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STABILITY OF BIMODAL SYSTEMS

GÖKHAN ŞAHAN

A THESIS SUBMITTED for THE DEGREE of

DOCTOR of PHILOSOPHY

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GEBZE YÜKSEK TEKNOLOJİ ENSTİTÜSÜ	**DOKTORA JÜRİ ONAY FORMU**

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İMZA/MÜHÜR

SUMMARY

In this dissertation, the structure and stability of bimodal systems in $\mathbb{R}^3$ are investigated. As a first step, it is shown that one of the assumptions being used reduces the stability problem in $\mathbb{R}^3$ to the stability problem in $\mathbb{R}^2$. Afterwards, this assumption is removed and apart from the results in $\mathbb{R}^2$, some interesting conclusions are obtained. However, structural analysis shows that the behavior of the trajectories changes radically upon the change of the parameters of individual subsystems (i.e. eigenvalues, system matrices entries,...). The approach taken is based on the classification of the trajectories of bimodal systems as

i) the trajectories which change mode finite number of times as $t \to \infty$,
ii) the trajectories which change mode infinite number of times as $t \to \infty$.

With the help of this classification, it is shown that the effect of mode changes strongly influence the global asymptotic stability of bimodal systems in $\mathbb{R}^3$. It is also shown that the trajectories which change mode infinite number of times converge to fixed directions (which may be attractive or repulsive) and stability cones.

Key Words: Switched Mode Systems, Bimodal Systems, Global Asymptotic Stability, Fixed Directions, Stability Cones.

ÖZET

Bu tezde, $\mathbb{R}^3$'teki iki modlu sistemlerin yapısı ve kararlılığı incelenmiştir. İlk aşamada, yapılan bir kabulle $\mathbb{R}^3$'teki kararlılık probleminin $\mathbb{R}^2$'ye indirgenebildiği gösterilmiştir. Sonrasında bu kabul kaldırılmış ve daha genel koşullar altında, $\mathbb{R}^2$'den farklı olarak ilginç sonuçlar elde edilmiştir. Elde edilen sonuçlar, sistem yörüngesinin davranışının bağımsız alt sistemlerdeki parametrelerin (alt sistem matrislerinin özdeğerlerinin ve bazı matris bileşenlerinin) değişikliklerine bağlı olarak hareket ettiğini göstermiştir. Kullanılan yaklaşım ikili sistemin yörüngelerinin aşağıdaki şekilde sınıflandırılmasıyla şekillenmiştir:

i) $t \to \infty$ için, sonlu defa mod değiştiren yörüngeler,
ii) $t \to \infty$ için, sonsuz defa mod değiştiren yörüngeler.

Bu yaklaşım yardımı ile, sonlu ya da sonsuz mod değişikliğinin $\mathbb{R}^3$'teki iki modlu sistemlerin global asimptotik kararlılığı üzerinde etkili olduğu ve bu davranış ile yörüngeleri çeken ya da iten sabit doğrultuların ya da konilerin oluştuğu kanıtlanmıştır.

Anahtar Kelimeler: Anahtarlamalı Sistemler, İki Modlu Sistemler, Global Asimptotik Kararlılık, Sabit Doğrultular, Kararlılık Konileri.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanation**
H	:	Seperating hyperplane.
H^+	:	The half space where the smooth continuation is possible to the 1st mode.
H^-	:	The half space where the smooth continuation is possible to the 2nd mode.
L_i	:	The line passing through the origin which divides H into two open half planes.
L_i^+, L_i^-	:	The half lines which is received via dividing line L_i by origin.
P_i^+, P_i^-	:	The half planes which is received via dividing plane H by the line L_i.
S_i	:	The set of initial conditions x_0 where smooth continuation is possible only in i^{th} mode.
τ_i	:	The time that the trajectory spends in one of the mode.
$\theta_i, V_i(\theta_i)$	:	The angle and the direction of the initial condition x_0
$\theta_i + w_i\tau_i$	:	The angle at the moment of the changing mode with regarding to the i^{th} mode.
r_i, x_i, y_i	:	Real and complex eigenvectors for the i^{th} mode.
CDC	:	Control and Decision Conference : An international conference organized by IEEE which is recognized as the premier scientific and engineering conference dedicated to the advancement of the theory and practice of systems and control.
GAS	:	Globally asymptotically stable .
GYTE		Gebze Yüksek Teknoloji Enstitüsü.
IEEE	:	Institute of Electrical and Electronics Engineers,one of the world's largest professional association dedicated to advancing technological innovation and excellence for the benefit of humanity

LIST of FIGURES

1. INTRODUCTION

Switched systems consists of a finite number of subsystem and a switching signal which organizes the transition of the trajectories from one subsystem to the other. Subsystems may be linear or nonlinear; continuous or discrete depending on the choice of the model. Switching signal may be time dependent or autonomous (state dependent). Since Linear Time Invariant (LTI) systems are well known in the literature, most of the research done on switched systems are on LTI switched systems. The reader may refer to books [Johansson, 2002], [Liberzon, 2003] recent survey papers [Shorten et al., 2007], [Lin and Antsaklis, 2009], [Sun, 2010] for the details of the research on switched systems.

Bimodal Systems is a subclass of switched linear systems where the switching is autonomous and there are only two subsystems. One of the important issues in bimodal systems is the existence and uniqueness of solutions. This issue, often called well-posedness, have been addressed by [Imura and Schaft, 2000] and necessary and sufficient conditions for bimodal systems to be well-posed have been derived. Another essential issue for bimodal systems is global asymptotic stability GAS. GAS of bimodal systems have been investigated both using direct methods and Lyapunov methods. Using direct methods, the necessary and sufficient conditions for global asymptotic stability of bimodal systems in $\mathbb{R}^2$ is provided by [Çamlıbel et al., 2003] and [Iwatani and Hara, 2006]. Closely related to bimodal systems in $\mathbb{R}^2$ are planar systems, which consist of more than two subsystems all in $\mathbb{R}^2$. The reader may refer to [Lin and Antsaklis, 2009], [Sun, 2010], [Iwatani and Hara, 2006], [Xu and Antsaklis, 2000] for the stability results on planar systems. Stability of bimodal systems have also been investigated via Lyapunov methods in [Mori et al., 1997], [Shorten et al., 2004], [Shorten and Narendra, 2002].

Bimodal systems exhibit a rich dynamic behavior which is demonstrated in [Iwatani and Hara, 2006] via various examples of bimodal systems in $\mathbb{R}^2$. In $\mathbb{R}^3$, their behavior becomes more complex and surprising. For instance, both modes could be stable but the bimodal system may be unstable. Conversely, both modes could be unstable, but the bimodal system may be stable. As pointed out in [Carmona et al., 2005], capturing such properties is a nontrivial problem. The following example demonstrates the complexity of the behavior of bimodal systems. Let

- $\dot{x}=\begin{cases} A_1 x(t) & \text{if} \quad c^T x(t) \geq 0 \\ A_2 x(t) & \text{if} \quad c^T x(t) \leq 0 \end{cases}$ where A_1 and A_2 are such 3x3 matrices

that $A_1=\begin{bmatrix} 2 & -13 & 0 \\ 1 & -4 & 0 \\ 0 & 1 & -0.8 \end{bmatrix}$, $A_2=\begin{bmatrix} 0 & 0 & -3.616 \\ 1 & 0 & -9.2 \\ 0 & 1 & -0.8 \end{bmatrix}$, $c^T=[0 \quad 0 \quad 1]$.

Note that A_2 is in observable canonical form with the spectrum $\{-\lambda_2, \sigma_2, w_2\} = \{-0.4, -0.2, 3\}$ and the spectrum of A_1 is $\{-\lambda_1, \sigma_1, w_1\}=\{-0.8, -1, 2\}$. This system is unstable. However, if we keep A_2 as the same but change some of the entries of A_1 as follows,

- $A_1=\begin{bmatrix} 1 & -8 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & -0.8 \end{bmatrix}$.

Then the spectrum of A_1 is still $\{-\lambda_1, \sigma_1, w_1\}=\{-0.8, -1, 2\}$. However, bimodal system with this new A_1 is globally asymptotically stable. This implies that eigenvalues of individual subsystems can not determine global asymptotic stability by themselves and other tools are necessary. We will elaborate more on this example towards the end of the thesis.

[Carmona et al., 2005] have also considered the stability of bimodal systems in $\mathbb{R}^3$ by transforming the system to the surface of the unit sphere in $\mathbb{R}^3$ centered at the origin. In this framework, they checked for periodic solutions which would be equivalent to the search for invariant cones for the original bimodal system. The main result of this paper (Theorem 2 in [Carmona et al., 2005]) gives sufficient conditions (in terms of the eigenvalues of both modes) for the existence of invariant cones and the stability of the trajectories dwelling in these cones. [Iwatani and Hara, 2006] provided separate necessary and sufficient conditions (Theorem 19 in [Iwatani and Hara, 2006]) for bimodal systems in $\mathbb{R}^n$ where n>2. The necessary conditions given in [Iwatani and Hara, 2006] are trivial. However, the sufficient conditions are very restrictive as they require that the observability index of one of the subsystems to be ≤ 2. This result can not be used in our case as we assume that the observable index is three for both subsystems. As far as we know, conditions which are both necessary and sufficient for global asymptotic stability of bimodal systems in $\mathbb{R}^n$ where n > 2 are not known yet.

Our objective in this dissertation is to investigate the structure and global asymptotic stability of bimodal systems in R^3 and provide the conditions which are both necessary and sufficient for global asymptotic stability. Along this line, we first study the geometric structure induced by eigenvalues and eigenvectors of individual subsystems on the plane separating two subsystems. This dissertation also yields an alternative (to the conditions given by [Imura and Schaft, 2000]) set of necessary and sufficient conditions for bimodal systems to be well-posed in our setup. Next, we investigate the behavior of trajectories as they start and evolve in one of the modes. We classify these trajectories as the ones which change mode in a finite time and the ones which do not change mode. The behavior of the trajectories which change mode in a finite time are further investigated after mode change. Firstly, this is made under an assumption which simplifies the geometric structure of the bimodal system and as a following step, this condition is relaxed. This yields a final classification of the trajectories as

i) the trajectories which change mode finite number of times as $t \to \infty$,
ii) the trajectories which change mode infinite number of times as $t \to \infty$.

Then, we show that the trajectories which change mode infinite number of times (as $t \to \infty$), converges to fixed directions on the switching plane and the trajectories which change mode finite number of times (as $t \to \infty$) are stable in our setup when the real eigenvalues are negative. Finally, we prove that bimodal system is globally asymptotically stable if and only if the trajectories starting from the fixed directions are stable (decay to the origin as $t \to \infty$). The stability of the trajectories starting from fixed directions depend on the rate of convergence defined later in the dissertation and can be calculated easily in our setup. It turns out that one of the assumptions which simplifies the structure of bimodal systems in $\mathbb{R}^3$, reduces the stability condition in $\mathbb{R}^3$ to the stability condition of bimodal systems in $\mathbb{R}^2$. However, the classification of the trajectories and attractiveness of the fixed directions changes substantially as subsystem parameters change.

An incomplete preliminary version of this paper with both modes stable, was presented at [Eldem and Sahan, 2009]. GAS, location and attractiveness of fixed directions for a special case (B_1=0) is accepted for publication [Eldem and Sahan, to appear]. This case reduces the stability conditions of $\mathbb{R}^3$ to $\mathbb{R}^2$ which is known well in the literature, [Çamlıbel et al., 2003], [Iwatani and Hara, 2006].

2. STRUCTURAL PROPERTIES

A bimodal LTI system in $\mathbb{R}^3$ can be defined as

$$\Sigma_0 := \dot{x}(t) = \begin{cases} A_1 x(t) & \text{if } c^T x(t) \geq 0 \\ A_2 x(t) & \text{if } c^T x(t) \leq 0 \end{cases} \tag{2.1}$$

where x, c ∈ $\mathbb{R}^3$ and A_1 and A_2 are matrices in $\mathbb{R}^{3\times 3}$. Furthermore, $\dot{x}(t) := \frac{d}{dt}(x(t))$. This system is said to be globally asymptotically stable (GAS) if every possible trajectory x(t) decays to the origin as $t \to \infty$.

2.1. Geometry of Initial Conditions in $\mathbb{R}^3$ and Well-Posedness

Note that the plane $H := \{x \mid c^T x = 0\}$ divides R^3 into two open half-spaces, H^+ and H^- defined as $H^+ := \{x \mid c^T x > 0\}$, $H^- := \{x \mid c^T x < 0\}$. Then, for any initial condition $x_0 \in H^+$ (H^-) only mode 1 (mode 2) is active, i.e., $\dot{x}(t) = A_1 x(t)$ ($\dot{x}(t) = A_2 x(t)$). In view of the theory of differential equations, for any initial condition $x_0 \in H^+$ (H^-), there exists $\varepsilon > 0$ and a local solution $x(t, x_0)$ such that $c^T x(t, x_0) > 0$ ($c^T x(t, x_0) < 0$) for all $t \in [0, \varepsilon]$. Since only one of the modes is active, the solution is unique and this is smooth continuation in H^+ (H^-) as defined in [Imura and Schaft, 2000]. However, for trajectories starting from H, we can not claim uniqueness of solutions because both modes are allowed to be active on H. Furthermore, if one of the pairs is unobservable, there will be trajectories which start on H and stay on H for all $t \geq 0$. In order to eliminate this case, we assume that

- A1: the pairs (c^T, A_1) and (c^T, A_2) are observable.

Observability of the pair (c^T, A_i) implies that $\dim(\ker(c^T) \cap \ker(c^T A_i)) = 1$. Thus, $L_i := \ker(c^T) \cap \ker(c^T A_i)$ is a line passing through the origin which divides H into two open half planes, P_i^+ and P_i^-. On one side of this line $c^T A_i x > 0$ (P_i^+) and on the other side $c^T A_i x < 0$ (P_i^-). Similarly, the origin $\ker(c^T) \cap \ker(c^T A_i) \cap \ker(c^T A_i^2)$)

divides L_i into two open half lines L_i^+ and L_i^-, where $c^T A_i^2 x > 0$ if x is on L_i^+ and $c^T A_i^2 x < 0$ if x is on L_i^-.

Let S_i denote the set of initial conditions x_0 where the solution $x(t,x_0)$ is unique and $x(t,x_0) \in S_i$ for all $t \in [0,\varepsilon]$ for some $\varepsilon > 0$. In view of Definition 2.3 and Lemma 2.1 given in [Imura and Schaft, 2000], S_1 (or S_2) is the set of initial conditions where smooth continuation is possible only in 1st (or 2nd) mode. It is easy to see that if $c^T x > 0$ ($c^T x < 0$), then $x \in S_1$ ($x \in S_2$). Hence, S_1 and S_2 are nonempty. In order to complete the characterization of the sets S_1 and S_2, we have to determine what happens on H. Towards this end, we first recall that well-posedness simply means the existence and the uniqueness of solutions for a given dynamical system. By existence, we mean existence of a solution in the sense of Carathéodory which is defined as

$$x(t)=x_0+\int_{t_0}^{t} f\big(x(\tau)\big)d\tau, \tag{2.2}$$

where $\dot{x}(t)=f(x(t))$ and $x(t_0)=x_0$. This leads to the following definitions of well-posedness given in [Imura, 2002].

Definition 2.1: If, for a given initial state $x(t_0)$, $x(t)$ satisfies Carathéodory equation given above on $[t_0,t_1)$ and there is no left-accumulation point of event times on $[t_0,t_1)$, then $x(t)$ is said to be a continuous-state solution (simply called a C-solution) of $\sum_0$ on $[t_0,t_1)$ in the sense of Carathéodory.

Definition 2.2: Bimodal system given by equation (2.1) is said to be well-posed in the continuous-state sense (simply, C-well-posed) if for every initial state $x(0) \in \mathbb{R}^n$ there exists a unique C-solution of $\sum_0$ on $[0,\infty)$.

Lemma 2.1: Suppose that A1 hold, (c^T, A_2) is in observable canonical form and, A_1, A_2 and c^T are as given below

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ A_2=\begin{bmatrix} 0 & 0 & m_0 \\ 1 & 0 & m_1 \\ 0 & 1 & m_2 \end{bmatrix}, c^T=[0 \quad 0 \quad 1]. \tag{2.3}$$

Then, bimodal system given by equation (2.3) is C-well-posed if and only if

- $\ker c^T \cap \ker(c^T A_1) = \ker c^T \cap \ker(c^T A_2)$ *(or equivalently* $a_{31} = 0$ *) and*
- $a_{32}, a_{21} > 0$.

Proof 2.1: (Necessity) Suppose that $\sum_0$ *is C-well-posed. Then, for any initial condition in H,* $\frac{d}{dt}(c^T x)\Big|_{t=0}$ *must have the same sign for both modes. Otherwise, either there will be two solutions (*$\frac{d}{dt}(c^T x)\Big|_{t=0} > 0$ *for the first mode and* $\frac{d}{dt}(c^T x)\Big|_{t=0} < 0$ *for the second mode) or there will be no solutions in the sense of Carathéodory* $\frac{d}{dt}(c^T x)\Big|_{t=0} < 0$ *for the first mode and* $\frac{d}{dt}(c^T x)\Big|_{t=0} > 0$ *for the second mode). In view of this observation let* $x(0) = [\,\gamma_1\ \gamma_2\ \ 0\,]^T$ *where* γ_1, γ_2 *are arbitrary real numbers. Calculating the derivative of* $c^T x$ *for both modes, we get*

$$\frac{d}{dt}(c^T x)\Big|_{t=0} = \begin{Bmatrix} c^T A_1 x(0) \\ c^T A_2 x(0) \end{Bmatrix} = \begin{Bmatrix} a_{31}\gamma_1 + a_{32}\gamma_2 \\ \gamma_2 \end{Bmatrix} \tag{2.4}$$

Note that the sign of $a_{31}\gamma_1 + a_{32}\gamma_2$ *can be changed arbitrarily by appropriate choice of the real number* γ_1*. This contradicts well-posedness. Thus, it is necessary that* $a_{31} = 0$*, or equivalently*

$$\ker c^T \cap \ker(c^T A_2) \subset \ker c^T \cap \ker(c^T A_1) \tag{2.5}$$

Since both modes are observable, it follows that $\dim(\ker c^T \cap \ker(c^T A_2)) = \dim(\ker c^T \cap \ker(c^T A_1))$*. Consequently, we get* $\ker c^T \cap \ker(c^T A_2) = \ker c^T \cap \ker(c^T A_1)$*. In view of this result,* $\frac{d}{dt}(c^T x)\Big|_{t=0}$ *can be written as follows*

$$\frac{d}{dt}(c^T x)\Big|_{t=0} = \begin{Bmatrix} c^T A_1 x(0) \\ c^T A_2 x(0) \end{Bmatrix} = \begin{Bmatrix} a_{32}\gamma_2 \\ \gamma_2 \end{Bmatrix}. \tag{2.6}$$

Note that if $a_{32} = 0$*, then the sign of* $c^T A_1 x(0)$ *is fixed and the sign of* $c^T A_2 x(0)$ *can be changed arbitrarily by an appropriate choice of* γ_2*. On the other hand, if*

a_{32}<0 and γ_2>0, then $c^T A_1 x(0)$<0 and $c^T A_2 x(0)$>0 which implies that there are no solutions in the sense of Carathéodory. Thus, $c^T A_1 x(0)$ and $c^T A_2 x(0)$ has the same sign if and only if a_{32}>0.

The above results guarantees well-posedness for every initial condition in H except in H_1:=kerc^T ∩ ker($c^T A_1$). It is clear that for any initial condition in H_1 we have to check the sign of second derivative. Towards this end, let

$$x(0)=[\,\gamma_1 \;\; 0 \;\; 0\,]^T \tag{2.7}$$

In this case, we have

$$\frac{d^2}{dt^2}(c^T x)\Big|_{t=0} = \begin{Bmatrix} c^T A_1^2 x(0) \\ c^T A_2^2 x(0) \end{Bmatrix} = \begin{Bmatrix} a_{21} a_{32} \gamma_1 \\ \gamma_1 \end{Bmatrix}. \tag{2.6}$$

If we set a_{21}=0, then as in the previous case, the sign of $c^T A_1^2 x(0)$ is fixed and the sign of $c^T A_2^2 x(0)$ can be changed arbitrarily. Similarly, if a_{21}<0 and γ_1>0, then $c^T A_1^2 x(0)$<0 and $c^T A_2^2 x(0)$>0 which again implies that there are no solutions in the sense of Carathéodory. Thus, $c^T A_1^2 x(0)$ and $c^T A_2^2 x(0)$ have the same sign if and only if a_{21}>0. Since both of the pairs (c^T, A_1) and (c^T, A_2) are observable, it follows that kerc^T ∩ ker ($c^T A_2$) ∩ ker ($c^T A_2^2$) = ker c^T ∩ ker ($c^T A_1$) ∩ ker ($c^T A_1^2$) = {0}.

The proof of sufficiency essentially follows similar lines in reverse order and therefore will be omitted here.

In view of the preceding Lemma, our next assumption is as follows.

- A2: kerc^T ∩ker($c^T A_1$) = kerc^T ∩ker($c^T A_2$) (or equivalently a_{31}=0) and a_{32}, a_{21}>0.

The geometry induced after the assumptions A1 and A2 and without the assumptions A1 and A2 are depicted in Fig.1 and 2 below.

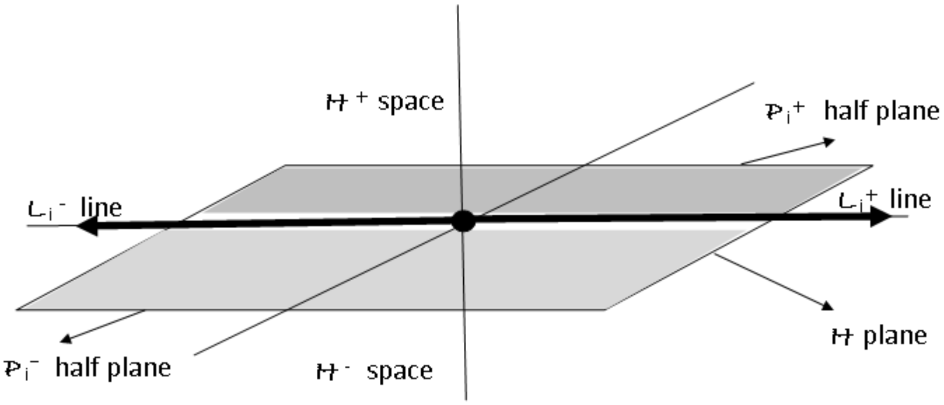

Figure 2.1: Plane geometry for half spaces.

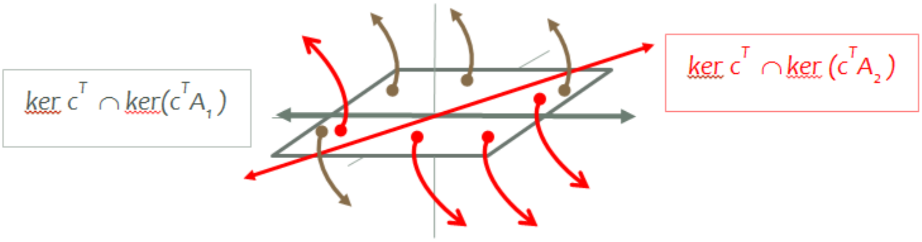

Figure 2.2 : Plane geometry without assumption 2.

Remark 2.1: The necessary and sufficient conditions for C-well-posedness of bimodal LTI systems is first given by [Imura and Schaft, 2000]. The conditions given above are equivalent to conditions given in [Imura and Schaft, 2000] and constitutes an alternative way of expressing the conditions of C-well-posedness. Furthermore, the above result is first proven for bimodal LTI systems in $\mathbb{R}^3$ in [Eldem and Sahan, to appear] and its extension to bimodal systems in $\mathbb{R}^n$ is given as follows.

Theorem 2.1: [Eldem and Sahan, to appear] A bimodal system in $\mathbb{R}^n$ where both of the pairs (c^T, A_1) and (c^T, A_2) are observable and the pair (c^T, A_2) is in observable canonical form, is well posed if and only if

$$\bigcap_{i=1}^{k} ker\left(c^T A_1^{i-1}\right) = \bigcap_{i=1}^{k} ker\left(c^T A_2^{i-1}\right) \quad for\ \ k=1,2,\ldots n \qquad (2.9)$$

and $a_{i+1,i} > 0$ for $i = 1,2, \ldots, n-1$ where $A_1 := \{ a_{ij} \}$.

Theorem 2.1 was given as a conjecture in [Eldem and Sahan, to appear] but the validity of this conjecture for bimodal systems in $\mathbb{R}^n$ is shown in [Sahan and Eldem, submitted]. The subspaces given in the equation above have also been used by [Ferrer et al., 2002] in obtaining reduced forms of bimodal systems.

Remark 2.2: We should also remark here that bimodal systems where both subsystems are in observable canonical form are also investigated in the literature. Assuming that both subsystems are in observable canonical form (as given above by A_2) is equivalent to assuming that the vector field is continuous on H. In our case, the vector field is not necessarily continuous on H.

In R^3, each mode has at least one real eigenvalue. If this eigenvalue is positive, then any trajectory that starts with an initial condition equal to the eigenvector of the real eigenvalue, will stay in the same mode for all $t \geq 0$ and go to infinity. Such bimodal systems are, thus not GAS. Therefore, our investigation in R^3 is restricted to the cases where the real eigenvalues are negative. We further exclude the case where all the eigenvalues are real, because in this case there will be a cone of initial conditions (generated by nonnegative linear combinations of the eigenvectors) such that any trajectory starting with an initial condition in this cone will decay to the origin without changing mode. Since our aim is to focus on the effect of mode changes on the dynamic behavior of the system, we leave this case for future work and consider the case where there is only one real eigenvalue and a conjugate pair of imaginary eigenvalues in both modes. Thus, we assume that

- A3: the eigenvalues of A_i's are $\{-\lambda_i, \sigma_i \pm jw_i\}$ where λ_i, σ_i and w_i are real numbers with $\lambda_i, w_i > 0$ for i=1,2.

In the remaining part of this thesis, it will be assumed that A1 - A3 always hold and (c^T, A_2) is in observable canonical form. In this case, system matrices are

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & -\lambda_2(\sigma_2^2 + w_2^2) \\ 1 & 0 & 2\sigma_2\lambda_2 - (\sigma_2^2 + w_2^2) \\ 0 & 1 & 2\sigma_2 - \lambda_2 \end{bmatrix}, c^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad (2.10)$$

where a_{32} and $a_{21}>0$.

2.2. Behavior of the Trajectories in One Mode

In this section, we investigate the behavior of the trajectories as they start and smoothly continue in one mode until they change mode.

Lemma 2.2: Let $\{r_i\}$ and $\{x_i \pm jy_i\}$ ($i=1,2$) denote the real and complex eigenvectors of A_i. Then, the eigenvectors can be uniquely chosen such that

$$c^T x_i = c^T r_i = (-1)^{i+1}, \quad c^T y_i = 0 , \; c^T A_1 y_1 > 0 \text{ and } c^T A_2 y_2 < 0, \; (i=1,2) \tag{2.11}$$

where

$$r_1 = \begin{bmatrix} \frac{(a_{11}-\sigma_1)^2 + w_1^2 + a_{12}a_{21}}{a_{32}a_{21}} \\ \frac{-\lambda_1 - a_{33}}{a_{32}} \\ 1 \end{bmatrix}, \quad [x_1 \quad y_1] = \begin{bmatrix} \frac{(a_{11}+\lambda_1)(a_{11}-\sigma_1)+a_{12}a_{21}}{a_{32}a_{21}} & \frac{w_1(a_{11}+\lambda_1)}{a_{32}a_{21}} \\ \frac{\sigma_1 - a_{33}}{a_{32}} & \frac{w_1}{a_{32}} \\ 1 & 0 \end{bmatrix} \tag{2.12}$$

$$r_2 = \begin{bmatrix} -(\sigma_2^2 + w_2^2) \\ 2\sigma_2 \\ 1 \end{bmatrix}, \quad [x_2 \quad y_2] = \begin{bmatrix} \sigma_2\lambda_2 & -w_2\lambda_2 \\ -(\lambda_2 - \sigma_2) & -w_2 \\ 1 & 0 \end{bmatrix} \tag{2.13}$$

Proof 2.2: Since the second mode is in observable canonical form as given in equation (2.10), using straightforward calculations, it can be easily shown that

$$A_2 r_2 = -\lambda_2 r_2 \text{ and } A_2[x_2 \quad y_2] = [x_2 \quad y_2]\begin{bmatrix} \sigma_2 & w_2 \\ -w_2 & \sigma_2 \end{bmatrix}, \tag{2.14}$$

and equations (2.11) and (2.13) hold for i=2. Similarly, using the following equality

$$A_1 [r_1 \quad x_1 \quad y_1] = [-\lambda_1 r_1 \;\; \sigma_1 x_1 - w_1 y_1 \;\; w_1 x_1 + \sigma_1 y_1] \tag{2.15}$$

where A_1 is as given in equation (2.10) and also recalling that $detA_1 = -\lambda_1(\sigma_1^2 + w_1^2)$ and $trace(A_1) = 2\sigma_1 - \lambda_1$, it can also be easily shown that equations (2.11) and (2.12) hold for i=1.

In view of the preceding result, it is clear that any initial condition in R^3 can be expressed as a linear combination of the eigenvectors of each mode as $\delta_i r_i + \beta_i x_i + \gamma_i y_i$ where δ_i, β_i and γ_i are real numbers. Thus, we can use two different bases for R^3. The set of trajectories that start out with initial conditions $\delta_i r_i$ where $\delta_i > 0$, will decay to the origin without going into the other mode. Since these are already stable trajectories in our setup, we need to investigate only the trajectories with nontrivial sinusoidal parts. More specifically, let $z_i(t)$ (i=1,2) denote the trajectories starting from S_i and smoothly continuing into S_i with initial conditions such that either $\beta_i \neq 0$ and/or $\gamma_i \neq 0$. Then, the behavior of such trajectories in the i^{th} mode can be written as

$$z_i(t) = K_i \{\alpha_i \exp(-\lambda_i t) r_i + \exp(\sigma_i t) [\sin(\theta_i + w_i t) x_i + \cos(\theta_i + w_i t) y_i]\}, \qquad (2.16)$$

for i = 1, 2 where $K_i := (\beta_i^2 + \gamma_i^2)^{1/2} > 0$, $\alpha_i := \frac{\delta_i}{(\beta_i^2+\gamma_i^2)^{1/2}} > 0$, $\sin\theta_i := \frac{\beta_i}{(\beta_i^2+\gamma_i^2)^{1/2}} > 0$, $\cos\theta_i := \frac{\gamma_i}{(\beta_i^2+\gamma_i^2)^{1/2}} > 0$. This implies that,

$$c^T z_i(t) = K_i \{\alpha_i \exp(-\lambda_i t) + \exp(\sigma_i t) \sin(\theta_i + w_i t)\} = K_i \exp(-\lambda_i t)\{f_i(t)\} \qquad (2.17)$$

$$\{f_i(t)\} := \alpha_i + \exp((\lambda_i + \sigma_i) t) \sin(\theta_i + w_i t) \qquad (2.18)$$

Lemma 2.3: Consider the trajectories given by equation (2.16). Let

$$\hat{x}_i := x_i - r_i \text{ and } b_i := \cot\phi_i := \frac{\lambda_i + r_i}{w_i} \text{ for } i = 1, 2. \qquad (2.19)$$

Suppose that $x_0 \in H$ *and* $z_i(0) = x_0$. *Then,* $z_i(t)$ *smoothly continues into* H^+ *for* i=1 (H^- *for* i=2) *if and only if* $z_i(0) = K_i (\sin(\theta_i) x_i + \cos(\theta_i) y_i)$ *where* $K_i > 0$ *as defined in equation (2.16) and* $-\varphi_i \leq \theta_i < \pi - \varphi_i$.

Proof 2.3: Since $x_0 \in H$, *it follows that* $c^T z_i(0) = 0$. *Then equations (2.17) and (2.18) imply that* $\alpha_i = -\sin(\theta_i)$. *Consequently, we get* $z_i(0) = K_i (\sin(\theta_i) x_i + \cos(\theta_i) y_i)$ *where* x_i *is as defined in equation (2.19). If* i = 1, *then the trajectory smoothly continues into the first mode if* $c^T A_1 z_1(0) > 0$. *Calculating* $c^T A_1 z_1(0)$ *we get*

$$0 \leq c^T A_1 z_1(0) = K_1 \, [(\lambda_1+\sigma_1)sin(\theta_1)+w_1cos(\theta_1)] = K_1M_1sin(\theta_1+\varphi_1) \qquad (2.20)$$

where φ_1 is defined as in equation (2.12) and

$$M_1 := [(\lambda_1+\sigma_1)^2+w_1^2]^{1/2}. \qquad (2.21)$$

Consequently, $c^T A_1 z_1(0) > 0 \Leftrightarrow -\phi_1 < \theta_1 < \pi - \phi_1$. Note that at $\theta_1 = -\phi_1$ we have $c^T A_1 z_1(0) = 0$. In this case we have to check the sign of $c^T A_1^2 z_1(0)$ for smooth continuation into H^+. Calculating $c^T A_1^2 z_1(0)$, we obtain

$$\frac{d^2}{dt^2}\left(c^T z_1(t)\right) = c^T A_1^2 z_1(0) = \left(a_{11}(\sigma_1+\lambda_1) - \sigma_1\lambda_1 - (\sigma_1^2+w_1^2)\right) sin\theta_1 \qquad (2.22)$$
$$+w_1(a_{11}+\lambda_1)cos(\theta_1) + M_1(2\sigma_1 - a_{11} - \lambda_1)sin(\theta_1+\phi_1)$$

This implies that at $\theta_1 = -\phi_1$ we have

$$\begin{aligned} c^T A_1^2 z_1(0) &= \left(a_{11}(\sigma_1+\lambda_1) - \sigma_1\lambda_1 - (\sigma_1^2+w_1^2)\right)sin(-\phi_1) \qquad (2.23)\\ &\quad +w_1(a_{11}+\lambda_1)cos(-\phi_1)\\ &= \left(-a_{11}(\sigma_1+\lambda_1) + \sigma_1\lambda_1 + (\sigma_1^2+w_1^2)\right)sin(-\phi_1)\\ &\quad +w_1(a_{11}+\lambda_1)cos(\phi_1)\\ &= \frac{(\lambda_1^2+w_1^2+\sigma_1^2)+2\sigma_1\lambda_1}{\left[\frac{(\lambda_1+\sigma_1)^2}{w_1^2}+1\right]^{1/2}} = \frac{(\lambda_1+\sigma_1)^2+w_1^2}{\left[\frac{(\lambda_1+\sigma_1)^2}{w_1^2}+1\right]^{1/2}} > 0 \end{aligned}$$

and hence, $c^T A_1^2 z_1(0) > 0$. Using similar reasoning with mode 2, it can be easily shown that

$$\begin{aligned} c^T A_2 z_2(0) &= -K_2[(\lambda_2+\sigma_2)sin(\theta_2) + w_2cos(\theta_2)] \qquad (2.24)\\ &= -K_2[(\lambda_2+\sigma_2)^2 + w_2^2]^{1/2} sin(\theta_2+\phi_2) \end{aligned}$$

and $c^T A_2 z_2(0) < 0 \Leftrightarrow -\phi_2 < \theta_2 < \pi - \phi_2$. It can also be shown that $c^T A_2^2 z_2(0) < 0$ at $\theta_2 = -\phi_2$. Consequently, we have $-\phi_2 \leq \theta_2 < \pi - \phi_2$ and this concludes the proof.

In view of preceding Lemma, any initial condition $x_0 \in H$ can be written as $x_0 = K_i V_i(\theta_i)$ where $K_i > 0$ and $V_i(\theta_i)\!: = \sin(\theta_i)\hat{x}_i + \cos(\theta_i) y_i$. This follows from the fact that x_i and y_i are linearly independent and they are two different bases of H for i=1,2.

Definition 2.3: In the sequel, we refer to $V_i(\theta_i)$ *as directions. It is clear that if* $V_1(\theta_1)$ *is a direction, then there exists* θ_2 *and a constant* $\eta_{12}(\theta_1)>0$ *such that*

$$V_1(\theta_1)=\eta_{12}(\theta_1)V_2(\theta_2). \tag{2.25}$$

We shall refer to $V_1(\theta_1)$ *and* $V_2(\theta_2)$ *as equivalent directions and use the notation* $V_1(\theta_1)\simeq V_2(\theta_2)$*. Furthermore, we shall also use the terminology a trajectory starting from direction* $Vi(\theta i)$*in order to refer to a trajectory starting on H with initial condition* $z_i(0) = K_i V_i(\theta_i)$ *where* $K_i > 0$ *is a real constant.*

2.3. Classification of Trajectories

In this section, we use the following definition in order to classify the trajectories as transitive and nontransitive. A slightly different version of this definition is originally given in [Iwatani and Hara, 2006] as transitive and weakly transitive. Here, we prefer to use transitive and nontransitive.

Definition 2.4: Let $z_i(t)$ *be a trajectory as given by equation (2.16). If there exists finite time* $\tau_i > 0$ *where* $c^T z_i(\tau_i) = 0$ *and the trajectory changes mode at* $t = \tau_i$*, then such trajectories are called transitive. Otherwise, they are called nontransitive.*

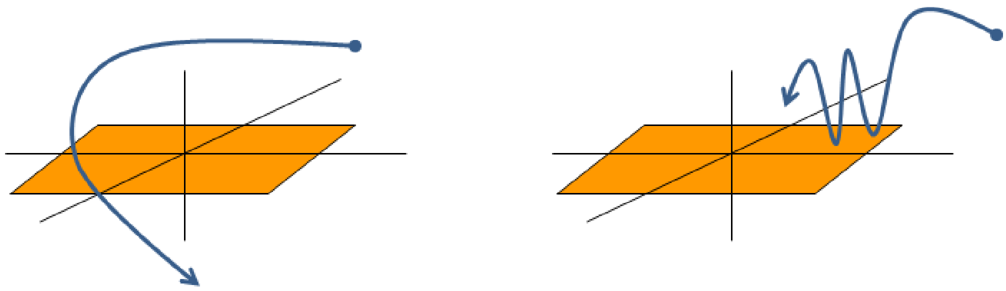

Figure 3: Transitive and nontranstive trajectories.

Lemma 2.4: Consider the bimodal system (2.10) and its trajectories given by equation (2.16). Then the following hold.

i) If $\lambda_i+\sigma_i > 0$, then all possible trajectories $z_i(t)$ are transitive. More precisely, there exists a finite $\tau_i > 0$ such that $c^T z_i(\tau_i)=0$ and the trajectory changes mode at $t=\tau_i$.

ii) If $\lambda_i+\sigma_i \leq 0$, then there exists a unique angle $\phi_i \in (0, \varphi_i)$ such that

- *If $\theta_i \in [-\varphi_i, -\phi_i]$, then the trajectories starting from $V_i(\theta_i)$ are nontransitive.*
- *If $\theta_i \in (-\phi_i, \pi-\varphi_i)$ and $\alpha_i \geq \alpha_{i0}$, then the trajectories starting from $V_i(\theta_i)$ are nontransitive and*

$$\lim_{\theta_i \to (-\varphi_i)^+} [\theta_i + w_i \tau_i(\theta_i)] = 2\pi - \phi_i \,. \tag{2.26}$$

- *If $\theta_i \in (-\phi_i, \pi-\varphi_i)$ and $\alpha_i < \alpha_{i0}$ then the trajectories starting from $V_i(\theta_i)$ are transitive, where $\alpha_{i0} := \sin(\varphi_i)\exp[b_i(2\pi - \varphi_i - \theta_i)]$.*

Proof 2.4: i) We only give the proof for trajectories starting from S_1. The proof for trajectories starting from S_2 follows similar lines. Using equation (2.18), the time derivatives of $f_1(t)$ are

$$\frac{df_1}{dt} = M_1 exp\big((\lambda_1+\sigma_1)t\big) sin\big(\theta_1 + w_1 t + \phi_1\big), \tag{2.27}$$

$$\frac{d^2 f_1}{dt^2} = M_1^2\, exp\big((\lambda_1+\sigma_1)t\big) sin\big(\theta_1 + w_1 t + 2\phi_1\big), \tag{2.28}$$

where M_1 is as given in equation (2.16). Since $\phi_1 < \frac{\pi}{2}$, the equations above imply that local minimums of $f_1(t)$ occur at time instances

$$t_k \coloneqq \frac{2\pi k - \phi_1 - \theta_1}{w_1},\ k = 1, 2, \ldots \ \text{where}\ \frac{d^2 f_{1(t_k)}}{dt^2} > 0. \tag{2.29}$$

Since $z_1(t)$ smoothly continues into S_1, there exists $\varepsilon > 0$ such that $c^T z_1(t) > 0$ for $(0,\varepsilon]$. Let k^* be the minimum integer such that $f_1(t_{k^*}) < 0$. Existence of t_{k^*} directly *follows from the fact that $f_1(t)$ is a sinusoidal function with an exponentially increasing amplitude. Since $f_1(t_k) \geq 0$ for $k < k^*$, it follows that $f_1(t) \geq 0$ over the interval* $(0, \frac{2\pi(k^*-1)+\pi-\phi_1-\theta_1}{w_1})$, *where* $\frac{2\pi(k^*-1)+\pi-\phi_1-\theta_1}{w_1}$ *is the local maximum nearest to* $\frac{2\pi k-\phi_1-\theta_1}{w_1}$. *Consequently, $f_1(t)$ decreases over the interval*

$$\left[\frac{2\pi(k^*-1)+\pi-\phi_1-\theta_1}{w_1}, \frac{2\pi k^*-\phi_1-\theta_1}{w_1}\right] \tag{2.30}$$

and there exists a unique $\tau_1>0$ in the same interval such that $f_1(\tau_1)=0$ and. $\frac{df_1}{dt} < 0$. *Thus, the trajectory changes mode at $t = \tau_1$. If $k^* = 1$ and $\theta_1 \leq \pi - \phi_1$, the same argument still holds. If $k^* = 1$ and $\theta_1 > \pi - \phi_1$, then $f_1(t)$ decreases over the interval* $[\varepsilon, \frac{2\pi-\phi_1-\theta_1}{w_1}]$ *and there exists a unique $\tau_1 > 0$ in the same interval such that $f_1(\tau_1)=0$ and* $\frac{df_1(\tau_1)}{dt} < 0$. *Thus, the trajectory changes mode at $t = \tau_1$.*

ii) Note that since $b_1 \leq 0$, this time $f_1(t)$ is a sinusoidal function with an exponentially nonincreasing amplitude. However, local minimums are still at the same points as in the previous case. Let $r_1(\theta_1)$ be defined as

$$r_1(\theta_1) := \sin(2\pi - \phi_1) \exp[\, b_1(2\pi - \phi_1 - \theta_1)\,] - \sin(\theta_1) \tag{2.31}$$

where $\theta_1 \in [-\phi_1, 0]$. Since $b_1 \leq 0$ and $\phi_1 \geq \frac{\pi}{2}$, we have $r_1(\theta_1) \geq 0$ for $\theta_1 \in [-\phi_1, \phi_1 - \pi]$ where $-\sin(\theta_1) \geq \sin(\phi_1)$. On the other hand, $r_1(\theta_1) < 0$ at $\theta_1=0$. Hence, since $r_1(\theta_1)$ is a continuous function, there exists $\phi_1 \in (0, \phi_1]$ such that $r_1(-\phi_1) = 0$. In order to prove uniqueness, we show that $r_1(\theta_1)$ is a decreasing function of θ_1 over the interval $[-\frac{\pi}{2}, 0]$. Towards this end, note that since $b_1 = \cot(\phi_1)$, we have

$$\begin{aligned}\frac{dr_1(\theta_1)}{dt} &= -b_1 \sin\left(2\pi-\phi_1\right) \exp\left[b_1\left(2\pi-\phi_1-\theta_1\right)\right] - \cos(\theta_1) \\ &= \cos\left(\phi_1\right) \exp\left[b_1\left(2\pi-\phi_1-\theta_1\right)\right] - \cos(\theta_1)\end{aligned} \tag{2.32}$$

Since $\pi > \phi_1 \geq \frac{\pi}{2}$, it follows that $\cos(\phi_1) \leq 0$. Furthermore, as $\cos(\theta_1) \geq 0$ over the interval $[-\frac{\pi}{2}, 0]$, it follows that $\frac{dr_1(\theta_1)}{dt} \leq 0$ *where the equality holds only for ϕ_1*

$= \theta_1 = \frac{\pi}{2}$. *This implies that* $r_1(\theta_1)$ *is a decreasing function of* θ_1 *over the interval* $[-\frac{\pi}{2}, 0]$. *Consequently, there exists a unique* $\phi_1 \in (0, \pi-\phi_1)$ *such that* $r_1(\phi_1) = 0$ *or equivalently*

$$\sin(2\pi - \phi_1)\exp[\, b_1(2\pi - \phi_1 + \varphi_1)] = \sin(-\varphi_1), \qquad (2.33)$$

$$\lim_{\theta_i \to (-\varphi_i)^+} [\theta_i + w_i \tau_i(\theta_i)] = 2\pi - \phi_i\,. \qquad (2.34)$$

If $\lambda_1 + \sigma_1 = 0$, *then* $b_1=0$ *and* $\phi_1=(\pi/2)$. *Hence,* $\varphi_1 = \frac{\pi}{2}$. *For the rest of the proof, it is enough to check if the first local minimum of* $c^T z_1(t)$ *is negative or not. Towards this end, let* $z_1(t)$ *be a trajectory starting from* S_1 *with initial condition* $z_1(0) = K_1(\alpha_1 + x_1\sin(\theta_1) + y_1\cos(\theta_1))$. *Since* $z_1(t)$ *smoothly continues into* S_1, *it follows that* $\alpha_1 \geq \sin(-\theta_1)$. *Furthermore, using equation (2.29) at the first local minimum, we have*

$$c^T z_1(t_1) = K_1 \exp(-\lambda_1 t_1)\,(\alpha_1 + \sin(2\pi - \phi_1)\exp[\, b_1(2\pi - \phi_1 - \theta_1)\,]). \qquad (2.35)$$

- *If* $\theta_1 \in [-\phi_1, -\varphi_1]$, *then since* $r_1(\theta_1) \geq 0$, *we have* $c^T z_1(t_1) \geq 0$. *Furthermore, since* $f_1(t)$ *has an exponentially decreasing (or constant if* $b_1=0$ *) amplitude and* $\frac{d^2 f_1(t_1)}{dt^2} > 0$, *it follows that* $c^T z_1(t_1) \geq 0$ *for all* $t \geq 0$ *or equivalently the trajectory is nontransitive.*
- *If* $\alpha_1 \geq \sin(\phi_1)\exp[\, b_1(2\pi - \phi_1 - \theta_1)\,]$ $(=\alpha_{10})$, *then similar to the previous case, the trajectory is nontransitive.*
- *If* $\alpha_1 < \sin(\phi_1)\exp[\, b_1(2\pi - \phi_1 - \theta_1)\,]$ $(=\alpha_{10})$, *then* $c^T z_1(t_1) < 0$ *and there exists* $0 < \tau_1 < t_1$ *such that* $c^T z_1(t_1) = 0$ *and the trajectory changes mode or equivalently it is transitive.*

Remark 2.3: Note that when $\lambda_i + \sigma_i \leq 0$, *we have* $\sigma_i < 0$ *and consequently corresponding mode is stable. Thus, any trajectory which starts with initial condition* $K_i V_i(\theta_i)$ *where* $\theta_i \in [-\phi_i, -\varphi_i]$ *and* $K_i > 0$ *will stay in the same mode and decay to the origin. Also note that the set of such initial conditions constitute a cone in H bounded* *by the rays along the directions* $V_i(-\phi_i)$ *and* $V_i(-\varphi_i)$*. We formalize this observation via the following definition.*

Definition 2.5: If $\lambda_i + \sigma_i \leq 0$ *for a mode i, then the closed convex cone in* $H \cap S_i$ *bounded by the rays along the directions* $V_i(-\phi_i)$ *and* $V_i(-\varphi_i)$ *will be called a stability cone. The stability cones are denoted by* C^+ *and* C^-*, for i=1,2, respectively. A stability cone is said to be attractive on an interval I which includes the cone* ($[-\phi_i, -\varphi_i] \subset I$)*, if for any trajectory starting from direction* $V_i(\theta_i)$ *where* $\theta_i \in I$*, enters the cone after some finite time or if* $T_i^k(\theta_i) \to -\varphi_i$ *as* $k \to \infty$*. Otherwise, the stability cone is called repulsive.*

In order to simplify the notation, we use τ_i instead of $\tau_i(\theta_i)$ to denote the time at which the trajectories change mode, in the rest of the work.

Lemma 2.5 Consider the bimodal system (2.10) and let $z_i(t)$ *be a transitive trajectory with* $z_i(0) \in S_i \cap H$*. Then, the following hold for the time* τ_i *at which the trajectory changes mode.*

i) τ_i *and* $\theta_i + w_i \tau_i$ *are functions of* θ_i *implicitly given as*

$$\exp((\lambda_i + \sigma_i)\,\tau_i)\,\sin(\theta_i + w_i \tau_i) = \sin(\theta_i) \tag{2.36}$$

ii) If $\lambda_i + \sigma_i > 0$ *and* $-\phi_i \leq \theta_i \leq 0$*, then*

$$\pi \leq \theta_i + w_i \tau_i < \pi + \phi_i . \tag{2.37}$$

iii) If $\lambda_i + \sigma_i \leq 0$ *and* $-\varphi_i \leq \theta_i \leq 0$*, then*

$$\pi \leq \theta_i + w_i \tau_i < 2\pi - \phi_i . \tag{2.38}$$

iv) If $0 \leq \theta_i < \pi - \phi_i$ *, then*

$$\pi - \phi_i < \theta_i + w_i \tau_i \leq \pi \tag{2.39}$$

for both $\lambda_i + \sigma_i > 0$ *and* $\lambda_i + \sigma_i \leq 0$.

v) τ_i *and* $\theta_i + w_i \tau_i$ *are decreasing functions of* θ_i *and*

$$\lim_{\theta_i \to \pi - \phi_i} \tau_i = 0 \ , \ \lim_{\theta_i \to 0 \mp} \frac{d(\theta_i + w_i \tau_i)}{d\theta_i} = -\exp(-b_i \pi) \qquad (2.40)$$

Proof 2.5: We only give the proof for a trajectory starting from $H \cap S_1$. *The proof for a trajectory starting from* $H \cap S_2$ *follows similar lines.*

i) Let $z_1(t)$ *be a transitive trajectory which starts from* $H \cap S_1$. *Then, it follows that*

$$c^T z_1(0) = K_1 \{ \alpha_1 + \sin(\theta_1) \} = 0 \Rightarrow \alpha_1 = -\sin(\theta_1) \,. \qquad (2.41)$$

Suppose that the trajectory changes mode at $t = \tau_1$. *Then* $c^T z_1(\tau_1) = 0$ *and we get*

$$\exp((\sigma_1 + \lambda_1)\tau_1) \sin(\theta_1 + w_1 \tau_1) = \sin(\theta_1). \qquad (2.42)$$

Note that this equation is satisfied trivially for $\theta_1 = 0$ *as* $\theta_1 + w_1 \tau_1 = \pi$.

ii) If $\lambda_1 + \sigma_1 > 0$ *and* $-\phi_1 \leq \theta_1 \leq 0$, *then since* $\sin(\theta_1) \leq 0$, *equation (2.42) implies that* $\pi \leq \theta_1 + w_1 \tau_1$. *Since* $\exp((\sigma_1 + \lambda_1)\tau_1) > 1$, *it also follows that* $|\sin(\theta_1 + w_1 \tau_1)| < |\sin\theta_1|$. *Consequently, we get* $\pi \leq \theta_1 + w_1 \tau_1 < \pi + \phi_1$.

iii) If $\lambda_1 + \sigma_1 \leq 0$ *and* $-\varphi_1 < \theta_1 \leq 0$, *then as in the previous case, we again have* $\pi \leq \theta_1 + w_1 \tau_1$. *In this case* $\exp((\sigma_1 + \lambda_1)\tau_1) < 1$ *and therefore* $|\sin(\theta_1 + w_1 \tau_1)| > |\sin\theta_1|$. *Consequently, it follows that* $\pi \leq \theta_1 + w_1 \tau_1 < 2\pi - \phi_1$.

iv) If $\theta_1 \in [0, \pi - \phi_1)$, *then since* $\sin(\theta_1) \geq 0$, *equation (2.42) implies that* $\theta_1 + w_1 \tau_1 \leq \pi$ *both for* $\lambda_1 + \sigma_1 > 0$ *and* $\lambda_1 + \sigma_1 \leq 0$. *On the other hand, the function* $f_1(t)$ *defined by equation (2.11) has a maximum at* $\theta_1 + w_1 t = \pi - \phi_1$ *and* $f\left(\frac{\pi - \phi_1 - \theta_1}{w_1}\right) > 0$. *Consequently,* $\pi - \phi_1 < \theta_1 + w_1 \tau_1 \leq \pi$.

v) Using the relation between $\sin(\theta_1 + w_1 \tau_1)$ *and* $\sin(\theta_1)$ *given by equation (2.42) above and assuming that* $\theta_1 \neq 0$, *we have* $\frac{\sin(\theta_1)}{\sin(\theta_1 + w_1 \tau_1)} = \exp((\lambda_1 + \sigma_1)\tau_1) \Rightarrow \ln\left(\frac{\sin(\theta_1)}{\sin(\theta_1 + w_1 \tau_1)}\right) = (\lambda_1 + \sigma_1)\, \tau_1$

$$\Rightarrow (\lambda_1+\sigma_1)\left(\frac{d\tau_1}{d\theta_1}\right) = cot(\theta_1) - cot(\theta_1+w_1\tau_1)\left(1+w_1\frac{d\tau_1}{d\theta_1}\right) \quad (2.43)$$

$$\Rightarrow \quad w_1\frac{d\tau_1}{d\theta_1} = \frac{cot(\theta_1)-cot(\theta_1+w_1\tau_1)}{cot(\theta_1+w_1\tau_1)+b_1}. \quad (2.44)$$

This also implies that

$$\frac{d(\theta_1+w_1\tau_1)}{d\theta_1} = 1 + w_1\frac{d\tau_1}{d\theta_1} = \frac{b_1+cot(\theta_1)}{b_1+cot(\theta_1+w_1\tau_1)}. \quad (2.45)$$

For $\theta_1=0$, we calculate the limit

$$lim_{\theta_1\to 0}\frac{b_1+cot(\theta_1)}{b_1+cot(\theta_1+w_1\tau_1)} = lim_{\theta_1\to 0}\frac{sin(\theta_1)b_1 + cos(\theta_1)}{sin(\theta_1)b_1 + \exp(b_1w_1\tau_1)cos(\theta_1+w_1\tau_1)} \quad (2.46)$$

and so

$$lim_{\theta_1\to 0}\frac{b_1+cot(\theta_1)}{b_1+cot(\theta_1+w_1\tau_1)} = -\exp(-b_1\pi). \quad (2.47)$$

Note that if $\theta_1 \in [0, \pi - \phi_1)$, we have $b_1+cot(\theta_1) > 0$ and $b_1+cot(\theta_1+w_1\tau_1) < 0$ as $\pi - \phi_1 < (\theta_1 + w_1\tau_1) \leq \pi$. This implies that

$$\frac{d(\theta_1+w_1\tau_1)}{d\theta_1}, \frac{d(w_1\tau_1)}{d\theta_1} < 0 \quad (2.48)$$

• *If $\lambda_1 + \sigma_1 > 0$ and $\theta_1 \in [-\phi_1, 0)$ we get $b_1 + cot(\theta_1) \leq 0$ (equality holds only at $\theta_1 = -\phi_1$) and $b_1 + cot(\theta_1+w_1\tau_1) > 0$ as $\pi < \theta_1+w_1\tau_1 \leq \pi+\phi_1$. Hence, the inequality above still holds.*

• *If $\lambda_1 + \sigma_1 \leq 0$ and $-\varphi_1 < \theta_1 \leq 0$, then again we have $b_1+cot(\theta_1) < 0$ as both b_1, $cot(\theta_1) < 0$. Since $\phi_1 > (\pi/2)$ and $\pi \leq \theta_1+w_1\tau_1 < 2\pi - \phi_1$, we again get $b_1 + cot(\theta_1+w_1\tau_1) > 0$. Thus, the inequality above still holds. Consequently, it follows that both $1+ w_1\frac{d(\tau_1)}{d\theta_1}$ and $w_1\frac{d(\tau_1)}{d\theta_1}$ are negative. Therefore, $(\theta_1+w_1\tau_1)$ and τ_1 are decreasing functions of θ_1.*

In order to complete the proof, we now show that $\lim_{\theta_1 \to \pi - \phi_1} (w_1\tau_1) = 0$*. Towards this end, note that the function* $f_1(t)$ *defined by equation (2.11) has a maximum at* $\theta_1 + w_1 t = \pi - \phi_1$ *and* $f_1(t_0) > 0$*, where* $t_0 := \frac{\pi - \phi_1 - \theta_1}{w_1}$ *. This implies that* $t_0 < \tau_1$ *and* $\frac{df_1(t)}{dt} < 0$ *over the interval* (t_0 , τ_1)*. This implies that*

$$f_1(t_0) > f_1(t) > 0 \tag{2.49}$$

if $t \in (t_0 , \tau_1)$*. Since,* $\lim_{\theta_1 \to \pi - \phi_1} f_1(t_0) = f_1(0) = 0$*, it follows that* $\lim_{\theta_1 \to \pi - \phi_1} f_1(t) = f_1(0) = 0$ *for any* $t \in [t_0, \tau_1]$*. Consequently, since* $f_1(t)$ *is a continuous function, as* $\theta_1 \to \pi - \phi_1$ *we get* $\tau_1 \to 0$ *and this completes the proof.*

Note that the movement of transitive trajectories on H is controlled by the following functions $F_i : \theta_i \to \theta_i + w_i \tau_i$ and $D_i : V_i(\theta_i) \to V_i(\theta_i + w_i \tau_i)$. These functions are well-defined on the interval $\theta_i \in [- \phi_i, \pi - \phi_i)$ if $\lambda_i + \sigma_i > 0$ and on the interval $\theta_i \in (- \varphi_i, \pi - \phi_i)$ if $\lambda_i + \sigma_i \leq 0$. Further note that, since $\lim_{\theta_i \to \pi - \phi_i} w_i \tau_i = 0$, we can continuously extend these functions as follows

$$F_i (\pi - \phi_i) := \pi - \phi_i \text{ and } D_i (V_i (\pi - \phi_i)) := V_i (\pi - \phi_i). \tag{2.50}$$

In view of this extension $F_i (\theta_i)$ and $D_i (V_i (\theta_i))$ are well-defined functions on the compact interval $[- \phi_i, \pi - \phi_i]$ if $\lambda_i + \sigma_i > 0$. If $\lambda_i + \sigma_i \leq 0$, then they are well-defined on $(- \varphi_i, \pi - \phi_i]$.

Lemma 2.6: Given the bimodal system (2.10), then the following hold for the trajectories starting from H.

i) If $\lambda_i + \sigma_i > 0$*, then* F_i *is a nonexpansive map i.e., for any* $\theta_i, \hat{\theta}_i \in [-\phi_i, \pi - \phi_i]$

$$\left|F_i(\hat{\theta}_i) - F_i(\theta_i)\right| \leq \left|\hat{\theta}_i - \theta_i\right| \tag{2.51}$$

and $\frac{dF_i(\theta_i)}{d\theta_i}$ *is a strictly decreasing function of* θ_i *over the interval* $[-\phi_i, \pi - \phi_i]$

where $\left.\frac{dF_i(\theta_i)}{d\theta_i}\right|_{\theta_i=-\phi_i} = 0$ *and* $\left.\frac{dF_i(\theta_i)}{d\theta_i}\right|_{\theta_i=\pi-\phi_i} = -1.$

ii) If $\lambda_i + \sigma_i < 0$*, then* F_i *is an expansive map i.e., for any* $\theta_i, \hat{\theta}_i \in [-\phi_i, \pi - \phi_i]$

$$\left|F_i(\hat{\theta}_i)\text{-}F_i(\theta_i)\right| \geq \left|\hat{\theta}_i\text{-}\theta_i\right| \tag{2.52}$$

and $\frac{dF_i(\theta_i)}{d\theta_i}$ *is a strictly increasing function of* θ_i *over the interval* $(-\phi_i, \pi - \phi_i]$

where $lim_{\theta_i\to(-\varphi_i)^+} \frac{dF_i(\theta_i)}{d\theta_i} = -\infty$ *and* $\left.\frac{dF_i(\theta_i)}{d\theta_i}\right|_{\theta_i=\pi-\phi_i} = -1.$

iii) If $\lambda_i+\sigma_i = 0$*, then* $\frac{dF_i(\theta_i)}{d\theta_i} = -1$ *for all* $\theta_i \in \left[-\frac{\pi}{2}, \pi\text{-}\phi_i\right]$ *and* $lim_{\theta_i\to\left(-\frac{\pi}{2}\right)^+} \frac{dF_i(\theta_i)}{d\theta_i} = -1.$

iv) $D_i\big(V_i(\theta_i)\big) = exp(\sigma_i\tau_i)(V_i(\theta_i + w_i\tau_i))$ *for any* $\theta_i \in [-\phi_i, \pi - \phi_i]$ *if* $\lambda_i + \sigma_i > 0$ *and for any* $\theta_i \in (-\varphi_i, \pi - \phi_i]$ *if* $\lambda_i + \sigma_i \leq 0.$

Proof 2.6: i) The proof of this item has two stages. We first show that if $\theta_i \in [-\varphi_i, 0]$*, then* $\left|\exp(b_i w_i \tau_i)\frac{dF_i(\theta_i)}{d\theta_i}\right| \leq 1$ *where the equality holds only at* $\theta_i = 0$*. Then, we show that if* $\theta_i \in [0,\ \pi - \phi_i] \Rightarrow \left|\exp(b_i w_i \tau_i)\frac{dF_i(\theta_i)}{d\theta_i}\right| \geq 1$ *where the equality hold only at* $\theta_i = 0$ *and at* $\theta_i = \pi - \phi_i$*. Towards this end, note that in view of equations (2.42) and (2.45) above* $\exp(b_i w_i \tau_i)\frac{dF_i(\theta_i)}{d\theta_i}$ *can be expressed as follows*

$$\exp(b_i w_i \tau_i)\frac{dF_i(\theta_i)}{d\theta_i} = \frac{sin(\theta_i)b_i + cos\,(\theta_i)}{sin(\theta_i+w_i\tau_i)b_i + cos(\theta_i+w_i\tau_i)} = \frac{\sin(\theta_i+\phi_i)}{\sin(\theta_i+w_i\tau_i+\phi_i)} \tag{2.53}$$

If $\theta_i \in [-\phi_i, 0)$*, then since* $\pi < \theta_i + w_i\tau_i < \pi + \phi_i$ *and* $\phi_i < \frac{\pi}{2}$*, it follows that* $cos(\theta_i + w_i\tau_i)$*,* $sin(\theta_i + w_i\tau_i)$*,* $sin(\theta_i) < 0$ *and* $cos(\theta_i) > 0$*. Furthermore, since* $\frac{\sin(\theta_i)}{\sin(\theta_i+w_i\tau_i)} > 1$ *by equation (2.42), it also follows that* $cos(\theta_i) \leq cos(\theta_i + w_i\tau_i)$*. Finally, since* $\sin(\theta_i + \phi_i) \geq 0$ *we also have* $|sin(\theta_i)|b_i < cos(\theta_i)$*. Consequently, equation (2.53) implies that if* $\theta_i \in [-\phi_i, 0)$*, we have*

$$\left|\frac{\sin(\theta_i+\phi_i)}{\sin(\theta_i+w_i\tau_i+\phi_i)}\right| = \frac{cos\,(\theta_i) - |sin(\theta_i)|b_i}{|cos(\theta_i+w_i\tau_i)| + |sin(\theta_i+w_i\tau_i)|b_i} < 1 \tag{2.54}$$

Furthermore, since $\theta_i + w_i\tau_i = \pi$ *at* $\theta_i = 0$, *we get* $\left|\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)}\right| = 1$ *at* $\theta_i = 0$. *On the other hand, if* $\theta_i \epsilon [0, \pi - \phi_i]$, *then* $\pi < \theta_i + w_i \tau_i + \phi_i < \pi + \phi_i$. *This implies that* $sin(\theta_i + w_i \tau_i + \phi_i) < - sin(\phi_i)$ *for any* $\theta_i \epsilon [0, \pi - \phi_i]$, *whereas* $sin(\theta_i + \phi_i) \geq sin(\phi_i)$ *if* $\theta_i \epsilon [0, \pi - 2\phi_i]$. *Consequently, it follows that*

$$1 + \frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)} \leq 0 \tag{2.55}$$

over the interval $0 \leq \theta_i \leq \pi - 2\phi_i$ *and the equality holds only at* $\theta_i = 0$. *We claim that*

$$1 + \frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)} < 0 \tag{2.56}$$

over the interval $(0, \pi - \phi_i)$. *If the claim is not true, then there exists* $\theta_i \epsilon (0, \pi - \phi_i)$ *such that*

$$1 + \frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)} = 0 \Rightarrow \sin(\theta_i + \phi_i) = -sin(\theta_i + w_i\tau_i + \phi_i) \tag{2.57}$$

which is possible only for $\theta_i = \pi - \phi_i$ *or* $\theta_i = 0$. *Consequently,*

$$1 + \frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)} \leq 0 \tag{2.58}$$

for any $\theta_i \epsilon [0, \pi - \phi_i]$ *where the equality holds only at the end points of the interval. Thus, we have*

$$\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)} \leq -1 \Leftrightarrow \left|\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)}\right| \geq 1 \tag{2.59}$$

where the equality holds only at the end points of the interval $[0, \pi - \phi_i]$. *Since*

$$\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)} = exp(b_i w_i \tau_i) \frac{dF_i(\theta_i)}{d\theta_i} \tag{2.60}$$

and $lim_{\theta_i \to \pi - \phi_i}(w_i\tau_i) = 0$ *by Lemma 2.5.v, the result given above also implies that*

$$lim_{\theta_i \to \pi - \phi_i} \frac{dF_i(\theta_i)}{d\theta_i} = -1 \qquad (2.61)$$

For the second stage of the proof let us assume that $sin(\theta_i) \neq 0$ *and calculate the second derivative*

$$\frac{d^2F_i(\theta_i)}{d\theta_i^2} = \frac{-\frac{b_i+cot(\theta_i+w_i\tau_i)}{sin^2(\theta_i)} + \frac{(b_i+cot(\theta_i))^2}{sin^2(\theta_i+w_i\tau_i)(b_i+cot(\theta_i+w_i\tau_i))}}{(b_i+cot(\theta_i+w_i\tau_i))^2} = \frac{sin(\phi_i)\left[\left(exp(b_iw_i\tau_i)\frac{dF_i(\theta_i)}{d\theta_i}\right)^2 - 1\right]}{sin(\theta_i)exp(b_iw_i\tau_i)sin(\theta_i+w_i\tau_i+\phi_i)} \qquad (2.62)$$

If $\theta_i \in [-\phi_i, 0)$*, then the term in the numerator of the above equation is* <0 *as* $\left|exp(b_iw_i\tau_i)\frac{dF_i(\theta_i)}{d\theta_i}\right| < 1$. *Since both* $sin(\theta_i)$ *and* $sin(\theta_i+w_i\tau_i+\phi_i)$ *are* < 0*, the term in the denominator is* > 0*. Consequently,* $\frac{d^2F_i(\theta_i)}{d\theta_i^2} < 0$. *Similarly, if* $\theta_i \in (0, \pi - \phi_i)$ *then then the term in the numerator of the above equation is* > 0 *whereas the term in the denominator is* < 0 *since* $sin(\theta_i + w_i\tau_i + \phi_i)$ *is* < 0*. Therefore, we again have* $\frac{d^2F_i(\theta_i)}{d\theta_i^2} < 0$. *This implies that* $\frac{dF_i(\theta_i)}{d\theta_i}$ *is a monotone decreasing function over* $[-\phi_i, 0) \cup (0, \pi-\phi_i)$*. Since* $\frac{dF_i(\theta_i)}{d\theta_i} = -exp(-b_1\pi)$ *at* $\theta_i = 0$ *and* $\frac{dF_i(\theta_i)}{d\theta_i} = -1$ *at* $\theta_i = \pi - \phi_i$ *by Lemma 2.5.v and Lemma 2.6.i, it follows that* $\frac{dF_i(\theta_i)}{d\theta_i}$ *is a monotone decreasing function over* $[-\phi_i, \pi-\phi_i]$*. Consequently, we have*

$$Sup_{\theta_i \in [0,\pi-\phi_i]} \left|\frac{dF_i(\theta_i)}{d\theta_i}\right| = 1 \qquad (2.63)$$

or equivalently $|F_i(\hat{\theta}_i) - F_i(\theta_i)| \leq |\hat{\theta}_i - \theta_i|$ *for any pair* $\theta_i, \hat{\theta}_i \in [-\phi_i, \pi - \phi_i]$.

ii) In this case, we first show that If $\theta_i \in (-\varphi_i, 0]$*, then* $\left|\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)}\right| \geq 1$ *where the equality holds only at* $\theta_i = 0$*. Then, we prove that if* $\theta_i \in [0, \pi - \phi_i]$*, then* $\left|\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)}\right| \leq 1$ *where the equality holds at* $\theta_i = 0$ *and at* $\theta_i = \pi-\phi_i$*. Towards this end, note that* $\phi_i > \frac{\pi}{2}$ *and* $b_i < 0$*. Therefore, if* $\theta_i \in (-\varphi_i, 0]$*, then by Lemma 2.5.3, it follows that* $\pi < \theta_i + w_i\tau_i < 2\pi - \phi_i < \frac{3\pi}{2}$ *and this mplies that* $cos(\theta_i + w_i\tau_i) < 0$*. Since* $|sin(\theta_i + w_i\tau_i)| > |sin(\theta_i)|$ *and* $sin(\theta_i + w_i\tau_i + \phi_i) < 0$*, we also have* $b_i sin(\theta_i + w_i\tau_i) < |cos(\theta_i + w_i\tau_i)| < cos(\theta_i)$*. Then, for any* $\theta_i \in (-\varphi_i, 0)$*, we get*

$$\left|\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)}\right| = \frac{b_i sin(\theta_i)+cos\,(\theta_i)}{|cos(\theta_i+w_i\tau_i)| - b_i sin(\theta_i+w_i\tau_i)} \tag{2.64}$$

which follows from the fact that $b_i sin(\theta_i), b_i sin(\theta_i + w_i\tau_i),\ cos\,(\theta_i) > 0$. *Thus, it follows that* $\left|\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)}\right| > 1$. *Note that* $\theta_i + w_i\tau_i = \pi$ *at* $\theta_i = 0$ *and we get* $\left|\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)}\right| = 1$ *at* $\theta_i = 0$. *Furthermore, in view of Lemma 2.4.2.ii, we have* $lim_{\theta_i\to(-\varphi_i)^+}(\theta_i+w_i\tau_i) = 2\pi-\phi_i$,*which implies that*

$$lim_{\theta_i\to(-\varphi_i)^+}\frac{dF_i(\theta_i)}{d\theta_i} = lim_{\theta_i\to(-\varphi_i)^+}\frac{b_i+cot\,(\theta_i)}{b_i+cot\,(\theta_i+w_i\tau_i)} = -\infty \tag{2.65}$$

Now we will show that $\left|\frac{cos(\theta_i+\phi_i)}{cos(\theta_i+w_i\tau_i+\phi_i)}\right| \geq 1$ *if* $\theta_i \in [0, \pi - \phi_i]$, *where the equality holds only at* $\theta_i = 0$ *and at* $\theta_i = \pi - \phi_i$ *. Towards this end, note that since* $sin(\theta_i)\, exp(-b_i w_i \tau_i) = sin(\theta_i + w_i \tau_i)$, *it follows that*

$$\frac{d}{d\theta_i}\left(\frac{\cos(\theta_i+\phi_i)}{\cos(\theta_i+w_i\tau_i+\phi_i)}\right) = \frac{sin(\theta_i+\phi_i)cos(\phi_i)}{cos^2(\theta_i+w_i\tau_i+\phi_i)}[\cos(\theta_i)\exp(-b_i w_i\tau_i) - \cos(\theta_i + w_i\tau_i)]. \tag{2.66}$$

Furthermore, since $\theta_i \in [0, \pi - \phi_i]$, *we have* $\pi - \phi_i \leq \theta_i + w_i\tau_i \leq \pi$ *and since* $\phi_i > \frac{\pi}{2}$ *this implies that* $|cos(\theta_i + w_i\tau_i)| < \cos(\theta_i) < \cos(\theta_i)\exp(-b_i w_i\tau_i)$. *Consequently,* $\frac{cos(\theta_i+\phi_i)}{cos(\theta_i+w_i\tau_i+\phi_i)}$ *is a decreasing function over the interval* $[0, \pi-\phi_i]$ *except across discontinuity at* $\theta_i + w_i\tau_i + \phi_i = \frac{3\pi}{2}$ *where*

$$lim_{\theta_i+w_i\tau_i+\phi_i\to\frac{3\pi}{2}}\ \frac{cos(\theta_i+\phi_i)}{cos(\theta_i+w_i\tau_i+\phi_i)} = \pm\infty \tag{2.67}$$

However, this does not change the fact that $\left|\frac{cos(\theta_i+\phi_i)}{cos(\theta_i+w_i\tau_i+\phi_i)}\right| \geq 1$, *because* $cos(\theta_i + w_i\tau_i + \phi_i) < cos(\phi_i)$ *for* $\frac{3\pi}{2} < \theta_i + w_i\tau_i + < \pi + \phi_i$ *whereas* $cos(\theta_i) \geq cos(\phi_i)$ *for any* $\theta_i \in [0, \pi - \phi_i]$. *Consequently,* $\left|\frac{sin(\theta_i+\phi_i)}{sin(\theta_i+w_i\tau_i+\phi_i)}\right| \leq 1$ *over the interval* $[0, \pi - \phi_i]$ *and the equality holds only at* $\theta_i = 0$ *and at* $\theta_i = \pi - \phi_i$. *Since* $w_i \tau_i = 0$ *at* $\theta_i = \pi - \phi_i$ *we also get*

$$lim_{\theta_i \to \pi\text{-}\phi_i} \frac{dF_i(\theta_i)}{d\theta_i} = \text{-}1 \quad (2.68)$$

The rest of the proof is similar to the previous case and we conclude that $\frac{d^2F_i(\theta_i)}{d{\theta_i}^2}$ *over the interval* $(\varphi_i, \pi - \phi_i\,]$. *Hence, we have* $\left|\frac{dF_i(\theta_i)}{d\theta_i}\right| \geq 1$ *where the equality holds only at* $\theta_i = \pi - \phi_i$ *. Equivalently,* $F_i(\theta_i)$ *is an expansive map.*

iii) Note that if $\lambda_i + \sigma_i = 0$*, then* $\phi_i = \frac{\pi}{2}$ *and in view of equation (2.42) we have* $sin(\theta_i + w_i \tau_i) = sin(\theta_i)$*. This implies that* $\theta_i + w_i \tau_i = \pi - \theta_i$ *. Thus, we have*

$$\frac{dF_i(\theta_i)}{d\theta_i} = \frac{cot(\theta_i)}{cot(\theta_i + w_i\tau_i)} = \frac{cos(\theta_i)}{cos(\pi\text{-}\theta_i)} = \text{-}1 \quad (2.69)$$

for all $\theta_i \in (\text{-}\phi_i, \pi\text{-}\phi_i]$*. Note that the trajectories starting from* $V_i\ (\text{-}\phi_i\ )$ *are nontransitive. Hence,* $F_i(\theta_i)$ *is not defined at* $\theta_i = -\phi_i$*. However,*

$$lim_{\theta_i \to \text{-}\frac{\pi}{2}^+} \frac{dF_i(\theta_i)}{d\theta_i} = \lim_{\theta_i \to \text{-}\frac{\pi}{2}^+} \frac{cos(\theta_i)}{cos(\pi\text{-}\theta_i)} = \lim_{\theta_i \to \text{-}\frac{\pi}{2}^+} \frac{cos(\theta_i)}{\text{-}cos(\theta_i)} = \text{-}1 \quad (2.70)$$

and by continuous extension at $\theta_i = -\phi_i$ *we have* $\frac{dF_i(\theta_i)}{d\theta_i} = -1$.

iv) Since $c^T z_i(\tau_i) = 0$*, it follows that* $0 = exp(\sigma_i \tau_i) sin(\theta_i + w_i \tau_i) - sin(\theta_i) exp(-\lambda_i \tau_i)$*. Then, as*

$$z_i(\tau_i) = -sin(\theta_i)exp(-\lambda_i \tau_i)\, r_i + exp(\sigma_i \tau_i)\{sin(\theta_i + w_i \tau_i)\, x_i + cos(\theta_i + w_i \tau_i) y_i\} \quad (2.71)$$

we get $z_i(\tau_i) = exp(\sigma_i \tau_i)\{ sin(\theta_i + w_i \tau_i)\hat{x}_i + cos(\theta_i + w_i \tau_i)\, y_i \}$ *and this concludes the proof.*

2.4. Mode Change

A transitive trajectory $z_1(t)$ starting from S_1 changes mode at $t=\tau_1$. From this point on the trajectory smoothly continues into S_2. In order to follow this change, we set $z_2(0) := z_1(\tau_1)$ and determine $z_2(0)$. This requires a change of basis from $\{\hat{x}_1, y_1\}$

to { $\hat{x}_2$, y_2 }. Towards this end, let $[\Gamma_i^T \quad 0]^T := [\hat{x}_i \quad y_i]$. Then using the results of Lemma 2.3, we get

$$\Gamma_1 = \begin{bmatrix} \frac{a_{11}(\sigma_1+\lambda_1)-\sigma_1\lambda_1-(\sigma_1^2+w_1^2)}{a_{32}a_{21}} & \frac{w_1(a_{11}+\lambda_1)}{a_{32}a_{21}} \\ \frac{\lambda_1+\sigma_1}{a_{32}} & \frac{w_1}{a_{32}} \end{bmatrix}, \Gamma_2 = \begin{bmatrix} \sigma_2\lambda_2 + (\sigma_2^2 + w_2^2) & -w_2\lambda_2 \\ -(\lambda_2+\sigma_2) & -w_2 \end{bmatrix}. \quad (2.72)$$

where Γ_1 and Γ_2 are clearly nonsingular. Let us now define B_1 as follows

$$B_1 := \frac{\lambda_1+a_{11}-a_{21}\lambda_2}{a_{21}w_2(1+{b_2}^2))} \quad (2.73)$$

Suppose that that $B_1>0$. It can be easily seen that geometric meaning of this expression is that the angle between y_1 and L_1^+ is strictly less than the angle between y_2 and L_2^-. If we consider the case $\lambda_1+a_{11}-a_{21}\lambda_2=0$ ($B_1=0$) , then the geometric meaning of this expression is that the vectors y_1 and y_2 are on the same line.

We can now state and prove the following result.

Lemma 2.7: Let $V_1(\theta_1)$ be a direction where $\theta_1 \in [-\phi_1, 2\pi-\phi_1)$ and b_i be as defined in equation (2.19) and $C_1 := \frac{w_1(1+{b_1}^2)}{a_{21}w_2(1+{b_2}^2)}$. Then, there exists a unique $\theta_2 \in [-\phi_2, 2\pi-\phi_2)$ and $\eta_{12}(\theta_1)>0$ such that $V_1(\theta_1)=\eta_{12}(\theta_1)V_2(\theta_2)$ where

$$\theta_2 = cot^{-1}\{\frac{cos\theta_1+b_1sin\theta_1}{C_1sin\theta_1-B_1(cos\theta_1+b_1sin\theta_1)} - b_2\} \quad (2.74)$$

$$\eta_{12}(\theta_1) = \frac{-w_1sin(\theta_1+\phi_1)}{a_{32}w_2sin(\theta_2+\phi_2)}\frac{sin\phi_2}{sin\phi_1} \text{ for } \theta_1 \in (-\varphi_1, \pi-\phi_1) \cup (\pi-\phi_1, 2\pi-\phi_1) \quad (2.75)$$

$$\eta_{12}(\theta_1) = C_1\frac{w_1}{a_{32}w_2}\frac{-sin(\phi_1)}{sin\phi_2} \text{ for } \theta_1 = -\phi_1, \pi-\phi_1.$$

Proof 2.7: Let $Q_i := \begin{bmatrix} 1 & 0 \\ b_i & 1 \end{bmatrix}$. The equation $V_1(\theta_1) = \eta_{12}(\theta_1) V_2(\theta_2)$ has to be solved for $\eta_{12}(\theta_1)$ and θ_2. Note that both $\eta_{12}(\theta_1)$ and θ_2 are functions of θ_1 and in view of equation (2.19) and the structure of $[\hat{x}_i \quad y_i]$ (equations (2.12) and (2.13)), we get

$$\eta_{12}(\theta_1)\begin{bmatrix} sin\theta_2 \\ cos\theta_2+b_2sin\theta_2 \end{bmatrix} = Q_2\,\Gamma_2^{-1}\,\Gamma_1\,Q_1^{-1}\begin{bmatrix} sin\theta_1 \\ cos\theta_1+b_1sin\theta_1 \end{bmatrix}. \tag{2.76}$$

Using straightforward calculations, it can be shown that equations (2.74) and (2.75) hold. Note that in view of our condition on well-posedness, L_1 *and* L_2 *are on the same line. Therefore, for any* $\theta_1 \in [-\phi_1, \pi-\phi_1]$ *it must hold that* $\theta_2 \in [\pi-\phi_2, 2\pi-\phi_2]$ *and this implies that* $sin(\theta_1+\phi_1) \geq 0$ *and* $sin(\theta_2+\phi_2) \leq 0$ *where the equalities hold at the end points of the intervals. Similarly, for any* $\theta_1 \in [\pi-\phi_1, 2\pi-\phi_1]$ *we have* $\theta_2 \in [-\phi_2, \pi-\phi_2]$*, which implies that* $sin(\theta_1+\phi_1) \leq 0$ *and* $sin(\theta_2+\phi_2) \geq 0$*. Consequently, it holds that* $\eta_{12}(\theta_1) > 0$*.*

Remark 2.4: In view of the above result, it is clear that for a given direction $V_2(\theta_2)$ *where* $\theta_2 \in [-\phi_2, 2\pi-\phi_2)$*, there exists a unique* $\theta_1 \in [-\phi_1, 2\pi-\phi_1)$ *and* $\eta_{21}(\theta_2) > 0$ *such that* $V_2(\theta_2) = \eta_{21}(\theta_2)\, V_1(\theta_1)$*. In fact, following the same lines as in the proof Lemma 2.7, we get*

$$\theta_1 = cot^{-1}\{\frac{cos\theta_2+b_2sin\theta_2}{C_2sin\theta_2+B_2(cos\theta_2+b_2sin\theta_2)} - b_1\} \tag{2.77}$$

$$\eta_{21}(\theta_2) = \frac{-a_{32}w_2sin(\theta_2+\phi_2)}{w_1sin(\theta_1+\phi_1)}\frac{sin\phi_1}{sin\phi_2},\ \ for\ \theta_2 \in (-\phi_2, \pi-\phi_2) \cup (\pi-\phi_2, 2\pi-\phi_2), \tag{2.78}$$

$$\eta_{21}(\theta_2) = C_2\frac{a_{32}w_2}{w_1}\frac{sin(\phi_2)}{sin(\phi_1)}\ \ for\ \ \theta_2 = -\phi_2, \pi-\phi_2.$$

where $C_2 := (C_1)^{-1}$ *and* $B_2 := C_2B_1$*. Let*

$$h_1(\theta_1) := \frac{cos\theta_1+b_1sin\theta_1}{C_1sin\theta_1-B_1(cos\theta_1+b_1sin\theta_1)} - b_2,\quad h_2(\theta_2) := \frac{cos\theta_2+b_2sin\theta_2}{C_2sin\theta_2+B_2(cos\theta_2+b_2sin\theta_2)} - b_1 \tag{2.79}$$

Remark 2.5: Let us define $G_i(\theta_i)$*,* $cot\psi_2$ *and* $cot\psi_1$ *as follows.*

$$G_i(\theta_i) := cot^{-1}\{h_i(\theta_i)\}\ and\ cot\psi_2 := B_1^{-1} + b_2,\ cot\psi_1 := B_2^{-1} - b_1. \tag{2.80}$$

Note that $lim_{\theta_1\to\pi+\psi_1} h_1(\theta_1) = \pm\infty$*. Since* $cot^{-1}(\cdot)$ *is multi-valued function, we define* $G_1(\pi+\psi_1)=0$*. Similarly,* $lim_{\theta_1\to\pi-\psi_2} h_2(\theta_2) = \pm\infty$ *and we define* $G_2(\pi-$

ψ_2)=0. With these definitions G_i (θ_i) is well-defined and continuous over the real line. Using the definitions of h_i (θ_i) and G_i (θ_i), the following equivalent directions can be obtained easily

$$V_1(-\phi_1) \simeq V_2(\pi-\phi_2),\ y_1=V_1(0)\simeq V_2(\pi-\psi_2),\ -y_2 = V_1(\psi_1)\simeq V_2(\pi), \qquad (2.81)$$
$$V_1(\pi-\phi_1)\simeq V_2(-\phi_2),\ -y_1 = V_1(\pi)\simeq V_2(-\psi_2),\ V_1(\pi+\psi_1)\simeq V_2(0)=y_2$$

The geometries for cases $B_1>0$ and $B_1=0$ are depicted in the following pictures.

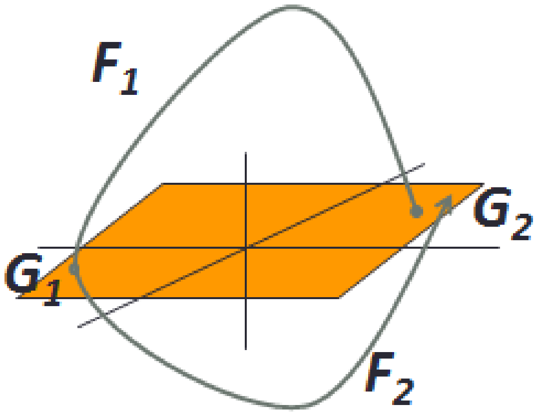

Figure 2.4: The functions which drive the trajectory.

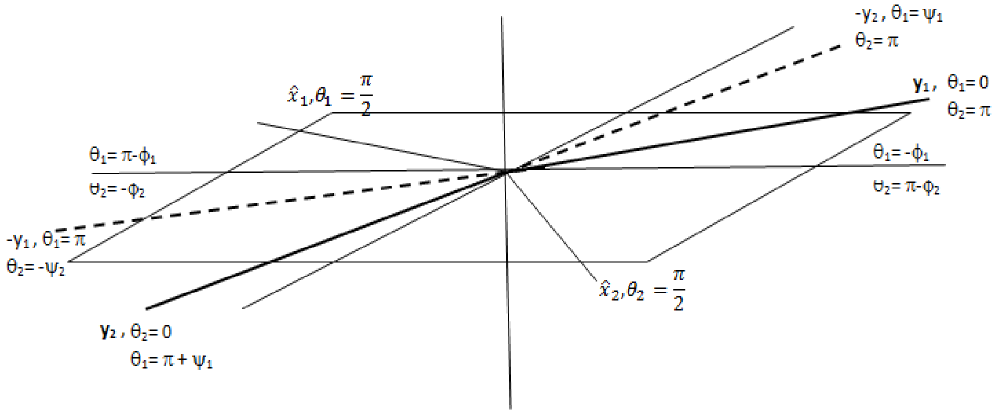

Figure 2.5: Plane geometry for $B_1>0$.

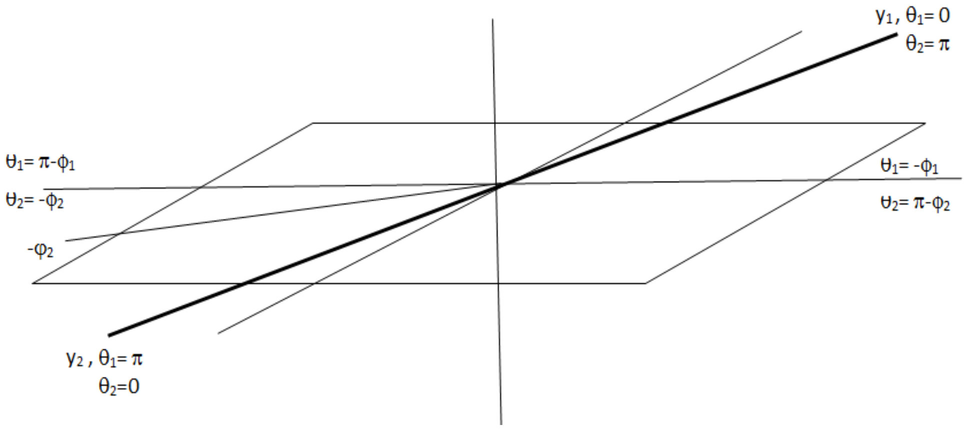

Figure 2.6: Plane geometry for B_1=0.

Before continuing further, we summarize the domain and the range of the composite functions G_i (F_i (θ_i)) by the following Corollary.

Corollary 2.1: Suppose that $B_1 \geq 0$. Then, the following hold for the domain and range of G_i (F_i (θ_i)).

i) If $\lambda_1+\sigma_1 > 0$, then the domain of G_1 ($F_1(\theta_1)$) is [-ϕ_1, π-ϕ_1].

- *G_1 ($F_1(\theta_1)$) maps [0, π-ϕ_1] onto [-ϕ_2, -ψ_2].*
- *G_1 ($F_1(\theta_1)$) maps [-ϕ_1, 0] into [-ψ_2, π-ϕ_2).*

ii) If $\lambda_2+\sigma_2 > 0$, then the domain of G_2 ($F_2(\theta_2)$) is [-ϕ_2, π-ϕ_2].

- *G_2 ($F_2(\theta_2)$) maps [0, π-ϕ_2] onto [-ϕ_1, ψ_1].*
- *G_2 ($F_2(\theta_2)$) maps [-ϕ_2,0] into [ψ_1,π-ϕ_1).*

iii) If $\lambda_1+\sigma_1 \leq 0$, then the domain of G_1 ($F_1(\theta_1)$) is (-φ_1, π-ϕ_1] where $\varphi_1 = \frac{\pi}{2}$ if $\lambda_1+\sigma_1=0$.

- *G_1 ($F_1(\theta_1)$) maps [0, π-ϕ_1] onto [-ϕ_2, -ψ_2].*
- *G_1 ($F_1(\theta_1)$) maps (-φ_1, 0] onto [-ψ_2, π-ϕ_2).*

iv) If $\lambda_2+\sigma_2 \leq 0$, then the domain of G_2 ($F_2(\theta_2)$) is (-φ_2, π-ϕ_2] where $\varphi_2 = \frac{\pi}{2}$ if $\lambda_2+\sigma_2=0$.

- *G_2 ($F_2(\theta_2)$) maps [0, π-ϕ_2] onto [-ϕ_1, ψ_1].*
- *G_2 ($F_2(\theta_2)$) maps (-φ_2, 0] into [ψ_1, π-ϕ_1).*

3. STABILITY AND FIXED DIRECTIONS

In the previous chapter we have investigated the properties of the trajectories as they start and evolve in one mode until they change mode. In this chapter we will focus on the properties as they change mode and afterwards. Along these lines we give a final classification of the trajectories of bimodal systems in $\mathbb{R}^3$. Namely,

i) the trajectories which change mode finite number of times as $t\rightarrow\infty$,
ii) the trajectories which change mode infinite number of times as $t\rightarrow\infty$.

The set of nontransitive trajectories in both modes are clearly in the class of trajectories which change mode finite number of times (namely zero times). Characterization of such trajectories are given in Lemma 2.4.2. Some of the transitive trajectories of both modes may also be in the class of trajectories which change mode finite number of times. This could happen if some trajectories change mode finite number of times and end up in [$-\phi_i$, $-\varphi_i$] (if $\lambda_i + \sigma_i < 0$) and always stay in the i^{th} mode as $t\rightarrow\infty$.

Since $\lambda_i + \sigma_i < 0$, it follows that $\sigma_i < -\lambda_i < 0$ and consequently i^{th} mode is stable. Thus, the class of trajectories which change mode finite number of times decay to the origin as $t\rightarrow\infty$, in our set up. In other words, such trajectories necessarily end up (after a finite period of time) in a stability cone of Definition 2.5.

In view of the above observations and Definition 2.5, we need to investigate only the stability of the class of trajectories which change mode infinite number of times as $t\rightarrow\infty$. Since such trajectories change mode at H, we can restrict our investigation to trajectories which start from H without loss of any generality.

In order to formalize what we have explained in the previous paragraph, let

$$T_1(\theta_1) := G_2(F_2(G_1(F_1(\theta_1)))). \tag{3.1}$$

Since F_i 's and G_i 's are continuous, it follows that T_1 is also continuous. Thus we have

$$\theta_1 \rightarrow \theta_1 + w_1\tau_1 \rightarrow \theta_2 \rightarrow \theta_2 + w_2\tau_2 \rightarrow T_1(\theta_1) \tag{3.2}$$

T_1 is defined for trajectories which change mode at least two times. Furthermore, let $T_1^k(\theta_1)$ denote

$$T_1^k(\theta_1) := T_1(.....T_1(T_1(\theta_1)).....) \quad (k \text{ times}). \tag{3.3}$$

It is clear that $T_1^k(\theta_1)$ is defined for trajectories which change mode at least 2k times. We can also define $T_2 : [-\phi_2, \pi - \phi_2] \rightarrow [-\phi_2, \pi - \phi_2]$ (or T_2: $(-\varphi_2, \pi - \phi_2] \rightarrow [-\phi_2, \pi - \phi_2]$ for $\lambda_2 + \sigma_2 \leq 0$) as

$$T_2(\theta_2) := G_1(F_1(G_2(F_2(\theta_2)))), \tag{3.4}$$

which is again a continuous function and use the notation $T_2^k(\theta_2)$ in a completely similar way.

Definition 3.1: A direction $V_i(\theta_i^*)$ in H is called a fixed direction if θ_i^* is a fixed point of $T_i(\theta_i)$, or equivalently $T_i(\theta_i^*) = \theta_i^*$. A fixed direction $V_i(\theta_i^*)$ is called attractive on the interval I which includes θ_i^*, if for any $\theta_i \in I$ and any $\varepsilon > 0$ there exists a positive integer k such that we have $\left|T_i^k(\theta_i) - \theta_i^*\right| < \varepsilon$. If a fixed direction is not attractive for any interval I which includes θ_i^*, then it is called repulsive.

If $V_1(\theta_1^*)$ is a fixed direction in $H \cap S_1$, then there exists $\theta_2^* := G_1(F_1(\theta_1^*))$ such that $V_2(\theta_2^*)$ is a fixed direction in $H \cap S_2$. Hence, fixed directions exist in pairs. In view of this observation, we shall also use the notation $V(\theta_1^*, \theta_2^*)$ to denote a pair of fixed directions in the sequel. Furthermore, if all the trajectories starting from a fixed direction are stable (decays to the origin as $t \rightarrow \infty$), then the fixed direction is called stable, or equivalently, fixed direction pair will be called stable.

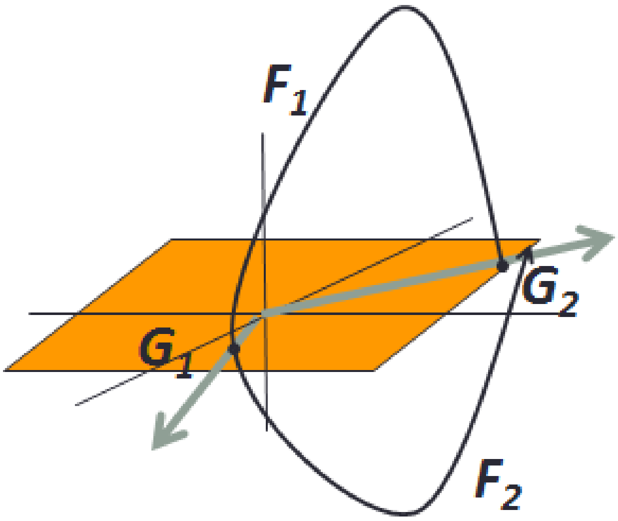

Figure 3.1: Fixed direction pairs.

3.1. Stability of fixed directions

Let θ_1 be a fixed point of $T_1(.)$, i.e., $T_1(\theta_1)=\theta_1$. Suppose that a trajectory starts with initial condition equal to $V_1(\theta_1)$. Then, in view of Lemma 2.6.iv, we have

$$D_1(V_1(\theta_1))=exp(\sigma_1\tau_1)V_1(\theta_1+w_1\tau_1). \quad (3.5)$$

Since the trajectory changes mode at this point, by Lemma 2.7 it follows that

$$exp(\sigma_1\tau_1)V_1(\theta_1+w_1\tau_1)=exp(\sigma_1\tau_1)\eta_{12}(\theta_1+w_1\tau_1)V_2(\theta_2). \quad (3.6)$$

From this point on, the trajectory smoothly continues into S_2 and by Lemma 2.6.iv we get

$$D_2(V_2(\theta_2))=exp(\sigma_2\tau_2)V_2(\theta_2+w_2\tau_2) \quad (3.7)$$

where the trajectory changes mode again. Since $T_1(\theta_1)=\theta_1$, we have

$$exp(\sigma_2\tau_2)V_2(\theta_2+w_2\tau_2)=exp(\sigma_2\tau_2)\eta_{21}(\theta_2+w_2\tau_2)V_1(\theta_1). \quad (3.8)$$

Thus, initial condition $V_1(\theta_1)$ is mapped to $exp(\sigma_1\tau_1+\sigma_2\tau_2)\eta_{12}(\theta_1+w_1\tau_1)\eta_{21}(\theta_2+w_2\tau_2)V_1(\theta_1)$. Let

$$\gamma_F(\theta_1) := \exp(\sigma_1\tau_1+\sigma_2\tau_2)\eta_{12}(\theta_1+w_1\tau_1)\eta_{21}(\theta_2+w_2\tau_2). \tag{3.9}$$

In the sequel, $\gamma_F(\theta_1)$ will be called rate of convergence of the fixed direction $V_1(\theta_1)$. It is clear that if $\gamma_F(\theta_1)<1$, then the fixed direction is stable, if $\gamma_F(\theta_1)>1$ then the fixed direction is unstable. If $\gamma_F(\theta_1)=1$, then bimodal system is marginally stable and has infinite number of periodic solutions with period $\tau_1+\tau_2$. These periodic solutions are closed space curves.

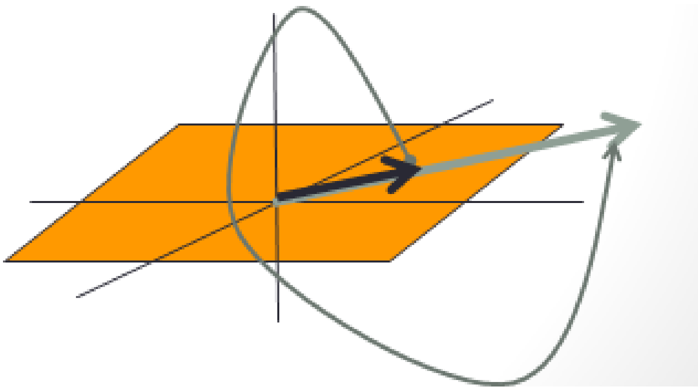

Figure 3.2: Comparision of $V_1(\theta_1)$ and $\gamma_F(\theta_1)V_1(\theta_1)$.

We summarize the stability result explained above as follows.

Lemma 3.1: A fixed direction $V_1(\theta_1)$ where $T_1(\theta_1)=\theta_1$ is stable if and only if its convergence rate $\gamma_F(\theta_1)<1$.

Corollary 3.1: Let $V_1(\theta_1)$ be a fixed direction. Then, γ_F can be expressed as

$$\gamma_F = \frac{\exp(-\lambda_1\tau_1-\lambda_2\tau_2)}{\frac{dF_1 dF_2}{d\theta_1 d\theta_2}} \tag{3.10}$$

Proof 3.1: Using equations (2.74)-(2.78), we get

$$\eta_{12}(\theta_1+w_1\tau_1)\eta_{21}(\theta_2+w_2\tau_2) = \frac{sin(\theta_1++w_1\tau_1\phi_1)}{sin(\theta_2+\phi_2)}\frac{sin(\theta_2++w_2\tau_2\phi_2)}{sin(\theta_1+\phi_1)} \tag{3.11}$$

$$= \frac{1}{\exp((\lambda_1+\sigma_1)\tau_1)\frac{dF_1}{d\theta_1}\exp((\lambda_2+\sigma_2)\tau_2)\frac{dF_2}{d\theta_2}}$$

which implies that equation (3.10) holds.

Theorem 3.1: Let $V_1(\theta_1^*)$ be a fixed direction which is attractive on an interval I and consider a trajectory with initial condition $V_1(\theta_1)$ where $\theta_1 \in I$. Then, the trajectory is stable (decays to origin as t→∞) if and only if the fixed direction $V_1(\theta_1^*)$ is stable.

Proof 3.1: Since $V_1(\theta_1^*)$ is attractive on I, it follows that for any ε>0 there exists an integer k such that $|T_1^k(\theta_1) - \theta_1^*| < \varepsilon$. In the proof we only consider the case where $T_1^k(\theta_1)$ converges to θ_1^* from right. The case where $T_1^k(\theta_1)$ converges to θ_1^* from left can be treated in a completely similar way. In view of this fact, we can assume without loss of generality that the initial condition is $K_1V_1(\theta_1^*+\varepsilon_0)$ where $T_1^k(\theta_1):= \theta_1^*+\varepsilon_0$ and ε_0 can be made arbitrarily small by increasing k. Then, in view of Lemma 2.6.iv, $D_1(\cdot)$ maps $V_1(\theta_1^*+\varepsilon_0)$ to

$$D_1 (V_1 (\theta_1^* + \varepsilon_0)) = exp (\sigma_1 (\tau_1^* - \varepsilon_1)) V_1 (\theta_1^* + \varepsilon_0 + w_1 (\tau_1^* - \varepsilon_1)) \qquad (3.12)$$

for some $\varepsilon_1>0$, as $\theta_1+w_1\tau_1$ is a decreasing function of θ_1. Here, $F_1(\theta_1^*):=\theta_1^*+w_1\tau_1^*$. Since the trajectory changes mode at this point, Lemma 2.7 implies that

$$V_1 (\theta_1^* + \varepsilon_0 + w_1(\tau_1^* - \varepsilon_1)) = \eta_{12} (\theta_1^* \varepsilon_0 + w_1 (\tau_1^*-\varepsilon_1)) V_2 (\theta_2^* - \varepsilon_2) \qquad (3.13)$$

for some $\varepsilon_2>0$ because θ_2 decreases as θ_1 increases. Similarly, $D_2(\cdot)$ maps $V_2(\theta_2^* - \varepsilon_2)$ to

$$D_2 (V_2 (\theta_2^* - \varepsilon_2)) = exp (\sigma_2 (\tau_2^* + \varepsilon_3)) V_2 (\theta_2^* - \varepsilon_2 + w_2 (\tau_2^* + \varepsilon_3)) \qquad (3.14)$$

for some $\varepsilon_3>0$, as $\theta_2+w_2\tau_2$ is a decreasing function of θ_2. Here, $F_2 (\theta_2^*) := \theta_2^* +w_1\tau_2^*$. Since the trajectory changes mode again at this point, Lemma 2.7 implies that

$$V_2(\theta_2^*-\varepsilon_2+w_2(\tau_2^*+ \varepsilon_3)) = \eta_{21}(\theta_2^*-\varepsilon_2+w_2(\tau_2^*+\varepsilon_3))V_1(\theta_1^*+\varepsilon_4) \qquad (3.15)$$

for some $\varepsilon_4>0$. Hence, $V_1(\theta_1^*+\varepsilon_0)$ is mapped to $\gamma(\theta_1^*+\varepsilon_0)V_1(\theta_1^*+\varepsilon_4)$ where

$$\gamma(\theta_1^*+\varepsilon_0):=\exp(\sigma_1(\tau_1^*-\varepsilon_1)+\sigma_2(\tau_2^*+\varepsilon_3)\eta_{12}(\theta_1^*+\varepsilon_0+w_1(\tau_1^*-\varepsilon_1))\eta_{21}(\theta_2^*-\varepsilon_2+w_2(\tau_2^*+\varepsilon_3)). \tag{3.16}$$

Since $D_1(\cdot)$ and $D_2(\cdot)$ are continuous functions and change of basis is also continuous, it follows that $\gamma(\theta_1^*+\varepsilon_0)\to\gamma_F(\theta_1^*)$ as $\varepsilon_0\to 0$. Hence, there exists $\varepsilon>0$ such that $\gamma(\theta_1^*+\varepsilon)<1$ if and only if $\gamma_F(\theta_1^*)<1$. Consequently, the trajectory decays to the origin if and only if $\gamma_F(\theta_1^*)<1$.

Lemma 3.2: Let $\theta_1\neq 0$ be in the domain of $T_1(.)$ and define $\theta_{11}:=T_1(\theta_1)$. Then

$$\frac{\cot(\theta_1)+b_1}{\cot(\theta_{11})+b_1}=\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1(\theta_1+w_1\tau_1)L_2(\theta_2+w_2\tau_2) \tag{3.17}$$

where $\theta_2:=G_1(F_1(\theta_1))$ is a decreasing function of θ_1 and

$$L_1(\theta_1+w_1\tau_1):=[1-B_2(\cot(\theta_1+w_1\tau_1)+b_1)] \tag{3.18}$$
$$L_2(\theta_2+w_2\tau_2):=[1+B_1(\cot(\theta_2+w_2\tau_2)+b_2)].$$

(i) If $\theta_1>0$, then

$$\frac{\cot(\theta_1)+b_1}{\cot(\theta_{11})+b_1}>1 \Leftrightarrow \theta_1<T_1(\theta_1)=\theta_{11}, \tag{3.19}$$

$$\frac{\cot(\theta_1)+b_1}{\cot(\theta_{11})+b_1}<1 \Leftrightarrow \theta_1>T_1(\theta_1)=\theta_{11}, \tag{3.20}$$

ii) If $\theta_1<0$, then

$$\frac{\cot(\theta_1)+b_1}{\cot(\theta_{11})+b_1}<1 \Leftrightarrow \theta_1<T_1(\theta_1)=\theta_{11}, \tag{3.21}$$

$$\frac{\cot(\theta_1)+b_1}{\cot(\theta_{11})+b_1}>1 \Leftrightarrow \theta_1>T_1(\theta_1)=\theta_{11}, \tag{3.22}$$

iii) For both cases above θ_1 is a fixed direction if and only if $\frac{\cot(\theta_1)+b_1}{\cot(\theta_{11})+b_1}=1.$

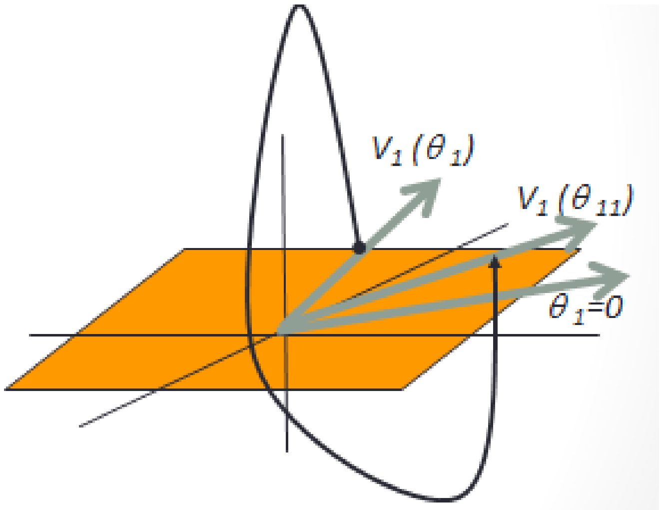

Figure 3.3: A sample trajectory for $\theta_1>0$; $\theta_1>T_1(\theta_1)=\theta_{11}$.

Proof 3.2: Suppose that a trajectory starts from $V_1(\theta_1)$ in $H\cap S_1$ and hits the plane at time τ_1 along the direction $V_2(\theta_2)$ in $H\cap S_2$. Then we have $G_1(F_1(\theta_1))=\theta_2$. In view of Lemma 2.7, we have

$$G_1(\theta_1+w_1\tau_1)=cot^{-1}\left\{\frac{cot(\theta_1+w_1\tau_1)+b_1}{C_1-B_1(cot(\theta_1+w_1\tau_1)+b_1)}-b_2\right\} \quad (3.23)$$

$$\Rightarrow cot(\theta_2)+b_2=\frac{cot(\theta_1+w_1\tau_1)+b_1}{C_1-B_1(cot(\theta_1+w_1\tau_1)+b_1)}.$$

In order to show that $\theta_2:=G_1(F_1(\theta_1))$ is a decreasing function of θ_1, we calculate the derivative of both sides of the above equation with respect to θ_1 and we get

$$\frac{-1}{sin^2(\theta_2)}\frac{d\theta_2}{d\theta_1}=\frac{-C_1}{sin^2(\theta_1+w_1\tau_1)[C_1-B_1(cot(\theta_1+w_1\tau_1)+b_1)]^2}\frac{dF_1}{d\theta_1} \quad (3.24)$$

since $\frac{dF_1}{d\theta_1}\leq 0$, *the claim holds.*

Also suppose that, the same trajectory hits back $H\cap S_1$ at time τ_2 along the direction $V_1(\theta_{11})$. Then, in view of Remark 2.4, we have

$$G_2(\theta_2+w_2\tau_2) = cot^{-1}\left\{\frac{cot(\theta_2+w_2\tau_2)+b_2}{C_2+B_2(cot(\theta_2+w_2\tau_2)+b_2)}-b_1\right\} \quad (3.25)$$

$$\Rightarrow cot(\theta_{11})+b_1=\frac{cot(\theta_2+w_2\tau_2)+b_2}{C_2+B_2(cot(\theta_2+w_2\tau_2)+b_2)}.$$

Solving the equations above for the ratio of $cot(\theta_1)+b_1$ *to* $cot(\theta_{11})+b_1$*, we get*

$$\frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1} = \frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2} L_1(\theta_1+w_1\tau_1)L_2(\theta_2+w_2\tau_2) \qquad (3.26)$$

i) In order to simplify the notation we use L_i *instead of* $L_i(\theta_i + w_i\tau_i)$ *here. Since the trajectory starts from* $H\cap S_1$*, it follows that* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ *is a function of* θ_1*. If* $\theta_1 \in [0,\pi-\phi_1]$*, then we have* $cot\theta_1+b_1\geq 0$ *where the equality holds at* $\theta_1=\pi-\phi_1$*. Furthermore, since* $B_1\geq 0$*, it also follows that* $\theta_{11} \in [0,\pi-\phi_1]$ *and* $cot\theta_{11}+b_1\geq 0$*. Hence, we have*

$$\frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1} < 1 \Leftrightarrow cot\theta_1 < cot\theta_{11} \Leftrightarrow \theta_1 > \theta_{11} \qquad (3.27)$$

This means that $\theta_1>T_1(\theta_1)=\theta_{11}$*. Continuing along the same lines we also have*

$$\frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1} > 1 \Leftrightarrow cot\theta_1 > cot\theta_{11} \Leftrightarrow \theta_1 < \theta_{11} \qquad (3.28)$$

and $\theta_1<T_1(\theta_1)=\theta_{11}$*.*

ii) Similarly, if $\theta_1<0$*, then it follows that* $cot\theta_1+b_1<0$*. Furthermore, if* $cot\theta_{11}+b_1<0$ *also holds, then*

$$\frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1} > 1 \Leftrightarrow |cot\,(\theta_1)| > |cot\,(\theta_{11})| \Leftrightarrow 0>\theta_1>\theta_{11} \qquad (3.29)$$

so $\theta_1>T_1(\theta_1)=\theta_{11}$*. Under the same conditions, it also holds that*

$$0 < \frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1} < 1 \Leftrightarrow |cot\theta_1| < |cot\theta_{11}| \Leftrightarrow 0 > \theta_{11} > \theta_1 \qquad (3.30)$$

and $\theta_1<T_1(\theta_1)=\theta_{11}$*.Note that, there is also the possibility of* $\frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1} \leq 0$*. For this case, we must have* $cot\theta_1+b_1\leq 0$ *but* $cot\theta_{11}+b_1>0$*, because* $cot\theta_1+b_1\leq 0$ *for* $\theta_1 \in [-\phi_1,0]$ *or* $\theta_1 \in [-\varphi_1,0]$*. This implies that* $\theta_{11}>0>\theta_1$ *or equivalently* $\theta_1<T_1(\theta_1)=\theta_{11}$ *and this concludes the proof.*

It is clear by the above result that the behavior of the function $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ plays a crucial role in determining the attractive regions for fixed directions. In the remaining part of this dissertation, our investigation is based on the trajectories starting from $H \cap S_1$. Therefore, we need to specify the domain of $T_1(\cdot)$ for various combination of subsystem parameters.

Corollary 3.1: If $B_1>0$, then the following hold for the domain of $T_1(\cdot)$.

i) If $\lambda_i + \sigma_i > 0$ for both modes, then the domain of $T_1(\cdot)$ is $[-\phi_1,\pi-\phi_1]$.
ii) If $\lambda_1+\sigma_1>0$ and $\lambda_2+\sigma_2<0$, there exists θ_{1r} in the domain of $G_1(F_1(\cdot))$ such that $G_1(F_1(\theta_{1r}))=-\varphi_2$. Then, the domain of $T_1(\cdot)$ is $[-\phi_1,\theta_{1r})$ or empty if $-\phi_1 = \theta_{1r}$.
iii) If $\lambda_1+\sigma_1>0$ and $\lambda_2+\sigma_2=0$, then the domain of $T_1(\cdot)$ is $[-\phi_1,\pi-\phi_1)$.
iv) If $\lambda_1+\sigma_1\leq 0$ and $\lambda_2+\sigma_2>0$, then the domain of $T_1(\cdot)$ is $(-\varphi_1,\pi-\phi_1]$, where $\varphi_1=\frac{\pi}{2}$ if $\lambda_1+\sigma_1=0$.
v) If $\lambda_1+\sigma_1\leq 0$ and $\lambda_2+\sigma_2<0$, then the domain of $T_1(\cdot)$ is $(-\varphi_1,\theta_{1r})$. If $\lambda_2+\sigma_2=0$, then the domain of $T_1(\cdot)$ is $(-\varphi_1,\pi-\phi_1)$.

The analysis of the behavior of the function $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ over the domain of $T_1(\cdot)$ becomes simpler when $B_1=0$. This is due to the fact that if $B_1=0$, then y_1 and y_2 are on the same line and consequently, $L_1L_2=1$ over the entire domain of $T_1(\cdot)$. If $B_1>0$, then $L_1(\theta_1+w_1\tau_1)L_2(\theta_2+w_2\tau_2)\neq 1$. Therefore, in order to determine the location of fixed directions, the behavior of this function over the domain of $T_1(\theta_1)$ has to be investigated first. The behavior of L_1L_2 is given in the following Lemma.

Lemma 3.3: Suppose that $B_1\geq 0$. Then, the following hold.

i) If $B_1 > 0$, then there exists $\theta_{2z}>0$ such that $G_2(F_2(\theta_{2z}))=0$. Furthermore, if $G_1(F_1(-\phi_1))> \theta_{2z}$ (in the case $\lambda_1+\sigma_1 > 0$), then there exist $\theta_{10}>\theta_{1z}>0$ such that $G_1(F_1(-\theta_{10}))=\theta_{2z}$ and $G_1(F_1(-\theta_{1z}))=0$. If $B_1=0$, then $\theta_{2z}=0$ and $\theta_{10}=\theta_{1z}=0$.
ii) Let θ_1 be in the domain of $T_1(\cdot)$. Then, the following hold.
• L_1L_2 is a decreasing function of θ_1 if $\theta_1>0$ or if $\theta_1 \in [-\phi_1,-\theta_{10}]$.
• $L_1L_2\leq 0$ over the interval $(-\theta_{10},0)$ and $\lim_{\theta_1\to 0^\pm} L_1L_2 = \pm\infty$.

• *At the end points of the domain of* $T_1(\cdot)$, L_1L_2 *takes the following values.*

$$\lim_{\theta_1\to\theta_{1r}} L_1L_2 \geq 1 \ \text{ or } L_1L_2<0 \text{ and } L_1L_2|_{\theta_1=-\phi_1}<1 \quad \text{for } \lambda_1+\sigma_1>0 \text{ and } \lambda_2+\sigma_2\leq 0,$$
$$L_1L_2|_{\theta_1=\pi-\phi_1} > 1 \ \text{ and } \lim_{\theta_1\to(-\varphi_1)^+} L_1L_2 = 1 \quad \text{for } \lambda_1+\sigma_1 \leq 0 \text{ and } \lambda_2+\sigma_2>0,$$
$$\lim_{\theta_1\to(\theta_{1r})^-} L_1L_2 \geq 1 \ \text{ or } L_1L_2<0 \text{ and } \lim_{\theta_1\to(-\varphi_1)^+} L_1L_2 = 1 \quad \text{for } \lambda_1+\sigma_1 \leq 0 \text{ and } \lambda_2+\sigma_2\leq 0, \tag{3.31}$$

where the equalities hold at the limit as $\lambda_2+\sigma_2\to 0$, *in which case* $\theta_{1r}\to\pi-\phi_1$.

• *If* $B_1>0$, *then* $\theta_1<T_1(\theta_1)=\theta_{11}$ *over the interval* $[-\theta_{10},\psi_1]$.

iii) If $\lambda_1+\sigma_1<0$ *and* $\lambda_2+\sigma_2>0$, *then* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *is a strictly decreasing function of* θ_1 *over the interval* $(-\varphi_1,\pi-\phi_1]$ *and*

$$\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=\pi-\phi_1} = 0 \quad \text{and} \quad \lim_{\theta_1\to(-\varphi_1)^+}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2} = \infty \tag{3.32}$$

If $\lambda_1+\sigma_1=0$ *and* $\lambda_2+\sigma_2>0$, *then we have* $\lim_{\theta_1\to\left(-\frac{\pi}{2}\right)^+}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2} = 1$

iv) If $\lambda_1+\sigma_1>0$, *and* $\lambda_2+\sigma_2\leq 0$, *then* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *is a strictly increasing function of* θ_1 *over the interval* $[-\phi_1, \theta_{1r})$. *Furthermore,*

$$\lim_{\theta_1\to(\theta_{1r})^-}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2} = \infty \quad \text{and} \quad \left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=-\phi_1} = 0 \quad \text{if } \lambda_2+\sigma_2<0. \tag{3.33}$$

If $\lambda_2+\sigma_2=0$, *then we have* $\lim_{\theta_1\to\pi-\phi_1}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2} = 1$.

Proof 3.3: i) Suppose that $B_1>0$. *Then* $G_2(F_2(.))$ *maps* $[0, \pi-\phi_2]$ *onto* $[-\phi_1, \psi_1]$ *if* $\lambda_1+\sigma_1>0$. *If* $\lambda_1+\sigma_1\leq 0$, *then* $G_2(F_2(.))$ *maps* $[0, \pi-\phi_2)$ *onto* $(-\phi_1, \psi_1]$ *as* $\pi-\phi_2$ *is not in the domain of* $G_2(F_2(.))$. *Since* $G_2(F_2(.))$ *is a continuous function, the existence of* $\theta_{2z} > 0$ *such that* $G_2(F_2(\theta_{2z})) = 0$ *is clear. In view of the hypothesis* $G_1(F_1(-\phi_\infty) > \theta_{2z}$, *if* $\lambda_1+\sigma_1>0$. *Then,* $G_1(F_1(.))$ *maps* $[-\phi_1,0]$ *onto* $[-\psi_2, G_1(F_1(-\phi_\infty))]$ *which implies that there exists* θ_{10} *such that* $G_1(F_1(-\theta_{10}))=\theta_{2z} > 0$. *Furthermore, in view of Lemma 3.2,*

θ_2 decreases as θ_1 increases which implies the existence of $\theta_{1z} < \theta_{10}$ such that $G_1(F_1(-\theta_{1z}))=0$. On the other hand, if $\lambda_1+\sigma_1\leq 0$, then $G_1(F_1(.))$ maps $(-\varphi_1,0]$ onto $[-\psi_2,\pi-\phi_2)$. Since $G_1(F_1(.))$ is a continuous function the existence of $\theta_{10}>\theta_{1z}>0$ with the given properties follows directly. Finally, if $B_1=0$, then y_1 and y_2 are on the same line. This implies that, $\theta_{10}=\theta_{1z}=0$ and $\theta_{2z}=0$.

ii) Since $L_1=[1-B_2(cot(\theta_1+w_1\tau_1)+b_1)]$, it follows that

$$\frac{dL_1}{d\theta_1} = \frac{B_2}{sin^2(\theta_1+w_1\tau_1)}\frac{dF_1}{d\theta_1} \tag{3.34}$$

Since $\frac{dF_1}{d\theta_1} \leq 0$ and $B_2 \geq 0$, L_1 is a decreasing function of θ_1 over the entire domain of $T_1(\cdot)$. Similarly, since $L_2=[1+B_1(cot(\theta_2+w_2\tau_2)+b_2)]$, it also follows that

$$\frac{dL_2}{d\theta_1} = \frac{-B_1}{sin^2(\theta_2+w_2\tau_2)}\frac{dF_2}{d\theta_2}\frac{d\theta_2}{d\theta_1} \tag{3.35}$$

Since $\frac{dF_2}{d\theta_2} \leq 0$ and $\frac{d\theta_2}{d\theta_1} < 0$, the equation above implies that L_2 is also a decreasing function of θ_1 over the entire domain of $T_1(\cdot)$.

- *Hence, if $\theta_1>0$, then $\pi-\phi_1 \leq \theta_1+w_1\tau_1 \leq \pi$ and by Lemma 2.5, we have $(cot(\theta_1+w_1\tau_1)+b_1) \leq 0$. Since $G_1(F_1(\theta_1)) \in (-\phi_2,-\psi_2]$, it also follows that $\pi \leq \theta_2+w_2\tau_2 < \pi+\phi_2$ by Lemma 2.5 that $(cot(\theta_2+w_2\tau_2)+b_2) \geq 0$. Consequently, $L_1\geq 1$, $L_2\geq 1$ and L_1L_2 is a decreasing function of θ_1. In fact, $lim_{\theta_1\to 0^+} L_1 = \infty$ as $lim_{\theta_1\to 0^+}(\theta_1 + w_1\tau_1) = \pi$. Thus, we have*

$$lim_{\theta_1\to 0^{\pm}} L_1L_2=\infty \tag{3.36}$$

On the other hand, if $\theta_1 \in [-\phi_1,-\theta_{10}]$, then $(cot(\theta_1+w_1\tau_1)+b_1) > 0$ and $0 \leq L_1 \leq 1$ as $\theta_1+w_1\tau_1 > \pi+\psi_1$. Similarly, since $G_1(F_1(\theta_{10}))=\theta_{2z} > 0$, it follows that $(cot(\theta_2+w_2\tau_2)+b_2) < 0$ and $0 \leq L_2 \leq 1$ as $cot(\theta_2+w_2\tau_2) \geq -cot\psi_2$. Consequently, we again get $L_1 \geq 0$, $L_2 \geq 0$ and L_1L_2 is a decreasing function of θ_1. In fact, since $G_1(F_1(\theta_{10}))=\theta_{2z}$, it follows that $cot(\theta_2+w_2\tau_2)=cot(\pi-\psi_2)$ at $\theta_1=\theta_{10}$ and $L_2(\pi-\psi_2)=0$. This implies that $L_1L_2=0$ at $\theta_1=\theta_{10}$.

• *If* $\theta_1 \in (-\theta_{10}, -\theta_{1z}]$, *then* $G_1(F_1(\theta_1)) \in [0, \theta_{2z})$ *which implies that* $\cot(\theta_2+w_2\tau_2) < -\cot\psi_2$. *Thus,* $L_2<0$.

Since $\theta_1+w_1\tau_1 \geq \pi+\psi_1$ *it follows that* L_1 *is still nonnegative. Consequently,* $L_1L_2<0$. *In fact, since* $G_1(F_1(-\theta_{1z})):=0$, *it turns out that* $F_1(-\theta_{1z})=\pi+\psi_1$ *and this implies that* $L_1=0$ *at* $\theta_1=-\theta_{1z}$. *However,since* $V_1(\pi+\psi_1) \simeq V_2(0)$, *we have* $\theta_2+w_2\tau_2=\pi$. *This implies that*

$$lim_{\theta_1+w_1\tau_1\to\pi+\psi_1} L_2 = -\infty \text{ and } lim_{\theta_1+w_1\tau_1\to\pi+\psi_1} L_1 = 0. \qquad (3.37)$$

Using L'Hopital's Rule, the limit of L_1L_2 *as* $\theta_1+w_1\tau_1\to\pi+\psi_1$ *can be calculated easily as follows.*

$$lim_{\theta_1+w_1\tau_1\to\pi+\psi_1} L_1L_2 = lim_{\theta_1+w_1\tau_1\to\pi+\psi_1}\{\frac{1-B_2(cot(\theta_1+w_1\tau_1)+b_1)}{sin(\theta_2+w_2\tau_2)} \qquad (3.38)$$

$$sin(\theta_2+w_2\tau_2)(1+B_1b_2)+B_1 cos(\theta_2+w_2\tau_2)\}$$

$$= lim_{\theta_1+w_1\tau_1\to\pi+\psi_1}\{\frac{B_2}{sin^2(\theta_1+w_1\tau_1)\,cos(\theta_2+w_2\tau_2)\frac{dF_2}{d\theta_2}\frac{d\theta_2}{d(\theta_1+w_1\tau_1)}}[-B_1] \in (-\infty,0),$$

which is a finite negative number as $cos(\theta_2+w_2\tau_2)\frac{dF_2}{d\theta_2}$ *and* $\frac{d\theta_2}{d(\theta_1+w_1\tau_1)}$ *are both > 0. Thus,* $L_1L_2 < 0$ *if* $\theta_1 \in (-\theta_{10}, -\theta_{1z}]$.

If $\theta_1 \in (-\theta_{1z}, 0]$, *since* $\theta_1+w_1\tau_1$ *is a decreasing function of* θ_1, *we have* $\theta_1+w_1\tau_1 < \pi+\psi_1$. *This implies that* $L_1<0$. *On the other hand, since* $G_1(F_1(-\theta_{1z})) = 0$ *and* $\theta_2+w_2\tau_2$ *is an increasing function of* θ_1, *it follows that* $(\cot(\theta_2+w_2\tau_2)+b_2) > 0$. *Thus,* $L_2 > 0$. *Consequently,* $L_1L_2 < 0$ *again. In fact,* $lim_{\theta_1\to 0^-} L_1 = -\infty$ *as* $lim_{\theta_1\to 0^-} \theta_1 + w_1\tau_1 = \pi$. *Thus, we have* $lim_{\theta_1\to 0^-} L_1L_2 = -\infty$.

• *Let us first consider the case* $\lambda_1+\sigma_1>0$ *and* $\lambda_2+\sigma_2\leq 0$. *In this case, the domain of* $T_1(.)$ *is* $[-\phi_1, \theta_{1r})$ *where* $G_1(F_1(\theta_{1r}))=-\varphi_2$. *Since* $-\varphi_2$ *is in the stability cone* C^-, *it is not in the domain of* $T_1(.)$. *However, the following limits exists*

$$lim_{\theta_1\to(\theta_{1r})^-}\ G_1(F_1(\theta_1)) = -\varphi_2 \text{ and } lim_{\theta_1\to(-\varphi_2)^+}\ \left(F_2(\theta_2)\right) = 2\pi - \phi_2 \qquad (3.39)$$

This implies that $lim_{\theta_2\to(-\varphi_2)^+}\ L_2 = 1$. *If* $\theta_{1r} > 0$, *then* $\cot(\theta_{1r}+w_1\tau_1(\theta_{1r}))+b_1 < 0$ *which implies that* $L_1>1$ *or equivalently* $lim_{\theta_1\to(\theta_{1r})^-} L_1L_2 > 1$. *If* $\theta_{1r} \leq 0$, *(this is* *equivalent to the fact that -ψ_2≤-φ_2) then by item 2.ii of this Lemma L_1 < 0 which implies that* $lim_{\theta_1\to(\theta_{1r})^-}\ L_1L_2$ *< 0. Finally, irrespective of the sign of θ_{1r}, $\theta_{1r}\to\pi-\phi_1$ as $\varphi_2\to\frac{\pi}{2}$ (equivalently $\lambda_2+\sigma_2\to0$). This implies that* $lim_{\theta_1\to(\theta_{1r})^-}\ L_1$ *= 1. Consequently, we have* $lim_{\theta_1\to(\theta_{1r})^-}\ L_1L_2$ *= 1. For the other end point of the domain of $T_1(.)$, note that by the proof of item 2.i of this Lemma 0 ≤ L_1 ≤ 1 and 0≤L_2<1 as $G_1(F_1(-\phi_1)) > \theta_{2z} > 0$ by the hypothesis. Thus, we get 0 ≤ L_1L_2 < 1.*

For the case where $\lambda_1+\sigma_1 \le 0$ and $\lambda_2+\sigma_2 > 0$, the domain of $T_1(.)$ is (-φ_1, π-ϕ_1]. At $\theta_1=\pi-\phi_1$, we have $\cot(\theta_1+w_1\tau_1)+b_1=0$ which implies that L_1=1 and $\cot(\theta_2+w_2\tau_2)+b_2>0$ as $\pi<\theta_2+w_2\tau_2<\pi+\phi_2$ at $\theta_2=-\phi_2$. Consequently, we get $L_1L_2|_{\theta_1=\pi-\phi_1}$ *> 1. For the other end of the interval, note that similar to the previous case we have*

$$lim_{\theta_1\to(-\varphi_1)^+}\ \left(F_1(\theta_1)\right) = 2\pi - \phi_1 \Rightarrow lim_{\theta_1\to(-\varphi_1)^+}\ L_1 = 1. \qquad (3.40)$$

Since $V_1(2\pi-\phi_1)\simeq V_2(\pi-\phi_2)$, this also implies that $lim_{\theta_1\to(-\varphi_1)^+}\ L_2 = 1$. *Consequently, we have* $lim_{\theta_1\to(-\varphi_1)^+}\ L_1L_2 = 1$. *This result also holds for the case where $\lambda_1+\sigma_1=0$ and $\varphi_1=\frac{\pi}{2}$.*

Finally, for the case $\lambda_1+\sigma_1\le0$ and $\lambda_2+\sigma_2\le0$ the domain of $T_1(.)$ is (-φ_1, θ_{1r}). Similar to the previous case, we have $lim_{\theta_1\to(-\varphi_1)^+}\ L_1L_2 = 1$ *which also holds for $\varphi_1=\frac{\pi}{2}$. For the other end point of the domain, the results of the case where $\lambda_1+\sigma_1>0$ and $\lambda_2+\sigma_2\le0$ again holds and we have*

$$lim_{\theta_1\to(\theta_{1r})^-}\ G_1(F_1(\theta_1)) = -\varphi_2 \text{ and } lim_{\theta_1\to(-\varphi_2)^+}\ \left(F_2(\theta_2)\right) = 2\pi - \phi_2 \qquad (3.41)$$

This implies that $lim_{\theta_1\to(\theta_{1r})^-}\ L_1 = 1$ *and* $lim_{\theta_1\to(\theta_{1r})^-}\ L_1 > 1$ *if θ_{1r} > 0. Thus,* $lim_{\theta_1\to(\theta_{1r})^-}\ L_1L_2 > 1$ *if θ_{1r} > 0. On the other hand, if θ_{1r} ≤ 0, then it follows that (as in the first case) L_1<0 and we get* $lim_{\theta_1\to(\theta_{1r})^-}\ L_1L_2$*<0. Furthermore, irrespective of the sign of θ_{1r}, $\theta_{1r}\to\pi-\phi_1$ as $\varphi_2\to\frac{\pi}{2}$ (equivalently $\lambda_2+\sigma_2\to0$). This implies that* $lim_{\theta_1\to(\theta_{1r})^-}\ L_1 = 1$. *Consequently, we have* $lim_{\theta_1\to(\theta_{1r})^-}\ L_1L_2 = 1$ *and this concludes the proof for part 2.*

• *If $[-\theta_{10},0)$ is in the domain of $T_1(.)$, then by the item 2.ii of this Lemma $L_1L_2\leq 0$ over the interval $[-\theta_{10},0)$. Since* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\geq 0$ *over the entire domain of $T_1(.)$, Lemma 2.5 implies that* $\frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1}<0$. *Since $\theta_1<0$, we have $cot(\theta_1)+b_1<0$. Consequently, $cot(\theta_{11})+b_1>0$ which implies that $\theta_{11}=T_1(\theta_1)>\theta_1$. If $[0,\psi_1]$ is also in the domain of $T_1(\cdot)$, then $G_1(F_1(\cdot))$ maps $[0,\psi_1]$ into $[-\phi_2, -\psi_2]$. Since for any $\theta_2\in[-\phi_2, -\psi_2]$, we have $\theta_2+w_2\tau_2>\pi$, it follows that $G_2(F_2(\cdot))$ maps into $(\psi_1,\pi-\phi_1)$. Consequently, $\theta_{11}=T_1(\theta_1)>\theta_1$ for any $\theta_1\in[-\theta_{10},\psi_1]$.*

iii) If $\lambda_1+\sigma_1\leq 0$ and $\lambda_2+\sigma_2>0$, then by Lemma 2.6, we get $\left.\frac{dF_2}{d\theta_2}\right|_{\theta_2=-\phi_2}=0$ *and* $lim_{\theta_1\to\pi-\phi_1}\frac{dF_1}{d\theta_1}=-1$. *Since $V_1(\pi-\phi_1)\simeq V_2(-\phi_2)$, it follows that* $\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=\pi-\phi_1}=0$. *If $\lambda_1+\sigma_1<0$, then* $lim_{\theta_1\to(-\varphi_1)^+}\frac{dF_1}{d\theta_1}=-\infty$ *by Lemma 2.6. Thus, as θ_1 increases over the interval $(-\varphi_1,\pi-\phi_1)$,* $\frac{dF_1}{d\theta_1}$ *increases from $-\infty$ to -1 or equivalently* $\left|\frac{dF_1}{d\theta_1}\right|$ *decreases from ∞ to 1. Furthermore, θ_2 decreases from $\pi-\phi_2$ to $-\phi_2$ and Lemma 2.6 implies that* $\frac{dF_2}{d\theta_2}$ *increases from -1 to zero or equivalently* $\left|\frac{dF_2}{d\theta_2}\right|$ *decreases from 1 to zero. Consequently,* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *decreases from ∞ to zero. If $\lambda_1+\sigma_1=0$, then* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *decreases from 1 to zero as* $\left.\frac{dF_1}{d\theta_1}\right|_{\theta_1=\pi-\phi_1}=-1$ *by Lemma 2.6.iii. Consequently,* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *is a decreasing function of θ_1 over the entire domain of $T_1(\cdot)$ and* $lim_{\theta_1\to-\varphi_1}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=\infty$.

iv) If $\lambda_1+\sigma_1>0$ and $\lambda_2+\sigma_2\leq 0$, then by Lemma 2.6.i, it follows that $\frac{dF_1}{d\theta_1}$ *decreases from zero to -1 over the interval $[-\phi_1,\pi-\phi_1]$. Furthermore, if $\lambda_2+\sigma_2<0$, then* $\frac{dF_2}{d\theta_2}$ *decreases from -1 to $-\infty$ as θ_2 decreases over the interval $[-\varphi_2,\pi-\phi_2]$ again by Lemma 2.6. If $\lambda_2+\sigma_2=0$, then* $\frac{dF_2}{d\theta_2}=-1$ *over the entire domain of $T_1(.)$. Consequently,* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *is an increasing function of θ_1 over the domain of $T_1(.)$ and using Lemma 2.6 again we get*

$$lim_{\theta_1\to(\theta_{1r})^-}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=\infty \text{ and } \left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=-\phi_1}=0 \qquad (3.42)$$

if $\lambda_2+\sigma_2<0$. If $\lambda_1+\sigma_1=0$ then we have $lim_{\theta_1\to(\theta_{1r})^-}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=1$.

Remark 3.1: The results given above are derived under the assumption that $G_1(F_1(-\phi_1)) > \theta_{2z}$ in the case $\lambda_1+\sigma_1>0$. If $G_1(F_1(-\phi_1))\leq \theta_{2z}$, then $G_1(F_1(\cdot))$ maps $[-\phi_1,\pi-\phi_1]$ into $[-\phi_2,-\psi_2]$. Therefore, θ_{10} and θ_{1z} do not exist. Since $\pi<\theta_2+w_2\tau_2<2\pi-\phi_2$, it follows that $T_1(\cdot)$ maps $[-\phi_1,\pi-\phi_1]$ into $(\psi_1,\pi-\phi_1)$. Consequently, we get $T_1(\theta_1)>\theta_1$ for all $\theta_1 \in [-\phi_1,\psi_1]$. This simplifies the analysis for every case because we have to consider the behavior of the functions L_1L_2 and $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ only over the interval $(\psi_1, \pi-\phi_1)$. Thus, our assumption does not cause any loss of generality.

3.2. The Case B_1=0

In this case y_1 and y_2 are on the same line and therefore V(0,0) is a natural fixed direction. Since $\frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1} = \frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ as $L_1L_2=1$, the other fixed directions occur if $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=1$. We formalize this observation as follows.

Theorem 3.2: Consider bimodal system (2.10) and suppose that $\lambda_i+\sigma_i$ has the same sign for both modes. Then, V(0,0) is a unique pair of fixed directions if $\lambda_i+\sigma_i \neq 0$. Furthermore, we have

i) If $\lambda_i+\sigma_i >0$ for both modes, then the fixed direction $V_i(0)$ is attractive on the interval $[-\phi_i, \pi-\phi_i]$ for i=1,2 and all possible trajectories $z_i(t)$ change mode infinite number of times as $t\to\infty$. Furthermore, bimodal system is GAS if and only if

$$\frac{\sigma_1}{w_1}+\frac{\sigma_2}{w_2}<0. \tag{3.43}$$

ii) If $\lambda_i+\sigma_i <0$ for both modes, then the fixed direction $V_i(0)$ is repulsive (stability cones are attractive on the intervals $[-\phi_i,0)$, respectively) and bimodal system is GAS.

ii) If $\lambda_i+\sigma_i=0$, then every direction is a fixed direction. More precisely, for every $\theta_1 \in (-\phi_1, \pi-\phi_1)$, $V(\theta_1,\theta_2)$ is a pair of fixed directions where $\theta_2:=G_1(F_1(\theta_1))$and bimodal system is GAS.

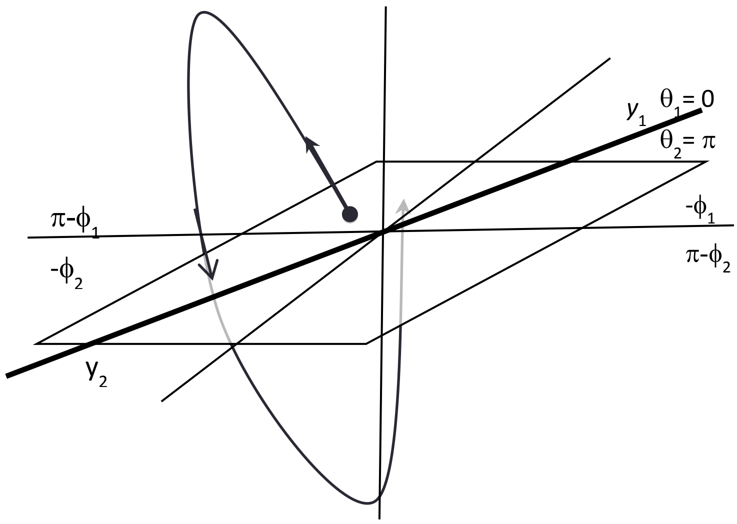

Figure 3.4: Theorem 10, Case 1.

Proof 3.2: Suppose that $z_1(t)$ starts from the direction $V_1(0)$. Then, in view of Lemma 2.4.i and Lemma 2.5.i, $z_1(t)$ changes mode at $\theta_1+w_1\tau_1=\pi$. Since $V_1(\pi)\simeq V_2(0)$ we set $z_1(\tau_1)=z_2(0)$. Thus, $z_2(t)$ starts from the direction $V_2(0)$ and changes mode again at $\theta_2+w_2\tau_2=\pi$, where $V_2(\theta_2+w_2\tau_2)\simeq V_1(0)$. Hence, $V_1(0)$ is a fixed direction or equivalently $V(0,0)$ is a pair of fixed directions. Note that in this case, we have

$$\gamma_F(0):=exp(\frac{\sigma_1}{w_1}\pi)\eta_{12}(\pi)exp\left(\frac{\sigma_2}{w_2}\pi\right)\eta_{21}(\pi) \tag{3.44}$$

Using Lemma 2.7 and Remark 2.4 we get $\eta_{12}(\pi)\eta_{21}(\pi)=1$. Consequently, $\gamma_F(0)<1$ if and only if $\frac{\sigma_1}{w_1}+\frac{\sigma_2}{w_2}<0.$

i) Note that, if $\lambda_i+\sigma_i>0$ for both modes, then using Corollary(3.1) we have

$$T_1([0,\pi-\phi_1])\subset[0,\pi-\phi_1]\ ;\ T_1([-\phi_1,0])\subset[-\phi_1,0]. \tag{3.45}$$

Recall that in view of Lemma 2.6.i, we have $1>\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\geq 0$ where the equality holds only at $\theta_1=\pi-\phi_1$. Then, we have $\frac{cot\,(\theta_1)+b_1}{cot\,(\theta_{11})+b_1}<1$. Therefore, Lemma 3.2.i-ii implies (since $L_1L_2=1$) that if $\theta_1>0$, then $\theta_1>T_1(\theta_1)$. Furthermore, if $\theta_1<0$, then $\theta_1<T_1(\theta_1)=\theta_{11}$. Hence, it follows that $T_1^k(\theta_1)\rightarrow 0$ as $k\rightarrow\infty$ for any $\theta_1\in[-\phi_1,\pi-\phi_1]$. Thus, since $G_1(F_1(0))=0$, it follows that $V(0,0)$ is a unique pair of fixed directions

and $V_1(0)$ is attractive on $[-\phi_1,\pi-\phi_1]$. (This also means that all the trajectories change mode infinite number of times as $t\to\infty$). Then, in view of Theorem 3.1 bimodal system is GAS if and only if the trajectories starting from the fixed directions are stable or equivalently if and only if equation (3.10) holds.

ii) Similar to the first case, we have $T_1[0,\theta_{1r})\subset[0,\pi-\phi_1)$ and $T_1((-\phi_1,0])\subset(-\phi_1,0]$. Since $\lambda_i+\sigma_i<0$ in view of Lemma 2.6.ii and Lemma 3.2, we get $1<\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=\frac{\cot(\theta_1)+b_1}{\cot(\theta_{11})+b_1}$. Hence, if $\theta_1>0$, we have $\theta_1<T_1(\theta_1)=\theta_{11}$ and if $\theta_1<0$, we have $\theta_1>T_1(\theta_1)=\theta_{11}$. This implies that if $\theta_1>0$, then there exists a finite integer $k\geq 0$ such that $G_1(F_1(T_1^k(\theta_1)))\in[-\phi_2,-\varphi_2]$ for any $\theta_1\in(0,\pi-\phi_1]$. Similarly, if $\theta_1<0$, then there exists a finite integer l such that $T_1^l(\theta_1)\in[-\phi_1,-\varphi_1]$. Therefore, all the trajectories change mode finite number of times as $t\to\infty$ and enter stability cones. Consequently, fixed direction pair $V(0,0)$ is repulsive and the stability cones $[-\phi_i,-\varphi_i]$ are attractive over the intervals $[-\phi_i,0)$. This means that bimodal system is GAS. Note that equation (3.10) is satisfied automatically as $\sigma_i<-\lambda_i<0$, for $i=1,2$.

iii) If $\lambda_i+\sigma_i=0$ for $i=1,2$, then we have $\frac{\cot(\theta_1)+b_1}{\cot(\theta_{11})+b_1}=1$ for $\theta_1\in(-\phi_1,\pi-\phi_1)$. Consequently, every direction is a fixed direction. Since $\varphi_i=\phi_i=\frac{\pi}{2}$ in this case, the directions $V_i(-\phi_i)$ are stability cones, $i=1,2$. Thus, every trajectory (except the ones starting along $V_i(-\phi_i)$ change mode infinite number of times as $t\to\infty$. At a fixed direction $V_1(\theta_1)$ where $\theta_1\neq 0$, we have $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=1$. Then, Corollary 3.1 implies that $\gamma_F(\theta_1)<1$. Since $\sigma_i<0$ for $i=1,2$ as in the previous case, the fixed direction $V_1(0)$ is also stable. Consequently, bimodal system is GAS.

Theorem 3.3: Let us consider bimodal system (2.10) where $\lambda_1+\sigma_1\leq 0$ and $\lambda_2+\sigma_2>0$. Then, the following hold.

i) If $b_2=|b_1|$, then $V_1(0)$ is a unique fixed direction which is attractive on $[0,\pi-\phi_1]$ and the stability cone $[-\phi_1,-\varphi_1]$ is attractive on $[-\phi_1,0)$.

ii) If $b_2>|b_1|$, then $V_1(0)$ is a fixed direction and there exists another fixed direction $V_1(\theta_1^*)$ where $\theta_1^*\in(-\varphi_1,0)$. Moreover, $V_1(0)$ is attractive on $(\theta_1^*,\pi-\phi_1]$, $V_1(\theta_1^*)$ is repulsive and the stability cone $[-\phi_1,-\varphi_1]$ is attractive on $[-\phi_1,\theta_1^*)$.

iii) If $b_2<|b_1|$, then $V_1(0)$ is a fixed direction and there exists another fixed *direction $V_1(\theta_1^*)$ where $\theta_1^* \in (0,\pi-\phi_1]$. Moreover, $V_1(\theta_1^*)$ is attractive on $(0,\pi-\phi_1]$, $V_1(0)$ is repulsive and the stability cone $[-\phi_1,-\varphi_1]$ is attractive on $[-\phi_1,0)$. For all the cases considered above, bimodal system is GAS if and only if* $\frac{\sigma_1}{w_1}+\frac{\sigma_2}{w_2}<0$.

Proof 3.3 : In view of Lemma 2.6.i-ii, $\left|\frac{dF_1}{d\theta_1}\right|$ *is a nonincreasing function of θ_1 as $\lambda_1+\sigma_1\leq 0$ and* $\left|\frac{dF_2}{d\theta_2}\right|$ *is an increasing function of θ_2 as $\lambda_2+\sigma_2>0$. However, since $\theta_2=G_1(F_1(\theta_1))$, it follows that θ_2 decreases as θ_1 increases over the interval $(-\varphi_1,\pi-\phi_1]$. Thus,* $\left|\frac{dF_2}{d\theta_2}\right|$ *is a decreasing function of θ_1. Since* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *is continuous on $(-\varphi_1,\pi-\phi_1]$, it follows that* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *is a decreasing function of θ_1. Consequently, there exists a unique $\theta_1^* \in (-\varphi_1,\pi-\phi_1]$ such that,*

$$\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=\theta_1^*} = 1 \tag{3.46}$$

i) If $b_2=|b_1|$, then since $b_2>0$ and $b_1\leq 0$ we get

$$\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=\theta_1^*} = exp(-b_1\pi - b_2\pi) = 1 \tag{3.47}$$

Thus, $V_1(0)=V_1(\theta_1^*)$ is a unique fixed direction. Moreover, since $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *is a decreasing function of θ_1, Lemma 3.2 implies that $\theta_1>T_1(\theta_1)$ over the intervals $[0,\pi-\phi_1]$ and $(-\varphi_1,0)$. Consequently, $V_1(0)$ is attractive on $[0,\pi-\phi_1]$ and the stability cone $[-\phi_1,-\varphi_1]$ is attractive on the interval $[-\phi_1,0)$. This also implies that the trajectories starting from the direction $V_1(\theta_1)$ where $\theta_1 \in [0,\pi-\phi_1]$ change mode infinite number of times as $t\rightarrow\infty$. The trajectories starting along the direction $V_1(\theta_1)$ where $\theta_1 \in (-\varphi_1,0)$ change mode finite number of times as $t\rightarrow\infty$ and enter the stability cone $[-\phi_1,-\varphi_1]$ if $\lambda_1+\sigma_1<0$. If $\lambda_1+\sigma_1=0$, then The trajectories starting along the direction $V_1(\theta_1)$ where $\theta_1 \in (-\varphi_1,0)$ also change mode infinite number of times and converge to the stability cone along the direction of $V_1(-\frac{\pi}{2})$ (In this case the stability cone behaves as a degenerate fixed direction). Since $\sigma_1<-\lambda_1<0$, such trajectories decay to the origin. Thus, it follows that bimodal system is GAS if and only if* $\frac{\sigma_1}{w_1}+\frac{\sigma_2}{w_2}<0$.

ii) If $b_2>|b_1|$, then

$$\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=0} = exp(-b_1\pi - b_2\pi) = 1 \qquad (3.48)$$

Thus, as $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ is a decreasing function on $(-\varphi_1,\pi-\phi_1]$, there exists a unique $\theta_1^* \in [-\varphi_1,0)$ such that $\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=\theta_1^*}=1$. Therefore, there are two fixed directions, $V_1(0)$ and $V_1(\theta_1^*)$. Since $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}<1$ on $(\theta_1^*,\pi-\phi_1]$, by Lemma 3.2 it follows that $\theta_1>T_1(\theta_1)$ on $(0,\pi-\phi_1]$ and $\theta_1<T_1(\theta_1)$ on $(\theta_1^*,0)$. Consequently, $V_1(0)$ is attractive on $(\theta_1^*,\pi-\phi_1]$. Since $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}>1$ on $[-\varphi_1,\theta_1^*)$ Lemma 3.2 implies that $\theta_1>T_1(\theta_1)$ on $[-\varphi_1,\theta_1^*)$. Consequently, the stability cone $[-\phi_1,-\varphi_1]$ is attractive on the interval $[-\phi_1,\theta_1^*)$. This implies that $V_1(\theta_1^*)$ is repulsive. Also note that Corollary 3.1 implies that $\gamma_F(\theta_1^*)<1$ or equivalently $V_1(\theta_1^*)$ is a stable fixed direction. Then it follows that bimodal system is GAS if and only if $\frac{\sigma_1}{w_1}+\frac{\sigma_2}{w_2}<0$, again.

iii) If $b_2<|b_1|$, then

$$\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=0} = exp(-b_1\pi - b_2\pi) > 1 \qquad (3.49)$$

Thus, there exists a unique $\theta_1^* \in [0,\pi-\phi_1]$ such that

$$\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=\theta_1^*} = exp(-b_1\pi - b_2\pi) > 1 \qquad (3.50)$$

Therefore, there are two fixed directions, $V_1(0)$ and $V_1(\theta_1^*)$. Since $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}<1$ on $(\theta_1^*,\pi-\phi_1]$ and $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}>1$ on $(-\phi_1,\theta_1^*)$, Lemma 3.2 implies that $\theta_1>T_1(\theta_1)$ on $(\theta_1^*,\pi-\phi_1]$ and $\theta_1<T_1(\theta_1)$ on $(0, \theta_1^*)$. This implies that $V_1(\theta_1^*)$ is attractive on $(0,\pi-\phi_1]$ and since $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}>1$ on $[-\varphi_1,0)$ again by Lemma 3.2 it follows that $\theta_1>T_1(\theta_1)$ on $(-\varphi_1,0)$. Consequently, the stability cone $[-\phi_1,-\varphi_1]$ is attractive on the interval $[-\phi_1,0)$. This implies that $V_1(0)$ is repulsive. Also note that by Corollary 3.1 we have $\gamma_F(\theta_1^*)<1$ or *equivalently $V_1(\theta_1^*)$ is a stable fixed direction. This implies that bimodal system is GAS if and only if $\frac{\sigma_1}{w_1}+\frac{\sigma_2}{w_2}<0$, again.*

Remark 3.2: The trajectories for bimodal systems (2.4) behave in a similar manner in case of $\lambda_1+\sigma_1>0$ and $\lambda_2+\sigma_2\leq 0$, that is,

i) If $b_1=|b_2|$, then $V_1(0)$ is a unique fixed direction which is attractive on $[-\phi_1,0)$ and the stability cone $[-\phi_2,-\varphi_2]$ is attractive on $(0,\pi-\phi_1]$.
ii) If $b_1<|b_2|$ then $V_1(0)$ is a fixed direction and there exists another fixed direction $V_1(\theta_1^*)$ where $\theta_1^*\in(-\varphi_1,0)$. Moreover, $V_1(\theta_1^*)$ is attractive on $[-\phi_1,0)$, $V_1(0)$ is repulsive and the stability cone $[-\phi_2,-\varphi_2]$ is attractive on $(0,\pi-\phi_1]$.
iii) If $b_1>|b_2|$, then $V_1(0)$ is a fixed direction and there exists another fixed direction $V_1(\theta_1^*)$ where $\theta_1^*\in(0,\pi-\phi_1]$. Moreover, $V_1(0)$ is attractive on $[-\phi_1,\theta_1^*)$, $V_1(\theta_1^*)$ is repulsive and the stability cone $[-\phi_2,-\varphi_2]$ is attractive on $(\theta_1^*,\pi-\phi_1]$.
For all the cases considered above, bimodal system is GAS $\Leftrightarrow \frac{\sigma_1}{w_1}+\frac{\sigma_2}{w_2}<0$.

3.3. The Case $B_1>0$

In this case, recall that by Lemma 3.3.ii $L_1L_2\geq 1$ and it is a decreasing function of θ_1 for $\theta_1>0$. If $\theta_1<0$, then in view of Lemma 3.3.ii we have $L_1L_2\leq 0$ over the interval $[-\theta_{10},0)$ and $0\leq L_1L_2\leq 1$ for $\theta_1\in(-\varphi_1,-\theta_{10}]$. Furthermore, $\lim_{\theta_1\to 0^{\pm}} L_1L_2=\pm\infty$. In this framework, we have the following results for the variants of this case.

Case 3.1: $\lambda_1+\sigma_1\leq 0$, $\lambda_2+\sigma_2>0$:

Recall from Lemma 3.3.iii that $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ is a strictly decreasing function of θ_1 over the interval $(-\varphi_1,\pi-\phi_1]$ where

$$\left.\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\right|_{\theta_1=\pi-\phi_1}=0 \quad and \quad lim_{\theta_1\to(-\varphi_1)^+}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=\infty \tag{3.51}$$

if $\lambda_1+\sigma_1<0$. *If* $\lambda_1+\sigma_1=0$, *then* $lim_{\theta_1\to(-\varphi_1)^+}\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=1$. *These facts imply that* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ *is a decreasing function of* θ_1 *over both of the intervals* $(-\varphi_1,-\theta_{10}]$ *and* $(0,\pi-\phi_1]$ *and there are two versions of this case.*

- *If* $\lambda_1+\sigma_1<0$, *then equations (3.51) hold. Thus,* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ *decreases from* ∞ *to zero over the interval* $(-\varphi_1,-\theta_{10}]$ *and* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ *decreases from* ∞ *to 0 over the interval* $[0,\pi-\phi_1]$. *Therefore, there exist unique* $\theta_1^{*1}\in[0,\pi-\phi_1]$ *and unique* $-\theta_1^{*2}\in(-\varphi_1,-\theta_{10}]$ *such that*

$$\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2\bigg|_{\theta_1=\theta_1^{*1}}=\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2\bigg|_{\theta_1=\theta_1^{*2}}=1 \qquad (3.52)$$

This implies that $V(\theta_1^{*1},-\theta_2^{*1})$ *and* $V(-\theta_1^{*2},\theta_2^{*2})$ *are two pairs of fixed directions where* $-\theta_2^{*1}:=G_1(F_1(\theta_1^{*1}))$ *and* $\theta_2^{*2}:=G_1(F_1(-\theta_1^{*2}))$. *Furthermore, Lemma 2.5 and the equations above imply that* $V(\theta_1^{*1},-\theta_2^{*1})$ *is an attractive pair and* $V(-\theta_1^{*2},\theta_2^{*2})$ *is a repulsive pair. Thus,* $V(\theta_1^{*1},-\theta_2^{*1})$ *is attractive in the interval* $(-\theta_1^{*2},\pi-\phi_1]\subset H\cap S_1$ *and* $[-\varphi_2,\theta_2^{*2})\subset H\cap S_2$; *where as the stability cone* $[-\phi_1,-\varphi_1]$ *is attractive for the regions* $[-\phi_1,-\theta_1^{*2})\subset H\cap S_1$. *Further note that since* $lim_{\theta_1\to(-\varphi_1)^+}L_1L_2=1$ *and decreases as* θ_1 *increases, it follows that* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\Big|_{\theta_1=-\theta_1^{*2}}\geq 1$. *Consequently, equation (3.4) implies that* $V(-\theta_1^{*2},\theta_2^{*2})$ *is a repulsive stable pair of fixed directions. But there is no guarantee for the stability of* $V(\theta_1^{*1},-\theta_2^{*1})$. *Consequently, the overall system is stable if and only if the trajectories starting from the fixed directions* $V(\theta_1^{*1},-\theta_2^{*1})$ *are stable or equivalently* $\gamma*(\theta_1^{*1},-\theta_2^{*1})<1$.

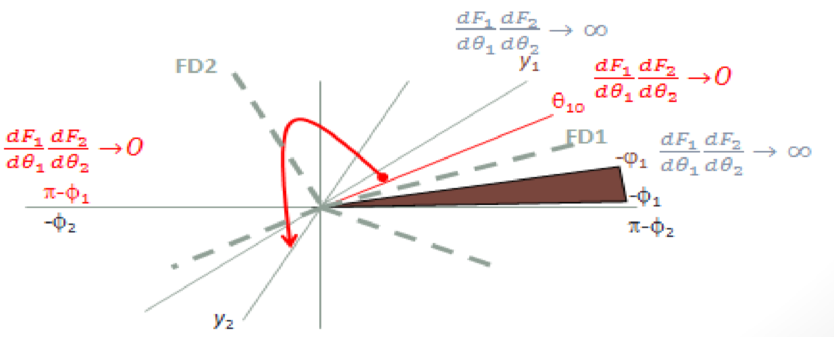

Figure 3.5: Geometry for $\lambda_1+\sigma_1>0$.

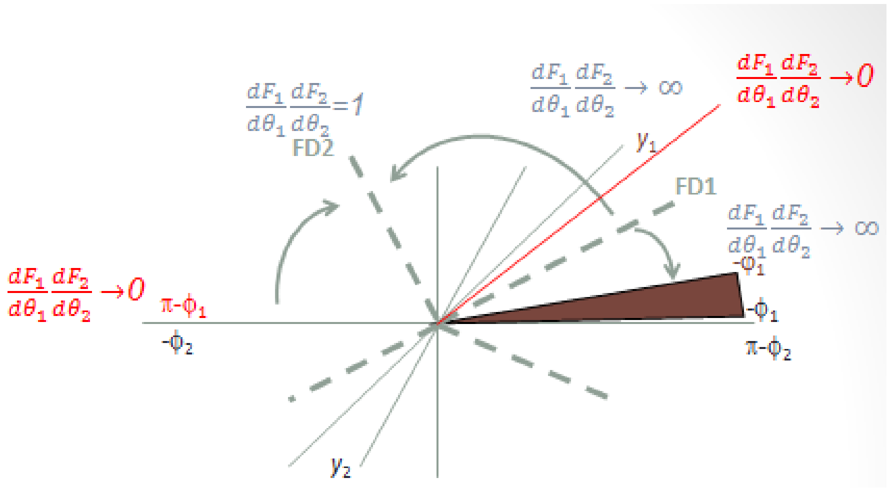

Figure 3.6: Attr./Rep. of fixed directions for $\lambda_1+\sigma_1>0$.

- *If $\lambda_1+\sigma_1=0$, then the fixed directions $V(-\theta_1^{*2}, \theta_2^{*2})$ merge into the stability cone which becomes a single line along the direction $V_1(-(\frac{\pi}{2}))\simeq V_2(\pi-\phi_2)$. Since $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ decreases from 1 to $-\infty$ over the interval $(-\phi_1,0)$ and from ∞ to zero over the interval$(0,\pi-\phi_1]$, Lemma 2.5 implies that the stability cone $V_1(-\frac{\pi}{2}))$ is repulsive and consequently $V(\theta_1^{*1},-\theta_2^{*1})$ is attractive for the whole H. Consequently, the overall system is GAS if and only if the trajectories starting from the fixed directions $V(\theta_1^{*1},-\theta_2^{*1})$ are stable.*

Case 3.2: $\lambda_1+\sigma_1\leq 0$, $\lambda_2+\sigma_2\leq 0$:

In view of Lemma 2.6, we have $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}\geq 1$ over the entire domain of $T_1(\cdot)$. Therefore, equation (3.10) implies that for any fixed direction that may exist, the convergence rate $\gamma_F < 1$. Then, the trajectories starting from this fixed direction will be stable. Consequently, for this case, the overall system is always GAS. Some of the variants of this case are itemized below.

- *If $\lambda_2+\sigma_2=0$ and $\lambda_1+\sigma_1<0$, then $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ decreases monotonically from ∞ to zero over the interval $(-\varphi_1,-\theta_{10}]$. Therefore, there exists a $-\theta_1^* \in(-\varphi_1,-\theta_{10}]$ such that*

$$\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2\Big|_{\theta_1=-\theta_1^*} = 1 \tag{3.53}$$

Since $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2>1$ *for* $\theta_1 < -\theta_1^*$ *and* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2<1$ *for* $\theta_1 > -\theta_1^*$*, it follows that* $V(-\theta_1^*,\ \theta_2^*)$ *(where* $\theta_2^*:=G_1(F_1(-\theta_1^*))$*) is a repulsive pair of fixed directions. Consequently, the stability cone represented by* $V_2(-\frac{\pi}{2})$ *is attractive for the region* $(-\theta_1^*,\pi-\phi_1]$ *in* $H\cap S_1$*, whereas the stability cone* $[-\phi_1,-\varphi_1]$ *is attractive for the region* $[-\phi_1,-\theta_1^*)$ *in* $H\cap S_1$*. Furthermore, the overall system GAS.*

- *If* $\lambda_1+\sigma_1=\lambda_2+\sigma_2=0$*, then* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}=1$ *over the domain of* $T_1(\cdot)$*. Therefore, Lemma 2.6 and Lemma 3.3.ii implies that* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ *decreases monotonically from one to zero over the interval* $[-\varphi_1,-\theta_{10}]$ *and decreases from* ∞ *to 1 over the interval* $[0,\pi-\phi_1]$*. Consequently, the stability cone* $V_1(-\frac{\pi}{2})$ *is repulsive and the stability cone* $V_2(-\frac{\pi}{2})$ *is attractive for the whole* $H\cap S_2$*. Consequently, overall system is GAS.*
- *If* $\lambda_2+\sigma_2<0$ *and* $\lambda_1+\sigma_1<0$*, then in view of Lemma 3.3.ii, we get* $1\geq L_1L_2\geq 0$ *and decreasing for* $\theta_1 \in (-\varphi_1,-\theta_{10}]$*, but* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ *is no longer a monotone (increasing or decreasing) function. This is due to the fact that* $\left|\frac{dF_1}{d\theta_1}\right|$ *is a decreasing function of* θ_1 *and* $\left|\frac{dF_2}{d\theta_2}\right|$ *is an increasing function of* θ_1 *over the domain of* $T_1(\cdot)$*. For instance,* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2=\infty$ *at* $\theta_1=-\varphi_1$ *and at* $\theta_1=-\theta_{10}$ *it is equal to zero. Thus, there exists at the least one fixed direction. But, depending on the existence and the values of the local minima and maxima of the function* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2$ *there may be more. Furthermore, by Lemma 3.3.ii, it follows that* $L_1L_2\leq 0$ *for* $\theta_1 \in [-\theta_{10},0)$ *if* $\theta_{1r} > 0$*. Thus,* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2<1$ *or equivalently* $T_1(\theta_1)>\theta_1$ *over the interval* $[-\theta_{10},0)$*. Similarly, in view of Lemma 2.6.ii and Lemma 3.3.ii, we get* $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}L_1L_2\geq 1$ *over the interval* $[0,\theta_{1r})$ *and this also implies that* $T_1(\theta_1)>\theta_1$ *over the interval* $[0,\theta_{1r})$*. Therefore, the stability cone* $[-\phi_2,-\varphi_2]$ *is attractive at least for the region* $[-\phi_2,\theta_{2z}]$ *in* $H\cap S_2$*. If* $\theta_{1r} \leq 0$*, then since* $G_1(F_1(\theta_{1r}))=-\varphi_1$*, the stability cone* $[-\phi_2,-\varphi_2]$ *is again attractive for the region* $[-\phi_2,\theta_{2z}]$*.*

Case 3.3: $\lambda_1+\sigma_1>0$*,* $\lambda_2+\sigma_2\leq 0$ *:*

In this case, Lemma 3.3.iv implies that $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ *is strictly increasing function of* θ_1 *over the domain of* $T_1(\theta_1)$*. Since* L_1L_2 *is strictly decreasing for both* $\theta_1>0$ *and* θ_1

$\in (-\phi_1,-\theta_{10}]$ by Lemma 3.3.iv, it follows that the function $\frac{dF_1}{d\theta_1}\frac{dF_2}{d\theta_2}$ is no longer monotone (decreasing or increasing) over these intervals. As in the previous case, this is left for future study.

Remark 3.3: In view of observations we get so far, one can write an algorithm which calculates both fixed directions and convergence rate $\gamma_F(-\theta_1^*)$, numerically, with the help of a computer algorithm. For this purpose, the equation $\sin(\theta_1+w_1\tau_1)=\exp(-b_1(w_1\tau_1))\sin(\theta_1)$ can be used for a trajectory that starts from the 1^{st} mode. This equation gives us the relation between the angle that we start at P_i^+ (i.e. the angle θ_1) and the angle at the moment the trajectory hits P_i^- (i.e. the angle $\theta_1+w_1\tau_1$). But, since this is a nonlinear equation, we need an estimated angle for $\theta_1+w_1\tau_1:=\theta_1$. To this end, we use $\pi-\phi_1$. Estimation can be improved constructing a loop and iteratively modifying with a command as follows $\theta_1 = \pi - \sin^{-1}(\sin(\theta_1)\exp(-b_1(\tilde{\theta}_1-\theta_1)))$

Applying the same procedure also for the 2^{nd} mode, the corresponding fixed direction angle θ_2^* and afterwards the convergence rate $\gamma_F(\theta_1^*)$ can also be calculated. The followings are examples which are evaluated in this way.

Example 3.1: Consider bimodal system (2.10) with the following spectrum

$$\{-\lambda_1,\sigma_1,w_1\} = \{-0.8,-1, 2\}, \{-\lambda_2,\sigma_2,w_2\} = \{-0.4, 0.2, 3\} \tag{3.54}$$

where

$$A_1=\begin{bmatrix} \frac{1}{2} & -\frac{25}{4} & 0 \\ 1 & -\frac{5}{2} & 0 \\ 0 & 1 & -0.8 \end{bmatrix} \tag{3.55}$$

Note that $\lambda_1+\sigma_1<0$, $\lambda_2+\sigma_2>0$, and $B_1=0.2885 > 0$. The fixed direction is $\theta_1^*\simeq 38°$ with the rate of convergence $\gamma_F(\theta_1^*)=0.9120$. Therefore, the system is GAS. We can change the entries of the system matrix A_1 without changing its spectrum A_1 as follows.

$$A_1 = \begin{bmatrix} 1 & -8 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & -0.8 \end{bmatrix} \tag{3.56}$$

Thus, we still have $\{-\lambda_1, \sigma_1, w_1\} = \{-0.8, -1, 2\}$*. However,* $B_1 = 0.4487$ *and the fixed direction is* $\theta_1^* \simeq 46°$ *with convergence rate* $\gamma_F(\theta_1^*) = 1.2637$*. Consequently, the system is not GAS. Further change of the entries of* A_1 *yields*

$$A_1 = \begin{bmatrix} 2 & -13 & 0 \\ 1 & -4 & 0 \\ 0 & 1 & -0.8 \end{bmatrix} \tag{3.57}$$

where the spectrum is again

$$\{-\lambda_1, \sigma_1, w_1\} = \{-0.8, -1, 2\}. \tag{3.58}$$

But $B_1 = 0.7692$ *and the fixed direction is* $\theta_1^* \simeq 55.5°$ *the rate of convergence* $\gamma_F(\theta_1^*) = 2.0111$*. Therefore, the system is unstable again.*

Thus, it's clear that not only the spectrum of the system matrices but also the coupling condition B_1 *is also very important for determining GAS. The graph of the unstable trajectory is as follows. It starts in red line and goes toward to infinity as* $t \to \infty$*.*

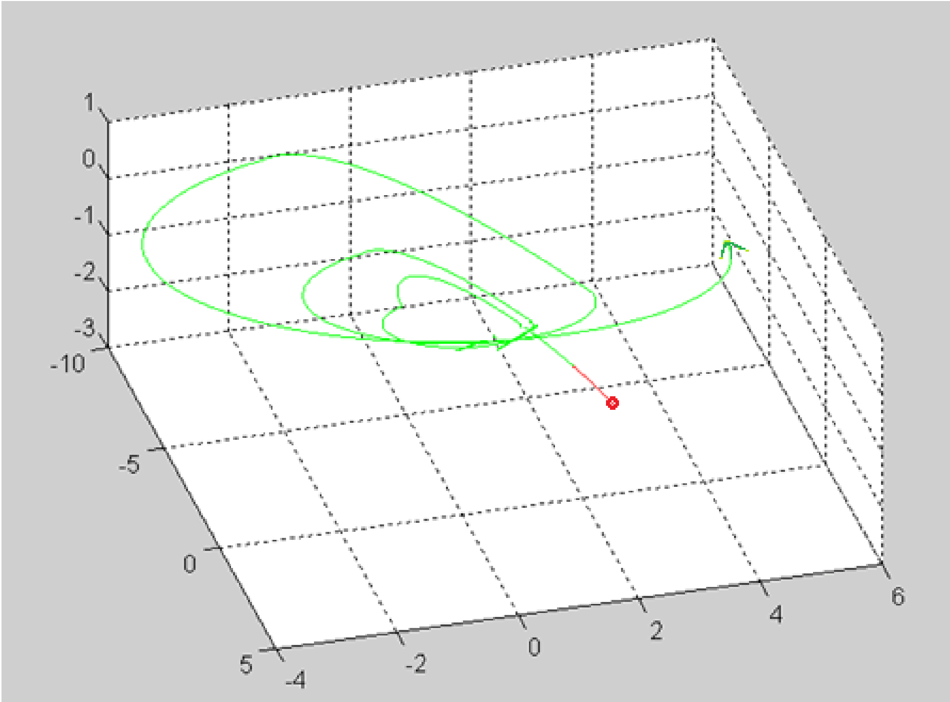

Figure 3.7: Unstable trajectory for Example 1.

This time let's assign both of the modes as stable, i.e. both $\sigma_i < 0$ for i=1,2. For this purpose, let's change only σ_2 and investigate the effects of the coupling condition.

Example 3.2: Consider the bimodal system (2.10) and assume that the spectrum of the system matrices as

$$\{-\lambda_1,\sigma_1,w_1\} = \{ -0.8, -1, 2\}, \{-\lambda_2,\sigma_2,w_2\} = \{ -0.4, -0.2, 3\} \qquad (3.59)$$

where

$$A_1 = \begin{bmatrix} 2 & -13 & 0 \\ 1 & -4 & 0 \\ 0 & 1 & -0.8 \end{bmatrix} \qquad (3.60)$$

So $\lambda_1+\sigma_1<0$, $\lambda_2+\sigma_2>0$, $B_1=0.7965>0$ and rate of convergence $\gamma_F(\theta_1^*)=1.1065$ which means the system is unstable. Also, fixed direction is $\theta_1^* \simeq 62,6°$. If we change *the entries of the system matrix A_1, remaining the spectrum same, rate of convergence changes. That is, choose the system matrix A_1 as*

$$A_1=\begin{bmatrix} 1 & -8 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & -0.8 \end{bmatrix} \tag{3.61}$$

so the spectrum is same : $\{-\lambda_1,\sigma_1,w_1\}=\{-0.8, -1, 2\}$. But, we changed also $a_{11} \in A_1$ which is efficient in B_1. Therefore, B_1 changed as 0.4646, so the rate of convergence changes : $\gamma_F(\theta_1^*)=0.7908$, that is, the system is stable, fixed direction is $\theta_1^* \simeq 56{,}6°$. Note that θ_1^* is decreasing as B_1 decreases. Remember that $\theta_1^*=0$ for $B_1=0$.

Thus, it's clear that not only the spectrum of the system matrices but also the coupling condition B_1 is also very important on the stability analysis.

Example 3.3: Let's consider the bimodal system (2.10) and assume that the spectrum of the system matrices as

$$\{-\lambda_1,\sigma_1,w_1\} = \{-0.4429, -0.5286, 2.9585\}, \{-\lambda_2,\sigma_2,w_2\} = \{-0.5, -0.4887, 5\} \tag{3.62}$$

where

$$A_1=\begin{bmatrix} 1 & -6 & 1 \\ 2 & -2 & 1 \\ 0 & 1 & -0.5 \end{bmatrix} \tag{3.63}$$

So $\lambda_1+\sigma_1<0$, $\lambda_2+\sigma_2>0$, $B_1=0.0426>0$ and rate of convergence $\gamma_F(\theta_1^*)=1$, that is the system is marginally stable. If $\sigma_2=-0.4887$ becomes to 0.5, γ_F also increases and reaches to the value 1.0061, then the system is unstable. Conversely, decreasing σ_2 to 0.4, γ_F also decreases and reaches to the value 0.9538, then the system is stable. Similar analysis should be made using the eigenvalue λ_2, but taking into account that marginal changing in λ_2 should make $B_1<0$.

4. CONCLUSIONS AND FUTURE WORKS

In this thesis, the structure and stability of bimodal systems in $\mathbb{R}^3$ are investigated under the assumptions (A1-A3). It is shown that the trajectories which change mode infinite number of times eventually converge to fixed directions or stability cones on H. Consequently, GAS is determined by the stability of the trajectories starting from the fixed directions. Thus, the existence of trajectories which change mode infinite number of times (as $t\to\infty$) is a crucial property in bimodal systems.

It's also shown that for the case $B_1=0$ conditions for GAS in $\mathbb{R}^3$ is reduced to the conditions in $\mathbb{R}^2$, which is well known in the literature [Iwatani and Hara, 2006], [Çamlıbel et.al., 2003]. That is, the necessary and sufficient condition for GAS is $\frac{\sigma_1}{w_1}+\frac{\sigma_2}{w_2}<0$ when $B_1=0$. However, as shown above, the behavior of trajectories change radically as the parameters of the subsystems change. There may be one, two or infinite number of fixed directions and one or two stability cones. Furthermore, attractiveness of the fixed directions and the stability cones also changes substantially.

It is also shown that the stability of the trajectories which change mode only a finite number of times as $t\to\infty$ is guaranteed in our setup. Since the negativity of the real eigenvalues is a necessary condition for GAS for any bimodal system, it follows that the necessary and sufficient condition for the stability of the trajectories which change mode only a finite number of times as $t\to\infty$ is the negativity of the real eigenvalues of both modes. On the other hand, it is also proved that the trajectories which change mode an infinite number of times as $t\to\infty$ are stable if and only if the rate of convergence γ_F of each fixed direction satisfies the inequality $\gamma_F<1$. With these observations we present the following conjecture.

Conjecture 4.1: Consider the bimodal system (2.4) under the assumptions A1-A3. Then, the following hold.

i) If all the trajectories change mode only finite number of times as $t\to\infty$, then bimodal system is GAS if and only if all the real eigenvalues of both modes are negative.

ii) If there are trajectories which change mode infinite number of times as $t \to \infty$, then bimodal system is GAS if and only if all the real eigenvalues of both modes are negative and all the trajectories starting from the fixed directions are stable.

The validity of the conjecture is shown for the class of bimodal systems considered in this thesis. In our future work, we plan to prove the validity of this conjecture for all bimodal systems under A1-A2 (both modes are observable and bimodal system is well-posed). If the conjecture is still valid under A1 and A2, then bimodal systems in $\mathbb{R}^n$ can be investigated by classifying the trajectories as in the above conjecture.

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Gökhan Şahan was born in 1978, in İstanbul. In 1998, he graduated from the Department of Mathematics at Ege University. Same year, he started working on his Master's degree in Algebra and Number Theory, again at the Department of Mathematics of Ege University. He received MS degree in 2002. During that time he was awarded with the Ege University Accomplishment Award and MS Success Scholarship. In 2005, he joined Department of Mathematics of Gebze Institute of Technology (GIT), for PhD studies in Applied Mathematics.

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