DIFFERENCE SCHEMES OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

In the present work the nonlocal boundary problem for a difference equation in a Banach space E with strongly positive operator A

$$-\frac{1}{\tau^2}\left[u_{k+1}-2u_k+u_{k-1}\right]+Au_k=\varphi_k, 1\le k\le N-1,$$
$$u_0=u_N, -u_2+4u_1-3u_0=u_{N-2}-4u_{N-1}+3u_N, N\tau=1,$$

is considered. Applying the operator approach we obtain the stability estimates, almost coercive stability estimates and coercive stability estimates for the solution of this nonlocal boundary problem. In applications this abstract result permits us to obtain the stability estimates, almost coercive stability estimates and coercive stability estimates for the solution of the difference schemes for elliptic equations. This result is based on the positivity of the difference operator generated by nonlocal boundary conditions. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Keywords: Elliptic Difference Equation, Difference Schemes, Stability.

ELİPTİK DENKLEMLERİN LOKAL OLMAYAN SINIR DEĞER PROBLEMLERİNİN FARK METODLARI

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ÖZ

Bu çalışmada Banach uzayında verilen fark denkleminin kuvvetli A pozitif operatörlü lokal olmayan sınır değer problemi

$$-\frac{1}{\tau^2}\left[u_{k+1}-2u_k+u_{k-1}\right]+Au_k=\varphi_k, 1\le k\le N-1,$$
$$u_0=u_N, -u_2+4u_1-3u_0=u_{N-2}-4u_{N-1}+3u_N, N\tau=1,$$

ele alınmıştır. Operatör yaklaşımını uygulayarak bu lokal olmayan sınır değer probleminin cözümünün kararlılık kestirimlerini, hemen hemen koersitif kestirimlerini ve koersif kestirimlerini elde ettik. Uygulamalarda bu sonuç, eliptik denklemlerin fark metodlarının çözümü için kararlılık kestirimlerini, hemen hemen koersif kestirimlerini ve koersif kestirimlerini elde etmemizi sağladı. Bu fark metodlarının çözümü için yapılan teorik sonuçların doğruluğu, yapılan numerik denemelerle desteklenmiştir.

Anahtar kelimeler: Eliptik Fark Denklemi, Fark Metodları, Kararlılık.

DEDICATION

To my parents

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INTRODUCTION

It is known that most problems in fluid mechanics (dynamics, elasticity) and other areas of physics lead to partial differential equations of the elliptic type. These equations can be derived as models of physical systems and are considered as the methods for solving boundary value problems.

A problem is called *well-posed* if for each set of data there exists exactly one solution and this solution is dependent on the data continuously. Our goal in this work is to investigate the well-posedness of difference schemes for well-posed of the nonlocal boundary value problems for equations of elliptic type.

It is known that the mixed problem for elliptic equations can be solved by Fourier series method, by Fourier transform method and by Laplace transform method.

Now let us give some examples.

First let us consider the simple nonlocal boundary value problem for elliptic equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x^2} = (-2t^3 + 3t^2 + 11t - 6)\sin x, \quad 0 \le t \le 1, \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0 \ , \ 0 \le t \le 1, \\ u(0,x) = u(1,x) \ , \ u_t(0,x) = u_t(1,x) \ , \quad 0 \le x \le \pi. \end{cases} \tag{1.1}$$

For the solution of the problem (1.1), we use the method of separation of variables or so called Fourier series method. In order to solve the problem we need to separate $u(t,x)$ into two parts

$$u(t,x) = v(t,x) + w(t,x),$$

where $v(t,x)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial x^2} = 0, \ 0 \le t \le 1, \ 0 < x < \pi, \\ v(t,0) = v(t,\pi) = 0 \ , \qquad 0 \le t \le 1, \\ v(0,x) = v(1,x) \ , \ v_t(0,x) = v_t(1,x) \ , \quad 0 \le x \le \pi, \end{cases} \tag{1.2}$$

and $w(t,x)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}+\dfrac{\partial^2 w}{\partial x^2}=(-2t^3+3t^2+11t-6)\sin x,\ 0\le t<1,\ 0<x<\pi, \\ w(t,0)=w(t,\pi)=0\ , \quad 0\le t<1, \\ w(0,x)=w(1,x)\ ,\ w_t(0,x)=w_t(1,x)\ ,\ \ 0\le x\le \pi. \end{cases} \tag{1.3}$$

First we will obtain the solution of the problem (1.2). By the method of separation of variables, we obtain

$$v(t,x)=T(t)X(x)$$

We have that

$$\frac{T''(t)}{T(t)}+\frac{X''(x)}{X(x)}=0$$

or

$$\frac{T''(t)}{T(t)}=-\frac{X''(x)}{X(x)}=-\lambda$$

and using the boundary conditions, we obtain

$$X(0)=X(\pi)\ =0.$$

We have that

$$X''(x)=-\lambda X(x),$$
$$X(0)=X(\pi)=0.$$

If $\lambda\ge 0$, then the boundary value problem

$$X''(x)-\lambda X(x)=0,\ X(0)=X(\pi)=0$$

has only trivial solution $X(x)=0$.

So, we consider only the case $\lambda<0$. The nontrivial solutions of the boundary value problem

$$X''(x)-\lambda X(x)=0,\ X(0)=X(\pi)=0$$

are

$$X_k(x)=A_k\cos(kx)+B_k\sin(kx)\ \ ,\quad \text{where } k=1,2,3,\cdots.$$

If we use the boundary conditions then we obtain

$$X_k(x)=\sin(kx)\,,\qquad \text{where } k=1,2,3,\cdots.$$

Now to get $T(t)$ we can write

$$\frac{T''(t)}{T(t)}=-\lambda$$

or

$$T''(t)+\lambda T(t)=0.$$

So,

$$T_k(t)=E_k\exp(kt)+F_k\exp(-kt).$$

Thus,

$$v(t,x)=\sum_{k=1}^{\infty}v_k(t,x)=\sum_{k=1}^{\infty}(E_k\exp(kt)+F_k\exp(-kt))\sin kx.$$

Using the nonlocal boundary conditions, we obtain

$$v_t(0,x) = v_t(1,x)$$

or

$$E_k = F_k = 0 \quad \text{for any } k.$$

Hence

$$v(t,x) = 0.$$

Second we will obtain the solution for (1.3). Let

$$w(t,x) = \sum_{k=1}^{\infty} D_k(t) \sin kx.$$

Then

$$\begin{aligned} w_{tt} + w_{xx} &= D_k''(t) \sin kx - k^2 D_k(t) \sin kx \\ &= \left(-2t^3 + 3t^2 + 11t - 6\right) \sin x. \end{aligned}$$

If $k \neq 1$, then

$$D_k''(t) - k^2 D_k(t) = 0.$$

So we obtain

$$D_k(t) = c_1 e^{kt} + c_2 e^{-kt}.$$

Using the boundary conditions we get $c_1 = c_2 = 0$.

So $D_k(t) = 0$ for any $k \neq 1$.

If $k = 1$, then

$$D_1''(t) - D_1(t) = -2t^3 + 3t^2 + 11t - 6.$$

$D_1(t)$ can be found as

$$D_1(t) = c_1 e^t + c_2 e^{-t} + (2t^3 - 3t^2 + t).$$

Now by using the nonlocal boundary conditions, we obtain that

$$w(t,x) = (2t^3 - 3t^2 + t) \sin x.$$

Thus,

$$\begin{aligned} u(t,x) &= v(t,x) + w(t,x) \\ &= 0 + \left(2t^3 - 3t^2 + t\right) \sin x \end{aligned}$$

or

$$u\,(t,x) = \left(2t^3 - 3t^2 + t\right) \sin x$$

is the solution of the given nonlocal boundary value problem (1.1).

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} + \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, \ldots, x_n) \in \overline{\Omega}, 0 \leq t \leq T, \\ u(0,x) = u\,(T,x) + \varphi(x), \ x \in \overline{\Omega}, \\ u_t(0,x) = u_t(T,x) + \psi(x), x \in \overline{\Omega} \\ u(t,x) = 0, x \in S \end{cases}$$

where α_r and $f(t,x)$ $(t \in [0,T],\ x \in \overline{\Omega})$, $\varphi(x), \psi(x)$ $(x \in \overline{\Omega})$ are given smooth functions. Here Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \le k \le n)$ with boundary

$$S,\ \overline{\Omega} = \Omega \cup S.$$

However, the method of separation of variables can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Second, we consider the nonlocal boundary value problem for elliptic equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x^2} = 2(2t^3 - 3t^2 + t) + (12t - 6)x^2, 0 < t < 1, 0 < x < \pi, \\ u(t,0) = u_x(t,0) = 0\ , \quad 0 \le t \le 1, \\ u(0,x) = u(1,x), \quad u_t(0,x) = u_t(1,x)\ , \quad 0 \le x \le \pi. \end{cases} \tag{1.4}$$

Here, we will use the Laplace transform method (in x) to solve the problem (1.4). Let

$$u(t,s) = L\{u(t,x)\}.$$

So our problem becomes

$$\frac{\partial^2 u(t,s)}{\partial t^2} + s^2 u(t,s) = \frac{2(2t^3 - 3t^2 + t)}{s} + \frac{2(12t - 6)}{s^3}.$$

Now the homogenous equation is

$$u_{tt} + s^2 u(t,s) = 0.$$

Then

$$u(t,s) = c_1 \sin st + c_2 \cos st\ .$$

And for the particular solution we will use the UC method. Let

$$u^p(t,s) = At^3 + Bt^2 + Ct + D$$

substituting in the equation we get

$$6At + 2B + s^2(At^3 + Bt^2 + Ct + D) = \frac{2(2t^3 - 3t^2 + t)s^2}{s^3} + \frac{2(12t - 6)}{s^3}$$

and

$$A = \frac{4}{s^3}, \quad B = -\frac{6}{s^3}, \quad C = \frac{2}{s^3}, \quad D = 0.$$

So

$$u(t,s) = c_1 \sin st + c_2 \cos st + \frac{4}{s^3}t^3 - \frac{6}{s^3}t^2 + \frac{2}{s^3}t.$$

Now, using the nonlocal boundary conditions we can write

$$u(t,s) = \frac{4}{s^3}t^3 - \frac{6}{s^3}t^2 + \frac{2}{s^3}t.$$

Finally taking the inverse of Laplace we obtain

$$u(t,x) = L^{-1}\{u(t,s)\} = 2L^{-1}\left\{\frac{2}{s^3}t^3 - \frac{3}{s^3}t^2 + \frac{1}{s^3}t\right\}$$

$$= 2t^3x^2 - 3t^2x^2 + tx^2.$$

Hence

$$u(t,x) = (2t^3 - 3t^2 + t)x^2$$

is the solution of the given nonlocal boundary value problem (1.4).

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} + \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, \dots, x_n) \in \overline{\Omega}^+, 0 \le t \le T, \\ u(0,x) = \varphi(x), \\ u_t(0,x) = \psi(x), \ x \in \overline{\Omega}^+, \\ u(t,x) = 0, \ x \in S^+, \end{cases}$$

where α_r and $f(t,x)$ $(t \in [0,T],\ x \in \overline{\Omega}^+)$, $\varphi(x), \psi(x)$ $(x \in \overline{\Omega}^+)$ are given smooth functions. Here Ω^+ is the open set in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with boundary

$$S^+, \ \overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However, Laplace transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Third, we consider a mixed nonlocal boundary value problem for elliptic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = (12t-6)\exp(-x^2) + (2t^3 - 3t^2 + t)(-2 + 4x^2)\exp(-x^2), \\ 0 < t < 1, \ -\infty < x < \infty, \\ u(t,-\infty) = u(t,\infty) = 0, \quad 0 \le t < 1, \\ u(0,x) = u(1,x), \ \ u_t(0,x) = u_t(1,x), \ \ -\infty < x < \infty. \end{cases} \tag{1.5}$$

Here we will use the Fourier transform method to solve the problem (1.5).

We take the Fourier transform of both sides of the equation

$$u_{tt} + u_{xx} = (12t-6)\exp(-x^2) + (2t^3 - 3t^2 + t)(-2 + 4x^2)\exp(-x^2).$$

Then, we have

$$F\{u_{tt}\} + F\{u_{xx}\} = F\{(12t-6)\exp(-x^2)\} + F\{(2t^3 - 3t^2 + t)(-2 + 4x^2)\exp(-x^2)\}.$$

From that it follows

$$(F\{u_{tt}(t,x)\}) + (is)^2 F\{u(t,x)\}$$
$$= (12t-6)F\{\exp(-x^2)\} + (2t^3 - 3t^2 + t)(-s^2 F\{\exp(-x^2)\}).$$

Now let $u(t,s) = F\{u(t,x)\}$. So our problem becomes

$$\frac{\partial^2 u(t,s)}{\partial t^2} - s^2 u(t,s)$$
$$= (12t-6)F\{\exp(-x^2)\} + (2t^3 - 3t^2 + t)(-s^2 F\{\exp(-x^2)\}).$$

Now the homogenous equation is

$$u_{tt}(t,s) - s^2 u(t,s) = 0.$$

Then

$$u(t,s) = c_1 e^{st} + c e^{-st}.$$

For the particular equation we will use the UC method. Let

$$u^p(t,s) = At^3 + Bt^2 + Ct + D.$$

Substituting in the equation we get

$$6At + 2B - s^2(At^3 + Bt^2 + Ct + D) = ((12t-6) - s^2(2t^3 - 3t^2 + t))F\{e^{-x^2}\}.$$

Hence

$$\begin{aligned} A &= 2F\{e^{-x^2}\}, \\ B &= -3F\{e^{-x^2}\}, \\ C &= F\{e^{-x^2}\}, \\ D &= 0. \end{aligned}$$

So the solution is

$$u(t,s) = c_1 e^{st} + c_2 e^{-st} + (2t^3 - 3t^2 + t)F\{e^{-x^2}\}.$$

Now using the nonlocal boundary conditions we get $u(t,s)$ as

$$u(t,s) = (2t^3 - 3t^2 + t)F\{e^{-x^2}\}.$$

Finally taking the inverse of Fourier transformation we obtain the solution for the problem (1.5) as

$$u(t,x) = (2t^3 - 3t^2 + t)e^{-x^2}.$$

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the 2m-th order multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} + \sum\limits_{|r|=2m} a_r \frac{\partial^{|r|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = f(t,x), \\ 0 \le t \le T, x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ u(0,x) = u(T,x) + \varphi(x), x \in \mathbb{R}^n, \\ u_t(0,x) = u_t(T,x) + \psi(x),\ x \in \mathbb{R}^n, \end{cases}$$

where a_r, $f(t,x)$ $(t \in [0,T],\ x \in R^n)$, $\varphi(x), \psi(x)$ $(x \in R^n)$ are given smooth functions. However, the Fourier transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

In the present work the nonlocal boundary problem for difference equation in a Banach space E with strongly positive operator A

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1} - 2u_k + u_{k-1}] + Au_k = \varphi_k, 1 \leq k \leq N-1, \\ u_0 = u_N, \quad -u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N, \quad N\tau = 1, \end{cases}$$

is considered. Applying the operator approach we obtain the stability estimates, almost coercive stability estimates and coercive stability estimates for solution of this nonlocal boundary problem. In applications this abstract result permits us to obtain the stability estimates, almost coercive stability estimates and coercive stability estimates for the solution of the difference schemes for elliptic equations.This result is based on the positivity of the difference operator generated by nonlocal boundary conditions. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Let us briefly describe the contents of the various sections. It consists of six chapters.

First chapter is the introduction.

Second chapter presents positive operators and fractional spaces, analytic semigroups that is needed for this work.

Third chapter consists of five sections. A brief survey of all investigations in this area can be found in the first section. In the second section the Green's function is constructed. Third section is devoted to the study of the positivity of the operator A with constant coefficients generated by the nonlocal boundary value problem in C_h Banach space. In the fourth section the positivity of the difference operator with variable coefficients generated by the nonlocal boundary value problem in C_h Banach space is studied. In the last section the positivity of difference operators in the C_h^α Holder space is established.

Fourth chapter is about the well posedness of the second order of accuracy difference schemes.

Fifth chapter is the applications. The first and second order of accuracy difference schemes are studied. A matlab program is given to conclude that the second order of accuracy is more accurate. The tables and figures are included.

Sixth chapter is the conclusions.

Chapter 2

POSITIVE OPERATORS AND FRACTIONAL SPACES ANALYTIC SEMIGROUPS

Now let us give the definition of positive operators and introduce the fractional spaces that will be needed in the sequel.

Definition 2.1. The operator A is said to be strongly positive if its spectrum $\sigma(A)$ lies in the interior of the sector of angle ϕ, $0 < 2\phi < \pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1(\phi) = \{\rho e^{i\phi} : 0 \leq \rho \leq \infty\}$ and $S_2(\phi) = \{\rho e^{-i\phi} : 0 \leq \rho \leq \infty\}$, and outside of the resolvent $(\lambda - A)^{-1}$ is subject to the bound

$$\left\|(\lambda - A)^{-1}\right\|_{E \to E} \leq \frac{M(\phi)}{1 + |\lambda|} \tag{2.1}$$

The infimum of all such angles ϕ is called the spectral angle of the strongly positive operator A and is denoted by $\phi(A) = \phi(A, E)$. Since the spectrum $\sigma(A)$ is a closed set, it lies inside the sector formed by the rays $S_1(\phi(A))$ and $S_2(\phi(A))$, and some neighborhood of the apex of this sector does not intersect $\sigma(A)$. We shall consider contours $\Gamma = \Gamma(\phi, r)$ composed by the rays $S_1(\phi)$, $S_2(\phi)$ and an arc of circle of radius r centered at the origin; ϕ and r will be chosen so that $\sigma(A) < |\sigma| < \pi/2$ and the arc of circle of radius r lies in the resolvent set $\rho(A)$ of the operator A.

Definition 2.2. A family $U(t)$, $t \geq 0$, of bounded linear operators is a *strongly continuous semigroup* if the following conditions are satisfied:

1. $U(t + \tau) = U(t)U(\tau) = U(\tau)U(t), \quad t \geq 0, \quad \tau \geq 0; \quad U(0) = I.$
2. For each fixed $v_0 \in E$ the function $U(t)v_0$ is continuous in t for $t \geq 0$.

From the strong continuity of the operator function $U(t)$ it follows that its norm is uniformly bounded on any bounded segment $[0, T]$. Next, from the semigroup property it follows that when $t \to \infty$ this norm grows no faster than an exponential. Specifically, one has the estimate

$$\|U(t)\|_{E \to E} \leq M e^{wt}, \quad t \geq 0. \tag{2.2}$$

Definition 2.3. The operator $U'(0)$, defined by the formula

$$U'(0)v_0 = \lim_{\Delta t \to +0} \Delta t^{-1}[U(\Delta t) - I]v_0$$

on the elements $v_0 \in E$ for which the limit on the right hand side exists, is called the *generator of the semigroup* $U(t)$.

The operator $U'(0)$ has a dense domain in E and for any $\mu > w$ in the case of a real space E, or any complex μ with $\operatorname{Re}\mu > w$ in the case of a complex space E, the operator $\mu I - U'(0)$ has a bounded inverse, i.e., $U'(0)$ is closed.

Theorem 2.1. *Let B be an operator with dense domain acting in a complex Banach space E. In order for B to be the generator of a strongly continuous semigroup $U(t)$ satisfying the estimate (2.2), it is necessary and sufficient that any complex number λ with* $\operatorname{Re}\lambda > w$*,belong to the resolvent set of B and that the following estimate holds:*

$$\left\|(\lambda I - B)^{-n}\right\|_{E\to E} \leq M(\operatorname{Re}\lambda - w)^{-n}, \quad n = 1, 2, \ldots \tag{2.3}$$

Note that one may assume that the estimates (2.3) are satisfied only for some sequence λ_m such that $\operatorname{Re}\lambda_m \to \infty$ as $m \to \infty$.

In the case of a real Banach space E the estimates (2.3) must hold for all real $\lambda > w$.

In what follows a semigroup with generator $-A$ will be denoted by $\exp\{-tA\}$. By passing from the problem with the operator A to the problem with the operator $A + \mu I$ one can ensure that the norm of this semigroup decreases exponentially, i.e., the following estimate holds:

$$\|\exp\{-tA\}\|_{E\to E} \leq Me^{-\delta t}, \quad M > 0, \quad \delta > 0. \tag{2.4}$$

Definition 2.4. A strongly continuous semigroup $U(t)$ acting in a complex Banach space E is said to be *analytic* if it can be continued from the half line $0 \leq t \leq \infty$ to an operator function $U(z)$ that is analytic in some sector

$$S_\alpha = \{z : |\arg z| < \alpha, \quad 0 < |z| < \infty\}, \quad 0 < \alpha \leq \frac{\pi}{2}$$

and is strongly continuous in its closure $\overline{S_\alpha}$.

Generators of analytic semigroups acting in a complex Banach space E admit the following characterization in terms of their resolvents.

Theorem 2.2. *Let B be an operator with dense domain acting in a complex Banach space E In order for B to be the generator of analytic semigroup it is necessary and sufficient that there exists real numbers w and $\tau > 0$ such that all complex λ satisfying* $\operatorname{Re}\lambda \geq w$ *and* $|\lambda| \geq \tau$

belong to the resolvent set of B and the following estimate holds:

$$\left\|(\lambda I - B)^{-1}\right\|_{E\to E} \leq M|\lambda|^{-1}.$$

Let $f(z)$ be an analytic function on the set bounded by such a contour Γ, and suppose that f satisfies estimate

$$|f(z)| \leq M|z|^{-\varepsilon}$$

for some $\varepsilon > 0$. Then the operator Cauchy-Riesz integral

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-A)^{-1}\,dz \tag{2.5}$$

converges in the operator norm and defines a bounded linear operator $f(A)$, a function of the strongly positive operator A. If $f(z)$ is continuous in a neighborhood of the origin, then in (2.5) we shall consider that $r = 0$, i.e., $\Gamma = S_1(\phi) \cup S_2(\phi)$.

As in the case of a bounded operator A one shows that $f(A)$ does not depend on the choice of the contour Γ in the domain of analyticity of the function $f(z)$, and that the correspondence between the function $f(z)$ and the operator $f(A)$ is linear and multiplicative.

The function $f(z) = z^{-\alpha}$ defines a bounded operator $A^{-\alpha}$ whenever $\alpha > 0$. Here the contour Γ is chosen with $r > 0$. By the multiplicativity property, $A^{-(\alpha+\beta)} = A^{-\alpha}A^{-\beta} = A^{-\beta}A^{-\alpha}$ for any powers of the strongly positive operator A, and not only for negative integer ones. From this identity it follows (when $\alpha+\beta$ is an integer) that the equation $A^{-\alpha}x = 0$ has the unique solution $x = 0$. Hence, the positive powers $A^{\alpha} = (A^{-\alpha})^{-1}$ of the strongly positive operator are defined. The operator A^{α} $(\alpha > 0)$ are unbounded if A is unbounded, they have dense domains $D(A^{\alpha})$ and one has the continuous embedding $D(A^{\alpha}) \subset D(A^{\beta})$ if $\beta < \alpha$.

The theory of fractional powers of operators can be constructed for a wider class of positive operators. For such operators the estimate (2.1) is required to hold for some ϕ and not only from the interval $[0, \pi/2]$, but from the larger interval $[0, \pi)$.

Now let us consider the function $f(z) = e^{-tz}$. For any $t > 0$ this function tends to zero faster than any power $z^{-\alpha}$ as $|z| \to \infty$ and its values lie inside any sector bounded by a contour Γ. Therefore, formula (2.5) can be used to define the function $\exp\{-tA\}$ of the strongly positive operator A. By multiplicativity, the semigroup property holds:

$$\exp\{-(t_1+t_2)A\} = \exp\{-t_1A\}\exp\{-t_2A\},\ t_1, t_2 > 0.$$

Consider the function $\Psi(z) = z^{\alpha}e^{-tz}$ for some $\alpha > 0$ and $t > 0$. Since, obviously, $\Psi(z) \to 0$ faster than any negative power of z as $|z| \to \infty$, $\Psi(z)$ defines the operator function

$$\Psi(A) = \frac{1}{2\pi i}\int_\Gamma z^{\alpha}e^{-tz}(z-A)^{-1}\,dz. \tag{2.6}$$

Let us show that the operator $\exp\{-tA\}$ maps E into $D(A^{\alpha})$ and $A^{\alpha}\exp\{-tA\} = \Psi(A)$. Let x be an arbitrary element of E. By the multiplicativity property, (2.5) implies that

$$A^{-\alpha}\Psi(A)x = \frac{1}{2\pi i}\int_\Gamma e^{-tz}(z-A)^{-1}x\,dz = \exp\{-tA\}x,$$

which proves our assertion. Thus, we have the formula

$$A^{\alpha}\exp\{-tA\} = \frac{1}{2\pi i}\int_\Gamma z^{\alpha}e^{-tz}(z-A)^{-1}\,dz. \tag{2.7}$$

In the above argument we must assume that the contour Γ contains an arc of radius r, since we applied the operator $A^{-\alpha}$, which corresponds to the function $z^{-\alpha}$. The final formula (2.7) is valid for any (small) $r > 0$. Since the integrand in (2.7) is continuous at the point $z = 0$, letting $z \to 0$ we obtain the formula

$$A^{\alpha}\exp\{-tA\} = \frac{1}{2\pi i}[\int_\infty^0 \rho^{\alpha}e^{i\alpha\phi}e^{-t\rho e^{i\phi}}\left(\rho e^{i\phi} - A\right)^{-1}d\rho$$

$$+\int_0^\infty \rho^\alpha e^{-i\alpha\phi} e^{-t\rho e^{-i\phi}} \left(\rho e^{-i\phi} - A\right)^{-1} d\rho]$$

for some $0 < \phi < \pi/2$. From this and the estimate (2.1) it follows that

$$\|A^\alpha \exp\{-tA\}\|_{E\to E} \le \frac{M(\phi)}{\pi} \int_0^\infty \rho^{\alpha-1} e^{-t\rho\cos\phi} d\rho = \frac{M(\phi)\,\Gamma(\alpha)}{\pi(\cos\phi)^\alpha} t^{-\alpha}. \tag{2.8}$$

In particular, we have the estimate

$$\|\exp\{-tA\}\|_{E\to E} \le \frac{M(\phi)}{\pi}. \tag{2.9}$$

Let us show that the estimate (2.8) can be sharpened by a factor that decays exponentially when $t \to +\infty$.

Let A be a strongly positive operator. We claim that for sufficiently small $\delta > 0$ the operator $A - \delta$ is also strongly positive, and $\phi(A-\delta) = \phi(A)$. Indeed, let $\lambda \in \Gamma(\phi)$. Consider the equation $\lambda x - (A-\delta)x = y$ for an arbitrary $y \in E$. The substitution $\lambda x - Ax = z$ yields the equation $z + \delta(\lambda - A)^{-1} z = y$. Since

$$\left\|\delta(\lambda - A)^{-1}\right\|_{E\to E} \le \delta M(\phi)$$

if $\lambda \in \Gamma(\phi)$, we see that for $\delta \le [2M(\phi)]^{-1}$the equation for z has a unique solution, and $\|z\| \le 2\|y\|$. Consequently, the equation for x has a unique solution, and

$$\|x\| \le M(\phi)\,[|\lambda| + 1]^{-1} \|z\| \le 2M(\phi)\,[|\lambda| + 1]^{-1} \|y\|.$$

This means that the operator $\lambda - (A+\delta)$ has a bounded inverse for $0 < \delta \le [2M(\phi)]^{-1}$ and

$$\left\|[\lambda - (A-\delta)]^{-1}\right\|_{E\to E} \le 2M(\phi)\,[|\lambda| + 1]^{-1}.$$

Thus, we have shown that $A - \delta$ is a strongly positive operator. Hence, by (2.9), we have the estimate

$$\|\exp\{-(A-\delta)t\}\|_{E\to E} \le \frac{2M(\phi)}{\pi}.$$

This obviously yields

$$\|\exp\{-At\}\|_{E\to E} \le \frac{2M(\phi)}{\pi} e^{-\delta t}, \tag{2.10}$$

where we can put $\delta = [2M(\phi)]^{-1}$.

Let $t > 1$. Then, using the semigroup property, we can write

$$\exp\{-tA\} = \exp\{-A\}\exp\{-(t-1)A\}.$$

Next, applying the estimates (2.8) with $t = 1$ and (2.10), we obtain

$$\|A^\alpha \exp\{-tA\}\|_{E\to E} \le \frac{M(\phi)}{\pi(\cos\phi)^\alpha} \frac{2M(\phi)}{\pi} e^{-\delta(t-1)}.$$

Hence, the following estimate holds for $t > 1$:

$$\|A^\alpha \exp\{-tA\}\|_{E\to E} \le M_1(\phi)\, e^{-\delta t}.$$

If $0 < t \le 1$, then estimate (2.8) prevails. Combining these two estimates, we conclude that

$$\|A^\alpha \exp\{-tA\}\|_{E\to E} \le \tilde{M}(\phi)\, e^{-\delta t} t^{-\alpha}. \tag{2.11}$$

for some $\tilde{M}(\phi) > 0$ and $\delta > 0$.

Further, formula (2.5) allows us to establish that the operator- valued function $\exp\{-tA\}$ is differentiable in the operator norm for $t > 0$ and

$$\frac{d}{dt}\exp\{-tA\} = -A\exp\{-tA\}. \tag{2.12}$$

In particular, this implies that $\exp\{-tA\}$ is continuous in the operator norm. Using the semigroup property we deduce that the derivative of $\exp\{-tA\}$ is also continuous in the operator norm for $t > 0$. Finally, formula (2.12) shows that the operator-valued function $\exp\{-tA\}$ has derivative of arbitrary order in the operator norm for $t > 0$.

Now let $x \in D(A)$. Then the ($E-$valued) function $\exp\{-tA\}x$ has a derivative for $t > 0$ and, by (2.12),

$$\frac{d}{dt}\exp\{-tA\}x = -\exp\{-tA\}Ax. \tag{2.13}$$

Next, for x as above we can write

$$(z-A)^{-1}x = z^{-1}x + z^{-1}(z-A)^{-1}Ax.$$

Using formula (2.5), we obtain $\exp\{-tA\}x = \frac{1}{2\pi i}\int_\Gamma e^{-tz}[z^{-1}x + z^{-1}(z-A)^{-1}]dz$.

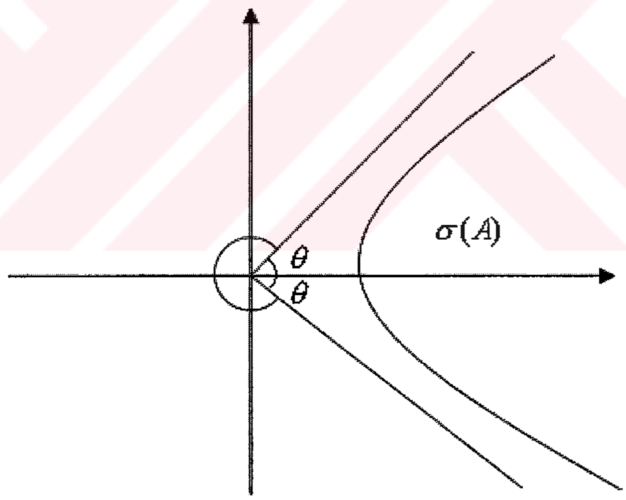

Figure 2.1: The contour

Using the Cauchy theorem, we get

$$\exp\{-tA\}x = \frac{1}{2\pi i}\int_\Gamma e^{-tz}z^{-1}(z-A)^{-1}Axdz + x.$$

The estimate (2.1) shows that in the last equality one can pass to the limit under the integral sign when $t \to +0$. Hence, the limit

$$\lim_{t\to+0}\exp\{-tA\}x = x + \frac{1}{2\pi i}\int_\Gamma z^{-1}(z-A)^{-1}Axdz.$$

exists (in the norm of E). By Cauchy's theorem, the integral

$$\vartheta = \frac{1}{2\pi i} \int_{\Gamma} z^{-1} (z - A)^{-1} Axdz = \frac{1}{2\pi i} \int_{-\sigma - i\infty}^{-\sigma + i\infty} z^{-1} (z - A)^{-1} Axdz.$$

for some $\sigma > 0$. Hence, by (2.1),

$$\|\vartheta\|_E \leq \frac{M}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{\sigma^2 + t^2} \|Ax\|_E .$$

Since ϑ does not depend on σ, it follows that $\vartheta \equiv 0$. Hence, we proved that

$$\lim_{t \to +0} \exp\{-tA\} x = x \tag{2.14}$$

for any $x \in D(A)$. Since the norm $\|\exp\{-tA\}\|_{E \to E}$ is uniformly bounded for $t > 0$, the limit relation (2.14) holds for any $x \in E$.

Thus, if we extend the operator- valued function $U(t) = \exp\{-tA\}$, $t > 0$, at $t = 0$ by $U(0) = I$, we obtain a strongly continuous semigroup. From the estimate (2.11) (with $\alpha = 0$) it follows that this semigroup is analytic. Finally, let us show that its generator is $U'(0) = -A$. From (2.10) and the estimate (2.10) we derive the identity

$$U(t)x - x = -\int_0^t U(s) Axds$$

for $x \in D(A)$. Since $U(t)$ is strongly continuous to the left at the point $t = 0$, this implies that $x \in D(U'(0))$ and $U'(0)x = -Ax$. Hence, $U'(0)$ is an extension of the operator $-A$. By the estimate (2.10), the operator $U'(0) + \lambda$ and $-A + \lambda$ have bounded inverses for any $\lambda < 0$. Therefore, $U'(0) = -A$.

We have shown that the operator valued function $\exp\{-tA\}$ is an analytic semigroup with generator $-A$ and with an exponentially decaying norm. Operators $-A$ that generate such semigroups were called strongly positive operators.

Let A be a strongly positive operator. With the help of A we introduce the fractional space $E_\alpha(E, A), 0 < \alpha < 1$, consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_\alpha = \sup_{\lambda > 0} \lambda^\alpha \left\|A(\lambda + A)^{-1} v\right\|_E + \|v\|_E .$$

Chapter 3

POSITIVITY OF DIFFERENCE OPERATORS GENERATED BY THE NONLOCAL BOUNDARY CONDITIONS

3.1 Introduction

Let us consider a differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{d^2u}{dx^2} + \delta u \tag{3.1}$$

with domain $D(A^x) = \left\{u \in C^{(2)}[0,1] : u(0) = u(1), u'(0) = u'(1)\right\}$. Here $a(x)$ is the smooth function defined on the segment $[0,1]$ and $a(x) > 0, \delta > 0$.

Let us define the grid space $[0,1]_h = \{x_k = kh, 0 \leq k \leq N, Nh = 1\}$, N is a fixed positive integer. The number h is called the step of the grid space. A function $\varphi^h = \{\varphi_k\}_0^N$ defined on $[0,1]_h$ will be called a grid function. To the operator A^x defined by the formula (3.1) we assign the difference operator A_h^x defined by the formula

$$A_h^x u^h = \left\{-a(x_k)\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k\right\}_1^{N-1}, \quad u^h = \{u_k\}_0^N, \tag{3.2}$$

which acts on grid functions defined on $[0,1]_h$ with

$$u_0 = u_N \ \text{ and} - u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N.$$

We denote $C_h = C[0,1]_h$ and $C_h^\alpha = C^\alpha[0,1]_h$ the Banach spaces of all grid functions $v^h = \{v_k\}_1^{N-1}$defined on $[0,1]_h$ equipped with the norms

$$\left\|v^h\right\|_{C_h} = \max_{1\leq k\leq N-1} |v_k|,$$

$$\left\|v^h\right\|_{C_h^\alpha} = \max_{1\leq k\leq N-1} |v_k| + \max_{1\leq k<k+r\leq N-1} \frac{|v_{k+r} - v_k|}{(r\tau)^\alpha}.$$

In the present chapter we will investigate the resolvent of the operator $-A_h^x$, *i.e.*, solving the equation

$$A_h^x u^h + \lambda u^h = f^h \tag{3.3}$$

or

$$-a_k\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}+\delta u_k+\lambda u_k=f_k,$$

$$a_k=a(x_k),\ f_k=f(x_k),\ 1\le k\le N-1,$$

$$u_0=u_N,\ \ -u_2+4u_1-3u_0=u_{N-2}-4u_{N-1}+3u_N.$$

The positivity of difference operator A_h^x defined by the formula (3.2) in C_h and Hölder spaces C_h^{α} is established.

3.2 Green's Function

In this section we will study the strong positivity in C_h of the operator A_h^x defined by formula (3.2) in the case $a(x)\equiv 1$.

Lemma 3.1. Let $\lambda\ge 0$. Then the equation (3.3) is uniquely solvable, and the following formula holds

$$u^h=(A_h^x+\lambda)^{-1}f^h=\left\{\sum_{j=1}^{N-1}J(k,j;\lambda+\delta)f_jh\right\}_0^N, \tag{3.4}$$

where

$$J(k,1;\lambda+\delta)=J(k,N-1;\lambda+\delta)$$

$$=-\frac{1+\mu h}{2+3\mu h}\frac{(R^{N-3}-4R^{N-2}+R-4)}{2\mu}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}$$

for $k=0$ and $k=N$;

$$J(k,j;\lambda+\delta)=-\frac{1+\mu h}{2+3\mu h}\frac{(R^2-4R+1)(R^{j-2}+R^{N-j-2})}{2\mu}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}$$

for $2\le j\le N-2$ and $k=0$, $k=N$;

$$J(k,1;\lambda+\delta)=\frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{R^{k-1}(2(R+3)+R^2(R-3))$$

$$+R^{N-k}(4-R)(1+R)+R^{N+k-3}(1-4R)(1+R)$$

$$+R^{2N-k-3}(3R-1-2R^2(3R+1))\}(1-R^N)^{-1}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1},$$

$$J(k,N-1;\lambda+\delta)=-\frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{R^k(R-4)(R+1)$$

$$+R^{N-k-1}(-2(R+3)+R^2(3-R))+R^{N+k-3}(1-3R+2R^2(3R+1))$$

$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1},$$

$$J(k,j;\lambda+\delta)=\frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{(R-1)^3(R^{j+k-2}+R^{2N-2-j-k})$$

$$+(-1+3R+R^2(3-R))(R^{N-k+j-2}+R^{N+k-j-2})+2(1-3R)(R^{2N-2+j-k}+R^{2N-2-j+k})$$

$$+2R^{|j-k|}(R^N-1)\left(R-3+R^{N-2}(-1+3R)\right)\}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}$$

for $2 \le j \le N-2$ and $1 \le k \le N-1$. Here

$$R=(1+\mu h)^{-1}, \mu=\frac{1}{2}\left(h(\lambda+\delta)+\sqrt{(\lambda+\delta)(4+h^2(\lambda+\delta))}\right).$$

Proof. We see that the problem (3.3) can be obviously rewritten as the equivalent nonlocal boundary value problem for the first order linear difference equations

$$\begin{cases} \frac{u_k-u_{k-1}}{h}+\mu u_k=z_k, \ 1\le k\le N, \\ u_0=u_N, \ -u_2+4u_1-3u_0=u_{N-2}-4u_{N-1}+3u_N, \\ -\frac{z_{k+1}-z_k}{h}+\mu z_k=(1+\mu h)f_k, \quad 1\le k\le N-1. \end{cases}$$

From that there follows the system of recursion formulas

$$\begin{cases} u_k=Ru_{k-1}+hRz_k, \quad 1\le k\le N, \\ z_k=Rz_{k-1}+hf_k, \quad 1\le k\le N-1. \end{cases}$$

Hence

$$\begin{cases} u_k=R^ku_0+\sum\limits_{i=1}^{k}R^{k-i+1}hz_i, \ 1\le k\le N, \\ z_k=R^{N-k}z_N+\sum\limits_{j=k}^{N-1}R^{j-k}hf_j, \ 1\le k\le N-1. \end{cases}$$

From the first formula and the condition $u_N=u_0$ it follows that

$$u_N=R^Nu_0+\sum_{i=1}^{N}R^{N-i+1}hz_i.$$

Since $1-R^N\neq 0$, it follows that

$$u_N=u_0=\frac{1}{1-R^N}\sum_{i=1}^{N}R^{N-i+1}hz_i=\frac{1}{1-R^N}\left\{hRz_N+\sum_{i=1}^{N}R^{N-i+1}hz_i\right\}$$

$$=\frac{1}{1-R^N}\left\{\left(hR+\sum_{i=1}^{N}R^{2N-2i+1}h\right)z_N+\sum_{i=1}^{N-1}hR^{N-i+1}\sum_{j=i}^{N-1}R^{j-i}hf_j\right\}$$

$$=\frac{1}{1-R^N}\left\{\frac{(R-R^{2N+1})}{1-R^2}hz_N+\sum_{j=1}^{N-1}h^2\sum_{i=1}^{j}R^{N+j-2i+1}f_j\right\}$$

$$=\frac{1}{(1-R^N)(1-R^2)}\left[R(1-R^{2N})hz_N+\sum_{j=1}^{N-1}h^2\left[R^{N-j+1}-R^{N+j+1}\right]f_j\right],$$

and for $k, 1 \le k \le N-1$:

$$u_k = \frac{1}{1-R^N}\left\{hR^{k+1}z_N + \sum_{i=1}^{N-1} R^{k+N-i+1}hz_i\right\} + \sum_{i=1}^{k} R^{k-i+1}hz_i$$

$$= \frac{R^k}{(1-R^N)}\left\{\frac{(R-R^{2N+1})}{1-R^2}hz_N + \sum_{j=1}^{N-1} h^2\left[R^{N-j+1} - R^{N+j+1}\right]f_j\right\}$$

$$+\sum_{i=1}^{k} R^{N+k-2i+1}hz_N + \sum_{i=1}^{k}\sum_{j=i}^{N-1} h^2 R^{k+j-2i+1}f_j$$

$$= \frac{1}{1-R^2}\left[R^{k+1} + R^{N-k+1}\right]hz_N$$

$$+\frac{1}{(1-R^N)(1-R^{N-1})}\sum_{j=1}^{N-1} h^2\left[R^{N-j+1} - R^{N+j+1}\right]f_j$$

$$+\sum_{j=1}^{k} h^2 \sum_{i=1}^{j} R^{k+j-2i+1}f_j + \sum_{j=k+1}^{N-1} h^2 \sum_{i=1}^{k} R^{k+j-2i+1}f_j$$

$$= \frac{1}{1-R^2}\left[R^{k+1} + R^{N-k+1}\right]hz_N$$

$$+\frac{R^k}{(1-R^N)(1-R^{N-1})}\sum_{j=1}^{N-1} h^2\left[R^{N-j+1} - R^{N+j+1}\right]f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1} h^2(R^{|k-j|+1} - R^{k+j+1})f_j$$

Now by using the formulas for u_N, u_0, u_k and the condition

$$-u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N$$

we can write

$$u_0 + u_N = 2\frac{R+R^{N+1}}{1-R^2}hz_N$$

$$+\frac{2}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1} h^2\left(R^{N-j+1} - R^{N+j+1}\right)f_j,$$

$$u_1 + u_{N-1} = \frac{2}{1-R^2}(R^2 + R^N)hz_N$$

$$+\frac{(R+R^{N-1})}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1} h^2\left(R^{N-j+1} - R^{N+j+1}\right)f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1} h^2\left(R^{|1-j|+1} + R^{|N-1-j|+1} - R^{2+j} - R^{N+j}\right)f_j,$$

$$u_2 + u_{N-2} = \frac{2}{1-R^2}(R^3 + R^{N-1})hz_N$$

$$+\frac{(R^2+R^{N-2})}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1} h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1} h^2\left(R^{|2-j|+1}+R^{|N-2-j|+1}-R^{j+3}-R^{N-1+j}\right)f_j$$

$$=\frac{2}{1-R^2}(R^3+R^{N-1})hz_N$$

$$+\frac{(R^2+R^{N-2})}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1} h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1} h^2\left(R^{j-1}+R^{N-1-j}-R^{j+3}-R^{N-1+j}\right)f_j$$

$$+\frac{1}{1-R^2}\left(\left(R^2-1\right)f_1h^2+(R^2-1)f_{N-1}h^2\right).$$

Since

$$u_2+u_{N-2}+3\left(u_0+u_N\right)=4(u_1+u_{N-1}),$$

we have that

$$\frac{2}{1-R^2}(R^3+R^{N-1})hz_N+\frac{(R^2+R^{N-2})}{(1-R^N)(1-R^2)}$$

$$\times\sum_{j=1}^{N-1} h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1} h^2\left(R^{j-1}+R^{N-1-j}-R^{j+3}-R^{N-1+j}\right)f_j$$

$$-h^2(f_1+f_{N-1})+6\frac{R+R^{N+1}}{1-R^2}hz_N$$

$$+\frac{6}{(1-R^N)(1-R^2)}\sum_{j=1}^{N-1} h^2\left(R^{N-j+1}-R^{N+j+1}\right)f_j$$

$$=\frac{8}{1-R^2}(R^2+R^N)hz_N+\frac{4(R+R^{N-1})}{(1-R^N)(1-R^2)}$$

$$\times\sum_{j=1}^{N-1} h^2(R^{N-j+1}-R^{N+j+1})f_j$$

$$+\frac{4}{1-R^2}\sum_{j=1}^{N-1} h^2(R^j+R^{N-j}-R^{j+2}-R^{N+j})f_j.$$

Hence from here z_N can be found as

$$z_N=\frac{-hR^2(-1+R^2)(-1+R^N)(f_1+f_{N-1})}{2(-1+R)R(-1+R^N)(-3R^2+R^3-R^N+3R^{N+1})}$$

$$+\frac{(6R^2-4R^3+R^4+R^N-4R^{1+N})\sum_{j=1}^{N-1} h(R^{N-j+1}-R^{N+j+1})f_j}{2(-1+R)R(-1+R^N)(-3R^2+R^3-R^N+3R^{N+1})}$$

$$-\frac{R^2(-1+R^N)\sum_{j=1}^{N-1} h\left(R^{j-1}+R^{N-1-j}-R^{j+3}-R^{N-1+j}\right) f_j}{2(-1+R)R(-1+R^N)(-3R^2+R^3-R^N+3R^{N+1})}$$

$$-\frac{4R^2(-1+R^N)\sum_{j=1}^{N-1} h(R^j+R^{N-j}-R^{j+2}-R^{N+j})f_j}{2(-1+R)R(-1+R^N)(-3R^2+R^3-R^N+3R^{N+1})}$$

Now using the formulas for z_N and $u_0 = u_N$ we obtain

$$u_N = u_0 \;= -\frac{h^2(-4R^3+R^4+R^N-4R^{N+1})(f_1+f_{N-1})}{2(R-1)(-3R^2+R^3-R^N+3R^{N+1})}$$

$$-\sum_{j=2}^{N-2} \frac{h^2R^{1-j}(R^2-4R+1)(R^{2j}+R^N)}{2(R-1)(-3R^2+R^3-R^N+3R^{N+1})} f_j.$$

$$= -\frac{1+\mu h}{2+3\mu h}\frac{\left(R^{N-3}-4R^{N-2}+R-4\right)}{2\mu}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}(f_1+f_{N-1})$$

$$-\sum_{j=2}^{N-2}\frac{1+\mu h}{2+3\mu h}\frac{(R^2-4R+1)(R^{j-2}+R^{N-j-2})}{2\mu}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1} f_j h.$$

The formula for u_k in the case $k=0$ and $k=N$ is proved. Now, consider $1 \le k \le N-1$.We have that

$$u_k = \frac{1}{1-R^2}\left[R^{k+1}+R^{N-k+1}\right] h z_N$$

$$+\frac{R^k}{(1-R^N)(1-R^{N-1})}\sum_{j=1}^{N-1} h^2\left[R^{N-j+1}-R^{N+j+1}\right] f_j$$

$$+\frac{1}{1-R^2}\sum_{j=1}^{N-1} h^2(R^{|k-j|+1}-R^{k+j+1})f_j$$

$$= (-h^2(-1+R)(R^k(3R^5-2R^4-R^6-6R^3+12R^{N+2}-R^{N+1}+4R^{N+3}$$
$$-4R^{N+4})+R^{N-k}(R^{N+1}+2R^{N+3}-4R^4-3R^5+R^6-3R^{N+2}+6R^{N+4}))f_1$$
$$+h^2(-1+R)(R^k(3R^5+R^4-3R^6-R^{N+1}+3R^{N+2}-2R^{N+3}-6R^{N+4}$$
$$+6R^{2N+2})+R^{N-k}(R^{N+1}-9R^{N+2}-4R^{N+3}+2R^4+6R^3+R^6-3R^5))f_{N-1})$$
$$\times(2(R-1)R(1-R^2)\left(-1+R^N\right)(-3R^2+R^3-R^N+3R^{N+1}))^{-1}$$
$$+\sum_{j=2}^{N-2} h^2(R^{2+j+k}(R-1)(-1+3R-3R^2+R^3+6R^{N+1})$$
$$-R^{N+2+j-k}(1+4R-4R^3+R^4-8R^{N+1}+6R^{N+2})-R^{N+2+k-j}(1+3R+3R^2$$
$$-6R^3+3R^4-8R^{N+1}+6R^{N+2})-R^{2N+2-j-k}(R-1)^4$$
$$+R^{|j-k|}(2(R-1)R^2(3R^2-R^3+R^N-R^{2N}-6R^{N+1}-3R^{N+2}+R^{N+3}$$
$$+3R^{2N+1}))f_j(\,2(R-1)R(1-R^2)\left(-1+R^N\right)(-3R^2+R^3-R^N+3R^{N+1}))^{-1}$$

$$= \frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{R^{k-1}(2(R+3)+R^2(R-3))$$
$$+R^{N-k}(4-R)(1+R)+R^{N+k-3}(1-4R)(1+R)$$
$$+R^{2N-k-3}(3R-1-2R^2(3R+1))\}(1-R^N)^{-1}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}f_1$$
$$-\frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\{R^k(R-4)(R+1)$$
$$+R^{N-k-1}(-2(R+3)+R^2(3-R))+R^{N+k-3}(1-3R+2R^2(3R+1))$$
$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}(I-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}f_{N-1}$$
$$+\frac{1+\mu h}{2+3\mu h}\frac{1+\mu h}{2+\mu h}(2\mu)^{-1}\sum_{j=2}^{N-2}\{(R-1)^3(R^{j+k-2}+R^{2N-2-j-k})$$
$$+(-1+3R+R^2(3-R))(R^{N-k+j-2}+R^{N+k-j-2})+2(1-3R)(R^{2N-2+j-k}+R^{2N-2-j+k})$$
$$+2R^{|j-k|}(R^N-1)\left(R-3+R^{N-2}(-1+3R)\right)\}$$
$$\times(1-R^N)^{-1}(1-\frac{2-\mu h}{2+3\mu h}R^{N-2})^{-1}\ f_jh.$$

Lemma 3.1 is proved.

The grid function $J(k,j;\lambda+\delta)$ is called the *Green's function* of the resolvent equation (3.3). Notice that

$$J(k,j;\lambda+\delta)=J(j,k;\lambda+\delta)\geq 0,$$
$$\sum_{j=1}^{N-1}J(k,j;\lambda+\delta)h=\frac{1}{\lambda+\delta},\quad 1\leq j<k\leq N.$$

Thus, we obtain the formula for the resolvent $(\lambda I+A_h^x)^{-1}$ in the case $\lambda\geq 0$. In the same way we can obtain a formula as (3.4) for the resolvent $(\lambda I+A_h^x)^{-1}$ in the case of complex λ. But we need to obtain that $1+2\mu h$, $2+3\mu h$, $1-R^N$, and $1-\frac{2-\mu h}{2+3\mu h}R^{N-2}$ are not equal to zero.

3.3 Positivity of Difference Operator A_h in C_h.

Theorem 3.1. *For all $\lambda,\lambda\in R_\varphi=\{\lambda:|\arg\lambda|\leq\varphi,0\leq\varphi\leq\pi/2\}$ the resolvent $(\lambda I+A_h)^{-1}$ defined by the formula (3.1) is subject to the bound*

$$\left\|(\lambda I+A_h)^{-1}\right\|_{C_h\to C_h}\leq M(\varphi,\delta)(1+|\lambda|)^{-1},$$

where $M(\varphi,\delta)$ does not depend on h.

The proof of this theorem is based on the following lemmas:

Lemma 3.2. If $\operatorname{Re}\lambda>0$, then $\operatorname{Re}\mu>0$.

Proof. (Ashyralyev and Kendirli, 2000) We have

$$\operatorname{Re}\mu = \operatorname{Re}(\lambda+\delta)\,\frac{h}{2} + \operatorname{Re}\sqrt{\frac{h^2}{4}(\lambda+\delta)^2 + (\lambda+\delta)}$$

$$= \frac{h}{2}(\operatorname{Re}(\lambda+\delta)) + \operatorname{Re}\sqrt{\lambda+\delta}\sqrt{\frac{h^2}{4}(\lambda+\delta)+1}.$$

We denote $\lambda+\delta = re^{i\varphi}$, here $r = |\lambda+\delta|$ and $|\varphi| \le \pi/2$. Then

$$\sqrt{\frac{h^2}{4}(\lambda+\delta)+1} = r_1 e^{i\varphi_1} \text{ with } r_1 = \left|\sqrt{\frac{h^2}{4}(\lambda+\delta)+1}\right|,$$

$|\varphi_1| < \pi/4$. Therefore

$$\operatorname{Re}\sqrt{\lambda+\delta}\sqrt{\frac{h^2}{4}(\lambda+\delta)+1} \ge 0.$$

Lemma 3.2 is proved.

Lemma 3.3. The following estimate holds

$$|\mu| \ge \sqrt{|\lambda+\delta|}.$$

Proof. (Ashyralyev and Kendirli, 2000) Using the formula for μ we have that

$$\left|\frac{\mu}{\sqrt{\lambda+\delta}}\right| = \left|\frac{h}{2}\sqrt{\lambda+\delta} + \sqrt{\frac{h^2}{4}(\lambda+\delta)+1}\right|.$$

Now, using the notations of Lemma 3.2, we have

$$\left|\frac{h}{2}\sqrt{\lambda+\delta} + \sqrt{\frac{h^2}{4}(\lambda+\delta)+1}\right| = \frac{h}{2}\sqrt{r}e^{i\varphi\,/\,2} + r_1 e^{i\varphi_1}$$

$$= \sqrt{(\frac{h}{2}\sqrt{r}\cos\frac{\varphi}{2} + r_1\cos\varphi_1)^2 + (\frac{h}{2}\sqrt{r}\sin\frac{\varphi}{2} + r_1\sin\varphi_1)^2}$$

$$= \sqrt{\frac{rh^2}{4} + r_1^2 + h\sqrt{rr_1}\cos(\varphi/2-\varphi_1)} \ge r_1 \ge 1$$

since $|\varphi/2 - \varphi_1| \le \pi/2$.

Lemma 3.3 is proved.

Lemma 3.4. The following estimate holds

$$|R| \le \frac{1}{1+\sqrt{|\lambda+\delta|}h\cos\varphi} < 1$$

where $|\varphi| < \pi/2$.

Proof. (Ashyralyev and Kendirli, 2000) We have

$$|R| = \frac{1}{\sqrt{(1 + h\operatorname{Re}\mu)^2 + (h\operatorname{Im}\mu)^2}}.$$

Since $\operatorname{Re}\mu = |\mu|\cos\varphi$,where $|\varphi| \le \pi/2$, we obtain

$$|R| \le \frac{1}{1 + |\mu|\, h\cos\varphi}.$$

Hence by Lemma 3.3 we obtain the proof of Lemma 3.4.

Lemma 3.5.

$$|1+\mu| - 1 > 0$$

if $\lambda \in R_\varphi = \{\lambda : |\arg\lambda| < \pi - \varphi, 0 < \varphi < \pi/2, 0 < |\lambda|\}$.

Proof. (Alibekov, 1978) It is enough to show that $\gamma(\lambda) \ne 0$. It is obvious that $\gamma(\lambda_0) > 0$ for large enough absolute values of λ_0.

Assume that $\gamma(\lambda) = 0$. Let $|1+\mu| = 1$. Denote $b = h^2\lambda$ where $b \in R_\varphi$.

First case: $\operatorname{Im} b = 0$. Then $\arg b = 0$. So $\arg\lambda = 0$. This implies that $\arg\mu = 0$. Hence μ is real. So if $|1+\mu| = 1$ then $\mu = 0$. This implies that $\lambda = 0$. But this is a contradiction since in the domain $\lambda \ne 0$.

Second case: $\operatorname{Im} b > 0$. From the formula

$$\mu = \frac{1}{2}(b + \sqrt{b(4+b)})$$

it follows that $\mu \in R_\varphi^{\pm}$ when $b \in R_\varphi^{\pm}$.

Now since

$$|1+\mu| = 1$$

then

$$\alpha - \varphi = \arg(1+\mu).$$

Since

$$\arg\mu = \alpha$$

we have that

$$\arg\mu^2 = 2\alpha.$$

Then from the formula

$$\mu^2 = b(1+\mu)$$

it follows that

$$\arg\ \mu^2 = \arg b + \arg(1+\mu).$$

Then

$$2\alpha = \arg b + (\alpha - \varphi).$$

So

$$\arg b = \alpha + \varphi = \pi.$$

However since $b \in R_\varphi^{+}$ this implies that $|\arg b| < \pi$.

Hence this makes a contradiction. So

$$|1+\mu| > 1.$$

Therefore we have proved that

$$1+\mu \neq 0,$$

$$2+\mu \neq 0,$$

$$\text{and } (1+\mu)^{2N} - 1 \neq 0.$$

Lemma 3.6. For all $\lambda \in R_\varphi \cup \{0\}$the following inequality holds

$$|1+\mu| \leq 2\left|\sin(\pi-\varphi)\right|^{-1}|2+\mu|$$

Proof. (Alibekov, 1978) Let

$$\mu = \frac{1}{2}\left(h^2\lambda + \sqrt{h^2\lambda\,(4+h^2\lambda)}\right) \in R_\varphi$$

since $\lambda \in R_\varphi$ then $1+\mu \in R_\varphi$. So

$$|\arg(1+\mu)| < \pi - \varphi.$$

Let

$$1+\mu = \rho\exp(i\beta)$$

then

$$|1+\mu| = |\rho\exp(i\beta)| = |\rho|\,.$$

Hence

$$2+\mu = 1+\mu+1 = \rho\exp(i\beta)+1 = \rho\cos\beta + 1 + i\beta\sin\beta$$

and

$$|2+\mu| = \sqrt{(\rho\cos\beta+1)^2 + (\rho\sin\beta)^2} = \sqrt{(\rho^2+1+2\rho\cos\beta}.$$

Let

$$K_0 = 2\left|\sin(\pi-\varphi)\right|^{-1} \geq 2,$$

then

$$\rho \leq K_0\sqrt{1+2\rho\cos\beta+\rho^2}$$

is obvious when $|\beta| < \pi/2$. Now let

$$\pi/2 < |\beta| < \pi - \varphi.$$

Then

$$\sin(\pi/2 < |\beta| < \pi - \varphi)$$

$$\sin(\pi-\varphi) < \sin|\beta| < 1.$$

Since

$$K_0 = 2(\sin(\pi-\varphi))^{-1}$$

then

$$\frac{2}{K_0} < \sin\beta < 1.$$

Taking the square we have

$$\frac{4}{K_0^2} < \sin^2\beta < 1$$

$$1 < 4 < K_0^2 \sin^2\beta < K_0^2$$

$$K_0^2 \sin^2\beta > 1$$

$$K_0^2 \sin^2\beta - 1 > 0$$

$$1 - K_0^2 \sin^2\beta < 0. \tag{3.5}$$

Now for the inequality

$$\rho^2 \leq K_0^2(1 + 2\rho\cos\beta + \rho^2) = K_0^2 + 2K_0^2\rho\cos\beta + K_0^2\rho^2$$

since $K_0 > 1$ then

$$K_0^2 + (2K_0^2\cos\beta)\rho + (K_0^2 - 1)\rho^2 \geq 0 \tag{3.6}$$

will hold for all $\rho = |1+\mu| > 1$ if the discriminant of (3.6) < 0. This condition is satisfied since the discriminant is

$$4K_0^2(-K_0^2\sin^2\beta + 1) < 0$$

from (3.5).

Hence the lemma is proved.

Lemma 3.7.

$$|\mu|\,(|1+\mu| - 1)^{-1} \leq M_{(\varphi)} = \text{constant} \qquad \lambda \in R_\varphi \tag{3.7}$$

Proof. (Alibekov, 1978) Consider this function

$v_{(\mu)} = |\mu|\,(|1+\mu| - 1)^{-1}$ is a continuous function since $|1+\mu| > 1$.

Let us consider the limits when $\mu \to 0$ and $\mu \to \infty$.

Now let $\mu \to 0$.

$$\mu = \frac{1}{2}(h^2\lambda + \sqrt{h^2\lambda(4 + h^2\lambda)})$$

Now

$$c_1\left|h^2\lambda\right|^{1/2} \leq |\mu| \leq c_2\left|h^2\lambda\right|^{1/2} \qquad \text{where } c_1, c_2 \text{ are constants.}$$

Since μ is too small from that follows

$$\arg\mu = \arg(h^2\lambda)^{1/2} = \left|\arg\lambda^{1/2}\right| = \frac{1}{2}\left|\arg\lambda\right|$$

hence when $\mu \to 0$ then $\lambda \to 0$.

Denote

$$\xi = \cos\left(\frac{(\pi - \varphi)}{2}\right) > 0$$

since

$$|\arg\mu| = 2^{-1}\,|\arg\lambda| < \frac{(\pi-\varphi)}{2}$$

$$\mu = \rho\exp(i\gamma)$$

$$\lim_{\mu\to 0} v_{(\mu)} = \lim_{\rho\to 0\ ,\ \gamma\to\frac{\arg\lambda}{2}} \frac{\rho}{\sqrt{1+2\rho\cos\gamma+\rho^2}-1} < \xi^{-1}$$

$$= \lim_{\rho\to 0} \frac{\rho\left(\sqrt{1+2\rho\cos\gamma+\rho^2}+1\right)}{1+2\rho\cos\gamma+\rho^2-1} = \frac{\sqrt{1+2\rho\cos\gamma+\rho^2}+1}{2\cos\gamma+\rho}$$

$$= \frac{2}{2\cos\gamma} = \frac{1}{\cos\gamma}$$

$$\lim_{\gamma\to\frac{\arg\lambda}{2}} \frac{1}{\cos\gamma} = \frac{1}{\cos\frac{\arg\lambda}{2}} = \frac{1}{\cos\left(\frac{\pi-\varphi}{2}\right)} = \frac{1}{\xi} = \xi^{-1}$$

Now consider limit when $\mu\to\infty$.

$$v_{(\mu)} = \frac{|\mu|}{|1+\mu|-1}$$

$$\lim_{\mu\to\infty} \frac{|\mu|}{|1+\mu|-1} = \frac{\mu}{1+\mu-1} \quad \text{since } \mu \text{ is too large.}$$

$$= \frac{\mu}{\mu} = 1.$$

Hence the inequality (3.7) is satisfied.

Lemma 3.8. The following inequality holds

$$\left|\frac{2-\mu h}{2+3\mu h}\right| \leq 1,$$

where h is sufficiently small.

Proof. Let $\mu = \rho e^{i\beta}$. h is sufficiently small, then μh is also small since

$$\arg\mu = 2^{-1}\arg(\lambda+\delta)\,,$$

$$\mu = \rho(\cos\beta + i\sin\beta).$$

Then

$$|\arg\mu| = |\beta| < \pi/2.$$

Now

$$\left|\frac{2-\rho h(\cos\beta+i\sin\beta)}{2+3\rho h(\cos\beta+i\sin\beta)}\right| = \frac{\sqrt{(2-\rho h\cos\beta)^2+(\rho h\sin\beta)^2}}{\sqrt{(2+3\rho h\cos\beta)^2+(3\rho h\sin\beta)^2}}$$

$$= \sqrt{\frac{4-4\rho h\cos\beta+\rho^2h^2\cos^2\beta+\rho^2h^2\sin^2\beta}{4+9\rho^2h^2\cos^2\beta+12\rho h\cos\beta+9\rho^2h^2\sin^2\beta}}$$

$$= \sqrt{\frac{4-4\rho h\cos\beta+\rho^2h^2}{4+9\rho^2h^2+12\rho h\cos\beta}} \leq 1.$$

Lemma 3.9. The following estimate holds

$$\left|\frac{1+\mu h}{2+3\mu h}\right| \le 1$$

Proof. Let $\mu = \rho e^{i\beta} = \rho(\cos\beta + i\sin\beta)$. Then

$$\left|\frac{1+\rho h(\cos\beta + i\sin\beta)}{2+3\rho h(\cos\beta + i\sin\beta)}\right| = \sqrt{\frac{(1+\rho h\cos\beta)^2 + (\rho h\sin\beta)^2}{(2+3\rho h\cos\beta)^2 + (3\rho h\sin\beta)^2}}$$

$$= \sqrt{\frac{1+2\rho h\cos\beta + \rho^2 h^2}{4+12\rho h\cos\beta + 9\rho^2 h^2}} \le 1$$

Lemma 3.10. The following inequalities hold:

$$\left|\left(1 - \frac{2-\mu h}{2+3\mu h} R^{N-2}\right)^{-1}\right| > 0,$$

$$\left|\left(1 - R^N\right)^{-1}\right| > 0,$$

where h is sufficiently small.

Proof. The proof of this lemma is based on the triangle inequality and on the estimates of lemmas 3.4 and 3.9.

In the sequel for the proof of strong positivity of the difference operator in C_h we will need to consider the following nonlocal boundary value problem

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{h^2} + (\delta+\lambda)u_k = f_k, \quad 1 \le k \le N-1, \\ u_0 = u_N, -u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N + 2h\phi. \end{cases} \tag{3.8}$$

Theorem 3.2. *Let $\lambda \in R_\varphi$. Then for the solution of the nonlocal boundary value problem the following inequality holds*

$$\max_{0\le k\le N} |u_k| \le M(\delta,\varphi)\left(\frac{1}{1+|\lambda|}\|f\|_{C_h} + M(\delta,\varphi)\,|\phi|\right),$$

where $M(\delta,\varphi)$ does not depend on f, ϕ and h.

Proof. Let u_k be a solution of the general nonlocal boundary value problem (3.1) and w_k be a solution of the nonlocal boundary value problem (3.1) in the case $a_k = 1$. Then we can write

$$u_k = w_k + v_k,$$

where v_k is the solution of the following nonlocal boundary value problem

$$\begin{cases} -\frac{v_{k+1}-2v_k+v_{k-1}}{h^2} + (\delta+\lambda)v_k = 0, \\ v_0 = v_N, \quad -v_2 + 4v_1 - 3v_0 = v_{N-2} - 4v_{N-1} + 3v_N + 2h\phi. \end{cases}$$

Using the formula

$$v_k = -\frac{2}{\mu}\frac{1+\mu h}{2+3\mu h}\frac{(R^{k-1} + R^{N-1-k})\phi}{(1 - \frac{2-\mu h}{2+3\mu h}R^{N-2})}, 1 \le k \le N-1$$

for the solution of v_k and by Lemmas 3.9 and 3.10, we obtain

$$\max_{1\leq k\leq N-1} |v_k| \leq \frac{M(\delta,\varphi)}{|\mu|} |\phi| .$$

Theorem 3.2 is proved.

3.4 Positivity of Difference Operator A_h^x in C_h

Now we will investigate the strong positivity of difference operator A_h^x defined by the formula (3.2) in C_h. In the sequel we will need the following difference analogue of Nirenberg's inequality which was obtained by Sobolevskii and Neginskii (Neginskii and Sobolevskii, 1970):

$$\max_{0\leq k\leq N-1} \left|\frac{u_{k+1}-u_k}{h}\right| \tag{3.9}$$
$$\leq K\left[\alpha \max_{1\leq k\leq N-1} \frac{|u_{k+1}-2u_k+u_{k-1}|}{h^2} + \alpha^{-1} \max_{0\leq k\leq N} |u_k|\right],$$

where K is a constant, $\alpha > 0$ is a small number.

We consider the difference operator A_h^x defined by the formula (3.2). If $a_k = a$ =const, then using the substitution $\lambda+\delta = a\lambda_1$ and the results of previous sections, we can obtain the estimate

$$\left\|(\lambda I + A_h^x)^{-1}\right\|_{C_h\to C_h} \leq M(\varphi,\delta)(1+|\lambda|)^{-1}$$

or

$$\max_{0\leq k\leq N} |u_k| \leq M(\delta,\varphi)\left(\frac{1}{1+|\lambda|}\|f\|_{C_h} + |\phi|\right)$$

and the coercive estimate

$$\max_{1\leq k\leq N-1} \frac{|u_{k+1}-2u_k+u_{k-1}|}{h^2} \leq M(\varphi,\delta) \max_{1\leq k\leq N-1} |f_k| \tag{3.10}$$

for the solutions of the difference equation with constant coefficients. Here $M(\varphi,\delta)$ does not depend on h and λ.

Now, let $a(x)$ be a continuous function on $[0,1]=\Omega$. Similarly to (Alibekov, 1978), using the method of frozen coefficients and the coercive estimate for the solutions of the difference equation with constant coefficients, we obtain the following theorem.

Theorem 3.3. *Let h be a sufficiently small number. Then for all $\lambda \in R_\varphi$ and $|\lambda| \geq K_0(\delta,\varphi) > 0$ the resolvent $(\lambda I + A_h^x)^{-1}$ is subject to the bound*

$$\left\|(\lambda I + A_h^x)^{-1}\right\|_{C_h\to C_h} \leq M(\varphi,\delta)(1+|\lambda|)^{-1},$$

where $M(\varphi,\delta)$ does not depend on h.

Proof. Given $\varepsilon > 0$, there exists a system $\{Q_j\}, j = 1,...,r$ of intervals and two half-intervals (containing 0 and 1, respectively) that covers the segment $[0,1]$ and such that $|a(x_1) - a(x_2)| < \varepsilon, x_1, x_2 \in Q_j$ because of the compactness of $[0,1]$. For this system we construct a partition of unity, that is, a system of smooth nonnegative functions $\xi_j(x)$ $(i=1,...,r)$ with $\sup\xi_j(x) \subset Q_j, \xi_j(0)=\xi_j(1), \xi_j'(0)=\xi_j'(1)$ and $\xi_1(x)+...+\xi_r(x)=1$ in $\bar{\Omega}=[0,1]$.

It is clear that for positivity of the difference operator (3.2) it suffices to establish the estimate

$$\max_{0\le k\le N} |u_k| \le \frac{M}{|\lambda|+1} \max_{1\le k\le N-1} |f_k| \tag{3.11}$$

for the solutions of difference equation (3.3).

Using $w_k = \xi_j(x_k)u_k,$ we obtain

$$w_0 = w_N, \quad -w_2 + 4w_1 - 3w_0 = w_{N-2} - 4w_{N-1} + 3w_N + \phi,$$

where

$$\phi = -(\xi_j(2h) - \xi_j(0))u_2 + 4(\xi_j(h) - \xi_j(0))u_1$$
$$-(\xi_j(1-2h) - \xi_j(1))u_{N-2} + 4(\xi_j(h) - \xi_j(1))u_{N-1}$$

and

$$(\delta+\lambda)\, w_k - a_k \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2}$$
$$= \xi_j(x_k) f_k - a_k \left\{ \frac{\xi_j(x_k) - \xi_j(x_{k-1})}{h} . \frac{u_k - u_{k-1}}{h} \right.$$
$$\left. + \frac{\xi_j(x_{k+1}) - 2\xi_j(x_k) + \xi_j(x_{k-1})}{h^2} u_k + \frac{\xi_j(x_{k+1}) - \xi_j(x_k)}{h} . \frac{u_{k+1} - u_k}{h} \right\}.$$

Then we have the following nonlocal boundary value problem

$$(\delta+\lambda)\, w_k - a^j \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} = F_k^j, \quad j = 1,\cdots,r, \tag{3.12}$$
$$w_0 = w_N, \quad -w_2 + 4w_1 - 3w_0 = w_{N-2} - 4w_{N-1} + 3w_N + \phi,$$

where $a^j = a(x^j)$ and

$$F_k^j = \xi_j(x_k) f_k - a_k \left\{ \frac{\xi_j(x_k) - \xi_j(x_{k-1})}{h} . \frac{u_k - u_{k-1}}{h} \right.$$

$$\left. + \frac{\xi_j(x_{k+1}) - 2\xi_j(x_k) + \xi_j(x_{k-1})}{h^2} u_k + \frac{\xi_j(x_{k+1}) - \xi_j(x_k)}{h} . \frac{u_{k+1} - u_k}{h} \right\}$$
$$+ \left[a_k - a^j\right] \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2}.$$

Since (3.12) is a difference equation with constant coefficients, we have the estimates

$$(1+|\lambda|) \max_{0\le k\le N} |w_k| \le K(\varphi,\delta) \left[\max_{1\le k\le N-1} \left|F_k^j\right| + (1+|\lambda|)\,|\phi| \right], \quad \lambda \in R_\varphi, \tag{3.13}$$

$$\max_{1\le k\le N-1} \left| \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} \right| \le M(\varphi,\delta) \left[\max_{1\le k\le N-1} \left|F_k^j\right| + (1+|\lambda|)\,|\phi| \right]. \tag{3.14}$$

Using the definition of Q_j and the continuity of $a(x)$ as well as the smoothness of $\xi_i(x)$, we obtain

$$\max_{1\le k\le N-1} \left|F_k^j\right| \le M(\varphi,\delta) \left[\max_{1\le k\le N-1} |f_k| + \max_{0\le k\le N} |u_k| + \max_{0\le k\le N-1} \frac{|u_{k+1} - u_k|}{h} \right]$$

$$+\varepsilon \max_{1\le k\le N-1}\left|\frac{w_{k+1}-2w_k+w_{k-1}}{h^2}\right|$$

and

$$|\phi| \le M(\varphi,\delta)h \max_{0\le k\le N}|u_k|.$$

Assume that $0<\varepsilon<\frac{1}{M(\varphi,\delta)}$, then from the last estimate it follows that

$$\max_{1\le k\le N-1}\left|\frac{\xi_j(x_{k+1})u_{k+1}-2\xi_j(x_k)u_k+\xi_j(x_{k-1})u_{k-1}}{h^2}\right|$$

$$\le \frac{M(\varphi,\delta)}{1-\varepsilon M(\varphi,\delta)}\left\{\max_{1\le k\le N-1}|f_k|+\max_{0\le k\le N}|u_k|\right.$$

$$\left.+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}+(1+|\lambda|)h\phi,\right.$$

$$\max_{1\le k\le N-1}\left|F_k^j\right| \le \frac{M(\varphi,\delta)}{1-\varepsilon M(\varphi,\delta)}\left[\max_{1\le k\le N-1}|f_k|+\max_{0\le k\le N}|u_k|\right.$$

$$\left.+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}\right]+(1+|\lambda|)h\max_{0\le k\le N}|u_k|.$$

From this and the estimate (3.13) it follows that, for any $j=1,\dots,r$,

$$(1+|\lambda|)\max_{0\le k\le N}\left|\xi_j(x_k)u_k\right|$$

$$\le K(\varphi,\delta)\frac{M(\varphi,\delta)}{1-\varepsilon M(\varphi,\delta)}\left[\max_{1\le k\le N-1}|f_k|\right.$$

$$\left.+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}+(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|\right].$$

With the triangle inequality, we have

$$\max_{1\le k\le N-1}\left|\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}\right| \le K_1(\varphi,\delta)\left[\max_{1\le k\le N-1}|f_k|\right. \tag{3.15}$$

$$\left.+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}+(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|,\right.$$

$$(1+|\lambda|)\max_{0\le k\le N}|u_k| \le M_1(\varphi,\delta)\left[\max_{1\le k\le N-1}|f_k|+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}\right. \tag{3.16}$$

$$\left.+(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|\right].$$

Now using the inequality (3.9) we obtain

$$F=\max_{1\le k\le N-1}|f_k|+(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|+\max_{0\le k\le N-1}\frac{|u_{k+1}-u_k|}{h}$$

$$\le K_2(\varphi,\delta)\left[\max_{1\le k\le N-1}|f_k|+\alpha^{-1}(1+(1+|\lambda|)h)\max_{0\le k\le N}|u_k|\right.$$

$$\left.+\alpha\max_{1\le k\le N-1}\left|\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}\right|\right]$$

Hence for small α from the last inequality and the inequality (3.15) it follows that

$$F \leq M_2(\varphi,\delta)\left[\alpha^{-1}\left(1+(1+|\lambda|)h\right)\max_{0\leq k\leq N}|u_k| + \max_{1\leq k\leq N-1}|f_k|\right].$$

Therefore from (3.16) it follows

$$(1+|\lambda|)\max_{0\leq k\leq N}|u_k| \leq M_2(\varphi,\delta)\left[\alpha^{-1}\left(1+(1+|\lambda|)h\right)\max_{0\leq k\leq N}|u_k| + \max_{1\leq k\leq N-1}|f_k|\right].$$

Hence for all λ,

$$|\lambda| > \frac{M_2(\varphi,\delta)}{2\alpha - M_2(\varphi,\delta)h} - 1 = K_0\left(\varphi,\delta\right)$$

we have the estimate (3.11). Theorem 3.3 is proved.

3.5 Structure of the Fractional Spaces and Positivity of Difference Operators in C_h^{α}

The operator A_h^x commutes with its resolvent $(\lambda + A_h^x)^{-1}$. Therefore, by Theorem 3.3 we obtain that the operator A_h^x is positive in the fractional spaces $E_\alpha(C_h, A_h^x)$ generated by the difference operator A_h^x. Recall that $E_\alpha(C_h, A_h^x)$ is the set of all grid functions u^h for which the following norm

$$\left\|u^h\right\|_{E_\alpha(C_h,A_h^x)} = \sup_{\lambda>0}\lambda^{\alpha}\left\|A_h^x(\lambda+A_h^x)^{-1}u^h\right\|_{C_h} + \left\|u^h\right\|_{C_h}$$

is finite. Since for fixed h the operator A_h^x is bounded, this norm is finite for all grid functions.

Let C_h^{β} $(0\leq\beta\leq 1)$ denote the Banach space of all grid functions $u^h = \{u_k\}_1^{N-1}$ with $u_1 = u_{N-1}$ equipped with the norm

$$\left\|u^h\right\|_{C_h^{\beta}} = \max_{1\leq k<k+j\leq N-1}\frac{|u_k - u_{k+j}|}{(j\tau)^{\beta}} + \left\|u^h\right\|_{C_h}.$$

The main result of this section is the following theorem on the structure of the fractional spaces $E_\alpha(C_h, A_h^x)$.

Theorem 3.4. *For $0<\alpha<1/2$ the norms of the spaces $E_\alpha(C_h, A_h^x)$ and $C_h^{2\alpha}$ are equivalent uniformly in h, $0<h<h_0$.*

The results of Theorems 3.3 and 3.4 permit us to obtain the positivity in $C_h^{2\alpha}$ norms of the operators A_h^x.

Theorem 3.5. *Let h be a sufficiently small number. Then for all $\lambda\in R_\varphi$, $|\lambda|\geq K_0\left(\delta,\varphi\right)>0$ and $0<\alpha<1/2$ the resolvent $(\lambda+A_h^x)^{-1}$ is subject to the bound*

$$\left\|(\lambda+A_h^x)^{-1}\right\|_{C_h^{2\alpha}\to C_h^{2\alpha}} \leq \frac{M(\varphi,\delta)}{\alpha(1-2\alpha)}(1+|\lambda|)^{-1} \tag{3.17}$$

where $M(\varphi,\delta)$ does not depend on h and α.

The proof of Theorem 3.4 relies on certain properties of Green's function $J(k,j;\lambda+\delta)$ of the resolvent equation (3.3). In the case $a(x) \equiv a^2$ we have that

$$(A_h^x + \lambda)^{-1} f^h = \left\{ \sum_{j=1}^{N-1} J(k,j;\lambda+\delta) f_j h_1 \right\}_0^N, \tag{3.18}$$

where

$$J(k,1;\lambda+\delta) = J(k,N-1;\lambda+\delta)$$

$$= -\frac{1+\mu h}{2+3\mu h} \frac{\left(R^{N-3} - 4R^{N-2} + R - 4\right)}{2\mu} \left(I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2}\right)^{-1}$$

for $k = 0$ and $k = N$;

$$J(k,j;\lambda+\delta) = -\frac{1+\mu h_1}{2+3\mu h_1} \frac{(R^2-4R+1)(R^{j-2}+R^{N-j-2})}{2\mu} \left(I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2}\right)^{-1}$$

for $2 \le j \le N-2$ and $k=0$, $k=N$;

$$J(k,1;\lambda+\delta) = \frac{1+\mu h_1}{2+3\mu h_1} \frac{1+\mu h_1}{2+\mu h_1} (2\mu)^{-1} \{R^{k-1}(2(R+3) + R^2(R-3))$$
$$+R^{N-k}(4-R)(1+R) + R^{N+k-3}(1-4R)(1+R)$$
$$+R^{2N-k-3}(3R-1-2R^2(3R+1))\}(1-R^N)^{-1}\left(I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2}\right)^{-1},$$

$$J(k,N-1;\lambda+\delta) = -\frac{1+\mu h_1}{2+3\mu h_1} \frac{1+\mu h_1}{2+\mu h_1} (2\mu)^{-1} \{R^k(R-4)(R+1)$$
$$+R^{N-k-1}(-2(R+3) + R^2(3-R)) + R^{N+k-3}(1-3R+2R^2(3R+1))$$
$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}\left(I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2}\right)^{-1},$$

$$J(k,j;\lambda+\delta) = \frac{1+\mu h_1}{2+3\mu h_1} \frac{1+\mu h_1}{2+\mu h_1} (2\mu)^{-1} \{(R-1)^3(R^{j+k-2} + R^{2N-2-j-k})$$
$$+(-1+3R+R^2(3-R))(R^{N-k+j-2} + R^{N+k-j-2}) + 2(1-3R)(R^{2N-2+j-k}$$
$$+R^{2N-2-j+k}) + 2R^{|j-k|}(R^N-1)\left(R-3+R^{N-2}(-1+3R)\right)\}$$
$$\times(1-R^N)^{-1}\left(1 - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2}\right)^{-1}$$

for $2 \le j \le N-2$ and $1 \le k \le N-1$. Here

$$R = (1+\mu h_1)^{-1}, h_1 = a^{-1}h,$$

$$\mu = \frac{1}{2}\left(h_1(\lambda+\delta) + \sqrt{(\lambda+\delta)(4+h_1^2(\lambda+\delta))}\right).$$

A direct consequence of the last formula is

$$\sum_{j=1}^{N-1} J(k,j;\lambda+\delta) h_1 = \frac{1}{\lambda+\delta}. \tag{3.19}$$

Now, we will give the proof of Theorem 3.4.

First we consider the case $a(x) = a^2$. Let $a > 0$. For any $\lambda > 0$ we have the obvious identity

$$A_h^x(\lambda + A_h^x)^{-1} f^h = \lambda \left[\frac{1}{\lambda+\delta} - (\lambda + A_h^x)^{-1}\right] f^h + \frac{\delta}{\lambda+\delta} f^h. \qquad (3.20)$$

By formulas (3.18), (3.19) and the identity (3.20) we can write

$$\{A_h^x\,(\lambda + A_h^x)^{-1}\, f^h\}_k = \lambda \sum_{j=1}^{N-1} J(k,j;\lambda+\delta)\,[f_m - f_j]\,h_1 + \frac{\delta}{\lambda+\delta} f_m. \qquad (3.21)$$

Let $k = 0$. Then using (3.21) for $m = 1$, we obtain

$$\{A_h^x\,(\lambda + A_h^x)^{-1}\, f^h\}_0 = \lambda \sum_{j=1}^{N-1} J(0,j;\lambda+\delta)\,[f_1 - f_j]\,h_1 + \frac{\delta}{\lambda+\delta} f_1$$

$$= -\lambda \sum_{j=2}^{N-2} \frac{1+\mu h_1}{2+3\mu h_1} \frac{(R^2-4R+1)(R^{j-2}+R^{N-j-2})}{2\mu} (I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2})^{-1}\,[f_1 - f_j]\,h_1$$

$$+\lambda \frac{1+\mu h_1}{2+3\mu h_1} \frac{(R^{N-3}+1)(4R-1)}{2\mu} (I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2})^{-1}\,[f_1 - f_{N-1}]\,h_1 + \frac{\delta}{\lambda+\delta} f_1$$

$$= -\lambda \sum_{j=3}^{N-3} \frac{1+\mu h_1}{2+3\mu h_1} \frac{(R^2-4R+1)R^{j-2}}{2\mu} (I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2})^{-1}\,[f_1 - f_j]\,h_1$$

$$= -\lambda \sum_{j=3}^{N-3} \frac{1+\mu h_1}{2+3\mu h_1} \frac{(R^2-4R+1)R^{N-j-2}}{2\mu} (I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2})^{-1}\,[f_{N-1} - f_j]\,h_1$$

$$-\lambda \frac{1+\mu h_1}{2+3\mu h_1} \frac{(R^2-4R+1)(1+R^{N-4})}{2\mu} (I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2})^{-1}\,[f_1 - f_2]\,h_1$$

$$-\lambda \frac{1+\mu h_1}{2+3\mu h_1} \frac{(R^2-4R+1)(R^{N-4}+1)}{2\mu} (I - \frac{2-\mu h_1}{2+3\mu h_1} R^{N-2})^{-1}\,[f_{N-1} - f_{N-2}]\,h_1 + \frac{\delta}{\lambda+\delta} f_1.$$

We have that

$$\lambda^\alpha \left|\{A_h^x\,(\lambda + A_h^x)^{-1}\, f^h\}_0\right| \le M[\lambda^{1+\alpha} \sum_{j=3}^{N-3} \frac{\frac{1}{(1+\sqrt{|\lambda+\delta|}h_1\cos\varphi)^{j-2}}}{|\mu|} |f_1 - f_j| h_1$$

$$+\lambda^{1+\alpha} \sum_{j=3}^{N-3} \frac{\frac{1}{(1+\sqrt{|\lambda+\delta|}h_1\cos\varphi)^{N-j-2}}}{|\mu|} |f_{N-1} - f_j| h_1$$

$$+\lambda^{1+\alpha} \frac{1}{|\mu|} |f_1 - f_2| h_1 + \lambda^{1+\alpha} \frac{1}{|\mu|} |f_{N-1} - f_{N-2}| h_1 + \frac{\lambda^\alpha \delta}{\lambda+\delta} |f_1|]$$

$$\le M(\varphi,\delta)[\sum_{j=3}^{N-3} \frac{1}{((j-2)h_1)^{\frac{1}{2}+\alpha}} ((j-1)h_1)^{2\alpha} h_1$$

$$+\lambda^{1+\alpha} \frac{1}{\sqrt{\lambda+\delta}} h_1^{1+2\alpha} + 1] \left\|f^h\right\|_{C_h^{2\alpha}} \le M_1(\varphi,\delta) \left\|f^h\right\|_{C_h^{2\alpha}}.$$

Thus,

$$\lambda^{\alpha}\left|\{A_h^x(\lambda+A_h^x)^{-1}f^h\}_0\right|\leq M_1(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}. \tag{3.22}$$

The proof of the estimate

$$\lambda^{\alpha}\left|\{A_h^x(\lambda+A_h^x)^{-1}f^h\}_N\right|\leq M_1(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}$$

follows the scheme of the proof of the estimate (3.22) and is based on the formula (3.21) for $m=N-1$. Let $1\leq k\leq N-1$. Then using (3.21) for $m=k$, we obtain

$$\{A_h^x(\lambda+A_h^x)^{-1}f^h\}_k=\lambda\sum_{j=1}^{N-1}J(k,j;\lambda+\delta)\left[f_k-f_j\right]h_1+\frac{\delta}{\lambda+\delta}f_k$$

$$=\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{R^{k-1}(2(R+3)+R^2(R-3))$$

$$+R^{N+k-3}(1-4R)(1+R)\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\left[f_k-f_1\right]h_1$$

$$+\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{+R^{N-k}(4-R)(1+R)$$

$$+R^{2N-k-3}(3R-1-2R^2(3R+1))\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\left[f_k-f_{N-1}\right]h_1$$

$$-\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{+R^{N-k-1}(-2(R+3)+R^2(3-R))$$

$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\left[f_k-f_{N-1}\right]h_1$$

$$-\lambda\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\{R^k(R-4)(R+1)$$

$$+R^{N+k-3}(1-3R+2R^2(3R+1))\}(1-R^N)^{-1}(I-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\left[f_k-f_1\right]h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}\{(R-1)^3R^{j+k-2}$$

$$+(-1+3R+R^2(3-R))R^{N-k+j-2}+2(1-3R)R^{2N-2+j-k}\}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ \left[f_1-f_j\right]h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}\{(R-1)^3R^{j+k-2}$$

$$+(-1+3R+R^2(3-R))R^{N-k+j-2}+2(1-3R)R^{2N-2+j-k}\}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ \left[f_k-f_1\right]h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}\{(R-1)^3R^{2N-2-j-k}$$

$$+(-1+3R+R^2(3-R))R^{N+k-j-2}+2(1-3R)(R^{2N-2-j+k}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ [f_{N-1}-f_j]\,h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}\{(R-1)^3R^{2N-2-j-k}$$

$$+(-1+3R+R^2(3-R))R^{N+k-j-2}+2(1-3R)(R^{2N-2-j+k}$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ [f_k-f_{N-1}]\,h_1$$

$$+\frac{1+\mu h_1}{2+3\mu h_1}\frac{1+\mu h_1}{2+\mu h_1}(2\mu)^{-1}\lambda\sum_{j=2}^{N-2}2R^{|j-k|}(R^N-1)\left(R-3+R^{N-2}(-1+3R)\right)$$

$$\times(1-R^N)^{-1}(1-\frac{2-\mu h_1}{2+3\mu h_1}R^{N-2})^{-1}\ [f_k-f_j]\,h_1+\frac{\delta}{\lambda+\delta}f_k.$$

The proof of the estimate

$$\lambda^\alpha\left|\{A_h^x\,(\lambda+A_h^x)^{-1}\,f^h\}_k\right|\leq M_1(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}$$

follows the scheme of the proof of the estimate (3.22) and is based on the last formula. Thus, for any $\lambda\geq 0$ and $k=0,...,N$ we establish the validity of the inequality

$$\left|\lambda^\alpha\{A_h^x\,(\lambda+A_h^x)^{-1}\,f^h\}_k\right|\leq M_2(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}.$$

This means that

$$\left\|f^h\right\|_{E_\alpha\left(C_h,A_h^x\right)}\leq M_2(\varphi,\delta)\left\|f^h\right\|_{C_h^{2\alpha}}.$$

Now let us prove the opposite inequality. For any positive operator A_h^x we can write

$$v=\int_0^\infty\sum_{j=1}^{N-1}J\left(k,j;t+\delta\right)A_h^x\left(t+A_h^x\right)^{-1}f_jh_1dt.$$

Consequently

$$f_k-f_{k+r}=\int_0^\infty\sum_{j=1}^{N-1}t^{-\alpha}\left[J\left(k,j;t+\delta\right)-J\left(k+r,j;t+\delta\right)\right]t^\alpha A_h^x\left(t+A_h^x\right)^{-1}f_jh_1dt,$$

whence

$$|f_k-f_{k+r}|\leq\int_0^\infty t^{-\alpha}\sum_{j=1}^{N-1}\left|J\left(k,j;t+\delta\right)-J\left(k+r,j;t+\delta\right)\right|h_1dt\left\|f^h\right\|_{E_\alpha\left(C_h,A_h^x\right)}.$$

Let

$$T_h=|rh_1|^{-2\alpha}\int_0^\infty t^{-\alpha}\sum_{j=1}^{N-1}\left|J\left(k,j;t+\delta\right)-J\left(k+r,j;t+\delta\right)\right|h_1dt.$$

The proof of estimate

$$\frac{|f_k - f_{k+r}|}{|rh_1|^{2a}} \leq T_h \left\| f^h \right\|_{E_\alpha(C_h, A_h^x)}$$

follows the scheme of the paper (Ashyralyev and Sobolevskii, 1994) and is based on the Lemmas 3.2 , 3.3 and 3.4. Thus, for any $1 \leq k < k + r \leq N - 1$ we have established the inequality

$$\frac{|f_k - f_{k+r}|}{|rh_1|^{2\alpha}} \leq \frac{M}{\alpha(1 - 2\alpha)} \left\| f^h \right\|_{E_\alpha(C_h, A_h^x)}.$$

This means that the following inequality holds:

$$\left\| f^h \right\|_{C_h^{2\alpha}} \leq \frac{M}{\alpha(1 - 2\alpha)} \left\| f^h \right\|_{E_\alpha(C_h A_h^x)}.$$

Theorem 3.2 in the case $a(x) = a^2$ is proved. Now, let $a(x)$ be a continuous functions and let x, $x_0 \in [0, 1]$ be arbitrary fixed points. It is easy to show that

$$\left|(A_h^x - A_h^{x_0})(A_h^{x_0})^{-1}\right| \leq M.$$

Therefore, using the formula

$$A_h^x(A_h^x + \lambda)^{-1} f^h = A_h^{x_0}\left(A_h^{x_0} + \lambda\right)^{-1} f^h$$
$$+\lambda(\lambda + A_h^x)^{-1}\left[A_h^x - A_h^{x_0}\right]\left(A_h^{x_0}\right)^{-1} A_h^{x_0}\left(A_h^{x_0} + \lambda\right)^{-1} f^h,$$

we derive

$$\left|\lambda^\alpha A_h^x(A_h^x + \lambda)^{-1} f^h\right| \leq \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})}$$
$$+M\lambda \left\|(\lambda + A_h^x)^{-1}\right\|_{C_h \to C_h} \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})} \leq M_1 \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})}.$$

From that it follows

$$\left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})} \leq M_1 \left\| f^h \right\|_{E_\alpha(C_h, A_h^{x_0})}.$$

Theorem 3.4 is proved.

The results of this chapter and the abstract results of papers (Ashyralyev and Kendirli, 2002), (Ashyralyev, 1992), (Ashyralyev, 2003), (Skubachevskii, 1997) permit us to investigate the well posedness of the nonlocal boundary value problems for elliptic differential and difference equations in the Banach spaces.

Chapter 4

THE WELL POSEDNESS OF THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEMES

4.1 Introduction and the Main Theorem

Coercivity inequalities in Hölder norms with a weight for the solutions of an abstract differential equation of elliptic type were established for the first time in (Sobolevskii, 1965). Further in (Sobolevskii, 1971), (Sobolevskii, 1977), (Grisvard, 1986), (Sobolevskii and Tiunchik, 1982), (Polichka and Tiunchik, 1982), (Primakova and Sobolevskii, 1974), (Ashyralyev, 1992), (Sobolevskii, 1997), (Ashyralyev, 1995), (Ashyralyev and Kendirli, 2002), (Ashyralyev and Sobolevskii, 2004), (Ashyralyev, 2003), (Gershteyn and Sobolevskii, 1974), (Ashyralyev, 1989), (Ashyralyev and Amanov, 1996), (Alibekov, 1978), (Skubachevskii, 1997) the coercive inequalities in Hölder norms with a weight and without a weight were obtained for the solutions of various local and nonlocal boundary value problems for differential and difference equations of elliptic type.

In this chapter we consider the nonlocal boundary value problem

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1}-2u_k+u_{k-1}]+Au_k=\varphi_k, 1\le k\le N-1, \\ u_0=u_N, \quad -u_2+4u_1-3u_0=u_{N-2}-4u_{N-1}+3u_N, \quad N\tau=1 \end{cases} \tag{4.1}$$

in an arbitrary Banach space E with a positive operator A.

It is known (Sobolevskii, 1977) that for a positive operator A it follows that $B=\frac{1}{2}(\tau A+\sqrt{A(4+\tau^2 A)})$ is strongly positive and $R=(I+\tau B)^{-1}$ which is defined on the whole space E is a bounded operator. Furthermore, we have that

$$\| R^k \|_{E\to E}\le M(1+\delta\tau)^{-k}, k\tau \| BR^k \|_{E\to E}\le M, k\ge 1, \delta>0, \tag{4.2}$$

$$\| B^\beta(R^{k+r}-R^k) \|_{E\to E}\le M\frac{(r\tau)^\alpha}{(k\tau)^{\alpha+\beta}}, 1\le k<k+r\le N,\ 0\le\alpha,\ \beta\le 1, \tag{4.3}$$

From (4.2) it follows that

$$||(I-R^N)^{-1}||_{E\to E}\le M, ||(I-(2I-\tau B)(2I+3\tau B)^{-1}R^{N-2})^{-1}||_{E\to E}\le M. \tag{4.4}$$

For any φ_k, $1 \le k \le N-1$ the solution of the problem (4.1) exists and the following formula holds

$$u_k = \sum_{j=1}^{N-1} G(k,j)\varphi_j\tau, \quad 0 \le k \le N, \tag{4.5}$$

where

$$G(k,1) = G(k,N-1)$$
$$= -C\left(R^{N-3} - 4R^{N-2} + R - 4\right)(2B)^{-1}(I - DR^{N-2})^{-1}$$

for $k = 0$ and $k = N$;

$$G(k,j) = -C(R^2 - 4R + 1)(R^{j-2} + R^{N-j-2})(2B)^{-1}(I - DR^{N-2})^{-1}$$

for $2 \le j \le N-2$ and $k = 0$, $k = N$;

$$G(k,1) = CC_1(2B)^{-1}\{R^{k-1}(2(R+3) + R^2(R-3))$$
$$+R^{N-k}(4-R)(1+R) + R^{N+k-3}(1-4R)(1+R)$$
$$+R^{2N-k-3}(3R - 1 - 2R^2(3R+1))\}(1-R^N)^{-1}(I - DR^{N-2})^{-1},$$
$$G(k,N-1) = -CC_1(2B)^{-1}\{R^k(R-4)(R+1)$$
$$+R^{N-k-1}(-2(R+3) + R^2(3-R)) + R^{N+k-3}(1 - 3R + 2R^2(3R+1))$$
$$+R^{2N-k-3}(4R-1)(R+1)\}(1-R^N)^{-1}(I - DR^{N-2})^{-1},$$
$$G(k,j) = CC_1(2B)^{-1}\{(R-1)^3(R^{j+k-2} + R^{2N-2-j-k})$$
$$+(-1 + 3R + R^2(3-R))(R^{N-k+j-2} + R^{N+k-j-2}) + 2(1-3R)(R^{2N-2+j-k}$$
$$+R^{2N-2-j+k}) + 2R^{|j-k|}(R^N - 1)\left(R - 3 + R^{N-2}(-1+3R)\right)\}(1-R^N)^{-1}(I - DR^{N-2})^{-1}$$

for $2 \le j \le N-2$ and $1 \le k \le N-1$. Here

$$C = (I + \tau B)(2I + 3\tau B)^{-1}, C_1 = (I + \tau B)(2I + \tau B)^{-1}, D = (2I - \tau B)(2I + 3\tau B)^{-1},$$

where I is the unit operator.

Really, we see that the problem (4.1) can be obviously rewritten as the equivalent nonlocal boundary value problem for the first order linear difference equations

$$\begin{cases} \frac{u_k - u_{k-1}}{\tau} + Bu_k = z_k, \quad 1 \le k \le N, \\ u_N = u_0, -u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N, \\ -\frac{z_{k+1} - z_k}{\tau} + Bz_k = (1 + \tau B)\varphi_k, \quad 1 \le k \le N-1. \end{cases}$$

From that there follows the system of recursion formulas

$$\begin{cases} u_k = Ru_{k-1} + \tau R z_k, \quad 1 \le k \le N, \\ z_k = Rz_{k+1} + \tau\varphi_k, \quad 1 \le k \le N-1. \end{cases}$$

Hence

$$\begin{cases} u_k = R^k u_0 + \sum_{i=1}^{k} R^{k-i+1}\tau z_i, 1 \le k \le N, \\ z_k = R^{N-k} z_N + \sum_{j=k}^{N-1} R^{j-k}\tau\varphi_j, 1 \le k \le N-1. \end{cases}$$

From the first formula and the condition $u_N = u_0$ it follows that

$$u_N = R^N u_0 + \sum_{i=1}^{N} R^{N-i+1} \tau z_i$$

and

$$\begin{aligned}
u_N = u_0 &= (I - R^N)^{-1} \sum_{i=1}^{N} R^{N-i+1} \tau z_i \\
&= \frac{1}{1 - R^N} \left\{ \tau R z_N + \sum_{i=1}^{N-1} R^{N-i+1} \tau z_i \right\} \\
&= (I - R^N)^{-1} \left\{ \left(\tau R + \sum_{i=1}^{N-1} R^{2N-2i+1} \tau \right) z_N \right. \\
&\quad \left. + \sum_{i=1}^{N-1} \tau R^{N-i+1} \sum_{j=i}^{N-1} R^{j-i} \tau \varphi_j \right\} \\
&= (I - R^N)^{-1} \left\{ (R - R^{2N+1})(I - R^2)^{-1} \tau z_N + \sum_{j=1}^{N-1} \tau^2 \sum_{i=1}^{j} R^{N+j-2i+1} \varphi_j \right\} \\
&= (I - R^N)^{-1} (I - R^2)^{-1} \Big[R(1 - R^{2N}) \tau z_N \\
&\quad + \sum_{j=1}^{N-1} \tau^2 \left[R^{N-j+1} - R^{N+j+1} \right] \varphi_j \Big],
\end{aligned} \tag{4.6}$$

and for k, $1 \leq k \leq N-1$:

$$u_k = (I - R^N)^{-1} \left\{ h R^{k+1} z_N + \sum_{i=1}^{N-1} R^{k+N-i+1} h z_i \right\} + \sum_{i=1}^{k} R^{k-i+1} h z_i \tag{4.7}$$

$$\begin{aligned}
&= (I - R^N)^{-1} (I - R^2)^{-1} R^k \left\{ (R - R^{2N+1})(I - R^2)^{-1} \tau z_N \right. \\
&\quad \left. + \sum_{j=1}^{N-1} \tau^2 \left[R^{N-j+1} - R^{N+j+1} \right] \varphi_j \right\} \\
&\quad + \sum_{i=1}^{k} R^{N+k-2i+1} \tau z_N + \sum_{i=1}^{k} \sum_{j=i}^{N-1} \tau^2 R^{k+j-2i+1} \varphi_j \\
&= (I - R^2)^{-1} \left[R^{k+1} + R^{N-k+1} \right] \tau z_N \\
&\quad + (I - R^N)^{-1} (I - R^{N-1})^{-1} \sum_{j=1}^{N-1} \tau^2 \left[R^{N-j+1} - R^{N+j+1} \right] \varphi_j \\
&\quad + \sum_{j=1}^{k} \tau^2 \sum_{i=1}^{j} R^{k+j-2i+1} f_j + \sum_{j=k+1}^{N-1} \tau^2 \sum_{i=1}^{k} R^{k+j-2i+1} \varphi_j \\
&= (I - R^2)^{-1} \left[R^{k+1} + R^{N-k+1} \right] \tau z_N
\end{aligned}$$

$$+(I-R^N)^{-1}(I-R^2)^{-1}R^k\sum_{j=1}^{N-1}\tau^2\left[R^{N-j+1}-R^{N+j+1}\right]\varphi_j$$

$$+(I-R^2)^{-1}\sum_{j=1}^{N-1}\tau^2\left(R^{|k-j|+1}-R^{k+j+1}\right)\varphi_j.$$

By using the formulas (4.6), (4.7), and the condition $-u_2+4u_1-3u_0=u_{N-2}-4u_{N-1}+3u_N$ we obtain (4.5).

4.2 Well-Posedness of the Nonlocal Difference Problem

Let $F_\tau(E)$ be the linear space of mesh functions $\phi^\tau=\{\phi_k\}_1^{N-1}$ with values in the Banach space E. Next on $F_\tau(E)$ we denote $C_\tau(E)$ and $C_\tau^\alpha(E)$ - Banach spaces with the norms

$$\|\varphi\|_{C_\tau(E)}=\max_{1\le k\le N-1}\|\varphi_k\|_E,$$

$$\|\varphi\|_{C_\tau^\alpha(E)}=\|\varphi\|_{C_\tau(E)}+\max_{1\le k<k+r\le N-1}\|\varphi_{k+r}-\varphi_k\|_E\,\frac{1}{(r\tau)^\alpha}.$$

The nonlocal boundary value problem (4.1) is said to be stable in $F_\tau(E)$ if we have the inequality

$$\|u^\tau\|_{F_\tau(E)}\le M\|\varphi^\tau\|_{F_\tau(E)},$$

where M is independent not only of φ^τ but also of τ.

Theorem 4.1. *The nonlocal boundary value problem (4.1) is stable in $C_\tau(E)$ and $C_\tau^\alpha(E)$ norms.*

The proof of Theorem 4.1 is based on the stability inequality in $C_\tau(E)$ and $C_\tau^\alpha(E)$ norms for the solutions of the second order of accuracy difference scheme for the boundary value problem

$$\begin{cases}-\frac{1}{\tau^2}[u_{k+1}-2u_k+u_{k-1}]+Au_k=\varphi_k,\\ 1\le k\le N-1, u_0=\varphi,\ u_N=\psi\end{cases}\tag{4.8}$$

for elliptic difference equations in an arbitrary Banach space E with a positive operator A and on the estimates

$$\|u_0\|_E\le M\|\varphi^\tau\|_{C_\tau(E)},$$

$$\|u_0\|_{E_\alpha}\le M\|\varphi^\tau\|_{C_\tau^\alpha(E)}\,.$$

The proof of these estimates is based on the formula

$$u_0=\tau B^{-1}C\left(R^{N-3}-4R^{N-2}+R-4\right)2^{-1}(I-DR^{N-2})^{-1}\{\varphi_1+\varphi_{N-1}\}$$

$$-(I+\tau B)B^{-2}C(R^2-4R+I)(I-R^{N-5})2^{-1}(I-DR^{N-2})^{-1}\,(\varphi_{N-1}+\varphi_1)$$

$$+\sum_{j=2}^{N-2}\tau B^{-1}C(R^2-4R+I)R^{j-2}2^{-1}(I-DR^{N-2})^{-1}\,(\varphi_1-\varphi_j)$$

$$+\sum_{j=2}^{N-2}\tau B^{-1}C(R^2-4R+I)R^{N-j-2}2^{-1}(I-DR^{N-2})^{-1}\left(\varphi_{N-1}-\varphi_j\right)$$

and on the estimates (4.2) and (4.3).

The nonlocal boundary value problem (4.1) is said to be *coercively stable* (well posed) in $F_\tau(E)$ if we have the coercive inequality

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{F_\tau(E)} + \left\|\{Au_k\}_1^{N-1}\right\|_{F_\tau(E)} \leq M\| \varphi^\tau \|_{F_\tau(E)},$$

where M is independent not only of φ^τ but also of τ.

Since the nonlocal boundary value problem

$$-u''(t)+Au(t)=f(t) \quad (0\leq t\leq 1), \quad u(0)=u(1),\ u'(0)=u'(1)$$

in the space $C(E)$ of continuous functions defined on $[0,1]$ and with values in E is not well-posed for the general positive operator A and space E , then the well-posedness of the difference nonlocal boundary value in $C_\tau(E)$ norm does not take place uniformly with respect to $\tau>0$. This means that the coercive norm

$$\| u^\tau \|_{K_\tau(E)} = \| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C(\tau,E)} + \| Au^\tau \|_{C_\tau(E)}$$

tends to ∞ as $\tau\to 0^+$. The investigation of the difference problem (4.1) permits us to establish the order of growth of this norm to ∞.

Theorem 4.2. *For the solution of the difference problem (4.1) we have almost coercive inequality*

$$\| u^\tau \|_{K_\tau(E)} \leq M\min\left\{\ln\frac{1}{\tau}, 1+|\ln\| B \|_{E\to E}|\right\}\| \varphi^\tau \|_{C_\tau(E)},$$

where M does not depend on φ_k, $1\leq k\leq N-1$ and τ.

The proof of Theorem 4.2 is based on the almost coercive stability inequality in $C_\tau(E)$ for the solution of the boundary value problem (4.8) and on the estimate

$$\| Au_0\|_E \leq M\min\left\{\ln\frac{1}{\tau}, 1+|\ln\| B \|_{E\to E}|\right\}\| \varphi^\tau \|_{C_\tau(E)}.$$

The proof of this estimate follows the scheme of the paper (Ashyralyev, 1992) and relies on the formula

$$Au_0=\tau BRC\left(R^{N-3}-4R^{N-2}+R-4\right)2^{-1}(I-DR^{N-2})^{-1}\{\varphi_1+\varphi_{N-1}\} \tag{4.9}$$

$$-\sum_{j=2}^{N-2}\tau BRC(R^2-4R+1)(R^{j-2}+R^{N-j-2})2^{-1}(I-DR^{N-2})^{-1}\varphi_j$$

and on the estimates (4.2) and (4.3).

Theorem 4.3. *Let $\varphi_{N-1}-\varphi_1\in E_\alpha$. Then the coercivity inequality holds:*

$$\| \left\{\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\}_1^{N-1} \|_{C_\tau^\alpha} + \| Au^\tau \|_{C_\tau^\alpha}$$

$$\leq \frac{M}{\alpha(1-\alpha)}\left[\| \varphi^\tau \|_{C_\tau^\alpha} + \| \varphi_1-\varphi_{N-1} \|_{E_\alpha}\right],$$

where M does not depend on φ_k, $1\leq k\leq N-1$, α and τ.

The proof of Theorem 4.3 is based on the coercive stability inequality in $C_\tau^\alpha(E)$ for the solution of the boundary value problem (4.8) and on the estimate

$$\| Au_0 - \varphi_1\|_{E_\alpha} \leq \frac{M}{\alpha(1-\alpha)}[\| \varphi^\tau \|_{C_\tau^\alpha} + \| \varphi_1 - \varphi_{N-1} \|_{E_\alpha}].$$

The proof of this estimate follows the scheme of the paper (Ashyralyev, 1992) and relies on the formula

$$Au_0 - \varphi_1 = \tau BRC(R^{N-3} - 4R^{N-2} + R - 4)2^{-1}(I - DR^{N-2})^{-1}\{-\varphi_1 + \varphi_{N-1}\}$$
$$-C(R^2 - 4R + 1)(I - R^{N-5})2^{-1}(I - DR^{N-2})^{-1}\ (\varphi_{N-1} - \varphi_1\ )$$

$$+\sum_{j=2}^{N-2} \tau BRC(R^2 - 4R + 1)R^{j-2}2^{-1}(I - DR^{N-2})^{-1}\ (\varphi_1 - \varphi_j)$$

$$+\sum_{j=2}^{N-2} \tau BRC(R^2 - 4R + 1)R^{N-j-2}2^{-1}(I - DR^{N-2})^{-1}\ (\varphi_{N-1} - \varphi_j)$$

and on the estimates (4.2) and (4.3).

Theorem 4.4. *The nonlocal boundary value problem* (4.1) *is well-posed in* $C_\tau(E_\alpha)$.

The proof of this theorem is based on the abstract theorem on the well-posedness in $C_\tau(E_\alpha)$ of the local boundary value difference problem (4.8) and on the estimate

$$\| Au_0 \|_{E_\alpha} \leq \frac{M}{\alpha(1-\alpha)} \| \varphi^\tau \|_{C_\tau(E_\alpha)}$$

for the solution of the problem (4.1). The proof of these estimates follows the scheme of the paper (Ashyralyev, 1992) and relies on the formula (4.9).

4.3 Applications

First, we consider the nonlocal boundary value problem for two-dimensional elliptic equations

$$\begin{cases} -\frac{\partial^2 u}{\partial y^2} - a(x)\frac{\partial^2 u}{\partial x^2} + \delta u = f(y,x), 0 < y < 1, 0 < x < 1, \\ u(0,x) = u(1,x), u_y(0,x) = u_y(1,x),\ 0 \leq x \leq 1, \\ u(y,0) = u(y,1),\ \ u_x(y,0) = u_x(y,1),\ \ 0 \leq y \leq 1, \end{cases} \tag{4.10}$$

where $a(x)$ and $f(y,x)$ are given sufficiently smooth functions and $a(x) > 0$, $\delta > 0$ is a sufficiently large number.

We introduce the Banach spaces $C^\beta[0,1]$ $(0 < \beta < 1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite:

$$\| \varphi \|_{C^\beta[0,1]} = \| \varphi \|_{C[0,1]} + \sup_{0 \leq x < x+\tau \leq 1} \frac{|\varphi(x+\tau) - \varphi(x)|}{\tau^\beta},$$

where $C[0,1]$ is the space of all continuous functions $\varphi(x)$ defined on $[0,1]$ with the usual norm

$$\| \varphi \|_{C[0,1]} = \max_{0\leq x\leq 1} |\varphi(x)|.$$

It is known that the differential expression

$$A^x v = -a(x)v''(x) + \delta v(x)$$

defines a positive operator A^x acting in $C^{\beta}[0,1]$ with domain $C^{\beta+2}[0,1]$ and satisfying the conditions $v(0) = v(1)$, $v_x(0) = v_x(1)$.

Let us associate with the nonlocal boundary value problem (4.10) the corresponding difference problem

$$\begin{cases} -\frac{1}{\tau^2}(u_{k+1}^n - 2u_k^n + u_{k-1}^n) - a^n \frac{1}{h^2}(u_k^{n+1} - 2u_k^n + u_k^{n-1}) + \delta u_k^n = \varphi_k^n, \\ \varphi_k^n = f(y_k, x_n), \quad a^n = a(x_n), \quad y_k = k\tau, \quad x_n = nh, \\ 1 \leq k \leq N-1, 1 \leq n \leq M-1, N\tau = 1, Mh = 1, \\ u_0^n = u_N^n, \quad -u_2^n + 4u_1^n - 3u_0^n = u_{N-2}^n - 4u_{N-1}^n + 3u_N^n, 0 \leq n \leq M, \\ u_k^0 = u_k^M, -u_k^2 + 4u_k^1 - 3u_k^0 = u_k^{M-2} - 4u_k^{M-1} + 3u_k^M, 0 \leq k \leq N. \end{cases} \tag{4.11}$$

We introduce the Banach spaces C_h, C_h^{β} of grid functions $\varphi^h = \{\varphi^n\}_1^{M-1}$ with norms

$$\| \varphi^h \|_{C_h} = \max_{1\leq n\leq M-1} | \varphi^n |, \| \varphi^h \|_{C_h^{\beta}} = \| \varphi^h \|_{C_h} + \max_{1\leq n<n+r\leq M-1} \frac{| \varphi^{n+r} - \varphi^n |}{(r\tau)^{\alpha}}.$$

The difference operator

$$A_h u^h = \left\{ -a^n \left(\frac{u^{n+1} - 2u^n + u^{n-1}}{h^2} \right) + \delta u^n \right\}_1^{M-1}$$

acting in the space of grid functions $\varphi^h = \{\varphi^n\}_0^M$ satisfying the conditions $\varphi^0 = \varphi^M$, $-\varphi^2 + 4\varphi^1 - 3\varphi^0 = \varphi^{M-2} - 4\varphi^{M-1} + 3\varphi^M$ is a positive operator. Therefore we can replace the difference problem (4.10) by the abstract boundary value difference problem (4.1). Using the results of the papers (Ashyralyev and Kendirli, 2000), (Ashyralyev and Kendirli, 2001), (Ashyralyev and Yenial-Altay, 2004) and of theorems 4.1,4.2, 4.3 and 4.4 we obtain that

Theorem 4.5. *Let τ and h be a sufficiently small numbers. For the solution of the difference problem* (4.11) *the following inequalities are valid:*

$$\| u^{\tau,h} \|_{C_\tau^\alpha\left(C_h^\beta\right)} \leq M(\beta) \| \varphi^{\tau,h} \|_{C_\tau^\alpha\left(C_h^\beta\right)}, 0 \leq \alpha < 1, \beta \geq 0,$$

$$\| \left\{ \frac{1}{\tau^2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h) \right\}_1^{N-1} \|_{C_\tau(C_h)} \leq M \ln \frac{1}{\tau + h} \| \varphi^{\tau,h} \|_{C_\tau(C_h)},$$

$$\| \left\{ \frac{1}{\tau^2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h) \right\}_1^{N-1} \|_{C_\tau^\alpha(C_h^\beta)}$$

$$\leq M(\alpha,\beta)[\| \varphi^{\tau,h} \|_{C_\tau^\alpha(C_h^\beta)} + \| \varphi_1^h - \varphi_{N-1}^h \|_{C_h^{\beta+2\alpha}}], 0 \leq 2\alpha + \beta < 1, \beta \geq 0,$$

where $M, M(\beta)$ and $M(\alpha,\beta)$ are independent of $\varphi^{\tau,h}$, h and τ.

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$ we consider the mixed boundary-value problem for the multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u(y,x)}{\partial y^2} - \sum\limits_{r=1}^{n} \alpha_r(x) \frac{\partial^2 u(y,x)}{\partial x_r^2} + \delta u(y,x) = f(y,x), \\ x = (x_1, \ldots, x_n) \in \Omega, 0 < y < 1, \\ u(0,x) = u(1,x), u_y(0,x) = u_y(1,x), f(0,x) - f(1,x) = 0, \ x \in \overline{\Omega}, \\ u(y,x) = 0, \ x \in S, \end{cases} \tag{4.12}$$

where $\alpha_r(x)$ $(x \in \Omega)$ and $f(y,x)$ $(y \in (0,1),\ x \in \Omega)$ are given smooth functions and $\alpha_r(x) > 0$, $\delta > 0$ is a sufficiently large number.

We introduce the Banach spaces $C_{01}^{\beta}(\overline{\Omega})$ $(\beta = (\beta_1, \ldots, \beta_n), 0 < x_k < 1, k = 1, \ldots, n)$ of all continuous functions satisfying a Hölder condition with the indicator $\beta = (\beta_1, \ldots, \beta_n), \beta_k \in (0,1)$, $1 \leq k \leq n$ and with weight $x_k^{\beta_k}(1 - x_k - h_k)^{\beta_k}$, $0 \leq x_k < x_k + h_k \leq 1$, $1 \leq k \leq n$ which is equipped with the norm

$$\| f \|_{C_{01}^{\beta}(\overline{\Omega})} = \| f \|_{C(\overline{\Omega})} + \sup_{0 \leq x_k < x_k + h_k \leq 1, 1 \leq k \leq n} |f(x_1, \ldots, x_n) - f(x_1 + h_1, \ldots, x_n + h_n)| \times \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k} (1 - x_k - h_k)^{\beta_k},$$

where $C(\overline{\Omega})$ stands for the Banach space of all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |f(x)|.$$

It is known that the differential expression

$$A^x v = -\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 v(y,x)}{\partial x^2} + \delta v(y,x)$$

defines a positive operator A^x acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $v = 0$ on S.

The discretization of problem (4.12) is carried out in two steps. In the first step let us define the grid sets

$$\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \cdots, h_n m_n), m = (m_1, \cdots, m_n),$$
$$0 \leq m_r \leq N_r, h_r N_r = L, r = 1, \cdots, n\},$$
$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.$$

We introduce the Banach spaces $C_h = C_h(\widetilde{\Omega}_h), C_h^{\beta} = C_{01}^{\beta}(\widetilde{\Omega}_h)$ of grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\| \varphi^h \|_{C(\overline{\Omega}_h)} = \max_{x \in \overline{\Omega}_h} |\varphi^h(x)|,$$

$$\| \varphi^h \|_{C^{\beta}_{01}(\overline{\Omega}_h)} = \| \varphi^h \|_{C(\overline{\Omega}_h)}$$

$$+ \sup_{0\le x_k < x_k+h_k \le 1, 1\le k\le n} |\varphi^h(x_1,\ldots,x_n) - \varphi^h(x_1+h_1,\ldots,x_n+h_n)|$$

$$\times \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k} (1-x_k-h_k)^{\beta_k},$$

To the differential operator A generated by the problem (4.12) we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{n} a_r(x)(u_{\bar{x}_r}^h)_{x_r,j_r} \tag{4.13}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$.

With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\begin{cases} -\frac{d^2 v^h(t,x)}{dy^2} + A_h^x v^h(y,x) = f^h(y,x), \ 0 \le y \le 1, \ x \in \widetilde{\Omega}_h, \\ f^h(0,x) = f^h(1,x), \ x \in \widetilde{\Omega}_h, \\ v^h(0,x) = v^h(1,x), v_y^h(0,x) = v_y^h(1,x), x \in \widetilde{\Omega}_h, \end{cases} \tag{4.14}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (4.14) by the difference scheme (4.1)

$$\begin{cases} -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h = \varphi_k^h(x), x \in \widetilde{\Omega}_h, \\ \varphi_k^h(x) = \{f(y_k, x_n)\}_1^{M-1}, y_k = k\tau, 1 \le k \le N-1, \ N\tau = 1, \\ \varphi_1^h(x) = \varphi_{N-1}^h(x), x \in \widetilde{\Omega}_h, \\ u_0^h(x) = u_N^h(x), -u_2^h + 4u_1^h - 3u_0^h = u_{N-2}^h - 4u_{N-1}^h + 3u_N^h, x \in \widetilde{\Omega}_h. \end{cases} \tag{4.15}$$

It is known that A_h^x is a positive operator in $C(\widetilde{\Omega}_h)$ and $C_{01}^{\beta}(\overline{\Omega}_h)$. Therefore we can replace the difference problem (4.15) by the abstract boundary value difference problem (4.1). Using the results of the papers (Sobolevskii, 1971), (Sobolevskii, 1977), (Alibekov, 1978) and of Theorems 4.1, 4.2, 4.3 and 4.4 we obtain that

Theorem 4.6. *Let τ and $|h|$ be a sufficiently small numbers. Then the solutions of difference scheme (4.15) satisfy the following estimates:*

$$\| u^{\tau,h} \|_{C_\tau^\alpha\left(C_h^\beta\right)} \le M(\beta) \| \varphi^{\tau,h} \|_{C_\tau^\alpha\left(C_h^\beta\right)}, 0 \le \alpha < 1, |\beta| \ge 0,$$

$$\| \left\{ \frac{1}{\tau^2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h) \right\}_1^{N-1} \|_{C_\tau(C_h)} \le M \ln \frac{1}{\tau + |h|} \| \varphi^{\tau,h} \|_{C_\tau(C_h)},$$

$$\| \left\{ \frac{1}{\tau^2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h) \right\}_1^{N-1} \|_{C_\tau^\alpha(C_h^\beta)}$$

$$\le M(\alpha,\beta) \| \varphi^{\tau,h} \|_{C_\tau^\alpha(C_h^\beta)}, 0 < 2\alpha + |\beta| < 1, \beta \ge 0,$$

where $M, M(\beta)$ and $M(\alpha,\beta)$ are independent of $\varphi^{\tau,h}$, h and τ.

The proof of theorem 4.6 is based on the abstract theorems 4.1, 4.2 and 4.3 and positivity of difference operator A_h^x defined by the formula (4.13) in $C(\widetilde{\Omega}_h)$ and $C_{01}^{\beta}(\overline{\Omega}_h)$.

Third, we consider the boundary value problem on the range $\{0 \leq y \leq 1,\ x \in \mathbb{R}^n\}$ for 2m-order multidimensional elliptic equations

$$\begin{cases} -\frac{\partial^2 u}{\partial y^2} + \sum\limits_{|r|=2m} a_r(x) \frac{\partial^{|r|} u}{\partial x_1^{r_1} ... \partial x_n^{r_n}} + \delta u(y,x) = f(y,x), \\ \\ 0 < y < 1,\ x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ \\ u(0,x) = u(1,x), \quad u_y(0,x) = u_y(1,x), x \in \mathbb{R}^n, \end{cases} \tag{4.16}$$

where $a_r(x)$ and $f(y,x)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number.

We will assume that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x) \left(i\xi_1\right)^{r_1} ... \left(i\xi_n\right)^{r_n}, \xi = (\xi_1, \cdots, \xi_n) \in R^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \tag{4.17}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1 |\xi|^{2m} \leq (-1)^m B^x(\xi) \leq M_2 |\xi|^{2m} < \infty$$

for $\xi \neq 0$.

The discretization of problem (4.16) is carried out in two steps. In the first step let us give the difference operator A_h^x by the formula

$$A_h^x u_x^h = \sum_{2m \leq |r| \leq S} b_r^x D_h^r u_x^h + \delta u_x^h. \tag{4.18}$$

The coefficients are chosen in such a way that the operator A_h^x approximates in a specified way the operator

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} ... \partial x_n^{r_n}} + \delta.$$

We shall assume that for $|\xi_k h| \leq \pi$ the symbol $A(\xi h, h)$ of the operator $A_h^x - \delta$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \geq M_1 |\xi|^{2m},\ |\arg A^x(\xi h, h)| \leq \phi < \phi_0 < \frac{\pi}{2}. \tag{4.19}$$

With the help of A_h^x we arrive at the boundary value problem

$$\begin{cases} -\frac{d^2 v^h(y,x)}{dy^2} + A_h^x v^h(y,x) = \varphi^h(y,x), 0 < y < 1, \\ \\ v^h(0,x) = v^h(1,x), v_y^h(0,x) = v_y^h(1,x), x \in \mathbb{R}_h^n \end{cases} \tag{4.20}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (4.20) by the difference scheme

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h] + A_h^x u_k^h = \varphi_k^h, 1 \le k \le N-1, \\ u_0^h = u_N^h, \quad -u_2^h + 4u_1^h - 3u_0^h = u_{N-2}^h - 4u_{N-1}^h + 3u_N^h, \quad N\tau = 1. \end{cases} \tag{4.21}$$

Let us give a number of corollaries of the abstract theorems given in the above.

Theorem 4.7. *Let τ and h be a sufficiently small numbers.Then the solutions of the difference schemes (4.21) satisfy the following stability estimates:*

$$\| u^{\tau,h} \|_{C_\tau^\alpha\left(C_h^\beta\right)} \le M \| \varphi^{\tau,h} \|_{C_\tau^\alpha\left(C_h^\beta\right)}, 0 \le \alpha < 1, 0 \le \beta < 1,$$

where M does not depend on $\varphi^{\tau,h}$, α, β, h and τ.

The proof of Theorem 4.7 is based on the abstract theorem 4.1, the positivity of the operator A_h^x in C_h^β and on the fact that for any $0 < \beta < \frac{1}{2m}$ the norms in the spaces $E_\beta(A_h^x, C_h)$ and $C_h^{2m\beta}$ are equivalent uniformly in h (see (Ashyralyev and Sobolevskii, 1994), (Smirnitskii and Sobolevskii, 1981)) and on the following theorem on the structure of the fractional spaces $E_\alpha((A_h^x)^{\frac{1}{2}}, C_h)$.

Theorem 4.8. ***(Ashyralyev, 1992)*** *Let A be a strongly positive operator in a Banach space E with spectral angle $\phi(A,E) < \frac{\pi}{2}$. Then for $0 < \alpha < \frac{1}{2}$ the norms of the spaces $E_\alpha(A^{\frac{1}{2}}, E)$ and $E_{\frac{\alpha}{2}}(A, E)$ are equivalent.*

Now, we consider the coercive stability of (4.21).

Theorem 4.9. *Let τ and h be a sufficiently small numbers. Then the solutions of the difference schemes (4.21) satisfy the following almost coercive stability estimate:*

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau(C_h)} \le M \ln \frac{1}{\tau + h} \| \varphi^{\tau,h} \|_{C_\tau(C_h)},$$

where M does not depend on $\varphi^{\tau,h}$, h and τ.

The proof of Theorem 4.9 is based on the abstract Theorem 4.2, the positivity of the operator A_h^x in C_h (Ashyralyev and Sobolevskii, 1994) and (Smirnitskii and Sobolevskii, 1981) and on the almost coercivity inequality for an elliptic operator A_h^x in C_h and on the estimate

$$\min\left\{\ln\frac{1}{\tau}, 1 + \left|\ln \| B_h^x \|_{C_h \to C_h}\right|\right\} \le M \ln \frac{1}{\tau + h}.$$

Theorem 4.10. *Let τ and h be a sufficiently small numbers. Then the solutions of the difference schemes (4.21) satisfy the coercivity estimates:*

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau^\alpha\left(C_h^\beta\right)}$$

$$\le M(\alpha,\beta)[\| \varphi^{\tau,h} \|_{C_\tau^\alpha\left(C_h^\beta\right)} + \| \varphi_1^h - \varphi_{N-1}^h \|_{C_h^{\beta+2\alpha}}], 0 \le \alpha < 1, 0 < \beta < 1,$$

where $M(\alpha,\beta)$ does not depend on $\varphi^{\tau,h}$, h and τ.

The proof of Theorem 4.10 is based on the abstract Theorems 4.3 and 4.4, the positivity of the operator A_h^x in C_h^β and the coercivity inequality for an elliptic operator A_h^x in C_h^β, $0 < \beta < 1$ and on the fact that for any $0 < \beta < \frac{1}{m}$ the norms in the spaces $E_\beta((A_h^x)^{\frac{1}{2}}, C_h)$ and $C_h^{m\beta}$ are equivalent uniformly in h (Ashyralyev and Sobolevskii, 1994) and (Smirnitskii and Sobolevskii, 1981).

Chapter 5

APPLICATIONS

5.1 Introduction

We consider the nonlocal boundary value problem for elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = [-12t^2 + 12t - 2 + t^2(1-t)^2]\sin x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = u(1,x),\ u_t(0,x) = u_t(1,x),\ 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \leq t \leq 1. \end{cases} \tag{5.1}$$

The exact solution is:

$$u(t,x) = t^2(1-t)^2 \sin x.$$

For approximate solutions of the nonlocal boundary value problem (5.1), we will use the first and second order of accuracy difference schemes. To solve this difference equations we have applied a procedure of modified Gauss elimination method. This method is explained in the following subsections.Two computer programs are written in Matlab and a table of analysis with the figures are given.

5.2 The First Order of Accuracy Difference Scheme

Consider the nonlocal boundary value problem (5.1) for elliptic equation. For approximate solution of the nonlocal boundary value problem (5.1), consider the grid space depending on the small parameters τ and h:

$$\begin{aligned} [0,1]_\tau \times [0,\pi]_h &= \{(t_k, x_n) : t_k = k\tau,\ 1 \leq k \leq N-1,\ N\tau = 1, \\ x_n &= nh,\ 1 \leq n \leq M-1, Mh = \pi\}. \end{aligned}$$

Applying the formulas

$$\begin{aligned} \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_k) &= O(\tau^2), \\ \frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) &= O(h^2), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \frac{u(1)-u(0)}{\tau} - u'(0) &= O(\tau), \\ \frac{u(1)-u(1-\tau)}{\tau} - u'(1) &= O(\tau) \end{aligned} \tag{5.3}$$

the first order of accuracy in t for approximate solutions of the nonlocal boundary value problem for elliptic equation (5.1) are obtained.

$$\begin{cases} \frac{U_n^{k+1}-2U_n^k+U_n^{k-1}}{\tau^2} + \frac{U_{n+1}^k-2U_n^k+U_{n-1}^k}{h^2} = \left[-12(k\tau)^2+12(k\tau)-2+(k\tau)^2(1-(k\tau))^2\right]\sin(nh), \\ 1 \le k \le N-1, \quad 1 \le n \le M-1, \\ U_0^k = U_M^k = 0, \quad 0 \le k \le N, \\ U_n^1 - U_n^0 = U_n^N - U_n^{N-1}, \ 0 \le n \le M. \end{cases} \tag{5.4}$$

Here we have$(N+1)\times(M+1)$ system of linear equations and we will write them in the matrix form. By resorting the system

$$\begin{cases} \left(\frac{1}{h^2}\right) U_{n+1}^k + \left(-\frac{2}{\tau^2}-\frac{2}{h^2}\right) U_n^k + \left[\frac{1}{h^2}\right] U_{n-1}^k + \left(\frac{1}{\tau^2}\right) U_n^{k+1} + \left[\frac{1}{\tau^2}\right] U_n^{k-1} = \varphi_n^k, \\ \varphi_n^k = \left[-12(k\tau)^2+12(k\tau)-2+(k\tau)^2(1-(k\tau))^2\right]\sin(nh), \\ 1 \le k \le N-1, \quad 1 \le n \le M-1, \\ U_0^k = U_M^k = 0, \quad 0 \le k \le N, \\ U_n^1 - U_n^0 = \left(U_n^N - U_n^{N-1}\right), U_n^0 = U_n^N, \ 0 \le n \le M \end{cases}$$

So,

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 0 \le n \le M, \\ U_0 = \widetilde{0}, \quad U_M = \widetilde{0}. \end{cases} \tag{5.5}$$

Denote

$$a = \left(\frac{1}{h^2}\right), \quad b = \left(\frac{1}{\tau^2}\right), \quad c = \left(\frac{-2}{\tau^2} - \frac{2}{h^2}\right),$$

$$\varphi_n^k = \left[-12(k\tau)^2+12(k\tau)-2+(k\tau)^2(1-(k\tau))^2\right]\sin(nh),$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \cdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & .. & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 \\ b & c & b & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & b & c & b & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & .. & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & c & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & b & c & b & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & b & c & b \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 & 1 & -1 \end{bmatrix}_{(N+1)\times(N+1)},$$

and $C = A$.

$$D = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ \dots \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s = n-1,\ n, n+1$$

For the solution of the last matrix equation, the modified variant Gauss elimination method is used. We seek a solution of the matrix equation by the following form:

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 2, 1, 0,$$

where α_j $(j = 1, ..., M-1)$ is a $(N+1) \times (N+1)$ square matrix and β_j $(j = 1, ..., M-1)$ is a $(N+1) \times (1)$ matrix. And α_1, β_1 are

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}_{(N+1)\times(1)} .$$

Using the equality

$$U_s = \alpha_{s+1} U_{s+1} + \beta_{s+1}, (\text{for } s = n,\ n-1)$$

and the equality

$$AU_{n+1} + B\, U_n + CU_{n-1} = D\varphi_n,$$

we can write

$$[A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1}]U_{n+1} + \left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n$$

The last equation is satisfied if it is to be selected

$$A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1} = 0,$$

$$\left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n, \quad 1 \le n \le M-1.$$

Formulas for $\alpha_{n+1},\ \beta_{n+1}$:

$$\begin{aligned} \alpha_{n+1} &= -\left(B + C\alpha_n\right)^{-1} A, \\ \beta_{n+1} &= \left(B + C\alpha_n\right)^{-1} \left(D\varphi_n - C\beta_n\right), n = 1, 2, 3, ...M-1. \end{aligned}$$

So,

$$U_M = \widetilde{0},$$

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 2, 1, 0.$$

Algorithm

1. **Step** Input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{\pi}{M}$.
2. **Step** Use the first order of accuracy difference scheme and write in matrix form:

$$A\, U_{n+1} + B\, U_n + C\, U_{n-1} = D\varphi_n, \quad 0 \le n \le M.$$

3. **Step** Determine the entries of the matrices $A,\ B,\ C$ and D.
4. **Step** Find α_1, β_1.
5. **Step** Compute $\alpha_{n+1},\ \beta_{n+1}$.
6. **Step** Compute U_n's $(n = M-1, ..., 2, 1)$, $\left(U_M = \widetilde{0}\right)$ using the following formula:

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}.$$

Matlab Implementation of the First Order of Accuracy Difference Scheme

```
function firstord
    close; close;
    N=50; M=50;
    tau=1/N;
    h=pi/M;
    a = 1/(h^2);
    b = 1/(tau^2);
    c = -2/(tau^2)-2/(h^2);
    for i=2:N; A(i,i)=a; end;
    A(N+1,N+1)=0; A;
    C=A;
    for i=2:N ; B(i,i-1)= b ; end;
    for i=2:N ; B(i,i)= c ; end;
    for i=2:N ; B(i,i+1)= b ; end;
    B(1,1)=1; B(1,N+1)=-1;
    B(N+1,1)=-1; B(N+1,2)=1; B(N+1,N) =1 ; B(N+1,N+1)=-1; B;
    for i=2:N; D(i,i)=1; end ;
    D(N+1,N+1)=0; D;
    for j=1:M+1;
    for k=1:N+1;
    t=(k-1)*tau;
    x=(j-1)*h;
    s=t^4-2*t^3-11*t^2+12*t-2;
    fii(k,j:j)=-s*sin(x) ;
    end;
    end;
    alpha(N+1,N+1,1:1)= 0 ;
    betha(N+1,1:1) = 0 ;
    for j=1:M-1;
    alpha( :, :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j))*(-A) ;
    betha( :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j ) )*(D*fii(:, j:j )- C * betha(:, j:j ) );
    end;
    U( N+1,1, M:M ) = 0;
    for z = M-1:-1:1 ;
    U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 ) + betha(:,z+1:z+1);
    end;
```

```
for z = 1:M ;
p(:,z+1:z+1)=U(:,:,z:z);
end;
'EXACT SOLUTION OF THIS PROBLEM' ;
for j=1:M+1;
for k=1:N+1 ;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=(t^2)*((1-t)^2)*sin(x);
end;
end;
figure ;
surf(es) ;
title('EXACT SOLUTION');
axis tight;
figure ;
surf(p) ;
title('THE DIFFERENCE SCHEMES SOLUTION');
axis tight;
```

5.3 The Second Order of Accuracy Difference Scheme

Consider again the nonlocal boundary-value problem (5.1). Applying the formulas (5.2), (5.3) and

$$u'(0) = \frac{-u(2\tau) + 4u(\tau) - 3u(0)}{2h} + O(\tau^2),$$
$$u'(1) = \frac{u(1-2\tau) - 4u\,(1-\tau) + 3u(1)}{2h} + O(\tau^2),$$

the difference scheme second order of accuracy in t and in x for approximate solutions of the nonlocal boundary value problem (5.1)

$$\begin{cases} \frac{U_n^{k+1}-2U_n^k+U_n^{k-1}}{\tau^2} + \frac{U_{n+1}^k-2U_n^k+U_{n-1}^k}{h^2} = \varphi_n^k \\ \varphi_n^k = [-12(k\tau)^2 + 12(k\tau) - 2 + (k\tau)^2(1-(k\tau))^2]\sin{(nh)} \\ U_0^k = U_M^k = 0, \quad 0 \le k \le N, \\ -3U_n^0 + 4U_n^1 - U_n^2 = U_n^{N-2} + 4U_n^{N-1} + 3U_n^N,\ 0 \le n \le M, \end{cases}$$

is obtained. Here

$$\begin{aligned} a &= \frac{1}{h^2}, \\ b &= \frac{1}{\tau^2}, \\ c &= -\frac{2}{\tau^2} - \frac{2}{h^2}, \end{aligned}$$

We have again the $(N+1) \times (M+1)$ system of linear equations.We will write them in the matrix form. By resorting the system we obtain the matrix equation with new data:

$$\varphi_n^k = [-12(k\tau)^2 + 12(k\tau) - 2 + (k\tau)^2(1-(k\tau))^2] \sin(nh),$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \dots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & .. & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 \\ b & c & b & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & b & c & b & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & .. & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & c & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & b & c & b & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & b & c & b \\ -3 & 4 & -1 & 0 & \dots & 0 & -1 & 4 & -3 \end{bmatrix}_{(N+1)\times(N+1)},$$

and $C = A$.

$$D = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ \cdots \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s = n-1,\ n,\ n+1.$$

For the solution of the last matrix equation, the same algorithm is used for the second order of accuracy difference scheme.

Matlab Implementation of the Second Order of Accuracy Difference Scheme

```
function secondord
    close; close;
    N=50; M=50;
    tau=1/N;
    h=pi/M;
    a = 1/(h^2);
    b = 1/(tau^2);
    c = -2/(tau^2)-2/(h^2);
    for i=2:N; A(i,i)=a; end;
    A(N+1,N+1)=0; A;
    C=A;
    for i=2:N ; B(i,i-1)= b ; end;
    for i=2:N ; B(i,i)= c ; end;
    for i=2:N ; B(i,i+1)= b ; end;
    B(1,1)=1; B(1,N+1)=-1;
    B(N+1,1)=-3; B(N+1,2)=4; B(N+1,3)=-1; B(N+1,N-1)=-1; B(N+1,N)=4;
    B(N+1,N+1)=-3; B;
    for i=2:N; D(i,i)=1; end ;
    D(N+1,N+1)=0; D;
    for j=1:M+1;
    for k=1:N+1;
    t=(k-1)*tau;
    x=(j-1)*h;
    s=t^4-2*t^3-11*t^2+12*t-2;
    fii(k,j:j)=-s*sin(x) ;
    end;
    end;
```

```
alpha(N+1,N+1,1:1)= 0 ;
betha(N+1,1:1) = 0 ;
for j=1:M-1;
alpha( :, :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j))*(-A) ;
betha( :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j ) )*(D*fii(:, j:j )- C * betha(:, j:j ) );
end;
U( N+1,1, M:M ) = 0;
for z = M-1:-1:1 ;
U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 ) + betha(:,z+1:z+1);
end;
for z = 1:M ;
p(:,z+1:z+1)=U(:,:,z:z);
end;
'EXACT SOLUTION OF THIS PROBLEM' ;
for j=1:M+1;
for k=1:N+1 ;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=(t^2)*((1-t)^2)*sin(x);
end;
end;
figure ;
surf(es) ;
title('EXACT SOLUTION');
axis tight;
figure ;
surf(p) ;
title('THE DIFFERENCE SCHEMES SOLUTION');
axis tight;
```

5.4 Numerical Analysis

Consider the nonlocal boundary value problem for elliptic equation (5.1). For the approximate solutions of the nonlocal boundary value problem (5.1), the first and the second order of accuracy difference schemes with $\tau = \frac{1}{50}$, $h = \frac{\pi}{50}$ will be used. The exact and numerical solutions are given in the table 5.1 and figures 5.1, 5.2 and 5.3.

TABLE

The first line is the exact solution, the second line is the solution of the first order of accuracy difference scheme and the third line is the solution of second order of accuracy difference scheme .

Table 5.1 Numerical analysis

$t_k \setminus x_n$	0	0.63	1.26	1.89	2.52	3.14
	0	0	0	0	0	0
0.0	0	0.0260	0.0412	0.0410	0.0248	0
	0	0.0036	0.0043	0.0036	0.0013	0
	0	0.0150	0.0243	0.0243	0.0150	0
0.2	0	0.0380	0.0621	0.0628	0.0392	0
	0	0.0172	0.0279	0.0281	0.0174	0
	0	0.0339	0.0548	0.0548	0.0339	0
0.4	0	0.0544	0.0905	0.0923	0.0586	0
	0	0.0343	0.0576	0.0590	0.0377	0
	0	0.0339	0.0548	0.0548	0.0339	0
0.6	0	0.0544	0.0905	0.0923	0.0586	0
	0	0.0343	0.0576	0.0590	0.0377	0
	0	0.0150	0.0243	0.0243	0.0150	0
0.8	0	0.0380	0.0621	0.0628	0.0392	0
	0	0.0172	0.0279	0.0281	0.0174	0
	0	0	0	0	0	0
1.0	0	0.0260	0.0412	0.0410	0.0248	0
	0	0.0036	0.0043	0.0036	0.0013	0

Thus, the second order of accuracy difference scheme is more accurate comparing the first order of accuracy difference scheme.

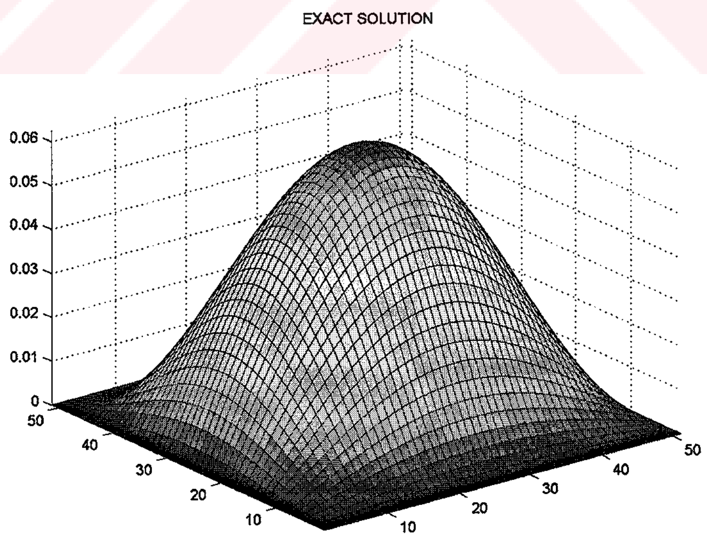

Figure 5.1: Exact solution

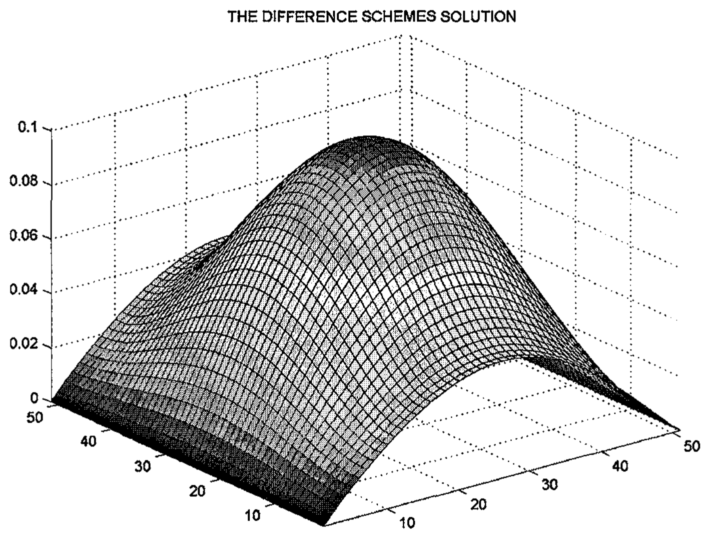

Figure 5.2: The first order of accuracy difference scheme

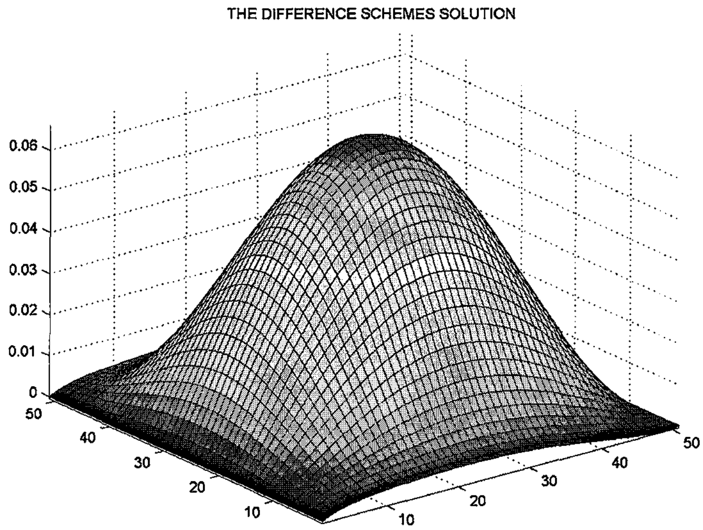

Figure 5.3: The second order of accuracy difference schemes

Now we will consider the special case of the nonlocal boundary value problem for elliptic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial^2 u(t,x)}{\partial x^2} = [-2t^3 + 3t^2 + 11t - 6]\sin x, \ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = u(1,x), \ u_t(0,x) = u_t(1,x), \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \ 0 \le t \le 1. \end{cases} \tag{5.6}$$

The exact solution is

$$u(t,x) = (2t^3 - 3t^2 + t)\sin x.$$

Applying the modified Gauss elimination method we obtain the following results for the first and second order of accuracy difference schemes of the problem (5.6).

TABLE

For the approximate solutions of the nonlocal boundary value problem (5.6), the first and second order of accuracy difference schemes with $\tau = \frac{1}{50}$, $h = \frac{\pi}{50}$ are used.

In the table the first line is the error of the first order of accuracy and the second line is the error of the second order of accuracy.

Table 5.2 Numerical analysis

$t_k \setminus x_n$	0	0.63	1.26	1.89	2.52	3.14
0.0	0	0	0	0	0	0
	0	0	0	0	0	0
0.2	0	0.0049	0.002	0.0017	0.0047	0
	0	0.0049	0.002	0.0017	0.0047	0
0.4	0	0.0024	0.001	0.0008	0.0023	0
	0	0.0024	0.001	0.0008	0.0023	0
0.6	0	0.0024	0.001	0.0008	0.0023	0
	0	0.0024	0.001	0.0008	0.0023	0
0.8	0	0.0049	0.002	0.0017	0.0047	0
	0	0.0049	0.002	0.0017	0.0047	0
1.0	0	0	0	0	0	0
	0	0	0	0	0	0

From table 5.2. we conclude that in this case the result values of the first and the second order of accuracy are almost the same. Instead, in this case the first difference scheme has also second order of accuracy. Really,

$$\frac{u(\tau,x_n)-u(0,x_n)}{\tau}-\frac{u(1,x_n)-u(1-\tau,x_n)}{\tau}$$

$$=-u_t(0,x_n)+u_t(1,x_n)$$

$$+\frac{\tau}{2}\left(u_{tt}(0,x_n)+u_{tt}(1,x_n)\right)+O\left(\tau^2\right).$$

We obtain that

$$u_{tt}(t,x_n)=(12t-6)\sin x_n,$$

$$u_{tt}(0,x_n)=-6\sin x_n,$$

$$u_{tt}(1,x_n)=6\sin x_n,$$

$$u_{tt}(0,x_n)+u_{tt}(1,x_n)=0.$$

Hence the error of approximation of this difference scheme is two. The values of first and second order of accuracy difference schemes are almost the same. Therefore the second order of accuracy is not more accurate than the first order of accuracy for this problem.

However for the problem (5.1) we have

$$u(t,x_n)=t^2(1-t)^2\sin x_n,$$

$$u_{tt}(t,x_n)=2(1-t)^2-8(1-t)t+2t^2,$$

$$u_{tt}(0,x_n)=2\sin x_n,$$

$$u_{tt}(1,x_n)=2\sin x_n,$$

$$u_{tt}(0,x_n)+u_{tt}(1,x_n)=4\sin x_n\neq 0.$$

Hence the error of approximation of this difference scheme is one. Therefore for this problem the second order of accuracy is more accurate.

Chapter 6

CONCLUSIONS

This work is devoted to the study of the stability of the nonlocal boundary value problem for elliptic equations. The following original results are obtained:

- the abstract theorems on the stability estimates, almost coercive stability estimates and coercive stability estimates for the solution of the nonlocal boundary problem for elliptic difference equation in the Banach space are proved,

- theorems on the stability estimates, almost coercive stability estimates and coercive stability estimates for the solution of difference schemes for elliptic equations are proved,

- the positivity of the second order difference operator in C_h and C_h^α is proved,

- the structure of the interpolation space is investigated,

- the theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments

- two papers from this work are submitted for publication in the journals.

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