Esra ŞİMŞEK

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HOW TO MAKE USE OF ROBUST METHODS TO BETTER ESTIMATE THE HCCMEs

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Esra ŞİMŞEK

Fatih University

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APPROVAL PAGE

Student : Esra ŞİMŞEK

Institute : Institute of Social Sciences

Department : Economics

Thesis Subject : How to Make Use of Robust Methods to Better Estimate the HCCMEs

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Arts.

Prof. Mehmet ORHAN

Head of Department

This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Arts.

Prof. Mehmet ORHAN

Supervisor

Examining Committee Members

Prof. Mehmet ORHAN ……………………….

Prof. Murat KARAGÖZ ……………………….

Assoc. Prof. Nizamettin BAYYURT ……………………….

It is approved that this thesis has been written in compliance with the formatting rules laid down by the Graduate Institute of Social Sciences.

Assoc. Prof. Mehmet KARAKUYU

Director

AUTHOR DECLARATIONS

1. The material included in this thesis has not been submitted wholly or in part for any academic award or qualification other than that for which it is now submitted.

2. The program of advanced study of which this thesis is part has consisted of:

i) Research Methods course during the undergraduate study

ii) Examination of several thesis guides of particular universities both in Turkey and abroad as well as a professional book on this subject.

Esra ŞİMŞEK

May, 2012

ABSTRACT

Esra ŞİMŞEK **May 2012**

How to Make Use of Robust Methods to Better Estimate the HCCMEs

With the help of simulation, sensitivity of the Heteroscedasticity-Consistent Covariant Matrix Estimators (HCCMEs) to the outliers is shown much more easily. In the past, there were two research streamlines: First one is Furno (1996, and 1997) whose suggestion is to decrease the impact of outliers and bad leverage points by Weighted Least Squares (WLS). The other is the Least Median of Squares (LMS) and the Least Trimmed Squares (LTS). But both techniques have some shortcomings. The purpose of this thesis is to reduce and remove the shortcomings of the past techniques which have the negative impact of bad leverage points and outliers and reduce the lack of information, by using robust regression techniques. The HCCMEs are calculated with and without the bad leverage points and outliers and document better results. The true covariance matrix is calculated with different settings of the variance of the error term and the design matrix and then the outliers and bad leverage points will be removed by using the MCD. Observations with high MCD distances are detected by the help of robust regression techniques. We also evaluate the improvement of the HCCMEs via Quasi-t statistics and the Symmetric Loss Function.

Keywords: Heteroscedasticity, MCD, HC0, HC1, HC2, HC3, HC4, HC5, Bias, Small Sample, HCCME.

KISA ÖZET

Esra ŞİMŞEK **Mayıs 2012**

Farklı-Varyans Uyumlu Kovaryans Tahmin Edicilerin daha iyi Tahmini için Dayanıklı Yöntemlerin Kullanılması

Kovaryans matrisinin sapmasının aykırı gözlemlere duyarlılığını hem simülasyonla hem de teorik olarak gösterilmiştir. Bunlardan ilki Furno'ya (1996 ve 1997) aittir ve aykırı gözlemlerin etkilerinin Ağırlaştırılmış En Küçük Kareler (WLS) yöntemi ile azaltılmasını öngörmektedir. Diğeri ise White'ın kullandığı hata terimlerinin kalıntıları yerine daha dayanıklı tahmin yöntemlerinin (En Küçük Kareler Medyanı (LMS), En Küçük Budanmış Kareler (LTS)) kalıntılarının kullanılması ile elde edilen tahmin edicilerin kullanılmasıdır. Ancak bu iki yöntemde de bazı eksiklikler mevcuttur. Bu tezin amacı, aykırı gözlemlere ve kötü kaldıraç noktalarına karşı dayanıklılığı sağlayarak eldeki gözlemlere ait verileri maksimum düzeyde kullanarak kovaryans matrisini tahmin etmektir. Aykırı gözlemler ve kötü kaldıraç noktaları En Küçük Varyans-Kovaryans Deteminantı (MCD) kullanılarak temizlendikten sonra kovaryans matrisi tahmin edicilerinin daha iyi sonuç verdiği farklı X ve hata terimleri varyansları için gösterilmiştir.

Anahtar Kelimeler

Farklı varyans, MCD, HC0, HC1, HC2, HC3, HC4, HC5, Sapma, kırpılmış örneklem, HCCME.

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LIST OF ABBREVIATIONS

HCCME	Heteroscedasticity-Consistent Covariance Matrix Estimator
LS	Least Squares
OLS	Ordinary Least Squares
RLS	Reweighted Least Squares
SRF	Sample Regression Function
PRF	Population Regression Function
LMS	Least Median of Squares
LTS	Least Trimmed Squares
MCD	Minimum Covariance Determinant

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CHAPTER 1

INTRODUCTION

The superiority of the OLS estimation in the Classical Linear Regression Model addressed in many textbooks is due to the Gauss-Markov Theorem with its roots introduced to the literature more than two centuries ago.

1.1 Gauss-Markov Theorem

The Gauss-Markov Theorem, states that under the assumptions of the classical linear regression model, the least-squares estimators, in the class of best, linear, unbiased estimators, i.e they are BLUE. Here, “best” means having the minimum variance.

1.2 Gauss-Markov Theorem Assumptions

The classical linear regression model (CLRM) assumptions (Greene, 2008):

Assumption 1: Linear regression model, as shown in (1) (k-variable). The regression model specifies a linear relationship between y and $X_1, \dots, X_2$.

$$Y_i = \beta_1 + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i \quad (1)$$

Assumption 2: Data generation. Values taken by the regressor X are considered fixed in repeated samples and can be any mixture of constants and random variables. More technically, X is assumed to be nonstochastic.

Assumption 3: Zero mean value of disturbance u_i, which means that the independent variables will not carry useful information for prediction of u_i. Symbolically, we have

$$E(u_i|X_i) = 0 \quad (2)$$

Assumption 4: Homoskedasticity and nonautocorrelation. Each disturbance u_i has the same variance σ^2 and is uncorrelated with every other disturbance u_j.

Symbolically, we have

$$var\ (u_i|X_i) = \sigma^2 \quad (3)$$

where var stands for variance and

$$\text{cov}\ (u_i, u_j) = 0 \quad (4)$$

where i and j are two different observations and where cov means covariance.

Assumption 5: No perfect multicollinearity.. There is no exact linear relationship among any independent variables in the model.

Assumption 6: Normal distribution**.** The disturbances are normally distributed. Symbolically,

$$u \sim N(0, \sigma^2) \quad (5)$$

We consider the linear model $Y_i = \beta_1 + \beta_2 X_i + u_i$ in which Y is the dependent variable; X is the independent variable; β is the coefficient vector and u is the error term.

The estimated model is;

(6)

where $\widehat{Y_i}$ is the estimated value of Y_i. The residual r_i is the difference between the observed value and the estimated value,

$$e_i = Y_i - \widehat{Y_i} \quad (7)$$

The most popular method of estimating the coefficients is the least-squares method, that is given by; (Gauss and Legendre, around 1800)

$$minimize \sum_{i=1}^{n} e_i^2 \qquad (8)$$

Many empirical studies have shown that assumptions of the Gauss-Markov Theorem are often not plausible. Although the method of the least squares still used, it is not efficient any more.

1.3 Leverage Points

In the terminology, regression outliers are observations that do not obey the linear pattern formed by the majority of the data. Because of the outliers effect on the trend of the data, it is difficult to make a good analysis. To be sure about the whole picture one must work on the outliers. In most cases outliers are not the mistakes, they represent the data that are coming from the extraordinary conditions. But some recording or reading errors of the data are also possible. To make correct inferences, one has to detect and work on them very carefully.

An observation (x_i, y_i) is said to be a leverage point when its regressor lie outside of the majority of the regressors. The term "leverage" comes from the mechanics, because such a point pulls the LS solution towards it.

The LS method estimates σ from its residuals, e_i using the formula:

$$\hat{\sigma} = \sqrt{\frac{1}{n-k-1}\sum_{i=1}^{n} {e_i}^2}$$

(9)

where k is the number of the regressors. Once the estimate for variance is calculated one can obtain the standardized residuals $\frac{e_i}{\hat{\sigma}}$. It is also common to calculate these values and label the observations for which this figure exceeds 2.5, or less than -2.5 as the regression ouliers. The logic behind is that values generated by Gaussian Distribution are rarely larger than 2.5 or less than -2.5, whereas the other observations are considered to obey the model.

In simple regression models, where the number of regressors is small, the detection of outliers may be possible by looking at the scatterplot of the regressors and the regressand, but in multiple regression, where k is large, the detection by eye is no longer possible and residual plots become an important source of information. Since most of the regressions are carried by routinely, many results must have been affected or even determined by outliers that remained unnoticed.

We can see these plots for looking the effects of the outliers.

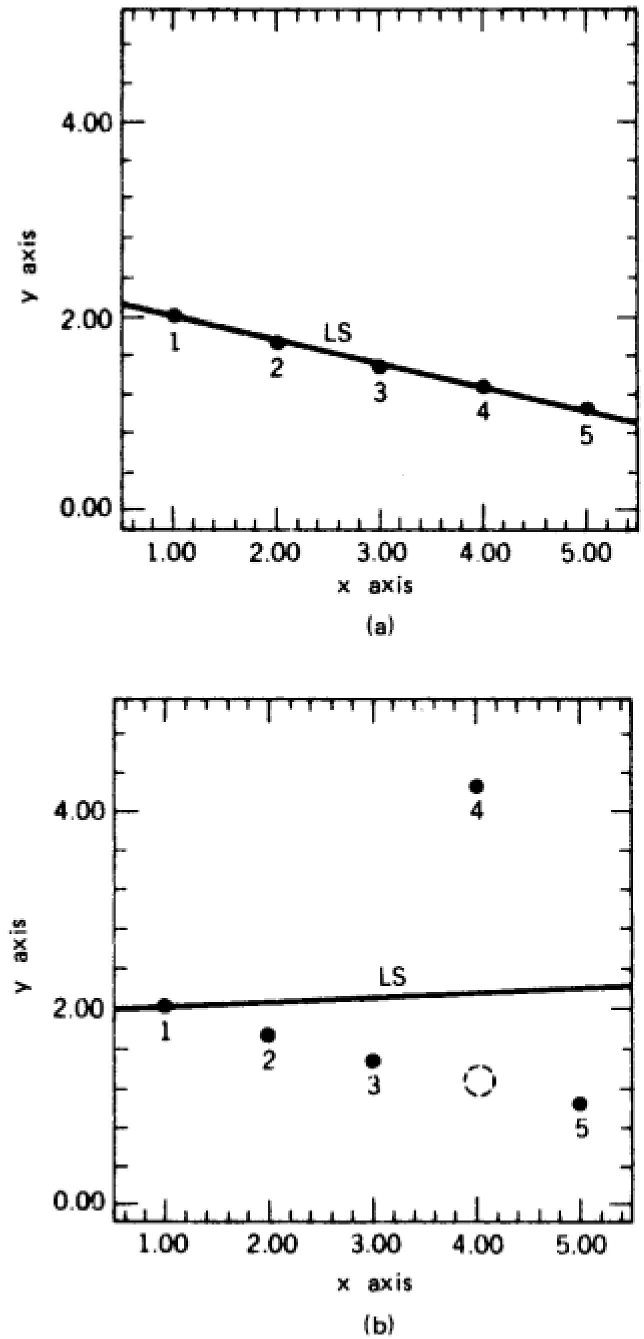
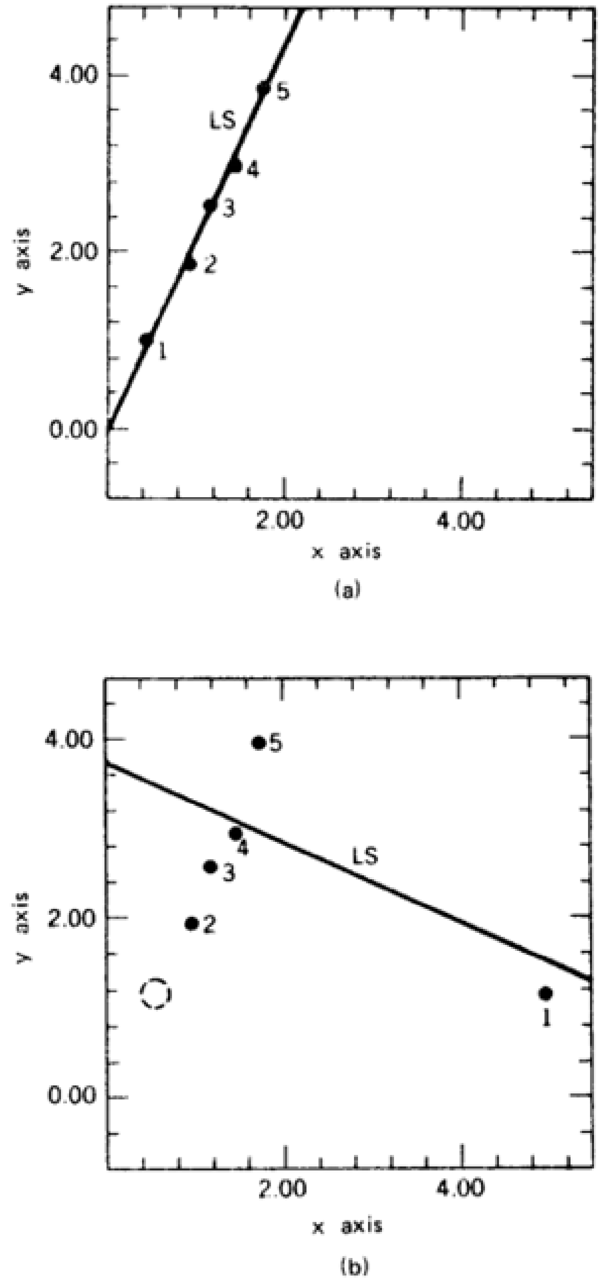


Figure 1.1a is the scatter of the 5 points. Here the least squares (LS) fits the data very well.

However, suppose that there is a wrong value of y_4 which is an outlier in the y direction. Figure 1.1b shows us the new LS line.

Figure 1.1: (a) Five points and their least squares regression line, (b) Same data with one outlier in the y-direction[1]

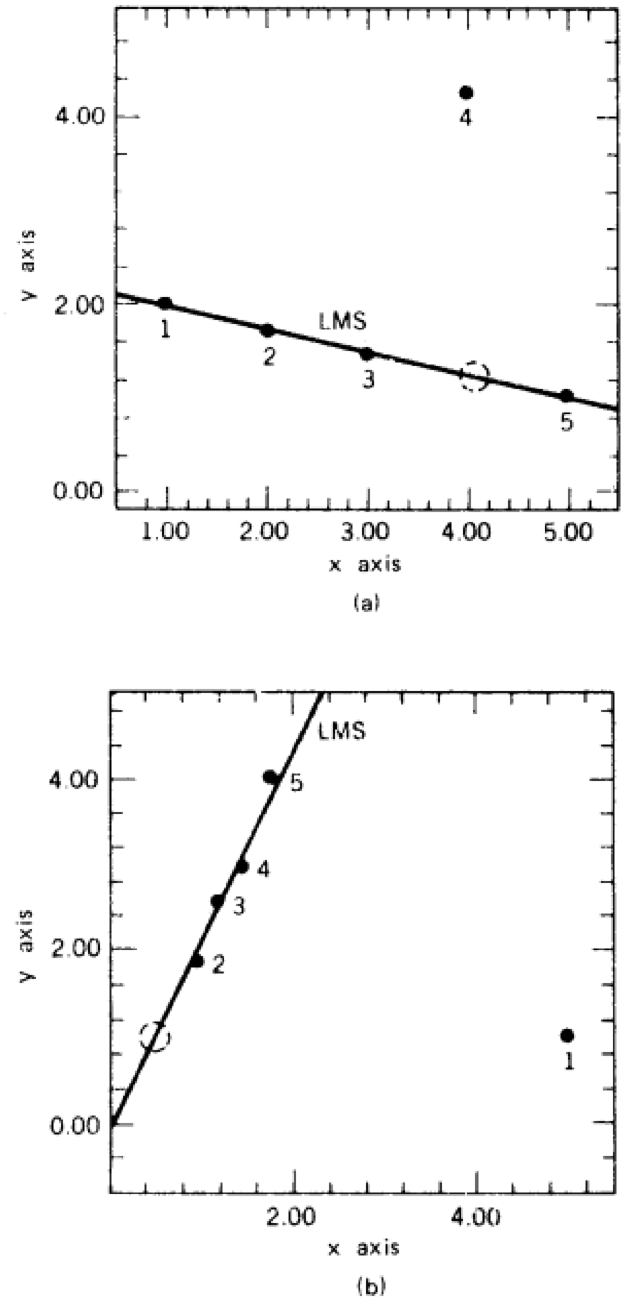
[1] **Source:** Rousseeuw , (1990): 16.11



Again the least squares (LS) fits the data very well as we see in 1.2a. The outlier can also occur in the x plane. And its effect on least squares is very large. Because it tilts the LS line. (1.2b)

Figure 1.2: (a) Five points with their least squares regression line, (b) Same data with one outlier in the x-direction[2]

[2] **Source:** Rousseeuw , (1990): 16.12



If we replace median instead of the sum in this equation:

$$minimize \sum_{i=1}^{n} r_i^2$$

we get:

$$minimize\ median\ r_i^2$$

which is the least median of squares (LMS) method (Rousseeuw, 1984).

In 1.3a and 1.3b are the desired fit for the examples in 1.1b and 1.2b. Its breakdown point is 0.50 that is the highest possible value, which means LMS can cope with the several ouliers.

Figure 1.3: Robustness of LMS regression with respect to (a) an outlier in the y direction, (b) an outlier in the x direction. [3]

[3] **Source:** Rousseeuw , (1990): 16.15

CHAPTER 2

HETEROSCEDASTICITY

One of the assumptions (Assumption 4) we have mentioned is that the errors u_i in the regression equation have a common variance σ^2. This is known as *homoscedasticity* assumption. If the errors do not have a common variance they are *heteroscedastic*. To make clear this definition we can see Figure 2.1 and Figure 2.2.

Figure 2.1 shows, the conditional variance of Y_i (which is equal to that u_i), remains the same regardless of the values taken by the variable X.

In contrast, consider Figure 2.2, which shows that the conditional variance of Y_i increases as X increases. The variances of Y_i are not the same, there is heteroskedasticity. Symbolically,

$$E(u_i^2) = \sigma_i^2 \tag{10}$$

To make the difference between homoskedasticity and heteroskedasticity clear, we can think the two-variable model $Y_i = \beta_1 + \beta_2 X_i + u_i$, Y represents savings and X representsincome. Figure 2.1 and Figure 2.2 show that as income increases, savings on the average increase. But in Figure 2.1 the variance of savings remains the same at all levels of income, whereas in Figure 2.2 it increases with income. It seems that in Figure 2.2 the higher-income families on the average save more than the lower-income families, but there is also more variability in their savings.

Figure 2.1: Homoscedastic Disturbances[4]

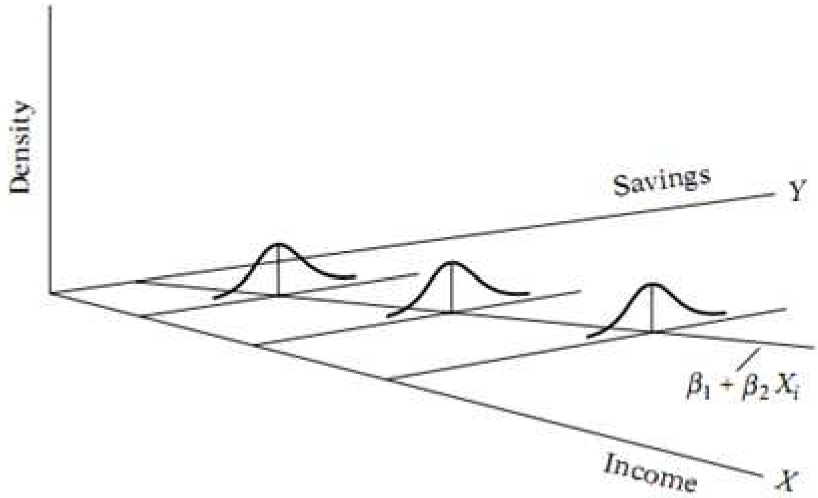
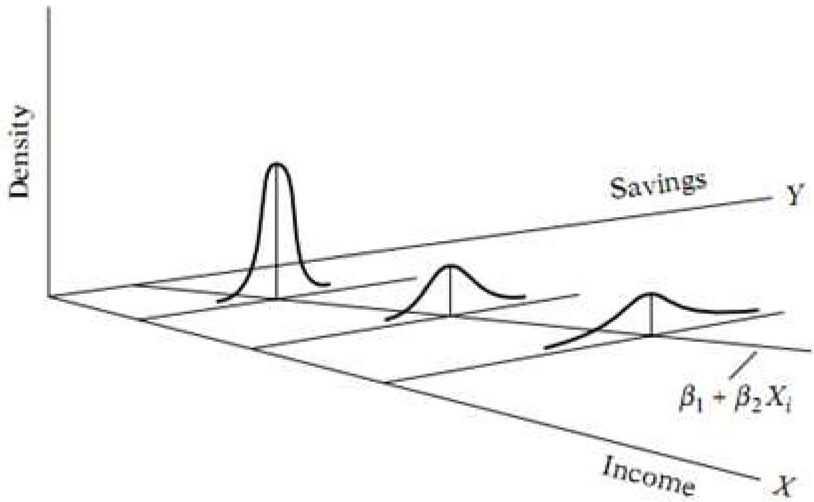


Figure 2.2: Heteroscedastic Disturbances[5]



[4] **Source:** Gujarati, 2004:73

[5] **Source:** Gujarati, 2004:74

2.1 Detection of Heteroscedasticity:

There are some formal methods of detecting heteroscedasticity.

2.1.1 Breusch-Pagan-Godfrey (BPG) test:

To illustrate this test, consider the k-variable linear regression model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i \,. \qquad (11)$$

Assume that the error variance σ_i^2 is decribed as

$$\sigma_i^2 = f(\alpha_1 + \alpha_2 Z_{2i} + \cdots + \alpha_m Z_{mi}) \qquad (12)$$

That is, σ_i^2 is some function of the nonstochastic variables Z's; some or all of the X's can serve as Z's. Specifically, assume that

$$\sigma_i^2 = \alpha_1 + \alpha_2 Z_{2i} + \cdots + \alpha_m Z_{mi} \qquad (13)$$

That is, σ_i^2 is a linear function of the Z's. If $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, $\sigma_i^2 = \alpha_1$, which is a constant. Therefore, to test whether σ_i^2 is homoscedastic, one can test the hypothesis that $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$. This is the basic idea behind the Breusch-Pagan test.

2.1.2 White's General Heteroscedasticity test:

The general test for the heteroscedasticity proposed by White does not rely on the normality assumption unlike the BPG. As an illustration of the basic idea, consider the following three-variable regression model:

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i \qquad (14)$$

The White test proceeds as follows:

Step 1. Given the data, estimate (14) and obtain the residuals, $\hat{u}_i$.

Step 2. We then run the following regression:

$$\hat{u}_i^2 = \alpha_1 + \alpha_2 X_{2i} + \alpha_3 X_{3i} + \alpha_4 X_{2i}^2 + \alpha_5 X_{3i}^2 + \alpha_6 X_{2i} X_{3i} + v_i \qquad (15)$$

That is, the squared residuals from the original regression are regressed on the original X variables or regressors, their squared values, and the cross product(s) of the regressors. Higher powers of regressors can also be introduced. Note that there is a constant term in this equation even though the original regression mayo r may not contain it. Obtain the R^2 from this regression.

Step 3. Under the null hypothesis that there is no heteroscedasticity, it can be shown that sample size (n) times the R^2 obtained from the auxiliary regression *asymptotically* follows the chi-square distribution with df equal to the number of regressors (excluding the constant term) in the auxiliary regression. That is,

$$n.R^2 \sim X_{df}^2 \qquad (16)$$

In our example, there are 5 df since there are 5 regressors in the auxiliary regression.

Step 4. If the chi-square value obtained in Eq. (16) exceeds the critical chi-square value at the chosen level of significance, the conclusion is that there is heteroscedasticity. If it does not exceed the critical chi-square value, there is no heteroscedasticity, which is to say that in the auxiliary regression (15), $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$.

2.2 HCCM Estimators:

2.2.1 Introduction:

The k-variable population regression model (PRF) involving the dependent variable y and k-1 explanatory variables may be written as:

$$y_i = \beta_1 + \beta_2 x_{2i} + \cdots + \beta_k x_{ki} + u_i \qquad (17)$$

$i = 1,2,3 \ldots, n$

where β_1 is the intercept term, the u_i is the stochastic disturbance term and n is the sample size.

Equation (17) is a shorthand expression for the following set of n simultaneous equations:

$$y_1 = \beta_1 + \beta_2 x_{21} + \cdots + \beta_k x_{k1} + u_1$$

$$y_2 = \beta_1 + \beta_2 x_{22} + \cdots + \beta_k x_{k2} + u_2 \qquad (18)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_n = \beta_1 + \beta_2 x_{2n} + \cdots + \beta_k x_{kn} + u_n$$

Let us write the system (18) in an alternative but more illuminating way as follows:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{21} & x_{31} & \cdots & x_{k1} \\ 1 & x_{22} & x_{32} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{2n} & x_{3n} & \cdots & x_{kn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad (19)$$

$$\underset{n \times 1}{\boldsymbol{y}} = \underset{n \times k}{\boldsymbol{X}} \; \underset{k \times 1}{\boldsymbol{\beta}} + \underset{n \times 1}{\boldsymbol{u}}$$

where column vector of observations on the dependent variable y

$\boldsymbol{X} = \boldsymbol{n} \times \boldsymbol{k}$ matrix giving n observations on k-1 variables X_2 to X_k, the first column of 1's representing the intercept term

$\boldsymbol{\beta} = \boldsymbol{k} \times \mathbf{1}$ column vector of the unknown parameters $\beta_1, \beta_2, \dots, \beta_k$

$\boldsymbol{u} = \boldsymbol{n} \times \mathbf{1}$ column vector of n disturbances u_i.

System (19) is known as the matrix representation of the general (k-variable) linear regression model.

Our aim is to estimate the OLS estimator of β , so firstly let us look at the k-variable sample regression function (SRF):

$$y_i = \hat{\beta}_1 + \hat{\beta}_2 x_{2i} + \cdots + \hat{\beta}_k x_{ki} + \hat{u}_i \tag{20}$$

Also it can be written in matrix notation as:

$$\boldsymbol{y} = \boldsymbol{X}\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{u}} \tag{21}$$

And in matrix form as:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{21} & x_{31} & \cdots & x_{k1} \\ 1 & x_{22} & x_{32} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{2n} & x_{3n} & \cdots & x_{kn} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \tag{22}$$

$$\begin{matrix} \boldsymbol{y} & = & \boldsymbol{X} & \hat{\boldsymbol{\beta}} & + & \boldsymbol{u} \\ n \times 1 & & n \times k & k \times 1 & & n \times 1 \end{matrix}$$

where $\hat{\beta}$ is a k-element column vector of the OLS estimators of the regression coefficients and $\hat{u}$ is an $n \times 1$ column vector of n residuals.

In the k-variable case, the OLS estimators are obtained by minimizing;

$$\sum \hat{u}_i^2 = \sum (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_{2i} - \cdots - \hat{\beta}_k x_{ki})^2$$

(23)

where $\sum \hat{u}_i^2$ is the residual sum of squres (RSS). In matrix notation, it is the same with minimizing $\hat{u}'\hat{u}$ since;

$$\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}} = [\hat{u}_1 \quad \hat{u}_2 \quad \ldots \quad \hat{u}_n] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \hat{u}_1^2 + \hat{u}_2^2 + \cdots + \hat{u}_n^2 = \sum \hat{u}_i^2 \quad (24)$$

Now from (14) we obtain

$$\hat{\boldsymbol{u}} = \boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}} \quad (25)$$

Therefore;

$$\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}} = (\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \quad (26)$$

$$= \boldsymbol{y}'\boldsymbol{y} - 2\hat{\boldsymbol{\beta}}'\boldsymbol{X}'\boldsymbol{y} - \hat{\boldsymbol{\beta}}'\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}}$$

where the properties of the transpose of a matrix is used.

Equation (26) is the matrix representation of (23). By differentiating (23) partially with respect to $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_k$ and setting the resulting expressions to zero yields k simultaneous equations in k unknowns as follows:

$$n\hat{\beta}_1 + \hat{\beta}_2 \sum x_{2i} + \hat{\beta}_3 \sum x_{3i} \ldots + \hat{\beta}_k \sum x_{ki} = \sum y_i$$

$$\hat{\beta}_1 \sum x_{2i} + \hat{\beta}_2 \sum x_{2i}^2 + \hat{\beta}_3 \sum x_{3i} x_{2i} + \cdots + \hat{\beta}_k \sum x_{2i} x_{ki} = \sum x_{2i} y_i$$

$$\hat{\beta}_1 \sum x_{3i} + \hat{\beta}_2 \sum x_{3i} x_{2i} + \hat{\beta}_3 \sum x_{3i}^2 \ldots + \hat{\beta}_k \sum x_{3i} x_{ki} = \sum x_{3i} y_i$$

… .

$$\hat{\beta}_1 \sum x_{ki} + \hat{\beta}_2 \sum x_{ki} x_{2i} + \hat{\beta}_3 \sum x_{3i} x_{ki} + \cdots + \hat{\beta}_k \sum x_{ki}^2 = \sum x_{ki} y_i \qquad (27)$$

Equation (27) can be represented by matrix form as follows:

[■(𝑛&∑▒𝑥↓2𝑖 &∑▒𝑥↓3𝑖 &…&∑▒𝑥↓𝑘𝑖 @∑▒𝑥↓2𝑖 &∑▒〖𝑥↓2𝑖〗↑2 &∑▒𝑥↓3𝑖 𝑥↓2𝑖&…&∑▒𝑥↓2𝑖 𝑥↓𝑘𝑖@∑ (28)

$$(X'X)\,\hat{\beta} = X'y \qquad (29)$$

In equation (29) the unique unknown is $\hat{\beta}$. By using the matrix algebra, if the inverse of $(X'X)$ exists, premultiplying both sides of (29) by this inverse, we obtain;

$$(X'X)^{-1}(X'X)\,\hat{\beta} = (X'X)^{-1}X'y$$

Since $(X'X)^{-1}(X'X) = I$, an identity matrix of order $k \times k$, we get;

$$\underset{k\times 1}{\hat{\beta}} = \underset{k\times k}{(X'X)^{-1}} \; \underset{k\times n}{X'} \; \underset{n\times 1}{y} \qquad (30)$$

Equation (30) is a fundamental result of the OLS theory in matrix notation.

2.2.2 Variance-Covariance Matrix of

An important assumption of the classical linear regression model is the homoscedasticity, that is, the variance of the disturbances are the same, σ^2. Many empirical studies have shown that this assumption is often not plausible.

Since this is an assumption of the classical linear regression model, it need not to be guaranteed in practice. So one has to be careful about the heteroscedasticity, its detection, consequences and the remedies to recover this problem.

When there is homoscedasticity, the variance-covariance matrix for the disturbance vector is

$$\Sigma = E(uu') = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix} \qquad (31)$$

Where σ^2 is the homoscedastic variance of u_i. Also this variance-covariance matrix can be obtained from the following formula

$$var - cov(\beta) = \sigma^2 (X'X)^{-1} \qquad (32)$$

When heteroscedasticity occurs, the variance-covariance matrix for the disturbance vector is

$$\Sigma = E(uu') = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix} \qquad (33)$$

The elements on the main diagonal of this matrix (σ_i) give the variances of the disturbances and the elements off the main diagonal give the covariances. Here, disturbances are pairwise uncorrelated.

2.3 Literature Survey

The correct covariance matrix for least squares estimatior is:

$$Cov(\beta_{OLS}) = (X'X)^{-1}X'\Sigma X(X'X)^{-1} \tag{34}$$

where the unique unknown is the Σ matrix whose diagonal elements are the variances of the error terms.

The most commonly used heteroscedasticity-consistent covariance matrix estimator (HCCME) is proposed by Halbert White in *Econometrica* paper. (White, 1980) White's estimator, which we shall refer to as HC0 is defined as:

$$HC0 = (X'X)^{-1}\ X'diag\ [e_i^2]X(X'X)^{-1} \tag{35}$$

The entries on the main diagonal of HC0 are the squared least-squares residuals. White's estimator is implemented into several statistical software and it is easy to compute and consistent under both homoscedasticity and heteroscedasticity of unknown form. However, this estimator can be biased in finite samples. (Mackinnon, & White, 1985; Cribari-Neto, & Zarkos, 1999; Cribari-Neto, & Zarkos, 2001)

Several variants of HC0 were proposed in the literature. Hinkley (1977) made a simple degrees of freedom adjustment to HC0 which is known as HC1, is derived from multiplying every squared residuals by the factor $\frac{n}{n-k}$.

$$HC1 = (X'X)^{-1}\ X'\frac{N}{N-k}diag(e_i^2)X(X'X)^{-1} \tag{36}$$

To understand the motivation for the second alternative, we will mention about the analysis of the outliers and influential observations. (Belsley, Kuh, & Welsch, 1980). In equation 35, OLS residuals e are used, not the error terms ε. Even if the error terms are homoscedastic, the residuals may not be. The "hat" matrix H and its diagonals H_{ii} that diagonals are also known as the leverage values, are defined like that $H = X(X'X)^{-1}X'$, $H_{ii} = H[i,i]$, then

$$var(e_i) = \sigma^2(1 - H_{ii}) \neq \sigma^2 \quad (37)$$

Since $\frac{1}{N} \leq H_{ii} \leq 1$, $var(e_i)$ underestimates σ^2. Equation (37) suggests that although e_i^2 is a biased estimator of σ^2, $\frac{e_i^2}{1 - H_{ii}}$ will be less biased. This led Mackinnon and White (1985) that based on work suggested by Horn, Horn, and Duncan (1975).

The bias is primarily due to the existence of the high leverage points in the X matrix. So, HC2 is defined as:

$$HC2 = (X'X)^{-1} X' diag\left(\frac{e_i^2}{1 - H_{ii}}\right) X(X'X)^{-1} \quad (38)$$

The estimator that has come to be widely known as HC3 is actually an approximation suggested by Davidson and Mackinnon, (1985), but this variation approximates a more complicated jackknife estimator of Efron (1982). The HC3 estimator is defined as:

$$HC3 = (X'X)^{-1} X' diag\left(\frac{e_i^2}{(1 - H_{ii})^2}\right) X(X'X)^{-1} \quad (39)$$

Notice that HC3 weights each squared OLS residuals by factor of . By the help of the simulations, Long and Ervin, (2000) evaluated the empirical power functions of the t-tests of the regression coefficients, by using both the OLS estimator and the four HCCME estimators that discussed so far. They recommended to use of HC3 estimator since it can keep the test size at the nominal level regardless of the presence or absence of heteroscedasticity. Also, Cribari-Neto, Ferrari and Oliveira's, (2005) simulation results shows the superiority of HC3 over its predecessors. The performance of HC3 does depend to some extent on the presence or absence of points of high leverage in X. (e.g. Chester, & Jewitt, 1987; Kauermann & Caroll, 2001; Wilcox, 2001). Also, research has shown (e.g., Long & Ervin, 2000, Sudmant & Kennedy, 1990) that HC3 can have a liberal bias in small samples. This leads to Cribari-Neto's (2004) HC4 estimator:

$$HC4 = (X'X)^{-1}X' diag\left(\frac{e_i^2}{(1 - H_{ii})^{\delta_i}}\right) X(X'X)^{-1} \quad (40)$$

where $\delta_i = Min\left(4, \frac{H_{ii}}{\overline{H}}\right)$, $\overline{H} = \frac{1}{N}\sum_i H_{ii}$.

Note that δ_i is same as , since $\sum_i H_{ii} = tr(H) = p.$ Note also that, high leverage of the ith observation makes $\hat{e}_i^2$ more inflated. The truncation at 4 correspond to twice that is used in the definition of HC3. That is, $\delta_i = 4$ when $H_{ii} > 4\overline{H} = \frac{4p}{n}$. The numerical results of Cribari-Neto's (2004) shows that HC4 performs better than the HC0 and HC3.

Cribari-Neto, Souza, and Vasconcellos (2007) propose a new heteroscedasticity-consistent covariance matrix estimator:

$$HC5 = (X'X)^{-1}X'diag\left(\frac{e_i^2}{(1-H_{ii})^{\alpha_i}}\right)X(X'X)^{-1}$$

$$\alpha_i = \min\left\{\frac{H_{ii}}{\overline{H}}, \max\left\{\frac{kH_{max}}{\overline{H}}\right\}\right\}.$$

Here, 0<k<1 is a pre-defined constant and

$$\alpha_i = \min\left\{\frac{H_{ii}}{\overline{H}}, \max\left\{\frac{kH_{max}}{\overline{H}}\right\}\right\} = \min\left\{\frac{nH_{ii}}{p}, \max\left\{4, \frac{nkH_{max}}{p}\right\}\right\}, \qquad (41)$$

where $H_{max} = \max\{H_{11}, \dots, H_{nn}\}$ is the maximal leverage. Again, $\overline{H} = n^{-1}\sum_i H_{ii} = \frac{p}{n}$. The constant α_i determines how much the ith squared residual should be inflated in order to account for the ith observation leverage; here, it is determined by the ratio between H_{max} and $\overline{H}$ and also y the ratio between H_{ii} and $\overline{H}$. Here, all α_i's are influenced by the maximal leverage and HC5 is the first heteroscedasticity-consistent covariance matrix estimator that; when the maximal leverage increases, all squared residuals are discounted more heavily.

CHAPTER 3

ROBUST ESTIMATION & OTHER ROBUST METHODS

To understand robust estimation better, there is an illustrative example, similar to example in Rousseuw, 1990.

Suppose that there is 5 measurements of the weight of a gold in grams, 1.18, 1.21, 1.12, 1.15, 1.25

We compute the sample mean for estimating the actual weight,:

$$\bar{x} = \frac{1.18 + 1.21 + 1.12 + 1.15 + 1.25}{5} = 1.18$$

Then we sort the observations from smallest to largest for finding the sample median,

1.12< 1.15<1.18<1.21< 1.25

The sample median which is the middle observation, is 1.18.

Now, let’s suppose that someone has made an error in this data set that is;

1.18, 1.21, 1.12, 11.5, 1.25

New sample mean is:

$$\bar{x} = \frac{1.18 + 1.21 + 1.12 + 11.5 + 1.25}{5} = 3.25$$

For the new median we sort the data again;

1.12< 1.18<1.21<1.25<11.5

And we find the median value 1.21 which is still reasonable. So, the outlier has changed the median very slightly that we say the sample median is a robust estimator. But sample median is very sensitive to outliers.

Let’s suppose we have a simple linear regression model:

$$y_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_k x_{ki} + \epsilon_i \tag{42}$$

for i=1,…,n where y stands for the response variable (dependent variable) and x stands for the independent regressors (explanatory variables). β_0 denotes the constant term, or the vertical intercept. The classical theory assumes that the error term, ϵ follows a Gaussian distribution with mean 0 and variance σ^2. The Ordinary Least Squares (OLS) residual for the i^{th} row of observations, e_i, is given by

$$e_i\left(\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_k\right) = y_i - \left(\hat{\beta}_0, \hat{\beta}_{1x_{1i}}, \dots, \hat{\beta}_{kx_{ki}}\right) \quad (43)$$

The objective of the LS method is to minimize the sum of squares of the residuals $e_i\left(\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_k\right)$. Formally, this can be written as

$$minimize_{(\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_k)} \sum_{i=1}^{n} e_i^2 \quad (44)$$

The basic idea is to make all of the residuals as small as possible so that the sum of their squares should be minimized. Indeed, the observations that deviate from the bulk of the data are panelized by taking the square of the distance from the line. LS simply wants to place a line among the regression points in such a way that the cumulative squares of the distances is minimized.

3.1 Breakdown Value:

In any data set, one can displace the LS fit as much as he wants by moving a single data point $[(x]_i, y_i)$ far enough away. This experiment can be carried out with statistical package by changing one of the observations. This statement is true for both single and multiple regression. On the other hand, it is possible to find some robust regression methods that can resist several outliers.

The breakdown value is a rough but useful measure of robustness. This value was first introduced by Hampel (1971) and is applied to the finite-sample setting by Donoho and Huber (1983).

Here is the latter version. Consider a data set $Z = (x_{1i}, \cdots, x_{pi}; i = 1, \ldots, n)$ and a regression estimator T. Applying T to Z yields a vector $(\hat{\beta}_0, \ldots, \hat{\beta}_p)$ of regression coefficients.

Now consider all possible contaminated data sets Z' obtained by replacing any m of the original observations by arbitrary points.

This yields the maximum bias:

$$maxbias(m; T, Z) = sup_{Z}' \| T(Z') - T(Z) \| \qquad (45)$$

Where $\| \cdots \|$ is the Euclidean norm. If m outliers can have an arbitrarily large effect on T it follows that $maxbias(m; T, Z) = \infty$, hence $T(Z')$ becomes useless. Therefore the breakdown value of the estimator T at the data set Z is defined as

$$\varepsilon_n^*(T, Z) = \min\left\{\frac{m}{n}; maxbias(m; T, Z) = \infty\right\} \qquad (46)$$

In other words, it is the smallest fraction of contamination that can cause the regression method T to run away arbitrarily far from T(Z). For many estimators $\varepsilon_n^*(T, Z)$ varies only slightly with Z and n, so that we can denote its limiting value (for $n \to \infty$) by $\varepsilon^*(T)$.

The breakdown point of an estimator is the smallest fraction of the observations that have to be replaced to carry the estimator over all bounds. (Rousseuw, 1991)

The breakdown of the sample is $\{x_1, x_2, \ldots, x_n\}$ of n observations is equal to 1/n because it is sufficient to replace a single observation by a large value.

Breakdown point of a sample average is 0% which means that it is very sensitive to outliers. But for the sample median breakdown point is 0.5%, which is the highest breakdown point attainable.

3.2 Positive Breakdown Regression:

Let us consider the simplest case (p=0) in which the model (42) reduces to the univariate location problem $y_i = \beta_0 + e_i$. The LS method (44) yields the sample average $T = \beta_0 = E_i(y_i)$, E standing for the expected value or the average, with again $\varepsilon^*(T) = 0\%$. On the other hand, it is easily verified that the sample median $T = med_i(y_i)$ has $\varepsilon^*(T) = 50\%$, which is the highest breakdown value attainable. Because for a larger fraction of contamination, no method can distinguish between the original data and the replaced data. Estimators T with

$\varepsilon^*(T) = 50\%$, like the univariate median, will be called high breakdown estimators.

The first high-breakdown regression method the repeated median estimator proposed by Siegel (1982). It computes univariate medians in a hierarchical way. For simple regression, its asymptotic behavior was obtained by Hössjer et al. (1994), and for algorithms and numerical results see Rousseeuw et al. (1993,1995). But in multiple regression where $k \geq 2$, the repeated median estimator is not equivariant, in the sense that it does not transform properly under linear transformations of $(x_{i1}, \cdots, x_{ip})$.

However, it is possible to construct a high-breakdown method which is still equivariant. It is instructive to look at (44). This criterion should logically be called least sum of squares, but for historic reasons (Legendre's terminology) the word "sum" is rarely mentioned. Now let us replace the sum by a median. This yields the least median of squares method **(LMS)**, defined by

$$minimize_{(\hat{\beta}_0,\dots,\hat{\beta}_p)} med_i r_i^2 \quad (47)$$

Rousseeuw, 1984 which has a 50% breakdown value. The LMS is clearly equivariant because (47) is based on residuals only.

Another method is the least trimmed squares method **(LTS)** proposed in Rousseeuw, 1983; Rousseeuw, 1984. It is given by

$$minimize_{(\hat{\beta}_0,\dots,\hat{\beta}_p)} \sum_{i}^{h} (r^2)_{i:n} \quad (48)$$

where $(r^2)_{1:n} \leq (r^2)_{2:n} \leq \cdots \leq (r^2)_{n:n}$ are the ordered squared residuals (note that the residuals are first squared and then ordered). Criterion (48) resembles that of LS but does not count the largest squared residuals, thereby allowing the LTS fit to steer clear of outliers. For the default setting $h \approx \frac{n}{2}$ we find $\varepsilon^* = 50\%$, whereas for larger h we obtain $\varepsilon^* \approx \frac{n-h}{n}$. For instance, putting $h \approx 0.75n$ yields $\varepsilon^* = 25\%$ which is often sufficient. The LTS is asymptotically normal unlike the LMS, but for $n \leq 1000$ the LMS still has the better finite-sample efficiency. Here we will focus on the LMS, the LTS results being similar.

When using the LMS regression, σ can be estimated by

$$\hat{\sigma} = 1.483\left(1 + \frac{5}{n-k-1}\right)\sqrt{med_i r_i^2}$$

(49)

where r_i are the residuals from the LMS fit, and $1.483 = \phi^{-1}\left(\frac{3}{4}\right)$ makes $\hat{\sigma}$ consistent at Gaussian error distributions. The finite–sample correction factor was obtained from simulations. Note that the LMS scale estimate $\hat{\sigma}$ is highy robust. Therefore, we can identify regression outliers by their standardized LMS residuals $\frac{r_i}{\hat{\sigma}}$.

In regression analysis inference is very important. The LMS by itself is not suited for inference because of its low finite-sample efficiency. This can be resolved by carrying out a reweighted least squares (RLS) step. To each observation i one assigns a weight w_i based on its standardized LMS residual $\frac{r_i}{\hat{\sigma}}$, e.g. by putting $w_i = w\left(\left|\frac{r_i}{\hat{\sigma}}\right|\right)$ where w is a decreasing continuous function. A simpler way, but still effective, is to put $w_i = 1$ if $\left|\frac{r_i}{\hat{\sigma}}\right| \leq 2.5$ and $w_i = 0$ otherwise. Either way, the RLS fit $(\hat{\beta}_0, \ldots, \hat{\beta}_p)$ is then defined by:

$$minimize_{(\hat{\beta}_0,\ldots,\hat{\beta}_p)} \sum_{i=1}^{n} w_i r_i^2 \qquad (50)$$

which can be computed quickly. The result inherits the breakdown value, but is more efficient and yields all the usual inferential output such as t-statistics, F statistics, and R^2 statistic, and the corresponding p-values. These p-values assume that the data with $w_i = 1$ come from the model (42) whereas the data with $w_i = 0$ do not. Another approach which avoids this assumption is to bootstrap the LMS as done by Efron and Tibshirani (1993). The LMS, LTS and RLS can be computed with the program PROGRESS by Rousseuw and Leroy (1987).

3.3 Masking Effect

"Outliers can not always be detectable by looking at residuals from the classical least-squares (LS), since the latter suffers from **the masking effect**. Masking means here that outliers affect the LS estimator in such a way that diagnostics based on LS are not capable of detecting them anymore." (Bramati & Croux, 2007, p. 521).

"When outliers are clustered, they 'mask' each other and sensitivity analysis fails to detect such outliers." (Zaman, Rousseeuw, & Orhan, 2001, p.2)

3.4 Detecting Leverage Points by Eye

In a linear regression model a data point $(x_1 1i, x_1 2i, \dots, x_1 ki, y_1 i)$ with outlying point $x_i = (x_{i1}, x_{i2}, \dots, x_{ik})$ plays a special role, because a slight change of the coefficients estimated may give case i a large residual. Therefore, the LS method gives priority to approaching such a point in minimizing the objective function.

In general, a point (x_i, y_i) is a leverage point if its if its x_i is an outlier relative to the majority of the set $X = \{x_i, \dots, x_n\}$. Detecting outliers in the p-dimensional data set X is not trivial, especially where p is greater than two when we can no longer have the opportunity of inspection by eye.

A classical approach to the solution of the problem is to compute the Mahalanobis Distance defined as:

$$MD(x_i) = \sqrt{(x_i - X)\left(Cov(X)\right)^{-1}(x_i - X)^t} \qquad (51)$$

for each x_i. Here $\overline{X}$ is the sample mean of the data set X, and Cov(X) is its sample covariance matrix. The distance $MD(x_i)$ tells us how far away x_i from the mass of the data relative to the size of the mass is. It is well known that this approach suffers from the masking effect, by which the multiple outliers do not necessarily have a large Mahalanobis Distance $MD(x_i)$.

One of the most commonly used statistic to discover the leverage points has been the diagonal entries of the hat matrix. Indeed, these entries are equivalent to the Mahalanobis Distance since,

$$h_{ii} = \frac{MD_{x_i}^2}{n-1} + \frac{1}{n} \qquad (52)$$

Therefore, the diagonal entries of the hat matrix are masked when the distances are masked.

One can play with the elements in the square root Formula of the (51) equation to have some more reliable diagnostics.

The Minimum Volume Ellipsoid proposed by Rousseuw (1983b, 1985) proposes an ellipsoid with the minimum volume to include some certain percentage of the data. One can refer to Rousseeuw and Leron (1987) in order to have some more detailed information about the technique.

3.5 Diagnostic Display

Combining the notions of regression outliers and leverage points, we see that four types of observations may occur in regression data:

Regular observations with interval x_i and well-fitting y_i;

Vertical outliers with interval x_i and non-fitting y_i;

Good leverage points with outlying x_i and well-fitting y_i;

Bad leverage points with outlying x_i and non-fitting y_i.

In general, good leverage points are beneficial, since they can improve the precision of regression coefficients. Bad leverage points are harmful because they can change the least squares fit drastically. In the coming applications one of the best techniques is to detect the regression outliers with standardized LMS residuals and leverage points which are diagnosed by robust distances. Indeed, Rousseeuw and van Zomeren (1990) proposed a display which plots robust residuals versus robust distances (Rousseeuw & Van Zomeren, 1992) where the cutoffs at the $[-2.5, 2.5]$ band and the $\sqrt{\chi^2_{p,0.975}}$ are bordered by horizontal and vertical lines. With the help of such a display, the four types of points categorized above are determined automatically. One can play with the band length and the critical values of the $\sqrt{x_p^2}$, to be more robust or loose to such points of outliers.

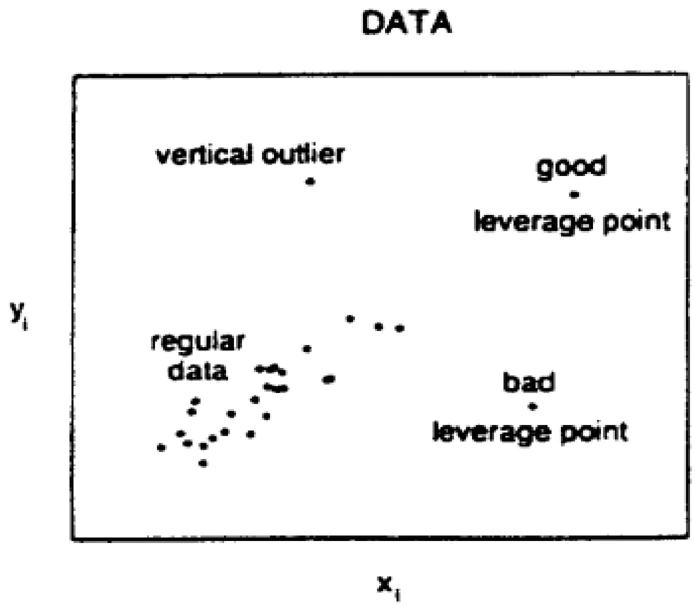
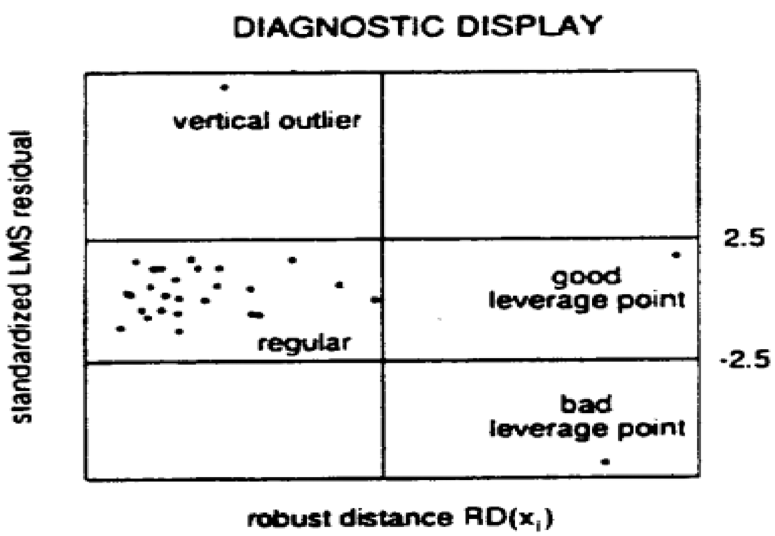


Figure 3.1: Simple regression data with points of all four types[6]



[6] **Source:** Rousseeuw, 1997: 111

Figure 3.2: Diagnostic display of these data with vertical line at the cut off $x^2_{1,0.975} = 2.24$

3.6 Applications

The main obstacle preventing the wide, common and frequent applications of high breakdown methods was the difficulty and slowness of computation, but the invention of powerful computers enabled such computations available. For example, there are several incentive users of LMS in financial markets, where profits can be made by finding majority patterns and detecting subgroups that behave in another way. In management science, the LMS has been applied to measures of production efficiency by Seaver and Triantis (1995). The LMS is used in chemistry after the publication of Massart, Kaufman, Rousseeuw, and Leroy (1986). Also, the LMS is an essential component of a new system for connecting optical fiber cables implemented at NIST. (Wang, Vecchıa, Young, and Brillant, 1994). In large electric power systems, Mili, Cheniae, Vichare, and Rousseeuw (1995) modified positive breakdown methods to estimate the system's state variables. Faster algorithms needed to be constructed to allow real- time estimation.

Positive-breakdown methods have opened new possibilities in the rapidly evolving field of computer vision. The LMS has been used for analyzing noisy images, (Meer, Mintz, Rosenfeld, & Kim, 1991) for interpreting color images (Drew, 1994), for discontinuity-preserving surface reconstruction, (Sinha, & Schunck, 1992), for extracting geometric primitives, (Roth & Levine, 1993), (Steward, 1994) for robot positioning (Kumar and Hanson, 1994), and for detecting moving objects in video from a mobile camera, (Thompson, Lechleider & Stuck, 1993), (Abdel-Mottaleb, Chellapa, & Rosenfeld, 1993). The MVE was applied to image segmentation (Jolion, Meer, & Banaouche, 1991). Chork (1990) used the MVE to analyze data on surface rocks in New South Wales, for which concentrations of several chemical elements were measured. Outliers in this multivariate data set revealed mineralizations, yielding targets for mining prospection. A larger study in

Finland carried out MVE-based factor analysis, Chork and Salminen (1993). The same methods apply to environmetrics, since mineralizations in geochemistry are similar to contaminations of the environment.

3.7 Other Robust Methods

Least median of squares **(LMS)** is defined (Rousseeuw, 1984) by;

$$minimize_{(\hat{\beta}_0,\dots,\hat{\beta}_p)} med_i r_i^2 \qquad (53)$$

which has a 50% breakdown value. Here p is the number of coefficients, including the intercept term. The LMS is clearly equivariant because (53) is based on residuals only.

Another method is the least trimmed squares method **(LTS)** (Rousseeuw, 1983a, 1984). It is given by:

$$minimize \sum_{i=1}^{h} (r^2)_{i:n} \qquad (54)$$

where $(r^2)_{1:n} \leq (r^2)_{2:n} \leq \cdots \leq (r^2)_{n:n}$ are the ordered squared residuals (note that the residuals are first squared and then ordered). Although it resembles the LS, LTS does not count the largest squared residuals thereby allowing the LTS fit to steer clear of outliers. For the default setting $h \approx \frac{n}{2}$, $\varepsilon^* = 50\%$ is found but for larger h $\varepsilon^* \approx \frac{n-h}{n}$. The LTS is asymptotically normal unlike the LMS, but for $n \leq 1000$ the LMS still has the better finite-sample efficiency.

The breakdown value of LTS with $h = \left[\frac{n+p+1}{2}\right]$ equals that of the LMS.

In spite of the advantages, its major disadvantage is their large computation time when the size of the data set increases.

Minimum covariance determinant estimator **(MCD)** looks for the observations whose empirical covariance matrix has the smallest possible determinant.

3.8 M- Estimators:

The earliest systematic theory of robust regression was based on M-estimators (Huber, 1973,1981) given by

$$minimize_{(\hat{\beta}_0,\dots,\hat{\beta}_p)} \sum_{i=1}^{n} \rho\left(\frac{r_i}{\hat{\sigma}}\right) \qquad (55)$$

where $\rho(t) = |t|$ yields least absolute values (L^1) regression as a special case. For general ρ one needs a robust $\hat{\sigma}$ to make the M-estimator equivariant under scale factors. This $\hat{\sigma}$ either needs to be fixed in advance or estimated jointly with $(\hat{\beta}_0, \dots, \hat{\beta}_p)$. (Huber, 1981) Scale equivariance holds automatically for R-estimators (Jureckova, 1971), and L-estimators (Koenker, & Portnoy, 1987). The breakdown value of all M-, L-, and R- estimators is 0% because their vulnerability to bad leverage points.

The next step was the development of generalized M-estimators (GM estimators), with the purpose of bounding the influence of outlying $(x_{1i}, x_{2i}, \dots x_{ki})$ by giving them a small weight. This is why GM-estimators are often called bounded influence methods. A survey is given in Hampel, Ronchetti, Rousseuw, and Stahel, 1986. Both M- and GM- estimators can be computed by iteratively reweighted LS or by the Newton-Raphson algorithm. Unfortunately, the breakdown value of all GM-estimators goes down to zero for increasing p, when there are more opportunitiesfor outliers to ocur.

In the special case of simple regression $(p = 1)$ several earlier methods exist, such as the Brown-Mood line, the robust-resistant line of Tukey and the Theil-Sen slope. These methods are reviewed by Rousseeuw and Leroy (1987) with their breakdown values.

3.3 Basic idea and C-step for LTS

One of the keys of the new algorithm is the fact that starting from any approximation to the LTS regression coefficients, it is possible to compute another approximation with an even lower objective function. (Rousseeuw, Van Driessen, 2006)

Property 1. Consider a data set $(x_1, y_1), \dots, (x_n, y_n)$ consisting of p-variate x_i (typically with $x_{ip} = 1$) and a response variable y_i. Let $H_1 \subset \{1, \dots, n\}$ with $|H_1| = h$, and put $Q_1 := \sum_{i \in H_1} (r_1(i))^2$ where $r_1(i) = y_i - \left(\hat{\theta}_1^1 x_{1i} + \hat{\theta}_2^1 x_{2i} + \dots + \hat{\theta}_{p1}^1 x_{pi}\right)$ for all $i = 1, \dots, n$ where $\hat{\theta}_1 = \left(\hat{\theta}_{11}, \dots, \hat{\theta}_{p1}\right)$ is any p-dimensional vector. Now take H_2 such that $\{|r_1(i)|; i \in H_2\} := \{|r_1|_{1:n}, \dots, |r_1|_{h:n}\}$ where $|r_1|_{1:n} \leq |r_1|_{2:n} \leq \dots \leq |r_1|_{n:n}$ are the ordered absolute values of the residuals, and compute the least squares (LS) fit $\hat{\theta}_2$ of the h observations in H_2. This yields $r_2(i)$ for all $i = 1, \dots, n$ and $Q_2 := \sum_{i \in H_2} (r_2(i))^2$. Then

$$Q_2 \leq Q_1.$$

Proof: Because H_2 corresponds to the h smallest absolute residuals out of n, we have $\sum_{i \in H_2} (r_1(i))^2 \le \sum_{i \in H_1} (r_1(i))^2 = Q_1$. Because the LS estimator $\hat{\theta}_2$ of these h observations is such that it minimizes Q_2 we find $Q_2 = \sum_{i \in H_2} (r_2(i))^2 \le \sum_{i \in H_2} (r_1(i))^2 \le Q_1$. □

Applying the above property to H_1 yields H_2 with $Q_2 \le Q_1$. In our algorithm we will call this a C-step, where C stands for 'concentration' since H_2 is more concentrated (has a lower sum of squared residuals) than H_1. In our algorithmic terms, the C-step can be described as follows.

Given the h-subset H_{old} then:

- Compute $\hat{\theta}_{old} \coloneqq$ LS regression estimator based on H_{old}
- Compute the residuals $r_{old}(i)$ for $i = 1, \ldots, n$
- Sort the absolute values of these residuals, which yields a permutation

π for which

$|r_{old}(\pi(1))| \le |r_{old}(\pi(2))| \le \cdots \le |r_{old}(\pi(n))|$

- Put $H_{new} \coloneqq \{\pi(1), \pi(2), \ldots, \pi(h)\}$
- Compute $\hat{\theta}_{new} \coloneqq$ LS regression estimator based on H_{new} .

Alternatively, any vector $\hat{\theta}_{old}$ may be given, in that case we do not need any H_{old} . For a fixed number of dimensions p, the C-step takes only O(n) time because H_{new} can be determined in O(n) operations without fully sorting the n absolute residuals $|r_{old}(i)|$.

Repeating C-steps yields an iteration process. If $Q_2 = Q_1$ we stop; otherwise we apply another C-step yielding Q_3, and so on. The sequence $Q_1 \ge Q_2 \ge Q_3 \ge \cdots$ is nonnegative and hence must converge. In fact, since there are only finitely many h-subsets there must be an index m such that $Q_m = Q_{m-1}$, hence convergence is always reached after a finite number of steps. (In practice, m is often below 10.) This is not sufficient for Q_m to be global minimum of the LTS objective function, but it is a necessary condition.

This provides a partial idea for an algorithm:

Take many initial choices of H_1 *and apply C-steps to each until convergence, and keep the solution with lowest value of* (54).

3.4 Basic idea and C-step for MCD

A key step of the new algorithm is the fact, starting from any approximation to the MCD, it is possible to compute another approximation with an even lower determinant. (Rousseeuw, Van Driessen, 1999)

Theorem 1. Consider a data set $X_n = \{x_1, \dots, x_n\}$ of p-variate observations. Let $H_1 \subset \{1, \dots, n\}$ with $|H| = h$, and put $T_1 := \left(\frac{1}{h}\right) \sum_{i \in H_1} x_i$ and $S_1 := \left(\frac{1}{h}\right) \sum_{i \in H_1} [\![(x_i -]\!] T_1)(x_i - T_1)'$.

If $\det S_1 \neq 0$, define the relative distances $d_1(i) = \sqrt{(x_i - T_1)' S_1^{-1} (x_i - T_1)}$ for $i = 1, \dots, n$.

Now take H_2 such that $\{d_1(i); i \in H_2\} := \{(d_1)_{1:n}, \ldots, (d_1)_{h:n}\}$, where $(d_1)_{1:n} \leq (d_1)_{2:n} \leq \cdots, (d_1)_{n:n}$ are the ordered distances, and compute T_2 and S_2 based on H_2. Then

$$\det(S_2) \leq \det(S_1)$$

with equality if and only if $T_2 = T_1$ and $S_2 = S_1$. The theorem requires that $\det(S_1) \neq 0$, which is no real restriction because if $\det(S_1) = 0$. We already have the minimal objective value.

In algorithmic terms, the C-step can be described as follows.

Given the h-subset H_{old} or the pair (T_{old}, S_{old}), perform the following:

- Compute the distances $d_{old}(i)$ for $i = 1, \ldots, n$.
- Sort these distances, which yields a permutation π for which $|d_{old}(\pi(1))| \leq |d_{old}(\pi(2))| \leq \cdots \leq |d_{old}(\pi(n))|$.
- Put $H_{new} := \{\pi(1), \pi(2), \ldots, \pi(h)\}$.
- Compute $T_{new} \coloneqq ave(H_{new})$ and $S_{new} \coloneqq cov(H_{new})$.

For a fixed number of dimensions p, the C-step takes only O(n) time because H_{new} can be determined in O(n) operations without sorting all the $d_{old}(i)$ distances.

Repeating C-steps yields an iteration process. If $\det(S_2) = 0$ or $\det(S_2) = \det(S_1)$, we stop; otherwise, we run another C-step yielding $\det(S_3)$, and so on. The sequence $\det(S_1) \geq \det(S_2) \geq \det(S_3) \geq \ldots$ is nonnegative and hence must be convergent. In fact, because there are only finitely many h-subsets, there must be an index m such that $\det(S_m) = 0$ or $\det(S_m) = \det(S_{m-1})$, hence convergence is reached. (In practice, m is often below 10.) Afterward, running the C-step on (T_m, S_m) no longer reduces the determinant. This is not sufficient for $\det(S_m)$ to be the global minimum of the MCD objective function, but it is a necessary condition.

Theorem 1 thus provides a partial idea for an algorithm:

Take many initial choices of H_1 and apply C-steps to each until convergence, and keep the solution with lowest determinant. (56)

CHAPTER 4

REMOVING HIGH LEVERAGE POINTS & ESTIMATING VARIANCE COVARIANCE OF BETAHATS

The Monte Carlo experiment was based on a simple regression model. In all evaluations, the full sample and the incomplete sample in which the leverage points are removed by MCD were compared. Prominent five different X settings and five sigma settings (the settings of the variance of the error terms) were used in the simulations, that is shown in Table 4.1 and Table 4.2 respectively.

In all cases the model is:

$$y_i = \beta_0 + \beta_1 x_{1i} + \sigma_i e_i, \quad e_i \sim N(0,1)$$

Table 4.1: X settings used in the simulations.

X SETTINGS
1) U(0,1)
2) N(0,1)
3) t distribution with 3 degrees of freedom.
4) Log-normal distribution (log(x)=normal)
5) Cauchy distribution (N(0,1)/N(0,1))

Table 4.2: Sigma settings used in the simulations.

SIGMA SETTİNGS **(The settings of the variance of the error terms)**
a) Homoscedasticity (equal variance of the error terms)

b) $c0 + (c1 * x_{1i}) + (c2 * (x_{1i})^2)$ (Cragg, 1983)
c) $(x_{1i})^2$ (Flachaire, 2005).
d) $\sqrt{exp(c0 * x_{1i})}$ (Lima, Souza, Cribari-Neto, & Fernandes, 2010).

The regression parameters were set at $\beta_0 = \beta_1 = 1$ in all.

The degree of heteroscedasticity, measured by

$$\lambda = \frac{\max(\sigma_i^2)}{\min(\sigma_i^2)}$$

remains constant when the sample size changes. Under homoscedastictiy, $\lambda = 1$; when the error terms not constant across observations, the more intense of heteroscedasticity, the larger value of λ.

The number of outliers, measured by

$$L = full\ sample\ size - short\ sample\ size$$

are shown in the tables for different cases.

Full sample sizes are adjusted to 20, 30, 40, 50, 60, 80 and 100 for all cases.

All simulations were performed by using GAUSS (Version 7.0) and were based on 10,000 replications.

We test the null hypothesis $\mathcal{H}_0 : \beta_i = 0$ against $\mathcal{H}_1 : \beta_i \neq 0$.

Test statistic is:

$$t: \frac{\hat{\beta}_i}{\sqrt{\widehat{var}(\hat{\beta}_i)}}$$

where 5 % critical values are used.

Besides, the symmetric loss function was analized and its formula is below. Here Ω is the covariance matrix. (Sun & Sun, 2005).

Symmetric Loss Function:

$$L_3(\hat{\Omega}, \Omega) = \mathrm{tr}\left(\hat{\Omega}\Omega^{-1}\right) + \mathrm{tr}\left(\hat{\Omega}^{-1}\Omega\right) - 2k$$

where k is dimension of the covariance matrix.

We also prepared the Stein, entropy and quadratic losses but preferred just reporting the symmetric loss because the losses are similar to each other.

The program first generates the covariates, and then fixing the error terms from predetermined distribution in each iteration. The MCD procedure detects the covariates with high leverages (with MCD distances larger than the critical χ^2 values). These selected observations are removed from the data set and getting the short sample. Then we estimate the covariance matrix (Ω) by the help of the specified HCCMEs with the original (full) sample and the sample without the high leverages (short).

CHAPTER 5

SIMULATION RESULTS

In Table 5.1, 5.2, 5.3, 5.4, and 5.5 the percentage deviations of the HCCMEs that is the percentage change between the estimated variances of the betahats with HC0, HC1, HC2, HC3, HC4, and HC5 respectively and the true variances of betahats for the full and short sample for different X settings (Table 4.1) and sigma settings (Table 4.2). Also, the number of outliers (L) and the degree of heteroscedasticity (λ) are shown in the tables for distinct cases. The column heads are F for full sample and S for the short sample that is free from the high leverage observations.

The results are different across the settings of the regressors and the variances of the disturbance term. Positive and negative results can be seen in Tables 5.1, 5.2, 5.3, 5.4, and 5.5 according to its underestimates or overestimates of the HCCMEs. In Table 5.1, there is the first simulation setting of the regressor, we name it Case 1, all regressors are uniformly distributed between 0 and 1. For the first sigma setting, Case a, in which the error term variances set equal to 1, removing the occasional leverage points does not change the results so much. When sample size (T) 20, huge percentage errors belong to HC4 and HC5. Also, HC0 has the next big difference as well as HC3.

Case b, in which we introduce heteroscedasticity, because of limited high leverage points or the leverages with low MCD distances, again short sample results have not better performance than the full sample when the sample size is larger than 50. But short sample HCCMEs are slightly better when $T<50$.For

Case c, short sample results are much better than the full sample results, and when sample size is getting lower, the difference become more significant. One point that deserves attention is the superior performance of HC2 and the inferior performances of HC4 and HC5.

In Case d, the short sample results are better the full sample results at almost all sample sizes. It is interesting to note that HC4 and HC5 that are claimed to have better performance than others are much worse than all others.

In Case 2, standard normal distribution is used for generating the covariates. Especially, in small samples, up to 50, short sample results have lower percentage differences than the full sample. When heteroscedasticity is introduced in Case b, percentage differences are large at T=20 and 30 and after T=50, the differences are getting smaller. Again the largest differences belong to HC4 and HC5. Similar comments are true for Cases c and d.

T distribution with 3 degrees of freedom generates the covariates in Case 3. The density of this distribution has thick tails to let high leverage covariates. The short sample results are generally better than the full sample results. When the sample size increases, the performance of the estimators become better. Best performance of HC2 and followed by HC3 stands out in Case c and Case d.

In Case 4, the lognormal distribution generates the covariates for increasing the number and degree of leverages. This time the differences in performances are obvious. The gap between HC4 and HC5 estimators and the true ones become huge and the larger values for which “>1000” is used, are observed. This means that the estimated value is more than ten times the true value. Here, we observe HC2’s failure because of high leverage covariates. Note also that, there is an extra failure in the performance of HC4 and HC5, which suggests that one should avoid from using these estimators especially when there is high leverage and heteroscedasticity simultaneously.

In Case 5, covariates are coming from the ratio of two standard normals to let very large and small values possible. Again, the failure of HC4 and HC5 can be observed. When the heteroscedasticity introduced in Case b, the short sample HCCMEs are much better than the full sample HCCMEs. The slope coefficient’s variance in Case d, stands out obviously since almost all estimators fail in full sample which is a good example of the benefit from detecting and eliminating the bad leverage points with high distances.

Quasi-t statistics are shown in Table 5.6, 5.7, 5.8, 5.9, and 5.10. The results can be comparable with the true ones for the full sample and the short sample. Again the short sample results generally match up better with the true ones. The results of Quasi-t statistics and the percentage deviations are similar, as we expect.

In Table 5.11, 5.12, 5.13, 5.14, and 5.15 the symmetric loss results are compared. Especially, for the bigger sample sizes and higher degrees of heteroscedasticity the gap between the full sample and the short sample estimates becomes clear that results in short sample’s better converge to zero.

Table 5.1: Percentage Deviations for the Case 1*

Case 1*		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**		λ=1; L=2		λ=1; L=1		λ=1; L=2		λ=1; L=2		λ=1; L=3		λ=1; L=8		λ=1; L=3	
beta 0	**HC0**	-25.4	-23.0	-7.6	-7.8	-6.0	-6.2	-4.7	-4.9	-3.9	-4.1	-3.4	-3.9	-3.1	-3.2
	HC1	-17.1	-13.4	-1.0	-1.0	-1.1	-1.0	-0.8	-0.7	-0.6	-0.6	-0.9	-1.1	-1.1	-1.1
	HC2	0.2	0.2	0.1	0.1	-0.1	-0.1	0.0	0.0	0.1	0.1	0.0	0.0	0.1	0.0
	HC3	37.8	34.8	8.6	8.8	6.3	6.6	5.0	5.2	4.2	4.4	3.5	4.0	3.4	3.3
	HC4	120.9	106.8	3.5	2.7	3.5	2.6	2.1	1.8	1.9	1.9	1.8	2.1	3.2	2.8
	HC5	120.9	106.8	3.5	2.7	3.5	2.6	2.1	1.8	1.9	1.9	1.8	2.1	3.2	2.8
beta 1	**HC0**	-23.6	-22.9	-10.1	-10.0	-8.4	-8.1	-6.2	-6.1	-5.3	-5.3	-3.4	-3.9	-3.0	-3.0
	HC1	-15.2	-13.3	-3.6	-3.3	-3.6	-3.0	-2.3	-2.0	-2.0	-1.8	-1.0	-1.1	-1.0	-1.0
	HC2	0.1	0.1	0.1	0.1	-0.1	0.0	-0.1	0.0	0.0	0.0	-0.1	-0.1	0.1	0.0
	HC3	34.3	33.8	11.6	11.3	9.3	8.9	6.5	6.6	5.7	5.7	3.5	3.9	3.2	3.1
	HC4	104.3	96.4	9.5	6.8	11.1	7.3	5.6	4.5	5.5	4.9	1.6	1.8	2.7	2.4
	HC5	104.3	96.4	9.5	6.8	11.1	7.3	5.6	4.5	5.5	4.9	1.6	1.8	2.7	2.4
b**		λ=4.2; L=1		λ=5.6; L=3		λ=5.5; L=4		λ=5.5; L=7		λ=5.7; L=1		λ=5.8; L=5		λ=5.5; L=3	
beta 0	**HC0**	-16.4	-13.4	-8.8	-8.0	-6.2	-6.1	-5.1	-5.1	-3.5	-3.5	-2.8	-3.0	-2.1	-2.3
	HC1	-7.1	-3.2	-2.3	-0.6	-1.3	-0.6	-1.1	-0.5	-0.2	-0.1	-0.4	-0.3	-0.1	-0.2
	HC2	-0.6	1.0	0.3	0.7	0.4	0.7	0.2	0.3	0.6	0.6	0.3	0.3	0.4	0.4
	HC3	18.9	18.0	10.6	10.4	7.5	8.0	5.9	6.0	4.8	4.9	3.6	3.8	3.0	3.1
	HC4	19.2	9.3	8.5	3.4	4.3	3.1	3.4	1.9	2.2	2.1	1.8	1.7	1.4	1.4
	HC5	19.2	9.3	8.5	3.4	4.3	3.1	3.4	1.9	2.2	2.1	1.8	1.7	1.4	1.4
beta 1	**HC0**	-19.7	-14.9	-14.5	-11.6	-8.8	-8.1	-7.3	-6.8	-4.7	-4.7	-4.1	-4.3	-2.6	-2.7
	HC1	-10.7	-4.9	-8.4	-4.5	-4.0	-2.7	-3.5	-2.3	-1.5	-1.3	-1.6	-1.6	-0.7	-0.7
	HC2	-3.1	-0.7	-2.4	-1.0	-1.0	-0.4	-0.8	-0.5	-0.2	-0.1	-0.4	-0.4	0.0	0.0
	HC3	17.8	16.1	11.7	10.9	7.5	8.0	6.2	6.3	4.6	4.7	3.4	3.6	2.7	2.8
	HC4	22.0	7.7	14.9	5.6	5.5	3.5	5.1	2.6	2.1	1.9	1.9	1.8	1.1	1.1
	HC5	22.0	7.7	14.9	5.6	5.5	3.5	5.1	2.6	2.1	1.9	1.9	1.8	1.1	1.1
c**		λ=157.6 L=2		λ=63466.7 L=1		λ=19886.5 L=2		λ=6811.3 L=3		λ=1236.4 L=4		λ=237053.3 L=4		λ=1203.8 L=2	
beta 0	**HC0**	-24.3	-15.9	-10.1	-9.3	-7.4	-6.3	-6.2	-5.8	-5.9	-5.9	-3.3	-3.1	-2.9	-2.7
	HC1	-15.9	-5.4	-3.7	-2.5	-2.5	-1.1	-2.3	-1.6	-2.6	-2.4	-0.8	-0.4	-0.9	-0.7
	HC2	-4.0	0.7	1.2	1.9	0.8	1.6	0.8	1.1	0.2	0.3	0.6	0.8	0.4	0.5
	HC3	22.8	21.1	14.4	14.7	10.0	10.3	8.3	8.7	6.7	6.9	4.6	4.8	3.9	3.8
	HC4	38.7	14.7	15.1	14.2	9.8	7.4	8.5	7.8	7.5	7.5	3.7	3.4	3.7	3.1
	HC5	38.7	14.7	15.1	14.2	9.8	7.4	8.5	7.8	7.5	7.5	3.7	3.4	3.7	3.1
beta 1	**HC0**	-28.6	-20.4	-14.4	-13.8	-10.4	-10.0	-8.7	-8.5	-7.7	-7.6	-4.8	-4.7	-4.0	-3.9
	HC1	-20.6	-10.4	-8.2	-7.4	-5.6	-5.0	-4.9	-4.5	-4.5	-4.2	-2.3	-2.1	-2.0	-1.9
	HC2	-7.8	-3.5	-2.8	-2.5	-1.5	-1.2	-1.4	-1.3	-1.4	-1.4	-0.7	-0.6	-0.5	-0.5
	HC3	19.8	17.4	10.5	10.6	8.2	8.5	6.5	6.5	5.3	5.3	3.5	3.7	3.1	3.0
	HC4	37.1	11.7	11.0	9.7	8.1	6.0	6.8	5.7	6.2	5.7	2.6	2.3	2.9	2.3
	HC5	37.1	11.7	11.0	9.7	8.1	6.0	6.8	5.7	6.2	5.7	2.6	2.3	2.9	2.3
d**		λ=1.1; L=3		λ=1.1; L=2		λ=1.1; L=2		λ=1.1; L=7		λ=1.1; L=2		λ=1.1; L=6		λ=1.1; L=5	
beta 0	**HC0**	-12.2	-14.0	-8.2	-8.3	-7.7	-7.2	-7.8	-7.0	-4.5	-4.7	-3.8	-3.8	-2.6	-2.8
	HC1	-2.4	-2.5	-1.7	-1.3	-2.9	-2.0	-4.0	-2.4	-1.2	-1.3	-1.3	-1.1	-0.6	-0.7
	HC2	-0.2	-0.1	0.0	0.1	0.2	0.0	0.2	0.2	0.0	0.0	0.1	0.0	0.1	0.0
	HC3	14.2	16.3	9.3	9.5	8.9	7.9	9.1	7.9	4.8	4.9	4.2	4.1	2.8	2.9
	HC4	10.5	4.5	6.4	3.7	9.6	4.5	17.2	6.3	2.8	2.5	3.8	2.6	1.4	1.3
	HC5	10.5	4.5	6.4	3.7	9.6	4.5	17.2	6.3	2.8	2.5	3.8	2.6	1.4	1.3
beta 1	**HC0**	-19.1	-15.6	-11.1	-10.0	-7.8	-7.4	-7.8	-7.4	-5.1	-5.0	-3.9	-4.0	-2.6	-2.8
	HC1	-10.1	-4.3	-4.7	-3.1	-2.9	-2.2	-4.0	-2.8	-1.8	-1.6	-1.4	-1.3	-0.7	-0.7
	HC2	-0.3	-0.1	-0.1	0.1	0.0	-0.1	0.1	0.2	0.0	-0.1	0.0	0.0	0.1	0.0
	HC3	24.2	18.7	12.7	11.6	8.6	7.8	9.0	8.4	5.4	5.2	4.1	4.1	2.9	2.9
	HC4	37.1	8.7	16.4	8.2	8.2	4.1	15.4	6.9	4.1	2.9	3.4	2.6	1.4	1.3
	HC5	37.1	8.7	16.4	8.2	8.2	4.1	15.4	6.9	4.1	2.9	3.4	2.6	1.4	1.3

Table 5.2: Percentage Deviations for the Case 2*

Case 2*		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**		**λ=1; L=1**		**λ=1; L=1**		**λ=1; L=3**		**λ=1; L=2**		**λ=1; L=2**		**λ=1; L=4**		**λ=1; L=2**	
beta 0	**HC0**	-10.4	-10.7	-7.0	-7.2	-4.7	-5.3	-4.1	-4.2	-3.2	-3.5	-2.5	-2.7	-1.9	-2.0
	HC1	-0.4	-0.2	-0.4	-0.4	0.3	0.1	-0.1	0.0	0.1	0.0	0.0	-0.1	0.1	0.0
	HC2	0.1	0.1	0.0	-0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1
	HC3	13.1	12.7	7.9	7.9	5.3	5.9	4.4	4.5	3.3	3.6	2.5	2.7	2.2	2.2
	HC4	16.6	3.9	6.0	3.2	3.3	3.1	3.8	2.7	1.2	1.2	1.4	1.3	1.2	0.9
	HC5	16.6	3.9	6.0	3.2	3.3	3.1	3.8	2.7	1.3	1.2	1.4	1.3	1.2	0.9
beta 1	**HC0**	-19.7	-16.0	-12.4	-11.9	-11.4	-10.3	-8.8	-7.7	-8.2	-5.9	-5.8	-5.1	-4.1	-3.4
	HC1	-10.8	-6.1	-6.2	-5.4	-6.7	-5.2	-5.0	-3.7	-5.0	-2.5	-3.4	-2.5	-2.1	-1.4
	HC2	0.3	0.3	0.1	-0.1	0.0	0.0	0.1	0.1	0.0	0.0	0.0	-0.1	0.1	0.1
	HC3	28.2	20.4	15.0	13.6	13.7	12.3	10.2	8.8	9.6	6.2	6.3	5.2	4.7	3.8
	HC4	73.0	16.7	23.8	14.5	34.5	27.2	22.5	14.4	23.9	6.2	12.6	7.0	8.0	3.8
	HC5	73.0	16.7	23.8	14.5	34.5	27.2	22.5	14.4	27.8	6.2	13.0	7.0	8.4	3.8
b**		**λ=62.3; L=1**		**λ=29.7; L=1**		**λ=35.1; L=2**		**λ=42; L=3**		**λ=31.3; L=3**		**λ=32.4; L=2**		**λ=26.7;L=2**	
beta 0	**HC0**	-15.9	-16.9	-9.9	-11.7	-11.6	-9.8	-7.1	-6.3	-3.9	-5.6	-6.0	-3.2	-3.1	-3.5
	HC1	-6.6	-7.1	-3.5	-5.2	-6.9	-4.8	-3.2	-2.1	-0.6	-2.2	-3.5	-0.7	-1.2	-1.5
	HC2	0.4	-1.2	0.7	-0.5	-1.3	-1.4	0.1	0.0	0.7	-0.4	-1.0	0.4	0.0	-0.2
	HC3	23.0	18.2	12.8	12.4	10.8	8.2	8.1	6.9	5.5	5.3	4.5	4.2	3.2	3.1
	HC4	71.6	14.6	12.5	12.0	27.5	12.9	14.6	6.5	4.3	5.5	10.4	4.0	3.0	3.0
	HC5	71.6	14.6	12.5	12.0	27.5	12.9	14.6	6.5	4.3	5.5	12.3	4.0	3.0	3.0
beta 1	**HC0**	-47.8	-24.4	-22.5	-17.6	-21.5	-18.7	-15.6	-11.5	-12.3	-10.3	-13.2	-7.1	-6.8	-5.6
	HC1	-42.0	-15.5	-16.9	-11.5	-17.4	-14.2	-12.0	-7.5	-9.2	-7.0	-10.9	-4.7	-4.9	-3.7
	HC2	-23.3	-6.1	-7.0	-4.1	-6.9	-6.1	-4.8	-2.8	-3.4	-2.7	-5.1	-1.9	-2.0	-1.5
	HC3	16.4	17.0	12.2	11.9	10.8	8.7	7.6	6.9	6.6	5.6	4.1	3.7	3.1	2.9
	HC4	177.7	16.8	39.6	15.7	45.7	24.7	26.0	10.8	20.7	10.2	20.3	6.2	8.2	4.3
	HC5	177.7	16.8	39.6	15.7	45.7	24.7	26.0	10.8	20.7	10.2	24.7	6.2	8.2	4.3
c**		**λ=32747.6 L=1**		**λ=77841.8 L=1**		**λ=30388.3 L=1**		**λ=13682 L=1**		**λ=1974.4 L=2**		**λ=603776 L=5**		**λ=5059.7 L=2**	
beta 0	**HC0**	-19.3	-20.8	-13.3	-11.0	-10.0	-9.9	-9.4	-7.2	-5.9	-6.2	-6.1	-4.6	-3.6	-3.7
	HC1	-10.4	-11.5	-7.1	-4.4	-5.3	-5.0	-5.6	-3.3	-2.7	-2.9	-3.7	-1.9	-1.6	-1.7
	HC2	-0.6	-2.6	-1.1	-0.1	-0.6	-0.5	-1.0	-0.2	0.1	-0.2	0.0	0.3	0.1	-0.2
	HC3	26.4	21.2	13.7	12.3	10.0	10.0	8.6	7.5	6.5	6.3	6.6	5.5	3.9	3.5
	HC4	92.6	31.7	31.0	9.7	14.3	12.6	19.7	8.3	8.6	7.5	15.0	7.0	5.5	4.0
	HC5	92.6	31.7	31.0	9.7	14.3	12.6	19.7	8.3	8.6	7.5	15.0	7.0	5.5	4.0
beta 1	**HC0**	-51.2	-32.1	-28.1	-16.8	-15.4	-14.6	-18.6	-11.1	-12.1	-11.4	-13.9	-9.1	-8.4	-6.3
	HC1	-45.7	-24.1	-22.9	-10.6	-10.9	-10.0	-15.2	-7.3	-9.1	-8.2	-11.7	-6.6	-6.5	-4.4
	HC2	-25.5	-11.4	-11.5	-4.4	-4.4	-4.0	-7.0	-2.9	-3.9	-3.6	-4.9	-2.6	-2.8	-1.9
	HC3	18.1	16.6	9.9	10.0	8.2	8.1	6.9	6.1	5.2	5.0	5.2	4.5	3.3	2.8
	HC4	209.6	35.5	60.3	10.2	16.9	13.0	33.6	8.9	15.5	11.9	24.8	10.0	11.5	5.2
	HC5	209.6	35.5	60.3	10.2	16.9	13.0	33.6	8.9	15.5	11.9	24.8	10.0	11.5	5.2
d**		**λ=1.5; L=1**		**λ=1.8; L=1**		**λ=1.9; L=1**		**λ=2.1; L=2**		**λ=1.9; L=3**		**λ=2; L=2**		**λ=2.2; L=2**	
beta 0	**HC0**	-15.3	-14.0	-6.6	-6.8	-5.0	-5.4	-4.6	-4.8	-3.4	-3.5	-2.7	-2.6	-2.0	-2.0
	HC1	-5.9	-3.9	0.1	0.1	0.0	-0.3	-0.6	-0.6	-0.1	0.0	-0.2	0.0	0.0	0.0
	HC2	-1.4	-0.7	0.1	0.0	-0.3	-0.2	-0.3	-0.3	0.0	0.0	0.0	0.0	0.0	0.0
	HC3	17.2	15.4	7.4	7.4	4.7	5.5	4.4	4.5	3.6	3.7	2.8	2.7	2.1	2.1
	HC4	41.9	10.5	4.1	1.8	1.1	2.4	4.4	2.1	3.0	1.6	2.3	1.2	1.2	0.9
	HC5	41.9	10.5	4.1	1.8	1.1	2.4	4.4	2.1	3.0	1.6	2.7	1.2	1.2	0.9
beta 1	**HC0**	-22.9	-17.2	-14.0	-10.3	-10.2	-9.0	-11.7	-7.5	-8.4	-6.5	-7.2	-4.5	-4.1	-3.7
	HC1	-14.4	-7.5	-7.9	-3.6	-5.5	-4.1	-8.0	-3.5	-5.3	-3.1	-4.8	-2.0	-2.2	-1.7
	HC2	-2.4	-0.4	-1.3	0.1	2.1	0.0	-1.5	0.0	-0.7	0.0	-0.8	0.1	0.1	0.0
	HC3	26.7	20.3	14.0	11.9	17.5	10.2	10.6	8.3	7.8	6.9	6.4	4.9	4.5	3.8
	HC4	80.8	16.3	26.9	7.5	46.7	12.3	31.0	11.3	18.5	7.9	16.3	5.2	7.8	4.4
	HC5	80.8	16.3	26.9	7.5	50.1	12.3	31.0	11.3	18.5	7.9	19.2	5.2	7.8	4.4

Table 5.3: Percentage Deviations for the Case 3*

Case 3*		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**		λ=1; L=2		λ=1; L=5		λ=1; L=7		λ=1; L=1		λ=1; L=4		λ=1; L=6		λ=1; L=3	
beta 0	HC0	-13.1	-13.2	-10.5	-10.5	-5.9	-7.4	-4.2	-4.3	-3.6	-4.1	-2.6	-2.9	-2.0	-2.0
	HC1	-3.5	-2.3	-4.1	-2.8	-0.9	-1.4	-0.2	-0.3	-0.2	-0.5	-0.1	-0.2	0.0	0.0
	HC2	0.0	0.0	-0.2	0.3	0.2	0.1	-0.1	-0.1	0.0	0.0	0.0	0.0	0.1	0.1
	HC3	16.3	15.4	12.2	12.7	6.7	8.3	4.1	4.3	3.7	4.3	2.6	3.0	2.2	2.3
	HC4	16.9	5.1	19.9	8.1	4.1	4.1	0.6	0.7	0.9	1.4	0.5	0.9	0.4	0.5
	HC5	16.9	5.1	19.9	8.1	4.1	4.1	0.6	0.7	0.9	1.4	0.5	0.9	0.4	0.5
beta 1	HC0	-16.4	-13.9	-13.7	-11.2	-7.6	-7.8	-5.3	-5.4	-4.2	-4.6	-3.2	-3.5	-2.5	-2.6
	HC1	-7.1	-3.2	-7.5	-3.5	-2.8	-1.9	-1.3	-1.4	-0.9	-1.1	-0.7	-0.8	-0.5	-0.5
	HC2	-0.1	-0.1	-0.3	0.1	0.1	0.0	-0.2	-0.2	0.0	0.0	0.1	0.1	0.1	0.1
	HC3	20.5	16.1	15.9	13.1	8.6	8.5	5.1	5.3	4.5	4.9	3.4	3.8	2.7	2.8
	HC4	23.8	5.1	28.4	7.8	6.4	3.8	1.6	1.7	1.7	1.9	1.4	1.8	0.9	1.0
	HC5	23.8	5.1	28.4	7.8	6.4	3.8	1.6	1.7	1.7	1.9	1.4	1.8	0.9	1.0
b**		λ=14.9; L=2		λ=17.3; L=4		λ=37.7; L=5		λ=9.6; L=6		λ=2.5; L=5		λ=5.3; L=3		λ=4.4; L=8	
beta 0	HC0	-23.0	-19.6	-11.0	-12.6	-6.9	-6.3	-5.3	-6.4	-3.2	-3.4	-3.1	-3.4	-2.0	-2.1
	HC1	-14.5	-9.5	-4.7	-5.3	-2.0	-0.6	-1.3	-2.0	0.2	0.2	-0.6	-0.8	0.0	0.1
	HC2	-5.3	-2.9	-0.8	-1.4	-0.1	1.3	-0.3	-0.6	0.2	0.3	-0.1	-0.1	0.0	0.1
	HC3	17.5	17.8	10.8	11.4	7.3	9.8	5.0	5.7	3.6	4.2	3.0	3.2	2.1	2.5
	HC4	25.8	12.8	8.4	7.2	4.0	6.7	1.9	2.0	0.7	1.2	0.9	1.2	0.3	0.7
	HC5	25.8	12.8	8.4	7.2	4.0	6.7	1.9	2.0	0.7	1.2	0.9	1.2	0.3	0.7
beta 1	HC0	-24.9	-18.1	-14.4	-14.5	-9.4	-9.3	-6.9	-7.3	-4.1	-4.5	-3.8	-4.1	-2.6	-2.7
	HC1	-16.6	-7.8	-8.3	-7.3	-4.6	-3.8	-3.0	-2.9	-0.8	-0.9	-1.3	-1.5	-0.6	-0.6
	HC2	-7.4	-3.4	-2.9	-3.1	-1.8	-1.5	-1.2	-1.2	0.0	0.1	-0.4	-0.5	-0.2	-0.2
	HC3	15.1	14.3	10.2	9.9	6.3	7.1	4.9	5.4	4.4	5.0	3.0	3.3	2.2	2.5
	HC4	22.6	5.8	8.9	5.3	3.5	2.9	2.1	1.9	1.4	1.8	1.0	1.3	0.4	0.5
	HC5	22.6	5.8	8.9	5.3	3.5	2.9	2.1	1.9	1.4	1.8	1.0	1.3	0.4	0.5
c**		λ=9810.1 L=3		λ=196809.8 L=2		λ=491.7 L=1		λ=2226.4 L=1		λ=253; L=3		λ=2316146.5 L=4		λ=5522.3 L=12	
beta 0	HC0	-30.4	-2.1	-8.7	-9.6	-6.6	-6.8	-5.1	-5.3	-4.5	-5.1	-3.3	-3.6	-3.0	-3.4
	HC1	-22.6	11.0	-2.2	-2.6	-1.7	-1.8	-1.1	-1.3	-1.2	-1.6	-0.8	-1.0	-1.0	-1.1
	HC2	-6.6	12.6	-0.2	-0.2	-0.3	-0.3	0.0	-0.1	-0.1	-0.2	0.0	0.0	-0.2	0.0
	HC3	27.0	29.8	9.2	10.3	6.6	6.8	5.3	5.5	4.4	4.8	3.4	3.7	2.8	3.6
	HC4	75.8	17.9	3.1	3.9	2.3	2.5	1.7	1.9	1.6	2.2	1.4	1.8	1.6	2.5
	HC5	75.8	17.9	3.1	3.9	2.3	2.5	1.7	1.9	1.6	2.2	1.4	1.8	1.6	2.5
beta 1	HC0	-26.4	-10.8	-11.1	-12.3	-8.1	-8.2	-6.3	-6.6	-5.6	-6.1	-4.3	-4.6	-3.7	-3.7
	HC1	-18.2	1.1	-4.8	-5.5	-3.3	-3.2	-2.4	-2.7	-2.3	-2.7	-1.9	-2.0	-1.7	-1.5
	HC2	-7.0	1.4	-2.0	-2.3	-1.5	-1.5	-1.0	-1.1	-1.0	-1.2	-0.8	-0.8	-0.7	-0.6
	HC3	19.6	15.4	8.0	8.9	5.6	5.7	4.7	4.7	3.8	4.0	2.9	3.1	2.3	2.6
	HC4	51.3	2.7	2.2	2.9	1.3	1.2	1.2	1.2	1.0	1.4	1.0	1.2	1.1	0.9
	HC5	51.3	2.7	2.2	2.9	1.3	1.2	1.2	1.2	1.0	1.4	1.0	1.2	1.1	0.9
d**		λ=1.4; L=2		λ=1.1; L=4		λ=1.6; L=1		λ=1.4; L=1		λ=1.8; L=2		λ=1.9; L=2		λ=1.3; L=4	
beta 0	HC0	-11.0	-12.5	-8.8	-8.6	-5.1	-5.4	-4.1	-4.2	-3.3	-3.5	-2.6	-2.8	-2.0	-2.2
	HC1	-1.1	-1.6	-2.3	-0.9	-0.1	-0.3	-0.1	-0.1	0.0	0.0	-0.1	-0.2	0.0	-0.1
	HC2	-0.1	0.0	0.3	0.1	0.0	0.0	-0.1	-0.1	0.0	0.0	0.0	0.0	0.0	0.0
	HC3	12.5	14.5	10.7	9.7	5.5	5.7	4.1	4.3	3.5	3.7	2.7	2.9	2.1	2.2
	HC4	4.2	3.7	11.1	2.5	1.0	1.2	0.5	0.6	0.4	0.6	0.5	0.7	0.3	0.4
	HC5	4.2	3.7	11.1	2.5	1.0	1.2	0.5	0.6	0.4	0.6	0.5	0.7	0.3	0.4
beta 1	HC0	-13.7	-13.5	-11.1	-10.3	-6.5	-6.8	-4.9	-5.0	-4.0	-4.2	-3.2	-3.3	-2.5	-2.6
	HC1	-4.1	-2.7	-4.8	-2.9	-1.6	-1.8	-0.9	-1.0	-0.7	-0.8	-0.7	-0.8	-0.5	-0.5
	HC2	-0.3	0.0	0.1	-0.1	0.0	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	HC3	15.4	15.7	13.1	11.3	7.0	7.2	5.1	5.3	4.1	4.3	3.3	3.5	2.5	2.7
	HC4	7.6	4.4	13.9	4.7	2.5	2.8	1.4	1.5	0.9	1.2	1.1	1.3	0.7	0.8
	HC5	7.6	4.4	13.9	4.7	2.5	2.8	1.4	1.5	0.9	1.2	1.1	1.3	0.7	0.8

Table 5.4: Percentage Deviations for the Case 4*

Case 4*		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**		λ=1; L=4		λ=1; L=4		λ=1; L=9		λ=1.; L=7		λ=1; L=9		λ=1; L=14		λ=1; L=12	
beta 0	HC0	-7.0	-16.4	-7.8	-8.9	-5.8	-9.2	-6.4	-6.0	-6.8	-4.7	-5.9	-4.3	-3.9	-2.7
	HC1	3.3	-4.5	-1.2	-1.3	-0.9	-3.0	-2.5	-1.4	-3.6	-0.8	-3.5	-1.3	-1.9	-0.4
	HC2	0.1	0.0	0.0	-0.1	-0.1	-0.1	0.0	-0.1	-0.2	0.0	0.0	0.0	0.0	0.0
	HC3	85.9	22.1	10.4	9.7	7.2	10.7	8.1	6.3	10.8	5.0	9.1	4.5	4.5	2.7
	HC4	>1000	36.8	31.1	3.8	18.7	18.7	25.2	5.3	60.0	3.8	44.3	4.6	12.1	1.2
	HC5	>1000	36.8	39.3	3.8	31.3	18.7	68.5	5.3	>1000	3.8	>1000	4.6	44.5	1.2
beta 1	HC0	-91.1	-25.4	-23.4	-12.5	-19.3	-13.4	-17.6	-8.5	-30.8	-7.4	-28.2	-6.0	-11.2	-3.6
	HC1	-90.1	-14.7	-17.9	-5.2	-15.0	-7.5	-14.2	-4.1	-28.5	-3.7	-26.3	-3.1	-9.4	-1.4
	HC2	0.6	-0.2	0.1	-0.2	-0.2	-0.1	0.0	-0.1	-0.9	0.1	0.0	-0.1	0.0	0.0
	HC3	>1000	39.1	36.3	14.2	26.8	16.4	24.6	9.4	53.6	8.4	46.5	6.3	13.6	3.8
	HC4	>1000	117.6	165.7	11.7	114.6	41.8	102.6	14.4	349.7	12.6	258.8	9.9	48.3	3.7
	HC5	>1000	117.6	206.4	11.7	185.7	41.8	265.5	14.4	>1000	12.6	>1000	9.9	166.8	3.7
b**		λ=60.2; L=3		λ=54.9; L=6		λ=59.7; L=6		λ=255.6; L=9		λ=1380.6; L=8		λ=206.6; L=10		λ=1439; L=19	
beta 0	HC0	-35.8	-16.7	-38.9	-24.6	-32.4	-12.0	-75.3	-10.7	-67.6	-7.3	-34.5	-5.7	-82.4	-6.9
	HC1	-28.7	-5.6	-34.5	-17.7	-28.8	-6.5	-74.3	-6.1	-66.5	-3.6	-32.8	-3.0	-82.0	-4.5
	HC2	-10.5	-0.2	-15.4	-5.7	-11.2	-1.0	-49.3	-1.4	-36.3	-0.4	-13.8	-0.5	-57.8	-1.3
	HC3	30.6	20.9	22.6	20.4	19.3	12.0	15.3	9.3	39.4	7.2	15.0	5.1	8.5	4.8
	HC4	190.9	18.4	179.8	88.5	125.0	22.0	574.9	16.9	687.9	11.5	111.5	8.0	671.4	12.8
	HC5	190.9	18.4	221.9	88.5	185.4	22.0	>1000	16.9	>1000	11.5	882.7	8.0	>1000	12.8
beta 1	HC0	-44.4	-25.4	-41.5	-29.4	-35.4	-15.9	-76.6	-13.8	-67.8	-9.9	-34.5	-7.6	-82.5	-8.6
	HC1	-38.2	-15.5	-37.3	-22.9	-32.0	-10.7	-75.6	-9.4	-66.7	-6.3	-32.8	-4.9	-82.1	-6.3
	HC2	-15.3	-4.5	-16.1	-8.3	-12.7	-3.0	-50.1	-2.9	-35.4	-1.4	-13.7	-1.3	-57.8	-2.0
	HC3	32.3	23.0	24.4	21.2	20.0	12.4	15.4	9.8	41.1	8.1	15.2	5.6	8.6	5.2
	HC4	216.4	25.8	186.6	102.8	133.2	26.4	581.3	20.7	685.9	14.3	110.7	10.0	671.4	14.9
	HC5	216.4	25.8	229.7	102.8	197.1	26.4	>1000	20.7	>1000	14.3	865.0	10.0	>1000	14.9
c**		λ=1211.1 L=2		λ=4135 L=5		λ=62114.1 L=8		λ=2313.8 L=10		λ=19995.2 L=6		λ=3655.2 L=14		λ=24314.2 L=14	
beta 0	HC0	-59.2	-7.8	-49.1	-26.3	-87.4	-18.3	-45.6	-17.8	-25.2	-7.4	-21.0	-9.7	-65.5	-6.9
	HC1	-54.7	3.7	-45.5	-19.9	-86.7	-12.8	-43.4	-13.5	-22.6	-3.9	-18.9	-6.8	-64.8	-4.6
	HC2	-29.3	7.4	-17.5	-6.7	-64.6	-3.0	-22.2	-3.7	-9.0	-0.2	-6.6	-1.7	-38.4	-1.1
	HC3	34.2	25.5	40.1	19.6	23.3	15.8	14.4	13.3	11.3	7.6	10.8	7.0	14.9	5.0
	HC4	461.4	15.5	343.2	68.4	>1000	41.2	159.8	47.8	67.2	9.7	56.7	18.5	329.3	12.1
	HC5	461.4	15.5	733.3	68.4	>1000	41.2	677.7	47.8	81.5	9.7	112.4	18.5	>1000	12.1
beta 1	HC0	-63.3	-14.3	-52.1	-28.7	-89.5	-21.0	-42.9	-19.6	-24.8	-9.5	-20.6	-10.7	-62.2	-7.5
	HC1	-59.2	-3.6	-48.6	-22.5	-89.0	-15.8	-40.5	-15.4	-22.2	-6.0	-18.5	-8.0	-61.5	-5.3
	HC2	-32.5	0.4	-19.0	-9.0	-66.4	-5.2	-20.4	-5.0	-9.1	-1.7	-6.5	-2.5	-35.9	-1.7
	HC3	32.7	17.8	40.6	17.2	22.9	14.2	14.2	12.8	10.5	6.9	10.5	6.5	15.6	4.6
	HC4	468.5	7.4	350.7	63.4	>1000	39.8	147.9	47.4	63.1	9.5	54.4	18.1	315.9	11.3
	HC5	468.5	7.4	739.5	63.4	>1000	39.8	619.4	47.4	76.5	9.5	107.0	18.1	>1000	11.3
d**		λ=2.8; L=2		λ=3.6; L=3		λ=2.4; L=6		λ=12.5; L=7		λ=4.3; L=10		λ=3.7; L=17		λ=46.7; L=17	
beta 0	HC0	-15.3	-11.3	-32.3	-8.8	-10.5	-7.2	-49.6	-5.9	-26.5	-5.4	-9.5	-4.7	-65.2	-3.0
	HC1	-5.8	-0.2	-27.5	-1.5	-5.8	-1.4	-47.5	-1.3	-24.0	-1.4	-7.2	-1.6	-64.5	-0.6
	HC2	-1.3	-0.1	-13.4	0.1	-1.1	0.0	-35.7	0.1	-11.9	0.1	-2.1	-0.1	-42.8	0.0
	HC3	18.4	12.8	31.7	10.1	11.2	8.0	36.8	6.5	16.0	6.0	7.2	5.0	9.9	3.2
	HC4	62.0	3.3	426.7	4.9	39.8	8.0	>1000	3.4	172.9	5.8	30.3	8.3	432.3	3.0
	HC5	62.0	3.3	>1000	4.9	44.7	8.0	>1000	3.4	>1000	5.8	114.6	8.3	>1000	3.0
beta 1	HC0	-35.7	-18.2	-60.3	-12.1	-24.5	-11.6	-91.3	-7.1	-48.8	-8.1	-21.1	-7.8	-75.1	-5.0
	HC1	-28.5	-8.0	-57.5	-5.0	-20.5	-6.0	-90.9	-2.5	-47.1	-4.3	-19.1	-4.8	-74.6	-2.6
	HC2	-6.1	-1.0	-27.1	-0.5	-3.6	-0.3	-67.2	0.1	-22.4	-0.2	-5.2	-0.3	-49.0	-0.2
	HC3	40.9	20.6	57.3	12.9	25.2	12.9	66.7	7.8	29.1	8.7	15.4	8.2	12.3	4.9
	HC4	206.7	18.1	838.4	11.8	114.9	22.6	>1000	5.6	328.9	13.3	74.1	20.1	503.8	8.5
	HC5	206.7	18.1	>1000	11.8	128.8	22.6	>1000	5.6	>1000	13.3	274.3	20.1	>1000	8.5

Table 5.5: Percentage Deviations for the Case 5*

Case 5*		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**		λ=1. L=5		λ=1. L=6		λ=1. L=9		λ=1. L=3		λ=1. L=7		λ=1. L=8		λ=1. L=17	
beta 0	HC0	-5.4	-14.3	-4.1	-7.9	-4.5	-7.5	-2.4	-4.2	-1.8	-3.7	-2.1	-2.7	-1.8	-2.6
	HC1	5.1	-1.1	2.7	0.5	0.5	-1.2	1.7	0.0	1.6	0.1	0.4	0.1	0.2	-0.2
	HC2	0.0	-0.2	-0.1	-0.2	0.1	0.0	0.1	0.0	0.0	0.1	0.0	0.1	-0.1	-0.1
	HC3	6.3	17.1	4.0	8.4	5.7	8.8	2.8	4.6	2.2	4.0	3.0	3.0	1.8	2.5
	HC4	>1000	7.4	-1.5	1.8	12.0	13.6	4.9	1.5	6.4	2.5	14.8	1.5	2.2	1.9
	HC5	>1000	7.4	-1.5	1.8	27.2	13.6	>1000	1.5	>1000	2.5	>1000	1.5	15.9	1.9
beta 1	HC0	-91.3	-23.8	-29.3	-15.6	-29.9	-14.9	-60.1	-6.7	-63.9	-8.2	-44.1	-5.0	-25.6	-5.5
	HC1	-90.4	-12.1	-24.2	-8.0	-26.2	-9.0	-58.4	-2.6	-62.7	-4.6	-42.6	-2.3	-24.1	-3.1
	HC2	0.3	0.0	0.2	-0.2	-0.2	-0.2	0.9	0.2	0.4	0.0	-0.3	0.1	0.3	-0.1
	HC3	>1000	34.1	46.8	19.1	44.8	17.9	270.8	7.6	290.2	9.1	120.1	5.6	43.8	5.6
	HC4	>1000	52.4	231.9	33.7	218.0	43.2	>1000	6.5	>1000	16.7	>1000	7.3	249.9	11.1
	HC5	>1000	52.4	254.2	33.7	505.8	43.2	>1000	6.5	>1000	16.7	>1000	7.3	>1000	11.1
b**		λ=41682.4; L=3		λ=29940.8; L=6		λ=463.6;L=5		λ=8723.2;L=8		λ=11836; L=6		λ=5880; L=13		λ=423933.2; L=17	
beta 0	HC0	-58.3	-20.7	-4.4	-23.2	-14.9	-13.0	39.0	-5.7	15.5	-7.1	-43.2	-4.9	-8.8	-5.0
	HC1	-53.7	-10.1	2.4	-16.2	-10.4	-7.8	44.8	-1.0	19.4	-3.5	-41.7	-2.0	-6.9	-2.6
	HC2	-48.4	-1.8	5.5	-4.6	-2.0	-1.0	48.7	0.7	31.3	0.0	-23.2	-0.2	17.7	-0.3
	HC3	74.9	23.1	66.6	19.9	13.7	13.2	59.2	7.6	68.4	7.8	22.8	4.7	91.6	4.8
	HC4	>1000	25.9	>1000	53.3	45.6	24.6	60.9	4.6	887.0	10.8	406.0	4.8	>1000	8.7
	HC5	>1000	25.9	>1000	53.3	45.6	24.6	>1000	4.6	>1000	10.8	>1000	4.8	>1000	8.7
beta 1	HC0	-99.2	-33.3	-98.6	-29.5	-23.0	-19.0	-83.9	-15.3	-92.8	-11.4	-76.8	-10.5	-89.7	-9.8
	HC1	-99.1	-24.5	-98.5	-23.1	-19.0	-14.1	-83.3	-11.1	-92.5	-8.0	-76.2	-7.8	-89.5	-7.6
	HC2	-91.7	-10.5	-88.3	-8.6	-7.3	-4.9	-58.4	-4.4	-72.9	-2.7	-51.0	-3.2	-63.5	-2.9
	HC3	67.1	20.9	32.1	19.0	12.1	11.9	23.4	8.2	29.5	7.0	13.1	4.9	82.3	4.6
	HC4	>1000	33.3	>1000	56.0	57.4	28.7	>1000	22.0	>1000	13.1	576.2	14.2	>1000	15.1
	HC5	>1000	33.3	>1000	56.0	57.4	28.7	>1000	22.0	>1000	13.1	>1000	14.2	>1000	15.1
c**		λ=101039 L=2		λ=1010567 L=5		λ=393244 L=6		λ=926006 L=11		λ=9408017 L=9		λ=10802473 L=11		λ=1830570.8 L=18	
beta 0	HC0	-35.6	-19.2	81.7	-10.6	-72.2	-11.4	-15.6	-13.0	-47.7	-8.5	4.9	-5.8	17.9	-5.3
	HC1	-28.4	-9.1	94.7	-2.9	-70.7	-5.9	-12.1	-8.3	-45.9	-4.7	7.6	-3.0	20.3	-2.9
	HC2	-2.8	-0.9	94.8	2.7	-47.4	0.1	6.2	-1.4	-9.7	-0.1	17.0	-0.1	30.5	0.1
	HC3	61.9	23.2	109.6	19.0	25.2	13.5	38.3	12.5	69.1	9.4	35.3	6.0	45.9	5.8
	HC4	477.5	32.9	611.2	26.9	897.5	19.7	161.1	34.5	601.1	17.4	124.2	9.3	91.8	11.2
	HC5	658.1	32.9	>1000	26.9	>1000	19.7	>1000	34.5	>1000	17.4	>1000	9.3	>1000	11.2
beta 1	HC0	-69.0	-37.5	-99.5	-30.1	-80.6	-18.7	-55.5	-22.1	-57.8	-15.0	-71.3	-10.5	-75.2	-11.0
	HC1	-65.6	-29.7	-99.5	-24.1	-79.6	-13.7	-53.6	-17.9	-56.3	-11.6	-70.6	-7.8	-74.7	-8.7
	HC2	-34.1	-15.1	-93.4	-10.1	-54.6	-5.4	-26.2	-7.7	-17.0	-4.2	-44.8	-3.1	-48.4	-3.4
	HC3	50.7	17.0	53.2	16.6	22.8	10.3	28.0	9.7	69.0	8.1	15.0	4.9	16.5	4.9
	HC4	797.4	53.8	>1000	57.9	956.6	21.0	326.1	42.5	664.6	23.7	459.5	13.9	570.2	18.6
	HC5	>1000	53.8	>1000	57.9	>1000	21.0	>1000	42.5	>1000	23.7	>1000	13.9	>1000	18.6
d**		λ=4.6; L=2		λ=314031510; L=3		λ=2294.6; L=7		λ=210.5; L=8		λ=48.5; L=11		λ=52585; L=15		λ=433.8; L=15	
beta 0	HC0	-8.3	-10.2	-4.7	-6.3	-10.5	-4.7	-3.0	-4.6	-8.8	-6.0	3.0	-3.1	-5.2	-2.5
	HC1	1.9	1.0	2.1	1.2	-5.8	1.5	1.0	0.2	-5.7	-2.0	5.7	0.0	-3.2	-0.1
	HC2	-0.6	0.7	0.6	0.6	0.7	0.9	2.6	0.2	-2.3	-0.7	30.6	0.0	-1.2	0.0
	HC3	8.2	13.7	58.1	8.3	14.7	6.9	15.3	5.3	6.3	5.2	82.5	3.3	3.2	2.5
	HC4	3.1	6.1	>1000	2.6	44.2	2.7	178.9	3.5	27.4	9.2	382.3	2.7	10.3	1.5
	HC5	3.1	6.1	>1000	2.6	53.2	2.7	>1000	3.5	119.0	9.2	>1000	2.7	24.7	1.5
beta 1	HC0	-15.5	-20.4	38.6	-14.8	-20.6	-14.0	26.0	-11.5	-39.3	-11.1	-20.7	-8.0	-10.4	-4.6
	HC1	-6.1	-10.4	48.5	-7.9	-16.4	-8.4	31.2	-7.1	-37.2	-7.3	-18.6	-5.1	-8.6	-2.3
	HC2	17.9	-0.6	>1000	-1.3	0.2	-0.9	150.6	-0.8	-17.9	-0.1	27.1	-0.2	2.4	0.0
	HC3	81.8	24.8	>1000	14.8	27.3	14.7	594.3	11.5	13.4	12.6	113.9	8.5	17.3	4.9
	HC4	446.5	25.8	>1000	18.1	103.5	27.0	>1000	23.5	124.8	35.4	597.2	20.7	54.6	7.2
	HC5	446.5	25.8	>1000	18.1	126.4	27.0	>1000	23.5	568.6	35.4	>1000	20.7	141.1	7.2

Table 5.6: Quasi-t Results for the Case 1*

Case 1*		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**	TRUE	**1,557**	**0,921**	**3,034**	**2,956**	**3,649**	**3,464**	**3,842**	**3,702**	**4,355**	**4,161**	**4,247**	**3,390**	**4,042**	**3,692**
beta 0	HC0	1,803	1,049	3,157	3,078	3,764	3,577	3,937	3,795	4,442	4,248	4,321	3,458	4,106	3,752
	HC1	1,710	0,989	3,050	2,970	3,669	3,482	3,857	3,715	4,367	4,173	4,267	3,409	4,065	3,713
	HC2	1,556	0,920	3,033	2,954	3,650	3,465	3,842	3,701	4,354	4,160	4,247	3,390	4,040	3,692
	HC3	1,327	0,793	2,911	2,834	3,538	3,355	3,749	3,609	4,267	4,072	4,175	3,324	3,975	3,633
	HC4	1,048	0,640	2,982	2,917	3,586	3,420	3,802	3,669	4,313	4,121	4,209	3,355	3,978	3,641
	HC5	1,048	0,640	2,982	2,917	3,586	3,420	3,802	3,669	4,313	4,121	4,209	3,355	3,978	3,641
	TRUE	**1,105**	**0,688**	**1,454**	**1,342**	**1,702**	**1,464**	**1,867**	**1,684**	**2,031**	**1,778**	**2,503**	**2,104**	**2,760**	**2,560**
beta 1	HC0	1,265	0,784	1,533	1,414	1,779	1,526	1,928	1,738	2,086	1,827	2,547	2,145	2,802	2,599
	HC1	1,200	0,739	1,481	1,365	1,734	1,486	1,889	1,701	2,051	1,795	2,515	2,115	2,773	2,572
	HC2	1,104	0,688	1,453	1,342	1,703	1,464	1,868	1,684	2,030	1,778	2,503	2,104	2,759	2,559
	HC3	0,953	0,595	1,376	1,272	1,628	1,403	1,809	1,632	1,975	1,729	2,460	2,064	2,717	2,520
	HC4	0,773	0,491	1,389	1,299	1,615	1,413	1,817	1,648	1,977	1,736	2,482	2,085	2,723	2,530
	HC5	0,773	0,491	1,389	1,299	1,615	1,413	1,817	1,648	1,977	1,736	2,482	2,085	2,723	2,530
b**	TRUE	**3,721**	**3,688**	**8,454**	**8,645**	**7,939**	**7,857**	**9,086**	**8,878**	**8,704**	**8,676**	**11,428**	**11,342**	**10,300**	**9,429**
beta 0	HC0	4,068	3,963	8,852	9,011	8,198	8,107	9,324	9,113	8,862	8,830	11,594	11,514	10,411	9,537
	HC1	3,860	3,748	8,552	8,670	7,990	7,878	9,136	8,899	8,713	8,679	11,448	11,359	10,306	9,438
	HC2	3,733	3,670	8,440	8,613	7,924	7,830	9,075	8,865	8,680	8,649	11,410	11,323	10,279	9,411
	HC3	3,413	3,395	8,037	8,229	7,656	7,561	8,830	8,622	8,502	8,471	11,227	11,134	10,148	9,287
	HC4	3,408	3,527	8,117	8,501	7,774	7,739	8,934	8,794	8,612	8,588	11,327	11,247	10,227	9,362
	HC5	3,408	3,527	8,117	8,501	7,774	7,739	8,934	8,794	8,612	8,588	11,327	11,247	10,227	9,362
	TRUE	**1,785**	**1,701**	**2,834**	**2,558**	**3,174**	**2,841**	**3,521**	**2,954**	**3,836**	**3,771**	**4,598**	**4,300**	**4,919**	**4,658**
beta 1	HC0	1,991	1,843	3,065	2,721	3,323	2,963	3,658	3,061	3,931	3,862	4,696	4,395	4,986	4,723
	HC1	1,889	1,744	2,962	2,618	3,239	2,880	3,584	2,989	3,865	3,796	4,636	4,336	4,936	4,674
	HC2	1,813	1,707	2,869	2,571	3,190	2,846	3,535	2,962	3,840	3,772	4,608	4,310	4,919	4,659
	HC3	1,644	1,578	2,681	2,429	3,061	2,734	3,416	2,866	3,751	3,685	4,521	4,226	4,854	4,595
	HC4	1,616	1,638	2,644	2,490	3,090	2,792	3,434	2,916	3,797	3,735	4,554	4,263	4,892	4,632
	HC5	1,616	1,638	2,644	2,490	3,090	2,792	3,434	2,916	3,797	3,735	4,554	4,263	4,892	4,632
c**	TRUE	**5,869**	**7,350**	**9,494**	**10,084**	**9,485**	**9,902**	**9,039**	**9,653**	**10,474**	**11,359**	**13,065**	**13,661**	**11,808**	**12,136**
beta 0	HC0	6,746	8,014	10,014	10,586	9,857	10,230	9,332	9,946	10,795	11,708	13,283	13,876	11,982	12,304
	HC1	6,400	7,556	9,674	10,214	9,607	9,957	9,143	9,732	10,613	11,497	13,116	13,692	11,862	12,178
	HC2	5,990	7,325	9,436	9,992	9,446	9,823	9,004	9,598	10,464	11,344	13,027	13,608	11,782	12,106
	HC3	5,295	6,680	8,878	9,417	9,045	9,428	8,684	9,260	10,139	10,988	12,774	13,344	11,584	11,910
	HC4	4,984	6,862	8,850	9,437	9,051	9,552	8,677	9,298	10,101	10,956	12,828	13,433	11,595	11,950
	HC5	4,984	6,862	8,850	9,437	9,051	9,552	8,677	9,298	10,101	10,956	12,828	13,433	11,595	11,950
	TRUE	**1,739**	**1,806**	**2,508**	**2,510**	**2,706**	**2,642**	**2,897**	**2,871**	**3,232**	**3,201**	**3,968**	**3,900**	**4,076**	**4,083**
beta 1	HC0	2,058	2,024	2,710	2,704	2,858	2,785	3,032	3,002	3,364	3,330	4,066	3,995	4,160	4,164
	HC1	1,953	1,908	2,618	2,609	2,785	2,711	2,970	2,937	3,307	3,270	4,015	3,942	4,118	4,121
	HC2	1,812	1,838	2,544	2,541	2,727	2,658	2,918	2,891	3,255	3,223	3,982	3,912	4,087	4,093
	HC3	1,589	1,666	2,385	2,387	2,601	2,537	2,808	2,782	3,150	3,119	3,900	3,830	4,015	4,023
	HC4	1,486	1,709	2,380	2,396	2,602	2,566	2,803	2,793	3,137	3,113	3,917	3,856	4,018	4,036
	HC5	1,486	1,709	2,380	2,396	2,602	2,566	2,803	2,793	3,137	3,113	3,917	3,856	4,018	4,036
d**	TRUE	**2,624**	**2,175**	**2,961**	**2,775**	**2,534**	**2,134**	**2,587**	**1,752**	**3,753**	**3,642**	**3,779**	**2,998**	**4,743**	**4,234**
beta 0	HC0	2,800	2,345	3,091	2,899	2,638	2,214	2,694	1,817	3,842	3,730	3,853	3,056	4,805	4,293
	HC1	2,656	2,203	2,986	2,793	2,571	2,155	2,640	1,774	3,777	3,665	3,805	3,015	4,756	4,248
	HC2	2,626	2,176	2,960	2,774	2,532	2,133	2,584	1,751	3,754	3,643	3,777	2,997	4,741	4,233
	HC3	2,455	2,017	2,832	2,652	2,428	2,054	2,476	1,687	3,667	3,557	3,702	2,938	4,677	4,173
	HC4	2,497	2,128	2,870	2,724	2,420	2,087	2,389	1,699	3,702	3,598	3,709	2,959	4,710	4,205
	HC5	2,497	2,128	2,870	2,724	2,420	2,087	2,389	1,699	3,702	3,598	3,709	2,959	4,710	4,205
	TRUE	**1,082**	**0,644**	**1,334**	**1,118**	**1,626**	**1,408**	**1,601**	**1,058**	**1,932**	**1,794**	**2,297**	**1,900**	**2,761**	**2,533**
beta 1	HC0	1,203	0,701	1,415	1,178	1,693	1,463	1,667	1,099	1,983	1,841	2,343	1,939	2,799	2,570
	HC1	1,141	0,658	1,367	1,135	1,650	1,424	1,634	1,073	1,950	1,809	2,314	1,912	2,770	2,543
	HC2	1,083	0,644	1,335	1,117	1,626	1,408	1,600	1,057	1,932	1,795	2,297	1,900	2,760	2,533
	HC3	0,971	0,591	1,256	1,058	1,560	1,356	1,533	1,016	1,882	1,750	2,251	1,862	2,723	2,497
	HC4	0,924	0,618	1,236	1,075	1,563	1,380	1,490	1,023	1,894	1,769	2,259	1,876	2,743	2,517
	HC5	0,924	0,618	1,236	1,075	1,563	1,380	1,490	1,023	1,894	1,769	2,259	1,876	2,743	2,517

Table 5.7: Quasi-t Results for the Case 2*

Case 2 *		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**	**TRUE**	**4,454**	**4,217**	**5,427**	**5,235**	**6,267**	**6,073**	**7,040**	**6,880**	**7,618**	**7,483**	**8,941**	**8,662**	**9,999**	**9,871**
beta 0	**HC0**	4,706	4,462	5,628	5,436	6,419	6,240	7,188	7,028	7,745	7,617	9,054	8,781	10,097	9,971
	HC1	4,464	4,221	5,437	5,245	6,257	6,069	7,043	6,880	7,614	7,484	8,940	8,664	9,995	9,868
	HC2	4,452	4,215	5,427	5,237	6,264	6,070	7,039	6,879	7,620	7,484	8,942	8,663	9,995	9,867
	HC3	4,189	3,973	5,224	5,040	6,109	5,901	6,889	6,730	7,495	7,352	8,829	8,546	9,892	9,764
	HC4	4,126	4,138	5,272	5,152	6,167	5,981	6,909	6,790	7,572	7,439	8,879	8,604	9,942	9,825
	HC5	4,126	4,138	5,272	5,152	6,167	5,981	6,909	6,790	7,570	7,439	8,877	8,604	9,940	9,825
	TRUE	**5,742**	**4,716**	**5,490**	**4,914**	**6,254**	**4,770**	**6,594**	**5,620**	**8,191**	**6,964**	**10,072**	**8,353**	**9,432**	**8,619**
beta 1	**HC0**	6,408	5,144	5,866	5,236	6,643	5,037	6,904	5,850	8,549	7,178	10,377	8,574	9,630	8,771
	HC1	6,079	4,866	5,667	5,052	6,475	4,899	6,764	5,727	8,406	7,053	10,246	8,460	9,533	8,682
	HC2	5,733	4,708	5,487	4,917	6,253	4,769	6,591	5,618	8,189	6,965	10,073	8,357	9,425	8,615
	HC3	5,071	4,297	5,120	4,611	5,866	4,502	6,281	5,389	7,823	6,757	9,769	8,143	9,220	8,461
	HC4	4,366	4,365	4,935	4,593	5,392	4,230	5,957	5,255	7,358	6,757	9,491	8,073	9,075	8,459
	HC5	4,366	4,365	4,935	4,593	5,392	4,230	5,957	5,255	7,246	6,757	9,476	8,073	9,061	8,459
b**	**TRUE**	**6,295**	**6,505**	**8,223**	**8,518**	**10,372**	**11,376**	**10,877**	**12,026**	**13,327**	**13,456**	**14,902**	**16,769**	**16,026**	**16,280**
beta 0	**HC0**	6,866	7,135	8,663	9,065	11,029	11,979	11,285	12,423	13,595	13,849	15,366	17,048	16,283	16,569
	HC1	6,513	6,749	8,369	8,747	10,750	11,660	11,057	12,156	13,366	13,604	15,173	16,828	16,119	16,399
	HC2	6,283	6,544	8,196	8,541	10,440	11,456	10,871	12,026	13,283	13,481	14,977	16,738	16,028	16,300
	HC3	5,676	5,984	7,744	8,036	9,853	10,935	10,460	11,634	12,974	13,116	14,578	16,429	15,776	16,033
	HC4	4,806	6,077	7,752	8,049	9,185	10,705	10,159	11,652	13,048	13,103	14,181	16,441	15,789	16,042
	HC5	4,806	6,077	7,752	8,049	9,185	10,705	10,159	11,652	13,048	13,103	14,060	16,441	15,789	16,042
	TRUE	**3,943**	**4,946**	**5,555**	**5,711**	**5,622**	**6,205**	**6,707**	**7,319**	**7,820**	**8,224**	**8,665**	**9,215**	**10,607**	**10,925**
beta 1	**HC0**	5,456	5,689	6,309	6,292	6,346	6,881	7,299	7,779	8,348	8,682	9,298	9,561	10,989	11,247
	HC1	5,176	5,382	6,095	6,071	6,185	6,697	7,152	7,611	8,208	8,528	9,181	9,438	10,879	11,132
	HC2	4,501	5,105	5,762	5,831	5,827	6,404	6,874	7,423	7,955	8,337	8,895	9,302	10,716	11,005
	HC3	3,654	4,573	5,245	5,400	5,340	5,951	6,467	7,080	7,574	8,003	8,494	9,048	10,447	10,768
	HC4	2,366	4,578	4,702	5,309	4,658	5,556	5,976	6,954	7,119	7,833	7,901	8,940	10,197	10,697
	HC5	2,366	4,578	4,702	5,309	4,658	5,556	5,976	6,954	7,119	7,833	7,759	8,940	10,197	10,697
c**	**TRUE**	**4,635**	**5,280**	**6,848**	**7,576**	**6,718**	**7,140**	**6,474**	**7,076**	**7,582**	**7,985**	**8,683**	**10,814**	**10,731**	**11,124**
beta 0	**HC0**	5,160	5,934	7,355	8,030	7,082	7,521	6,802	7,347	7,816	8,246	8,962	11,069	10,927	11,335
	HC1	4,896	5,613	7,105	7,748	6,903	7,326	6,664	7,195	7,685	8,102	8,849	10,920	10,817	11,219
	HC2	4,649	5,351	6,887	7,581	6,739	7,158	6,507	7,082	7,579	7,994	8,685	10,796	10,727	11,135
	HC3	4,122	4,796	6,423	7,151	6,405	6,807	6,212	6,824	7,346	7,747	8,410	10,528	10,527	10,936
	HC4	3,340	4,600	5,984	7,234	6,282	6,730	5,917	6,801	7,276	7,702	8,098	10,456	10,450	10,910
	HC5	3,340	4,600	5,984	7,234	6,282	6,730	5,917	6,801	7,276	7,702	8,098	10,456	10,450	10,910
	TRUE	**2,168**	**2,639**	**3,128**	**3,562**	**3,808**	**3,835**	**3,983**	**4,504**	**4,599**	**4,745**	**4,404**	**5,018**	**5,807**	**6,329**
beta 1	**HC0**	3,102	3,204	3,688	3,905	4,140	4,150	4,415	4,777	4,906	5,041	4,746	5,263	6,066	6,539
	HC1	2,943	3,030	3,563	3,768	4,035	4,043	4,326	4,678	4,824	4,953	4,686	5,192	6,005	6,472
	HC2	2,512	2,804	3,326	3,643	3,894	3,914	4,130	4,571	4,692	4,833	4,516	5,083	5,889	6,390
	HC3	1,995	2,444	2,983	3,397	3,660	3,689	3,854	4,372	4,484	4,632	4,294	4,909	5,715	6,243
	HC4	1,232	2,268	2,471	3,394	3,522	3,608	3,446	4,317	4,280	4,487	3,942	4,783	5,501	6,170
	HC5	1,232	2,268	2,471	3,394	3,522	3,608	3,446	4,317	4,280	4,487	3,942	4,783	5,501	6,170
d**	**TRUE**	**4,080**	**3,654**	**5,463**	**5,352**	**6,118**	**6,124**	**7,021**	**6,759**	**7,705**	**7,477**	**8,860**	**8,738**	**9,975**	**9,879**
beta 0	**HC0**	4,433	3,941	5,652	5,543	6,277	6,298	7,187	6,927	7,840	7,611	8,982	8,854	10,075	9,980
	HC1	4,206	3,727	5,460	5,348	6,118	6,134	7,042	6,781	7,708	7,477	8,869	8,740	9,973	9,878
	HC2	4,109	3,667	5,461	5,353	6,127	6,129	7,031	6,769	7,705	7,477	8,861	8,738	9,974	9,878
	HC3	3,769	3,401	5,271	5,166	5,979	5,962	6,871	6,613	7,569	7,344	8,739	8,622	9,872	9,776
	HC4	3,426	3,476	5,353	5,305	6,085	6,053	6,870	6,688	7,592	7,416	8,759	8,686	9,917	9,834
	HC5	3,426	3,476	5,353	5,305	6,084	6,053	6,870	6,688	7,592	7,416	8,743	8,686	9,917	9,834
	TRUE	**3,234**	**2,740**	**5,422**	**5,011**	**5,984**	**4,869**	**6,623**	**5,820**	**7,671**	**6,666**	**7,549**	**6,840**	**9,640**	**8,834**
beta 1	**HC0**	3,684	3,012	5,848	5,290	6,315	5,106	7,048	6,051	8,015	6,893	7,838	6,999	9,844	9,001
	HC1	3,495	2,849	5,649	5,104	6,155	4,973	6,905	5,924	7,881	6,771	7,739	6,908	9,745	8,909
	HC2	3,273	2,746	5,458	5,007	5,924	4,869	6,673	5,820	7,699	6,667	7,579	6,838	9,637	8,834
	HC3	2,873	2,499	5,079	4,738	5,520	4,639	6,298	5,593	7,387	6,447	7,317	6,679	9,432	8,668
	HC4	2,405	2,541	4,814	4,833	4,941	4,595	5,787	5,517	7,047	6,418	6,999	6,669	9,283	8,644
	HC5	2,405	2,541	4,814	4,833	4,885	4,595	5,787	5,517	7,047	6,418	6,915	6,669	9,283	8,644

Table 5.8: Quasi-t Results for the Case 3*

Case 3*		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**	TRUE	**4,047**	**3,169**	**4,876**	**2,928**	**6,014**	**4,003**	**7,024**	**6,922**	**7,533**	**6,984**	**8,857**	**8,261**	**9,927**	**9,703**
beta 0	HC0	4,342	3,401	5,154	3,096	6,198	4,159	7,176	7,077	7,671	7,131	8,973	8,385	10,026	9,803
	HC1	4,119	3,206	4,979	2,969	6,041	4,031	7,031	6,931	7,542	7,003	8,860	8,271	9,925	9,702
	HC2	4,047	3,169	4,879	2,924	6,009	4,001	7,028	6,927	7,532	6,984	8,857	8,261	9,923	9,698
	HC3	3,753	2,949	4,603	2,758	5,822	3,846	6,883	6,779	7,396	6,838	8,742	8,139	9,821	9,594
	HC4	3,744	3,091	4,453	2,816	5,895	3,924	7,004	6,899	7,498	6,935	8,833	8,224	9,906	9,678
	HC5	3,744	3,091	4,453	2,816	5,895	3,924	7,004	6,899	7,498	6,935	8,833	8,224	9,906	9,678
	TRUE	**13,558**	**10,032**	**2,628**	**1,496**	**10,540**	**6,417**	**14,702**	**14,336**	**2,900**	**2,603**	**5,673**	**5,063**	**3,955**	**3,801**
beta 1	HC0	14,830	10,813	2,829	1,588	10,967	6,684	15,105	14,739	2,964	2,665	5,766	5,155	4,005	3,851
	HC1	14,069	10,195	2,733	1,523	10,689	6,479	14,800	14,436	2,914	2,617	5,694	5,085	3,965	3,811
	HC2	13,563	10,038	2,632	1,495	10,534	6,418	14,718	14,352	2,900	2,603	5,671	5,061	3,954	3,800
	HC3	12,351	9,311	2,441	1,407	10,114	6,160	14,340	13,973	2,837	2,541	5,578	4,969	3,903	3,749
	HC4	12,185	9,784	2,320	1,441	10,216	6,300	14,590	14,217	2,876	2,579	5,635	5,019	3,936	3,781
	HC5	12,185	9,784	2,320	1,441	10,216	6,300	14,590	14,217	2,876	2,579	5,635	5,019	3,936	3,781
b**	TRUE	**6,426**	**7,416**	**8,179**	**8,611**	**6,798**	**8,355**	**13,417**	**12,748**	**23,690**	**22,674**	**20,748**	**20,065**	**27,295**	**26,947**
beta 0	HC0	7,325	8,269	8,672	9,211	7,044	8,631	13,785	13,179	24,073	23,074	21,078	20,414	27,577	27,235
	HC1	6,949	7,796	8,378	8,849	6,866	8,381	13,507	12,876	23,669	22,651	20,813	20,147	27,300	26,938
	HC2	6,604	7,526	8,212	8,673	6,801	8,299	13,436	12,786	23,669	22,638	20,759	20,078	27,294	26,927
	HC3	5,928	6,832	7,770	8,159	6,563	7,974	13,093	12,402	23,270	22,207	20,444	19,748	27,012	26,620
	HC4	5,729	6,983	7,855	8,317	6,665	8,088	13,294	12,620	23,612	22,536	20,659	19,950	27,250	26,856
	HC5	5,729	6,983	7,855	8,317	6,665	8,088	13,294	12,620	23,612	22,536	20,659	19,950	27,250	26,856
	TRUE	**5,292**	**5,373**	**6,493**	**6,393**	**8,399**	**7,356**	**8,724**	**8,327**	**4,106**	**3,619**	**9,436**	**9,107**	**9,459**	**8,449**
beta 1	HC0	6,108	5,935	7,019	6,913	8,822	7,725	9,042	8,649	4,194	3,704	9,621	9,299	9,585	8,567
	HC1	5,794	5,596	6,781	6,641	8,599	7,501	8,860	8,450	4,123	3,636	9,500	9,177	9,489	8,474
	HC2	5,500	5,465	6,590	6,495	8,478	7,411	8,776	8,375	4,106	3,617	9,457	9,128	9,469	8,456
	HC3	4,933	5,025	6,185	6,099	8,145	7,108	8,517	8,109	4,019	3,533	9,296	8,961	9,355	8,346
	HC4	4,780	5,223	6,223	6,232	8,257	7,252	8,635	8,250	4,079	3,587	9,390	9,048	9,439	8,426
	HC5	4,780	5,223	6,223	6,232	8,257	7,252	8,635	8,250	4,079	3,587	9,390	9,048	9,439	8,426
c**	TRUE	**7,947**	**8,654**	**13,276**	**13,283**	**5,729**	**5,828**	**22,942**	**22,924**	**6,295**	**6,350**	**13,930**	**14,283**	**8,074**	**9,416**
beta 0	HC0	9,523	8,746	13,895	13,970	5,929	6,038	23,544	23,558	6,442	6,518	14,167	14,545	8,199	9,579
	HC1	9,034	8,215	13,424	13,462	5,779	5,881	23,069	23,072	6,333	6,402	13,989	14,352	8,116	9,469
	HC2	8,225	8,157	13,287	13,294	5,736	5,836	22,941	22,933	6,299	6,358	13,930	14,282	8,080	9,414
	HC3	7,052	7,595	12,702	12,647	5,549	5,639	22,352	22,323	6,159	6,202	13,697	14,023	7,963	9,251
	HC4	5,994	7,971	13,076	13,030	5,664	5,756	22,746	22,712	6,245	6,282	13,832	14,157	8,012	9,299
	HC5	5,994	7,971	13,076	13,030	5,664	5,756	22,746	22,712	6,245	6,282	13,832	14,157	8,012	9,299
	TRUE	**3,080**	**2,217**	**4,310**	**4,102**	**5,197**	**5,144**	**5,744**	**5,659**	**6,139**	**5,907**	**7,003**	**6,784**	**7,739**	**7,189**
beta 1	HC0	3,590	2,347	4,572	4,380	5,421	5,369	5,935	5,857	6,319	6,098	7,160	6,944	7,886	7,326
	HC1	3,406	2,204	4,417	4,220	5,284	5,230	5,816	5,736	6,212	5,990	7,070	6,852	7,807	7,243
	HC2	3,195	2,201	4,355	4,150	5,237	5,184	5,773	5,691	6,171	5,943	7,031	6,811	7,767	7,211
	HC3	2,816	2,064	4,147	3,931	5,058	5,004	5,615	5,530	6,026	5,792	6,904	6,681	7,650	7,097
	HC4	2,504	2,187	4,263	4,044	5,164	5,113	5,711	5,624	6,108	5,867	6,970	6,745	7,697	7,155
	HC5	2,504	2,187	4,263	4,044	5,164	5,113	5,711	5,624	6,108	5,867	6,970	6,745	7,697	7,155
d**	TRUE	**4,258**	**3,715**	**4,722**	**3,194**	**6,227**	**6,109**	**7,046**	**6,959**	**7,693**	**7,554**	**8,694**	**8,513**	**9,959**	**9,685**
beta 0	HC0	4,513	3,973	4,946	3,341	6,393	6,279	7,194	7,110	7,824	7,689	8,811	8,634	10,061	9,790
	HC1	4,282	3,746	4,778	3,210	6,231	6,116	7,049	6,963	7,692	7,555	8,700	8,522	9,960	9,688
	HC2	4,260	3,716	4,716	3,193	6,226	6,109	7,048	6,961	7,693	7,553	8,693	8,513	9,959	9,684
	HC3	4,015	3,472	4,487	3,050	6,063	5,942	6,904	6,815	7,563	7,419	8,577	8,393	9,857	9,578
	HC4	4,172	3,648	4,480	3,155	6,195	6,072	7,028	6,939	7,678	7,532	8,671	8,485	9,944	9,664
	HC5	4,172	3,648	4,480	3,155	6,195	6,072	7,028	6,939	7,678	7,532	8,671	8,485	9,944	9,664
	TRUE	**4,183**	**3,519**	**1,679**	**1,114**	**7,726**	**7,498**	**6,794**	**6,626**	**12,699**	**12,251**	**14,159**	**13,748**	**7,549**	**7,174**
beta 1	HC0	4,504	3,784	1,781	1,177	7,991	7,768	6,968	6,799	12,959	12,515	14,390	13,984	7,645	7,270
	HC1	4,272	3,567	1,720	1,131	7,789	7,566	6,827	6,658	12,741	12,298	14,209	13,803	7,568	7,194
	HC2	4,190	3,519	1,678	1,115	7,726	7,500	6,796	6,627	12,702	12,253	14,158	13,748	7,550	7,175
	HC3	3,893	3,271	1,579	1,056	7,470	7,241	6,627	6,458	12,450	11,996	13,930	13,516	7,456	7,080
	HC4	4,032	3,444	1,573	1,089	7,631	7,397	6,747	6,575	12,641	12,180	14,081	13,661	7,523	7,144
	HC5	4,032	3,444	1,573	1,089	7,631	7,397	6,747	6,575	12,641	12,180	14,081	13,661	7,523	7,144

Table 5.9: Quasi-t Results for the Case 4*

Case 4*		20		30		40		50		60		80		100	
		F* **	**S*** **	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**
a **	**TRUE**	**4,179**	**2,399**	**4,065**	**2,825**	**4,860**	**2,824**	**5,078**	**3,628**	**6,331**	**4,091**	**7,377**	**4,005**	**7,272**	**5,184**
beta 0	**HC0**	4,334	2,625	4,234	2,959	5,009	2,964	5,248	3,743	6,557	4,191	7,605	4,094	7,417	5,254
	HC1	4,111	2,455	4,090	2,843	4,882	2,867	5,142	3,655	6,447	4,108	7,509	4,031	7,343	5,194
	HC2	4,178	2,399	4,065	2,827	4,862	2,825	5,080	3,631	6,337	4,092	7,377	4,005	7,274	5,185
	HC3	3,065	2,171	3,869	2,698	4,694	2,684	4,884	3,519	6,016	3,993	7,063	3,917	7,116	5,116
	HC4	0,102	2,051	3,550	2,773	4,461	2,591	4,539	3,536	5,006	4,016	6,141	3,916	6,868	5,154
	HC5	0,000	2,051	3,443	2,773	4,241	2,591	3,912	3,536	0,852	4,016	0,816	3,916	6,050	5,154
	TRUE	**45,170**	**3,004**	**11,024**	**3,723**	**14,442**	**2,726**	**10,411**	**4,062**	**24,851**	**6,468**	**21,941**	**4,297**	**15,175**	**6,089**
beta 1	**HC0**	151,603	3,477	12,596	3,979	16,074	2,929	11,469	4,248	29,882	6,724	25,888	4,432	16,103	6,203
	HC1	143,823	3,253	12,169	3,823	15,667	2,833	11,238	4,148	29,379	6,590	25,562	4,364	15,941	6,132
	HC2	45,043	3,007	11,017	3,726	14,458	2,726	10,410	4,065	24,959	6,465	21,939	4,299	15,175	6,089
	HC3	6,943	2,547	9,441	3,484	12,823	2,526	9,327	3,885	20,048	6,212	18,130	4,168	14,237	5,976
	HC4	0,154	2,037	6,763	3,522	9,859	2,289	7,315	3,797	11,718	6,096	11,584	4,099	12,462	5,979
	HC5	0,001	2,037	6,298	3,522	8,544	2,289	5,445	3,797	1,427	6,096	1,038	4,099	9,291	5,979
b **	**TRUE**	**3,008**	**4,278**	**2,741**	**2,951**	**3,391**	**4,591**	**2,109**	**5,049**	**1,752**	**5,927**	**3,407**	**6,908**	**1,851**	**7,902**
beta 0	**HC0**	3,755	4,689	3,506	3,398	4,123	4,895	4,245	5,342	3,078	6,157	4,210	7,115	4,413	8,189
	HC1	3,562	4,404	3,388	3,253	4,018	4,749	4,159	5,210	3,026	6,037	4,157	7,012	4,368	8,087
	HC2	3,180	4,282	2,981	3,040	3,598	4,614	2,961	5,084	2,196	5,939	3,670	6,925	2,850	7,954
	HC3	2,632	3,891	2,476	2,690	3,105	4,337	1,964	4,830	1,484	5,723	3,176	6,737	1,777	7,719
	HC4	1,764	3,932	1,639	2,150	2,260	4,156	0,812	4,668	0,624	5,612	2,343	6,646	0,666	7,441
	HC5	1,764	3,932	1,528	2,150	2,007	4,156	0,042	4,668	0,012	5,612	1,087	6,646	0,000	7,441
	TRUE	**2,672**	**2,673**	**2,807**	**2,205**	**3,140**	**3,205**	**2,554**	**3,673**	**2,594**	**4,621**	**3,848**	**5,130**	**2,648**	**5,210**
beta 1	**HC0**	3,583	3,095	3,669	2,623	3,908	3,495	5,280	3,958	4,570	4,869	4,756	5,338	6,323	5,449
	HC1	3,399	2,907	3,545	2,512	3,809	3,391	5,174	3,860	4,494	4,775	4,696	5,261	6,260	5,382
	HC2	2,903	2,736	3,065	2,303	3,360	3,254	3,616	3,727	3,229	4,653	4,142	5,162	4,077	5,263
	HC3	2,323	2,410	2,517	2,003	2,866	3,023	2,378	3,506	2,184	4,444	3,585	4,991	2,541	5,081
	HC4	1,502	2,383	1,658	1,548	2,056	2,851	0,979	3,344	0,925	4,323	2,651	4,891	0,953	4,860
	HC5	1,502	2,383	1,546	1,548	1,822	2,851	0,051	3,344	0,018	4,323	1,239	4,891	0,000	4,860
c **	**TRUE**	**1,549**	**2,957**	**1,487**	**3,190**	**0,733**	**2,594**	**1,618**	**2,840**	**2,084**	**3,365**	**2,348**	**4,141**	**1,384**	**5,035**
beta 0	**HC0**	2,426	3,080	2,085	3,716	2,062	2,869	2,195	3,133	2,410	3,498	2,641	4,356	2,357	5,217
	HC1	2,301	2,904	2,014	3,564	2,010	2,778	2,150	3,053	2,369	3,433	2,608	4,290	2,333	5,156
	HC2	1,842	2,854	1,637	3,303	1,233	2,634	1,834	2,895	2,185	3,370	2,429	4,177	1,764	5,064
	HC3	1,337	2,640	1,257	2,918	0,660	2,410	1,513	2,668	1,975	3,244	2,230	4,003	1,291	4,913
	HC4	0,654	2,752	0,706	2,459	0,173	2,182	1,004	2,336	1,611	3,213	1,875	3,803	0,668	4,755
	HC5	0,654	2,752	0,515	2,459	0,002	2,182	0,580	2,336	1,547	3,213	1,611	3,803	0,007	4,755
	TRUE	**1,280**	**1,817**	**1,512**	**1,933**	**1,217**	**2,114**	**1,983**	**2,125**	**2,438**	**2,912**	**2,631**	**2,939**	**1,746**	**3,458**
beta 1	**HC0**	2,111	1,963	2,183	2,288	3,762	2,379	2,624	2,371	2,811	3,061	2,952	3,111	2,842	3,595
	HC1	2,003	1,851	2,109	2,195	3,666	2,303	2,571	2,311	2,763	3,004	2,915	3,063	2,813	3,553
	HC2	1,557	1,814	1,680	2,026	2,100	2,172	2,222	2,180	2,557	2,937	2,721	2,977	2,180	3,487
	HC3	1,111	1,675	1,275	1,785	1,098	1,978	1,855	2,001	2,319	2,817	2,503	2,847	1,624	3,382
	HC4	0,537	1,754	0,712	1,512	0,285	1,788	1,259	1,750	1,909	2,783	2,118	2,705	0,856	3,278
	HC5	0,537	1,754	0,522	1,512	0,004	1,788	0,739	1,750	1,835	2,783	1,829	2,705	0,009	3,278
d **	**TRUE**	**2,820**	**2,539**	**3,087**	**2,641**	**4,013**	**3,150**	**4,313**	**2,912**	**4,580**	**3,502**	**5,753**	**4,249**	**3,433**	**5,033**
beta 0	**HC0**	3,063	2,695	3,752	2,766	4,242	3,271	6,073	3,003	5,342	3,600	6,047	4,352	5,819	5,112
	HC1	2,906	2,541	3,625	2,662	4,135	3,173	5,950	2,932	5,253	3,527	5,971	4,283	5,761	5,050
	HC2	2,838	2,540	3,317	2,640	4,035	3,151	5,379	2,911	4,878	3,500	5,815	4,251	4,541	5,034
	HC3	2,592	2,391	2,690	2,517	3,806	3,031	3,688	2,822	4,252	3,401	5,556	4,147	3,275	4,955
	HC4	2,216	2,498	1,345	2,578	3,395	3,032	0,821	2,864	2,772	3,404	5,040	4,084	1,488	4,959
	HC5	2,216	2,498	0,427	2,578	3,337	3,032	0,000	2,864	0,329	3,404	3,927	4,084	0,001	4,959
	TRUE	**6,154**	**4,162**	**5,827**	**3,828**	**6,697**	**3,345**	**5,676**	**2,183**	**7,497**	**3,548**	**9,930**	**3,832**	**6,192**	**6,326**
beta 1	**HC0**	7,674	4,602	9,249	4,083	7,706	3,557	19,253	2,265	10,480	3,702	11,179	3,991	12,416	6,489
	HC1	7,280	4,339	8,935	3,928	7,511	3,451	18,864	2,211	10,304	3,627	11,038	3,928	12,292	6,410
	HC2	6,352	4,184	6,824	3,839	6,821	3,350	9,909	2,183	8,511	3,551	10,198	3,838	8,675	6,332
	HC3	5,184	3,790	4,646	3,604	5,984	3,148	4,396	2,103	6,599	3,403	9,243	3,685	5,842	6,176
	HC4	3,514	3,829	1,902	3,621	4,568	3,020	0,796	2,124	3,620	3,334	7,526	3,497	2,520	6,074
	HC5	3,514	3,829	0,578	3,621	4,427	3,020	0,000	2,124	0,392	3,334	5,133	3,497	0,002	6,074

Table 5.10: Quasi-t Results for the Case 5*

Case 5*		20		30		40		50		60		80		100	
		F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**	**TRUE**	**4,368**	**3,161**	**5,204**	**4,772**	**6,298**	**5,456**	**6,945**	**6,855**	**7,627**	**7,272**	**8,938**	**8,415**	**9,890**	**9,038**
beta 0	HC0	4,492	3,414	5,315	4,971	6,444	5,674	7,029	7,004	7,698	7,408	9,032	8,532	9,982	9,159
	HC1	4,261	3,178	5,134	4,760	6,281	5,488	6,887	6,853	7,568	7,267	8,918	8,413	9,882	9,048
	HC2	4,369	3,164	5,208	4,776	6,295	5,456	6,942	6,853	7,625	7,269	8,936	8,413	9,894	9,044
	HC3	4,237	2,921	5,103	4,584	6,127	5,231	6,851	6,704	7,546	7,129	8,807	8,293	9,801	8,928
	HC4	1,125	3,050	5,244	4,730	5,951	5,120	6,781	6,803	7,395	7,181	8,340	8,353	9,781	8,952
	HC5	0,005	3,050	5,244	4,730	5,583	5,120	0,021	6,803	0,002	7,181	0,014	8,353	9,187	8,952
	TRUE	**67,877**	**3,366**	**82,286**	**5,806**	**92,416**	**11,478**	**31,113**	**9,899**	**89,081**	**9,020**	**45,374**	**12,064**	**68,965**	**13,866**
beta 1	HC0	230,701	3,857	97,828	6,321	110,375	12,443	49,242	10,250	148,274	9,412	60,675	12,376	79,938	14,262
	HC1	218,862	3,591	94,511	6,052	107,580	12,034	48,247	10,029	145,782	9,233	59,912	12,202	79,134	14,089
	HC2	67,777	3,366	82,199	5,813	92,510	11,491	30,977	9,891	88,903	9,020	45,432	12,055	68,876	13,875
	HC3	10,193	2,906	67,923	5,321	76,800	10,573	16,158	9,542	45,097	8,636	30,584	11,739	57,510	13,493
	HC4	0,215	2,727	45,165	5,022	51,828	9,591	3,712	9,591	9,942	8,350	11,661	11,646	36,866	13,155
	HC5	0,001	2,727	43,721	5,022	37,549	9,591	0,003	9,591	0,001	8,350	0,007	11,646	1,184	13,155
b**	**TRUE**	**0,706**	**4,404**	**1,773**	**5,818**	**3,202**	**4,541**	**3,801**	**9,834**	**3,334**	**7,568**	**2,969**	**12,859**	**0,836**	**12,078**
beta 0	HC0	1,095	4,945	1,813	6,639	3,471	4,869	3,224	10,125	3,103	7,851	3,938	13,188	0,876	12,391
	HC1	1,038	4,645	1,752	6,357	3,383	4,728	3,159	9,881	3,051	7,704	3,889	12,990	0,867	12,241
	HC2	0,984	4,444	1,726	5,957	3,234	4,563	3,117	9,800	2,910	7,567	3,388	12,875	0,771	12,094
	HC3	0,534	3,970	1,374	5,313	3,002	4,268	3,012	9,482	2,569	7,289	2,679	12,565	0,604	11,798
	HC4	0,030	3,925	0,194	4,699	2,653	4,069	2,996	9,615	1,061	7,191	1,320	12,559	0,174	11,587
	HC5	0,000	3,925	0,000	4,699	2,653	4,069	0,329	9,615	0,000	7,191	0,004	12,559	0,000	11,587
	TRUE	**1,953**	**4,430**	**1,984**	**4,478**	**5,505**	**5,769**	**2,565**	**6,911**	**2,238**	**7,441**	**2,984**	**8,565**	**2,182**	**8,861**
beta 1	HC0	21,650	5,426	17,014	5,333	6,274	6,411	6,399	7,511	8,332	7,905	6,191	9,055	6,789	9,330
	HC1	20,539	5,097	16,437	5,106	6,115	6,225	6,270	7,330	8,192	7,757	6,113	8,919	6,720	9,217
	HC2	6,782	4,682	5,801	4,684	5,716	5,916	3,978	7,069	4,297	7,543	4,262	8,706	3,611	8,993
	HC3	1,511	4,029	1,726	4,105	5,198	5,454	2,309	6,645	1,966	7,195	2,806	8,364	1,616	8,663
	HC4	0,071	3,837	0,148	3,585	4,388	5,085	0,726	6,255	0,381	6,998	1,147	8,014	0,287	8,260
	HC5	0,001	3,837	0,000	3,585	4,388	5,085	0,006	6,255	0,000	6,998	0,003	8,014	0,000	8,260
c**	**TRUE**	**1,483**	**3,564**	**0,804**	**3,994**	**0,891**	**4,241**	**0,773**	**4,105**	**0,616**	**4,692**	**1,863**	**4,779**	**2,000**	**6,075**
beta 0	HC0	1,847	3,965	0,596	4,225	1,689	4,507	0,841	4,402	0,852	4,904	1,819	4,923	1,842	6,242
	HC1	1,753	3,738	0,576	4,053	1,647	4,372	0,824	4,287	0,838	4,807	1,796	4,851	1,823	6,165
	HC2	1,504	3,581	0,576	3,941	1,229	4,239	0,750	4,134	0,648	4,693	1,722	4,782	1,751	6,073
	HC3	1,165	3,211	0,555	3,662	0,796	3,981	0,657	3,871	0,474	4,487	1,601	4,642	1,656	5,905
	HC4	0,617	3,091	0,301	3,546	0,282	3,877	0,478	3,540	0,233	4,330	1,244	4,572	1,444	5,762
	HC5	0,538	3,091	0,000	3,546	0,014	3,877	0,158	3,540	0,009	4,330	0,015	4,572	0,002	5,762
	TRUE	**1,520**	**2,521**	**1,043**	**2,703**	**1,542**	**3,328**	**1,826**	**3,325**	**1,511**	**3,757**	**1,721**	**4,766**	**1,618**	**4,721**
beta 1	HC0	2,731	3,188	14,573	3,235	3,504	3,692	2,737	3,767	2,326	4,076	3,213	5,037	3,247	5,004
	HC1	2,590	3,006	14,079	3,102	3,415	3,582	2,681	3,669	2,287	3,995	3,172	4,963	3,214	4,942
	HC2	1,872	2,736	4,069	2,850	2,287	3,422	2,126	3,461	1,659	3,839	2,315	4,842	2,254	4,804
	HC3	1,238	2,331	0,843	2,503	1,392	3,169	1,614	3,174	1,162	3,614	1,605	4,653	1,499	4,611
	HC4	0,507	2,033	0,035	2,151	0,474	3,025	0,885	2,785	0,546	3,378	0,727	4,466	0,625	4,335
	HC5	0,426	2,033	0,000	2,151	0,024	3,025	0,194	2,785	0,021	3,378	0,006	4,466	0,000	4,335
d**	**TRUE**	**4,257**	**4,202**	**4,405**	**5,124**	**4,298**	**5,620**	**6,628**	**6,466**	**6,967**	**6,494**	**6,630**	**8,035**	**8,492**	**9,082**
beta 0	HC0	4,445	4,435	4,513	5,295	4,543	5,755	6,729	6,618	7,297	6,699	6,533	8,162	8,720	9,197
	HC1	4,216	4,181	4,360	5,095	4,428	5,578	6,593	6,459	7,174	6,561	6,450	8,035	8,633	9,088
	HC2	4,268	4,187	4,391	5,109	4,283	5,595	6,542	6,461	7,049	6,517	5,801	8,035	8,544	9,083
	HC3	4,092	3,941	3,503	4,925	4,014	5,435	6,172	6,302	6,756	6,331	4,908	7,907	8,360	8,969
	HC4	4,192	4,079	0,142	5,059	3,579	5,544	3,969	6,355	6,172	6,215	3,019	7,928	8,085	9,015
	HC5	4,192	4,079	0,000	5,059	3,473	5,544	0,015	6,355	4,708	6,215	0,061	7,928	7,605	9,015
	TRUE	**13,548**	**5,054**	**222,183**	**8,706**	**19,841**	**7,744**	**37,549**	**5,899**	**14,969**	**9,357**	**11,965**	**8,564**	**29,541**	**11,616**
beta 1	HC0	14,740	5,664	188,725	9,430	22,263	8,350	33,456	6,270	19,217	9,923	13,433	8,927	31,212	11,894
	HC1	13,984	5,340	182,325	9,074	21,699	8,093	32,780	6,119	18,894	9,718	13,264	8,789	30,898	11,754
	HC2	12,477	5,069	48,233	8,762	19,817	7,778	23,721	5,923	16,521	9,362	10,612	8,572	29,195	11,616
	HC3	10,049	4,523	7,618	8,127	17,587	7,230	14,250	5,586	14,057	8,817	8,181	8,223	27,271	11,342
	HC4	5,795	4,505	0,182	8,011	13,907	6,871	4,102	5,309	9,983	8,041	4,531	7,796	23,755	11,221
	HC5	5,795	4,505	0,000	8,011	13,186	6,871	0,012	5,309	5,789	8,041	0,091	7,796	19,027	11,221

Table 5.11: Symmetric Loss for the Case 1*

Case 1*	**Symmetric Loss**													
	20		**30**		**40**		**50**		**60**		**80**		**100**	
	F***	**S*****	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**
a**														
HC0	0.094	0.083	0.016	0.016	0.011	0.01	0.006	0.006	0.004	0.004	0.002	0.002	0.001	0.001
HC1	0.036	0.021	0.002	0.001	0.002	0.001	7E-04	5E-04	6E-04	5E-04	1E-04	1E-04	1E-04	1E-04
HC2	1E-05	1E-05	2E-06	1E-06	2E-06	1E-06	1E-06	1E-07	4E-07	5E-07	1E-06	1E-06	4E-07	9E-08
HC3	0.111	0.104	0.017	0.017	0.011	0.01	0.006	0.006	0.004	0.004	0.002	0.002	0.001	0.001
HC4	0.565	0.663	0.006	0.011	0.007	0.015	0.003	0.004	0.003	0.004	4E-04	3E-04	8E-04	0.001
HC5	0.663	0.565	0.011	0.006	0.015	0.007	0.004	0.003	0.004	0.003	3E-04	4E-04	0.001	8E-04
b**														
HC0	0.055	0.036	0.027	0.019	0.01	0.009	0.007	0.007	0.003	0.003	0.002	0.002	1E-03	0.001
HC1	0.014	0.003	0.009	0.003	0.002	0.001	0.001	7E-04	4E-04	3E-04	4E-04	4E-04	8E-05	8E-05
HC2	0.002	8E-04	0.001	3E-04	3E-04	2E-04	2E-04	1E-04	1E-04	1E-04	8E-05	8E-05	4E-05	4E-05
HC3	0.042	0.04	0.018	0.018	0.009	0.01	0.006	0.006	0.003	0.004	0.002	0.002	0.001	0.001
HC4	0.008	0.041	0.003	0.02	0.001	0.003	7E-04	0.003	5E-04	5E-04	4E-04	4E-04	2E-04	2E-04
HC5	0.041	0.008	0.02	0.003	0.003	0.001	0.003	7E-04	5E-04	5E-04	4E-04	4E-04	2E-04	2E-04
c**														
HC0	0.116	0.052	0.025	0.023	0.012	0.012	0.009	0.008	0.006	0.006	0.002	0.002	0.002	0.002
HC1	0.071	0.026	0.017	0.017	0.008	0.009	0.006	0.006	0.004	0.004	0.002	0.002	0.001	0.001
HC2	0.018	0.01	0.008	0.009	0.004	0.004	0.003	0.003	0.002	0.002	8E-04	9E-04	5E-04	5E-04
HC3	0.07	0.063	0.031	0.032	0.015	0.017	0.01	0.011	0.007	0.007	0.003	0.004	0.002	0.002
HC4	0.023	0.118	0.021	0.023	0.007	0.01	0.007	0.008	0.006	0.006	0.001	0.002	0.001	0.001
HC5	0.118	0.023	0.023	0.021	0.01	0.007	0.008	0.007	0.006	0.006	0.002	0.001	0.001	0.001
d**														
HC0	0.061	0.045	0.019	0.016	0.009	0.009	0.008	0.008	0.004	0.004	0.002	0.002	0.001	0.001
HC1	0.016	0.002	0.003	0.001	9E-04	5E-04	0.002	9E-04	3E-04	3E-04	2E-04	2E-04	5E-05	6E-05
HC2	2E-05	7E-06	3E-06	4E-06	9E-06	7E-06	3E-06	3E-06	5E-07	2E-06	3E-06	2E-06	9E-07	1E-07
HC3	0.063	0.045	0.02	0.018	0.01	0.009	0.009	0.009	0.004	0.004	0.002	0.002	0.001	0.001
HC4	0.009	0.134	0.008	0.03	0.002	0.008	0.005	0.026	9E-04	0.002	7E-04	0.001	2E-04	2E-04
HC5	0.134	0.009	0.03	0.008	0.008	0.002	0.026	0.005	0.002	9E-04	0.001	7E-04	2E-04	2E-04

Table 5.12: Symmetric Loss for the Case 2*

Case 2*	Symmetric Loss													
	20		30		40		50		60		80		100	
	F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**														
HC0	0.061	0.0425	0.0226	0.0212	0.0176	0.0149	0.0101	0.0083	0.0086	0.0049	0.0043	0.0035	0.0021	0.0016
HC1	0.0147	0.004	0.0043	0.0031	0.0052	0.0029	0.0026	0.0014	0.0028	0.0007	0.0013	0.0007	0.0005	0.0002
HC2	1E-05	1E-05	7E-07	2E-06	1E-06	1E-06	2E-06	2E-06	6E-07	2E-07	4E-07	1E-06	3E-06	1E-06
HC3	0.0775	0.0482	0.0248	0.0216	0.0199	0.017	0.0113	0.0089	0.0099	0.0049	0.0045	0.0033	0.0026	0.0018
HC4	0.0264	0.3686	0.0195	0.0507	0.0669	0.0971	0.0186	0.0435	0.0039	0.0506	0.0048	0.0158	0.0015	0.0067
HC5	0.3686	0.0264	0.0507	0.0195	0.0971	0.0669	0.0435	0.0186	0.0666	0.0039	0.0167	0.0048	0.0072	0.0015
b**														
HC0	0.438	0.0963	0.0909	0.0515	0.0637	0.0518	0.0337	0.0188	0.0196	0.0148	0.0298	0.0068	0.0065	0.0043
HC1	0.3156	0.03	0.0468	0.0172	0.038	0.0251	0.0174	0.0065	0.0101	0.0057	0.0208	0.0025	0.003	0.0015
HC2	0.0844	0.0046	0.0071	0.0018	0.0062	0.004	0.0025	0.0009	0.0013	0.0007	0.0039	0.0004	0.0004	0.0002
HC3	0.0784	0.0531	0.0291	0.027	0.0224	0.015	0.0129	0.0106	0.0075	0.0061	0.0037	0.0035	0.0021	0.0019
HC4	0.0348	1.1606	0.0365	0.1855	0.0672	0.1635	0.0174	0.0798	0.0134	0.0432	0.0073	0.0615	0.0027	0.0099
HC5	1.1606	0.0348	0.1855	0.0365	0.1635	0.0672	0.0798	0.0174	0.0432	0.0134	0.0875	0.0073	0.0099	0.0027
c**														
HC0	0.5931	0.2076	0.1303	0.0476	0.0385	0.0356	0.0508	0.0192	0.0215	0.0189	0.0263	0.0114	6.5389	0.0057
HC1	0.4323	0.0946	0.0788	0.0148	0.0167	0.0139	0.032	0.0069	0.0107	0.0083	0.0168	0.0051	6.4718	0.0023
HC2	0.1037	0.0163	0.0175	0.0021	0.0022	0.0017	0.0061	0.0009	0.0019	0.0014	0.0026	0.0007	6.3898	0.0004
HC3	0.087	0.0604	0.0258	0.0224	0.0151	0.0151	0.0108	0.0087	0.0065	0.006	0.0066	0.0048	6.2432	0.0019
HC4	0.1729	1.5747	0.018	0.2911	0.0293	0.0428	0.0136	0.1106	0.0188	0.031	0.0138	0.0683	6.1699	0.0148
HC5	1.5747	0.1729	0.2911	0.018	0.0428	0.0293	0.1106	0.0136	0.031	0.0188	0.0683	0.0138	6.1699	0.0043
d**														
HC0	0.0831	0.0488	0.0274	0.017	0.0143	0.0117	0.0178	0.008	0.0089	0.0057	0.0063	0.0027	0.0021	0.0018
HC1	0.0281	0.0065	0.0069	0.0015	0.0033	0.0018	0.0078	0.0014	0.0031	0.001	0.0026	0.0004	0.0005	0.0003
HC2	0.0008	8E-05	0.0004	5E-05	0.0008	6E-05	0.0003	6E-05	0.0001	4E-05	1E-04	1E-05	2E-05	5E-06
HC3	0.0691	0.047	0.0224	0.019	0.033	0.0126	0.0117	0.0082	0.0069	0.006	0.0046	0.003	0.0024	0.0019
HC4	0.0252	0.4097	0.0081	0.0594	0.015	0.1894	0.0123	0.0767	0.0063	0.0301	0.0027	0.0234	0.002	0.0059
HC5	0.4097	0.0252	0.0594	0.0081	0.213	0.015	0.0767	0.0123	0.0301	0.0063	0.0316	0.0027	0.0059	0.002

Table 5.13: Symmetric Loss for the Case 3*

Case 3*	Symmetric Loss													
	20		30		40		50		60		80		100	
	F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**														
HC0	0.045	0.037	0.029	0.021	0.009	0.011	0.005	0.005	0.003	0.004	0.002	0.002	0.001	0.001
HC1	0.007	0.001	0.008	0.001	0.001	4E-04	2E-04	2E-04	1E-04	2E-04	6E-05	9E-05	3E-05	3E-05
HC2	9E-06	9E-06	1E-05	9E-06	7E-06	3E-06	6E-06	7E-06	4E-07	5E-07	6E-07	8E-07	2E-06	2E-06
HC3	0.049	0.036	0.029	0.023	0.01	0.011	0.004	0.004	0.003	0.004	0.002	0.002	0.001	0.001
HC4	0.003	0.065	0.007	0.085	0.002	0.006	4E-04	3E-04	6E-04	4E-04	4E-04	2E-04	1E-04	1E-04
HC5	0.065	0.003	0.085	0.007	0.006	0.002	3E-04	4E-04	4E-04	6E-04	2E-04	4E-04	1E-04	1E-04
b**														
HC0	0.113	0.071	0.03	0.036	0.016	0.015	0.007	0.008	0.003	0.003	0.002	0.002	0.001	0.001
HC1	0.046	0.013	0.008	0.007	0.003	0.002	0.001	1E-03	8E-05	9E-05	2E-04	3E-04	4E-05	3E-05
HC2	0.007	0.002	0.001	0.001	4E-04	5E-04	2E-04	1E-04	3E-05	4E-05	3E-05	3E-05	2E-05	3E-05
HC3	0.039	0.041	0.018	0.019	0.009	0.015	0.005	0.005	0.003	0.004	0.002	0.002	0.001	0.001
HC4	0.015	0.067	0.006	0.01	0.009	0.005	6E-04	6E-04	9E-04	4E-04	2E-04	1E-04	3E-04	2E-04
HC5	0.067	0.015	0.01	0.006	0.005	0.009	6E-04	6E-04	4E-04	9E-04	1E-04	2E-04	2E-04	3E-04
c**														
HC0	0.188	0.019	0.022	0.027	0.012	0.013	0.007	0.008	0.005	0.007	0.003	0.004	0.002	0.003
HC1	0.101	0.03	0.003	0.004	0.002	0.002	8E-04	1E-03	8E-04	0.001	5E-04	6E-04	5E-04	4E-04
HC2	0.009	0.031	5E-04	6E-04	3E-04	3E-04	1E-04	1E-04	1E-04	2E-04	7E-05	7E-05	6E-05	4E-05
HC3	0.07	0.11	0.014	0.017	0.007	0.007	0.005	0.005	0.003	0.004	0.002	0.002	0.001	0.002
HC4	0.053	0.388	0.002	0.001	0.001	1E-03	6E-04	5E-04	8E-04	4E-04	5E-04	3E-04	9E-04	4E-04
HC5	0.388	0.053	0.001	0.002	1E-03	0.001	5E-04	6E-04	4E-04	8E-04	3E-04	5E-04	4E-04	9E-04
d**														
HC0	0.033	0.035	0.019	0.018	0.007	0.008	0.004	0.004	0.003	0.003	0.002	0.002	0.001	0.001
HC1	0.002	0.001	0.003	1E-03	3E-04	4E-04	1E-04	1E-04	7E-05	1E-04	8E-05	1E-04	3E-05	4E-05
HC2	4E-05	8E-06	2E-05	9E-06	1E-05	1E-05	7E-06	8E-06	8E-06	8E-06	2E-06	2E-06	8E-07	8E-07
HC3	0.032	0.036	0.021	0.018	0.007	0.008	0.004	0.004	0.003	0.003	0.002	0.002	0.001	0.001
HC4	0.003	0.007	0.002	0.024	8E-04	7E-04	4E-04	3E-04	5E-04	3E-04	2E-04	1E-04	1E-04	8E-05
HC5	0.007	0.003	0.024	0.002	7E-04	8E-04	3E-04	4E-04	3E-04	5E-04	1E-04	2E-04	8E-05	1E-04

Table 5.14: Symmetric Loss for the Case 4*

Case 4*	Symmetric Loss													
	20		30		40		50		60		80		100	
	F***	S***	F	S	F	S	F	S	F	S	F	S	F	S
a**														
HC0	21.47	0.112	0.09	0.026	0.056	0.027	0.044	0.011	0.152	0.008	0.12	0.005	0.016	0.002
HC1	19.13	0.032	0.052	0.004	0.034	0.008	0.028	0.002	0.126	0.002	0.103	0.001	0.011	3E-04
HC2	4E-05	6E-06	3E-06	3E-06	5E-06	7E-07	5E-07	2E-06	8E-05	2E-06	2E-07	2E-06	2E-07	3E-07
HC3	42.57	0.14	0.115	0.025	0.067	0.03	0.056	0.011	0.2	0.009	0.156	0.005	0.018	0.002
HC4	0.788	91173	0.018	1.174	0.155	0.693	0.024	0.578	0.02	2.849	0.012	1.946	0.002	0.174
HC5	5E+09	0.788	1.568	0.018	1.349	0.155	2.096	0.024	311.6	0.02	457.3	0.012	1.121	0.002
b**														
HC0	0.361	0.09	0.308	0.124	0.201	0.031	2.759	0.023	1.588	0.011	0.2	0.006	4.348	0.008
HC1	0.259	0.035	0.239	0.072	0.16	0.014	2.596	0.011	1.499	0.005	0.18	0.003	4.235	0.004
HC2	0.038	0.005	0.037	0.009	0.022	0.002	0.551	0.001	0.248	5E-04	0.027	3E-04	0.873	5E-04
HC3	0.112	0.064	0.061	0.048	0.043	0.019	0.047	0.012	0.168	0.008	0.027	0.004	0.046	0.004
HC4	0.054	1.527	0.522	1.262	0.056	0.782	0.036	5.071	0.018	6.216	0.009	0.616	0.02	5.965
HC5	1.527	0.054	1.662	0.522	1.342	0.056	2539	0.036	21231	0.018	8.141	0.009	5E+08	0.02
c**														
HC0	1.114	0.028	0.598	0.121	8.503	0.057	0.394	0.049	0.091	0.01	0.06	0.013	1.256	0.006
HC1	0.908	0.024	0.512	0.081	8.016	0.038	0.347	0.033	0.075	0.006	0.05	0.009	1.209	0.004
HC2	0.181	0.02	0.074	0.018	1.445	0.009	0.072	0.005	0.013	0.002	0.008	0.002	0.258	8E-04
HC3	0.138	0.085	0.167	0.054	0.144	0.035	0.035	0.023	0.018	0.009	0.016	0.007	0.048	0.004
HC4	0.035	3.959	0.29	2.831	0.129	16.48	0.164	0.998	0.01	0.276	0.031	0.212	0.014	2.568
HC5	3.959	0.035	6.741	0.29	1E+05	0.129	5.925	0.164	0.373	0.01	0.607	0.031	41473	0.014
d**														
HC0	0.22	0.057	0.947	0.023	0.086	0.02	9.653	0.008	0.469	0.009	0.058	0.008	2.289	0.003
HC1	0.129	0.011	0.805	0.004	0.058	0.005	9.195	8E-04	0.422	0.003	0.046	0.003	2.215	9E-04
HC2	0.004	2E-04	0.103	7E-05	0.001	1E-05	1.378	7E-07	0.065	1E-05	0.003	1E-05	0.48	5E-06
HC3	0.133	0.05	0.216	0.021	0.055	0.02	0.269	0.008	0.067	0.009	0.022	0.008	0.019	0.003
HC4	0.038	1.475	0.016	7.57	0.053	0.645	0.004	48.91	0.02	2.53	0.04	0.32	0.008	4.209
HC5	1.475	0.038	100.7	0.016	0.76	0.053	3E+09	0.004	364	0.02	2.028	0.04	9E+06	0.008

Table 5.15: Symmetric Loss for the Case 5*

Case 5*	Symmetric Loss													
	20		30		40		50		60		80		100	
	F***	**S*****	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**	**F**	**S**
a**														
HC0	22.7	0.098	0.151	0.038	0.134	0.032	0.992	0.007	1.236	0.009	0.365	0.003	0.09	0.004
HC1	20.24	0.019	0.101	0.008	0.097	0.01	0.904	7E-04	1.151	0.002	0.334	5E-04	0.078	0.001
HC2	1E-05	8E-06	8E-06	9E-06	6E-06	9E-06	8E-05	6E-06	2E-05	2E-06	7E-06	3E-06	8E-06	3E-06
HC3	44.7	0.114	0.175	0.04	0.146	0.033	2.044	0.007	2.219	0.009	0.674	0.004	0.137	0.004
HC4	0.222	1E+05	0.107	1.828	0.155	1.553	0.004	70.11	0.026	80.02	0.005	13.48	0.011	1.824
HC5	6E+09	0.222	2.054	0.107	4.41	0.155	1E+08	0.004	2E+10	0.026	4E+07	0.005	3468	0.011
b**														
HC0	237.1	0.215	75.83	0.171	0.094	0.064	4.925	0.032	15.68	0.021	2.985	0.016	14.91	0.013
HC1	213.3	0.088	70.76	0.088	0.057	0.03	4.694	0.015	15.13	0.009	2.883	0.007	14.59	0.007
HC2	11.12	0.013	7.111	0.01	0.007	0.003	1.029	0.002	2.419	8E-04	0.668	0.001	1.693	0.001
HC3	0.742	0.082	0.545	0.067	0.03	0.029	0.264	0.012	0.429	0.011	0.137	0.005	0.845	0.005
HC4	0.137	760.8	0.335	178.1	0.111	0.341	0.046	10.89	0.032	33.44	0.026	5.232	0.026	59.05
HC5	3E+06	0.137	2E+08	0.335	0.341	0.111	2E+05	0.046	7E+10	0.032	1E+06	0.026	5E+20	0.026
c**														
HC0	2.332	0.281	295.4	0.145	4.976	0.057	0.802	0.079	1.421	0.034	1.869	0.016	2.381	0.017
HC1	1.925	0.147	275.7	0.088	4.66	0.025	0.719	0.045	1.314	0.017	1.797	0.008	2.312	0.009
HC2	0.318	0.032	14.21	0.016	0.887	0.003	0.132	0.007	0.106	0.002	0.438	0.001	0.551	0.001
HC3	0.44	0.069	0.76	0.053	0.16	0.025	0.179	0.022	0.576	0.014	0.129	0.006	0.17	0.005
HC4	0.279	9.563	0.251	901.6	0.067	9.348	0.202	3.319	0.07	9.48	0.025	4.01	0.042	5.208
HC5	14.04	0.279	2E+11	0.251	4381	0.067	97.58	0.202	7524	0.07	95002	0.025	3E+07	0.042
d**														
HC0	0.04	0.065	0.686	0.031	0.279	0.027	0.154	0.018	0.258	0.015	0.788	0.008	0.033	0.003
HC1	0.006	0.013	0.814	0.009	0.299	0.009	0.185	0.006	0.227	0.006	0.807	0.003	0.032	6E-04
HC2	0.054	6E-04	53.95	5E-04	0.388	4E-04	1.511	2E-04	0.048	5E-04	1.791	1E-04	0.036	5E-05
HC3	0.579	0.066	2303	0.025	0.692	0.023	7.596	0.015	0.033	0.017	4.485	0.008	0.082	0.003
HC4	0.057	5.324	0.03	4E+06	0.061	1.996	0.049	117.4	0.096	0.723	0.036	23.35	0.005	0.306
HC5	5.324	0.057	6E+16	0.03	2.573	0.061	1E+07	0.049	4.888	0.096	80792	0.036	1.284	0.005

represents the X settings. (see Table 4.1)

** represents the sigma settings. (see Table 4.2)

*** F stands for Full sample

S stands for Short sample

CONCLUSION

OLS is very sensitive to the high leverage observations that one can change the OLS estimate of a coefficient with a small play in an observation. By the help of the Robust Regression techniques, estimation can be protected from misguiding results due to outliers and bad leverage points. These outliers can be coming from the recording errors or the measurement errors or from the original data set. In all these situations one has to get rid of the bad effects of these outliers. Although the good leverage points are valuable for the OLS for pulling the regression line to the target, bad leverage points are extremely harmful because of pulling the estimated regression line to the wrong directions. Here, MCD method is used to detect the observations with high distances from the whole group.

Earlier studies in the literature help us for choosing the 5 different settings of X and the 4 variances of the error terms that we run them in the simulations settings. The Monte Carlo experiment was based on a simple regression model, with sample size to 10,000 replications.

Firstly, the program which is coded in Gauss generates the covariates, and then fixing the error terms from predetermined distribution in each iteration. The detection of high leverages (with MCD distances larger than the critical χ^2 values) is satisfied by the MCD procedure. The selected observations are removed from the data set and short sample is obtained. We estimate the covariance matrix (Ω) with the help of the specified HCCMEs by the full and the short sample.

Under homoscedasticity in which case there is no need to use of the HCCMEs, OLS is the best performer. And for the remaining cases, the results of the short sample in which the outliers are removed by the method of MCD are generally much better than the full sample results, expectedly. However, in some cases, good leverage points can cause slightly worse results in short sample than the other. When there is an increase in the sample size or in the degrees of heteroscedasticity, the gap between the full sample and the short sample estimates becomes huge that results in short sample's converge to true ones.

The other point to mention that HC4 and HC5 which have been introduced recently, have worse performance than other HCCMEs especially when the high leverage points are not removed.

It should be borne in mind that simulation studies rely on certain types of Xs and sigma settings, therefore we can't generalize them. The performances of the HCCMEs can be better or worse according to these settings. Although proof-type studies is a better way to making analysis, this area of study does not yet include such proof-types and simulations give us a good idea about making the comparisons.

Note that, the program codes and the article submitted can be made available to the interested readers as well as the list of covariates and error term variances for each case and sample size.

APPENDIX A

PROGRAM CODES

```
/* ************ MCD PROCEDURE STARTS HERE *********** */

proc (1) = mymcd(mcdmat,hperc,chicri);
/* mcdmat includes the covariates X, with dimension nXk, k is the number of
independent variables. */
local olddet, newdet, Tproc, kproc, Xproc, Xshort, Lnew, Snew, dists, sortedmat,
newcov, outmat, boolvec, Hproc;
/* chicrit is the Chi-square critical value */
/* initializations */
Tproc=rows(mcdmat); kproc=cols(mcdmat);
Hproc=floor(Tproc*hperc+.1);   /* number of trimmed observations */
Dists=mcdmat[.,1]; /* just to initiate, starting value not important */
Xshort=mcdmat[1:Hproc,.]; Lnew=(meanc(Xshort))';
newdet=det((Xshort-Lnew)'*(Xshort-Lnew));
olddet=2*newdet;
xproc=mcdmat;
mcdmat=(seqa(1,1,Tproc))~mcdmat;
sortedmat=mcdmat~dists; /* for initialization only */
do while olddet>newdet;
     olddet=newdet;
     Xshort=sortedmat[1:Hproc,2:(1+kproc)];
     Lnew=(meanc(Xshort))'; /* L denoting the location */
     Snew=(1/Hproc)*((Xshort-Lnew)'*(Xshort-Lnew));
     /* S denoting the scatter */
     dists=sqrt(diag((Xproc-Lnew)*inv(Snew)*(Xproc-Lnew)'));
     sortedmat=sortc((mcdmat~dists),(kproc+2));
     /* Xs rows are sorted according to the distances */
     if rank(snew)==kproc;
       newdet=det(snew);
       else; newdet=olddet;
     endif;
endo;
outmat= sortc(sortedmat,1); /* obs with inital sorts and distances */
boolvec=(outmat[.,cols(outmat)]).<chicri; /* large dist obs marked 0*/
outmat=outmat~boolvec;
retp(outmat);
```

```
endp;
/* This outmat includes the index of obs~X~distances~booleans */
/* ************ MCD PROCEDURE ENDS HERE *********** */
/******************* SETTINGS *****************/
output file=out1aT20.out RESET;
Tp=20; /*n*/
kp=2;    /*k+1*/
bigmat=zeros(Tp,Tp);
/**********  Design matrix definition  BEGIN ***********/
/*Case 1*/
x2=rndu(Tp,1);
xp=ones(Tp,1)~X2;
/*Case 2*/
/*x2=rndn(Tp,1);
xp=ones(Tp,1)~x2;*/
/*Case 3*/
/*/* this procedure generates t random numbers*/
proc(1)=tstudent(r,c,f);
/* r:number of rows, c: number of columns, f:degrees of freedom*/
local k1,k2,k3,k4;
k1=2*pi*(rndu(r,1));
k2=sqrt(f*(((rndu(1,c))^(-2/f))-1));
k3=sin(k1);
k4=k3*k2;
retp(k4);
endp;
f=3;   /*degrees of freedom*/
x2=tstudent(Tp,1,f);
xp=ones(Tp,1)~x2;*/
/*Case 4*/
/*x2=exp(rndn(Tp,1));
xp=ones(Tp,1)~x2;*/
/*Case 5*/
/*x2=rndn(Tp,1)./rndn(Tp,1);
xp=(ones(Tp,1))~x2;*/
/**********  Design matrix definition  END ***********/
/********** Variances of the error terms definition  BEGIN **********/
/*Case a*/
varvec=ones(Tp,1);  /*homoscedasticity*/
```

```
/*Case b*/
/*c0=3.1;
c1=3.1;
varvec=exp(c0*(x2)+c1*((x2)^2));     */
/*Case c*/
/*c0=0.1;
c1=0.2;
c2=0.3;
varvec=c0+c1*(x2)+c2*((x2)^2);     */
/*Case d*/
/*varvec=(x2)^2;     */
/*Case e*/
/*c1=0.285;
varvec=sqrt(exp(c1*x2));     */
/********** Variances of the error terms definition  END **********/
betap=ones(kp,1);
consta=0.7;
perc=0.50;
hpercen=0.75;
criti=2.71 /* 10 percent*/;
/*criti=3.84; /* 5 percent*/*/
ROBmat=mymcd(xp[.,2:(cols(xp))],hpercen,criti);
bools=ROBmat[.,cols(ROBmat)];
xpR=selif(ROBmat,bools);
TpR=rows(xpR);
xpR=(ones(TpR,1))~(xpR[.,2:kp]);
bigmatR=zeros(Tpr,Tpr);
varvecR=selif(varvec,bools);

T1=invpd((xp)'*(xp))*(xp)';
T2=(xp)*invpd((xp)'*(xp));
hat=(xp)*invpd((xp)'*(xp))*(xp)';     /*Hat matrix*/
chat=diag(hat);
T3=diagrv(zeros(Tp,Tp),varvec);
covarTR=T1*T3*T2;    /* True covariance matrix of betas */
T3R=diagrv(zeros(TpR,TpR),varvecR);
T1R=invpd((xpR)'*(xpR))*(xpR)';
T2R=(xpR)*invpd((xpR)'*(xpR));
hatR=(xpR)*invpd((xpR)'*(xpR))*(xpR)';     /*Hat matrix*/
```

```
chatR=diag(hatR);
covarRTR=T1R*T3R*T2R;
/* Terms for HC1 */
factor1=1/(Tp/(Tp-kp));
factor1R=1/(TpR/(TpR-kp));
/* Terms for HC2 */
factor2=1-chat;
factor2R=1-chatR;
/* Terms for HC3 */
factor3=(1-chat)^2;
factor3R=(1-chatR)^2;
/* Terms for HC4 */
hatm=meanc(chat);
newhat=chat./hatm;
mat1=seqa(4,0,Tp)~newhat;
mat2=(mat1)';
mat3=minc(mat2);
factor4=(1-chat)^(mat3);
hatmR=meanc(chatR);
newhatR=chatR./hatmR;
mat1R=seqa(4,0,TpR)~newhatR;
mat2R=(mat1R)';
mat3R=minc(mat2R);
factor4R=(1-chatR)^(mat3R);
/* Terms for HC5 */
hmax=maxc(chat);
term1=(consta*hmax)/hatm;
term2=4~term1;
term3=(term2)';
term4=maxc(term3);
term5=newhat~seqa(term4,0,Tp);
term6=(term5)';
term7=minc(term6);
factor5=(1-chat)^(term7);
hmaxR=maxc(chatR);
term1R=(consta*hmaxR)/hatmR;
term2R=4~term1R;
term3R=(term2R)';
term4R=maxc(term3R);
```

```
term5R=newhatR~seqa(term4R,0,TpR);
term6R=(term5R)';
term7R=minc(term6R);
factor5R=(1-chatR)^(term7R);
mcss=100000;
counter=1;
sum0=0; sum1=0; sum2=0; sum3=0; sum4=0; sum5=0;
sumR0=0; sumR1=0; sumR2=0; sumR3=0; sumR4=0; sumR5=0;

do while counter<mcss;
    /*counter;*/
    eps=rndn(Tp,1).*sqrt(varvec);
    eps=eps-meanc(eps);   /*epsilons*/
    yp=(xp)*betap+eps;      /*population regression function*/
    ypR=selif(yp,bools);
    betaest=yp/xp;              /*betahats*/
    resp=yp-xp*betaest;     /*residuals*/
    respsq=(resp)^2;          /*residuals square*/
    betaestR=ypR/xpR;
    respR=ypR-xpR*betaestR;
    respsqR=(respR)^2;        /*Ols residuals*/
    hco0=respsq;              /*HC0 full and short samples*/
    hcR0=respsqR;
    hco1=respsq./factor1;     /*HC1 full and short samples*/
    hcR1=respsqR./factor1R;
    hco2=respsq./factor2;     /*HC2 full and short samples*/
    hcR2=respsqR./factor2R;
    hco3=respsq./factor3;     /*HC3 full and short samples*/
    hcR3=respsqR./factor3R;
    hco4=respsq./factor4;     /*HC4 full and short samples*/
    hcR4=respsqR./factor4R;
    hco5=respsq./factor5;     /*HC5 full and short samples*/
    hcR5=respsqR./factor5R;
    sum0=sum0+hco0;
    sum1=sum1+hco1;
    sum2=sum2+hco2;
    sum3=sum3+hco3;
    sum4=sum4+hco4;
    sum5=sum5+hco5;
```

```
        sumR0=sumR0+hcR0;
        sumR1=sumR1+hcR1;
        sumR2=sumR2+hcR2;
        sumR3=sumR3+hcR3;
        sumR4=sumR4+hcR4;
        sumR5=sumR5+hcR5;
        counter=counter+1;
endo;
/********* THIS PART CALCULATES THE DESIRED STATISTICS OUT OF
THE LOOP BEGIN **************/
me0=sum0/mcss;
me1=sum1/mcss;
me2=sum2/mcss;
me3=sum3/mcss;
me4=sum4/mcss;
me5=sum5/mcss;
F0=diagrv(bigmat,me0);
F1=diagrv(bigmat,me1);
F2=diagrv(bigmat,me2);
F3=diagrv(bigmat,me3);
F4=diagrv(bigmat,me4);
F5=diagrv(bigmat,me5);
me0R=sumR0/mcss;
me1R=sumR1/mcss;
me2R=sumR2/mcss;
me3R=sumR3/mcss;
me4R=sumR4/mcss;
me5R=sumR5/mcss;
F0R=diagrv(bigmatR,me0R);
F1R=diagrv(bigmatR,me1R);
F2R=diagrv(bigmatR,me2R);
F3R=diagrv(bigmatR,me3R);
F4R=diagrv(bigmatR,me4R);
F5R=diagrv(bigmatR,me5R);
omega0=T1*(F0)*T2;
omega1=T1*(F1)*T2;
omega2=T1*(F2)*T2;
omega3=T1*(F3)*T2;
omega4=T1*(F4)*T2;
```

```
omega5=T1*(F5)*T2;
omegaR0=T1R*(F0R)*T2R;
omegaR1=T1R*(F1R)*T2R;
omegaR2=T1R*(F2R)*T2R;
omegaR3=T1R*(F3R)*T2R;
omegaR4=T1R*(F4R)*T2R;
omegaR5=T1R*(F5R)*T2R;
/*Operations for Quasi-t test calculations*/
b0=1; b1=1;
u0=(diag(omega0))';u1=(diag(omega1))';u2=(diag(omega2))';u3=(diag(omega3))';u4
=(diag(omega4))';u5=(diag(omega5))';
seb00=sqrt(u0[1,1]);seb01=sqrt(u1[1,1]);seb02=sqrt(u2[1,1]);seb03=sqrt(u3[1,1]);se
b04=sqrt(u4[1,1]);seb05=sqrt(u5[1,1]);
t0=b0/(seb00);t1=b0/(seb01);t2=b0/(seb02);t3=b0/(seb03);t4=b0/(seb04);t5=b0/(seb
05);
/*Quasi-t results for beta 0-full sample*/
uR0=(diag(omegaR0))';uR1=(diag(omegaR1))';uR2=(diag(omegaR2))';uR3=(diag(o
megaR3))';uR4=(diag(omegaR4))';uR5=(diag(omegaR5))';
seRb00=sqrt(uR0[1,1]);seRb01=sqrt(uR1[1,1]);seRb02=sqrt(uR2[1,1]);seRb03=sqrt(
uR3[1,1]);seRb04=sqrt(uR4[1,1]);seRb05=sqrt(uR5[1,1]);
tR0=b0/(seRb00);tR1=b0/(seRb01);tR2=b0/(seRb02);tR3=b0/(seRb03);tR4=b0/(seR
b04);tR5=b0/(seRb05);
/*Quasi-t stats for beta 0-Short sample*/
seb10=sqrt(u0[1,2]);seb11=sqrt(u1[1,2]);seb12=sqrt(u2[1,2]);seb13=sqrt(u3[1,2]);se
b14=sqrt(u4[1,2]);
seb15=sqrt(u5[1,2]);
tt0=b1/(seb10);tt1=b1/(seb11);tt2=b1/(seb12);tt3=b1/(seb13);tt4=b1/(seb14);tt5=b1/(
seb15);
/*Quasi-t results for beta 1-full sample*/
seRb10=sqrt(uR0[1,2]);seRb11=sqrt(uR1[1,2]);seRb12=sqrt(uR2[1,2]);seRb13=sqrt(
uR3[1,2]);
seRb14=sqrt(uR4[1,2]);seRb15=sqrt(uR5[1,2]);
ttR0=b1/(seRb10);ttR1=b1/(seRb11);ttR2=b1/(seRb12);ttR3=b1/(seRb13);ttR4=b1/(
seRb14);ttR5=b1/(seRb15);
/*Quasi-t results for beta 1-Short sample*/
/*Quadratic Loss Functions by Full Sample*/;
lossQF0=(covarTR-omega0)'*(covarTR-omega0);lossQF1=(covarTR-
omega1)'*(covarTR-omega1);
lossQF2=(covarTR-omega2)'*(covarTR-omega2);lossQF3=(covarTR-
omega3)'*(covarTR-omega3);
```

lossQF4=(covarTR-omega4)'*(covarTR-omega4);lossQF5=(covarTR-omega5)'*(covarTR-omega5);

/*Quadratic Loss Functions by Short sample*/;

lossQR0=(covarRTR-omegaR0)'*(covarRTR-omegaR0);lossQR1=(covarRTR-omegaR1)'*(covarRTR-omegaR1);

lossQR2=(covarRTR-omegaR2)'*(covarRTR-omegaR2);lossQR3=(covarRTR-omegaR3)'*(covarRTR-omegaR3);

lossQR4=(covarRTR-omegaR4)'*(covarRTR-omegaR4);lossQR5=(covarRTR-omegaR5)'*(covarRTR-omegaR5);

/*Entropy Loss Functions by Full Sample*/;

lossEF0=sumc(diag((inv(omega0))*covarTR))-log(abs(inv(omega0)*covarTR))-kp;

lossEF1=sumc(diag((inv(omega1))*covarTR))-log(abs(inv(omega1)*covarTR))-kp;

lossEF2=sumc(diag((inv(omega2))*covarTR))-log(abs(inv(omega2)*covarTR))-kp;

lossEF3=sumc(diag((inv(omega3))*covarTR))-log(abs(inv(omega3)*covarTR))-kp;

lossEF4=sumc(diag((inv(omega4))*covarTR))-log(abs(inv(omega4)*covarTR))-kp;

lossEF5=sumc(diag((inv(omega5))*covarTR))-log(abs(inv(omega5)*covarTR))-kp;

/*Entropy Loss Functions by Short sample*/;

lossER0=sumc(diag((inv(omegaR0))*covarRTR))-log(abs(inv(omegaR0)*covarRTR))-kp;

lossER1=sumc(diag((inv(omegaR1))*covarRTR))-log(abs(inv(omegaR1)*covarRTR))-kp;

lossER2=sumc(diag((inv(omegaR2))*covarRTR))-log(abs(inv(omegaR2)*covarRTR))-kp;

lossER3=sumc(diag((inv(omegaR3))*covarRTR))-log(abs(inv(omegaR3)*covarRTR))-kp;

lossER4=sumc(diag((inv(omegaR4))*covarRTR))-log(abs(inv(omegaR4)*covarRTR))-kp;

lossER5=sumc(diag((inv(omegaR5))*covarRTR))-log(abs(inv(omegaR5)*covarRTR))-kp;

/*Symmetric Loss Functions by Full Sample*/;

lossSF0=sumc(diag((inv(omega0))*covarTR))+sumc(diag(omega0*inv(covarTR)))-2*kp;

lossSF1=sumc(diag((inv(omega1))*covarTR))+sumc(diag(omega1*inv(covarTR)))-2*kp;

lossSF2=sumc(diag((inv(omega2))*covarTR))+sumc(diag(omega2*inv(covarTR)))-2*kp;

lossSF3=sumc(diag((inv(omega3))*covarTR))+sumc(diag(omega3*inv(covarTR)))-2*kp;

lossSF4=sumc(diag((inv(omega4))*covarTR))+sumc(diag(omega4*inv(covarTR)))-2*kp;

lossSF5=sumc(diag((inv(omega5))*covarTR))+sumc(diag(omega5*inv(covarTR)))-2*kp;

```
/*Symmetric Loss Functions by Short sample*/;
lossSR0=sumc(diag((inv(omegaR0))*covarRTR))+sumc(diag(omegaR0*inv(covarRTR)))-2*kp;
lossSR1=sumc(diag((inv(omegaR1))*covarRTR))+sumc(diag(omegaR1*inv(covarRTR)))-2*kp;
lossSR2=sumc(diag((inv(omegaR2))*covarRTR))+sumc(diag(omegaR2*inv(covarRTR)))-2*kp;
lossSR3=sumc(diag((inv(omegaR3))*covarRTR))+sumc(diag(omegaR3*inv(covarRTR)))-2*kp;
lossSR4=sumc(diag((inv(omegaR4))*covarRTR))+sumc(diag(omegaR4*inv(covarRTR)))-2*kp;
lossSR5=sumc(diag((inv(omegaR5))*covarRTR))+sumc(diag(omegaR5*inv(covarRTR)))-2*kp;
/*Stein Loss Functions by Full Sample*/;
lossStF0=sumc(diag(omega0*(inv(covarTR))))-log(abs((omega0)*(inv(covarTR))))-kp;
lossStF1=sumc(diag(omega1*(inv(covarTR))))-log(abs((omega1)*(inv(covarTR))))-kp;
lossStF2=sumc(diag(omega2*(inv(covarTR))))-log(abs((omega2)*(inv(covarTR))))-kp;
lossStF3=sumc(diag(omega3*(inv(covarTR))))-log(abs((omega3)*(inv(covarTR))))-kp;
lossStF4=sumc(diag(omega4*(inv(covarTR))))-log(abs((omega4)*(inv(covarTR))))-kp;
lossStF5=sumc(diag(omega5*(inv(covarTR))))-log(abs((omega5)*(inv(covarTR))))-kp;
/*Stein Loss Functions by Short sample*/;
lossStR0=sumc(diag(omegaR0*(inv(covarRTR))))-log(abs((omegaR0)*(inv(covarRTR))))-kp;
lossStR1=sumc(diag(omegaR1*(inv(covarRTR))))-log(abs((omegaR1)*(inv(covarRTR))))-kp;
lossStR2=sumc(diag(omegaR2*(inv(covarRTR))))-log(abs((omegaR2)*(inv(covarRTR))))-kp;
lossStR3=sumc(diag(omegaR3*(inv(covarRTR))))-log(abs((omegaR3)*(inv(covarRTR))))-kp;
lossStR4=sumc(diag(omegaR4*(inv(covarRTR))))-log(abs((omegaR4)*(inv(covarRTR))))-kp;
lossStR5=sumc(diag(omegaR5*(inv(covarRTR))))-log(abs((omegaR5)*(inv(covarRTR))))-kp;
/********* THIS PART CALCULATES THE DESIRED STATISTICS OUT OF THE LOOP END **************/
/********* OUTPUT PREPARATION BEGIN **************/
```

```
cls;
output on;
"case 1a T is" tp;
"X is";
(xp[.,2])';
"number of obs removed";
Tp-TpR;
"degrees of heteroscedasticity";
(maxc(varvec))/(minc(varvec));
"true variance of epsilons";
(varvec)';
"True Covariance followed by HC0, HC1, HC2, HC3, HC4, HC5 FULL SAMPLE";
(diag(covarTR))'|(diag(omega0))'|(diag(omega1))'|(diag(omega2))'|(diag(omega3))'|(d
iag(omega4))'|(diag(omega5))';
"True Covariance followed by HC0, HC1, HC2, HC3, HC4, HC5 SHORT
SAMPLE";
(diag(covarRTR))'|(diag(omegaR0))'|(diag(omegaR1))'|(diag(omegaR2))'|(diag(omeg
aR3))'|(diag(omegaR4))'|(diag(omegaR5))';
"HC0, HC1, HC2, HC3, HC4, HC5 Quadratic LossES FULL sample ";
(diag(lossQF0))'|(diag(lossQF1))'|(diag(lossQF2))'|(diag(lossQF3))'|(diag(lossQF4))'|(
diag(lossQF5))';
"HC0, HC1, HC2, HC3, HC4, HC5 Quadratic LossES SHORT sample ";
(diag(lossQR0))'|(diag(lossQR1))'|(diag(lossQR2))'|(diag(lossQR3))'|(diag(lossQR4))'
|(diag(lossQR5))';
"HC0, HC1, HC2, HC3, HC4, HC5 Entropy Losses FULL sample ";
(diag(lossEF0))'|(diag(lossEF1))'|(diag(lossEF2))'|(diag(lossEF3))'|(diag(lossEF4))'|(d
iag(lossEF5))';
"HC0, HC1, HC2, HC3, HC4, HC5 Entropy Losses SHORT sample ";
(diag(lossER0))'|(diag(lossER1))'|(diag(lossER2))'|(diag(lossER3))'|(diag(lossER4))'|(
diag(lossER5))';
"HC0, HC1, HC2, HC3, HC4, HC5 Symmetric Losses FULL sample ";
(lossSF0)|(lossSF1)|(lossSF2)|(lossSF3)|(lossSR4)|(lossSF5);
"HC0, HC1, HC2, HC3, HC4, HC5 Symmetric Losses SHORT sample ";
(lossSR0)|(lossSR1)|(lossSR2)|(lossSR3)|(lossSF4)|(lossSR5);
"HC0, HC1, HC2, HC3, HC4, HC5 STEIN Losses FULL sample ";
(diag(lossStF0))'|(diag(lossStF1))'|(diag(lossStF2))'|(diag(lossStF3))'|(diag(lossStF4))'
|(diag(lossStF5))';
"HC0, HC1, HC2, HC3, HC4, HC5 STEIN Losses SHORT sample ";
(diag(lossStR0))'|(diag(lossStR1))'|(diag(lossStR2))'|(diag(lossStR3))'|(diag(lossStR4)
)'|(diag(lossStR5))';
```

```
"True Quasi-t stat for beta0 full sample is";
1/sqrt((covartr[1,1]));
"HC0, HC1, HC2, HC3, HC4, HC5 Quasi-t STATS FULL SAMPLE for beta0";
t0|t1|t2|t3|t4|t5;
"True Quasi-t stat for beta0 short sample is";
1/sqrt((covarRTR[1,1]));
"HC0, HC1, HC2, HC3, HC4, HC5 Quasi-t STATS SHORT SAMPLE for beta0";
tR0|tR1|tR2|tR3|tR4|tR5;
"True Quasi-t stat for beta1 full sample is";
1/sqrt((covartr[2,2]));
"HC0, HC1, HC2, HC3, HC4, HC5 Quasi-t STATS FULL SAMPLE for beta1";
tt0|tt1|tt2|tt3|tt4|tt5;
"True Quasi-t stat for beta1 short sample is";
1/sqrt((covarRtr[2,2]));
"HC0, HC1, HC2, HC3, HC4, HC5 Quasi-t STATS SHORT SAMPLE for beta1";
ttR0|ttR1|ttR2|ttR3|ttR4|ttR5;
"percentage deviations FULL";
100*((diag(omega0)-diag(covarTR))./(diag(covarTR)))';
100*((diag(omega1)-diag(covarTR))./(diag(covarTR)))';
100*((diag(omega2)-diag(covarTR))./(diag(covarTR)))';
100*((diag(omega3)-diag(covarTR))./(diag(covarTR)))';
100*((diag(omega4)-diag(covarTR))./(diag(covarTR)))';
100*((diag(omega5)-diag(covarTR))./(diag(covarTR)))';
"percentage deviations SHORT";
100*((diag(omegaR0)-diag(covarRTR))./(diag(covarRTR)))';
100*((diag(omegaR1)-diag(covarRTR))./(diag(covarRTR)))';
100*((diag(omegaR2)-diag(covarRTR))./(diag(covarRTR)))';
100*((diag(omegaR3)-diag(covarRTR))./(diag(covarRTR)))';
100*((diag(omegaR4)-diag(covarRTR))./(diag(covarRTR)))';
100*((diag(omegaR5)-diag(covarRTR))./(diag(covarRTR)))';
"number of obs removed";
Tp-TpR;
output off;
/********* OUTPUT PREPARATION BEGIN **************/
```

APPENDIX B

SOME EXAMPLES OF X & SIGMA SETTINGS THAT WERE USED IN THE SIMULATIONS

<u>Case 1b T20</u>

X is

0.78667715	0.012676156	0.014068436	0.49968462
0.78390375	0.62613287	0.053984818	0.31571816
0.011010702	0.82559118	0.39032000	0.63640308
0.080682491	0.25893282	0.38682087	0.23964318
0.30769482	0.95708474	0.71299373	0.11844922

number of obs removed (L);

1.0000000

degrees of heteroscedasticity (λ);

321.20784

true variance of epsilons;

78.035827	1.0405966	1.0452182	10.206706
76.330104	23.484562	1.1929002	3.6245421
1.0351114	106.94291	5.3777823	25.238664
1.3103518	2.7470702	5.2751065	2.5116084
3.4811053	332.48589	44.088400	1.5078522

Case 4c T40

X is;

0.22938746	0.63366499	1.3820438	7.6052041
4.0130835	19.148250	0.67998069	4.3984843
0.48960753	0.36804470	2.0910404	0.89006850
1.7610061	1.2784532	2.4077469	0.68918268
1.7101567	0.73399660	1.2834323	2.1462074
1.3364725	0.67480130	0.10060327	0.28946444
4.0206580	0.66105598	3.6982176	1.6457247
3.0657919	4.9081569	0.18241108	1.1998420
2.5027104	3.1334276	0.75724971	0.65923017
1.2367816	0.94862803	4.4135908	0.076830523

number of obs removed;

8.0000000

degrees of heteroscedasticity;

62114.157

true variance of epsilons;

0.052618605	0.40153132	1.9100450	57.839130
16.104839	366.65548	0.46237374	19.346664
0.23971553	0.13545690	4.3724501	0.79222194
3.1011424	1.6344427	5.7972450	0.47497277
2.9246359	0.53875101	1.6471984	4.6062063
1.7861586	0.45535679	0.010121017	0.083789665
16.165691	0.43699501	13.676813	2.7084096
9.3990802	24.090005	0.033273803	1.4396209
6.2635592	9.8183683	0.57342712	0.43458442
1.5296288	0.89989515	19.479784	0.0059029293

Case 5d T60

X is;

-1.8782138	-2.0442186	-6.9892272	-1.2213341
-1.9624709	0.40333014	0.059312801	-4.2440208
-1.1352397	-0.44574163	-0.70775680	1.1816206
3.7101551	-0.14362663	-0.10513524	0.32185545
4.2379683	1.6114211	-1.4474397	0.49617025
6.5850926	-9.8014936	0.32272008	-0.84408043
-3.8317838	-0.42463207	-1.8345097	-1.3428783
2.9046793	-0.53971552	-4.0687116	0.086784310
3.3870421	-1.6854313	0.26330347	-0.55347980
-1.4914349	-1.3362404	-0.38440883	0.23078413
-0.87996832	0.73971898	-0.55090003	-0.98055163
2.1425748	-11.763302	6.1814962	0.41240309
-0.96226851	3.2545907	-0.26622912	0.90192779
-1.2760617	0.95672633	-0.19272527	3.9222556
0.15542203	15.479210	-0.62001449	-0.74121844

number of obs removed

11.000000

degrees of heteroscedasticity

48.523971

true variance of epsilons

0.76517901	0.74729060	0.36936688	0.84026320
0.75604672	1.0591583	1.0084879	0.54619842
0.85063542	0.93845706	0.90406380	1.1833873
1.6967202	0.97974123	0.98512990	1.0469324
1.8292581	1.2581313	0.81362138	1.0732638
2.5558266	0.24740890	1.0470614	0.88667084
0.57924532	0.94128429	0.76995929	0.82583510
1.5127313	0.92597369	0.56001517	1.0124435
1.6203683	0.78649103	1.0382335	0.92415925
0.80853648	0.82661663	0.94669505	1.0334335
0.88214796	1.1111660	0.92449905	0.86959421
1.3570550	0.18707034	2.4129817	1.0605286
0.87186275	1.5900718	0.96277297	1.1371495
0.83373573	1.1460640	0.97291034	1.7487853
1.0223947	9.0773959	0.91543853	0.89976323

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