its image under Θ is compact since Θ is a holomorphic map. On the other hand, we know that exceptional set of f may contain at most two elements. If it is not empty set then $\widehat{\mathbb{C}} \setminus \epsilon_f$ will be noncompact. But this is impossible since Θ is onto. Hence, we do not have any exceptional point of f if f is a Lattès map.

Definition 4.2. Let $\Theta : S \to \widehat{\mathbb{C}}$ be a holomorphic function defined on a compact Riemann surface S. A **critical point** or **ramification point** of Θ is the point whose derivative is 0. A **critical value** or **ramification value** is the image of the critical point under Θ. The set of critical values is denoted by V_f. The **postcritical set** P_f is the union of the (forward) orbits of the critical values.

Lemma 4.3 ([12]). *Let f be a Lattès map. Then postcritical set $P_f = \bigcup_{n=0}^{\infty} f^n(V_f)$ is exactly equal to the set of critical values of Θ. Hence the postcritical set is a finite set.*

Proof. We will show that w_1 is critical value of Θ if and only if it is either critical value of f or $f^{-1}(w_1)$ is critical value of θ. Let $w_1 \in \widehat{\mathbb{C}}$. Since Θ is onto, we have some element $z_1 \in \mathbb{C}/\Lambda$ such that $\Theta(z_1) = w_1$. Similarly, we have some element $z_2 \in \mathbb{C}/\Lambda$ with $L(z_2) = z_1$. Then we have the following diagram :

$$\begin{array}{ccc} \mathbb{C}/\Lambda \ni z_2 & \xmapsto{L} & z_1 \in \mathbb{C}/\Lambda \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Theta} \\ \widehat{\mathbb{C}} \ni w_2 & \xmapsto{f} & w_1 \in \widehat{\mathbb{C}} \end{array}$$

Since f is Lattès map, we have $f \circ \Theta = \Theta \circ L$. Then, $f'(\Theta(z)).\Theta'(z) = \Theta'(L(z)).L'(z)$ for all $z \in \mathbb{C}/\Lambda$. So we have $f'(w_2).\Theta'(z_2) = \Theta'(z_1).L'(z_2)$. By using this equality we can show that w_1 is critical value of Θ iff $\Theta'(z_1) = 0$ iff either $\Theta'(z_2) = 0$ or $f'(w_2) = 0$ iff $w_2 = f^{-1}(w_1)$ is critical value of Θ or w_1 is critical value of f. Hence we have proved the first claim. As a result of this, $V_\Theta = V_f \cup f(V_\Theta)$. By using induction, we can observe that $f^n(V_f) \subset V_\Theta$ for all n. Therefore we have $P_f \subset V_\Theta$. But we want to prove that $P_f = V_\Theta$. Suppose that this equality is not true i.e. V_Θ is not subset of P_f. Then there exist $w_1 \in V_\Theta$ but $w_1 \notin P_f$. Since w_1 is critical value of Θ and Θ is onto, then there exist some $z_1 \in \mathbb{C}/\Lambda$ such that $\Theta'(z_1) = 0$. Look at the following diagram :

$$\begin{array}{ccccccc}
\cdots\cdots \overset{L}{\longmapsto} & z_3 & \overset{L}{\longmapsto} & z_2 & \overset{L}{\longmapsto} & z_1 \\
& \downarrow \Theta & & \downarrow \Theta & & \downarrow \Theta \\
\cdots\cdots \overset{f}{\longmapsto} & w_3 & \overset{f}{\longmapsto} & w_2 & \overset{f}{\longmapsto} & w_1
\end{array}$$

Since w_1 is not in P_f, then w_2 is not critical point of f. This observation gives us $\Theta'(z_2) = 0$. By an induction argument, we can show that $\Theta'(z_i) = 0$ for all $i \in \mathbb{N}^+$. Then we have infinitely many critical points for Θ. But this is impossible as Θ is holomorphic, non-constant function and $\mathbb{C}/\Lambda$ is compact domain. Then P_f must be equal to V_Θ. Consequently, P_f is finite set since V_Θ is finite set.

□

Lemma 4.4. *Periodic points of Lattès map f are dense set on the Riemann Sphere.*

Proof. Let $w \in \widehat{\mathbb{C}}$. Then there exist $z \in \mathbb{C} \setminus \Lambda$ such that $\Theta(z) = w$. Since periodic poinnts of L are dense in the torus, then there exist sequences $\{z_n\}_{n\in\mathbb{N}}$ in the set of periodic points of L such that $z_n \longrightarrow z$ as $n \longrightarrow \infty$. Because of the continuity of Θ, $\Theta(z_n) \longrightarrow \Theta(z) = w$. But $\Theta(z_n)$'s are periodic point of f as z_n's are periodic point of L. So periodic points of f are dense on the Riemann Sphere.

□

Remark 4.5. Let f be a rational map with fixed point w and λ be a multiplier of w. If $|\lambda|$ is not zero or one, then f is linearizable. This means that there exists linearizing holomorphic map θ from a neighboorhod U of w to $\mathbb{C}$ with $\theta(w) = 0$ and the following diagram commutes:

$$\begin{array}{ccc}
U & \xrightarrow{f} & U \\
\theta \downarrow & & \downarrow \theta \\
\mathbb{C} & \xrightarrow{z \mapsto \lambda z} & \mathbb{C}
\end{array}$$

θ is called as Kœnigs linearizing map. Kœnigs linearizing maps are unique up to multiplication by a constant, see Theorem 8.2 in [11].

Example 4.6. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a map given by $f(z) = 2z+5$ if $z \in \mathbb{C}$ and $f(\infty) = \infty$. ∞ is fixed point of f and the multiplier $\lambda_\infty = \lim_{z \to 0} \frac{z^{-2}.2}{(\frac{2}{z}+5)^2} = \frac{1}{2}$. Then f is linearizable. We have that $\widehat{\mathbb{C}} \backslash \{0\}$ is an open neighboorhod of ∞. Define $\theta : \widehat{\mathbb{C}} \backslash \{0\} \to \mathbb{C}$ be a map given by $\theta(z) = \frac{2}{z+5}$ if $z \in \mathbb{C} \backslash \{0\}$ and $\theta(\infty) = 0$. Then θ is a holomorphic map and we have the following diagram;

$$\begin{array}{ccc} \widehat{\mathbb{C}} \backslash \{0\} & \xrightarrow{f} & \widehat{\mathbb{C}} \backslash \{0\} \\ \theta \big\downarrow & & \big\downarrow \theta \\ \mathbb{C} & \xrightarrow{z \mapsto \frac{1}{2} z} & \mathbb{C} \end{array}$$

So we have $\theta \circ f(z) = \theta(2z+5) = \frac{1}{z+5} = \frac{1}{2}(\frac{2}{z+5}) = \frac{1}{2}\theta(z)$. As a result, the diagram commutes and θ is a Kœnigs linearizing map.

Theorem 4.7 ([12]). *Let f be a Lattès map. Then Θ induces a canonical homeomorphism from the quotient space $\mathcal{T}/G_n$ to $\widehat{\mathbb{C}}$ where G_n of rigid rotation of the torus about some base point is a finite cyclic group such that $\Theta(z) = \Theta(z')$ if and only if $z = gz'$ for some $g \in G_n$.*

Proof. Let U be a simply connected open subset of $\widehat{\mathbb{C}} \setminus P_f = \widehat{\mathbb{C}} \setminus V_\Theta$ and n be a degree of Θ. Then preimage of U under Θ is disjoint union of n many open subsets $U_1, \ldots, U_n$ each of which are diffeomorphic to U.

Let Θ_i be the restriction of Θ to U. We will prove that the map $\Theta_i^{-1} \circ \Theta_j : U_j \longrightarrow U_i$ is an isometry, using the standard flat metric on the torus.

Since periodic points of f are everywhere dense and U is open subset, we can find some periodic point w_0 in U. Thus there exist some $m \in \mathbb{N}$ such that $f^m(w_0) = w_0$. This means that w_0 is fixed point of f^m. On the other hand we have $\Theta^{-1}(w_0) = \{z_1, \ldots, z_n\}$ and this set is L^m-invariant. All points in this set are peridic points of L. Hence we can find some iteration $\widehat{f}$ of f such that w_0 is fixed point of this iteration. Moreover, all points in $\Theta^{-1}(w_0)$ are fixed point of corresponding iteration $\widehat{L}$ of L. Let $z_j = \Theta^{-1}(w_0)$. Define the map L' from complex numbers to itself as $L'(z - z_j) = \widehat{L}(z) - z_j$. It is well defined because for all complex numbers we can find unique complex number z such that it is equal to $z - z_j$. On the other hand, $\widehat{L}(t) = \widehat{a}t + \widehat{b}$ for some complex numbers $\widehat{a}$ and $\widehat{b}$ since $L(t) = at + b$ for some $a, b \in \mathbb{C}$. Then $L'(z - z_j) = \widehat{a}(z - z_j)$ and it is linear map.

Define the map Φ_j such that $\Phi_j(z) = \Theta_j^{-1}(z) - z_j$. If we take a small open neihgborhood V of z_j, then $\Phi_j : V \to \mathbb{C}$ with $\Phi_j(z) = \Theta_j^{-1}(z) - z_j$ is well defined map. Hence we have the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\widehat{f}} & V \\ \downarrow{\scriptstyle \Phi_j} & & \downarrow{\scriptstyle \Phi_j} \\ \mathbb{C} & \xrightarrow{L'} & \mathbb{C} \end{array}$$

We have $\Phi_j(w_0) = \Theta_j^{-1}(w_0) - z_j = \Theta_j^{-1}(w_0) - \Theta_j^{-1}(w_0) = 0$. Moreover, $\widehat{a} = (L')'$ is multiplier of $\widehat{f}$ at w_0 and $|\widehat{a}|$ is bigger than 1 since $|a|$ is bigger than 1. So we have that if this diagram commutes then Φ_j will be a Koenigs linearizing map. For all $w \in V$, we have $\Phi_j \circ \widehat{f}(w) = \Theta_j^{-1}(\widehat{f}(w)) - z_j = \widehat{L}(\Theta_j^{-1}(w)) - z_j = \widehat{a}(\Theta_j^{-1}(w) - z_j) = L' \circ \Phi_j(w)$. This means that the diagram is commute. So Φ_j is Koenigs linearizing map. We know that Kœnigs linearizing map is unique up to multiplication by a nonzero constant. Therefore, $\Phi_i(z)$ must be equal to $a_{ij}\Phi_j(z)$ for some nonzero constant a_{ij} and for all $z \in V$. It follows that $\Theta_i^{-1} \circ \Theta_j(z) = a_{ij}z + b_{ij}$ for some constant a_{ij}, b_{ij} and for all $z \in V$. On the other hand, U_j is simply connected domain, V is open subset of U and $\Theta_i^{-1} \circ \Theta_j$ is homomorphism. Then $\Theta_i^{-1} \circ \Theta_j(z) = a_{ij}z + b_{ij}$ for all $z \in U_j$. Now we want to show that $|a_{ij}| = 1$.

Look at the following commutative diagram:

$$\begin{tikzcd}
U_j \arrow[r, "\Theta_i^{-1}\circ\Theta_j"] \arrow[dr, "\Theta"'] & U_i \arrow[d, "\Theta"] \\
& \widehat{\mathbb{C}}
\end{tikzcd}$$

Then $\Theta(a_{ij}z+b_{ij}) = \Theta(f_{ij}(z)) = \Theta(z) = \Theta(z+\lambda) = \Theta(f_{ij}(z+\lambda)) = \Theta(a_{ij}z+b_{ij}+a_{ij}\lambda)$ for all $z \in U_j$ and for all $\lambda \in \Lambda$ where $f_{ij} = \Theta_i^{-1} \circ \Theta_j$. It follows that $a_{ij}\Lambda \subset \Lambda$ because $\Theta(z) = \Theta(z + \lambda)$ for each $z \in \mathbb{C}$ if and only if $\lambda \in \Lambda$.

Choosing local lifting of $\Theta_i^{-1} \circ \Theta_j$ to the universal covering space $\mathbb{C}$ with the covering map p and continuing analytically we obtain an affine self map A_{ij} of $\mathbb{C}$ with derivative $A'_{ij} = a_{ij}$ which satisfy the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "A_{ij}"] \arrow[d, "p"] & \mathbb{C} \arrow[d, "p"] \\
U_j \arrow[r, "\Theta_i^{-1}\circ\Theta_j"] \arrow[dr, "\Theta"'] & U_i \arrow[d, "\Theta"] \\
& \widehat{\mathbb{C}}
\end{tikzcd}$$

Let $\tilde{G}$ be the group consisting of all affine transformation $\tilde{g}$ of $\mathbb{C}$ which satisfy the equality $\Theta \circ p \circ A_{ij} = \Theta \circ p$. We can observe that all of these transformations send U_j to U_i for all i, j. Now, let $\tilde{\Lambda}$ be the subgroup of $\tilde{G}$ consisting of all translations with elements of Λ. We can observe that $\tilde{\Lambda}$ is the normal subgroup of $\tilde{G}$. Thus the quotient space $G = \tilde{G}/\tilde{\Lambda}$ has exactly n elements since it contains only one transformation which sends U_1 to any U_i. Let $h : G \to \mathbb{C}\backslash\{0\}$ such that $h(g) = g'$. Then h is one-to-one and homomorphism. Hence it must carry G isomorphically onto the unique subgroup of $\mathbb{C}\backslash\{0\}$ of order n, namely the group G_n of n-th roots of unity. Furthermore, a generator of G must have a fixed point in the torus, so G can be considered as a group of rotations about this fixed point. In fact, we can identify G with the group G_n of n-th roots of unity, acting by multipication on $\mathbb{C}/\Lambda$, by translating coordinates.

$\square$

Proposition 4.8 ([12]). *If we have a cyclic group of rotation G_n of the torus with order n, then n is either $2, 3, 4$ or 6 and $\mathcal{T}/G_n \cong \widehat{\mathbb{C}}$.*

Proof. Let a rotation through angle $\frac{2\pi}{n}$ as a real linear map. The trace of such a rotation is $2\cos(2\pi/n)$. Since such a rotation carries the lattice to itself, then its trace must be an integer. Then $2\cos(2\pi/n)$ may be equal to $-2, -1, 0$ or 1. As a result, n is necessarily either $2, 3, 4$ or 6.

□

Proposition 4.9 ([12]). *If f is Lattès map then it is conformally conjugate to a map of the form $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ where L is an affine map from torus to itself which commutes with a generator of G_n and L/G_n is the induced holomorphic map from the quotient surface to itself.*

Proof. Since we have a commutative diagram and Θ induces a canonical homeomorphism from the $\mathcal{T}/G_n$ to $\widehat{\mathbb{C}}$, then induced holomorphic map L/G_n must be well defined. Let z be a n-th root of unity, then the rotation $g(t) = zt$ generates G_n. We will show that it commutes with L. We have $azt + b \equiv z^k(at + b) \mod \Lambda$ for some power z^k since the points $L(t)$ and $L(g(t))$ represents the same element of $\mathcal{T}/G_n$. If this equation is true for some generic choice of t , then it will be true identically for all t. Now differentiating with respect to t we see that $z^k = z$, and substituing $t = 0$ we see that $b \equiv zb \mod \Lambda$. As a result of this, we have $g \circ L = L \circ g$. □

4.3 POWER AND CHEBYSHEV MAPS

Let Λ be lattice of rank 1 and f, Θ and L as stated in the definition of a finite quotient of an affine map. Similar considerations in Lemma 3.19 and Lemma 4.4 yield that periodic points of f are dense on $\widehat{\mathbb{C}}/\varepsilon_f$. In this case, the postcritical set of f may not be finite but must be a discrete set. We can observe that the postcritical set P_f is subset of the set V_Θ of critical value of Θ by using same considerations in Lemma 4.3. Since Θ is holomorphic, V_Θ must be a discrete. Otherwise, we can find compact subset K of $\mathbb{C}/\Lambda$ such that the restriction $\Theta|_K$ is holomorphic and has infinitely many critical point. However, this is impossible. Hence P_f must be a discrete set. As a result of these, we can prove Theorem 4.7 for these maps.

Theorem 4.10. *Let f be a finite quotient of an affine map, Λ be a lattice of rank 1 and Θ and L as stated above. Then:*

- *Θ induces a canonical homeomorphism from the quotient space $\mathcal{C}/\Lambda$ to $\widehat{\mathbb{C}}/\varepsilon_f$ where G_n of rigid rotation of the cylinder about some base point is a finite cyclic group. Furthermore, we can identify G_n with the group of n-th roots of unity by translating coordinates,*
- *n is either 1 or 2,*
- *f is conformally conjugate to an induced holomorphic map $L/G_n : \mathcal{C}/G_n \to \mathcal{C}/G_n$ and L commutes with a generator of G_n.*

Proof. We can prove first and third parts of the theorem by using same considerations in Theorem 4.7 and Lemma 4.9. Now, let a rotation through angle $\frac{2\pi}{n}$ as a real linear map. Since such a rotation of cylinder carries the lattice to itself, its trace must be either 2 or -2 by Proposition 3.22. This yields that n is either 1 or 2. This means that the degree of Θ is either 1 or 2.

□

Lemma 4.11. *Let Λ be a lattice of rank 1 and f, Θ and L as stated above, then f is holomorphically conjugate either to a power map or to a Chebyshev map.*

Proof. We can assume that $\Lambda = \mathbb{Z}$ by replacing z by wz where w is the generator of Λ. Suppose that the degree of Θ is 1. So Θ is $1-1$ and onto. We know that if we have a map on $\mathbb{C}/\Lambda$ where Λ of rank 1 is a discrete additive subgroup of $\mathbb{C}$, then it is equal to

$h \circ g$ where h is a rational map and $g : \mathbb{C} \to \mathbb{C}$ be a map given by $g(z) = e^{2\pi iz}$, see [7] for details. Then $\Theta = h \circ g$ where h is a rational map and $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = e^{2\pi iz}$. Observe that if $\Theta(z) = e^{2\pi iz}$ for each $z \in \mathbb{C}/\Lambda$, then $f(z) = \Theta \circ L \circ \Theta^{-1}(z) = e^{-2\pi ib} z^a$ for all z on the domain of f. On the other hand, $e^{-2\pi ib} z^a = K \circ z^a \circ K^{-1}$ where $K(z) = (e^{2\pi ib})^{\frac{1}{a-1}}$. As a conclusion, $f = \Theta \circ L \circ \Theta^{-1} = h \circ g \circ L \circ g^{-1} \circ h^{-1}$ and $g \circ L \circ g^{-1}(z) = e^{-2\pi ib} z^a = K \circ z^a \circ K^{-1}$ for each z. This means that f is holomorphically conjugate to a power map. Finally, we can observe that $\varepsilon_f = \{0, \infty\}$.

A rational map g is called a *Chebyshev map* if g is holomorphically conjugate either to a Chebyshev polynomial or to a negative Chebyshev polynomial. Suppose that the degree of Θ is 2 and Θ is given by $\Theta(z) = e^{2\pi iz} + e^{-2\pi iz} = 2cos(2\pi z)$ for all $z \in \mathbb{C}/\Lambda$. This map degree 2 holomorphic map on $\mathbb{C}/\Lambda$. By using third part of Theorem 4.10, we can observe that $2b \in \Lambda$. Firstly, assume that $b \in \Lambda$. Then we have the following identities $f(2cos(z)) = f(e^{iz} + e^{iz}) = f(\Theta(\frac{z}{2\pi})) = \Theta(L(\frac{z}{2\pi})) = \Theta(a\frac{z}{2\pi}) = cos(az)$. On the other hand, a is an integer by Lemma 3.15. Then f is the degree n Chebyshev polynomial. Now, suppose that $b \notin \Lambda$. Since $2b \in \Lambda$, we have the following identities $f(2cos(z)) = \Theta(a\frac{z}{2\pi} + b) = e^{2\pi ib}(2cos(z)) = -2cos(z)$. So $-f$ is the degree n Chebyshev polynomial. As an example, look at the following commutative diagrams;

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{z \mapsto 2z} & \mathbb{C}/\Lambda \\ \Theta \downarrow & & \downarrow \Theta \\ \widehat{\mathbb{C}} \setminus \epsilon_f & \xrightarrow{z \mapsto z^2 - 2} & \widehat{\mathbb{C}} \setminus \epsilon_f \end{array} \qquad \begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{z \mapsto 2z + \frac{1}{2}} & \mathbb{C}/\Lambda \\ \Theta \downarrow & & \downarrow \Theta \\ \widehat{\mathbb{C}} \setminus \epsilon_f & \xrightarrow{z \mapsto 2 - z^2} & \widehat{\mathbb{C}} \setminus \epsilon_f \end{array}$$

Finally, suppose that the degree of Θ is 2. Since second roots of unity G_2 acts by multiplication on $\mathcal{C}$ and $\mathcal{C}/G_2 \cong \widehat{\mathbb{C}} \backslash \varepsilon_f$, Θ is a rational map of $e^{2\pi iz} + e^{-2\pi iz} = 2cos(2\pi z)$. As a result of this and previous paragraph, we can observe that f is holomorphically conjugate to a Chebyshev map. $\square$

5 RAMIFICATION BEHAVIOUR OF LATTÈS MAPS

In this section, we will obtain the explicit form of conjugating holomorphisms of Lattès maps and calculate the ramification behaviour of Lattès maps. Throughout the section, let Λ be a lattice of $\mathbb{C}$ with generators 1 and τ.

5.1 WEIERSTRASS' ELLIPTIC FUNCTION

Weierstrass' elliptic function on $\widehat{\mathbb{C}}$ with respect to Λ is defined as;

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in\Lambda\setminus\{0\}} \frac{1}{(z-w)^2} - \frac{1}{w^2} \tag{5.1}$$

This function is convergent, its degree is 2 and its derivative is $\wp'(z) = -2\sum_{w\in\Lambda} \frac{1}{(z-w)^3}$ whose degree is 3, see [4].

Theorem 5.1 ([6]). *Let Λ be a lattice of $\mathbb{C}$ and $\wp$ be the Weierstrass' elliptic function with respect to Λ. Then we have the followings;*

- *the equation $\wp(z_1) = \wp(z_2)$ holds if and only if either $z_1 + z_2 \equiv 0 \mod \Lambda$ or $z_1 - z_2 \equiv 0 \mod \Lambda$,*
- $\wp(0) = \infty$,
- *the only solution of the equation $\wp'(z) = 0$ are $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$ i.e. half-periods,*
- *for any distinct $z_1, z_2 \in \mathbb{C}/\Lambda$, we have*
$$\wp(z_1+z_2) + \wp(z_1) + \wp(z_2) = \frac{1}{4}\Big(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\Big)^2 \tag{5.2}$$
whenever $z_1 + z_2 \notin \Lambda$.

Recall the algebraic differential equation for $\wp$;

Theorem 5.2 ([4], Theorem V.3.4). *We have;*

$$\wp'(z)^2 = 4\wp(z)^3 + a\wp(z) + b \tag{5.3}$$

where $a = -60\sum_{w\in\Lambda\setminus\{0\}}\frac{1}{w^4}$ *and* $b = -140\sum_{w\in\Lambda\setminus\{0\}}\frac{1}{w^6}$.

Let f be a Lattès map and Θ and L as stated in the definition of a finite quotient of an affine map. Then Θ can be written as a rational function of $\wp$ and $\wp'$ because of the following theorem;

Theorem 5.3 ([4], Theorem V.3.3). *Let* $\Theta : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ *be a meromorphic function where* $\Lambda = \langle w_1, w_2\rangle$. *Then* Θ *can be written as a rational function of* $\wp$ *and* $\wp'$ *with respect to* Λ.

In section 3.1, we showed that Θ induces a canonical homeomorphism from the quotient space $\mathcal{T}/G_n$ to $\widehat{\mathbb{C}}$ where G_n of rigid rotation of the torus about some base point is a finite cyclic group and we can identify G_n with the group of n-th roots of unity. Since n is either $2, 3, 4$ or 6, we have the following theorem;

Theorem 5.4 ([14]). *Let* Θ *and* G_n *as stated above and* $\wp$ *be a Weierstrass' elliptic function with respect to* Λ *which is in the domain of* Θ. *Then we have;*

$$\Theta(z) = \begin{cases} \wp(z) & , \text{if } \ G_n \simeq \mathbb{Z}/2\mathbb{Z} \\ \wp'(z) & , \text{if } \ G_n \simeq \mathbb{Z}/3\mathbb{Z} \\ (\wp(z))^2 & , \text{if } \ G_n \simeq \mathbb{Z}/4\mathbb{Z} \\ (\wp(z))^3 & , \text{if } \ G_n \simeq \mathbb{Z}/6\mathbb{Z} \end{cases}$$

5.2 RAMIFICATION BEHAVIOUR OF LATTÈS MAPS

In this section, we will study on ramification behavıour of Lattès maps. Let us start with some basic examples.

Example 5.5. Look at the Weierstrass' elliptic function. Ramification values of $\wp$ are 0 and half-periods of corrosponding lattice by Theorem 5.1. This means that inverses of 0 and each half-period have contain only one element.

Example 5.6. Let $L : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ be a holomorphic map given by $L(z) = az + b$ where $a, b \in \mathbb{C}\backslash\{0\}$ and Λ be a discrete additive subgroup of $\mathbb{C}$ with rank 2. Then L has not any ramification point since $L'(z) = a$ for all $z \in \mathbb{C}$. This means that $L^{-1}(z)$ has exactly $|a|^2$ many distinct elements for each $z \in \mathbb{C}$.

We know that if $f : X \to Y$ is a holomorphic map whose degree is n between two Riemann surfaces X and Y, then $f^{-1}(y)$ has n elements with counting multiplicity for each $y \in Y$. If $y \in Y$ is ramification value of f, then the inverses of y are not distinct i.e. it has inverses less than the degree of f. So we can define the ramification index of ramification value $y \in Y$. Define $e_f(y) = \{x \in X : f^{-1}(y) = x\}$ and $d =$ degree of f. The **ramification index** $r_f(y)$ of $y \in Y$ is equal to the number $d - e_f(y)$.

Theorem 5.7. *(Riemann-Hurwitz Formula for $\widehat{\mathbb{C}}$) Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic map whose degree is d. Then we have the following equation;*

$$2d - 2 = \sum_{z \in \widehat{\mathbb{C}}} r_f(z) \tag{5.4}$$

Now we can find the ramification values of Lattès maps. Let $\Lambda = \langle 1, \tau \rangle$ be a discrete additive subgroup of $\mathbb{C}$ and f be a Lattès map. Then we have the following commutative diagram;

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{L} & \mathbb{C}/\Lambda \\ \Theta\big\downarrow & & \big\downarrow\Theta \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

where $L(z) = az + b$ for some complex numbers a and b and Θ be a finite-to-one holomorphic map. We know that the degree of Θ is either $2, 3, 4$ or 6 and the degree of f is bigger than 1. We can start with Θ whose degree is 2 to find the ramification values.

Theorem 5.8. *If the degree of* Θ *is* 2 *and* $|a|^2$ *is an odd integer, then* f *ramified over* 4 *points i.e.* f *has* 4 *ramification values.*

Proof. If the degree of Θ is 2, then $\Theta = \wp$ by Theorem 5.4. We know that the ramification values of $\wp$ are $\wp(\frac{1}{2})$, $\wp(\frac{\tau}{2})$, $\wp(\frac{1+\tau}{2})$ and $\wp(0)$ by Theorem 5.1. Then ramification values of f should be equal to these values by Lemma 4.3. This means that f has at most 4 ramification values. We can show that $2b \equiv 0 \mod \Lambda$ by using Lemma 4.9. By using first part of Theorem 5.1, we can observe that if $z \in L^{-1}(\frac{1}{2})$ and $z + w \equiv 0 \mod \Lambda$, then $w \in L^{-1}(\frac{1}{2})$. Same observation can be done for $0, \frac{\tau}{2}$ and $\frac{1+\tau}{2}$. This means that if inverses of half-period and zero are not half-period or zero, then they must be a pair. Now, we will show that the set of inverses of each half-period and 0 contains exactly one half-period or 0. Let $\frac{\tau}{2} \in \mathbb{C}/\Lambda$. Suppose that $L^{-1}(\frac{\tau}{2})$ does not contain 0 and any half-period. Since the degree of L is $|a|^2$, then $L^{-1}(\frac{\tau}{2})$ has $|a|^2$ many distinct inverses and they must be a pair as they are not equal to either 0 or half-period. However, this is impossible since $|a|^2$ is an odd integer. Therefore, $L^{-1}(\frac{\tau}{2})$ has either at least one half-period or 0. Suppose that $L^{-1}(\frac{\tau}{2})$ has two distinc elements $z_1, z_2 \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. Then we have $az_1 \equiv az_2 \mod \Lambda$. So $|a|^2 z_1 \equiv \overline{a}az_1 \equiv \overline{a}az_2 \equiv |a|^2 z_2 \mod \Lambda$. Since $|a|^2$ is an odd integer, we have $z_1 \equiv z_2 \mod \Lambda$. However, this is impossible. As a conclusion, $L^{-1}(\frac{\tau}{2})$ has exactly one element in the set $\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. On the other hand, we can observe that $f^{-1}(\wp(\frac{\tau}{2})) = \wp(L^{-1}(\frac{\tau}{2}))$ as $\frac{\tau}{2}$ is ramification point of $\wp$. Then $|f^{-1}(\wp(\frac{\tau}{2}))| = \frac{|a|^2-1}{2} + 1$. Since the degree of f is $|a|^2 > 1$, then $\wp(\frac{\tau}{2})$ is ramification value of f and the ramification index $r_f(\frac{\tau}{2})$ is $\frac{|a|^2-1}{2}$. Similar considerations hold for

$\frac{1}{2}, \frac{1+\tau}{2}$ and 0 and we have $r_f(\frac{1}{2}) = r_f(\frac{1+\tau}{2}) = r_f(0) = \frac{|a|^2-1}{2}$. Consequently, f has exactly 4 ramification values at $\wp(0), \wp(\frac{1}{2}), \wp(\frac{\tau}{2})$ and $\wp(\frac{1+\tau}{2})$. Finally, we can check these solutions by using Riemann-Hurwitz formula for $\widehat{\mathbb{C}}$; $2|a|^2 - 2 = 4\frac{|a|^2-1}{2}$.

□

Theorem 5.9. *If the degree of* Θ *is* 2 *and* $|a|^2$ *is even integer, then:*

(1) If $|a|^2 = 2$*, then* f *has exactly two ramification values,*

(2) If $|a| = 2$*, then* f *has exactly* 3 *ramification values and if* $|a|^2 = 4$ *but* $|a| \neq 2$*, then* f *has either three or four ramification values,*

(2) If $|a|^2 > 4$*, then* f *has exactly four ramification values.*

Proof. By the same argument in the proof of Proposition 5.8, we have that $\Theta = \wp$, f can be ramified over only $\wp(0), \wp(\frac{1}{2}), \wp(\frac{\tau}{2})$ and $\wp(\frac{1+\tau}{2})$ and if inverses of half-period and zero are not zero or half-period, then they must be pair. Since $2b \equiv 0 \bmod \Lambda$, b is equal either 0 or half-peirod. Now, we can start to find ramification values of f.

(1) Let $|a|^2 = 2$. Then $2|a|^2 - 2 = 2 = 1 + 1$. By the Riemann-Hurwitz formula, f has exactly two ramification values. Although we can not find what they are, we can make some observation about these points. Since $2b \equiv 0 \bmod \Lambda$, $L(0) \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. Suppose that $L(0) = 0$. Since the degree of L is 2, then 0 has two inverses. One of these is 0. Since the inverses of 0 distinct from half-periods and 0 are pair, other inverse of 0 must be half-peirod. Then $|f^{-1}(\wp(0))| = 2$. So $\wp(0)$ is not ramification value of f. Suppose that $\frac{\tau}{2} \in L^{-1}(0)$. Look at the $L^{-1}(\frac{\tau}{2})$. We have that $0, \frac{\tau}{2} \notin L^{-1}(\frac{\tau}{2})$. We can observe that $L^{-1}(\frac{\tau}{2})$ consist of other two half-period or none of them. If it consist of $\frac{1}{2}$ and $\frac{\tau}{2}$, then $\wp(\frac{\tau}{2})$ is not ramification value of f. Since f has two ramification value, $\frac{1}{2}$ and $\frac{1+\tau}{2}$ must be ramification values of f. If $L^{-1}(\frac{\tau}{2})$ does not contain any half-period, then their inverses must be pair. So $|f^{-1}(\wp(\frac{\tau}{2}))| = 1$. Hence, $\wp(\frac{\tau}{2})$ is ramification value of f. To find the other ramification value, we can look at the inverses of $\frac{1}{2}$ and $\frac{1+\tau}{2}$. Inverses of one of these elements can not contain any half-period. Hence, this element and $\frac{\tau}{2}$ is ramification values of f. Up to now in this proof, we have assumed that

$L(0) = 0$ and $L^{-1}(0) = \{0, \frac{\tau}{2}\}$. We can make similar considerations if $L(0)$ is equal to one of the half-periods. Consequently, we have the following observations;

- $\wp(L(0))$ is not ramification value of f and other inverse of $L(0)$ must be half-period,
- If inverse of $L(0)$ other than 0 is not ramification value, then remaining half-periods must be ramification values of f,
- If inverse of $L(0)$ other than 0 is ramification value of f, then one of the remaining half-periods must be ramification value. For this, we can look at the inverses of these elements.

(2) Let $|a| = 2$. Since $2b \equiv 0 \mod \Lambda$, $L(0) \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. Suppose that $L(0) = 0$. Then $b \equiv 0 \mod \Lambda$ and $L^{-1}(0) = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. This gives us $|f^{-1}(\wp(0))| = 4$. Therefore, $\wp(0)$ is not ramification value of f. Since L sends all half-periods and 0 to 0, then the set of inverses of half-periods can not contain any half-periods and 0 and their inverses must be pair. Then we have if w is half-period, $|f^{-1}(\wp(w))| = 2 < 4$. This means that half-periods are ramification values of f. So f has three ramification values. Similar considerations hold other possible value of 0 under L. In each cases, $\wp(L(0))$ is not ramification values of f and remaining two half-periods and 0 is ramification values of f. Then f has exactly three ramification values for each L. We can check that these values satisfy the Riemann-Hurwitz formula.

Let $|a|^2 = 4$ but $|a| \neq 2$. We have that $L(0) \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. Suppose that $L(0) = 0$. Then $L^{-1}(0)$ contains either one half-period or three half period. If it contains all half-periods, then f ramified over half-periods. Assume that $L^{-1}(0)$ contains only one half-period $w \in \{\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. Then inverses of 0 distinct from 0 and w under L must be pair. So $|f^{-1}(\wp(0))| = 3$ and the ramification index $r_f(0) = 1$. Then $\wp(0)$ is ramification values of f. Look at the inverses of half-periods. Sinde degree of L is 4, the set of inverses of half-periods can contain either two half-periods distinct from w or nothing. Suppose that the set of inverses of half-periods does not contain any half-period for each half-period. Then their ramification indexes are $4-2=2$. On the other hand, we have $1+2+2+2=7>2.4-2$. However, this is impossible by the Riemann-Hurwitz formula for $\widehat{\mathbb{C}}$. So the set of inverses of one of the half-periods must contain two remaining half-periods. Consequently, f has four ramification values, ramification indexes of two of them is 1 and two of them is 2. Similar considerations holds for other values of $L(0)$ and Riemann-Hurwitz formula is satisfied in each cases.

(3) Let $|a|^2>4$. We know that f can be ramified over four points; $0, \frac{1}{2}, \frac{\tau}{2}$ and $\frac{1+\tau}{2}$. Let $w \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. Then $|f^{-1}(\wp(w))| \leq \frac{|a|^2-4}{2}+4$. Since $|a|^2>4$, we have $|f^{-1}(\wp(w))|<|a|^2$. So $\wp(w)$ is ramification value of f as the degree of f is $|a|^2$. Consequently, half-periods and 0 are ramification values of f i.e. f ramified over 4 points.

□

Theorem 5.10. *If the degree of* Θ *is* 3*, then* f *has* 2 *ramification values whenever* $|a|^2=3$ *and* f *has* 3 *ramification values whenever* $|a|^2 \neq 3$.

Proof. Since the degree of Θ is 3, then $\Theta=\wp'$ by Theorem 5.4. We have that $\wp''(z)=6(\wp(z))^2$ for each $z \in \mathbb{C}/\Lambda$. So the ramification values of Θ are d_1, d_2 and 0 where d_1 and d_2 are roots of $\wp$. Then f can be ramified over $\wp'(d_1), \wp'(d_2)$ and $\wp'(0)$. By using Lemma 4.9, we can show that $3b \equiv 0 \mod \Lambda$. On the other hand, we can observe that if the inverses of d_1, d_2 and 0 are not d_1, d_2 or 0, then they must be a triple since the multiplicities of d_1, d_2 and 0 are 3. Suppose that $b \equiv 0 \mod \Lambda$. Then L sends d_1 and d_2 to same value and $0 \in L^{-1}(0)$. If $|a|^2=3k+1$ for some integer $|k|>0$, then $L^{-1}(0)$ can not contain d_1 and d_2. Look at the inverses of d_1 and d_2. These set can contain either both of them or nothing. These two cases are impossible since the inverses distinct from d_1, d_2 and 0 must be a trible. So $|a|^2 \neq 3k+1$ for all $k \in \mathbb{Z}$. Similarly, we can show that $|a|^2$ can not be equal $3k+2$ for all $k \in \mathbb{Z}$. So $|a|^2=3k$ for some integer $|k|>0$. Then $L^{-1}(0)$ must contain d_1 and d_2 and $L^{-1}(d_1)$ and $L^{-1}(d_2)$ can not contain d_1, d_2 and 0. Hence we have the followings;

- If $|a|^2 = 3$, then $|f^{-1}(\wp'(0))| = 3$, $|f^{-1}(\wp'(d_1))| = 1 < 3$ and $f^{-1}(\wp'(d_2)) = 1 < 3$. So ramification values of f are only $\wp'(d_1)$ and $\wp'(d_2)$. Since $2.3 - 2 = 2 + 2$, Riemann-Hurwitz formula is satisfied.
- If $|a|^2 > 3$, then $|f^{-1}(\wp'(0))| = \frac{|a|^2-3}{3} + 3 < |a|^2$, $|f^{-1}(\wp'(d_1))| = \frac{|a|^2}{3} < |a|^2$ and $f^{-1}(\wp'(d_2)) = \frac{|a|^2}{3} < |a|^2$. So ramification values of f are $\wp'(d_1)$, $\wp'(d_2)$ and $\wp'(0)$. We can check that these values satify the Riemann-Hurwitz formula.

Suppose that $b \not\equiv 0 \mod \Lambda$. Let $|a|^2 = 3k$ for some $|k| > 0$. Then either the set of inverses one of the d_1, d_2 and 0 should contain all these points or the sets of inverses d_1, d_2 and 0 can not contain these points. If the sets of inverses d_1, d_2 and 0 can not contain these points, then we have $r_f(\wp'(d_1)) = r_f(\wp'(d_2)) = r_f(\wp'(0)) = \frac{2|a|^2}{2}$. However, this is impossible by the Riemann-Hurwitz formula for $\widehat{\mathbb{C}}$. Then we have $L(d_1) = L(d_2) = L(0)$. Hence, if $|a|^2 = 3$, then $\wp'(L(0))$ is not ramification value of f and values of other two points under $\wp'$ are ramification values of f. If $|a|^2 > 3$, then $\wp'(d_1)$, $\wp'(d_2)$ and $\wp'(0)$ are ramification values of f.

Let $|a|^2 = 3k + 1$ for some integer $|k| > 0$. Then the set of inverses of d_1, d_2 and 0 contains only one of these elements. By using this, we can show that $\wp'(d_1)$, $\wp'(d_2)$ and $\wp'(0)$ are ramification values of f. Finally, we can observe that $|a|^2 \neq 3k + 2$ for each integer $|k| > 0$ by looking possible inverses of d_1, d_2 and 0 under L.

□

Theorem 5.11. *If the degree of* Θ *is* 4*, then* f *has* 2 *ramification values whenever* $|a|^2 = 2$ *and* f *has* 3 *ramification values whenever* $|a|^2 \neq 2$*. Furthermore,* $|a|^2 \neq 4k+3$ *for all integers* k*.*

Proof. If the degree of Θ is 4, then $\Theta = \wp^2$ by Theorem 5.4. So the ramification values of f must be equal to either image of 0 under $\wp^2$, images of half periods under $\wp^2$ or zeros of $\wp$ by Theorem 5.1. On the other hand, we have the equation $\wp'^3 = 4\wp^3 + 4a\wp$ in this case, see [14] for details. Then images of two half periods under $\wp^2$ are same and image of remaining half period is 0 by Theorem 5.1. So f can be ramified over at most 3 points; $\wp^2(0) = \infty$, $\wp^2(e_1) = \wp^2(e_2)$ and $\wp^2(e_3) = 0$ where e_1, e_2, e_3 are half periods.

Let $|a|^2 = 2$. By the Riemann-Hurwitz formula, f has exactly two ramification values. We can observe that the inverses of 0 and ∞ under $\wp^2$ contains only e_1 and 0, respectively. Then their images under L must be contain either both e_1 and e_2 or both 0 and e_3 since the degree of $\wp^2$ is 4. So $|f^{-1}(\wp^2(e_2))| = 1$ and $\wp^2(e_2)$ is ramification value of f. The other ramification value which contains e_1 and e_2 as inverses under L is one of the $\wp^2(e_3)$ and $\wp^2(0)$.

Let $|a| = 3$. Then inverses of 0 and e_3 must contain e_1 and e_2 since the degree of $\wp^2$ is 4 and inverses of 0 and ∞ under $\wp^2$ contains only e_1 and 0 respectively. However, this is impossible as L is well-defined map. So $|a|^2$ can not be equal to 3.

Let $|a|^2 \geq 4$. Look at $L^{-1}(0)$. Since the degree of Θ is 4 and the inverses of ∞ under $\wp^2$ contains only 0, the inverses of 0 under L distinct from half periods and 0 must be quartet and if the image of one of e_1 and e_2 is 0, then the image of the other must be 0. Now we can examine all possibilities of $L^{-1}(0)$. Suppose that $L^{-1}(0)$ contains 0 and half periods. Then we have $|f^{-1}(\infty)| = \frac{|a|^2-4}{4} + 3$, $|f^{-1}(\wp(e_1))| = \frac{2|a|^2}{4}$ and $|f^{-1}(0)| = \frac{|a|^2}{4}$. Since the degree of f is $|a|^2 \geq 4$, ∞, 0 and $\wp(e_1)$ are ramification values of f. Suppose that $L^{-1}(0)$ contains e_1, e_2 and one of e_3 and 0. Then $|f^{-1}(\infty)| = \frac{|a|^2-3}{4} + 2$ and either $|f^{-1}(0)| = \frac{|a|^2}{4}$ or $|f^{-1}(\wp(e_1))| = \frac{2|a|^2}{4}$. Since all these numbers must be an integer, this case is impossible. Suppose that $L^{-1}(0)$ contains e_1 and e_2. Then 0 and e_3 must be in either $L^{-1}(e_1) \cup L^{-1}(e_2)$ or $L^{-1}(e_3)$ by the Riemann Hurwitz formula and the fact that number of set of inverses must be an integer. Similarly, if $L^{-1}(0)$ contains 0 and e_3, then 0 and e_3 must be in either $L^{-1}(e_1) \cup L^{-1}(e_2)$ or $L^{-1}(e_3)$. In each two cases, ∞, 0 and $\wp(e_1)$ are ramification values of f. Suppose that $L^{-1}(0)$ contains one of the 0 and e_3. Then the inverses of e_1 and e_2 must contain e_1 and e_2 and the inverses of e_3 must contain either 0 or e_3 which is not in $L^{-1}(0)$ by the same reasons in previous case. So ∞, 0 and $\wp(e_1)$ are again ramification values of f. Finally, suppose that $L^{-1}(0)$ does not contain 0, e_1, e_2 and e_3. Then these four points must be in either $L^{-1}(e_1) \cup L^{-1}(e_2)$ or $L^{-1}(e_3)$ by the same reasons in last two cases and ∞, 0 and $\wp(e_1)$ are ramification values of f. As a summary, if $|a|^2 \geq 4$, we have the following table;

	$\|f^{-1}(\infty)\|$	$\|f^{-1}(\wp(e_1))\|$	$\|f^{-1}(0)\|$
$\|L^{-1}(0) \cap \{0, e_1, e_2, e_3\}\| = 4$	$\frac{\|a\|^2-4}{4} + 3$	$\frac{2\|a\|^2}{4}$	$\frac{\|a\|^2}{4}$
$\|L^{-1}(0) \cap \{0, e_1, e_2, e_3\}\| = 3$	impossible		
$\|L^{-1}(0) \cap \{0, e_1, e_2, e_3\}\| = 2$	$\frac{\|a\|^2-2}{4} + 1$	$\frac{2\|a\|^2}{4}$	$\frac{\|a\|^2-2}{4} + 2$
	$\frac{\|a\|^2-2}{4} + 2$	$\frac{2\|a\|^2}{4}$	$\frac{\|a\|^2-2}{4} + 1$
$\|L^{-1}(0) \cap \{0, e_1, e_2, e_3\}\| = 1$	$\frac{\|a\|^2-1}{4} + 1$	$\frac{2\|a\|^2-2}{4} + 1$	$\frac{\|a\|^2-1}{4} + 1$
$\|L^{-1}(0) \cap \{0, e_1, e_2, e_3\}\| = 0$	$\frac{\|a\|^2}{4}$	$\frac{2\|a\|^2}{4}$	$\frac{\|a\|^2-4}{4} + 3$
	$\frac{\|a\|^2}{4}$	$\frac{2\|a\|^2-4}{4} + 3$	$\frac{\|a\|^2}{4}$

Table 5.1.: Ramification behaviour of Lattès map arising from $\wp^2$

In each cases, f has exactly 3 ramification values and $|a|^2 \neq 4k + 3$ for each $k \in \mathbb{Z}$.

□

Theorem 5.12. *If the degree of Θ is 6, then f has 2 ramification values whenever $|a|^2 = 2$ and f has 3 ramification values whenever $|a|^2 \neq 2$. Furthermore, $|a|^2 \neq 6k+5$ for all integers k.*

Proof. If the degree of Θ is 6, then $\Theta = \wp^3$ by Theorem 5.4. So the ramification values of f must be equal to either image of 0 under $\wp^3$, images of half periods under $\wp^3$ or zeros of $\wp$ by Theorem 5.1. On the other hand, we have the equation $\wp'^3 = 4\wp^3 + 4b$ in this case, see [14] for details. Then images of the half periods under $\wp^3$ are same. So f can be ramified over at most 3 points; $\wp^3(0) = \infty$, $\wp^3(e_1) = \wp^3(e_2) = \wp^3(e_3)$ and $\wp^3(d_1) = \wp^3(d_2) = 0$ where e_1, e_2, e_3 are half periods and d_1, d_2 are roots of $\wp$.

By using similar considerations in the proof of previous theorem, we can obtain the following table;

	$\lvert f^{-1}(\infty)\rvert$	$\lvert f^{-1}(\wp(e_1))\rvert$	$\lvert f^{-1}(0)\rvert$
$\lvert L^{-1}(0)\cap\{0,e_1,e_2,e_3,d_1,d_2\}\rvert = 6$	$\frac{\lvert a\rvert^2-6}{6}+3$	$\frac{3\lvert a\rvert^2}{6}$	$\frac{2\lvert a\rvert^2}{6}$
$\lvert L^{-1}(0)\cap\{0,e_1,e_2,e_3,d_1,d_2\}\rvert = 5$	impossible		
$\lvert L^{-1}(0)\cap\{0,e_1,e_2,e_3,d_1,d_2\}\rvert = 4$	$\frac{\lvert a\rvert^2-4}{6}+2$	$\frac{3\lvert a\rvert^2}{6}$	$\frac{2\lvert a\rvert^2-2}{6}+1$
$\lvert L^{-1}(0)\cap\{0,e_1,e_2,e_3,d_1,d_2\}\rvert = 3$	$\frac{\lvert a\rvert^2-3}{6}+1$	$\frac{3\lvert a\rvert^2-3}{6}+2$	$\frac{2\lvert a\rvert^2}{6}$
	$\frac{\lvert a\rvert^2-3}{6}+2$	$\frac{3\lvert a\rvert^2-3}{6}+1$	$\frac{2\lvert a\rvert^2}{6}$
$\lvert L^{-1}(0)\cap\{0,e_1,e_2,e_3,d_1,d_2\}\rvert = 2$	$\frac{\lvert a\rvert^2-2}{6}+1$	$\frac{3\lvert a\rvert^2}{6}$	$\frac{2\lvert a\rvert^2-4}{6}+2$
$\lvert L^{-1}(0)\cap\{0,e_1,e_2,e_3,d_1,d_2\}\rvert = 1$	$\frac{\lvert a\rvert^2-1}{6}+1$	$\frac{3\lvert a\rvert^2-3}{6}+1$	$\frac{2\lvert a\rvert^2-2}{6}+1$
$\lvert L^{-1}(0)\cap\{0,e_1,e_2,e_3,d_1,d_2\}\rvert = 0$	$\frac{\lvert a\rvert^2}{6}$	$\frac{3\lvert a\rvert^2}{6}$	$\frac{2\lvert a\rvert^2-6}{6}+3$
	$\frac{\lvert a\rvert^2}{6}$	$\frac{3\lvert a\rvert^2-6}{6}+3$	$\frac{2\lvert a\rvert^2}{6}$

Table 5.2.: Ramification behaviour of Lattès map arising from $\wp^3$

By using this table, we can observe that if $|a|^2 = 2$, then the ramification values of f are ∞ and $\wp^3(e_1)$ and if $|a|^2 \neq 2$, then the ramification values of f are ∞, $\wp^3(e_1)$ and 0. Furthermore, $|a|^2$ can not be equal to $6k+5$ for each $k \in \mathbb{Z}$. □

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APPENDIX A: TOPOLOGY OF RIEMANN SURFACES

Let S and T be a Riemann surface. Consider the function space $C(S,T)$ consisting of all continuous maps from S to T. We want to define a topology on $C(S,T)$ which is known as **compact-open topology**.

Definition A.1. Let $C(S,T)$ be a function space consisting of all continuous maps from S to T where S is a locally compact space and (T,d) is a metric space. For all $f \in C(S,T)$, we define a **basic neighborhood** $U_{K,\varepsilon}(f)$ where K is compact subset of f of f as follows: $g \in U_{K,\varepsilon}(f)$ if and only if $d(f(s),g(s)) < \varepsilon$ for all $s \in K$.

Let $U \subseteq C(S,T)$. We say that U is **open** if and only if for every $f \in U$, there exist a compact subset K of S and $\varepsilon > 0$ such that the basic neighborhood $U_{K,\varepsilon} \subseteq U$.

Theorem A.2 ([11]). *$C(S,T)$ is a well-defined Hausdorff space with compact-open topology. A sequence of $f_i \in C(S,T)$ converges to the limit g if and only if*

(a) for every compact $K \subseteq S$, the sequence of maps $f_i|_K$ converges uniformly to $g|_K$,

or equivalently if and only if

(b) every element $s \in S$ has a neighborhood U such that the sequence of maps $f_i|_U$ converges uniformly to $g|U$.

This topology on $C(S,T)$ depends only on the topologies of S and T and not on the particular choice of metric for T. Furthermore, if S is σ-compact, then $C(S,T)$ is itself a metrizable topological space.

If we study on Riemann surfaces, $C(S,T)$ is always a metrizable topological space because of the following proposition;

Proposition A.3 ([11]). *Every Riemann surface admits a countable conformal metric with constant curvature.*

Other important property of Riemann surfaces is the anfollowing;

Proposition A.4 ([11]). *Every Riemann surface can be expressed as a countable union of compact subsets.*