DIFFERENCE SCHEMES OF NONLOCAL BOUNDARY VALUE PROBLEMS
FOR HYPERBOLIC EQUATIONS

by

Necmettin AĞGEZ

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ABSTRACT

It is known that various nonlocal boundary value problem for the hyperbolic equations can be reduced to the nonlocal boundary problem

$$\frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \;\; u(0) = \alpha u(1) + \varphi, \; u'(0) = \beta u'(1) + \psi$$

for differential equation in a Hilbert space H with self - adjoint positive operator A. Applying the operator approach we obtain the stability estimates for solution of this nonlocal boundary problem. In applications this abstract result permit us to obtain the stability estimates for the solution of nonlocal boundary value problem for hyperbolic equations. The first and second order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem are presented. The stability estimates for the solution of these difference schemes are obtained. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

ÖZET

Bilindiği gibi çeşitli local olmayan hiperbolik tip sınır-değer denklemleri, Hilbert uzayındaki kendi kendine eş positive operatör A ile local olmayan sınır-değer problemine

$$\frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \;\; u(0) = \alpha u(1) + \varphi, \;\; u'(0) = \beta u'(1) + \psi$$

dönüştürülebilir. Operator metod kullanarak, bu locol olmayan problemin kararlılığı elde edilmiştir.

Yapılan soyut uygulamalar bize local olmayan iki hiperbolik tip sınır-değer problemlerinin kararlılığını elde etmemizi sağlamıştır. Bu local olmayan hiperbolik tip sınır-değer problemleri için A- nın tamsayı değerli üslerinin oluşturduğu birinci ve ikinci mertebeden yaklaşımlı sonlu farklar metodlarıyla kurulmuştur. Bu sonlu farklar metodları ile çözümün kararlı olup olmadığı incelenmiştir ve yapılan nümerik denemelerle, elde edilen teorik sonuçların doğruluğu desteklenmiştir.

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INTRODUCTION

It is known that most problems in fluid mechanics (dynamics,elasticity)and other areas of physics lead to partial differential equations of the hyperbolic type. These equations can be derived as models of physical systems and consider methods for solving boundary value problems.

A problem is called well-posed if for each set of data there exists exactly one solution and dependence of the solution on the data continuous. Our goal in this work is to show that various types of the nonlocal boundary value problems for equations of hyperbolic type are well-posed. Also, consider the difference method for solving these problems.

Let us consider the simple problems for wave equations. First, consider the initial-boundary value problem for wave equations

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le t < \infty, \ \ 0 < x < L, \tag{0.1}$$

$$\begin{aligned} u(t,0) &= u(t,L) = 0 \ , \qquad\qquad 0 \le t < \infty, \\ u(0,x) &= \varphi(x) \ , \ u_t(0,x) = \psi(x) \ , \ \ 0 \le x \le L. \end{aligned}$$

The mixed problem (0.1) can be solved using the so-called a Laplace transform method (in t), or method of separation of variables.

The one-dimensional wave equation can be solved by separation of variables using a trial solution

$$u(x,t) = X(x)T(t). \tag{0.2}$$

This gives

$$\frac{1}{c^2 T(t)} \frac{\partial^2 T(t)}{\partial t^2} = \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} = -k^2 \quad , \quad k \text{ is constant.}$$

So, the solution for X is

$$X(x) = A\cos(kx) + B\sin(kx).$$

So, the solution for T is

$$T(t) = E\cos(ckt) + F\sin(ckt).$$

Applying the boundary conditions

$$u(t,0) = u(t,L) = 0,$$

we obtain

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

Therefore

$$A = 0, \quad kL = n\pi \text{ where n is integer.}$$

Then

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t,x) = \sum_{n=1}^{\infty} (E_n \cos\lambda_n t + F_n \sin\lambda_n t) \sin\frac{n\pi}{L}x, \ \lambda_n = \frac{cn\pi}{L}.$$

Using the initial conditions we have that

$$u(0,x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi}{L}x = \varphi(x).$$

Hence we must choose the E_n's so that $u(0,x)$ becomes the Fourier sine series of $\varphi(x)$

$$E_n = \frac{2}{L}\int_0^L \varphi(x)\sin\frac{n\pi x}{L}dx.$$

In the same manner, we can obtain

$$u_t|_{t=0} = \left[\sum_{n=1}^{\infty}(-E_n\lambda_n \sin\lambda_n t + F_n\lambda_n\cos\lambda_n t)\sin\frac{n\pi x}{L}\right] t=0$$

$$= \sum_{n=1}^{\infty} F_n\lambda_n \sin\frac{n\pi x}{L} = \psi(x).$$

Hence we must choose F_n's for $t = 0$ $\dfrac{\partial u}{\partial t}$ becomes the Fourier sine series of $\psi(x)$

$$F_n\lambda_n = \frac{2}{L}\int_0^L \psi(x)\sin\frac{n\pi x}{L}dx$$

or since $\lambda_n = \dfrac{cn\pi}{L}$

$$F_n = \frac{2}{cn\pi}\int_0^L \psi(x)\sin\frac{n\pi x}{L}dx.$$

Thus

$$u(t,x) = \sum_{n=1}^{\infty}\left\{\left[\frac{2}{L}\int_0^L \varphi(x)\sin\frac{n\pi x}{L}dx\right]\cos\lambda_n t + \left[\frac{2}{cn\pi}\int_0^L \psi(x)\sin\frac{n\pi x}{L}dx\right]\sin\lambda_n t\right\}\sin\frac{n\pi}{L}x. \tag{0.3}$$

Note that using the same manner one obtains the solution of the simply nonlocal boundary value problem for wave equations

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, 0 < t < T,\ 0 < x < L,$$

$$\begin{aligned} u(t,0) &= u(t,L) = 0, \quad 0 \le t < \infty, \\ u(0,x) &= \alpha u(T,x) + \varphi(x), \quad |\alpha| \ne 1, \\ u_t(0,x) &= \alpha u_t(1,x) + \psi(x), \quad 0 \le x \le L. \end{aligned}$$

However, the method of separation of variables and other classical methods can be used only in the case when c^2 =constant. It is a well-known the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Second, we will consider the initial value problem for wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, 0 < t < \infty,\ -\infty < x < \infty, \tag{0.4}$$
$$u(0,x) = \varphi(x), u_t(0,x) = \psi(x), -\infty < x < \infty.$$

It is a well-known that this initial value problem can be solved using the so-called d'Alembert's solution, a Laplace transform method (in t), or a Fourier transform method (in x).

Here consider d'Alembert's solution.

Let

$$v = x + ct \quad , \quad z = x - ct, \text{ then}$$
$$v_x = 1 \quad , \quad z_x = 1.$$

Let's denote $u(t,x)$ as a functions of v, z. By the chain rule,

$$u_x = u_v v_x + u_z z_x = u_v + u_z,$$

$$u_{xx}\left(u_v + u_z\right)_x = \left(u_v + u_z\right)_v v_x + \left(u_v + u_z\right)_z z_x = u_{vv} + 2u_{vz} + u_{zz},$$

$$u_{tt} = c^2(u_{vv} - 2u_{vz} + u_{zz}).$$

The wave equation then becomes

$$u_{vz} = \frac{\partial^2 u}{\partial z \partial v} = 0.$$

We have that

$$\frac{\partial u}{\partial v} = h(v),$$
$$u = \int h(v)dv + B(z).$$

So, any solution of this equation is of the form

$$u(t,x) = A(x+ct) + B(x-ct),$$

where A and B are any functions. This solution known as d'Alembert's solution of the wave equation. Using the initial conditions it can be written

$$u(x,0) = A(x) + B(x) = \varphi(x), \tag{0.5}$$
$$u_t(0,x) = cA'(x) - cB'(x) = \psi(x). \tag{0.6}$$

Dividing (0.6) by c and integrating with respect to x, we obtain

$$A(x) - B(x) = k(x_0) + \frac{1}{c}\int_{x_0}^{x} \psi(s)ds \quad k(x_0) = A(x_0) + B(x_0). \tag{0.7}$$

If we add this to (0.5), then ψ drops out and division by 2 gives

$$A(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_{x_0}^{x} \psi(s)ds + \frac{1}{2}k(x_0). \tag{0.8}$$

Subtraction of (0.7) from (0.5) and divided by 2

$$B(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_{x_0}^{x} \psi(s)ds - \frac{1}{2}k(x_0). \tag{0.9}$$

In the equation (0.8) $x \to x+ct$ integral from x_0 to $x+ct$.

In the equation (0.9) $x \to x-ct$ and get mines an integral from x_0 to $x-ct$ or plus integral from $x-ct$ to x_0. Thus, our final result is

$$u(t,x) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{0.10}$$

However, this method and other methods can be used only in the case when c^2=constant. Most difference schemes for solving the wave equations are conditionally stable. In this work it studied the unconditionally stable difference schemes.

It is known that various nonlocal boundary value problem for the hyperbolic equations can be reduced to the boundary value problem

$$\frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \;\; u(0) = \alpha u(1) + \varphi, \;\; u'(0) = \beta u'(1) + \psi$$

for differential equation in a Hilbert space H with self - adjoint positive operator A.

In the present work the stability estimates for solution of the last nonlocal boundary problem are obtained. In applications this abstract result permit us to obtain the stability estimates for the solution of nonlocal boundary value problem for the hyperbolic equations. The first and second order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem are presented. The stability estimates for the solution of these difference schemes are obtained. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Let us briefly describe the contents of the various sections.It consists of an introduction ,conclusions and three sections.First section presents all the elementary Hilbert space theory that is needed for this work. Second section consists of four subsections. A brief survey of all investigations in this area can be found in the first subsection. Second subsection is devoted to the study of the stability of this nonlocal boundary value problem.In last two subsections the first and second order of accuracy difference schemes are presented. The stability estimates for the solution of these difference schemes are obtained.Third section is devoted to the numerical analysis.

1 ELEMENTS OF HILBERT SPACE

This section is the selected concepts of the elementary Hilbert space theory as developed in [24]-[26]. Several examples and problems are given to illustrate concepts and their applications to differential equations.

1.1 Hilbert Space

Definition 1.1. A complex linear space H is called an inner product space if there is a complex-valued function $\langle .,. \rangle : H \times H \to C$ with the properties

i. $\langle x,x \rangle \geq 0$ and $\langle x,x \rangle = 0 \iff x = \sigma,$

ii. $\langle x,y \rangle = \overline{\langle y,x \rangle}$ for all $x, y \in H$,

iii. $\langle \alpha x,y \rangle = \alpha \langle x,y \rangle,$ for all $x, y \in H$ and $\alpha \in C,$

iv. $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$ for all $x, y, z \in H.$

The function $\langle x,y \rangle$ is called the inner product of x and y. A Hilbert space is a complete inner product space. An inner product on H defines a norm on H given by $\|x\| = \langle x,x \rangle^{\frac{1}{2}}$. Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

Example 1.1. *The space $C_2[-1,1]$ of all defined and continuous functions on a given closed interval $[-1,1]$ is an inner product space with the inner product given by*

$$\langle x,y \rangle = \int_{-1}^{1} x(t)\overline{y(t)}dt. \tag{1.1}$$

Solution.

$$\langle x,x \rangle = \int_{-1}^{1} |x(t)|^2 \, dt \geq 0 \quad and \qquad \langle x,x \rangle = 0 \iff x(t) = 0,$$

$$\langle x,y \rangle = \int_{-1}^{1} x(t)\overline{y(t)}dt = \int_{-1}^{1} \overline{\overline{x(t)\overline{y(t)}}}dt = \overline{\int_{-1}^{1} \overline{x(t)\overline{\overline{y(t)}}}dt}$$

$$= \overline{\int_{-1}^{1} \overline{x(t)}y(t)dt} = \overline{\int_{-1}^{1} y(t)\overline{x(t)}dt} = \overline{\langle y,x \rangle},$$

$$\langle \alpha x, z\rangle = \int_{-1}^{1} \alpha x(t)\overline{z(t)}dt = \alpha \int_{-1}^{1} x(t)\overline{z(t)}dt = \alpha \langle x, z\rangle ,$$

$$\langle x+y, z\rangle = \int_{-1}^{1} [x(t)+y(t)]\,\overline{z(t)}dt = \int_{-1}^{1} x(t)\overline{z(t)}dt + \int_{-1}^{1} y(t)\overline{z(t)}dt$$

$$= \langle x, z\rangle + \langle y, z\rangle .$$

So, the space $C_2[-1,1]$ is an inner product space.

Note that the space $C_2[-1,1]$ is not complete.

Let

$$x_n(t) = \begin{cases} -1, & -1 \le t < -\frac{1}{n}, \\ nt, & -\frac{1}{n} \le t \le \frac{1}{n}, \\ 1, & \frac{1}{n} < t \le 1. \end{cases}$$

Then

$$x(t) = \lim_{n\to\infty} x(t) = \begin{cases} -1, & -1 \le t < 0, \\ 0, & t = 0, \\ 1, & 0 < t \le 1. \end{cases}$$

Let us prove that $x_n(t)$ is a Cauchy sequence

$$\|x_n - x_m\|_{C_2[-1,1]} = \left(\int_{-1}^{1} |x_n(t) - x_m(t)|^2 dt\right)^{\frac{1}{2}}$$

$$\le \left(\int_{-1}^{1} |x_n(t) - x(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int_{-1}^{1} |x(t) - x_m(t)|^2 dt\right)^{\frac{1}{2}} .$$

Since

$$\left(\int_{-1}^{1} |x_n(t) - x(t)|^2 dt\right)^{\frac{1}{2}} \le \sqrt{\frac{2}{3n}}, \text{ we obtain}$$

$$\|x_n - x_m\|_{C_2[-1,1]} \le \sqrt{\frac{2}{3n}} + \sqrt{\frac{2}{3m}}.$$

$$\sqrt{\frac{2}{3n}} + \sqrt{\frac{2}{3m}} \to 0 \quad \textit{when} \quad n\to\infty \quad , m\to\infty.$$

So, $\{x_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence but not convergent , since $x(t) \notin C_2[-1,1]$. Then, $C_2[-1,1]$ is incomplete, that is, $C_2[-1,1]$ is not a Hilbert space.

Example 1.2. *The space* $L_2[-1,1] = \overline{C_2[-1,1]}$ *with the inner product (1.1) is a Hilbert space.*

Theorem 1.1. *Let* x, y *be any two vectors in a Hilbert space, then*

$$|\langle x,y\rangle| \leq \|x\| \, \|y\| \quad (\textit{ Schwartz inequality}). \tag{1.2}$$

Proof. If $y = 0$, then the last inequality holds since $\langle x, 0\rangle = 0$. Let $y \neq 0$ for any scaler λ we have

$$0 \leq \|x-\lambda y\|^2 = \langle x-\lambda y, x-\lambda y\rangle = \langle x,x\rangle - \langle x,\lambda y\rangle - \langle \lambda y, x\rangle + \langle \lambda y, \lambda y\rangle$$

$$= \|x\|^2 - \overline{\lambda \langle y,x\rangle} - \lambda \langle y,x\rangle + |\lambda|^2 \|y\|^2 .$$

We see that if

$$\overline{\lambda} = \frac{\overline{\langle x,y\rangle}}{\|y\|^2}, \text{ then}$$

$$0 \leq \|x\|^2 - \frac{\langle y,x\rangle}{\|y\|^2}\overline{\langle y,x\rangle} - \frac{\overline{\langle y,x\rangle}}{\|y\|^2}\langle y,x\rangle + \frac{\left|\overline{\langle y,x\rangle}\right|^2}{\|y\|^4}\|y\|^2 \quad \text{or}$$

$$0 \leq \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} - \frac{|\langle x,y\rangle|^2}{\|y\|^2} + \frac{|\langle x,y\rangle|^2}{\|y\|^2} \quad \text{or}$$

$$0 \leq \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} \quad \text{or}$$

$$|\langle x,y\rangle|^2 \leq \|x\|^2 \|y\|^2 \quad \text{or}$$

$$|\langle x,y\rangle| \leq \|x\| \, \|y\| .$$

Note that the inner product is related to the norm by the following identity

$$\langle x,y\rangle = \frac{1}{4}\left[\left(\|x+y\|^2 - \|x-y\|^2\right) + i\left(\|x+iy\|^2 - \|x-iy\|^2\right)\right]. \tag{1.3}$$

A norm on an inner product space satisfies the important Parallelogram law

Theorem 1.2. *If* H *is a Hilbert space, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 , \quad \forall\, x,y \in H. \quad (\textit{Parallelogram law}) \tag{1.4}$$

Conversely, H *is a complex complete normed space with the norm* $\|\cdot\|$ *satisfying the equation* (1.4) *then* H *is a Hilbert space with the scalar product* $\langle \cdot,\cdot \rangle$ *satisfying* $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$.

Example 1.3. *The space l^p of all sequence, $x = (\xi_i) = (\xi_1, \xi_2, ...)$ such that $|\xi_1|^p + |\xi_2|^p + ...$ converges with $p \neq 2$ is not an inner product space,hence not a Hilbert space.*

Solution. Our statement means that the norm of l^p with $p \neq 2$ cannot be obtained from an inner product. We prove this by showing that the norm does not satisfies the parallelogram law (1.4). Infect let us take $x = (1, 1, 0, 0, ...) \in l^p$ and $y = (1, -1, 0, 0, ...) \in l^p$ and calculate

$$\|x\| = \|y\| = 2^{1/p}, \quad \|x + y\| = \|x - y\| = 2.$$

We now see that (1.4) is not satisfied if $p \neq 2$.

l^p is complete. Hence l^p with $p \neq 2$ with is a Banach space which is not a Hilbert space.

Example 1.4. *The space $C[a, b]$ is not an inner product space,hence not a Hilbert space.*

Solution. We show that the norm defined by

$$\|x\| = \max_{t \in [a,b]} |x(t)|$$

cannot be obtained from an inner product since this norm does not satisfy the parallelogram law (1.4). Indeed, if we take $x(t) = 1$ and $y(t) = (t - a) / (b - a)$, we have $\|x\| = 1, \ \|y\| = 1$ and

$$x(t) + y(t) = 1 + \frac{t - a}{b - a},$$

$$x(t) - y(t) = 1 - \frac{t - a}{b - a}.$$

Hence

$$\|x + y\| = 2, \|x - y\| = 1$$

and

$$\|x + y\|^2 + \|x - y\|^2 = 5 \quad but \quad 2\left(\|x\|^2 + \|y\|^2\right) = 4.$$

We now see that (1.4) is not satisfied. So, the space $C[a, b]$ is not an inner product space.

1.2 Bounded Linear Operators in H

Definition 1.2. Let H_1 and H_2 are two Hilbert space. A linear operator A is an operator such that $A : H_1 \to H_2$

$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \quad$ for all $\quad \alpha, \beta \in C \;$ and $x, y \in H_1$.

The domain of A $D(A) = \{x \in H_1, \exists Ax \in H_2\}$ is a vector space and

$R(A) = \{y = Ax, \forall x \in D(A)\}$ denotes the range of A.

A linear operator $A : H \to H$ is said to be bounded if there exist a real number $M > 0$ such that

$$\|Ax\|_H \leq M \|x\|_H \quad for\ all\ x \in H. \tag{1.5}$$

If A linear operator $A : H \to H$ is bounded with M, then

$$\|A\| = \inf M \tag{1.6}$$

is called the norm of operator A.

Example 1.5. *A bounded linear operator from $H = L_2[0,1]$ into itself is defined by*

$$Ax = tx(t)\ ,\ 0 \leq t \leq 1. \tag{1.7}$$

Solution.

$$A(\alpha x_1 + \beta x_2) = t(\alpha x_1(t) + \beta x_2(t)) = \alpha t x_1(t) + \beta t x_2(t) = \alpha A x_1 + \beta A x_2.$$

So, A is a linear operator.

$$\|Ax\|_{L_2[0,1]} = \left(\int_0^1 t^2 |x(t)|^2 dt\right)^{\frac{1}{2}} \leq \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}} = \|x\|_{L_2[0,1]} .$$

So, A is a bounded operator.

Example 1.6. *Another bounded linear operator $L_2[0,1]$ into itself is defined by*

$$Ax(t) = \int_0^1 tsx(s)ds. \tag{1.8}$$

Solution.

$$\begin{aligned} A(\alpha x_1 + \beta x_2) &= \int_0^1 ts[\alpha x_1(s) + \beta x_2(s)]\,ds \\ &= \alpha \int_0^1 tsx_1(s)ds + \beta \int_0^1 tsx_2(s)ds \\ &= \alpha A x_1 + \beta A x_2. \end{aligned}$$

So, A is the linear operator. Using the Schwartz inequality, we obtain

$$\begin{aligned} \|Ax\|_{L_2[0,1]} &= \left(\int_0^1 |Ax(t)|^2 dt\right)^{\frac{1}{2}} = \left(\int_0^1 \left|\int_0^1 tsx(s)ds\right|^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 t^2 dt \left|\int_0^1 sx(s)ds\right|^2\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \left|\int_0^1 sx(s)ds\right|^2 \end{aligned}$$

$$\leq \frac{1}{\sqrt{3}} \left(\int_0^1 s^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 |x(s)|^2 ds \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{3}} \left(\frac{1}{3} \right)^{\frac{1}{2}} \|x\|_{L_2[0,1]}$$

$$\leq \frac{1}{3} \|x\|_{L_2[0,1]} .$$

So, A is the bounded operator.

Theorem 1.3. *The norm of the bounded linear operator A is*

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{x \neq \sigma} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| . \tag{1.9}$$

Example 1.7. *A is an operator defined by $Ax = \alpha x(t)$, $A : L_2[0,1] \longrightarrow L_2[0,1]$. Show that $\|Ax\| = |\alpha|$.*

Solution.

$$\|Ax\|_{L_2[0,1]} = \left(\int_0^1 |\alpha x(t)|^2 dt \right)^{\frac{1}{2}} = |\alpha| \|x\|_{L_2[0,1]} .$$

So,

$$\|A\| = \sup_{\|x\|_{L_2[0,1]}=1} \|Ax\|_{L_2[0,1]} = |\alpha| \sup_{\|x\|_{L_2[0,1]}=1} \|x\|_{L_2[0,1]} = |\alpha| .$$

1.3 Adjoint of an Operator

Definition 1.3. Let $A : H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert space. Then the Hilbert adjoint operator A^* of A is the operator

$$A^* : H_2 \to H_1,$$

such that for all $x \in H_1$ and $y \in H_2$

$$\langle Ax, y \rangle = \langle x, A^* y \rangle .$$

Theorem 1.4. *The Hilbert adjoint operator A^* of A is unique and bounded linear operator with the norm*

$$\|A^*\| = \|A\| . \tag{1.10}$$

Definition 1.4. A bounded linear operator $A : H \longrightarrow H$ on a Hilbert space H is said to be self-adjoint if $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in H$.

Definition 1.5. A self-adjoint operator A is said to be positive if $A \geq 0$, that is $(Ax, x) \geq 0$ for all $x \in H$.

Example 1.8. *A is an operator defined on the example 1.5. Show that if $\alpha \in R^1$, then A is a self-adjoint operator.*

Solution. If $\alpha \in R^1$, then

$$\begin{aligned}\langle Ax, y\rangle &= \int_0^1 Ax(t)\overline{y(t)}dt = \int_0^1 \alpha x(t)\overline{y(t)}dt \\ &= \int_0^1 x(t)\alpha\overline{y(t)}dt = \langle x, Ay\rangle .\end{aligned}$$

A is the self adjoint operator. If $\alpha \in R^1$ and $\alpha > 0$, then the operator T is positive operator. Actually

$$\langle Ax, x\rangle = \int_0^1 \alpha x(t)\overline{x(t)}dt = \alpha \langle x, x\rangle \geq 0.$$

Example 1.9. *A is an operator defined on the example (1.7). Show that A is a self-adjoint positive operator.*

Solution.

$$\langle Au, v\rangle = \int_0^1 su(s)\overline{v(s)}ds = \int_0^1 u(s)s\overline{v(s)}ds = \langle u, Av\rangle .$$

So, A is the self-adjoint operator.

$$\langle Au, u\rangle = \int_0^1 su(s)\overline{u(s)}ds = \int_0^1 s\,|u(s)|^2\,ds \geq 0.$$

Definition 1.6. Let $A : D(A) \longrightarrow H$ be a linear operator with $\overline{D(A)} = H$. Then A is called a symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D(A)$.

If A is symmetric and $D(A) = D(A^*)$, then A is a self-adjoint operator.

Example 1.10. *Let* $Au = -\dfrac{d^2u}{dx^2} + u$, $u(a) = u(b) = 0$ *and* $H = L_2[a, b]$. *Show that A is a self-adjoint positive operator.*

Solution.

$$\begin{aligned}A(\alpha u + \beta v) &= -\frac{d^2(\alpha u + \beta v)}{dx^2} + \alpha u + \beta v \\ &= \alpha\left(-\frac{d^2u}{dx^2} + u\right) + \beta\left(-\frac{d^2u}{dx^2} + u\right) \\ &= \alpha Au + \beta Av.\end{aligned}$$

So, A is the linear operator.

Since

$$\begin{aligned}\|u\|_{L_2[a,b]} &= \left(\int_a^b \frac{dx}{\sqrt{x-a}}\right)^{1/2} = \left(2\sqrt{x-a}\Big]_a^b\right)^{1/2} = \left(2\sqrt{b-a}\right)^{1/2}, \\ u(x) &= (x-a)^{1/4} \in L_2[a,b]. \text{ But}\end{aligned}$$

$$\|Au\|_{L_2[a,b]} = \sqrt{\int_a^b \left(-\frac{5}{16}(x-a)^{-9/4} + (x-a)^{-1/4}\right)^2 dx}$$

$$= \sqrt{\int_a^b \left(\frac{25}{256}(x-a)^{-9/2} - \frac{5}{8}(x-a)^{-5/2} + (x-a)^{-1/2}\right) dx}$$

$$= \sqrt{-\frac{25}{756}(x-a)^{-7/2} - \frac{5}{12}(x-a)^{-3/2} + 2(x-a)^{1/2}}\Bigg]_a^b = \infty.$$

So, A is the unbounded operator.

$$\begin{aligned}\langle Au, v\rangle &= \int_a^b \left\{-\frac{d^2u}{dx^2} + u(x)\right\}\overline{v(x)}dx \\ &= \int_a^b u(x)\left\{\overline{-\frac{d^2v}{dx^2} + v(x)}\right\}dx \\ &= \langle u, Av\rangle\end{aligned}$$

$$D(A) = \left\{u : \left(-\frac{d^2u}{dx^2} + u\right) \in L_2[a,b], u(a) = u(b) = 0\right\}.$$

$$\begin{aligned}\|u\|_{D(A)} &= \left(\int_a^b \left|-\frac{d^2u}{dx^2} + u\right|^2 dt\right)^{\frac{1}{2}} + \|u\|_{L_2[a,b]} \\ &\le 2\|u\|_{L_2[a,b]} + \|u''\|_{L_2[a,b]}\end{aligned}$$

$$\begin{aligned}&\le 3\left(\int_a^b \left(|u(t)|^2 + |u'(t)|^2 + |u''(t)|^2\right) dt\right)^{\frac{1}{2}} \\ &= 3\|u\|_{\overset{0}{W}{}_2^{(2)}[a,b]}.\end{aligned}$$

Since

$$\int_a^b |u'(t)|^2 dt = \int_a^b u'(t)d\overline{u(t)} = -\int_a^b u''(t)\overline{u(t)}dt.$$

$$\|u'\|_{L_2[a,b]} = \sqrt{\left|\int_a^b u''(t)\overline{u(t)}dt\right|}$$

$$\leq \sqrt{\left(\int_a^b |u''(t)|^2 dt\right)^{1/2} \left(\int_a^b \left|\overline{u(t)}\right|^2 dt\right)^{1/2}}$$

$$\leq \frac{\left(\int_a^b |u''(t)|^2 dt\right)^{1/2} + \left(\int_a^b |u(t)|^2 dt\right)^{1/2}}{2}$$

$$\leq \|u\|_{D(A)} .$$

Really, we have that

$$D(A) = \overset{0}{W}{}_2^{(2)} [a,b] \quad \left(\overset{0}{W}{}_2^{(2)} [a,b] \text{ is the Sobolev's space}\right). \tag{1.11}$$

Since

$$\overline{\overset{0}{W}{}_2^{(2)} [a,b]} = L_2[a,b]$$

it follows that $D(A)$ dense in $L_2[a,b]$. Then, A is the symmetric operator in $L_2[a,b]$.

Since $D(A^*) = \overset{0}{W}{}_2^{(2)} [a,b]$. We have that $D(A^*) = D(A)$. Therefore A is the self-adjoint operator.

$$\langle Au, u\rangle = \int_a^b \left\{-\frac{d^2u}{dx^2} + u(x)\right\} \overline{u(x)}dx \tag{1.12}$$

$$= \int_a^b -\frac{d^2u}{dx^2}\overline{u(x)}dx + \int_a^b |u(x)|^2 dx$$

$$= \int_a^b \left|\frac{du}{dx}\right|^2 dx + \int_a^b |u(x)|^2 dx \geq 0.$$

So, A is the positive operator.

As a result, the operator A is positive defined self adjoint linear operator in $L_2[a,b]$.

1.4 Spectrum

Definition 1.7. Let H be a Hilbert space and $A : H \longrightarrow H$ be a linear operator with $D(A) \subset H$. We associate the operator $A_\lambda = A - \lambda I$, where $\lambda \in C$ and I is the identity operator on $D(A)$.

If A_λ has an inverse, we denote it by $R_\lambda(A)$ and we call it the resolvent operator of A, or simply, resolvent of A.

$$R_\lambda(A) = (A - \lambda I)^{-1}. \tag{1.13}$$

Definition 1.8. (Regular value, resolvent set, spectrum)

Let A be a linear operator with the $D(A) \subset H$ and H is a Hilbert space. A regular value λ of A is a complex number such that

($R1$) $R_\lambda(A)$ exists.

($R2$) $R_\lambda(A)$ is bounded.

($R3$) $R_\lambda(A)$ is defined on a set which is dense in H.

The resolvent set $\rho(A)$ of A is the set of all regular values of A. Its complement $\sigma(A) = C - \rho(A)$ is called spectrum of A , and a $\lambda \in \sigma(A)$ is called spectral value of A. Furthermore, the spectrum $\rho(A)$ is partitioned into three disjoint sets as follows.

The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_\lambda(A)$ does not exist. A $\lambda \in \sigma(A)$ is called an eigenvalue of A.

The continuous spectrum $\sigma_c(A)$ is the set such that $R_\lambda(A)$ exists and satisfies ($R3$) but not ($R2$), that is $R_\lambda(T)$ unbounded.

The residual spectrum $\sigma_r(A)$ is the set such that $R_\lambda(A)$ exists (and may be bounded or not) but does not satisfy ($R3$) , that is the domain of $R_\lambda(A)$ is not dense in H.

If $A_\lambda x = (A - \lambda I)x = 0$ for some $x \neq 0$, then $\lambda \in \sigma_p(A)$, by definition, that is, λ is an eigenvalue of A.

The vector x is called an eigenvector of A corresponding to eigenvalue λ. The subspace of $D(A)$ consisting of 0 and all eigenvectors of A corresponding to an eigenvalue λ of A is called the eigenspace of A corresponding to that eigenvalue λ.

$$\sigma(A) = \sigma_c(A) \cup \sigma_p(A) \cup \sigma_r(A), \tag{1.14}$$

$$\sigma(A) \cup \rho(A) = C.$$

Definition 1.9. Let H be a Hilbert space over the field of real numbers and for any $x \in H$, let $\|x\|$ denote the norm of x. Let J be any interval of the real line R. A function $x : J \to H$ is called an abstract function.

A function $x(t)$ is said to be continuous at the point $t_0 \in J$,

if

$$\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0;$$

if $x : J \to H$ is continuous at each point of J, Then we say that x is continuous on J and we write $x \in C[J, H]$.

Definition 1.10. The Stieltjes integral of a function $x : [a, b] \to H$ with respect to a function $y : [a, b] \to H_1$. Let H, H_1, H_2 be three Hilbert space. A bilinear operator $P : H \times H_1 \to H_2$ whose norm is less than or equal to 1, that is,

$$\|P(x, y)\| \leq \|x\| \|y\|, \tag{1.15}$$

is called a product operator. We shall agree to write $P(x,y) = xy$. Let $x : [a,b] \to H$ and $y : [a,b] \to H_1$ be two bounded functions such that the product $x(t)y(t) \in H_2$, for each $t \in [a,b]$ is linear in both x and y and

$$\|x(t)y(t)\| \leq \|x(t)\| \, \|y(t)\|$$

(for example, $x(t) = A(t)$ is an operator with domain $D[A(t)] \supset H_1$, or one of the function x,y is a scalar function). We denote the partition $(a = t_0 < t_1 < t_2 < ... < t_n = b)$ together with the points τ_i $(t_i < \tau_1 < t_{i+1}, i = 0, 1, 2, ..., n-1)$ by π

and set $|\pi|=\max_i |t_{i+1} - t_i|$. We form the Stieltjes sum

$$S_\pi = \sum_{i=1}^{n-1} x(\tau_i)\left[y(t_{i+1}) - y(t_i)\right]. \tag{1.16}$$

If the $\lim S_\pi$ exist as $|\pi| \to 0$ and defines an element I in H_2 independent of π, then I is called the Stieltjes integral of the function $x(t)$ by the function $y(t)$, and is denoted by

$$\int_a^b x(t)\, dy(t). \tag{1.17}$$

Theorem 1.5. *If $x \in C[[a,b], H]$ and $y : [a,b] \to H_1$ is of bounded variation on $[a,b]$, then the Stieltjes integral (1.17) exists.*

Consider the function $y : [a,b] \to H_1$ and the partition

$$\pi : a = t_0 < t_1 < t_2 < ... < t_n = b.$$

Form the sum

$$V = \sum_{i=1}^{n-1} \|y(t_{i+1}) - y(t_i)\|. \tag{1.18}$$

The least upper bound of the set of all possible sums V is called the (strong) total variation of the function $y(t)$ on the interval $[a,b]$ and is denoted by $V_a^b(y)$. If $V_a^b(y) < \infty$, then $y(t)$ is called an abstract function of bounded variation on $[a,b]$.

Example 1.11. *If $x \in C[[a,b], H]$ and $y : [a,b] \to H_1$ is of bounded variation on $[a,b]$, then*

$$\left\| \int_a^b x(t)\, dy(t) \right\| \leq \int_a^b \|x(t)\|\, dV_a^t[y(t)] \leq \max_{t\in[a,b]} \|x(t)\|\, V_a^b[y(t)]. \tag{1.19}$$

1.5 Application of Spectral Representation of the Unit Matrix

Consider the initial value problem for the system of linear differential equations

$$\frac{dx}{dt} + Ax = 0, t > 0, x(0) = x_0, \tag{1.20}$$

where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.21}$$

The solution of the given initial value problem (1.2) is

$$x(t) = \exp(-At)x_0 \Longleftrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \exp(-At) \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}. \tag{1.22}$$

The characteristic equation $(1-\lambda)^2 + 1 = 0$ has roots $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$. The eigenvalue $\lambda_1 = 1+i$ has the vector $\begin{pmatrix} i \\ 1 \end{pmatrix}$ as an eigenvector. The eigenvalue $\lambda_2 = 1-i$ yields an eigenvector $\begin{pmatrix} 1 \\ i \end{pmatrix}$. Therefore, the spectral representation of unit matrix

$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = -\frac{1}{2}(ix_{10} - x_{20}) \begin{pmatrix} i \\ 1 \end{pmatrix} - \frac{1}{2}(ix_{20} - x_{10}) \begin{pmatrix} 1 \\ i \end{pmatrix}, \tag{1.23}$$

and

$$\exp(-tA) \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} =$$

$$-\frac{1}{2}\exp(-(1+i)t)(ix_{10} - x_{20}) \begin{pmatrix} i \\ 1 \end{pmatrix} - \frac{1}{2}\exp(-(1-i)t)(ix_{20} - x_{10}) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} (\frac{1}{2}\exp(-(1+i)t) + \frac{1}{2}\exp(-(1-i)t))x_{10} + (\frac{i}{2}\exp(-(1+i)t) - \frac{i}{2}\exp(-(1-i)t))x_{20} \\ (-\frac{i}{2}\exp(-(1+i)t) + \frac{i}{2}\exp(-(1-i)t))x_{10} + (\frac{1}{2}\exp(-(1+i)t) + \frac{1}{2}\exp(-(1-i)t))x_{20} \end{pmatrix}$$

$$= \begin{pmatrix} \exp(-t)\cos(t)x_{10} + \exp(-t)\sin(t)x_{20} \\ -\exp(-t)\sin(t)x_{10} + \exp(-t)\cos(t)x_{20} \end{pmatrix}.$$

Hence using the formula (1.22)

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \exp(-t)\cos(t)x_{10} + \exp(-t)\sin(t)x_{20} \\ -\exp(-t)\sin(t)x_{10} + \exp(-t)\cos(t)x_{20} \end{pmatrix} \Longleftrightarrow$$

$$\begin{cases} x_1(t) = \exp(-t)\cos(t)x_{10} + \exp(-t)\sin(t)x_{20}, \\ x_2(t) = -\exp(-t)\sin(t)x_{10} + \exp(-t)\cos(t)x_{20}. \end{cases}$$

Now let us return to the problem (1.21). It is known that Euler's method for approximate solution of the initial value problem (1.21) is

$$\frac{u_k - u_{k-1}}{\tau} + Au_k = 0, k \geq 1, u_0 = x_0. \tag{1.24}$$

The solution of this problem is

$$u_k = (I + \tau A)^{-k} u_0, k \geq 1. \tag{1.25}$$

Using the formula (1.23) the solution of difference method (1.24) is obtained.

$$u_k = -\frac{1}{2}\frac{1}{(1+\tau(1+i))^k}(iu_{10} - u_{20}) \begin{pmatrix} i \\ 1 \end{pmatrix} - \frac{1}{2}\frac{1}{(1+\tau(1-i))^k}(iu_{20} - u_{10}) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1}{2(1+\tau(1+i))^k} + \frac{1}{2(1+\tau(1-i))^k}\right) u_{10} + \left(\frac{i}{2(1+\tau(1+i))^k} - \frac{i}{2(1+\tau(1-i))^k}\right) u_{20} \\ \left(-\frac{i}{2(1+\tau(1+i))^k} + \frac{i}{2(1+\tau(1-i))^k}\right) u_{10} + \left(\frac{1}{2(1+\tau(1+i))^k} + \frac{1}{2(1+\tau(1-i))^k}\right) u_{20} \end{pmatrix}.$$

Thus, representation of unit matrix permits us to obtain the solution of initial value problem for system differential equations and for difference schemes.

1.6 Projection Operator. Spectral Family

Definition 1.11. A Hilbert space H is represented as the direct sum of a closed subspace Y and its orthogonal complement $Y^{\perp}$:

$$H = Y \oplus Y^{\perp} \tag{1.26}$$

$$x = y + z \quad , \quad where \ \ y \in Y, z \in Y^{\perp}.$$

Since the sum is direct, y is unique for any given $x \in H$. Hence(1.26) defines a linear operator

$$P : H \longrightarrow H,$$

$$x \longrightarrow y = Px.$$

P is called an orthogonal projection or projection on H.

Theorem 1.6. *A bounded linear operator $P : H \longrightarrow H$ on a Hilbert space H is projection if and only if P is self-adjoint and idempotent that is, $P^2 = P$.*

Spectral family from dimensional case as follows: If matrix A has n different eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 ... < \lambda_n$. then A has an orthogonal set of n vectors $x_1, x_2, x_3, ..., x_n$, where x_j corresponds to λ_j and we write these vectors as column vectors, for convenience. This basis for H, has a unique representation:

$$x = \sum_{j=1}^{n} \gamma_j x_j \ , \quad \gamma_j = (x, x_j) = x^T \overline{x_j} \ , \tag{1.27}$$

x_j is an eigenvector of A, so that we have $Ax_j = \lambda_j x_j$.

$$Ax = \sum_{j=1}^{n} \lambda_i \gamma_j x_j. \tag{1.28}$$

We can define an operator

$$P_j : H \longrightarrow H, \tag{1.29}$$

$$x \longrightarrow \gamma_j x_j \ .$$

Obviously, P_j is the projection (orthogonal projection) of H onto the eigenspace of A corresponding to λ_j . From the equation (1.27) can be written

$$x = \sum_{j=1}^{n} P_i x \qquad hence \qquad I = \sum_{j=1}^{n} P_i, \tag{1.30}$$

where I is an identity operator on H.

Formula (1.28) becomes

$$Ax = \sum_{j=1}^{n} \lambda_i P_i x \quad hence \quad A = \sum_{j=1}^{n} \lambda_i P_i. \tag{1.31}$$

This is a representation of A in terms of projections.

Theorem 1.7. ***Spectral Theorem:*** *Let $A : H \longrightarrow H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then there exists a family of orthogonal projection $\{E(\lambda)\}, \lambda \in R$ such that*

$$\lambda_1 \leq \lambda_2 \text{ implies that } E(\lambda_1) E(\lambda_1) = E(\lambda_2) E(\lambda_1) = E(\lambda_1);$$

$$E(\lambda + \varepsilon) \to E(\lambda) \quad (strongly) \text{ as } \varepsilon \to 0^+;$$

$$E(\lambda) \to 0 \quad (strongly) \text{ as } \lambda \to -\infty,$$

and

$$E(\lambda) \to I \quad (strongly) \text{ as } \lambda \to +\infty;$$

a) A has the spectral representation

$$A = \int_{m-0}^{M} \lambda dE_\lambda, \tag{1.32}$$

where E_λ is the spectral family associate with A; the integral is to be understood in the sense of uniform operator convergence, and for all $x, y \in H$.

$$\langle Ax, y \rangle = \int_{m-0}^{M} \lambda dw(\lambda) \qquad w(\lambda) = \langle E_\lambda x, y \rangle \tag{1.33}$$

where the integral is an ordinary Riemann-Stieltjes integral.

b) If P is a polynomial in λ with real coefficients,

$$P(\lambda) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_0$$

then the operator $P(A)$ defined by

$$P(A) = \alpha_n A^n + \alpha_{n-1} A^{n-1} + \ldots + \alpha_0 I$$

has the spectral representation

$$P(A) = \int_{m-0}^{M} P(\lambda) dE_\lambda \tag{1.34}$$

and for all $x, y \in H$.

Theorem 1.8. *Let $A : D(A) \longrightarrow H$ be a self-adjoint linear operator, where H is a complex Hilbert space and $D(A)$ is dense in H. Then A has the spectral representation*

$$A = \int_m^\infty \lambda dE_\lambda \qquad and \qquad I = \int_m^\infty dE_\lambda. \tag{1.35}$$

If F is the continuously bounded function on $[m, \infty]$, then

$$F(A) = \int_m^M F(\lambda)\, dE_\lambda. \tag{1.36}$$

Note that, from theorem 1.8 and property of E_λ and Stieltjes integral it follows

$$\|F(A)x\| \leq \int_m^\infty |f(\lambda)|\, d\|E_\lambda x\| \leq \int_m^\infty |f(\lambda)|\, dE_\lambda \|x\|$$

$$\leq \sup_{m\leq\lambda<\infty} |f(\lambda)| \int_m^\infty dE_\lambda \|x\|$$

$$\|F(A)x\| \leq \sup_{m\leq\lambda<\infty} |f(\lambda)|\, \|x\|$$

$$\|F(A)\| \leq \sup_{m\leq\lambda<\infty} |f(\lambda)|. \tag{1.37}$$

Example 1.12. *A is an operator defined on the example 1.10. Show that*

$$\|\exp(-At)\| \leq e^{-t}, \tag{1.38}$$

$$\left\|\cos\left(A^{1/2}t\right)\right\| \leq 1, \;\; \left\|A^{1/2}\sin\left(A^{1/2}t\right)\right\| \leq 1.$$

Solution. *Using the spectral representation of the self-adjoint positive defined operators we can write*

$$\exp(-At)\varphi = \int_1^\infty \exp(-\mu t) dE_\mu \varphi,$$

where (E_μ) is the spectral family associated with A. Therefore, for any $t \geq 0$ we have that

$$\|\exp(-At)\|_{H\to H} \leq \sup_{1\leq\mu<\infty} |\exp(-\mu t)| = \exp(-t).$$

The estimate (1.38) is proved. Using the spectral representation of the self-adjoint positive defined operators it can be writen

$$e^{\pm iA^{1/2}t}\varphi = \int_1^\infty e^{\pm it\mu^{1/2}} dE_\mu \varphi.$$

Therefore, using the last theorem

$$\left\|e^{\pm iA^{1/2}t}\right\| \leq \sup_{1\leq\mu<\infty} |e^{\pm it\mu^{1/2}}| = 1$$

is obtained.

So,

$$\left\|\cos\left(A^{1/2}t\right)\right\| = \left\|\frac{e^{iA^{1/2}t}+e^{-iA^{1/2}t}}{2}\right\|$$

$$\leq \frac{1}{2}\left[\left\|e^{iA^{1/2}t}\right\| + \left\|e^{-iA^{1/2}t}\right\|\right] \leq 1$$

and

$$\left\|A^{1/2}\sin\left(A^{1/2}t\right)\right\| = \left\|\frac{e^{iA^{1/2}t}-e^{-iA^{1/2}t}}{2i}\right\|$$

$$\leq \frac{1}{2\left|i\right|}\left[\left\|e^{iA^{1/2}t}\right\| + \left\|e^{-iA^{1/2}t}\right\|\right] \leq 1.$$

2 DIFFERENCE SCHEMES OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

2.1 The Problem

It is known (see, for example, [1]-[2]) that various initial boundary value problems for the hyperbolic equations can be reduced to the initial value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \\ u(0) = \varphi, u'(0) = \psi \end{cases} \tag{2.1}$$

for differential equation in a Hilbert space H with self -adjoint positive defined operator A.

In the paper [3] the first order of accuracy difference scheme for approximately solving problem (2.1)

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \\ f_k = f(t_k),\ t_k = k\tau,\ 1 \leq k \leq N-1, N\tau = 1, \\ \tau^{-1}(u_1 - u_0) + iA^{1/2}u_1 = iA^{1/2}u_0 + \psi, \qquad u_0 = \varphi \end{cases}$$

was considered. The stability estimates for the solution of this difference scheme were obtained. In the papers [4]-[5] the similar results for the solutions of the second order of accuracy of the following difference schemes

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k + \frac{\tau^2}{4}A^2u_{k+1} = f_k, \\ f_k = f(t_k), 1 \leq k \leq N-1, N\tau = 1, \\ \tau^{-1}(u_1 - u_0) + iA^{1/2}(I + \frac{i\tau}{2}A^{1/2})u_1 = z_1, \\ z_1 = (I + i\tau A^{1/2})\psi + \frac{\tau}{2}f_0 + (iA^{\frac{1}{2}} - \tau A)u_0, f_0 = f(0), u_0 = \varphi, \end{cases}$$

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{4}A(u_{k+1} + 2u_k + u_{k-1}) = f_k, \\ f_k = f(t_k), 1 \leq k \leq N-1, N\tau = 1, \\ \tau^{-1}(u_1 - u_0) + \frac{i}{2}A^{1/2}(u_1 + u_0) = z_1, \\ z_1 = (I + \frac{i\tau}{2}A^{1/2})\psi + \frac{\tau}{2}f_0 + (iA^{\frac{1}{2}} - \frac{\tau A}{2})u_0,\ f_0 = f(0), u_0 = \varphi \end{cases}$$

for approximately solving initial value problem (2.1) were obtained. However, for practical realization of these difference schemes it is necessary to construct an operator $A^{1/2}$.This is the difficult action for a computer.Therefore, in spite of theoretical results the role of their application of a numerical solution for an initial value problem is not great. In the paper [6] the first and second order of accuracy difference schemes generated by the integer power of A approximately solving initial boundary value problem (2.1) were presented. The stability estimates for the solution of these difference schemes were obtained.

The well-posedness of the nonlocal boundary value problems for parabolic, elliptic equations and equations of mixed types have been studied extensively, see for instance [7]-[23] and the references therein.

In the present work the nonlocal boundary value problem for hyperbolic equations are considered.

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \\ u(0) = \alpha u(1) + \varphi, u'(0) = \beta u'(1) + \psi \end{cases} \tag{2.2}$$

in a Hilbert space H with self -adjoint positive defined operator A. The stability estimates for solution of the nonlocal boundary problem (2.2) are obtained. In applications this abstract result permit us to obtain the stability estimates for the solution of nonlocal boundary value problem for the hyperbolic equations. The first and second order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem are presented. The stability estimates for the solution of these difference schemes are obtained. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

2.2 The Differential Hyperbolic Equation

A function $u(t)$ is called a *solution* of the problem (2.2) if the following conditions are satisfied:

i) $u(t)$ is twice continuously differentiable on the interval (0.1) and continuously differentiable on the segment $[0, 1]$. The derivatives at the endpoints of the segment are understood as the corresponding unilateral derivatives.

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0, 1]$, and the function $Au(t)$ is continuous on the segment $[0, 1]$.

iii) $u(t)$ satisfies the equations and the nonlocal boundary condition (2.2).

If the function $f(t)$ is not only continuous, but also continuously differentiable on [0,1],$\varphi \in D(A)$ and $\psi \in D(A^{\frac{1}{2}})$, it is known that (see,for example,[2]) the formula

$$u(t) = c(t)T\{(1 - \beta c(1))[\alpha \int_0^1 s(1 - \lambda)f(\lambda)d\lambda + \varphi] \tag{2.3}$$

$$+\alpha s(1)[\beta \int_0^1 c(1 - \lambda)f(\lambda)d\lambda + \psi]\} + s(t)T\{(1 - \alpha c(1))[\beta \int_0^1 c(1 - \lambda)f(\lambda)d\lambda + \psi]$$

$$-\beta As(1)[\alpha \int_0^1 s(1 - \lambda)f(\lambda)d\lambda + \varphi]\} + \int_0^t s(t - \lambda)f(\lambda)d\lambda$$

gives a solution of problem (2.2). Here

$$T = (1 + \alpha\beta - (\alpha + \beta)c(1))^{-1},$$

$$c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}, s(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.$$

Theorem 2.1. *Suppose that* $\varphi \in D(A)$,$\psi \in D(A^{\frac{1}{2}})$ *and* $f(t)$ *are continuously differentiable on* $[0,1]$ *function and* $|1+\alpha\beta| > |\alpha+\beta|$. *Then there is a unique solution of the problem (2.2) and the stability inequalities*

$$\max_{0\leq t\leq 1} \| u(t) \|_H \leq M \left[\| \varphi \|_H + \| A^{-1/2}\psi \|_H + \max_{0\leq t\leq 1} \| A^{-1/2} f(t) \|_H\right], \tag{2.4}$$

$$\max_{0\leq t\leq 1} \| A^{1/2}u(t) \|_H \leq M \left[\| A^{1/2}\varphi \|_H + \| \psi \|_H + \max_{0\leq t\leq 1} \| f(t) \|_H\right], \tag{2.5}$$

$$\max_{0\leq t\leq 1} \| \frac{d^2u(t)}{dt^2} \|_H + \max_{0\leq t\leq 1} \| Au(t) \|_H \leq M[\| A\varphi \|_H + \| A^{1/2}\psi \|_H \tag{2.6}$$
$$+ \| f(0) \|_H + \int_0^1 \| f'(t) \|_H \, dt]$$

hold, where M *does not depend on* $f(t)$, $t \in [0,1]$, *and* φ, ψ.

Proof. From the symmetry and positivity properties of the operator A it follows that

$$\|T\|_{H\to H} \leq \frac{1}{|1+\alpha\beta| - |\alpha+\beta|} \tag{2.7}$$

$$\|c(t)\|_{H\to H} \leq 1, \|A^{\frac{1}{2}}s(t)\|_{H\to H} \leq 1. \tag{2.8}$$

Using the formula (2.3)and estimates (2.7) and (2.8) we obtain

$$\|u(t)\|_H \leq \|c(t)\|_{H\to H}\|T\|_{H\to H}\{(1+|\beta|\|c(1)\|_{H\to H})$$
$$\times[|\alpha| \int_0^1 \|A^{\frac{1}{2}}s(1-\lambda)\|_{H\to H}\|A^{-\frac{1}{2}}f(\lambda)\|_H d\lambda + \|\varphi\|_H]$$
$$+|\alpha|\|A^{\frac{1}{2}}s(1)\|_{H\to H}[|\beta| \int_0^1 \|c(1-\lambda)\|_{H\to H}\|A^{-\frac{1}{2}}f(\lambda)\|_H d\lambda + \|A^{-\frac{1}{2}}\psi\|_H]\}$$
$$+\|A^{\frac{1}{2}}s(t)\|_{H\to H}\|T\|_{H\to H}\{(1+|\alpha|\|c(1)\|_{H\to H})$$
$$\times[|\beta| \int_0^1 \|c(1-\lambda)\|_{H\to H}\|A^{-\frac{1}{2}}f(\lambda)\|_H d\lambda + \|A^{-\frac{1}{2}}\psi]\|_H + |\beta|\|A^{\frac{1}{2}}s(1)\|_{H\to H}$$
$$\times[|\alpha| \int_0^1 \|A^{\frac{1}{2}}s(1-\lambda)\|_{H\to H}\|A^{-\frac{1}{2}}f(\lambda)\|_H d\lambda + \|\varphi\|_H]\}$$
$$+ \int_0^t \|A^{\frac{1}{2}}s(t-\lambda)\|_{H\to H}\|A^{-\frac{1}{2}}f(\lambda)\|_H d\lambda$$

$$\leq M\left[\| \varphi \|_H + \| A^{-1/2}\psi \|_H + \max_{0\leq t\leq 1} \| A^{-1/2}f(t) \|_H\right].$$

Applying $A^{\frac{1}{2}}$ to the formula (2.3)and using the estimates (2.7) and (2.8)in a similar manner we obtain

$$\|A^{\frac{1}{2}}u(t)\|_H \leq \|c(t)\|_{H\to H}\|T\|_{H\to H}\{(1+|\beta|\|c(1)\|_{H\to H})$$

$$\times[|\alpha|\int_0^1 \|A^{\frac{1}{2}}s(1-\lambda)\|_{H\to H}\|f(\lambda)\|_H d\lambda + \|A^{\frac{1}{2}}\varphi\|_H]$$

$$+|\alpha|\|A^{\frac{1}{2}}s(1)\|_{H\to H}[|\beta|\int_0^1 \|c(1-\lambda)\|_{H\to H}\|f(\lambda)\|_H d\lambda + \|\psi\|_H]\}$$

$$+\|A^{\frac{1}{2}}s(t)\|_{H\to H}\|T\|_{H\to H}\{(1+|\alpha|\|c(1)\|_{H\to H})$$

$$\times[|\beta|\int_0^1 \|c(1-\lambda)\|_{H\to H}\|f(\lambda)\|_H d\lambda + \|\psi]\|_H + |\beta|\|A^{\frac{1}{2}}s(1)\|_{H\to H}$$

$$\times[|\alpha|\int_0^1 \|A^{\frac{1}{2}}s(1-\lambda)\|_{H\to H}\|f(\lambda)\|_H d\lambda + \|A^{\frac{1}{2}}\varphi\|_H]\}$$

$$+\int_0^t \|A^{\frac{1}{2}}s(t-\lambda)\|_{H\to H}\|f(\lambda)\|_H d\lambda$$

$$\leq M\left[\| A^{\frac{1}{2}}\varphi \|_H + \| \psi \|_H + \max_{0\leq t\leq 1} \| f(t) \|_H\right].$$

Now, we obtain the estimate for $\| Au(t) \|_H$. Applying A to the formula (2.3) and using an integration by parts, we can write the formula

$$Au(t) = c(t)T\{(1-\beta c(1))[\alpha[f(1) - c(1)f(0) - \int_0^1 c(1-\lambda)f'(\lambda)d\lambda] + A\varphi]$$

$$+\alpha As(1)[\beta[-s(1)f(1) + \int_0^1 s(1-\lambda)f'(\lambda)d\lambda] + \psi]\}$$

$$+As(t)T\{(1-\alpha c(1))[\beta[-s(1)f(1) + \int_0^1 s(1-\lambda)f'(\lambda)d\lambda] + \psi]$$

$$-\beta As(1)[\alpha[f(1) - c(1)f(0) - \int_0^1 c(1-\lambda)f'(\lambda)d\lambda] + A\varphi]\}$$

$$+f(1)-c(t)f(0)-\int_0^t c(t-\lambda)f'(\lambda)d\lambda.$$

Using the last formula and estimates (2.7), (2.8),we can obtain

$$\|Au(t)\|_H \le \|c(t)\|_{H\to H}\|T\|_{H\to H}\{(1+|\beta|\|c(1)\|_{H\to H})$$

$$\times[|\alpha|[\|f(1)\|_H+\|c(1)\|_{H\to H}\|f(0)\|_H+\int_0^1 \|c(1-\lambda)\|_{H\to H}\|f'(\lambda)\|_H d\lambda]+\|A\varphi\|_H]$$

$$+\alpha\|A^{\frac{1}{2}}s(1)\|_{H\to H}[|\beta|[\|A^{\frac{1}{2}}s(1)\|f(1)\|_H+\int_0^1 \|A^{\frac{1}{2}}s(1-\lambda)\|_{H\to H}\|f'(\lambda)\|_H d\lambda]$$

$$+\|A^{\frac{1}{2}}\psi\|_H]\}+\|A^{\frac{1}{2}}s(t)\|_{H\to H}\|T\|_{H\to H}\{(1+|\alpha|\|c(1)\|_{H\to H})[|\beta|[\|A^{\frac{1}{2}}s(1)\|_{H\to H}\|f(1)\|_H$$

$$+\int_0^1 \|A^{\frac{1}{2}}s(1-\lambda)\|_{H\to H}\|f'(\lambda)\|_H d\lambda]+\|A^{\frac{1}{2}}\psi\|_H]$$

$$+|\beta|\|A^{\frac{1}{2}}s(1)\|_{H\to H}[|\alpha|[\|f(1)\|_H+\|c(1)\|_{H\to H}\|f(0\|_H)+$$

$$\int_0^1 \|c(1-\lambda)\|_{H\to H}\|f'(\lambda)\|_H d\lambda]+\|A\varphi\|_H]\}+\|f(1)\|_H+\|c(t)\|_{H\to H}\|f(0)\|_H$$

$$+\int_0^t \|c(t-\lambda)\|_{H\to H}\|f'(\lambda)\|_H d\lambda \le M\,[\|\ A\varphi\ \|_H$$

$$+\|\ A^{1/2}\psi\ \|_H+\|\ f(0)\ \|_H+\int_0^1 \|\ f'(t)\ \|_H\ dt.$$

Then from the last estimate, it follows that

$$\max_{0\le t\le 1}\|\ Au(t)\ \|_H\le M\left[\|\ A\varphi\ \|_H+\|\ A^{1/2}\psi\ \|_H+\|\ f(0)\ \|_H+\max_{0\le t\le 0}\|\ f'(t)\ \|_H\right].$$

By the last estimate and the triangular inequality there follows the estimate (2.6). Theorem 2.1 is proved.

First,for application of Theorem 2.1 we consider the mixed problem for wave equation

$$\begin{cases} u_{tt}-(a(x)u_x)_x+u=f(t,x), 0<t<1,\ 0<x<1,\\ u(0,x)=\alpha u(1,x)+\varphi(x), u_t(0,x)=\beta u_t(1,x)+\psi(x), 0\le x\le 1,\\ u(t,0)=u(t,1), u_x(t,0)=u_x(t,1), 0\le t\le 1.\end{cases} \tag{2.9}$$

The problem (2.9) has a unique smooth solution $u(t,x)$ for $|1+\alpha\beta|>|\alpha|+|\beta|$ and the smooth $a(x)\ge 0(x\in(0,1))$, $\varphi(x)$, $\psi(x)$ $(x\in[0,1])$ and $f(t,x)$ $(t,x\in(0,1))$ functions. This allows us to reduce the mixed problem (2.9) to the nonlocal boundary value problem (2.2) in Hilbert space H with a self- adjoint positive defined operator A defined by (2.9). Let us give a number of corollaries of the abstract theorem 2.1.

Theorem 2.2. *For solutions of the mixed problem (2.9) the stability inequalities*

$$\max_{0\le t\le 1} \| u \|_{W_2^1[0,1]} \le M \left[\max_{0\le t\le 1} \| f \|_{L_2[0,1]} + \| \varphi \|_{W_2^1[0,1]} + \| \psi \|_{L_2[0,1])} \right],$$

$$\max_{0\le t\le 1} \| u \|_{W_2^2[0,1]} + \max_{0\le t\le 1} \| u_{tt} \|_{L_2[0,1]}$$

$$\le M \left[\max_{0\le t\le 1} \| f_t \|_{L_2[0,1]} + \| f(0) \|_{L_2[0,1]} + \| \varphi \|_{W_2^2[0,1]} + \| \psi \|_{W_2^1[0,1])} \right]$$

hold, where M does not depend on $f(t,x)$ and $\varphi(x), \psi(x)$.

The proof of this theorem is based on the abstract Theorem 2.1 and the symmetry properties of the space operator generated by the problem (2.9).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space R^n $(0 < x_k < 1,\ \ 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$ we consider the mixed boundary value problem for the multi-dimensional hyperbolic equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = f(t,x), \tag{2.10}$$

$$x = (x_1, \dots, x_n) \in \Omega, \quad 0 < t < 1, \ ,$$

$$u(0,x) = \alpha u(1,x) + \varphi(x), \frac{\partial u(0,x)}{\partial t} = \beta \frac{\partial u(1,x)}{\partial t} + \psi(x), \quad x \in \overline{\Omega},$$

$$u(t,x) = 0, \quad x \in S, \quad 0 \le r \le n,$$

where $a_r(x), (x \in \Omega), \varphi(x), \psi(x)$ $(x \in \overline{\Omega})$ and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions and $a_r(x) \ge 0$.

We introduce the Hilbert spaces $L_2(\overline{\Omega})$-is the space of the all integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \}^{\frac{1}{2}}.$$

The problem (2.10) has a unique smooth solution $u(t,x)$ for $|1 + \alpha\beta| > |\alpha| + |\beta|$ and the smooth $a_r(x) \ge 0$ and $f(t,x)$ functions. This allows us to reduce the mixed problem (2.10) to the nonlocal boundary value problem (2.2) in Hilbert space H with a self- adjoint positive defined operator A defined by (2.10). Let us give a number of corollaries of the abstract theorem 2.1.

Theorem 2.3. *For solutions of the mixed problem (2.10) the stability inequalities*

$$\max_{0\le t\le 1} \| u \|_{W_2^1(\overline{\Omega})} \le M \left[\max_{0\le t\le 1} \| f \|_{L_2(\overline{\Omega})} + \| \varphi \|_{W_2^1(\overline{\Omega})} + \| \psi \|_{L_2(\overline{\Omega})} \right],$$

$$\max_{0\le t\le 1} \| u \|_{W_2^2(\overline{\Omega})} + \max_{0\le t\le 1} \| u_{tt} \|_{L_2(\overline{\Omega})}$$

$$\le M \left[\max_{0\le t\le 1} \| f_t \|_{L_2(\overline{\Omega})} + \| f(0) \|_{L_2(\overline{\Omega})} + \| \varphi \|_{W_2^2(\overline{\Omega})} + \| \psi \|_{W_2^1(\overline{\Omega})} \right]$$

hold, where M does not depend on $f(t,x)$ and $\varphi(x), \psi(x)$.

The proof of this theorem is based on the abstract Theorem 2.1 and the symmetry properties of the space operator generated by the problem (2.10).

Note that the stability estimate (2.4)is not satisfied for the general α and β. Let us give an example. Let A be the operator acting in $H = L_2[0,1]$ defined by the formula $Av(x) = -v''(x)$, with the domain $D(A) = \{v(x) : v''(x) \in L_2[0,1],\ v(0) = 0,\ v(1) = 0\}$. Evidently, the operator A is a positive defined self-adjoint operator in the Hilbert space $H = L_2[0,1]$. Now let $\alpha = 1, \beta = 1, f(t,x) = 0$. Then the problem (2.2) turns into the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < t < 1, \ \ 0 < x < 1, \tag{2.11}$$

$$u(0,x) = u(1,x), \quad u_t(0,x) = u_t(1,x), \ \ 0 \le x \le 1,$$

$$u(t,0) = 0, \ \ u_x(t,1) = 0, \ \ 0 \le t \le 1.$$

From the stability estimate (2.4) it follows that

$$u(t,x) = 0.$$

But,the corresponding counterexample of the nontrivial solution of the mixed problem (2.10) can be given by

$$u(t,x) = \sum_{k=1}^{\infty} (b_k \sin 2k\pi t + a_k \cos 2k\pi t) \sin 2k\pi x,$$

where $a_k, b_k, k = 1, 2, \cdots$ are an arbitrary numbers.

2.3 The First Order of Accuracy Difference Schemes

We consider the first order of accuracy difference scheme for approximately solving the boundary value problem (2.2)

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \quad f_k = f(t_{k+1}),\ t_{k+1} \\ = (k+1)\tau,\ 1 \le k \le N-1, N\tau = 1, \\ u_0 = \alpha u_N + \varphi, \tau^{-1}(u_1 - u_0) = \beta \tau^{-1}(u_N - u_{N-1}) + \psi. \end{cases} \tag{2.12}$$

A study of discretization ,over time only, of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables,A, is replaced by the difference operators A_h that act in the Hilbert spaces and are uniformly self-adjoint positive defined in h for $0 < h \le h_0$.

We are interested to study the stability of solutions of the difference scheme (2.12) under the assumption that

$$|1 + \alpha\beta| > |\alpha + \beta|. \tag{2.13}$$

We have not been able to obtain the discrete analogue of estimates (2.4),(2.5)and (2.6) under the assumption (2.13) for the solution of the difference scheme (2.12).Neverthe-less, we can established the discrete analogue of estimates (2.4),(2.5)and (2.6) under the more strong assumption than (2.13).

Theorem 2.4. *Let $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$ and $1 > |\alpha||\beta| + |\alpha| + |\beta|$. Then for the solution of the difference scheme (2.12) the stability inequalities*

$$\|u_k\|_H \le M\{\sum_{s=1}^{N-1} \|A^{-1/2} f_s\|_H \tau + \| A^{-1/2}\psi \|_H + \| \varphi \|_H\}, k = 0, 2, ...N, \tag{2.14}$$

$$\|u_1\|_H \le M[\| \varphi \|_H + \| (I + i\tau A^{1/2}) A^{-1/2} \psi \|_H],$$

$$\|A^{1/2} u_k\|_H \le M\{\sum_{s=1}^{N-1} \|f_s\|_H \tau + \| \psi \|_H + \| A^{1/2}\varphi \|_H\}, k = 0, 2, ...N, \tag{2.15}$$

$$\|A^{1/2} u_1\|_H \le M[\| A^{1/2}\varphi \|_H + \| (I + i\tau A^{1/2}) \psi \|_H],$$

$$\|Au_k\|_H \le M\{\sum_{s=2}^{N-1} \| f_s - f_{s-1} \|_H] + \| f_1 \|_H + \| A^{1/2}\psi \|_H + \| A\varphi \|_H\}, \tag{2.16}$$

$$\|Au_1\|_H \le M[\| A\varphi \|_H + \| (I + i\tau A^{1/2}) A^{1/2} \psi \|_H]$$

hold ,where M does not depend on $f_s, 1 \le s \le N-1$ and φ, ψ.

Proof. We will write the formula for the solution of the difference scheme (2.12). It is easy to show that(see [6]) there are unique solution of the problem

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \\ \quad f_k = f(t_{k+1}),\ t_{k+1} = (k+1)\tau,\ 1 \le k \le N-1, N\tau = 1, \\ u_0 = \mu, \tau^{-1}(u_1 - u_0) = \omega \end{cases} \tag{2.17}$$

and for the solutions of these problems the following formulas hold:

$$u_0 = \mu, u_1 = \mu + \tau\omega,$$

$$u_k = \frac{1}{2}\left[R^{k-1} + \tilde{R}^{k-1}\right]\mu + (R - \tilde{R})^{-1}\tau(R^k - \tilde{R}^k)\omega$$

$$- \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2}\left[R^{k-s} - \tilde{R}^{k-s}\right] f_s, 2 \le k \le N, \tag{2.18}$$

where $R = \left(I + i\tau A^{1/2}\right)^{-1}$, $\tilde{R} = \left(I - i\tau A^{1/2}\right)^{-1}$. Applying the last formula and the nonlocal boundary conditions $u_0 = \alpha u_N + \varphi, \tau^{-1}(u_1 - u_0) = \beta\tau^{-1}(u_N - u_{N-1}) + \psi$,we can write

$$\mu = \alpha[\frac{1}{2}\left[R^{N-1} + \tilde{R}^{N-1}\right]\mu + (R - \tilde{R})^{-1}\tau(R^N - \tilde{R}^N)\omega$$

$$- \sum_{s=1}^{N-1} \frac{\tau}{2i} A^{-1/2}\left[R^{N-s} - \tilde{R}^{N-s}\right] f_s] + \varphi,$$

$$\omega = \beta[\frac{A^{1/2}}{2i}\left[R^{N-1} - \tilde{R}^{N-1}\right]\mu + (R - \tilde{R})^{-1}\frac{\tau A^{1/2}}{i}(R^N + \tilde{R}^N)\omega$$

$$- \sum_{s=1}^{N-2} \frac{\tau}{4}\left[R^{N-s} + \tilde{R}^{N-s}\right] f_s + \tau R\tilde{R} f_{N-1}] + \psi.$$

Using the last two formulas, we obtain

$$\mu = T_\tau[1-\beta(R-\tilde{R})^{-1}\frac{\tau A^{1/2}}{i}(R^N+\tilde{R}^N)][-\alpha\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s]+\varphi] \quad (2.19)$$

$$+\alpha(R-\tilde{R})^{-1}\tau(R^N-\tilde{R}^N)[-\beta\sum_{s=1}^{N-2}\frac{\tau}{4}\left[R^{N-s}+\tilde{R}^{N-s}\right]f_s+\beta\tau R\tilde{R}f_{N-1}]+\psi],$$

$$\omega = T_\tau\{[1-\alpha[\frac{1}{2}\left[R^{N-1}+\tilde{R}^{N-1}\right]]][-\beta\sum_{s=1}^{N-2}\frac{\tau}{4}\left[R^{N-s}+\tilde{R}^{N-s}\right]f_s+\beta\tau R\tilde{R}f_{N-1} \quad (2.20)$$

$$+\psi]+\beta\frac{A^{1/2}}{2i}\left[R^{N-1}-\tilde{R}^{N-1}\right][-\alpha\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s+\varphi]\},$$

where

$$T_\tau = (I-\frac{\alpha}{2}(R^{N-1}+\tilde{R}^{N-1})-\frac{\beta}{2}(\tilde{R}^{-1}R^{N-1}+R^{-1}\tilde{R}^{N-1})+\alpha\beta R^{N-1}\tilde{R}^{N-1})^{-1}.$$

Hence, for the formal solution of the nonlocal boundary value problem (2.12) can be use the formulas (2.18),(2.19),(2.20). For substantiation of these formulas can be need to obtain the estimates (2.14),(2.15),(2.16).From the symmetry and positivity properties of the operator A it follows that

$$\| T_\tau \|_{H\mapsto H} \leq \frac{1}{1-|\alpha||\beta|-|\alpha|-|\beta|} \quad (2.21)$$

and the estimates

$$\begin{cases} \|R\|_{H\mapsto H}\leq 1, & \|\tilde{R}\|_{H\mapsto H}\leq 1, \\ \|R\tilde{R}^{-1}\|_{H\mapsto H}\leq 1, & \|\tilde{R}R^{-1}\|_{H\mapsto H}\leq 1, \\ \|\tau A^{1/2}R\|_{H\mapsto H}\leq 1, & \|\tau A^{1/2}\tilde{R}\|_{H\mapsto H}\leq 1. \end{cases} \quad (2.22)$$

Using the formulas (2.19),(2.20)and estimates (2.21) and (2.22) we obtain

$$\|\mu\|_H \leq \|T_\tau\|_{H\to H}[1+|\beta|\|\tilde{R}^{-1}R^{N-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{N-1}\|_{H\to H}] \quad (2.23)$$

$$\times[|\alpha|\sum_{s=1}^{N-1}\frac{\tau}{2}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H]+\|\varphi\|_H]$$

$$+|\alpha|\frac{1}{2}(\|\tilde{R}^{-1}R^{N-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{N-1}\|_{H\to H})$$

$$\times[|\beta|\sum_{s=1}^{N-2}\frac{\tau}{4}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H$$

$$+|\beta|\tau\|R\tilde{R}\|_{H\to H}\|A^{-1/2}f_{N-1}\|_H]+\|A^{-1/2}\psi\|_H]$$

$$\leq M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\parallel A^{-1/2}\psi\parallel_H+\parallel\varphi\parallel_H\}.$$

$$\|A^{-\frac{1}{2}}\omega\|_H\leq\|T_\tau\|_{H\to H}\{[1+|\alpha|[\frac{1}{2}\left[\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H}\right]] \tag{2.24}$$

$$\times[|\beta|\sum_{s=1}^{N-2}\frac{\tau}{4}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H$$

$$+|\beta|\tau\|R\tilde{R}\|_{H\to H}\|A^{-1/2}f_{N-1}\|_H+\|A^{-1/2}\psi\|_H]$$

$$+|\beta|\frac{1}{2}\left[\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H}\right]$$

$$\times[|\alpha|\sum_{s=1}^{N-1}\frac{\tau}{2}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H+\|\varphi\|_H]\}$$

$$\leq M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\parallel A^{-1/2}\psi\parallel_H+\parallel\varphi\parallel_H\}.$$

Applying $A^{\frac{1}{2}}$ to the formulas (2.19),(2.20)and using the estimates (2.21) and (2.22) in a similar manner we obtain

$$\|A^{\frac{1}{2}}\mu\|_H\leq\|T_\tau\|_{H\to H}[1+|\beta|[\|\tilde{R}^{-1}R^{N-1}\|_{H\to H}+\|R\tilde{R}^{N-1}\|_{H\to H}] \tag{2.25}$$

$$\times[|\alpha|\sum_{s=1}^{N-1}\frac{\tau}{2}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_s\|_H]+\|A^{\frac{1}{2}}\varphi\|_H]$$

$$+|\alpha|\frac{1}{2}\|\tilde{R}^{-1}R^{N-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{N-1}\|_{H\to H})$$

$$\times[|\beta|\sum_{s=1}^{N-2}\frac{\tau}{4}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_s\|_H$$

$$+|\beta|\tau\|R\tilde{R}\|_{H\to H}\|f_{N-1}\|_H]+\|\psi\|_H]$$

$$\leq M\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\parallel\psi\parallel_H+\parallel A^{\frac{1}{2}}\varphi\parallel_H\}.$$

$$\|\omega\|_H\leq\|T_\tau\|_{H\to H}\{[1+|\alpha|\frac{1}{2}[\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H}]] \tag{2.26}$$

$$\times[|\beta|\sum_{s=1}^{N-2}\frac{\tau}{4}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_s\|_H$$

$$+|\beta|\tau\|R\tilde{R}\|_{H\to H}\|f_{N-1}\|_H+\|\psi\|_H]$$

$$+|\beta|\frac{1}{2}\left[\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H}\right]$$

$$\times[|\alpha|\sum_{s=1}^{N-1}\frac{\tau}{2}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_s\|_H+\|A^{\frac{1}{2}}\varphi\|_H]\}$$

$$\leq M\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \| \psi \|_H + \| A^{\frac{1}{2}}\varphi \|_H\}.$$

Now, we obtain the estimates for $\| A\mu \|_H, \|A^{\frac{1}{2}}\omega\|_H$. Applying A to the formulas (2.19), (2.20)and using the Abel's formula, we can write

$$A\mu = T_\tau[1-\beta(\tilde{R}^{-1}R^{N-1}+R^{-1}\tilde{R}^{N-1})][-\alpha[\sum_{s=2}^{N-1}\frac{1}{2}\left[R^{N-s}+\tilde{R}^{N-s}\right](f_{s-1}-f_s)$$

$$+(R^{N-1}+\tilde{R}^{N-1})f_1-(R+\tilde{R})f_{N-1}]+A\varphi]$$

$$+\alpha i(\tilde{R}^{-1}R^{N-1}-R^{-1}\tilde{R}^{N-1})[i\beta[\sum_{s=2}^{N-2}\frac{1}{4}\left[R^{N-s}-\tilde{R}^{N-s}\right](f_{s-1}-f_s)$$

$$-(R^N-\tilde{R}^N)f_1-(R^2-\tilde{R}^2)f_{N-1}+R\tilde{R}f_{N-1}]+A^{\frac{1}{2}}\psi],$$

$$A^{\frac{1}{2}}\omega = T_\tau\{[1-\alpha[\frac{1}{2}\left[R^{N-1}+\tilde{R}^{N-1}\right]]][i\beta[\sum_{s=2}^{N-2}\frac{1}{4}\left[R^{N-s}-\tilde{R}^{N-s}\right](f_{s-1}-f_s)$$

$$-(R^N-\tilde{R}^N)f_1-(R^2-\tilde{R}^2)f_{N-1}+R\tilde{R}f_{N-1}]+A^{\frac{1}{2}}\psi]$$

$$+\beta\frac{1}{2i}\left[R^{N-1}-\tilde{R}^{N-1}\right][-\alpha[\sum_{s=2}^{N-1}\frac{1}{2}\left[R^{N-s}+\tilde{R}^{N-s}\right](f_{s-1}-f_s)$$

$$+(R^{N-1}+\tilde{R}^{N-1})f_1-(R+\tilde{R})f_{N-1}]+A\varphi]\}.$$

Using the last two formulas and the estimates (2.21) and (2.22) we obtain

$$\|A\mu\|_H \leq \|T_\tau\|_{H\to H}[1+|\beta|(\|\tilde{R}^{-1}R^{N-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{N-1}\|_{H\to H})] \tag{2.27}$$

$$\times[|\alpha|[\sum_{s=2}^{N-1}\frac{1}{2}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_{s-1}-f_s\|_H$$

$$+(\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H})\|f_1\|_H+(\|R+\tilde{R}\|_{H\to H})\|f_{N-1}\|_H]+\|A\varphi\|_H]$$

$$+|\alpha|(\|\tilde{R}^{-1}R^{N-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{N-1}\|_{H\to H})$$

$$\times[|\beta|[\sum_{s=2}^{N-2}\frac{1}{4}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_{s-1}-f_s\|_H$$

$$+(\|R^N\|_{H\to H}+\|\tilde{R}^N\|_{H\to H})\|f_1\|_H+(\|R^2-\tilde{R}^2\|_{H\to H})\|f_{N-1}\|_H$$

$$+\|R\tilde{R}\|_{H\to H}\|f_{N-1}\|_H]+\|A^{\frac{1}{2}}\psi\|_H]$$

$$\leq M\{\sum_{s=2}^{N-1}\| f_s-f_{s-1} \|_H+\| f_1 \|_H+\| A^{1/2}\psi \|_H+\| A\varphi \|_H\}.$$

$$\|A^{\frac{1}{2}}\omega\|_H \leq \|T_\tau\|_{H\to H}\{[1+|\alpha|[\frac{1}{2}\left[\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H}\right]]] \tag{2.28}$$

$$\times[|\beta|[\sum_{s=2}^{N-2}\frac{1}{4}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_{s-1}-f_s\|_H$$

$$+(\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H})\|f_1\|_H+(\|R^2-\tilde{R}^2\|_{H\to H})\|f_{N-1}\|_H$$

$$+\|R\tilde{R}\|_{H\to H}\|f_{N-1}\|_H]+\|A^{\frac{1}{2}}\psi\|_H]$$

$$+|\beta|\frac{1}{2}\left[\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H}\right]$$

$$[|\alpha|[\sum_{s=2}^{N-1}\frac{1}{2}\times\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_{s-1}-f_s\|_H$$

$$+(\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H})\|f_1\|_H+(\|R+\tilde{R}\|_{H\to H})\|f_{N-1}\|_H]+\|A\varphi\|_H]$$

$$\leq M\{\sum_{s=2}^{N-1}\| f_s-f_{s-1}\|_H+\| f_1\|_H+\| A^{1/2}\psi\|_H+\| A\varphi\|_H\}.$$

Now, we will prove the estimates (2.14),(2.15),(2.16).Let $k\geq 2$.Then using the formula (2,18) and estimates(2.22),(2.23),(2.24),(2,25) and (2.26)we obtain that

$$\|u_k\|_H \leq \frac{1}{2}\left[\|R^{k-1}\|_H+\|\tilde{R}^{k-1}\|_H\right]\|\mu\|_H+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1}\|_{H\to H}$$

$$+\|R^{-1}\tilde{R}^{k-1}\|_{H\to H})\|A^{-\frac{1}{2}}\omega\|_H+\sum_{s=1}^{k-1}\frac{\tau}{2}\left[\|R^{k-s}\|_H+\|\tilde{R}^{k-s}\|_H\right]\|A^{-\frac{1}{2}}f_s\|_H$$

$$\leq M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\| A^{-1/2}\psi\|_H+\|\varphi\|_H\}.$$

$$\|A^{\frac{1}{2}}u_k\|_H \leq \frac{1}{2}\left[\|R^{k-1}\|_H+\|\tilde{R}^{k-1}\|_H\right]\|A^{\frac{1}{2}}\mu\|_H+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1}\|_{H\to H}$$

$$+\|R^{-1}\tilde{R}^{k-1}\|_{H\to H})\|\omega\|_H+\sum_{s=1}^{k-1}\frac{\tau}{2}\left[\|R^{k-s}\|_H+\|\tilde{R}^{k-s}\|_H\right]\|f_s\|_H$$

$$\leq M\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\psi\|_H+\| A^{\frac{1}{2}}\varphi\|_H\}.$$

Now, we obtain the estimates for $\| Au_k\|_H$ for $k\geq 2$. Applying A to the formula (2.18)and using the Abel's formula, we can write

$$Au_k=\frac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]A\varphi+(R-\tilde{R})^{-1}\tau(R^k-\tilde{R}^k)A\psi$$

$$+\frac{1}{2}(\sum_{s=2}^{k-1}(\left[R^{k-s}+\tilde{R}^{k-s}\right](f_{s-1}-f_s)+2f_{k-1}-\left[R^{k-1}+\tilde{R}^{k-1}\right]f_1).$$

Using the last formula and estimates (2.22),(2.27),(2.28) we obtain

$$\|Au_k\|_H \leq \frac{1}{2}\left[\|R^{k-1}\|_{H\to H} + \|\tilde{R}^{k-1}\|_{H\to H}\right]\|A\varphi\|_H$$

$$+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1} + R^{-1}\tilde{R}^{k-1})\|A^{\frac{1}{2}}\psi\|_H$$

$$+\frac{1}{2}(\sum_{s=2}^{k-1}(\left[\|R^{k-s}\|_{H\to H} + \|\tilde{R}^{k-s}\|_{H\to H}\right](\|f_{s-1} - f_s\|_H)$$

$$+2\|f_{k-1}\|_H + \left[\|R^{k-1}\|_{H\to H} + \|\tilde{R}^{k-1}\|_{H\to H}\|f_1\|_H\right)$$

$$\leq M\{\sum_{s=2}^{N-1} \| f_s - f_{s-1} \|_H\right] + \| f_1 \|_H + \| A^{1/2}\psi \|_H + \| A\varphi \|_H\}.$$

Thus, the estimates (2.14),(2.15),(2.16)for any $k \geq 2$ are obtained.From $u_0 = \mu$ and (2.23),(2.25),(2.27) it follows the estimates (2.14),(2.15),(2.16)for $k = 0$.Note that in a similar manner with the estimates (2.24),(2.26),(2.28)we obtain

$$\|\tau R\omega\|_H \leq \|\tau AR\|_{H\to H}\|T_\tau\|_{H\to H}\{[1 + |\alpha|[\frac{1}{2}\left[\|R^{N-1}\|_{H\to H} + \|\tilde{R}^{N-1}\|_{H\to H}\right]]$$

$$\times[|\beta|\sum_{s=1}^{N-2}\frac{\tau}{4}\left[\|R^{N-s}\|_{H\to H} + \|\tilde{R}^{N-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H$$

$$+|\beta|\tau\|R\tilde{R}\|_{H\to H}\|A^{-1/2}f_{N-1}\|_H + \|A^{-1/2}\psi\|_H]$$

$$+|\beta|\frac{1}{2}\left[\|R^{N-1}\|_{H\to H} + \|\tilde{R}^{N-1}\|_{H\to H}\right]$$

$$\times[|\alpha|\sum_{s=1}^{N-1}\frac{\tau}{2}\left[\|R^{N-s}\|_{H\to H} + \|\tilde{R}^{N-s}\|_{H\to H}\right]\|A^{-1/2}f_s\|_H + \|\varphi\|_H]\}$$

$$\leq M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau + \| A^{-1/2}\psi \|_H + \| \varphi \|_H\},$$

$$\|\tau R\omega\|_H \leq \|\tau AR\|_{H\to H}\|T_\tau\|_{H\to H}\{[1 + |\alpha|\frac{1}{2}[\|R^{N-1}\|_{H\to H} + \|\tilde{R}^{N-1}\|_{H\to H}]]$$

$$\times[|\beta|\sum_{s=1}^{N-2}\frac{\tau}{4}\left[\|R^{N-s}\|_{H\to H} + \|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_s\|_H$$

$$+|\beta|\tau\|R\tilde{R}\|_{H\to H}\|f_{N-1}\|_H + \|\psi\|_H]$$

$$+|\beta|\frac{1}{2}\left[\|R^{N-1}\|_{H\to H} + \|\tilde{R}^{N-1}\|_{H\to H}\right]$$

$$\times[|\alpha|\sum_{s=1}^{N-1}\frac{\tau}{2}\left[\|R^{N-s}\|_{H\to H} + \|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_s\|_H + \|A^{\frac{1}{2}}\varphi\|_H]\}$$

$$\leq M\{\sum_{s=1}^{N-1} \|f_s\|_H\tau + \| \psi \|_H + \| A^{\frac{1}{2}}\varphi \|_H\},$$

$$\|A^{\frac{1}{2}}\tau R\omega\|_H \leq \|\tau AR\|_{H\to H}\|T_\tau\|_{H\to H}\{[1+|\alpha|[\frac{1}{2}\left[\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H}\right]]]$$

$$\times[|\beta|[\sum_{s=2}^{N-2}\frac{1}{4}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_{s-1}-f_s\|_H$$

$$+(\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H})\|f_1\|_H+(\|R^2-\tilde{R}^2\|_{H\to H})\|f_{N-1}\|_H$$

$$+\|R\tilde{R}\|_{H\to H}\|f_{N-1}\|_H]+\|A^{\frac{1}{2}}\psi\|_H]$$

$$+|\beta|\frac{1}{2}\left[\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H}\right]$$

$$\times[|\alpha|[\sum_{s=2}^{N-1}\frac{1}{2}\left[\|R^{N-s}\|_{H\to H}+\|\tilde{R}^{N-s}\|_{H\to H}\right]\|f_{s-1}-f_s\|_H$$

$$+(\|R^{N-1}\|_{H\to H}+\|\tilde{R}^{N-1}\|_{H\to H})\|f_1\|_H+(\|R+\tilde{R}\|_{H\to H})\|f_{N-1}\|_H]+\|A\varphi\|_H]$$

$$\leq M\{\sum_{s=2}^{N-1}\| f_s - f_{s-1} \|_H + \| f_1 \|_H + \| A^{1/2}\psi \|_H + \| A\varphi \|_H\}.$$

Using the formula $u_1 = \mu + \tau\omega$ and the last estimates, the triangle inequality and the estimates (2.23),(2.25),(2.27),we obtain the estimates (2.14),(2.15),(2.16)for $k = 1$. Theorem 2.4 is proved.

Note that these stability estimates (2.14),(2.15),(2.16)in the case $k = 1$ are weaker than respective estimates in the cases $k = 0, 2, ...N$.However, obtaining this type of estimate is important for applications. We denote by $a^\tau = (a_k)$ the mesh function of approximation. Then $\|(I + i\tau A^{-1/2})a_1\|_H \sim \|a_1\|_H = o(\tau)$ if we assume that $\tau\|Aa_1\|_H$ tends to 0 as $\tau \to 0$ not slower than $\|a_1\|_H$. It takes place in applications by supplementary restriction of the smooth property of the dates of space variables. It is clear that the uniformity in τ estimate

$$\|u_1\|_H \leq \|\varphi\|_H + \|A^{-1/2}\psi\|_H$$

is absent. However, estimates for the solution of first order of accuracy modification difference scheme for approximately solving the boundary value problem (2.2)

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_{k+1}=f_k, f_k=f(t_k),\ t_k=k\tau,\ 1\leq k\leq N-1, N\tau=1, \\ u_0=\alpha u_N+\varphi, (I+\tau^2 A)\tau^{-1}(u_1-u_0)=\beta\tau^{-1}(u_N-u_{N-1})+\psi. \end{cases} \tag{2.29}$$

are better than the estimates for the solution of difference scheme (2.12).

Theorem 2.5. *Let $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$ and $1 > |\alpha||\beta| + |\alpha| + |\beta|$. Then for the solution of the difference scheme (2.29) the stability inequalities*

$$\max_{0\leq k\leq N} \|u_k\|_H \leq M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau + \| A^{-1/2}\psi \|_H + \| \varphi \|_H\},$$

$$\max_{0\le k\le N}\|A^{1/2}u_k\|_H \le M\{\sum_{s=1}^{N-1}\| f_s \|_H \tau + \| A^{1/2}\varphi \|_H + \| \psi \|_H\},$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\|_H + \max_{0\le k\le N}\|Au_k\|_H$$

$$\le M\{\sum_{s=2}^{N-1}\| f_s - f_{s-1}\|_H + \| f_1 \|_H + \| A^{1/2}\psi \|_H + \| A\varphi \|_H\}$$

hold ,where M does not depend on $f_s, 1\le s\le N-1$ and φ,ψ.

The proof of this theorem follows the scheme of the proof of theorem 2.5 and it is based on the following formulas

$$u_0=\mu,\; u_1=\mu+\tau R\tilde{R}\omega,$$

$$u_k=\tfrac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]\mu+(R-\tilde{R})^{-1}\tau(R^k-\tilde{R}^k)R\tilde{R}\omega-\sum_{s=1}^{k-1}\tfrac{\tau}{2i}A^{-1/2}\left[R^{k-s}-\tilde{R}^{k-s}\right]f_s$$

$$=\tfrac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]\mu+(R-\tilde{R})^{-1}\tau(R^k-\tilde{R}^k)R\tilde{R}\omega$$

$$+A^{-1}\tfrac{1}{2}(\sum_{s=2}^{k-1}(\left[R^{k-s}+\tilde{R}^{k-s}\right](f_{s-1}-f_s)+2f_{k-1}-\left[R^{k-1}+\tilde{R}^{k-1}\right]f_1),\quad 2\le k\le N$$

and on the estimate (2.22) and

$$\| \tilde{T}_\tau \|_{H\mapsto H}\le \frac{1}{1-|\alpha||\beta|-|\alpha|-|\beta|}.$$

Here

$$\tilde{T}_\tau=(I-\frac{\alpha}{2}(R^{N-1}+\tilde{R}^{N-1})-\frac{\beta}{2}(R^N+\tilde{R}^N)+\alpha\beta R^N\tilde{R}^N)^{-1},$$

$$\mu = \tilde{T}_\tau[1-\beta(R-\tilde{R})^{-1}\frac{\tau A^{1/2}}{i}(R^N+\tilde{R}^N)R\tilde{R}]$$

$$\times[-\alpha\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s]+\varphi]$$

$$+\alpha(R-\tilde{R})^{-1}\tau(R^N-\tilde{R}^N)R\tilde{R}[-\beta\sum_{s=1}^{N-2}\frac{\tau}{4}\left[R^{N-s}+\tilde{R}^{N-s}\right]f_s+\beta\tau R\tilde{R}f_{N-1}]+\psi],$$

$$\omega = \tilde{T}_\tau\{[1-\alpha[\frac{1}{2}\left[R^{N-1}+\tilde{R}^{N-1}\right]]]$$

$$\times[-\beta\sum_{s=1}^{N-2}\frac{\tau}{4}\left[R^{N-s}+\tilde{R}^{N-s}\right]f_s+\beta\tau R\tilde{R}f_{N-1}+\psi]$$

$$+\beta\frac{A^{1/2}}{2i}\left[R^{N-1}-\tilde{R}^{N-1}\right][-\alpha\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s+\varphi]\}.$$

Note that theorems 2.4 and 2.5 permit us to establish the stability of the difference schemes of the first order of approximation over time and of an arbitrary order of approximation over space variables of approximate solutions of boundary value problems (2.9)and (2.10).

2.4 The Second Order of Accuracy Difference Schemes

Now, we consider the second order accuracy difference schemes for approximately solving the boundary value problem (2.2)

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k, \\ f_k=f(t_k), \quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ (I+\frac{\tau^2 A}{2})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0) \\ =\beta[\tau^{-1}(u_N-u_{N-1})+\frac{\tau}{2}(f_N-Au_N)]+\psi, \\ f_0=f(0), f_N=f(1), u_0=\alpha u_N+\varphi \end{cases} \tag{2.30}$$

and

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k, \\ f_k=f(t_k), \quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ (I+\frac{\tau^2 A}{4})[(I+\frac{\tau^2 A}{4})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)] \\ =\beta[\tau^{-1}(u_N-u_{N-1})+\frac{\tau}{2}(f_N-Au_N)]+\psi, \\ f_0=f(0), f_N=f(1), u_0=\alpha u_N+\varphi. \end{cases} \tag{2.31}$$

Theorem 2.6. *Let $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$ and $1>|\alpha||\beta|+|\alpha|+|\beta|$. Then for the solution of the difference scheme (2.30) the stability inequalities*

$$\max_{0\le k\le N}\|u_k\|_H\le M\{\sum_{s=0}^{N}\|A^{-1/2}f_s\|_H\tau+\|A^{-1/2}\psi\|_H+\|\varphi\|_H\}, \tag{2.32}$$

$$\max_{0\le k\le N}\|A^{1/2}u_k\|_H\le M\{\sum_{s=0}^{N}\|f_s\|_H\,\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H\}, \tag{2.33}$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\|_H+\max_{0\le k\le N}\|Au_k\|_H \tag{2.34}$$

$$\le M\{\sum_{s=1}^{N}\|f_s-f_{s-1}\|_H+\|f_0\|_H+\|A^{1/2}\psi\|_H+\|A\varphi\|_H\}$$

hold ,where M does not depend on $f_s, 0\le s\le N$ and φ,ψ.

Proof. We will write the formula for the solution of the difference scheme (2.30). It is easy to show that(see [6]) there are unique solution of the problem

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k, \\ f_k=f(t_k), \quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ (I+\frac{\tau^2 A}{2})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)=\omega, f_0=f(0), u_0=\mu \end{cases} \tag{2.35}$$

and for the solutions of these problems the following formulas hold:

$$u_0=\mu, u_1=\left(I+\frac{\tau^2 A}{2}\right)^{-1}\left(\mu+\tau\omega+\frac{\tau^2}{2}f_0\right],$$

$$u_k=\left(I+\frac{\tau^2 A}{2}\right)^{-1}[\frac{R^k(I+\frac{\tau^2 A}{2}+\frac{i\tau^3 A^{\frac{3}{2}}}{2})+\tilde{R}^{k-1}(I+i\tau A^{\frac{1}{2}})}{2}\mu$$

$$+(iA^{\frac{1}{2}})^{-1}\frac{R^{\tilde{k}-1}R^{-1}-R^{k-1}\tilde{R}^{-1}}{2}(\omega+\frac{\tau}{2}f_0)]$$

$$-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}-\tilde{R}^{k-s}\right]f_s, 2\le k\le N, \tag{2.36}$$

where $R=\left(I+i\tau A^{1/2}-\frac{\tau^2}{2}A\right)^{-1}$, $\tilde{R}=\left(I-i\tau A^{1/2}-\frac{\tau^2}{2}A\right)^{-1}$. Applying the last formula and the nonlocal boundary conditions

$$\begin{aligned} u_0 &= \alpha u_N+\varphi, (I+\frac{\tau^2A}{2})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)\\ &= \beta[\tau^{-1}(u_N-u_{N-1})+\frac{\tau}{2}(f_N-Au_N)]+\psi, \end{aligned}$$

we can write

$$\mu=\alpha[\left(I+\frac{\tau^2A}{2}\right)^{-1}[\frac{R^N(I+\frac{\tau^2A}{2}+\frac{i\tau^3A^{\frac{3}{2}}}{2})+\tilde{R}^{N-1}(I+i\tau A^{\frac{1}{2}})}{2}\mu$$

$$+(iA^{\frac{1}{2}})^{-1}\frac{R^{\tilde{N}-1}R^{-1}-R^{N-1}\tilde{R}^{-1}}{2}(\omega+\frac{\tau}{2}f_0)]$$

$$-\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s]+\varphi,$$

$$\omega=\beta[iA^{1/2}\left(I+\frac{\tau^2A}{2}\right)^{-1}[\frac{-R^N(I+\frac{\tau^2A}{2}+\frac{i\tau^3A^{\frac{3}{2}}}{2})+\tilde{R}^{N-1}(I+i\tau A^{\frac{1}{2}})}{2}\mu$$

$$+\left(I+\frac{\tau^2A}{2}\right)^{-1}\frac{R^{\tilde{N}-1}R^{-1}+R^{N-1}\tilde{R}^{-1}}{2}(\omega+\frac{\tau}{2}f_0)]$$

$$-\left(I+\frac{\tau^2A}{2}\right)^{-1}\sum_{s=1}^{N-2}\frac{\tau}{2}\left[-R^{N-s}-\tilde{R}^{N-s}\right]f_s$$

$$+\left(I+\frac{\tau^2A}{2}\right)^{-1}\left(I+\frac{\tau^2A}{4}\right)^{-1}f_{N-1}]+\psi.$$

Using the last two formulas, we obtain

$$\mu=T_\tau[1-\beta\left(I+\frac{\tau^2A}{2}\right)^{-1}\frac{R^{\tilde{N}-1}R^{-1}+R^{N-1}\tilde{R}^{-1}}{2}] \tag{2.37}$$

$$\times[\alpha(iA^{\frac{1}{2}})^{-1}\frac{R^{\tilde{N}-1}R^{-1}-R^{N-1}\tilde{R}^{-1}}{2}\frac{\tau}{2}f_0-\alpha\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s]+\varphi]$$

$$+\alpha(iA^{\frac{1}{2}})^{-1}\frac{R^{\tilde{N}-1}R^{-1}-R^{N-1}\tilde{R}^{-1}}{2}\beta\left(I+\frac{\tau^2A}{2}\right)^{-1}\frac{R^{\tilde{N}-1}R^{-1}+R^{N-1}\tilde{R}^{-1}}{2}\frac{\tau}{2}f_0$$

$$-\beta\left(I+\frac{\tau^2A}{2}\right)^{-1}\sum_{s=1}^{N-2}\frac{\tau}{2}\left[-R^{N-s}-\tilde{R}^{N-s}\right]f_s$$

$$+\beta\left(I+\frac{\tau^2A}{2}\right)^{-1}\left(I+\frac{\tau^2A}{4}\right)^{-1}f_{N-1}+\psi],$$

$$\omega=T_\tau\{[1-\alpha\left(I+\frac{\tau^2A}{2}\right)^{-1}\frac{R^N(I+\frac{\tau^2A}{2}+\frac{i\tau^3A^{\frac{3}{2}}}{2})+\tilde{R}^{N-1}(I+i\tau A^{\frac{1}{2}})}{2}] \tag{2.38}$$

$$\times[\beta\left(I+\frac{\tau^2A}{2}\right)^{-1}\frac{R^{\tilde{N}-1}R^{-1}+R^{N-1}\tilde{R}^{-1}}{2}\frac{\tau}{2}f_0-\beta\left(I+\frac{\tau^2A}{2}\right)^{-1}$$

$$\times\sum_{s=1}^{N-2}\frac{\tau}{2}\left[-R^{N-s}-\tilde{R}^{N-s}\right]f_s+\beta\left(I+\frac{\tau^2A}{2}\right)^{-1}\left(I+\frac{\tau^2A}{4}\right)^{-1}f_{N-1}+\psi]$$

$$+\beta iA^{1/2}\left(I+\frac{\tau^2A}{2}\right)^{-1}\frac{-R^N(I+\frac{\tau^2A}{2}+\frac{i\tau^3A^{\frac{3}{2}}}{2})+\tilde{R}^{N-1}(I+i\tau A^{\frac{1}{2}})}{2}$$

$$\times[\alpha(iA^{\frac{1}{2}})^{-1}\frac{R^{\tilde{N}-1}R^{-1}-R^{N-1}\tilde{R}^{-1}}{2}\frac{\tau}{2}f_0-\alpha\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s]+\varphi]\},$$

where

$$T_\tau=(I-\left(I+\frac{\tau^2A}{2}\right)^{-1}\frac{\alpha}{2}(R^N(I+\frac{\tau^2A}{2}+\frac{i\tau^3A^{\frac{3}{2}}}{2})+\tilde{R}^{N-1}(I+i\tau A^{\frac{1}{2}})))$$

$$-\left(I+\frac{\tau^2A}{2}\right)^{-1}\frac{\beta}{2}(\tilde{R^{-1}}R^{N-1}+R^{-1}\tilde{R}^{N-1})+\left(I+\frac{\tau^2A}{2}\right)^{-1}\alpha\beta R^{N-1}\tilde{R}^{N-1})^{-1}.$$

Hence, for the formal solution of the difference scheme (2.30) we have the formula (2.36),(2.37),(2.38). For substantiation of these formulas we need to obtain the stability estimates (2.32),(2.33),(2.34) for solutions of the difference scheme (2.30). From the symmetry and positivity properties of the operator A it follows that

$$\| T_\tau \|_{H\mapsto H}\leq\frac{1}{1-|\alpha||\beta|-|\alpha|-|\beta|} \tag{2.39}$$

and the estimates

$$\begin{cases} \|R\|_{H\mapsto H}\leq 1, & \|\tilde{R}\|_{H\mapsto H}\leq 1,\\ \|R\tilde{R}^{-1}\|_{H\mapsto H}\leq 1, & \|\tilde{R}R^{-1}\|_{H\mapsto H}\leq 1,\\ \|\tau A^{1/2}R\|_{H\mapsto H}\leq 1, & \|\tau A^{1/2}\tilde{R}\|_{H\mapsto H}\leq 1. \end{cases} \tag{2.40}$$

The proof of estimates (2.32),(2.33),(2.34) uses the outline of the proof theorem 2.4 and is based on the formulas (2.33),(2.34),(2.35)and estimates (2.36) and (2.37).Theorem 2.6 is proved.

Theorem 2.7. *Let $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$ and $1 > |\alpha||\beta| + |\alpha| + |\beta|$. Then for the solution of the difference scheme (2.31) the stability inequalities*

$$\max_{0\leq k\leq N} \|u_k\|_H \leq M\{\sum_{s=0}^{N} \|A^{-1/2} f_s\|_H \tau + \| A^{-1/2}\psi \|_H + \| \varphi \|_H\}, \tag{2.41}$$

$$\max_{0\leq k\leq N} \|A^{1/2} u_k\|_H \leq M\{\sum_{s=0}^{N} \| f_s \|_H \tau + \| A^{1/2}\varphi \|_H + \| \psi \|_H\}, \tag{2.42}$$

$$\max_{1\leq k\leq N-1} \|\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\|_H + \max_{0\leq k\leq N} \|Au_k\|_H \tag{2.43}$$

$$\leq M\{\sum_{s=1}^{N} \| f_s - f_{s-1} \|_H + \| f_0 \|_H + \| A^{1/2}\psi \|_H + \| A\varphi \|_H\}$$

hold ,where M does not depend on $f_s, 0 \leq s \leq N$ and φ, ψ.

Proof. We will write the formula for the solution of the difference scheme (2.31). It is easy to show that(see [6]) there are unique solution of the problem

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2}Au_k + \frac{1}{4}A(u_{k+1} + u_{k-1}) = f_k, \\ f_k = f(t_k), \quad t_k = k\tau, \ 1 \leq k \leq N-1, N\tau = 1, \\ (I + \frac{\tau^2 A}{4})\tau^{-1}(u_1 - u_0) - \frac{\tau}{2}(f_0 - Au_0) = \omega, f_0 = f(0), u_0 = \mu \end{cases} \tag{2.44}$$

and for the solutions of these problems the following formulas hold:

$$u_0 = \mu, u_1 = \left(I + \frac{\tau^2 A}{4}\right)^{-1} \left[(I - \frac{\tau^2 A}{4})\mu + \tau\omega + \frac{\tau^2}{2} f_0\right],$$

$$u_k = \frac{R^k + \tilde{R}^k}{2}\mu + (iA^{\frac{1}{2}})^{-1}\frac{\tilde{R}^k - R^k}{2}(\omega + \frac{\tau}{2} f_0)$$

$$-\sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} \left[R^{k-s} - \tilde{R}^{k-s}\right] f_s, 2 \leq k \leq N, \tag{2.45}$$

where $R = \left(I - \frac{i\tau A^{1/2}}{2}\right)\left(I + \frac{i\tau A^{1/2}}{2}\right)^{-1}$, $\tilde{R} = \left(I + \frac{i\tau A^{1/2}}{2}\right)\left(I - \frac{i\tau A^{1/2}}{2}\right)^{-1}$. Applying the last formula and the nonlocal boundary conditions

$$u_0 = \alpha u_N + \varphi,$$

$$\begin{aligned} &(I + \frac{\tau^2 A}{4})[(I + \frac{\tau^2 A}{4})\tau^{-1}(u_1 - u_0) - \frac{\tau}{2}(f_0 - Au_0)] \\ = \ &\beta[\tau^{-1}(u_N - u_{N-1}) + \frac{\tau}{2}(f_N - Au_N)] + \psi, \end{aligned}$$

we can write

$$\mu = \alpha[\frac{R^N + \tilde{R}^N}{2}\mu + (iA^{\frac{1}{2}})^{-1}\frac{\tilde{R}^N - R^N}{2}(\omega + \frac{\tau}{2} f_0)$$

$$-\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s]+\varphi,$$

$$\omega=\beta[iA^{1/2}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}[\frac{-R^{N-1}+\tilde{R}^{N}}{2}\mu+\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}$$

$$\times\frac{\tilde{R}^{N}+R^{N-1}}{2}(\omega+\frac{\tau}{2}f_0)]-\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\sum_{s=1}^{N-2}\frac{\tau}{2}\left[R^{N-1-s}-\tilde{R}^{N-s}\right]f_s$$

$$+\left(I+\frac{\tau^2A}{4}\right)^{-2}f_{N-1}]+\left(I+\frac{\tau^2A}{4}\right)^{-1}\psi.$$

Using the last two formulas, we obtain

$$\mu=T_\tau\{[1-\beta\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\frac{\tilde{R}^{N}+R^{N-1}}{2}] \tag{2.46}$$

$$\times[\alpha[(iA^{\frac{1}{2}})^{-1}\frac{\tilde{R}^{N}-R^{N}}{2}\frac{\tau}{2}f_0-\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s]+\varphi]$$

$$+\alpha(iA^{\frac{1}{2}})^{-1}\frac{\tilde{R}^{N}-R^{N}}{2}[\beta[\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\frac{\tilde{R}^{N}+R^{N-1}}{2}\frac{\tau}{2}f_0$$

$$-\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\sum_{s=1}^{N-2}\frac{\tau}{2}\left[R^{N-1-s}-\tilde{R}^{N-s}\right]f_s$$

$$+\left(I+\frac{\tau^2A}{4}\right)^{-2}f_{N-1}]+\left(I+\frac{\tau^2A}{4}\right)^{-1}\psi]\},$$

$$\omega=T_\tau\{[1-\alpha\frac{R^{N}+\tilde{R}^{N}}{2}][\beta[\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\frac{\tilde{R}^{N}+R^{N-1}}{2}\frac{\tau}{2}f_0$$

$$-\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\sum_{s=1}^{N-2}\frac{\tau}{2}\left[R^{N-1-s}-\tilde{R}^{N-s}\right]f_s+\left(I+\frac{\tau^2A}{4}\right)^{-2}f_{N-1}]$$

$$+\left(I+\frac{\tau^2A}{4}\right)^{-1}\psi]+\beta iA^{1/2}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\frac{-R^{N-1}+\tilde{R}^{N}}{2}$$

$$\times[\alpha[(iA^{\frac{1}{2}})^{-1}\frac{\tilde{R}^{N}-R^{N}}{2}\frac{\tau}{2}f_0-\sum_{s=1}^{N-1}\frac{\tau}{2i}A^{-1/2}\left[R^{N-s}-\tilde{R}^{N-s}\right]f_s]+\varphi]\},$$

where

$$T_\tau = (I - \frac{\alpha}{2}(R^N + \tilde{R}^N) - \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1} \frac{\beta}{2}(R^{N-1} + \tilde{R}^N) + \left(I + \frac{\tau^2 A}{4}\right)^{-1} \alpha\beta)^{-1}.$$

Hence, for the formal solution of the difference scheme (2.31) we have the formula (2.27),(2.46),(2.47). For substantiation of these formulas we need to obtain the stability estimates (2.41),(2.42),(2.43) for solutions of the difference scheme (2.31).From the symmetry and positivity properties of the operator A it follows that

$$\| T_\tau \|_{H \mapsto H} \leq \frac{1}{1 - |\alpha||\beta| - |\alpha| - |\beta|} \tag{2.47}$$

and the estimates

$$\begin{cases} \|R\|_{H \mapsto H} \leq 1, \qquad \|\tilde{R}\|_{H \mapsto H} \leq 1, \\ \| \left(I + \frac{\tau^2 A}{4}\right)^{-1} \|_{H \mapsto H} \leq 1. \end{cases} \tag{2.48}$$

The proof of estimates (2.41),(2.42),(2.43) uses the outline of the proof theorem 2.4 and is based on the formulas (2.45),(2.46),(2.47) and estimates (2.40) and (2.49). Theorem 2.7 is proved.

Note that theorems 2.6 and 2.7 permit us to establish the stability of the difference schemes of the second order of approximation over time and of an arbitrary order of approximation over space variables of approximate solutions of boundary value problems (2.9)and (2.10).

3 APPLICATIONS

3.1 The First Order of Accuracy in Time Difference Scheme

Consider the nonlocal boundary-value problem for wave equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = 2[1 + \frac{\alpha^2+\alpha}{(1-\alpha)^2} + \frac{2\alpha}{1-\alpha}t + t^2]\sin x, \\ (0 < t < 1, \quad 0 < x < \pi), \\ u(0,x) = \alpha u(1,x), \; u_t(0,x) = \alpha u_t(1,x) \quad 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le 1. \end{cases} \tag{3.1}$$

The exact solution is:

$$u(t,x) = \left(\frac{\alpha^2+\alpha}{(1-\alpha)^2} + \frac{2\alpha}{1-\alpha}t + t^2\right)\sin x.$$

For approximate solution of the nonlocal boundary-value problem (3.1), consider the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the 'small' parameters τ and h:
$[0,1]_\tau \times [0,\pi]_h = \{(t_k, x_n) : t_k = k\tau, \;\; 0 \le k \le N, \;\; N\tau = 1,$
$x_n = nh, \;\; 0 \le n \le M, \; Mh = \pi\}$.

Applying the formulas

$$\frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_{k+1}) = O(\tau),$$

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O(h^2),$$

and

$$\frac{u(1) - u(0)}{\tau} - u'(0) = O(\tau),$$

$$\frac{u(1) - u(1-\tau)}{\tau} - u'(1) = O(\tau)$$

and using the first order of accuracy in t implicit difference scheme for wave equation, the difference scheme first order of accuracy in t and second order of accuracy in x for approximate solutions of the nonlocal boundary value problem (3.1) are obtained.

$$\begin{cases} \frac{U_n^{k+1} - 2U_n^k + U_n^{k-1}}{\tau^2} - \frac{U_{n+1}^{k+1} - 2U_n^{k+1} + U_{n-1}^{k+1}}{h^2} + U_n^{k+1} \\ = 2[1 + \frac{\alpha^2+\alpha}{(1-\alpha)^2} + \frac{2\alpha}{1-\alpha}(k+1)\tau + ((k+1)\tau)^2]\sin(nh), \\ 1 \le k \le N-1, \quad 1 \le n \le M-1, \\ U_0^k = U_M^k = 0, \quad 0 \le k \le N, \\ U_n^1 - U_n^0 = \alpha\left(U_n^N - U_n^{N-1}\right), U_n^0 = \alpha U_n^N, \; 0 \le n \le M. \end{cases} \tag{3.2}$$

$(N+1) \times (M+1)$ system of linear equations are obtained and written in the matrix form. By resorting the system

$$\begin{cases} \left(\frac{-1}{h^2}\right) U_{n+1}^{k+1} + \left(\frac{1}{\tau^2} + \frac{2}{h^2} + 1\right) U_n^{k+1} + \left[-\frac{2}{\tau^2}\right] U_n^k + \left(\frac{1}{\tau^2}\right) U_n^{k-1} + \left[-\frac{1}{h^2}\right] U_{n-1}^{k+1} = \varphi_n^k, \\ \varphi_n^k = 2[1 + \frac{\alpha^2+\alpha}{(1-\alpha)^2} + \frac{2\alpha}{1-\alpha}(k+1)\tau + ((k+1)\tau)^2] \sin(nh) \quad , \\ 1 \le k \le N-1, \ 1 \le n \le M-1 \\ U_0^k = U_M^k = 0, \quad 0 \le k \le N, \\ U_n^1 - U_n^0 = \alpha \left(U_n^N - U_n^{N-1}\right), U_n^0 = \alpha U_n^N, \ 0 \le n \le M \end{cases}$$

So,

$$\begin{cases} A\, U_{n+1} + B\, U_n + C\, U_{n-1} = D\varphi_n, \quad 0 \le n \le M, \\ U_0 = 0, \ \ U_M = 0. \end{cases} \tag{3.3}$$

Denote

$$\begin{aligned} a &= \left(-\frac{1}{h^2}\right), \qquad b = \left(\frac{1}{\tau^2}\right), \\ c &= \left(\frac{-2}{\tau^2}\right), \qquad d = \left(\frac{1}{\tau^2} + \frac{2}{h^2} + 1\right), \end{aligned}$$

$$\varphi_n^k = 2[1 + \frac{\alpha^2+\alpha}{(1-\alpha)^2} + \frac{2\alpha}{1-\alpha}(k+1)\tau + ((k+1)\tau)^2] \sin(nh),$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \dots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & .. & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & -\alpha \\ b & c & d & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & b & c & d & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & .. & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & b & c & d & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & b & c & d \\ -1 & 1 & 0 & 0 & ... & 0 & 0 & \alpha & -\alpha \end{bmatrix}_{(N+1)\times(N+1)},$$

and $C = A$.

$$D = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ ... \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s = n-1,\ n,\ n+1.$$

For the solution of the last matrix equation, the modified variant Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 2, 1, 0,$$

where α_j , β_j, $(j = 1, ..., M-1)$ are $(N+1)\times(N+1)$ square matrices . And α_1, β_1 :

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_1 = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Using the equality $U_s = \alpha_{s+1}U_{s+1} + \beta_{s+1}$, (for $s = n,\ n-1$) and the equality $AU_{n+1} + B\,U_n + CU_{n-1} = D\varphi_n$,

$$[A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1}]U_{n+1} + \left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n$$

can be written.

The last equation is satisfied if it is to be selected:

$$\begin{cases} A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1} = 0, \\ \left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n, \\ 1 \le n \le M-1. \end{cases}$$

Formulas for $\alpha_{n+1},\ \beta_{n+1}$:

$$\begin{aligned} \alpha_{n+1} &= -\left(B + C\alpha_n\right)^{-1} A, \\ \beta_{n+1} &= \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), n = 1,2,3,...M-1. \end{aligned}$$

So,

$$\begin{aligned} &U_M = \sigma, \\ &U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}, \quad n = M-1,...,2,1,0. \end{aligned}$$

Algorithm

1. **Step** Input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{\pi}{M}$.
2. **Step** Use the first order of accuracy difference scheme and write in matrix form:

$$A\,U_{n+1} + B\,U_n + C\,U_{n-1} + = D\varphi_n, \quad 0 \le n \le M.$$

3. **Step** Determine the entries of the matrices $A,\ B,\ C$ and D.
4. **Step** Find α_1, β_1.
5. **Step** Compute $\alpha_{n+1},\ \beta_{n+1}$.
6. **Step;** Compute U_n-s $(n = M-1,...,2,1), (U_M = 0)$ using the following formula:

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}.$$

Matlab Implementation of the First Order of Accuracy Difference Scheme

```
function firstord
    close; close;
    N=20 ; M=20;
```

```
tau=1/N;
h=pi/M;
al=0.1;
a = -1/(h^2);
b = 1/(tau^2);
c = -2/(tau^2);
d = 1+ (1/(tau^2)) + (2/(h^2));
for i=2:N; A(i,i+1)=a; end;
A(N+1,N+1)=0; A;
C=A;
for i=2:N ; B(i,i-1)= b ; end;
for i=2:N ; B(i,i)= c ; end;
for i=2:N ; B(i,i+1)= d ; end;
B(1,1)=1; B(1,N+1)=-al;
B(N+1,1)=-1; B(N+1,2)=1; B(N+1,N) =al ; B(N+1,N+1)=-al; B;
for i=2:N; D(i,i)=1; end ;
D(N+1,N+1)=0; D;
for j=1:M+1;
for k=1:N+1;
s=1+((al^2)+al)/(1-al)^2+ (2*al/(1-al))*(k-1)*tau + ((k-1)*tau)^2;
fii(k,j:j) =2*s * sin((j)*h) ;
end;
end;
alpha(N+1,N+1,1:1)= 0 ;
betha(N+1,1:1) = 0 ;
for j=1:M-1;
alpha( :, :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j))*(-A) ;
betha( :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j ) )*(D*fii(:, j:j )...
- C * betha(:, j:j ) );
end;
U( N+1,1, M:M ) = 0;
for z = M-1:-1:1 ;
U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 ) + betha(:,z+1:z+1);
end;
```

```
for z = 1:M ;
p(:,z+1:z+1)=U(:,:,z:z);
end;
'EXACT SOLUTION OF THIS PROBLEM' ;
for j=1:M+1 ;
for k=1:N+1 ;
ss=((al^2)+al)/(1-al)^2+ (2*al/(1-al))*(k-1)*tau + ((k-1)*tau)^2;
es( k, j:j ) = ss*sin( (j-1)*h);
end;
end;
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;
title('EXACT SOLUTION');
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p) ; rotate3d ;
```

3.2 The Second Order of Accuracy in Time Difference Scheme

Consider again the nonlocal boundary-value problem (3.1). Applying the formula

$$\frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_k) = O(\tau^2),$$

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O(h^2),$$

and using the second order of accuracy in t implicit difference scheme for (2.31) wave equation, the difference scheme second order of accuracy in t and in x for approximate solutions of the nonlocal boundary value problem (3.1) are obtained.

$$\begin{cases} \frac{U_n^{k+1}-2U_n^k+U_{n-1}^{k-1}}{\tau^2} - \frac{U_{n+1}^k-2U_n^k+U_{n-1}^k}{2h^2} - \frac{U_{n+1}^{k+1}-2U_n^{k+1}+U_{n-1}^{k+1}}{4h^2} \\ -\frac{U_{n+1}^{k-1}-2U_n^{k-1}+U_{n-1}^{k-1}}{4h^2} + \frac{1}{2}U_n^k + \frac{1}{4}\left(U_n^{k+1}+U_n^{k-1}\right) = \varphi_n^k \\ \varphi_n^k = 2[1+\frac{\alpha^2+\alpha}{(1-\alpha)^2}+\frac{2\alpha}{1-\alpha}k\tau+(k\tau)^2]\sin(nh) \\ U_0^k = U_M^k = 0, \quad 0 \le k \le N, \\ a/2U_{n+1}^0 + eU_n^0 + a/2U_{n+1}^1 + (e+2)\,U_n^1 \;+ a/2U_{n-1}^1 + c\varphi_n^0 \\ = \alpha\ \left(-aU_{n+1}^N - dU_n^N - U_n^{N-1} - aU_{n-1}^N - ca\varphi_n^N\right), \\ U_n^0 = \alpha U_n^N,\ 0 \le n \le M, \end{cases}$$

where

$$\begin{aligned} a &= -\frac{\tau^2}{2h^2}, \\ b &= 1+\frac{\tau^2}{h^2}+\frac{\tau^2}{2}, \\ c &= \frac{\tau^2}{2}, \\ d &= -1+\frac{\tau^2}{h^2}+\frac{\tau^2}{2}, \\ e &= -1+\frac{\tau^2}{2h^2}+\frac{\tau^2}{4}, \end{aligned}$$

Again the $(N+1)\times(M+1)$ system of linear equations are obtained and written in the matrix form. By resorting the system

$$\begin{cases} \left(\frac{-1}{4h^2}\right)U_{n+1}^{k+1} + \left(-\frac{1}{2h^2}\right)U_{n+1}^k + \left[-\frac{1}{4h^2}\right]U_{n+1}^{k-1} \\ +\left(\frac{1}{\tau^2}+\frac{1}{2h^2}+\frac{1}{4}\right)U_n^{k+1} + \left[-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}\right]U_n^k + \left(\frac{1}{\tau^2}+\frac{1}{2h^2}+\frac{1}{4}\right)U_n^{k-1} \\ +\left[-\frac{1}{4h^2}\right]U_{n-1}^{k+1} + \left(-\frac{1}{2h^2}\right)U_{n-1}^k + \left[-\frac{1}{4h^2}\right]U_{n-1}^{k-1} = \varphi_n^k, \\ \quad 1 \le k \le N-1, \quad 1 \le n \le M-1 \\ \varphi_n^k = [2\frac{2\alpha^2-\alpha+1}{\alpha(\alpha-1)}+4(k+1)\tau+2(\frac{1-\alpha}{\alpha})\left((k+1)\tau\right)^2]\sin(nh) \\ U_0^k = U_M^k = 0, \quad 0 \le k \le N, \\ a/2U_{n+1}^0 + eU_n^0 + a/2U_{n+1}^1 + (e+2)\,U_n^1 \;+ a/2U_{n-1}^1 + c\varphi_n^0 \\ = \alpha\ \left(-aU_{n+1}^N - dU_n^N - U_n^{N-1} - aU_{n-1}^N - ca\varphi_n^N\right), \\ U_n^0 = \alpha U_n^N,\ \ 0 \le n \le M. \end{cases},$$

Again the matrix equation (3.3) is obtain with new data:

$$x = \left(\frac{1}{\tau^2}+\frac{1}{2h^2}+\frac{1}{4}\right),$$

$$y = \left(-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}\right),$$

$$z = \left(\frac{-1}{4h^2}\right),$$

$$w=\left(-\frac{1}{2h^2}\right).$$

$$\varphi_n^k=2[1+\frac{\alpha^2+\alpha}{(1-\alpha)^2}+\frac{2\alpha}{1-\alpha}k\tau+(k\tau)^2]\sin(nh),$$

$$\varphi_n=\begin{bmatrix}\varphi_n^0\\ \varphi_n^1\\ \varphi_n^2\\ \dots\\ \varphi_n^N\end{bmatrix}_{(N+1)\times 1},$$

$$A=\begin{bmatrix}0&0&0&0&\dots&0&0&0&0\\ z&w&z&0&\dots&0&0&0&0\\ o&z&w&z&\dots&0&0&0&0\\ 0&0&z&w&\dots&0&0&0&0\\ \dots&\dots&\dots&\dots&\dots&..&\dots&\dots&\dots\\ 0&0&0&0&\dots&w&z&0&0\\ 0&0&0&0&\dots&z&w&z&0\\ 0&0&0&0&\dots&c&z&w&z\\ a/2&a/2&0&0&\dots&0&0&0&a\alpha\end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix}-1&0&0&0&\dots&0&0&0&\alpha\\ x&y&x&0&\dots&0&0&0&0\\ 0&x&y&x&\dots&0&0&0&0\\ 0&0&x&y&\dots&0&0&0&0\\ \dots&\dots&\dots&\dots&\dots&..&\dots&\dots&\dots\\ 0&0&0&0&\dots&y&x&0&0\\ 0&0&0&0&\dots&x&y&x&0\\ 0&0&0&0&\dots&0&x&y&x\\ e&e+2&0&0&\dots&0&0&\alpha&d\alpha\end{bmatrix}_{(N+1)\times(N+1)},$$

and $C=A$.

$$D=\begin{bmatrix}0&0&0&\dots&0&0\\ 0&1&0&\dots&0&0\\ 0&0&1&\dots&0&0\\ \dots&\dots&\dots&\dots&\dots&\dots\\ 0&0&0&\dots&1&0\\ c&0&0&\dots&0&c\alpha\end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ \cdots \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s = n-1,\ n,\ n+1.$$

For the solution of the last matrix equation, the same algorithm is used for the first order of accuracy difference scheme.

Matlab Implementation of the Second Order of Accuracy Difference Scheme

```
function secondorderA
    close; close
    N=20; M=20;
    al=0.1;
    tau=1/N; h=pi/M;
    x = (1 / (tau^2))+(1/(2*(h^2)))+1/4;
    y = (- 2 / tau^2)+(1/(h^2))+1/2 ;
    z= -1/(4*h^2);
    w= -1/(2*h^2);
    a=-(tau^2)/(2*h^2);
    b=1+(tau^2)/h^2+(tau^2)/2;
    c=tau^2/2;
    d=-1+(tau^2)/h^2+(tau^2)/2;
    e=-1+(tau^2)/(2*h^2)+(tau^2)/4;
    for i=1:N-1 ; A(i+1,i)=z ; end;
    for i=1:N-1 ; A(i+1,i+1) =w ; end;
    for i=1:N-1 ; A(i+1,i+2)= z ; end;
    A(N+1,1)=a/2; A(N+1,2)=a/2; A(N+1,N+1)=a*al;
    for i=1:N-1 ; B(i+1,i)= x ; end;
    for i=1:N-1 ; B(i+1,i+1)= y ; end;
    for i=1:N-1 ; B(i+1,i+2)= x ; end;
    B(1,1)=-1; B(1,N+1)=al;
    B(N+1,1)=e; B(N+1,2)=e+2;
    B(N+1,N)=al; B(N+1,N+1)=d*al; B;
```

```
C=A;
for i=2:N; D(i,i)= 1; end ;
D(N+1,1)=c; D(N+1,N+1)=c*al;
'fi(j) finding ' ;
for j=1:M+1;
for k=1:N+1;
s=1+((al^2)+al)/(1-al)^2+ (2*al/(1-al))*(k-1)*tau + ((k-1)*tau)^2;
fii(k,j:j) =2*s * sin((j)*h) ;
end;
end;
alpha(N+1,N+1,1:1)= 0 ;
betha(N+1,1:1) = 0 ;
for j=1:M;
alpha(:,:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(-A) ;
betha(:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(D*fii(:,j:j)- C*betha(:,j:j));
end;
U( N+1,1, M:M ) = 0;
for z = M-1:-1:1 ;
U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 ) + betha(:,z+1:z+1);
end;
for z = 1:M ;
p(:,z+1:z+1)=U(:,:,z:z);
end;
figure;
surf(p) ;
rotate3d ;
```

3.3 The Second Order of Accuracy in Time Difference Scheme Generated by A^2

Applying the formulas

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - v''(x_n) = O(h^2),$$

$$\frac{u(x_{n+2}) - 4u(x_{n+1}) + 6u(x_n) - 4u(x_{n-1}) + u(x_{n-2})}{h^4} - u^{(iv)}(x_n) = O(h^2),$$

and

$$\frac{2u(0)-5u(h)+4u(2h)-u(3h)}{h^2}-u''(0)=O(h^2),$$
$$\frac{2u(1)-5u(1-h)+4u(1-2h)-u(1-3h)}{h^2}-u''(1)=O(h^2)$$

and using the second order of accuracy in t implicit difference scheme (2.30) for wave equation, the difference scheme second order 0f accuracy in t and in x for approximate solutions of the nonlocal boundary value problem (3.1) are obtained.

$$\begin{cases} \frac{U_n^{k+1}-2U_n^k+U_n^{k-1}}{\tau^2}-\frac{U_{n+1}^k-2U_n^k+U_{n-1}^k}{h^2}+U_n^k+\frac{\tau^2}{4}\times \\ \left[\frac{U_{n+2}^{k+1}-4U_{n+1}^{k+1}+6U_n^{k+1}-4U_{n-1}^{k+1}+U_{n-2}^{k+1}}{h^4}-2\frac{U_{n+1}^{k+1}-2U_n^{k+1}+U_{n-1}^{k+1}}{h^2}+U_n^{k+1}\right] \\ =\varphi_n^k,\ \ 1\le k\le N-1,\ 2\le n\le M-2, \\ \quad \varphi_n^k=2[1+\frac{\alpha^2+\alpha}{(1-\alpha)^2}+\frac{2\alpha}{1-\alpha}k\tau+(k\tau)^2]\sin(nh), \\ U_0^k=U_M^k=0,\ \ U_n^0=\alpha U_n^N,\quad 0\le k\le N,\quad 0\le n\le M, \\ aU_{n+1}^1-U_n^0+bU_n^1\ +aU_{n-1}^1+c\varphi_n^0 \\ \quad =\alpha\left(-aU_{n+1}^N-dU_n^N-U_n^{N-1}-aU_{n-1}^N-c\varphi_n^N\right),\ 1\le n\le M-1, \\ U_3^k=4U_2^k-5U_1^k,\quad U_{M-3}^k=4U_{M-2}^k-5U_{M-1}^k,\quad 0\le k\le N, \end{cases}$$

where

$$\begin{aligned} a &= \frac{-\tau^2}{2h^2}, \\ b &= 1+\frac{\tau^2}{h^2}+\frac{\tau^2}{2}, \\ c &= -\frac{\tau^2}{2}, \\ d &= -1+\frac{\tau^2}{h^2}+\frac{\tau^2}{2}. \end{aligned}$$

Again the $(N+1)\times(M+1)$ system of linear equations are obtain and written in the matrix form. By resorting the system

$$\begin{cases} \frac{\tau^2}{4h^4}U_{n+2}^{k+1}+\left[-\frac{1}{h^2}\right]U_{n+1}^k+\left[-\frac{\tau^2}{h^4}-\frac{\tau^2}{2h^2}\right]U_{n+1}^{k+1}+\left[\frac{1}{\tau^2}\right]U_n^{k-1}+\left[-\frac{2}{\tau^2}+\frac{2}{h^2}+1\right]U_n^k \\ +\left[\frac{1}{\tau^2}+\frac{3\tau^2}{2h^4}+\frac{\tau^2}{h^2}+\frac{\tau^2}{4}\right]U_n^{k+1}+\left[-\frac{\tau^2}{h^4}-\frac{\tau^2}{2h^2}\right]U_{n-1}^{k+1}+\left[-\frac{1}{h^2}\right]U_{n-1}^k+\frac{\tau^2}{4h^4}U_{n-2}^{k+1}=\varphi_n^k, \\ \varphi_n^k=2[1+\frac{\alpha^2+\alpha}{(1-\alpha)^2}+\frac{2\alpha}{1-\alpha}k\tau+(k\tau)^2]\sin(nh),1\le k\le N-1,\ 2\le n\le M-2, \\ U_0^k=U_M^k=0,\ \ U_n^0=\alpha U_n^N,\quad 0\le k\le N,\ \ 0\le n\le M, \\ aU_{n+1}^1-U_n^0+bU_n^1\ +aU_{n-1}^1+c\varphi_n^0 \\ =\alpha\left(-aU_{n+1}^N-dU_n^N-U_n^{N-1}-aU_{n-1}^N-c\varphi_n^N\right),\ 1\le n\le M-1, \\ U_3^k=4U_2^k-5U_1^k,\quad U_{M-3}^k=4U_{M-2}^k-5U_{M-1}^k,\quad 0\le k\le N, \end{cases}$$

following matrix equations are obtained

$$\begin{cases} A\,U_{n+2}+B\,U_{n+1}+C\,U_n+DU_{n-1}+EU_{n-2}=R\varphi_n,\quad 2\le n\le M-2, \\ U_0=\sigma,\ \ U_M=\sigma,\ \ U_3^k=4U_2^k-5U_1^k,\ \ U_{M-3}^k=4U_{M-2}^k-5U_{M-1}^k, 0\le k\le N. \end{cases}$$

Denote

$$\begin{aligned}
x &= \tfrac{\tau^2}{4h^4},\\
y &= -\tfrac{\tau^2}{h^4} - \tfrac{\tau^2}{2h^2},\\
z &= \tfrac{1}{\tau^2} + \tfrac{3\tau^2}{2h^4} + \tfrac{\tau^2}{h^2} + \tfrac{\tau^2}{4},\\
t &= -\tfrac{2}{\tau^2} + \tfrac{2}{h^2} + 1,\\
v &= \tfrac{1}{\tau^2},\\
w &= -\tfrac{1}{h^2}.
\end{aligned}$$

Here

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \ldots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\
0 & 0 & x & 0 & \ldots & 0 & 0 & 0\\
0 & 0 & 0 & x & \ldots & 0 & 0 & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
0 & 0 & 0 & 0 & \ldots & 0 & x & 0\\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & x\\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\
0 & w & y & 0 & \ldots & 0 & 0 & 0\\
o & 0 & w & y & \ldots & 0 & 0 & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\
0 & 0 & 0 & 0 & \ldots & w & y & 0\\
0 & 0 & 0 & 0 & \ldots & 0 & w & y\\
0 & a & 0 & 0 & \ldots & 0 & 0 & a\alpha
\end{bmatrix}_{(N+1)\times(N+1)}$$

$$C = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\alpha\\
v & t & z & 0 & \ldots & 0 & 0 & 0 & 0\\
o & v & t & z & \ldots & 0 & 0 & 0 & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots & .. & \ldots & \ldots & \ldots\\
0 & 0 & 0 & 0 & \ldots & v & t & z & 0\\
0 & 0 & 0 & 0 & \ldots & 0 & v & t & z\\
-1 & b & 0 & 0 & \ldots & 0 & 0 & \alpha & d\alpha
\end{bmatrix}_{(N+1)\times(N+1)}$$

and $D = B, \quad E = A.$

$$R = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 0\\
0 & 1 & 0 & \ldots & 0 & 0\\
0 & 0 & 1 & \ldots & 0 & 0\\
. & . & . & \ldots & . & .\\
0 & 0 & 0 & \ldots & 1 & 0\\
-c & 0 & 0 & \ldots & 0 & -c\alpha
\end{bmatrix}_{(N+1)\times(N+1)}$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ \dots \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)} , \quad s = n-2,\ n-1,\ n,\ n+1,\ n+2.$$

For the solution of the last matrix equation, the modified variant Gauss elimination method is used. We seek a solution of the matrix equation by the following form

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}, \quad n = M-2, ..., 2, 1, 0,$$

where α_j , β_j, $(j = 1 : M-1)$ are $(N+1)\times(N+1)$ square matrices and γ_j-s are $(N+1)\times 1$ column matrices. And α_1, β_1, γ_1, α_2, β_2, γ_2 :

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$\beta_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$\gamma_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}_{(N+1)\times(1)} , \quad \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}_{(N+1)\times(1)} ,$$

$$\alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & 0 & \dots & 0 \\ 0 & \frac{4}{5} & 0 & \dots & 0 \\ 0 & 0 & \frac{4}{5} & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & \frac{4}{5} \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$\beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & 0 & \dots & 0 \\ 0 & -\frac{1}{5} & 0 & \dots & 0 \\ 0 & 0 & -\frac{1}{5} & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & -\frac{1}{5} \end{bmatrix}_{(N+1)\times(N+1)} .$$

Using the equality $U_s = \alpha_{s+1}U_{s+1} + \beta_{s+1}U_{s+2} + \gamma_{s+1}$, (for $s = n,\ n-1,\ n-2$) and the equality
$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n$, gives

$$\begin{aligned}
&[A + C\beta_{n+1} + D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1}]U_{n+2}\\
&+[B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n + E\alpha_{n-1}\alpha_{n+1} + E\alpha_{n-1}\beta_n\\
&+E\beta_{n-1}\alpha_{n+1}]U_{n+1}\\
&+C\gamma_{n+1} + D\alpha_n\gamma_{n+1} + D\gamma_n + E\alpha_{n-1}\alpha_n\gamma_{n+1} + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\gamma_{n+1}\\
&+E\gamma_{n-1} = R\varphi_n.
\end{aligned}$$

The last equation is satisfied if it is to be selected:

$$\begin{cases}
A + C\beta_{n+1} + D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1} = 0,\\
B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n + E\alpha_{n-1}\alpha_{n+1} + E\alpha_{n-1}\beta_n + E\beta_{n-1}\alpha_{n+1} = 0,\\
C\gamma_{n+1} + D\alpha_n\gamma_{n+1} + D\gamma_n + E\alpha_{n-1}\alpha_n\gamma_{n+1} + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\gamma_{n+1}\\
+E\gamma_{n-1} = R\varphi_n,\\
2 \le n \le M-2.
\end{cases}$$

Formulas for α_{n+1}, β_{n+1}, γ_{n+1}:

$$\begin{aligned}
\alpha_{n+1} &= \left[C + D\alpha_n + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\right]^{-1}\left[-B - D\beta_n - E\alpha_{n-1}\beta_n\right],\\
\beta_{n+1} &= -\left[C + D\alpha_n + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\right]^{-1}A,\\
\gamma_{n+1} &= \left[C + D\alpha_n + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\right]^{-1}\left[R\varphi_n - D\gamma_n + E\alpha_{n-1}\gamma_n + E\gamma_{n-1}\right].
\end{aligned}$$

For solution of the last difference equation, obtain U_M, U_{M-1}, U_{M-2} :

$$\begin{aligned}
U_M &= \sigma,\\
U_{M-1} &= \left(\left(\beta_{M-2} + 5I\right) - \left(4I - \alpha_{M-2}\right)\alpha_{M-1}\right)^{-1}\left(\left(4I - \alpha_{M-2}\right)\right)\gamma_{M-1} - \gamma_{M-2},\\
U_{M-2} &= \alpha_{M-1}U_{M-1} + \gamma_{M-1}.
\end{aligned}$$

Applying the formulas

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1},\ n = M-3, ..., 2, 1, 0,$$

obtain $U_n - s$ $(n = M-3, ..., 2, 1, 0)$.

Algorithm

1. **Step** Input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$
2. **Step** Use the second order of accuracy difference scheme and write in matrix form;

$$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \quad 2 \le n \le M-2,$$

3. **Step** Determine the entries of the matrices A, B, C, D, E and R.

4. **Step** Find $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$.

5. **Step** Compute α_{n+1}, β_{n+1}, γ_{n+1}.

6. **Step** Find U_M, U_{M-1}, U_{M-2}.

7. **Step;** Compute U_n-s (n=M-3,...,2,1), using the following formula:
$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}.$$

Matlab Implementation of the Second Order of Accuracy Difference Scheme Generated by $\mathbf{A}^2$

```
function secondorderAA
    close; close;
    N=20; M=20;
    al=0.1;
    tau=1/N; h=pi/M;
    x= (tau^2)/(4*(h^4));
    y= -1*(tau^2)*(1/(h^4)+(1/(2*(h^2)))) ;
    z= (1/tau^2)+(3*(tau^2))/(2*(h^4))+(tau^2)/h^2+(tau^2)/4;
    t= (-2/tau^2)+(2/h^2)+1;
    v= 1/tau^2;
    w= -1/h^2;
    a=-(tau^2)/(2*h^2);
    b=1+(tau^2)/h^2+(tau^2)/2;
    c=-tau^2/2;
    d=-1+(tau^2)/h^2+(tau^2)/2;
    for i=2:N; A(i,i+1) = x ; end ; A(N+1,N+1)=0;
    E=A;
    for i=2:N ; B(i,i) =w ; end;
    for i=2:N ; B(i,i+1)= y ; end;
    B(N+1,2)=a; B(N+1,N+1)=a*al;
    D=B;
    for i=2:N ; C(i,i-1)=v ; end;
    for i=2:N ; C(i,i) =t ; end;
    for i=2:N ; C(i,i+1)= z ; end;
    C(1,1)=1 ; C(1,N+1)=-al;
    C(N+1,1)=-1; C(N+1,2)=b; C(N+1,N)=al; C(N+1,N+1)=d*al;
```

```
for i=2:N; R(i,i)= 1 ; end; R;
R(N+1,1)=-c;R(N+1,N+1)=-c*al;
R;
alpha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
betha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;
gamma(N+1,1:1)= 0 ;
alpha(1:N+1,1:N+1,2:2) = (4/5)*eye(N+1) ;
betha(1:N+1,1:N+1,2:2) = (-1/5)*eye(N+1);
gamma(N+1,2:2)= 0 ;
for j=2:M-2;
for k=1:N+1;
s=1+((al^2)+al)/(1-al)^2+ (2*al/(1-al))*(k-1)*tau + ((k-1)*tau)^2;
fii(k,j:j) =2*s * sin((j)*h) ;
end;
end;
fii;
for n = 2:M-2 ;
bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)...
+ E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;
betha(:,:,n+1:n+1 ) = -inv( bebek )*(A) ;
alpha(:,:,n+1:n+1) = -inv(bebek )*(B +D*betha(:,:,n:n) ...
+ E * alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*...
(R*fii(:,n:n) - D * gamma(:,n:n)...
-E * alpha(:,:,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
U(1:N+1,1:N+1)=nan;
U( 1:N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2) + 5*eye(N+1)) ...
- (4*eye(N+1)-alpha(:,:,M-2:M-2) )*alpha(:,:,M-1:M-1))...
*((4*eye(N+1)-alpha(:,:,M-2:M-2))*gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) =alpha(:,:,M-1:M-1)*U(:,M-1:M-1)+gamma(:,M-1:M-1);
'INITIAL VALUES OF U IS OBTAINED HERE' ;
for z = M-3:-1:1 ;
```

```
U(:,z:z )=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)+ ...
betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamma(:,z+1:z+1);
end;
for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;
title('SECOND-ORDER SOLUTION');
rotate3d ;
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p);
rotate3d ;
```

3.4 Numerical Analysis

Consider the nonlocal boundary-value problem for wave equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = 2[1 + \frac{\alpha^2+\alpha}{(1-\alpha)^2} + \frac{2\alpha}{1-\alpha}t + t^2]\sin x, \\ (0 < t < 1, \quad 0 < x < \pi), \\ u(0,x) = \alpha u(1,x), \ u_t(0,x) = \alpha u_t(1,x), \quad 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \quad 0 \leq t \leq 1. \end{cases} \tag{3.4}$$

The exact solution is:

$$u(t,x) = \left(\frac{\alpha^2+\alpha}{(1-\alpha)^2} + \frac{2\alpha}{1-\alpha}t + t^2\right)\sin x.$$

For approximate solutions of the nonlocal boundary-value problem (3.4), the first and the second order of accuracy difference schemes with $\tau = \frac{1}{15}$, $h = \frac{\pi}{45}$, $\alpha = 0.1$ will be used. The second order or fourth order difference equations to respect in n with matrix coefficients have been taken. To solve this difference equations have been applied a procedure of modification Gauss elimination method. The exact and numerical solutions are given in the following table 3.1 and figures(3.1, 3.2, 3.3, 3.4)

TABLE

The first line is the exact solution,the second line is the solution of the first order of accuracy difference scheme, the third line is the solution of second order of accuracy difference scheme and the fourth line is the solution of second order of accuracy difference scheme generated by A^2.

$t_k \setminus x_n$	0	0.63	1.26	1.89	2.52	3.14
0.0	0	0.0798	0.1292	0.1292	0.0798	0.0000
	0	0.0732	0.1185	0.1185	0.0732	0.0000
	0	0.0797	0.1289	0.1289	0.0797	0.0000
	0	0.0797	0.1290	0.1290	0.0797	0.0000
0.2	0	0.1295	0.2095	0.2095	0.1295	0.0000
	0	0.1127	0.1823	0.1823	0.1127	0.0000
	0	0.1292	0.2090	0.2090	0.1292	0.0000
	0	0.1292	0.2090	0.2090	0.1292	0.0000
0.4	0	0.2261	0.3659	0.3659	0.2261	0.0000
	0	0.1986	0.3213	0.3213	0.1986	0.0000
	0	0.2256	0.3651	0.3651	0.2256	0.0000
	0	0.2257	0.3651	0.3651	0.2257	0.0000
0.6	0	0.3698	0.5983	0.5983	0.3698	0.0000
	0	0.3307	0.5350	0.5350	0.3307	0.0000
	0	0.3690	0.5971	0.5971	0.3690	0.0000
	0	0.3692	0.5973	0.5973	0.3692	0.0000
0.8	0	0.5605	0.9069	0.9069	0.5605	0.0000
	0	0.5085	0.8228	0.8228	0.5085	0.0000
	0	0.5594	0.9052	0.9052	0.5594	0.0000
	0	0.5597	0.9055	0.9055	0.5597	0.0000
1.0	0	0.7982	1.2916	1.2916	0.7982	0.0000
	0	0.7321	1.1845	1.1845	0.7321	0.0000
	0	0.7968	1.2893	1.2893	0.7968	0.0000
	0	0.7971	1.2898	1.2898	0.7971	0.0000

Table 3.1 Numerical analysis

Thus, the second order of accuracy difference schemes were more accurate compare with the first order of accuracy difference scheme.

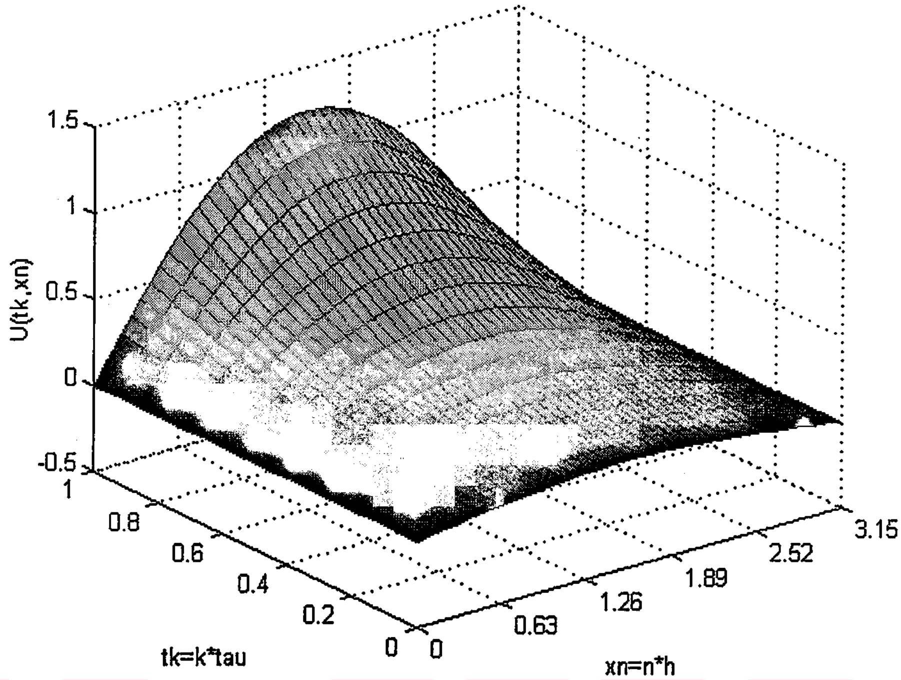

Figure 3.1 The exact solution

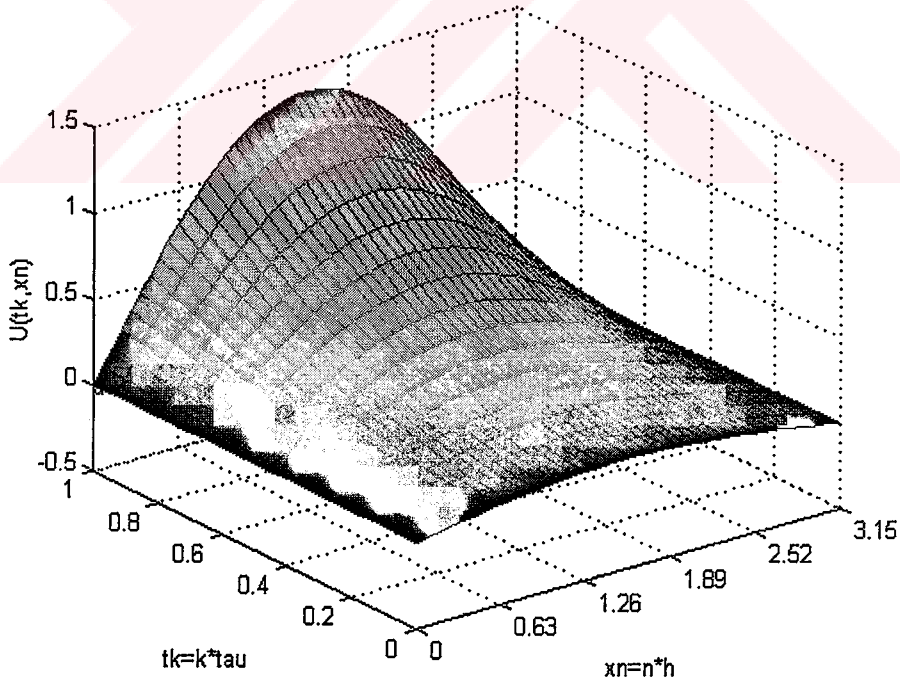

Figure 3.2 The first order of accuracy difference scheme

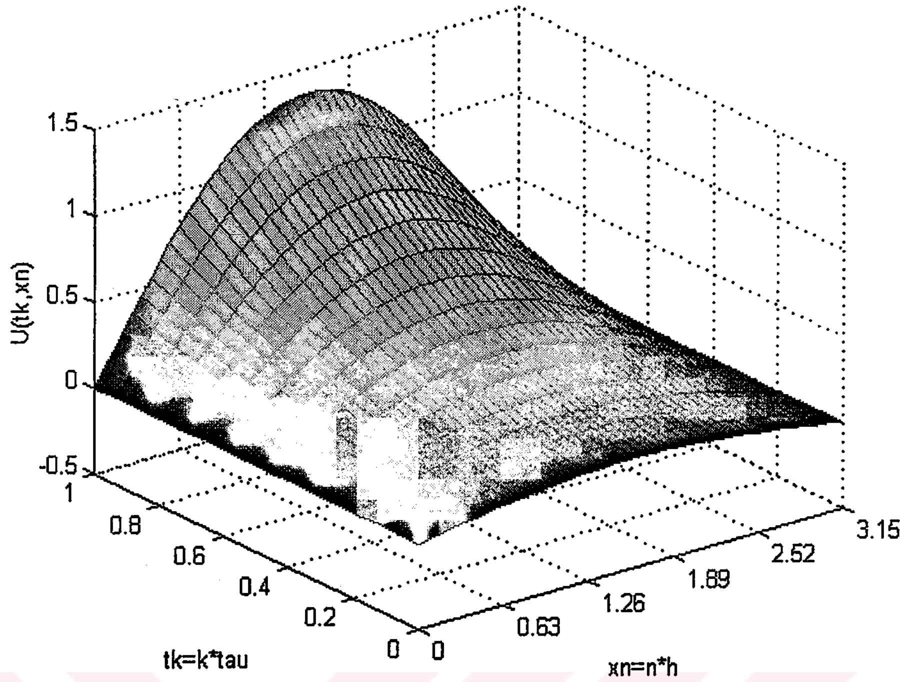

Figure 3.3 The second order of accuracy difference scheme

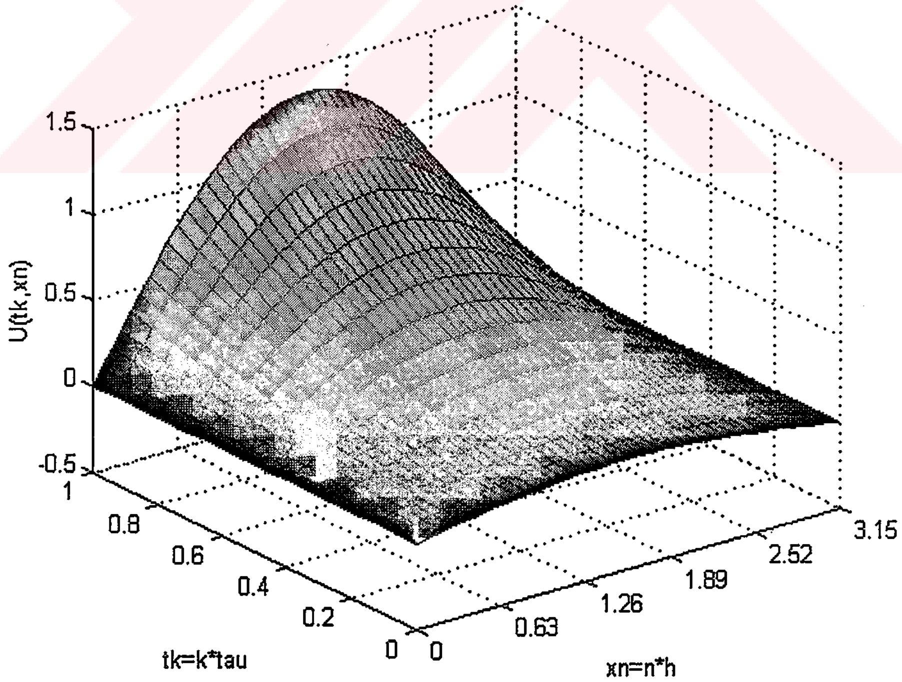

Figure 3.4 The second order of accuracy difference scheme generated by A^2

The first figure is the exact solution,the second figure is the solution of the first order of accuracy difference scheme, the third figure is the solution of second order of accuracy difference scheme and the fourth figure is the solution of second order of accuracy difference scheme generated by A^2.

CONCLUSIONS

This work is devoted to the study of the stability of the nonlocal boundary value problem for hyperbolic equations. The following original results are obtained:

- the abstract theorem on the stability estimates for solution of the nonlocal boundary problem for hyperbolic equations in the Hilbert space are proved,

- the stability estimates for the solutions of the two types of nonlocal boundary value problems for hyperbolic equations are obtained,

- the first and second order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem are described ,

-three theorems on the stability estimates for the solutions of these difference schemes are proved,

- the numerical analysis is given. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Finally,note that, this work is a part of project : "Theory of positive operators and the stability of difference schemes of approximate solutions of the nonlocal boundary value problem for partial differential equations" supported by Fatih University in 2001-2002(P50040101).

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