DIFFERENCE SCHEMES OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC-PARABOLIC EQUATIONS

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Yıldırım ÖZDEMİR

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assist. Prof. Ali ŞAHİN
Head of Department

This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Prof. Dr. Allaberen AHSYRALYEV
Supervisor

Examining Committee Members

Prof. Dr. Allaberen ASHYRALYEV

Prof. Dr. Barış KENDİRLİ

Prof. Dr. Muhammet KÖKSAL

It is approved that this thesis has been written in compliance with the formatting rules laid down by the Graduate Institute of Sciences and Engineering.

Assist. Prof. Nurullah ARSLAN
Director

January 2005

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ABSTRACT

In the present work the nonlocal boundary-value problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + Au(t) = f(t),\ 0 \le t \le 1, \\ \dfrac{du(t)}{dt} + Au(t) = g(t), -1 \le t \le 0, \\ u(-1) = \alpha u(\mu) + \beta u'(\lambda) + \varphi,\ |\alpha|,\ |\beta| \le 1,\ 0 < \mu, \lambda \le 1 \end{cases}$$

for the differential equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. Applying the operator approach the stability estimates for solution of this nonlocal boundary value problem are obtained. The stability of difference schemes for approximately solving this nonlocal boundary value problem are presented. In applications this abstract results permit to obtain the stability estimates for the solution of the difference schemes for hyperbolic-parabolic equations. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Keywords: Hyperbolic-Parabolic Difference Equation, Difference Schemes, Stability.

HİPERBOLİK-PARABOLİK DENKLEMLERİN LOKAL OLMAYAN SINIR DEĞER PROBLEMLERİ İÇİN FARK ŞEMALARI

Yıldırım ÖZDEMİR

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Tez Yöneticisi: Prof. Dr. Allaberen ASHYRALYEV

ÖZ

Bu çalışmada Hilbert uzayında verilen fark denklemlerinin self-adjoint pozitif tanımlı A operatörlü lokal olmayan sınır değer problemi

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + Au(t) = f(t),\ 0 \le t \le 1, \\ \dfrac{du(t)}{dt} + Au(t) = g(t), -1 \le t \le 0, \\ u(-1) = \alpha u(\mu) + \beta u'(\lambda) + \varphi,\ |\alpha|,\ |\beta| \le 1,\ 0 < \mu, \lambda \le 1 \end{cases}$$

ele alınmıştır. Operatör yaklaşımını uygulayarak bu lokal olmayan sınır değer probleminin çözümünün karalılık kestirimleri elde edilmiştir. Bu lokal olmayan sınır değer problemlerinin yaklaşık çözümleri için fark şemalarının kararlılığı gösterilmiştir. Uygulamalarda bu sonuç, hiperbolik-parabolik denklemlerin fark şemalarının çözümü için kararlılık kestirimlerini elde etmemizi sağlamıştır. Bu fark şemalarının çözümü için yapılan teorik sonuçların doğruluğu, sayısal denemelerde desteklenmiştir.

Anahtar Kelimeler: Hiperbolik-Parabolik Fark Denklemi, Fark Şemaları, Kararlılık.

DEDICATION

To my parents

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INTRODUCTION

It is known that most problems in fluid mechanics (dynamics, elasticity) and other areas of physics and mathematical biology lead to partial differential equations of the hyperbolic-parabolic type. These equations can be derived as models of physical systems or mathematical biology are considered as the methods for solving boundary value problems.

Methods of solutions of the boundary value problems for hyperbolic-parabolic differential equations have been studied extensively by many researches (see [Salahatdinov, M. S., 1974], [Djuraev, T. D., 1978], [Bazarov, D. and Soltanov, H., 1995], [Krein, S. G., 1966], [Ashyralyev, A. and Yurtsever, A., 2001], [Ashyralyev, A. and Yurtsever, A., 2001], [Ashyralyev, A. and Yurtsever, A., 2000], [Ashyralyev, A. and Yurtsever, A., 2004], [Ashyralyev, A. and Yurtsever, A., 2004], [Ashyralyev, A. and Muradov, I., 1995], [Ashyralyev, A. and Muradov, I., 1996], [Ashyralyev, A. and Muradov, I., 1998], [Ashyralyev, A. and Orazov, M. B., 1999], [Ashyralyev, A. and Orazov, M. B., 1999], [Ashyralyev, A., 1999], [Ashyralyev, A. and Ozdemir, Y., 2004], [Ashyralyev, A., Akca, H. and Biszewski, L., 2000], [Ashyralyev, A. and Sobolevskii, P. E., 1994], [Samarskii, A. and Andr'eev, V., 1978], [Ashyralyev, A. and Karatay, I., 2003], [Ashyralyev, A., Karatay, I. and Sobolevskii, P. E., 2004], [Samarskii, A. and Gulin, A. V., 1973], [Rannacher, R., 1982], [Ashyralyev, A. and Sobolevskii, P. E., 2001], [Ashyralyev, A. and Aggez, N., 2004], [Ashyralyev, A. and Sobolevskii, P. E., 1995], [Ashyralyev, A. and Sobolevskii, P. E., 1988], [Ashyralyev, A., Piskarev, S. and Wei, S., 2002], [Guidetti, D., Karasozen, B. and Piskarev, S., 2004], [Gavrilyuk, I. P. and Makarov, V. L., 2000], [Gavrilyuk, I. P., 1999], [Wang, Y. G. and Oberguggenberger, M., 1999], [Jurgen, B. W. and Garay, B. M., 2002], [Rautmann, R., 1997], [Ashyralyev, A. and Sobolevskii, P. E., 2004], [Ferreira, J., 1996], [Antman, S. S. and Seidman, T. F., 1999], [French, D. A., Jensen, S. and Seidman, T. I., 1999], [Evje, S. and Karlsen, K. H., 1999], [Doronin, G. G., Lar'kin, N. A. and Souza, A. J., 2000], [Clark, M. R., 1994], [Gerish, A., Kotshote, M. and Zacher, R.], [Gerish, A., Griffiths, D. F., Weiner, R. and Chaplain, M. A. J., 2001], [Gripenberg, G., 1994], [Gripenberg, G., 1994], [Milani, A., 1995], [Milani, A., 1997], [Chen, G. Q. and DiBenedetto, E., 2001], [Matsuma, A., 1981], [Ferreira, J. and Lar'kin, N. A., 1996], [Rausset, F., 2003] and the references therein).

Our goal in this work is to investigate the stability of difference schemes of the approximate solutions of the nonlocal boundary value problems for equations of hyperbolic-parabolic type.

It is known that the mixed problem for hyperbolic-parabolic equations can be solved by Fourier series method, by Fourier transform method and by Laplace transform method.

Now let us give some examples.

First let us consider the simple nonlocal boundary value problem for hyperbolic-parabolic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = \cos t \sin x, \ 0 \leq t \leq 1, \ 0 < x < \pi, \\ \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = (2\cos t - \sin t) \sin x, \ \ -1 \leq t \leq 0, \ 0 < x < \pi, \\ u(-1, x) = u(1, x), \ 0 \leq x \leq \pi, \\ u(t, 0) = u(t, \pi) = 0, \ \ -1 \leq t \leq 1. \end{cases} \tag{1.1}$$

For the solution of the problem (1.1), we use the method of separation of variables or so called Fourier series method. In order to solve the problem we need to separate $u(t,x)$ into two parts

$$u(t,x) = v(t,x) + w(t,x)$$

where

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + v = 0, \ 0 \leq t \leq 1, \ 0 < x < \pi, \\ \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + v = 0, \ \ -1 \leq t \leq 0, \ 0 < x < \pi, \\ v(-1, x) = v(1, x), \ 0 \leq x \leq \pi, \\ v(t, 0) = v(t, \pi) = 0, \ \ -1 \leq t \leq 1. \end{cases} \tag{1.2}$$

and

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + w = \cos t \sin x, \ 0 \leq t \leq 1, \ 0 < x < \pi, \\ \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + w = (2\cos t - \sin t) \sin x, \ \ -1 \leq t \leq 0, \ 0 < x < \pi, \\ w(-1, x) = w(1, x), \ 0 \leq x \leq \pi, \\ w(t, 0) = w(t, \pi) = 0, \ \ -1 \leq t \leq 1. \end{cases} \tag{1.3}$$

First we will obtain the solution of the problem (1.2).Now, we consider if $0 \leq t \leq 1$. By the method of separation of variables, we obtain

$$v(t,x) = T(t)X(x) \neq 0.$$

We have that

$$\frac{T''(t) + T(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0$$

or

$$\frac{T''(t) + T(t)}{T(t)} = \frac{X''(x)}{X(x)} = -k^2 = \lambda.$$

Since

$$\sqrt{-\lambda} = \frac{k\pi}{L} = \frac{k\pi}{\pi} = k,$$

we have that

$$X''(x) + k^2 X(x) = 0,$$

or

$$X_k(x) = \sin kx.$$

Now to get $T(t)$ we can write

$$T''(t)+T\left(t\right)=-k^{2}T\left(t\right),$$

or

$$T''(t)+\left(1+k^{2}\right)T(t)=0.$$

So the solution for $T(t)$ is

$$T_k(t)=A_k\sin\sqrt{1+k^2}t+B_k\cos\sqrt{1+k^2}t.$$

Thus,

$$v(t,x)=\sum_{k=1}^{\infty}v_k(t,x)=\sum_{k=1}^{\infty}\left(A_k\sin\sqrt{1+k^2}t+B_k\cos\sqrt{1+k^2}t\right)\sin kx.$$

Now, we will consider if $-1\leq t\leq 0$. By the method of separation of variables, we obtain

$$v\left(t,x\right)=T\left(t\right)X\left(x\right)\neq 0.$$

We have that

$$\frac{T'\left(t\right)+T\left(t\right)}{T\left(t\right)}-\frac{X''\left(x\right)}{X\left(x\right)}=0,$$

or

$$\frac{T'\left(t\right)+T\left(t\right)}{T\left(t\right)}=\frac{X''\left(x\right)}{X\left(x\right)}=-k^{2}=\lambda.$$

So,

$$X''\left(x\right)+k^{2}X\left(x\right)=0,$$

or

$$X_k\left(x\right)=\sin kx.$$

Now to get $T(t)$ we can write

$$T'\left(t\right)+\left(1+k^{2}\right)T\left(t\right)=0.$$

So the solution for $T(t)$ is

$$T_k\left(t\right)=C_ke^{-\left(1+k^2\right)t}.$$

Thus,

$$v\left(t,x\right)=\sum_{k=1}^{\infty}C_ke^{-\left(1+k^2\right)t}\sin kx.$$

Using the nonlocal boundary conditions

$$\begin{cases} v\left(1,x\right)=v\left(-1,x\right), \\ v\left(0_{+},x\right)=v\left(0_{-},x\right), \\ v'\left(0_{+},x\right)=v'\left(0_{-},x\right), \end{cases}$$

we can obtain

$$A_k=B_k=C_k=0 \qquad \forall k.$$

Hence

$$v\left(t,x\right)\equiv 0.$$

Second we will obtain the solution for (1.3). First, we consider if $0 \leq t \leq 1$. Let

$$f(t,x) = \sum_{k=1}^{\infty} D_k(t) \sin kx = \cos t \sin x.$$

Then

$$D_1(t) = \cos t,$$

and

$$D_k(t) = 0 \text{ for } k \neq 1.$$

Let

$$w(t,x) = \sum_{k=1}^{\infty} E_k(t) \sin kx.$$

Then

$$w_{tt} - w_{xx} + w = \sum_{k=1}^{\infty} \left(E_k''(t) + \left(1+k^2\right) E_k(t) \right) \sin kx = \cos t \sin x.$$

If $k \neq 1$, and $0 \leq t \leq 1$,then

$$E_k''(t) + \left(1+k^2\right) E_k(t) = 0.$$

So we obtain

$$E_k(t) = c_1 \cos \sqrt{1+k^2} t + c_2 \sin \sqrt{1+k^2} t.$$

If $k \neq 1$ and $-1 \leq t \leq 0$,then

$$G_k'(t) + \left(1+k^2\right) G_k(t) = 0.$$

So we obtain

$$G_k(t) = c_3 e^{-\left(1+k^2\right)t}.$$

Using the nonlocal boundary conditions

$$\begin{cases} E_k(1) = G_k(-1), \\ E_k(0_+) = G_k(0_-), \\ E_k'(0_+) = G_k'(0_-), \end{cases}$$

we get $c_1 = c_2 = c_3 = 0$.

So $E_k(t) = G_k(t) = 0 \quad \forall k \neq 1$.

If $k = 1$, then

$$E_1''(t) + 2E_1(t) = \cos t.$$

$E_1(t)$ can be found as

$$E_1(t) = c_1 \cos \sqrt{2} t + c_2 \sin \sqrt{2} t + \cos t.$$

Thus

$$w(t,x) = \left(c_1 \cos \sqrt{2} t + c_2 \sqrt{2} t + \cos t \right) \sin x.$$

Now, we consider if $-1 \leq t \leq 0$. Let

$$g(t,x) = \sum_{k=1}^{\infty} F_k(t) \sin kx = (2 \cos t - \sin t) \sin x.$$

Then

$$F_1(t) = 2\cos t - \sin t,$$

and

$$F_k(t) = 0 \text{ for } k \neq 1.$$

Let

$$w(t,x) = \sum_{k=1}^{\infty} G_k(t,x) \sin kx.$$

Then

$$w_t - w_{xx} + w = \sum_{k=1}^{\infty} \left(G_k'(t) + \left(1+k^2\right) G_k(t) \right) \sin kx = (2\cos t - \sin t) \sin x.$$

If $k = 1$, then

$$G_1'(t) + 2G_1(t) = 2\cos t - \sin t.$$

$G_1(t)$ can be found as

$$G_1(t) = e^{-2t}(c_3 - 1) + \cos t.$$

Thus

$$w(t,x) = \left\{ e^{-2t}(c_3 - 1) + \cos t \right\} \sin x.$$

Using the nonlocal boundary conditions

$$\begin{cases} E_1(1) = G_1(-1), \\ E_1(0_+) = G_1(0_-), \\ E_1'(0_+) = G_1'(0_-), \end{cases}$$

we obtain that

$$w(t,x) = \cos t \sin x.$$

Therefore $u(t,x)$ is found to be

$$u(t,x) = v(t,x) + w(t,x),$$

then

$$u(t,x) = \cos t \sin x.$$

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the multidimensional hyperbolic-parabolic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, \ldots, x_n) \in \overline{\Omega}, 0 \leq t \leq T, \\ \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = g(t,x), \\ x = (x_1, \ldots, x_n) \in \overline{\Omega}, -T \leq t \leq 0, \\ u_t(0+,x) = u_t(0-,x), x \in \overline{\Omega} \\ u(-T,x) = u(T,x) + \varphi(x),\ x \in \overline{\Omega}, \\ u(t,x) = 0, x \in S \end{cases}$$

where $\alpha_r > 0$ and $f(t,x)$ $(t \in [0,T]\,,\ x \in \overline{\Omega})$, $g(t,x)$ $(t \in [-T,0]\,,\ x \in \overline{\Omega}), \varphi(x), \psi(x)$ $(x \in \overline{\Omega})$ are given smooth functions. Here Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with boundary

$$S,\ \overline{\Omega} = \Omega \cup S.$$

However, the method of separation of variables can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Now another example for an hyperbolic-parabolic equation is a mixed problem given below. It can be solved by Laplace transformation method (in x).

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = -2e^{-x} + 1,\ 0 \leq t \leq 1,\ 0 < x < \infty, \\ \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = -2e^{-x} + 1,\ \ -1 \leq t \leq 0,\ 0 < x < \infty, \\ u(1,x) = u(-1,x)\,,\ 0 < x < \infty, \\ u(t,0) = u_x(t,0) = 0,\ \ -1 \leq t \leq 1. \end{cases} \tag{1.4}$$

Let $0 \leq t \leq 1$. Then, taking the Laplace transform of both sides of the differential equation

$$u_{tt} - u_{xx} + u = -2e^{-x} + 1,$$

we can write

$$\mathbf{L}\{u_{tt}\} - \mathbf{L}\{u_{xx}\} + \mathbf{L}\{u\} = \mathbf{L}\left\{(-2e^{-x} + 1)\right\}$$

or

$$(\mathbf{L}\{u(t,x)\})_{tt} - s^2\mathbf{L}\{u(t,x)\} + su(t,0) + u_x(t,0) + \mathbf{L}\{u(t,x)\} = -\frac{2}{s+1} + \frac{1}{s}.$$

Let

$$\mathbf{L}\{u(t,x)\} = v(t,s).$$

So our problem becomes

$$v_{tt}(t,s) - s^2 v(t,s) + v(t,s) = \frac{1-s}{s(s+1)}$$

or

$$v_{tt}(t,s) + \left(1 - s^2\right) v(t,s) = \frac{1-s}{s(s+1)}.$$

Now the complementary solution is

$$v_{tt}(t,s) + \left(1 - s^2\right) v(t,s) = 0$$

$$v_c(t,s) = c_1 e^{\sqrt{-1+s^2}t} + c_2 e^{-\sqrt{-1+s^2}t}.$$

And for the particular solution we can write

$$v_p(t,s) = \frac{1-s}{s(s+1)} \cdot \frac{1}{1-s^2} = \frac{1}{s(s+1)^2}.$$

So

$$v(t,s) = c_1 e^{\sqrt{-1+s^2}t} + c_2 e^{-\sqrt{-1+s^2}t} + \frac{1}{s(s+1)^2}.$$

Now, let $-1 \le t \le 0$. Then,

$$u_t - u_{xx} + u = -2e^{-x} + 1.$$

By taking the Laplace transform of both sides of the last differential equation, we obtain

$$\mathbf{L}\{u_t\} - \mathbf{L}\{u_{xx}\} + \mathbf{L}\{u\} = \mathbf{L}\left\{\left(-2e^{-x} + 1\right)\right\}$$

or

$$(\mathbf{L}\{u(t,x)\})_t - s^2\mathbf{L}\{u(t,x)\} + su(t,0) + u_x(t,0) + \mathbf{L}\{u(t,x)\} = -\frac{2}{s+1} + \frac{1}{s}.$$

Let

$$\mathbf{L}\{u(t,x)\} = v(t,s).$$

So our problem becomes

$$v_t(t,s) - s^2 v(t,s) + v(t,s) = \frac{1-s}{s(s+1)}$$

or

$$v_t(t,s) + \left(1-s^2\right)v(t,s) = \frac{1-s}{s(s+1)}.$$

So

$$v(t,s) = c_3 e^{\left(-1+s^2\right)t} + \frac{1}{s(s+1)^2}$$

by using the nonlocal boundary conditions, we have that

$$v(t,s) = \frac{1}{s(s+1)^2}.$$

Hence taking the inverse of Laplace

$$\begin{aligned} u(t,x) &= \mathbf{L}^{-1}\{v(t,s)\} \\ &= \mathbf{L}^{-1}\left\{\frac{1}{s}\right\} - \mathbf{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathbf{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\ &= 1 - e^{-x} - xe^{-x} \\ &= 1 - (1+x)e^{-x}. \end{aligned}$$

So

$$u(t,x) = 1 - (1+x)e^{-x}$$

is the solution of the given nonlocal boundary value problem (1.4).

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the multidimensional hyperbolic-parabolic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, \dots, x_n) \in \overline{\Omega}^+, 0 \le t \le T, \\ \frac{\partial u(t,x)}{\partial t} - \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = g(t,x), \\ x = (x_1, \dots, x_n) \in \overline{\Omega}^+, -T \le t \le 0, \\ u(-T,x) = u(T,x) + \varphi(x), \\ u_t(0+,x) = u_t(0-,x) + \varphi(x), \;\; x \in \overline{\Omega}^+, \\ u(t,x) = 0, \;\; x \in S^+, \end{cases}$$

where $\alpha_r > 0$ and $f(t,x)$ $\left(t \in [0,T], \, x \in \overline{\Omega}^+\right)$, $g(t,x)$ $\left(t \in [-T,0], \, x \in \overline{\Omega}^+\right)$, $\varphi(x), \psi(x)$ $\left(x \in \overline{\Omega}^+\right)$ are given smooth functions. Here Ω^+ is the open set in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with boundary

$$S^+, \;\; \overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However, Laplace transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

And the last example is a mixed problem solved by using Fourier Transformation method.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = \left(3 - 4x^2\right) e^{-x^2}, \;\; 0 \le t \le 1, \;\; -\infty < x < \infty, \\ \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = \left(3 - 4x^2\right) e^{-x^2}, \;\; -1 \le t \le 0, \;\; -\infty < x < \infty, \\ u(1,x) = u(-1,x), \;\; -\infty < x < \infty. \end{cases} \tag{1.5}$$

Let, $0 \le t \le 1$. By taking the Fourier transform of both sides, we obtain

$$\mathbf{F}\{u_{tt}\} - \mathbf{F}\{u_{xx}\} + \mathbf{F}\{u\} = \mathbf{F}\left\{\{\left(3 - 4x^2\right) e^{-x^2}\}\right\}$$

or

$$(\mathbf{F}\{u(t,x)\})_{tt} - (is)^2 \mathbf{F}\{u(t,x)\} + \mathbf{F}\{u(t,x)\} = \mathbf{F}\left\{\left(3 - 4x^2\right) e^{-x^2}\right\}.$$

Let

$$\mathbf{F}\{u(t,x)\} = v(t,s).$$

So our problem becomes

$$v_{tt}(t,s) + \left(s^2 + 1\right) v(t,s) = \mathbf{F}\left\{\left(3 - 4x^2\right) e^{-x^2}\right\}.$$

Now the solution for the complementary equation is

$$v_c(t,s) = c_1 \cos\sqrt{s^2+1}t + c_2 \sin\sqrt{s^2+1}t.$$

And for the particular equation

$$v_p(t,s) = A.$$

Substituting in the equation we get

$$\left(s^2+1\right)A = \mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\}.$$

Hence

$$A = \frac{1}{s^2+1}\mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\}.$$

So, the particular solution is

$$v_p(t,s) = \frac{1}{s^2+1}\mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\}.$$

By using the nonlocal boundary conditions we get $v(t,s)$ as

$$v(t,s) = c_1 \cos\sqrt{s^2+1}t + c_2 \sin\sqrt{s^2+1}t + \frac{1}{s^2+1}\mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\}.$$

Now, we consider if $-1 \leq t \leq 0$. By taking the Fourier transform of both, we obtain

$$\mathbf{F}\{u_t\} - \mathbf{F}\{u_{xx}\} + \mathbf{F}\{u\} = \mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\}$$

or

$$\left(\mathbf{F}\{u(t,x)\}\right)_t - (is)^2\,\mathbf{F}\{u(t,x)\} + \mathbf{F}\{u(t,x)\} = \mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\}.$$

Let

$$\mathbf{F}\{u(t,x)\} = v(t,s).$$

So our problem becomes

$$v_t(t,s) + \left(s^2+1\right)v(t,s) = \mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\}.$$

So, the solution is

$$v(t,s) = c_3 e^{-\left(s^2+1\right)(t+1)} + \frac{1}{s^2+1}\mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\}.$$

Since

$$\begin{aligned}
\mathbf{F}\left\{\left(3-4x^2\right)e^{-x^2}\right\} &= \mathbf{F}\left\{e^{-x^2}\right\} + \mathbf{F}\left\{\left(2-4x^2\right)e^{-x^2}\right\} \\
&= \mathbf{F}\left\{e^{-x^2}\right\} \cdot \mathbf{F}\left\{\left(e^{-x^2}\right)''\right\} \\
&= \left(1+s^2\right)F\left\{e^{-x^2}\right\},
\end{aligned}$$

we have that

$$v(t,s) = c_3 e^{-\left(s^2+1\right)(t+1)} + \mathbf{F}\left\{e^{-x^2}\right\}.$$

By using the nonlocal boundary conditions we get $v(t,s)$ as

$$v(t,s) = \mathbf{F}\left\{e^{-x^2}\right\}.$$

Finally taking the inverse of Fourier transformation we obtain

$$u(t,x) = e^{-x^2}$$

is the solution of the given nonlocal boundary value problem.

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the 2m-th order multidimensional hyperbolic-parabolic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \sum\limits_{|r|=2m} a_r \frac{\partial^{|r|}u}{\partial x_1^{r_1}...\partial x_n^{r_n}} = f(t,x), \\ 0 \le t \le T, x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ \frac{\partial u}{\partial t} - \sum\limits_{|r|=2m} a_r \frac{\partial^{|r|}u}{\partial x_1^{r_1}...\partial x_n^{r_n}} = f(t,x), \\ -T \le t \le 0, x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ u(-T,x) = u(T,x) + \varphi(x), x \in \mathbb{R}^n, \\ u_t(0+,x) = u_t(0-,x), \;\; x \in \mathbb{R}^n, \end{cases}$$

where α_r, $f(t,x)$ $(t \in [0,T],\ x \in R^n)$, $g(t,x)$ $(t \in [-T,0], x \in R^n), \varphi(x), \psi(x)$ $(x \in R^n)$ are given smooth functions.

However, the Fourier transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

In the present work the nonlocal boundary value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t), \;\; 0 \le t \le 1, \\ \frac{du(t)}{dt} + Au(t) = g(t), \;\; -1 \le t \le 0, \\ u(-1) = \alpha u(\mu) + \beta u'(\lambda) + \varphi, \\ |\alpha|, |\beta| \le 1, 0 < \mu, \lambda \le 1 \end{cases}$$

for differential equation in a Hilbert space H, with the self-adjoint positive definite operator A is considered. Applying the operator approach the stability estimates for solution of this nonlocal boundary problem are obtained. In applications this abstract result permits to obtain the stability estimates for the solution of the difference schemes for hyperbolic-parabolic equations.This result is based on the positivity of the difference operator generated by nonlocal boundary conditions. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Let us briefly describe the contents of the various sections. It consists of six chapters.

First chapter is the introduction.

Second chapter presents elementary statements in a Hilbert space that is needed for this work.

Third chapter consists of three sections. A brief survey of all investigations in this area can be found in the first section. In the second section the main theorem about stability of the nonlocal boundary value problem for hyperbolic-parabolic equations in a Hilbert space is proved. In applications this abstract result permits to obtain the stability estimates for the solution of the difference schemes for hyperbolic-parabolic equations. It is given in the last section.

Fourth chapter consists of two sections.The stable first and second order of accuracy difference schemes approximately solving the nonlocal boundary value problem for hyperbolic-parabolic equation in a Hilbert space H with self-adjoint positive definite operator A are presented. The stability estimates for the solutions of the difference schemes of the mixed type boundary value problems for hyperbolic-parabolic equations are obtained.

Fifth chapter is the applications. The first and second order of accuracy difference schemes are studied. A matlab program is given to conclude that the second order of accuracy is more accurate. The figures and table are included.

Sixth chapter is the conclusions.

CHAPTER 2

ELEMENTS OF HILBERT SPACE

This section is the selected concepts of the elementary Hilbert space theory as developed in [Krein, S. G., 1966].

2.1 Hilbert Space

Definition 2.1. A complex linear space H is called an inner product space if there is a complex-valued function $\langle .,. \rangle : H \times H \to C$ with the properties

i. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = \sigma$,

ii. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in H$,

iii. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$, for all $x, y \in H$ and $\alpha \in C$,

iv. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in H$.

The function $\langle x, y \rangle$ is called the inner product of x and y. A Hilbert space is a complete inner product space. An inner product on H defines a norm on H given by $\|x\| = \langle x, x \rangle^{1/2}$. Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

Example 2.1. *The space $C_2[-1,1]$ of all defined and continuous functions on a given closed interval $[-1,1]$ is an inner product space with the inner product given by*

$$\langle x, y \rangle = \int_{-1}^{1} x(t)\overline{y(t)}dt. \tag{2.1}$$

Note that the space $C_2[-1,1]$ is not complete. So, $C_2[-1,1]$ is not a Hilbert space.

Example 2.2. *The space $L_2[-1,1] = \overline{C_2[-1,1]}$ with the inner product (2.1) is a Hilbert space.*

Theorem 2.1. *Let x, y be any two vectors in a Hilbert space, then*

$$|\langle x, y \rangle| \leq \|x\| \, \|y\| \quad (\textit{ Schwartz inequality}). \tag{2.2}$$

Note that the inner product is related to the norm by the following identity

$$\langle x,y\rangle = \frac{1}{4}\left[\left(\|x+y\|^2 - \|x-y\|^2\right) + i\left(\|x+iy\|^2 - \|x-iy\|^2\right)\right]. \tag{2.3}$$

A norm on an inner product space satisfies the important Parallelogram law

Theorem 2.2. *If H is a Hilbert space, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \ , \quad \forall\, x,y \in H. \ \textit{(Parallelogram law)} \tag{2.4}$$

Conversely, H is a complex complete normed space with the norm $\|\cdot\|$ *satisfying the equation (2.4) then H is a Hilbert space with the scalar product* $\langle\cdot,\cdot\rangle$ *satisfying* $\|x\| = \langle x,x\rangle^{1/2}$.

Example 2.3. *The space* l^p *of all sequence,* $x = (\xi_i) = (\xi_1, \xi_2, ...)$ *such that* $|\xi_1|^p + |\xi_2|^p + ...$ *converges with* $p \neq 2$ *is not an inner product space,hence not a Hilbert space.*

Example 2.4. *The space* $C[a,b]$ *is not an inner product space,hence not a Hilbert space.*

2.2 Bounded Linear Operators in H

Definition 2.2. Let H_1 and H_2 are two Hilbert spaces. A linear operator A is an operator such that $A : H_1 \to H_2$

$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all $\alpha, \beta \in C$ and $x, y \in H_1$.

The domain of A $D(A) = \{x \in H_1, \exists Ax \in H_2\}$ is a vector space and

$R(A) = \{y = Ax, \forall x \in D(A)\}$ denotes the range of A.

A linear operator $A : H \to H$ is said to be bounded if there exist a real number $M > 0$ such that

$$\|Ax\|_H \leq M \ \|x\|_H \quad for\ all \ \ x \in H. \tag{2.5}$$

If A linear operator $A : H \to H$ is bounded with M, then

$$\|A\| = \inf M \tag{2.6}$$

is called the norm of operator A.

Example 2.5. *A bounded linear operator from* $H = L_2[0,1]$ *into itself is defined by*

$$Ax = tx(t) \ , \ 0 \leq t \leq 1. \tag{2.7}$$

Example 2.6. *Another bounded linear operator* $L_2[0,1]$ *into itself is defined by*

$$Ax(t) = \int_0^1 tsx(s)ds. \tag{2.8}$$

Theorem 2.3. *The norm of the bounded linear operator A is*

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{x\neq\sigma} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|. \tag{2.9}$$

Example 2.7. *A is an operator defined by* $Ax = \alpha x(t)$, $A : L_2[0,1] \longrightarrow L_2[0,1]$. *Show that* $\|Ax\| = |\alpha|$.

2.3 Adjoint of an Operator

Definition 2.3. Let $A : H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the Hilbert adjoint operator A^* of A is the operator

$$A^* : H_2 \to H_1,$$

such that for all $x \in H_1$ and $y \in H_2$

$$\langle Ax, y\rangle = \langle x, A^*y\rangle .$$

Theorem 2.4. *The Hilbert adjoint operator A^* of A is unique and bounded linear operator with the norm*

$$\|A^*\| = \|A\| . \tag{2.10}$$

Definition 2.4. A bounded linear operator $A : H \longrightarrow H$ on a Hilbert space H is said to be self-adjoint if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$.

Definition 2.5. A self-adjoint operator A is said to be positive if $A \geq 0$, that is $(Ax, x) \geq 0$ for all $x \in H$.

Example 2.8. *A is an operator defined on the example 2.5. Show that if $\alpha \in R^1$, then A is a self-adjoint operator.*

Example 2.9. *A is an operator defined on the example 2.7. Show that A is a self-adjoint positive operator.*

Definition 2.6. Let $A : D(A) \longrightarrow H$ be a linear operator with $\overline{D(A)} = H$. Then A is called a symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D(A)$. If A is symmetric and $D(A) = D(A^*)$, then A is a self-adjoint operator.

Example 2.10. *Let* $Au = -\dfrac{d^2u}{dx^2} + u \quad , \quad u(a) = u(b) = 0$ *and $H = L_2[a, b]$. Show that A is a self-adjoint positive operator.*

2.4 Spectrum

Definition 2.7. Let H be a Hilbert space and $A : H \longrightarrow H$ be a linear operator with $D(A) \subset H$. We associate the operator $A_\lambda = A - \lambda I$, where $\lambda \in C$ and I is the identity operator on $D(A)$.

If A_λ has an inverse, we denote it by $R_\lambda(A)$ and we call it the resolvent operator of A, or simply, resolvent of A.

$$R_\lambda(A) = (A - \lambda I)^{-1} . \tag{2.11}$$

Definition 2.8. (Regular value, resolvent set, spectrum)

Let A be a linear operator with the $D(A) \subset H$ and H is a Hilbert space. A regular value λ of A is a complex number such that

- $(R1)$ $R_\lambda(A)$ exists.
- $(R2)$ $R_\lambda(A)$ is bounded.
- $(R3)$ $R_\lambda(A)$ is defined on a set which is dense in H.

The resolvent set $\rho(A)$ of A is the set of all regular values of A. Its complement $\sigma(A) = C - \rho(A)$ is called spectrum of A , and $\lambda \in \sigma(A)$ is called spectral value of A. Furthermore, the spectrum $\rho(A)$ is partitioned into three disjoint sets as follows.

The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_\lambda(A)$ does not exist. A $\lambda \in \sigma(A)$ is called an eigenvalue of A.

The continuous spectrum $\sigma_c(A)$ is the set such that $R_\lambda(A)$ exists and satisfies $(R3)$ but not $(R2)$, that is $R_\lambda(A)$ unbounded.

The residual spectrum $\sigma_r(A)$ is the set such that $R_\lambda(A)$ exists (and may be bounded or not) but does not satisfy $(R3)$, that is the domain of $R_\lambda(A)$ is not dense in H.

If $A_\lambda x = (A - \lambda I)x = 0$ for some $x \neq 0$, then $\lambda \in \sigma_p(A)$, by definition, that is, λ is an eigenvalue of A.

The vector x is called an eigenvector of A corresponding to eigenvalue λ. The subspace of $D(A)$ consisting of 0 and all eigenvectors of A corresponding to an eigenvalue λ of A is called the eigenspace of A corresponding to that eigenvalue λ.

$$\sigma(A) = \sigma_c(A) \cup \sigma_p(A) \cup \sigma_r(A), \tag{2.12}$$

$$\sigma(A) \cup \rho(A) = C.$$

Definition 2.9. Let H be a Hilbert space over the field of real numbers and for any $x \in H$, let $\|x\|$ denote the norm of x. Let J be any interval of the real line R. A function $x : J \to H$ is called an abstract function. A function $x(t)$ is said to be continuous at the point $t_0 \in J$, if

$$\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0;$$

if $x : J \to H$ is continuous at each point of J, Then we say that x is continuous on J and we write $x \in C[J, H]$.

Definition 2.10. The Stieltjes integral of a function $x : [a, b] \to H$ with respect to a function $y : [a, b] \to H_1$. Let H, H_1 and H_2 be three Hilbert spaces. A bilinear operator $P : H \times H_1 \to H_2$ whose norm is less than or equal to 1, that is,

$$\|P(x, y)\| \leq \|x\| \|y\|, \tag{2.13}$$

is called a product operator. We shall agree to write $P(x, y) = xy$. Let $x : [a, b] \to H$ and $y : [a, b] \to H_1$ be two bounded functions such that the product $x(t)y(t) \in H_2$, for each $t \in [a, b]$ is linear in both x and y and

$$\|x(t)y(t)\| \leq \|x(t)\| \, \|y(t)\|$$

(for example, $x(t) = A(t)$ is an operator with domain $D[A(t)] \supset H_1$, or one of the function x,y is a scalar function). We denote the partition $(a = t_0 < t_1 < t_2 < ... < t_n = b)$ together with the points τ_i $(t_i < \tau_1 < t_{i+1}, i = 0, 1, 2, ..., n-1)$ by π and set $|\pi|=\max_i |t_{i+1} - t_i|$. We form the Stieltjes sum

$$S_\pi = \sum_{i=1}^{n-1} x(\tau_i) [y(t_{i+1}) - y(t_i)]. \tag{2.14}$$

If the $\lim S_\pi$ exist as $|\pi| \to 0$ and defines an element I in H_2 independent of π, then I is called the Stieltjes integral of the function $x(t)$ by the function $y(t)$, and is denoted by

$$\int_a^b x(t) \, dy(t). \tag{2.15}$$

Theorem 2.5. *If $x \in C[[a,b],H]$ and $y:[a,b] \to H_1$ is of bounded variation on $[a,b]$, then the Stieltjes integral (2.15) exists.*

Consider the function $y:[a,b] \to H_1$ and the partition

$$\pi : a = t_0 < t_1 < t_2 < \ldots < t_n = b.$$

Form the sum

$$V = \sum_{i=1}^{n-1} \| y(t_{i+1}) - y(t_i) \| . \tag{2.16}$$

The least upper bound of the set of all possible sums V is called the (strong) total variation of the function $y(t)$ on the interval $[a,b]$ and is denoted by $V_a^b(y)$. If $V_a^b(y) < \infty$, then $y(t)$ is called an abstract function of bounded variation on $[a,b]$.

Example 2.11. *If $x \in C[[a,b],H]$ and $y:[a,b] \to H_1$ is of bounded variation on $[a,b]$, then*

$$\left\| \int_a^b x(t)\, dy(t) \right\| \leq \int_a^b \|x(t)\|\, dV_a^t[y(t)] \leq \max_{t\in[a,b]} \|x(t)\|\, V_a^b[y(t)] . \tag{2.17}$$

2.5 Projection Operator. Spectral Family

Definition 2.11. A Hilbert space H is represented as the direct sum of a closed subspace Y and its orthogonal complement $Y^{\perp}$:

$$H = Y \oplus Y^{\perp} \tag{2.18}$$

$$x = y + z \quad , \quad \textit{where } \ y \in Y, z \in Y^{\perp}.$$

Since the sum is direct, y is unique for any given $x \in H$. Hence (2.18) defines a linear operator

$P : H \longrightarrow H,$

$x \longrightarrow y = Px.$

P is called an orthogonal projection or projection on H.

Theorem 2.6. *A bounded linear operator $P : H \longrightarrow H$ on a Hilbert space H is projection if and only if P is self-adjoint and idempotent that is, $P^2 = P$.*

Spectral family from dimensional case as follows: If matrix A has n different eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 ... < \lambda_n$. Then A has an orthogonal set of n vectors $x_1, x_2, x_3, ..., x_n$,where x_j corresponds to λ_j and we write these vectors as column vectors, for convenience. This basis for H, has a unique representation:

$$x = \sum_{j=1}^{n} \gamma_j x_j \ , \quad \gamma_j = (x, x_j) = x^T \overline{x_j} \ , \tag{2.19}$$

x_j is an eigenvector of A, so that we have $Ax_j = \lambda_j x_j$.

$$Ax = \sum_{j=1}^{n} \lambda_i \gamma_j x_j. \tag{2.20}$$

We can define an operator

$$P_j : H \longrightarrow H, \tag{2.21}$$

$$x \longrightarrow \gamma_j x_j \ .$$

Obviously, P_j is the projection (orthogonal projection) of H onto the eigenspace of A corresponding to λ_j . From the equation (2.19) can be written

$$x = \sum_{j=1}^{n} P_i x \ \text{ hence } \ I = \sum_{j=1}^{n} P_i, \tag{2.22}$$

where I is an identity operator on H. Formula (2.20) becomes

$$Ax = \sum_{j=1}^{n} \lambda_i P_i x \ \text{ hence } \ A = \sum_{j=1}^{n} \lambda_i P_i. \tag{2.23}$$

This is a representation of A in terms of projections.

Theorem 2.7. *(Spectral Theorem) A family of an orthogonal projection operators E_λ $(-\infty < \lambda < \infty)$ is said to be spectral representation identity if:*

1) E_λ is strongly left-continuous in λ;

2) $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$ for $\lambda < \mu$;

3) $E_{-\infty} = \lim\limits_{\lambda \to -\infty} E_\lambda = 0$ and $E_{+\infty} = \lim\limits_{\lambda \to \infty} E_\lambda = I$, where the limits are understood in the sense of strong convergence. For every bounded function $F(\lambda)$ give to entire real axis, one can define the Stieltjes operator integral

$$\int_a^b F(\lambda)\, dE_\lambda. \tag{2.24}$$

This integral is defined as the limit in the norm of integral sums of the form

$$\sum_{k=0}^{N} F(\lambda_k)\left(E_{\lambda_{k+1}} - E_\lambda\right),$$

if the segment $[a, b]$ is finite, and as an important integral if $a = -\infty$ or $b = \infty$. The integral (2.24) is a bounded operator with

$$\left\| \int_a^b F(\lambda)\, dE_\lambda \right\| \leq \sup_{a \leq \lambda \leq b} |F(\lambda)|.$$

If the function $F(\lambda)$ takes on only real values, the operator (2.24) is self-adjoint. If the function $F(\lambda)$ is real and unbounded, then formula (2.24), after assigning an appropriate meaning to the integral, yields a self-adjoint and generally speaking unbounded operator whose domain consists of only those elements x for which

$$\int_{-\infty}^{\infty} |F(\lambda)|^2\, d(E_\lambda x, x) < \infty.$$

It turns out that to every self-adjoint operator A there corresponds some spectral representation E_λ of the identity with

$$Ax = \int_{-\infty}^{\infty} \lambda dE_\lambda x$$

for $x \in D(A)$. The operators E_λ commute with any operator commuting with A.

If A is bounded, and m and M are the greatest lower bound and least upper bound of its spectrum then $E_\lambda = I$ for $\lambda > M$, so that

$$Ax = \int_m^{M+0} \lambda dE_\lambda x.$$

If the operator A is positive definite and $\langle Ax, x\rangle \geq a\langle x, x\rangle$, then

$$Ax = \int_a^{\infty} \lambda dE_\lambda x.$$

The real regular points of A are characterized by the fact that in their neighborhoods the operator E_λ is constant. Thus, the points of the spectrum of A coincide with the points of growth of the operator function E_λ.

By using the spectral representation one may bring into consideration a wide class of functions of an unbounded self-adjoint operator. Thus, for example, for any continuous function $F(\lambda)$ it is natural to put

$$F(A)x = \int_a^{\infty} F(\lambda)\, dE_\lambda x.$$

where E_λ is the spectral resolution of the identity corresponding to operator A. Here, to the operations of addition and multiplication of the corresponding operators.

Example 2.12. *A is a self-adjoint positive definite operator. Show that*

$$\|\exp(-At)\| \leq e^{-\delta t}, \tag{2.25}$$

$$\left\|\cos\left(A^{1/2}t\right)\right\| \leq 1, \quad \left\|A^{1/2}\sin\left(A^{1/2}t\right)\right\| \leq 1.$$

Solution. *Using the spectral representation of the self-adjoint positive defined operators we can write*

$$\exp(-At)\varphi = \int_\delta^{\infty} \exp(-\mu t) dE_\mu \varphi,$$

where (E_μ) is the spectral family associated with A. Therefore, for any $t \geq 0$ we have that

$$\|\exp(-At)\|_{H\to H} \leq \sup_{\delta\leq\mu<\infty} |\exp(-\mu t)| = \exp(-\delta t).$$

The estimate (2.25) is proved. Using the spectral representation of the self-adjoint positive defined operators it can be written

$$e^{\pm iA^{1/2}t}\varphi = \int_\delta^{\infty} e^{\pm it\mu^{1/2}} dE_\mu \varphi.$$

Therefore, using the last theorem

$$\left\|e^{\pm iA^{1/2}t}\right\| \leq \sup_{\delta\leq\mu<\infty} |e^{\pm it\mu^{1/2}}| = 1$$

is obtained.

So,

$$\left\|\cos\left(A^{1/2}t\right)\right\| = \left\|\frac{e^{iA^{1/2}t} + e^{-iA^{1/2}t}}{2}\right\| \leq \frac{1}{2}\left[\left\|e^{iA^{1/2}t}\right\| + \left\|e^{-iA^{1/2}t}\right\|\right] \leq 1$$

and

$$\left\|A^{1/2}\sin\left(A^{1/2}t\right)\right\| = \left\|\frac{e^{iA^{1/2}t} - e^{-iA^{1/2}t}}{2i}\right\| \leq \frac{1}{2|i|}\left[\left\|e^{iA^{1/2}t}\right\| + \left\|e^{-iA^{1/2}t}\right\|\right] \leq 1.$$

CHAPTER 3

NONLOCAL BOUNDARY VALUE PROBLEM FOR HYPERBOLIC PARABOLIC DIFFERENTIAL EQUATIONS

3.1 Introduction

We consider the nonlocal boundary value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t), \ 0 \leq t \leq 1, \\ \frac{du(t)}{dt} + Au(t) = g(t), \ -1 \leq t \leq 0, \\ u(-1) = \alpha u(\mu) + \beta u'(\lambda) + \varphi, \\ |\alpha|, |\beta| \leq 1, 0 < \mu, \lambda \leq 1 \end{cases} \tag{3.1}$$

for differential equations of mixed type in a Hilbert space H with self-adjoint positive definite operator A. It is known (see, for example, [Salahatdinov, M. S., 1974], [Djuraev, T. D., 1978], [Bazarov, D. and Soltanov, H., 1995], [Krein, S. G., 1966]) that various nonlocal boundary value problems for the hyperbolic-parabolic equations can be reduced to the nonlocal boundary value problem (3.1).

A function $u(t)$ is called a *solution* of the problem (3.1) if the following conditions are satisfied:

i) $u(t)$ is twice continuously differentiable on the interval $(0, 1]$ and continuously differentiable on the segment $[-1, 1]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [-1, 1]$, and the function $Au(t)$ is continuous on the segment $[-1, 1]$.

iii) $u(t)$ satisfies the equations and nonlocal boundary condition (3.1).

Methods for numerical solutions of the nonlocal boundary value problem (3.1) in the case $\beta = 0$ have been studied extensively (see [Ashyralyev, A. and Yurtsever, A., 2001], [Ashyralyev, A. and Yurtsever, A., 2001], [Ashyralyev, A. and Yurtsever, A., 2000], [Ashyralyev, A. and Yurtsever, A., 2004], [Ashyralyev, A. and Yurtsever, A., 2004], [Ashyralyev, A. and Muradov, I., 1995], [Ashyralyev, A. and Muradov, I., 1996], [Ashyralyev, A. and Muradov, I., 1998], [Ashyralyev, A. and Orazov, M. B., 1999] and the references therein). For example, in the paper [Ashyralyev, A. and Yurtsever, A., 2001] it's proved the following theorem on the stability.

Theorem 3.1. *Suppose that $\varphi \in D(A)$ and $f(t)$ be continuously differentiable on $[0,1]$ and $g(t)$ be continuously differentiable on $[-1,0]$ functions and $\beta = 0$. Then there is a unique solution of the problem (3.1) and the stability inequalities*

$$\max_{-1\le t\le 1} \|u(t)\|_H \le M \left[\|\varphi\|_H + \max_{-1\le t\le 0} \|g(t)\|_H + \max_{0\le t\le 1} \left\|A^{-1/2} f(t)\right\|_H\right],$$

$$\max_{-1\le t\le 1} \left\|A^{1/2} u(t)\right\|_H \le M \left[\left\|A^{1/2}\varphi\right\|_H + \|g(0)\|_H + \int_{-1}^{0} \|g'(t)\|_H \, dt + \max_{0\le t\le 1} \|f(t)\|_H\right],$$

$$\max_{-1\le t\le 0} \left\|\frac{du(t)}{dt}\right\|_H + \max_{0\le t\le 1} \left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{-1\le t\le 1} \|Au(t)\|_H$$

$$\le M \left[\|A\varphi\|_H + \left\|A^{1/2} g(0)\right\|_H + \|f(0)\|_H + \max_{-1\le t\le 0} \|g'(t)\|_H + \int_0^1 \|f'(t)\|_H \, dt\right]$$

hold, where M does not depend on $f(t), g(t)$, and φ.

We are interested in studying the stability of solutions of the problem (3.1) for $\beta \neq 0$. We have not been able to obtain the same stability estimates for the solutions of the problem (3.1) for $\beta \neq 0$. Nevertheless, in the present paper the stability estimates for the solution of the problem (3.1) under a stronger assumption than $f(t)$ be continuously differentiable on $[0,1]$ and $g(t)$ be continuously differentiable on $[-1,0]$ functions are established. In applications, the stability estimates for the solutions of the mixed type boundary value problems for the hyperbolic-parabolic equations are obtained.

3.2 The Main Theorem

Theorem 3.2. *Suppose that $\varphi \in D(A)$, $g(0) \in D\left(A^{1/2}\right), g'(0) \in H, f(0) \in D\left(A^{1/2}\right)$ and $f'(0) \in H$. Let $f(t)$ be twice continuously differentiable on $[0,1]$ and $g(t)$ be twice continuously differentiable on $[-1,0]$ functions. Then there is a unique solution of the problem (3.1) and the stability inequalities hold:*

$$\max_{-1\le t\le 1} \|u(t)\|_H \le M \left[\|\varphi\|_H + \max_{-1\le t\le 0} \left\|A^{-1/2} g'(t)\right\|_H \right. \tag{3.2}$$

$$\left. + \left\|A^{-1/2} g(0)\right\|_H + \left\|A^{-1/2} f(0)\right\|_H + \max_{0\le t\le 1} \left\|A^{-1/2} f'(t)\right\|_H\right],$$

$$\max_{-1\le t\le 1} \left\|\frac{du}{dt}\right\|_H + \max_{-1\le t\le 1} \left\|A^{1/2} u(t)\right\|_H \le M \left[\left\|A^{1/2}\varphi\right\|_H \right. \tag{3.3}$$

$$\left. + \|g(0)\|_H + \max_{-1\le t\le 0} \|g'(t)\|_H + \|f(0)\|_H + \max_{0\le t\le 1} \|f'(t)\|_H\right],$$

$$\max_{-1\le t\le 0} \left\|\frac{du}{dt}\right\|_H + \max_{0\le t\le 1} \left\|\frac{d^2u}{dt^2}\right\|_H + \max_{-1\le t\le 1} \|Au(t)\|_H \tag{3.4}$$

$$\le M \left[\|A\varphi\|_H + \left\|A^{1/2} g(0)\right\|_H + \|g'(0)\|_H + \max_{-1\le t\le 0} \|g''(t)\|_H \right.$$

$$\left. + \left\|A^{1/2} f(0)\right\|_H + \|f'(0)\|_H + \max_{0\le t\le 1} \|f''(t)\|_H\right],$$

where M does not depend on $f(t)$, $t \in [0,1]$, $g(t)$, $t \in [-1,0]$ and φ.

Proof: We will obtain the formula for solution of the problem (3.1). It is known that for smooth data of the initial value problems

$$\begin{cases} u''(t) + Au(t) = f(t), \quad (0 \le t \le 1), \\ u(0) = u_0, \; u'(0) = u_0', \end{cases} \tag{3.5}$$

$$\begin{cases} u'(t) + Au(t) = g(t), \quad (-1 \le t \le 0), \\ u(-1) = u_{-1}, \end{cases} \tag{3.6}$$

there are unique solutions of the problems (3.5), (3.6) and the following formulas hold:

$$u(t) = c(t)\,u(0) + s(t)\,u'(0) + \int_0^t s(t-y)\,f(y)\,dy, \quad 0 \le t \le 1, \tag{3.7}$$

where

$$c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}, \qquad s(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i},$$

and

$$u(t) = e^{-(t+1)A}u_{-1} + \int_{-1}^t e^{-(t-y)A}g(y)\,dy, \quad -1 \le t \le 0. \tag{3.8}$$

Using formulas (3.7), (3.8) and equation (3.1) we can write

$$u(t) = [c(t) - As(t)]\left\{e^{-A}u_{-1} + \int_{-1}^0 e^{yA}g(y)\,dy\right\} \tag{3.9}$$

$$+s(t)\,g(0) + \int_0^t s(t-y)\,f(y)\,dy.$$

Now, using the condition $u(-1) = \alpha u(\mu) + \beta u'(\lambda) + \varphi$, we obtain the operator equation

$$\left\{I - \alpha\left[c(\mu) - As(\mu)\right]e^{-A} + \beta\left[s(\lambda) + c(\lambda)\right]Ae^{-A}\right\}u_{-1} \tag{3.10}$$

$$= \alpha\left\{c(\mu)\int_{-1}^0 e^{yA}g(y)\,dy + s(\mu)\left[g(0) - A\int_{-1}^0 e^{yA}g(y)\,dy\right]\right.$$

$$\left. + \int_0^\mu s(\mu-y)\,f(y)\,dy\right\}$$

$$+\beta\left\{-As(\lambda)\int_{-1}^0 e^{yA}g(y)\,dy + c(\lambda)\left[g(0) - A\int_{-1}^0 e^{yA}g(y)\,dy\right]\right.$$

$$\left. + \int_0^\lambda c(\lambda-y)\,f(y)\,dy\right\} + \varphi.$$

The operator

$$I - \alpha\left[c(\mu) - As(\mu)\right]e^{-A} + \beta\left[s(\lambda) + c(\lambda)\right]Ae^{-A}$$

has an inverse

$$T = \left(I - \alpha\left[c(\mu) - As(\mu)\right]e^{-A} + \beta\left[s(\lambda) + c(\lambda)\right]Ae^{-A}\right)^{-1}$$

and the estimate holds:

$$\|T\|_{H \to H} \le M. \tag{3.11}$$

Actually, the proof of this estimate is based on the estimate

$$\left\|-\alpha\left[c(\mu) - As(\mu)\right]e^{-A} + \beta\left[s(\lambda) + c(\lambda)\right]Ae^{-A}\right\|_{H \to H} < 1.$$

Using the definitions of $c(\mu)$ and $s(\mu)$ and positivity self-adjointness property of A, we obtain

$$\left\|-\alpha\left[c(\mu)-As(\mu)\right]e^{-A}+\beta\left[s(\lambda)+c(\lambda)\right]Ae^{-A}\right\|_{H\to H}$$

$$\leq \sup_{\delta\leq\rho<\infty}\left|-\alpha\left[\cos\left(\sqrt{\rho}\mu\right)-\sqrt{\rho}\sin\left(\sqrt{\rho}\mu\right)\right]e^{-\rho}\right.$$

$$\left.+\beta\left[\sqrt{\rho}\sin\left(\sqrt{\rho}\lambda\right)+\rho\cos\left(\sqrt{\rho}\lambda\right)\right]\right|e^{-\rho}.$$

Since

$$\cos\left(\sqrt{\rho}\mu\right)-\sqrt{\rho}\sin\left(\sqrt{\rho}\mu\right)=\sqrt{\rho+1}\cos\left(\sqrt{\rho}\mu-\mu_0\right),$$

$$\sqrt{\rho}\sin\left(\sqrt{\rho}\lambda\right)+\rho\cos\left(\sqrt{\rho}\lambda\right)=\sqrt{\rho}\sqrt{\rho+1}\cos\left(\sqrt{\rho}\mu-\mu_1\right),$$

we have that

$$\left\|-\alpha\left[c(\mu)-As(\mu)\right]e^{-A}+\beta\left[s(\lambda)+c(\lambda)\right]Ae^{-A}\right\|_{H\to H}$$

$$\leq \sup_{\delta\leq\rho<\infty}\sqrt{\rho+1}\left(1+\sqrt{\rho}\right)e^{-\rho}.$$

It is easy to show that $\sup\limits_{\delta\leq\rho<\infty}\sqrt{\rho+1}\left(1+\sqrt{\rho}\right)e^{-\rho}<1$. So, the estimate (3.11) is proved.

Therefore, for the solution of the operator equation (3.10) we have the formula

$$\begin{aligned} u_{-1}=T\Bigg[\alpha\Bigg\{&c(\mu)\int_{-1}^{0}e^{yA}g(y)\,dy \\ &+s(\mu)\left[g(0)-A\int_{-1}^{0}e^{yA}g(y)\,dy\right] \\ &+\int_0^{\mu}s(\mu-y)f(y)\,dy\Bigg\} \\ +\beta\Bigg\{-As(\lambda)\int_{-1}^{0}e^{yA}g(y)\,dy&+c(\lambda)\left[g(0)-A\int_{-1}^{0}e^{yA}g(y)\,dy\right] \\ &+\int_0^{\lambda}c(\lambda-y)f(y)\,dy\Bigg\}+\varphi\Bigg]. \end{aligned} \tag{3.12}$$

Hence, for the solution of the nonlocal boundary value problem (3.1) we have the formulas (3.8), (3.9) and (3.12).

Now, we will establish estimates (3.2), (3.3) and (3.4) for the solution of the nonlocal boundary value problem (3.1). From the symmetry properties of the operator A it follows that

$$\|c(t)\|_{H\to H}\leq 1,\ \left\|A^{1/2}s(t)\right\|_{H\to H}\leq 1, t\geq 0, \tag{3.13}$$

$$\left\|A^{\gamma}e^{-tA}\right\|_{H\to H}\leq Mt^{-\gamma}e^{-\delta t}, t>0, 0\leq\gamma\leq 1, \delta>0, M>0. \tag{3.14}$$

First, we obtain estimate (3.2). Using formula (3.12) and an integration by parts, we obtain

$$\begin{aligned} u_{-1}=T\Bigg[\alpha\Bigg\{&c(\mu)\left[A^{-1}\left(g(0)-e^{-A}g(-1)-\int_{-1}^{0}e^{yA}g'(y)\right)dy\right] \\ &+s(\mu)\left(e^{-A}g(-1)+\int_{-1}^{0}e^{yA}g'(y)\,dy\right) \end{aligned} \tag{3.15}$$

$$+A^{-1}\left[f(\mu)-c(\mu)f(0)-\int_0^{\mu}c(\mu-y)f'(y)\,dy\right]\Bigg\}$$

$$+\beta\left\{-s(\lambda)\left(g(0)-e^{-A}g(-1)-\int_{-1}^{0}e^{yA}g'(y)\,dy\right)\right.$$

$$+c(\mu)\left(e^{-A}g(-1)+\int_{-1}^{0}e^{yA}g'(y)\,dy\right)$$

$$\left.+s(\lambda)f(0)+\int_0^{\lambda}s(\lambda-y)f'(y)\,dy\right\}+\varphi\Bigg].$$

Using estimates (3.11), (3.13) and (3.14), we obtain

$$\|u_{-1}\|_H\le\|T\|_{H\to H}\Big[|\alpha|\Big\{\|c(\mu)\|_{H\to H}\Big[\left\|A^{-1/2}\right\|_{H\to H}\Big(\left\|A^{-1/2}g(0)\right\|_H$$

$$+\|e^{-A}\|_{H\to H}\left[\left\|A^{-1/2}g(0)\right\|_H+\left\|A^{-1/2}[g(-1)-g(0)]\right\|_H\right]$$

$$+\int_{-1}^{0}\|e^{yA}\|_{H\to H}\left\|A^{-1/2}g'(y)\right\|_H dy\Big)\Big]$$

$$+\left\|A^{1/2}s(\mu)\right\|_{H\to H}\Big(\|e^{-A}\|_{H\to H}\left[\left\|A^{-1/2}g(0)\right\|_H+\left\|A^{-1/2}[g(-1)-g(0)]\right\|_H\right]$$

$$+\int_{-1}^{0}\|e^{yA}\|_{H\to H}\left\|A^{-1/2}g'(y)\right\|_H dy\Big)+\left\|A^{-1/2}\right\|_{H\to H}$$

$$\times\Big[\left\|A^{-1/2}f(0)\right\|_H+\left\|A^{-1/2}(f(\mu)-f(0))\right\|_H+\|c(\mu)\|_{H\to H}\left\|A^{-1/2}f(0)\right\|_H$$

$$+\int_0^{\mu}\|c(\mu-y)\|_{H\to H}\left\|A^{-1/2}f'(y)\right\|_H dy\Big]\Big\}$$

$$+|\beta|\Big\{\left\|A^{1/2}s(\lambda)\right\|_{H\to H}\Big(\|e^{-A}\|_{H\to H}\left[\left\|A^{-1/2}g(0)\right\|_H+\left\|A^{-1/2}[g(-1)-g(0)]\right\|_H\right]$$

$$+\int_{-1}^{0}\|e^{yA}\|_{H\to H}\left\|A^{-1/2}g'(y)\right\|_H dy\Big)+\|c(\lambda)\|_{H\to H}$$

$$\times\Big(\left\|A^{1/2}e^{-A}\right\|_{H\to H}\left[\left\|A^{-1/2}g(0)\right\|_H+\left\|A^{-1/2}[g(-1)-g(0)]\right\|_H\right]$$

$$+\int_{-1}^{0}\left\|A^{1/2}e^{yA}\right\|_{H\to H}\left\|A^{-1/2}g'(y)\right\|_H dy\Big)+\left\|A^{1/2}s(\lambda)\right\|_{H\to H}\left\|A^{-1/2}f(0)\right\|_H$$

$$+\int_0^{\lambda}\left\|A^{1/2}s(\lambda-y)\right\|_{H\to H}\left\|A^{-1/2}f'(y)\right\|_H\Big\}$$

$$+\|\varphi\|_H]\le M\Big[\|\varphi\|_H+\max_{-1\le t\le 0}\left\|A^{-1/2}g'(t)\right\|_H$$

$$+\left\|A^{-1/2}g(0)\right\|_H+\left\|A^{-1/2}f(0)\right\|_H+\max_{0\le t\le 1}\left\|A^{-1/2}f'(t)\right\|_H\Big]. \qquad (3.16)$$

Using formulas (3.8), (3.9) and an integration by parts, we obtain

$$u(t)=e^{-(t+1)A}u_{-1}+A^{-1}\Big(g(t)-e^{-A}g(-1) \qquad (3.17)$$

$$-\int_{-1}^{t}e^{yA}g'(y)\,dy,\ \ -1\le t\le 0,\Big)$$

$$u(t) = [c(t) - As(t)] \{e^{-A} u_{-1} \tag{3.18}$$
$$+A^{-1} \left(g(0) - e^{-A} g(-1) - \int_{-1}^{0} e^{yA} g'(y)\, dy \right) \}$$
$$+s(t) g(0) + A^{-1} [f(t) - c(t) f(0)$$
$$- \int_{0}^{t} c(t-y) f'(y)\, dy, \ 0 \le t \le 1.]$$

Using estimates (3.13) and (3.14), we obtain

$$\|u(t)\|_H \le \left\| e^{-(t+1)A} \right\|_{H\to H} \|u_{-1}\|_H + \left\| A^{-1/2} \right\|_{H\to H}$$
$$\times \left(\left\| A^{-1/2} g(0) \right\|_H + \|e^{-A}\|_{H\to H} \left\| A^{-1/2} g(0) \right\|_H + \left\| A^{-1/2} [g(t) - g(0)] \right\|_H \right.$$
$$\left. + \|e^{-A}\|_{H\to H} \left\| A^{-1/2} [g(-1) - g(0)] \right\|_H + \int_{-1}^{t} \|e^{yA}\|_{H\to H} \left\| A^{-1/2} g'(y) \right\|_H dy \right)$$
$$\le M \left[\|u_{-1}\|_H + \max_{-1\le t\le 0} \left\| A^{-1/2} g'(t) \right\|_H + \left\| A^{-1/2} g(0) \right\|_H \right], -1 \le t \le 0,$$
$$\|u(t)\|_H \le \|c(t)\|_{H\to H} \|e^{-A}\|_{H\to H} \|u_{-1}\|_H$$
$$+ \left\| A^{1/2} s(t) \right\|_{H\to H} \left\| A^{1/2} e^{-A} \right\|_{H\to H} \|u_{-1}\|_H$$
$$+ \left[\|c(t)\|_{H\to H} \left\| A^{-1/2} \right\|_{H\to H} + \left\| A^{1/2} s(t) \right\|_{H\to H} \right]$$
$$\times \left(\left\| A^{-1/2} g(0) \right\|_H + \|e^{-A}\|_{H\to H} \left\| A^{-1/2} g(0) \right\|_H \right.$$
$$+ \|e^{-A}\|_{H\to H} \left\| A^{-1/2} [g(-1) - g(0)] \right\|_H$$
$$\left. + \int_{-1}^{0} \|e^{yA}\|_{H\to H} \left\| A^{-1/2} g'(y) \right\|_H dy \right)$$
$$+ \left\| A^{1/2} s(t) \right\|_{H\to H} \left\| A^{-1/2} g(0) \right\|_H + \left\| A^{-1/2} \right\|_{H\to H}$$
$$\times \left[\left\| A^{-1/2} f(0) \right\|_H + \left\| A^{-1/2} [f(t) - f(0)] \right\|_H + \|c(t)\|_{H\to H} \right.$$
$$\left. \times \left\| A^{-1/2} f(0) \right\|_H + \int_{0}^{t} \|c(t-y)\|_{H\to H} \left\| A^{-1/2} f'(y) \right\|_H dy \right]$$
$$\le M \left[\max_{0\le t\le 1} \left\| A^{-1/2} f'(t) \right\|_H + \left\| A^{-1/2} f(0) \right\|_H \right.$$
$$\left. + \|u_{-1}\|_H + \max_{-1\le t\le 0} \left\| A^{-1/2} g'(t) \right\|_H + \left\| A^{-1/2} g(0) \right\|_H \right], \quad 0 \le t \le 1.$$

Then from (3.16) and the last two estimates, it follows (3.2).

Second, we obtain estimate (3.3). Applying $A^{1/2}$ to the formula (3.15) and using estimates (3.11), (3.13) and (3.14), we obtain

$$\left\| A^{1/2} u_{-1} \right\|_H \le \|T\|_{H\to H} \left[|\alpha| \left\{ \|c(\mu)\|_{H\to H} \left[\left\| A^{-1/2} \right\|_{H\to H} (\|g(0)\|_H \right. \right. \right.$$

$$+\left\|e^{-A}\right\|_{H\to H}\left[\left\|g\left(0\right)\right\|_{H}+\left\|g\left(-1\right)-g(0)\right\|_{H}\right]+\int_{-1}^{0}\left\|e^{yA}\right\|_{H\to H}\left\|g'\left(y\right)\right\|_{H}dy\Big)\Big]$$

$$+\left\|A^{1/2}s\left(\mu\right)\right\|_{H\to H}\left(\left\|e^{-A}\right\|_{H\to H}\left[\left\|g\left(0\right)\right\|_{H}+\left\|g\left(-1\right)-g(0)\right\|_{H}\right]\right.$$

$$\left.+\int_{-1}^{0}\left\|e^{yA}\right\|_{H\to H}\left\|g'\left(y\right)\right\|_{H}dy\right)+\left\|A^{-1/2}\right\|_{H\to H}$$

$$\times\left[\left\|f\left(0\right)\right\|_{H}+\left\|f\left(\mu\right)-f\left(0\right)\right\|_{H}+\left\|c\left(\mu\right)\right\|_{H\to H}\left\|f\left(0\right)\right\|_{H}\right.$$

$$\left.+\int_{0}^{\mu}\left\|c\left(\mu-y\right)\right\|_{H\to H}\left\|f'\left(y\right)\right\|_{H}dy\right]\Big\}$$

$$+|\beta|\left\{\left\|A^{1/2}s\left(\lambda\right)\right\|_{H\to H}\left(\left\|e^{-A}\right\|_{H\to H}\left[\left\|g\left(0\right)\right\|_{H}+\left\|\left[g\left(-1\right)-g(0)\right]\right\|_{H}\right]\right.\right.$$

$$\left.+\int_{-1}^{0}\left\|e^{yA}\right\|_{H\to H}\left\|g'\left(y\right)\right\|_{H}dy\right)+\left\|c\left(\lambda\right)\right\|_{H\to H}$$

$$\times\left(\left\|A^{1/2}e^{-A}\right\|_{H\to H}\left[\left\|g\left(0\right)\right\|_{H}+\left\|g\left(-1\right)-g(0)\right\|_{H}\right]\right.$$

$$\left.+\int_{-1}^{0}\left\|A^{1/2}e^{yA}\right\|_{H\to H}\left\|g'\left(y\right)\right\|_{H}dy\right)$$

$$\left.+\left\|A^{1/2}s\left(\lambda\right)\right\|_{H\to H}\left\|f\left(0\right)\right\|_{H}+\int_{0}^{\lambda}\left\|A^{1/2}s\left(\lambda-y\right)\right\|_{H\to H}\left\|f'\left(y\right)\right\|_{H}\right\}$$

$$\left.+\left\|A^{1/2}\varphi\right\|_{H}\right]\leq M\left[\left\|A^{1/2}\varphi\right\|_{H}+\max_{-1\leq t\leq 0}\left\|g'(t)\right\|_{H}\right.$$

$$\left.+\left\|g(0)\right\|_{H}+\left\|f(0)\right\|_{H}+\max_{0\leq t\leq 1}\left\|f'(t)\right\|_{H}\right]. \tag{3.19}$$

Applying $A^{1/2}$ to the formulas (3.17), (3.18) and using estimates (3.13) and (3.14), we obtain

$$\left\|A^{1/2}u\left(t\right)\right\|_{H}\leq\left\|e^{-(t+1)A}\right\|_{H\to H}\left\|A^{1/2}u_{-1}\right\|_{H}+\left\|A^{-1/2}\right\|_{H\to H}$$

$$\times\left(\left\|g\left(0\right)\right\|_{H}+\left\|e^{-A}\right\|_{H\to H}\left\|g\left(0\right)\right\|_{H}+\left\|g\left(t\right)-g\left(0\right)\right\|_{H}\right.$$

$$\left.+\left\|e^{-A}\right\|_{H\to H}\left\|g\left(-1\right)-g\left(0\right)\right\|_{H}+\int_{-1}^{t}\left\|e^{yA}\right\|_{H\to H}\left\|g'\left(y\right)\right\|_{H}dy\right)$$

$$\leq M\left[\left\|A^{1/2}u_{-1}\right\|_{H}+\max_{-1\leq t\leq 0}\left\|g'(t)\right\|_{H}+\left\|g(0)\right\|_{H}\right],\quad -1\leq t\leq 0,$$

$$\left\|A^{1/2}u\left(t\right)\right\|_{H}\leq\left\|c\left(t\right)\right\|_{H\to H}\left\|e^{-A}\right\|_{H\to H}\left\|A^{1/2}u_{-1}\right\|_{H}$$

$$+\left\|A^{1/2}s\left(t\right)\right\|_{H\to H}\left\|A^{1/2}e^{-A}\right\|_{H\to H}\left\|A^{1/2}u_{-1}\right\|_{H}$$

$$+\left[\left\|c\left(t\right)\right\|_{H\to H}\left\|A^{-1/2}\right\|_{H\to H}+\left\|A^{1/2}s(t)\right\|_{H\to H}\right]$$

$$\times\left(\left\|g\left(0\right)\right\|_{H}+\left\|e^{-A}\right\|_{H\to H}\left\|g\left(0\right)\right\|_{H}\right.$$

$$\left.+\left\|e^{-A}\right\|_{H\to H}\left\|g\left(-1\right)-g\left(0\right)\right\|_{H}+\int_{-1}^{0}\left\|e^{yA}\right\|_{H\to H}\left\|g'\left(y\right)\right\|_{H}dy\right)$$

$$+\left\|A^{1/2}s\left(t\right)\right\|_{H\to H}\left\|g\left(0\right)\right\|_{H}+\left\|A^{-1/2}\right\|_{H\to H}$$

$$\times\left[\|f(0)\|_H+\|f(t)-f(0)\|_H+\|c(t)\|_{H\to H}\|f(0)\|_H\right.$$

$$\left.+\int_0^t\|c(t-y)\|_{H\to H}\left\|f'(y)\right\|_H dy\right]\leq M\left[\max_{0\leq t\leq 1}\left\|f'(t)\right\|_H+\|f(0)\|_H\right.$$

$$\left.+\left\|A^{1/2}u_{-1}\right\|_H+\max_{-1\leq t\leq 0}\left\|g'(t)\right\|_H+\|g(0)\|_H\right],\quad 0\leq t\leq 1.$$

Then from (3.19) and the last two estimates, it follows (3.3).

Third, we obtain estimate (3.4). Using formula (3.15) and an integration by parts, we obtain

$$u_{-1}=T\left[\alpha\left\{c(\mu)\left[A^{-1}\left(g(0)-e^{-A}g(-1)\right.\right.\right.\right.$$

$$\left.\left.-A^{-1}\left[g'(0)-e^{-A}g'(-1)-\int_{-1}^0 e^{yA}g''(y)\,dy\right]\right)\right.$$

$$+s(\mu)\left(e^{-A}g(-1)+A^{-1}\left[g'(0)-e^{-A}g'(-1)-\int_{-1}^0 e^{yA}g''(y)\,dy\right]\right)$$

$$\left.+A^{-1}\left[f(\mu)-c(\mu)f(0)-\left[s(\mu)f'(0)+\int_0^\mu s(\mu-y)f''(y)\,dy\right]\right]\right\}$$

$$+\beta\left\{-s(\lambda)\left(g(0)-e^{-A}g(-1)-A^{-1}\left[g'(0)-e^{-A}g'(-1)-\int_{-1}^0 e^{yA}g''(y)\,dy\right]\right)\right.$$

$$+c(\lambda)\left(e^{-A}g(-1)+A^{-1}\left[g'(0)-e^{-A}g'(-1)-\int_{-1}^0 e^{yA}g''(y)\,dy\right]\right)$$

$$\left.\left.+s(\lambda)f(0)+A^{-1}\left[f'(\lambda)-c(\lambda)f'(0)-\int_0^\lambda c(\lambda-y)f''(y)\,dy\right]\right\}+\varphi\right].$$

Using estimates (3.11), (3.13) and (3.14), we obtain

$$\|Au_{-1}\|_H\leq\|T\|_{H\to H}\left[|\alpha|\left\{\|c(\mu)\|_{H\to H}\left[(\|g(0)\|_H\right.\right.\right.$$

$$+\left\|e^{-A}\right\|_{H\to H}\left[\|g(0)\|_H+\|g(-1)-g(0)\|_H\right]$$

$$\left.\left.+\left\|A^{-1}\right\|_{H\to H}\int_{-1}^0\left\|e^{yA}\right\|_{H\to H}\left\|g''(y)\right\|_H dy\right)\right]$$

$$+\left\|A^{-1}\right\|_{H\to H}\left[\left\|g'(0)\right\|_H+\left\|e^{-A}\right\|_{H\to H}\left[\left\|g'(0)\right\|_H+\left\|g'(-1)-g'(0)\right\|_H\right]\right])$$

$$+\left\|A^{1/2}s(\mu)\right\|_{H\to H}\left(\left\|A^{1/2}e^{-A}\right\|_{H\to H}\left[\|g(0)\|_H+\|g(-1)-g(0)\|_H\right]\right.$$

$$+\left\|A^{1/2}s(\mu)\right\|_{H\to H}\left\|A^{-1/2}\right\|_{H\to H}\left(\left\|g'(0)\right\|_H+\left\|e^{-A}\right\|_{H\to H}\right.$$

$$\left.\times\left[\left\|g'(0)\right\|_H+\left\|g'(-1)-g'(0)\right\|_H\right]+\int_{-1}^0\left\|e^{yA}\right\|_{H\to H}\left\|g''(y)\right\|_H dy\right)$$

$$+\left[\|f(0)\|_H+\|f(\mu)-f(0)\|_H+\|c(\mu)\|_{H\to H}\|f(0)\|_H+\right.$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|A^{1/2}s(\mu)\right\|_{H\to H}\left\|f'(0)\right\|_H\right.$$

$$\left.\left.+\int_0^\mu\left\|A^{1/2}s(\mu-y)\right\|_{H\to H}\left\|f''(y)\right\|_H dy\right]\right\}$$

$$+|\beta|\left\{\left\|A^{1/2}s(\lambda)\right\|_{H\to H}\left(\left\|A^{1/2}g(0)\right\|_H+\left\|A^{1/2}e^{-A}\right\|_{H\to H}\right.\right.$$

$$\times\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g(0)\right\|_{H}+\|g(-1)-g(0)\|_{H}\right]$$
$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\|g'(0)\|_{H}+\|e^{-A}\|_{H\to H}\|g'(0)\|_{H}\right.$$
$$\left.\left.+\|e^{-A}\|_{H\to H}\|g'(0)-g'(-1)\|_{H}+\int_{-1}^{0}\|e^{yA}\|_{H\to H}\|g''(y)\|_{H}dy\right]\right)$$
$$+\|c(\lambda)\|_{H\to H}\left(\|Ae^{-A}\|_{H\to H}\left[\left\|A^{-1/2}\right\|_{H\to H}\right.\right.$$
$$\left.\left.\times\left\|A^{1/2}g(0)\right\|_{H}+\|g(-1)-g(0)\|_{H}\right]\right)$$
$$+\|g'(0)\|_{H}+\|e^{-A}\|_{H\to H}\left[\|g'(0)\|_{H}+\|g'(0)-g'(-1)\|_{H}\right]$$
$$\left.+\int_{-1}^{0}\|e^{yA}\|_{H\to H}\|g''(y)\|_{H}dy\right)$$
$$+\left\|A^{1/2}s(\lambda)\right\|_{H\to H}\left\|A^{1/2}f(0)\right\|_{H}+\|f'(0)\|_{H}+\|f'(\lambda)-f'(0)\|_{H}$$
$$\left.+\|c(\lambda)\|_{H\to H}\|f'(0)\|_{H}+\int_{0}^{\lambda}\|c(\lambda-y)\|_{H\to H}\|f''(y)\|_{H}\right\}$$
$$+\|A\varphi\|_{H}]\leq M\left[\|A\varphi\|_{H}+\max_{-1\leq t\leq 0}\|g''(t)\|_{H}+\|g'(0)\|_{H}\right.$$
$$\left.+\left\|A^{1/2}g(0)\right\|_{H}+\left\|A^{1/2}f(0)\right\|_{H}+\|f'(0)\|_{H}+\max_{0\leq t\leq 1}\|f''(t)\|_{H}\right]. \tag{3.20}$$

Using formulas (3.17), (3.18) and an integration by parts, we obtain

$$u(t)=e^{-(t+1)A}u_{-1}+A^{-1}\left(g(t)-e^{-A}g(-1)\right.$$
$$\left.-A^{-1}\left[g'(t)-e^{-A}g'(-1)-\int_{-1}^{t}e^{yA}g''(y)dy\right]\right),\ -1\leq t\leq 0,$$
$$u(t)=[c(t)-As(t)]\left\{e^{-A}u_{-1}+A^{-1}\left(g(0)-e^{-A}g(-1)\right.\right.$$
$$\left.\left.-A^{-1}\left[g'(0)-e^{-A}g'(-1)-\int_{-1}^{0}e^{yA}g''(y)dy\right]\right)\right\}$$
$$+s(t)g(0)+A^{-1}f(t)-c(t)f(0)$$
$$-\left[s(t)f'(0)+\int_{0}^{t}s(t-y)f''(y)dy\right],\ 0\leq t\leq 1.$$

Applying A to the last two formulas and using estimates (3.13) and (3.14), we obtain

$$\|Au(t)\|_{H}\leq\left\|e^{-(t+1)A}\right\|_{H\to H}\|Au_{-1}\|_{H}$$
$$+\left(\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g(0)\right\|_{H}+\|e^{-A}\|_{H\to H}\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g(0)\right\|_{H}\right.$$
$$+\|g(t)-g(0)\|_{H}+\|e^{-A}\|_{H\to H}\|g(-1)-g(0)\|_{H}+\|A^{-1}\|_{H\to H}$$
$$\times\left[\|g'(0)\|_{H}+\|g'(t)-g'(0)\|_{H}+\|e^{-A}\|_{H\to H}\right.$$
$$\left.\left.\times\left[\|g'(0)\|_{H}+\|g'(-1)-g'(0)\|_{H}\right]+\int_{-1}^{t}\|e^{yA}\|_{H\to H}\|g''(y)\|_{H}dy\right]\right)$$

$$\leq M\left[\|Au_{-1}\|_H+\max_{-1\leq t\leq 0}\|g''(t)\|_H+\left\|A^{1/2}g(0)\right\|_H+\|g'(0)\|_H\right],$$

$$-1\leq t\leq 0,$$

$$\|Au(t)\|_H\leq\|c(t)\|_{H\to H}\left\|e^{-A}\right\|_{H\to H}\|Au_{-1}\|_H$$

$$+\left\|A^{1/2}s(t)\right\|_{H\to H}\left\|A^{1/2}e^{-A}\right\|_{H\to H}\|Au_{-1}\|_H$$

$$+\left[\|c(t)\|_{H\to H}\left\|A^{-1/2}\right\|_{H\to H}+\left\|A^{1/2}s(t)\right\|_{H\to H}\right]$$

$$\times\left(\left\|A^{1/2}g(0)\right\|_H+\left\|e^{-A}\right\|_{H\to H}\left\|A^{1/2}g(0)\right\|_H\right.$$

$$+\left[\|c(t)\|_{H\to H}\left\|A^{-1/2}\right\|_{H\to H}+\left\|A^{1/2}s(t)\right\|_{H\to H}\right]\left\|A^{-1/2}\right\|_{H\to H}$$

$$\left[\|g'(0)\|_H+\left\|e^{-A}\right\|_{H\to H}\left[\|g'(0)\|_H+\|g'(-1)-g'(0)\|_H\right]\right.$$

$$\left.+\int_{-1}^{0}\left\|e^{yA}\right\|_{H\to H}\|g''(y)\|_H\,dy\right]$$

$$+\left\|e^{-A}\right\|_{H\to H}\left\|A^{-1/2}\left[g(-1)-g(0)\right]\right\|_H$$

$$\left.+\int_{-1}^{0}\left\|e^{yA}\right\|_{H\to H}\left\|A^{-1/2}g'(y)\right\|_H\,dy\right)$$

$$+\left\|A^{1/2}s(t)\right\|_{H\to H}\left\|A^{1/2}g(0)\right\|_H$$

$$+\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}f(0)\right\|_H+\|f(t)-f(0)\|_H\right.$$

$$+\|c(t)\|_{H\to H}\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}f(0)\right\|_H$$

$$+\left[\left\|A^{1/2}s(t)\right\|_{H\to H}\left\|A^{-1/2}\right\|_{H\to H}\|g'(0)\|_H\right.$$

$$\left.\left.+\int_0^t\left\|A^{1/2}s(t-y)\right\|_{H\to H}\left\|A^{-1/2}\right\|_{H\to H}\|f''(y)\|_H\,dy\right]\right]$$

$$\leq M\left[\max_{0\leq t\leq 1}\|f''(t)\|_H+\left\|A^{1/2}f(0)\right\|_H+\|f'(0)\|_H\right.$$

$$\left.+\|Au_{-1}\|_H+\max_{-1\leq t\leq 0}\|g''(t)\|_H+\left\|A^{1/2}g(0)\right\|_H+\|g'(0)\|_H\right],\quad 0\leq t\leq 1.$$

Then from (3.20) and the last two estimates, it follows (3.4). Theorem 3.2 is proved.

Remark 3.1. We can obtain the same results for the solution of the following multipoint boundary value problem

$$\begin{cases}\frac{d^2(t)}{dt^2}+Au(t)=f(t)\ \ (0\leq t\leq 1),\\ \frac{d(t)}{dt}+Au(t)=g(t)\ \ (-1\leq t\leq 0),\\ u(-1)=\sum_{i=1}^{N}\alpha_i u(\mu_i)+\sum_{i=1}^{L}\beta_i u'(\lambda_i)+\varphi,\\ \sum_{i=1}^{N}|\alpha_i|,\sum_{i=1}^{L}|\beta_i|\leq 1,\\ 0<\mu_i\leq 1,1\leq i\leq N,0<\lambda_i\leq 1,1\leq i\leq L\end{cases}$$

for differential equations of mixed type in a Hilbert space H with self-adjoint positive definite operator A.

3.3 Applications

Now, we will consider the application of Theorem 3.2. First, we consider the mixed problem for hyperbolic- parabolic equation

$$\begin{cases} v_{yy} - (a(x)v_x)_x + \delta v = f(y,x), 0 < y < 1,\ 0 < x < 1, \\ v_y - (a(x)v_x)_x + \delta v = g(y,x), -1 < y < 0,\ 0 < x < 1, \\ v(-1,x) = v(1,x) + v_y(1,x) + \varphi(x), 0 \le x \le 1, \\ v(y,0) = v(y,1), v_x(y,0) = v_x(y,1), -1 \le y \le 1, \\ v(0+,x) = v(0-,x), v_y(0+,x) = v_y(0-,x), 0 \le x \le 1. \end{cases} \tag{3.21}$$

Problem (3.21) has a unique smooth solution $v(y,x)$ for the smooth $a(x) > 0$ $(x \in (0,1))$, $\varphi(x)$ $(x \in [0,1])$ and $f(y,x)$ $(y \in [0,1], x \in [0,1])$, $g(y,x)$ $(y \in [-1,0], x \in [0,1])$ functions and $\delta = \text{const} > 0$. This allows us to reduce the mixed problem(3.21) to the nonlocal boundary value problem (3.1) in Hilbert space H with a self-adjoint positive definite operator A defined by (3.21). Let us give a number of corollaries of the abstract Theorem 3.2.

Theorem 3.3. *The solutions of the nonlocal boundary value problem (3.21) satisfy the stability estimates*

$$\max_{-1\le y\le 1} \|v(y)\|_{L_2[0,1]} \le M \Big[\|f(0)\|_{L_2[0,1]} + \max_{0\le y\le 1} \|f_y(y)\|_{L_2[0,1]}$$

$$+ \|g(0)\|_{L_2[0,1]} + \max_{-1\le y\le 0} \|g_y(y)\|_{L_2[0,1]} + \|\varphi\|_{L_2[0,1]} \Big],$$

$$\max_{-1\le y\le 1} \|v(y)\|_{W_2^1[0,1]} \le M \Big[\|f(0)\|_{L_2[0,1]} + \max_{0\le y\le 1} \|f_y(y)\|_{L_2[0,1]}$$

$$+ \|g(0)\|_{L_2[0,1]} + \max_{-1\le y\le 0} \|g_y(y)\|_{L_2[0,1]} + \|\varphi\|_{W_2^1[0,1]} \Big],$$

$$\max_{-1\le y\le 1} \|v(y)\|_{W_2^2[0,1]} + \max_{-1\le y\le 0} \|v_y(y)\|_{L_2[0,1]} + \max_{0\le y\le 1} \|v_{yy}(y)\|_{L_2[0,1]}$$

$$\le M \Big[\|\varphi\|_{W_2^1[0,1]} + \|f(0)\|_{W_2^1[0,1]} + \|f_y(0)\|_{L_2[0,1]} + \max_{0\le y\le 1} \|f_{yy}(y)\|_{L_2[0,1]}$$

$$+ \|g(0)\|_{W_2^2[0,1]} + \|g_y(0)\|_{L_2[0,1]} + \max_{-1\le y\le 0} \|g_{yy}(y)\|_{L_2[0,1]} \Big]$$

where M does not depend on $f(y,x)$ $(y \in [0,1], x \in [0,1])$, $g(y,x)$ $(y \in [-1,0], x \in [0,1])$ and $\varphi(x)$ $(x \in [0,1])$.

The proof of this theorem is based on the abstract Theorem 3.2 and the symmetry properties of the space operator generated by the problem (3.21).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space R^n $(0 < x_k < 1,$

$1 \leq k \leq n$) with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$ we consider the mixed boundary value problem for the multidimensional hyperbolic-parabolic equation

$$\begin{cases} v_{yy} - \sum\limits_{r=1}^{n} (a_r(x) v_{x_r})_{x_r} = f(y,x), 0 \leq y \leq 1, \\ x = (x_1, \ldots, x_n) \in \Omega, \\ v_y - \sum\limits_{r=1}^{n} (a_r(x) v_{x_r})_{x_r} = g(y,x), -1 \leq y \leq 0, \\ x = (x_1, \ldots, x_n) \in \Omega, \\ v(-1,x) = v(1,x) + v_y(1,x) + \varphi(x), x \in \overline{\Omega}, \\ u(y,x) = 0, x \in S, -1 \leq y \leq 1. \end{cases} \tag{3.22}$$

where $a_r(x), (x \in \Omega), \varphi(x)$ $(x \in \overline{\Omega})$ and $f(y,x)$ $(y \in (0,1),\ x \in \Omega), g(y,x)$ $(y \in (-1,0),\ x \in \Omega)$ are given smooth functions and $a_r(x) \geq a > 0$.

We introduce the Hilbert spaces $L_2(\overline{\Omega})$ of the all integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \right\}^{1/2} .$$

Problem (3.22) has a unique smooth solution $v(y,x)$ for the smooth $a_r(x) > 0$ and $f(y,x)$,$g(y,x)$ functions. This allows us to reduce the mixed problem (3.22) to the nonlocal boundary value problem (3.1) in Hilbert space H with a self- adjoint positive definite operator A defined by (3.22). Let us give a number of corollaries of the abstract Theorem 3.2.

Theorem 3.4. *The solutions of the nonlocal boundary value problem (3.22) satisfy the stability estimates*

$$\max_{-1 \leq y \leq 1} \|v(y)\|_{L_2(\overline{\Omega})} \leq M \Big[\|f(0)\|_{L_2(\overline{\Omega})} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L_2(\overline{\Omega})}$$

$$+ \|g(0)\|_{L_2(\overline{\Omega})} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L_2(\overline{\Omega})} + \|\varphi\|_{L_2(\overline{\Omega})} \Big] ,$$

$$\max_{-1 \leq y \leq 1} \|v(y)\|_{W_2^1(\overline{\Omega})} \leq M \Big[\|f(0)\|_{L_2(\overline{\Omega})} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L_2(\overline{\Omega})}$$

$$+ \|g(0)\|_{L_2(\overline{\Omega})} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L_2(\overline{\Omega})} + \|\varphi\|_{W_2^1(\overline{\Omega})} \Big] ,$$

$$\max_{-1 \leq y \leq 1} \|v(y)\|_{W_2^2(\overline{\Omega})} + \max_{-1 \leq y \leq 0} \|v_y(y)\|_{L_2(\overline{\Omega})} + \max_{0 \leq y \leq 1} \|v_{yy}(y)\|_{L_2(\overline{\Omega})}$$

$$\leq M| \Big[\|\varphi\| |_{W_2^1(\overline{\Omega})} + \|f(0)\|_{W_2^1(\overline{\Omega})} + \|f_y(0)\|_{L_2(\overline{\Omega})} + \max_{0 \leq y \leq 1} \|f_{yy}(y)\|_{L_2(\overline{\Omega})}$$

$$+ \|g(0)\|_{W_2^2(\overline{\Omega})} + \|g_y(0)\|_{L_2(\overline{\Omega})} + \max_{-1 \leq y \leq 0} \|g_{yy}(y)\|_{L_2(\overline{\Omega})} \Big]$$

where M does not depend on $f(y,x)$ $(y \in [0,1], x \in [0,1])$, $g(y,x)$ $(y \in [-1,0], x \in [0,1])$ and $\varphi(x)$ $(x \in [0,1])$.

The proof of this theorem is based on the abstract Theorem 3.2 and the symmetry properties of the space operator generated by the problem (3.22).

CHAPTER 4

THE HYPERBOLIC-PARABOLIC DIFFERENCE EQUATION

4.1 The First Order of Accuracy Difference Scheme

Let us associate the boundary-value problem (3.1) with the corresponding first order of accuracy difference scheme

$$\begin{cases} \tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_{k+1}=f_k, \\ f_k=f\left(t_{k+1}\right),\ t_{k+1}=(k+1)\tau, 1\le k\le N-1, \\ \tau^{-1}\left(u_1-u_0\right)=-Au_0+g_0, \\ \tau^{-1}\left(u_k-u_{k-1}\right)+Au_k=g_k, \quad g_k=g(t_k), \\ t_k=k\tau,\ -(N-1)\le k\le 0, \\ u_{-N}=\alpha u_0+\beta\left(-Au_0+g_0\right)+\varphi,\ \mu\le 2\tau,\ \lambda\le 2\tau, \\ u_{-N}=\alpha u_{[\mu/\tau]}+\beta\left(-Au_0+g_0\right)+\varphi, 2\tau<\mu, \lambda\le 2\tau, \\ u_{-N}=\alpha u_0+\beta\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+\varphi, \mu\le 2\tau, 2\tau<\lambda, \\ u_{-N}=\alpha u_{[\mu/\tau]}+\beta\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+\varphi,\ 2\tau<\mu,\ 2\tau<\lambda. \end{cases} \tag{4.1}$$

A study of discretization, over time only, of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A, is replaced by the difference operators A_h that act in the Hilbert spaces H_h and are uniformly self-adjoint positive definite in h for $0<h\le h_0$.

First of all let us give some lemmas that will be needed below.

Lemma 4.1. The estimates hold:

$$\left\|R\left(\pm\tau A^{1/2}\right)\right\|_{H\to H}\le 1, \left\|\tau A^{1/2}R\left(\pm\tau A^{1/2}\right)\right\|_{H\to H}\le 1, \tag{4.2}$$

$$\left\|R^k\right\|_{H\to H}\le M\left(1+\delta\tau\right)^{-k}, \left\|AR^k\right\|_{H\to H}\le M\left(k\tau\right)^{-1}, k\ge 1, \tag{4.3}$$

where M does not depend on τ. Here $R\left(\pm\tau A^{1/2}\right) = \left(I \pm i\tau A^{1/2}\right)^{-1}$,

$R = R(\tau A) = (I + \tau A)^{-1}$.

Lemma 4.2. The operators

$$I - (\alpha - \beta A)e^{-A}$$
$$I - (\alpha\left[c(\mu) - As(\mu)\right] - \beta A)e^{-A}$$
$$I - (\alpha - \beta\left[s(\lambda) + c(\lambda)\right]A)e^{-A}$$

have inverses

$$(I - (\alpha - \beta A)e^{-A})^{-1},$$
$$\left(I - (\alpha\left[c(\mu) - As(\mu)\right] - \beta A)e^{-A}\right)^{-1},$$
$$\left(I - (\alpha - \beta\left[s(\lambda) + c(\lambda)\right]A)e^{-A}\right)^{-1}$$

and the estimates hold:

$$\left\|(I - (\alpha - \beta A)e^{-A})^{-1}\right\|_{H\to H} \le M, \tag{4.4}$$

$$\left\|\left(I - (\alpha\left[c(\mu) - As(\mu)\right] - \beta A)e^{-A}\right)^{-1}\right\|_{H\to H} \le M, \tag{4.5}$$

$$\left\|\left(I - (\alpha - \beta\left[s(\lambda) + c(\lambda)\right]A)e^{-A}\right)^{-1}\right\|_{H\to H} \le M. \tag{4.6}$$

Proof: First, we obtain the estimate (4.4). The proof of this estimate is based on the estimate

$$\left\|-(\alpha - \beta A)e^{-A}\right\|_{H\to H} < 1.$$

Using the positivity self-adjointness property of A, we obtain

$$\left\|-(\alpha - \beta A)e^{-A}\right\|_{H\to H} \le \sup_{\delta\le\rho<\infty}\left|(\alpha - \beta\rho)e^{-\rho}\right| \le \sup_{\delta\le\rho<\infty}(1+\rho)e^{-\rho}.$$

It is easy to show that $\sup\limits_{\delta\le\rho<\infty}(\rho+1)e^{-\rho} < 1$. So, the estimate (4.4) is proved. Second, we obtain the estimate (4.5). The proof of this estimate is based on the estimate

$$\left\|-(\alpha\left[c(\mu) - As(\mu)\right] - \beta A)e^{-A}\right\|_{H\to H} < 1.$$

Using the definitions of $c(\mu)$ and $s(\mu)$ and positivity self-adjointness property of A, we obtain

$$\left\|-(\alpha\left[c(\mu) - As(\mu)\right] - \beta A)e^{-A}\right\|_{H\to H}$$
$$\le \sup_{\delta\le\rho<\infty}\left|\alpha\left[\cos(\sqrt{\rho}\mu) - \sqrt{\rho}\sin(\sqrt{\rho}\mu)\right] - \beta\rho\right| e^{-\rho}.$$

Since

$$\cos(\sqrt{\rho}\mu) - \sqrt{\rho}\sin(\sqrt{\rho}\mu) = \sqrt{\rho+1}\cos(\sqrt{\rho}\mu - \mu_0),$$

we have that

$$\left\|-(\alpha\left[c(\mu) - As(\mu)\right] - \beta A)e^{-A}\right\|_{H\to H}$$

$$\le \sup_{\delta\le\rho<\infty}\left(\sqrt{\rho+1} + \rho\right)e^{-\rho}.$$

It is easy to show that $\sup\limits_{\delta\leq\rho<\infty}\left(\sqrt{\rho+1}+\rho\right)e^{-\rho}<1$. So, the estimate (4.5) is proved. Third, we obtain the estimate (4.6). The proof of this estimate is based on the estimate

$$\left\|-\left(\alpha-\beta\left[s\left(\lambda\right)+c\left(\lambda\right)\right]A\right)e^{-A}\right\|_{H\to H}<1.$$

Using the definitions of $c\left(\mu\right)$ and $s\left(\mu\right)$ and positivity self-adjointness property of A, we obtain

$$\begin{aligned}&\left\|-(\alpha-\beta\left[s\left(\lambda\right)+c\left(\lambda\right)\right]A)e^{-A}\right\|_{H\to H}\\ &\leq\sup_{\delta\leq\rho<\infty}\left|\alpha-\beta\left[\sqrt{\rho}\sin\left(\sqrt{\rho}\lambda\right)+\rho\cos\left(\sqrt{\rho}\lambda\right)\right]\right|e^{-\rho}.\end{aligned}$$

Since

$$\sqrt{\rho}\sin\left(\sqrt{\rho}\lambda\right)+\rho\cos\left(\sqrt{\rho}\lambda\right)=\sqrt{\rho}\sqrt{\rho+1}\cos\left(\sqrt{\rho}\mu-\mu_1\right),$$

we have that

$$\left\|-(\alpha-\beta\left[s\left(\lambda\right)+c\left(\lambda\right)\right]A)e^{-A}\right\|_{H\to H}\leq\sup_{\delta\leq\rho<\infty}\left(1+\sqrt{\rho+1}\sqrt{\rho}\right)e^{-\rho}.$$

It is easy to show that $\sup\limits_{\delta\leq\rho<\infty}\left(1+\sqrt{\rho+1}\sqrt{\rho}\right)e^{-\rho}<1$. So, the estimate (4.6) is proved. Lemma 4.2 is established.

Lemma 4.3. The operator

$$Q_\tau=\begin{cases}
I-\left(\alpha-\beta A\right)R^N \text{ if } \quad \mu\leq 2\tau,\lambda\leq 2\tau,\\[2ex]
\begin{array}{c}
I-\left(\alpha\left\{\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\\
\left.+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}\\
\left.\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\right\}-\beta A\right)R^N\\
\text{if} \quad 2\tau<\mu,\lambda\leq 2\tau,
\end{array}\\[2ex]
\begin{array}{c}
I-\left[\alpha-\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\right.\\
\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}\\
\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}\\[1ex]
-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.\\
\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}\\
\left.\left.\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}\right)\right]R^N\\
\text{if} \quad \mu\leq 2\tau,2\tau<\lambda,
\end{array}\\[2ex]
\begin{array}{c}
I-\left[\alpha\left\{\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\\
\left.+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}\\
\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\right\}R^N\\[1ex]
-\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\\
\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}\\
\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}\\[1ex]
-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.\\
\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}\\[1ex]
\left.\left.\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}\right)R^N\right]\\
\text{if} \quad 2\tau<\mu,2\tau<\lambda
\end{array}
\end{cases}$$

has an inverse

$$
T_\tau = \begin{cases}
\left(I - (\alpha - \beta A)\, R^N\right)^{-1} \ \text{if} \quad \mu \le 2\tau, \lambda \le 2\tau, \\[2mm]
\big(I - \big(\alpha \big\{\big\{\tfrac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right] \\
\quad + \tfrac{1}{2i} A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\big\} \\
\quad - \tfrac{1}{2i} A^{1/2}\left(I + \tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\big\} - \beta A\big)\, R^N\big)^{-1} \\
\quad \text{if} \quad 2\tau < \mu, \lambda \le 2\tau, \\[2mm]
\big(I - \big[\alpha - \beta\tau^{-1}\big(\big\{\big\{\tfrac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right] \\
\quad + \tfrac{1}{2i} A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\big\} \\
\quad - \tfrac{1}{2i} A^{1/2}\left(I + \tau^2 A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right) - R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\big\} \\
\quad - \big\{\big\{\tfrac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right] \\
\quad + \tfrac{1}{2i} A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\big\} \\
\quad + \tfrac{1}{2i} A^{1/2}\left(I + \tau^2 A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right) - R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\big\}\big)\big]\, R^N\big)^{-1} \\
\quad \text{if} \quad \mu \le 2\tau, 2\tau < \lambda, \\[2mm]
\big(I - \alpha \big\{\big\{\tfrac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right] \\
\quad + \tfrac{1}{2i} A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\big\} \\
\quad - \tfrac{1}{2i} A^{1/2}\left(I + \tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\big\}\, R^N \\
\quad + \beta\tau^{-1}\big(\big\{\big\{\tfrac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right] \\
\quad + \tfrac{1}{2i} A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\big\} \\
\quad - \tfrac{1}{2i} A^{1/2}\left(I + \tau^2 A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right) - R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\big\} \\
\quad - \big\{\big\{\tfrac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right] \\
\quad + \tfrac{1}{2i} A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\big\} \\
\quad + \tfrac{1}{2i} A^{1/2}\left(I + \tau^2 A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right) - R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\big\}\big)\, R^N\big]\big)^{-1} \\
\quad \text{if} \quad 2\tau < \mu, 2\tau < \lambda
\end{cases}
$$

and the estimate holds:

$$
||T_\tau||_{H \to H} \le M, \tag{4.7}
$$

where M does not depend on τ.

Proof. Note that if $\mu \le 2\tau, \lambda \le 2\tau$, then

$$
T_\tau - \left(I - (\alpha - \beta A)\, e^{-A}\right)^{-1}
$$

$$
= T_\tau \left(I - (\alpha - \beta A) e^{-A}\right)^{-1} (\alpha - \beta A)\left[R^N - e^{-A}\right].
$$

If $2\tau < \mu, \lambda \le 2\tau$, then

$$
T_\tau - \left(I - (\alpha\left[c\left([\mu/\tau]\tau\right) - As\left([\mu/\tau]\tau\right)\right] - \beta A) e^{-A}\right)^{-1}
$$

$$
= T_\tau \left(I - (\alpha\left[c\left([\mu/\tau]\tau\right) - As\left([\mu/\tau]\tau\right)\right] - \beta A) e^{-A}\right)^{-1}
$$

$$
\times \left(\left(\alpha\left\{\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\right.
$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Big\}-\beta A\Big)R^N$$

$$-(\alpha\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta A)e^{-A}).$$

If $\mu \leq 2\tau, 2\tau < \lambda$, then

$$T_\tau-\left(I-\left(\alpha-\beta\left[s\left([\lambda/\tau]\,\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A\right)e^{-A}\right)^{-1}$$

$$=T_\tau\left(I-\left(\alpha-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A\right)e^{-A}\right)^{-1}$$

$$\times\Big[\Big[\alpha-\beta\tau^{-1}\Big(\Big\{\Big\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$-\Big\{\Big\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$+\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\Big\}\Big)\Big]R^N$$

$$-(\alpha-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A)e^{-A}].$$

If $2\tau < \mu$, $2\tau < \lambda$, then

$$T_\tau-\left(I-\alpha\left(\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A\right)e^{-A}\right)^{-1}$$

$$=T_\tau\left(I-\alpha\left(\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A\right)e^{-A}\right)^{-1}$$

$$\times\Big(\Big[\alpha\Big\{\Big\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$-\beta\tau^{-1}\Big(\Big\{\Big\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$
$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}$$
$$\left.\left.\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}\right)\right]R^N$$
$$-\alpha\left(\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A\right)e^{-A}\right).$$

Using the last formulas and the estimates (4.4), (4.5), (4.6) and

$$\left\|AR^N-Ae^{-A}\right\|_{H\to H}\leq M\tau,$$
$$\left\|(\alpha-\beta A)\left[R^N-e^{-A}\right]\right\|_{H\to H}\leq M\tau,$$

$$\left\|\left(\alpha\left\{\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\right.$$
$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$
$$\left.\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\right\}-\beta A\right)R^N$$
$$\left.-(\alpha\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta A)e^{-A}\right\|_{H\to H}\leq M\tau,$$
$$\left\|\left[\alpha-\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\right.\right.$$
$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$
$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$
$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$
$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}$$
$$\left.\left.\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}\right)\right]R^N$$
$$\left.-(\alpha-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A)e^{-A}\right\|_{H\to H}\leq M\tau,$$
$$\left\|\left[\alpha\left\{\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\right.$$
$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$
$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$
$$-\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$-\Big\{\Big\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$+\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\Big\}\Big)\Big]R^N$$

$$-\alpha\left(\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A\right)e^{-A}\Big\|_{H\to H}\leq M\tau.$$

we can obtain the estimate (4.7). The proof of these estimates are based on estimates (3.13), (3.14), (3.11), (4.2) and (4.3). Lemma 2.3 is proved.

Theorem 4.1. *Suppose that $\varphi\in D(A)$, $g_0\in D(A^{1/2})$ and $f_1\in D(A^{1/2})$. Then for the solution of the difference scheme (4.1) the stability estimates hold:*

$$\max_{-N\leq k\leq N}\|u_k\|_H \tag{4.8}$$

$$\leq M\Big[\|\varphi\|_H+\left\|A^{-1/2}f_1\right\|_H+\max_{2\leq k\leq N-1}\left\|A^{-1/2}\left(f_k-f_{k-1}\right)\tau^{-1}\right\|_H$$

$$+\left\|A^{-1/2}g_0\right\|_H+\max_{-(N-1)\leq k\leq 0}\left\|A^{-1/2}\left(g_k-g_{k-1}\right)\tau^{-1}\right\|_H\Big],$$

$$\max_{-N\leq k\leq N}\left\|A^{1/2}u_k\right\|_H \tag{4.9}$$

$$\leq M\Big[\left\|A^{1/2}\varphi\right\|_H+\|f_1\|_H+\max_{2\leq k\leq N-1}\left\|\left(f_k-f_{k-1}\right)\tau^{-1}\right\|_H$$

$$+\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|\left(g_k-g_{k-1}\right)\tau^{-1}\right\|_H\Big],$$

$$\max_{1\leq k\leq N-1}\left\|\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)\right\|_H \tag{4.10}$$

$$+\max_{-(N-1)\leq k\leq 0}\left\|\tau^{-1}\left(u_k-u_{k-1}\right)\right\|_H+\max_{-N\leq k\leq N}\|Au_k\|_H$$

$$\leq M\Big[\|A\varphi\|_H+\left\|A^{1/2}f_1\right\|_H+\left\|\left(f_2-f_1\right)\tau^{-1}\right\|_H$$

$$+\max_{2\leq k\leq N-2}\left\|\left(f_{k+1}-2f_k+f_{k-1}\right)\tau^{-2}\right\|_H+\left\|A^{1/2}g_0\right\|_H$$

$$+\left\|\left(g_0-g_{-1}\right)\tau^{-1}\right\|_H+\max_{-(N-1)\leq k\leq -1}\left\|\left(g_{k+1}-2g_k+g_{k-1}\right)\tau^{-2}\right\|_H\Big],$$

where M does not depend on τ, $f_k, 1\leq k\leq N-1$, $g_k, -N+1\leq k\leq 0$ and φ.

Proof. We will obtain the formula for the solution of the difference scheme (4.1). It is known that there are unique solutions of the initial value difference problems

$$\begin{cases} \tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_{k+1}=f_k, \\ f_k=f\left(t_{k+1}\right),\ t_{k+1}=(k+1)\tau,\ 1\le k\le N-1, \\ u_0=\xi,\ \tau^{-1}\left(u_1-u_0\right)=\psi, \end{cases} \tag{4.11}$$

$$\begin{cases} \tau^{-1}\left(u_k-u_{k-1}\right)+Au_k=g_k, \quad g_k=g(t_k), \\ t_k=k\tau,\ -(N-1)\le k\le 0,\ u_{-N} \text{ is given} \end{cases} \tag{4.12}$$

and the following formulas hold:

$$\begin{cases} u_1=\xi+\tau\psi, \\ u_k=\left\{\frac{1}{2}\left[R^{k-1}\left(\tau A^{1/2}\right)+R^{k-1}\left(-\tau A^{1/2}\right)\right]\right. \\ \left.+\frac{1}{2i}A^{1/2}\left[R^{k-1}\left(\tau A^{1/2}\right)+R^{k-1}\left(-\tau A^{1/2}\right)\right]\right\}\xi \\ +\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^k\left(-\tau A^{1/2}\right)-R^k\left(\tau A^{1/2}\right)\right]\psi \\ +\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}\left(-\tau A^{1/2}\right)-R^{k-s}\left(\tau A^{1/2}\right)\right]f_s,\ 2\le k\le N, \end{cases} \tag{4.13}$$

$$u_k=R^{N+k}u_{-N}+\tau\sum_{s=-N+1}^{k}R^{k-s+1}g_s,\quad -(N-1)\le k\le 0. \tag{4.14}$$

Using (4.13) , (4.14) and the formulas

$$\tau^{-1}\left(u_1-u_0\right)=-Au_0+g_0, u_0=\xi, \tau^{-1}\left(u_1-u_0\right)=\psi,$$

we obtain

$$\xi=u_0=R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s,$$

$$\psi=\tau^{-1}\left(u_1-u_0\right)=-A\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s\right]+g_0.$$

Therefore

$$\begin{cases} u_1=(I-\tau A)\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s\right]+\tau g_0, \\ u_k=\left\{\left\{\frac{1}{2}\left[R^{k-1}\left(\tau A^{1/2}\right)+R^{k-1}\left(-\tau A^{1/2}\right)\right]\right.\right. \\ \left.+\frac{1}{2i}A^{1/2}\left[R^{k-1}\left(\tau A^{1/2}\right)+R^{k-1}\left(-\tau A^{1/2}\right)\right]\right\} \\ \left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^k\left(-\tau A^{1/2}\right)-R^k\left(\tau A^{1/2}\right)\right]\right\} \\ \times\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s\right] \\ +\frac{1}{2i}A^{-1/2}(I+\tau^2A)\left[R^k\left(-\tau A^{1/2}\right)-R^k\left(\tau A^{1/2}\right)\right]g_0 \\ +\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}\left(-\tau A^{1/2}\right)-R^{k-s}\left(\tau A^{1/2}\right)\right]f_s,\ 2\le k\le N. \end{cases} \tag{4.15}$$

If $\mu \leq 2\tau$, $\lambda \leq 2\tau$, then using the condition

$$u_{-N} = \alpha u_0 + \beta(-Au_0 + g_0) + \varphi,$$

we obtain the operator equation

$$u_{-N} = (\alpha - \beta A)\left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} R^{-s+1} g_s\right] + \beta g_0 + \varphi.$$

The operator

$$I - (\alpha - \beta A)\, R^N$$

has an inverse

$$T_\tau = \left(I - (\alpha - \beta A)\, R^N\right)^{-1}$$

and the formula holds

$$u_{-N} = T_\tau \left[(\alpha - \beta A)\,\tau + \sum_{s=-N+1}^{0} R^{-s+1} g_s + \beta g_0 + \varphi\right]. \quad (4.16)$$

If $2\tau < \mu$, $\lambda \leq 2\tau$, then using the condition

$$u_{-N} = \alpha u_{[\mu/\tau]} + \beta\,(-Au_0 + g_0) + \varphi,$$

we obtain the operator equation

$$\begin{aligned} u_{-N} = \alpha \Bigg[\Bigg\{ \Bigg\{ & \frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right] \\ & + \frac{1}{2i} A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg\} \\ & - \frac{1}{2i} A^{1/2}\left(I + \tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Bigg\} \\ & \times \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} R^{-s+1} g_s\right] \\ & + \frac{1}{2i} A^{-1/2}\left(I + \tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right] g_0 \\ & + \sum_{s=1}^{[\mu/\tau]-1} \frac{\tau}{2i} A^{-1/2}\left[R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right] f_s \Bigg] \\ & - \beta A\left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} R^{-s+1} g_s\right] + \beta g_0 + \varphi. \end{aligned}$$

The operator

$$\begin{aligned} I - \Bigg(\alpha \Bigg[\Bigg\{ \Bigg\{ & \frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right] \\ & + \frac{1}{2i} A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg\} \\ & - \frac{1}{2i} A^{1/2}\left(I + \tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Bigg\} + \beta A\Bigg)\Bigg] R^N \end{aligned}$$

has an inverse

$$T_\tau = \Bigg(I - \Bigg(\alpha \Bigg[\Bigg\{\Bigg\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Bigg\} + \beta A\Bigg)\Bigg] R^N\Bigg)^{-1}$$

and the formula holds

$$u_{-N} = T_\tau \Bigg(\alpha \Bigg[\Bigg\{\Bigg\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right] \tag{4.17}$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Bigg\}$$

$$\times \tau \sum_{s=-N+1}^{0} R^{-s+1} g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\tau^2 A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right] g_0$$

$$+\sum_{s=1}^{[\mu/\tau]-1} \frac{\tau}{2i}A^{-1/2}\left[R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right] f_s\Bigg]$$

$$-\beta A\tau \sum_{s=-N+1}^{0} R^{-s+1} g_s + \beta g_0 + \varphi\Bigg).$$

If $\mu \leq 2\tau$, $2\tau < \lambda$, then using the condition

$$u_{-N} = \alpha u_0 + \beta \frac{u_{[\lambda/\tau]} - u_{[\lambda/\tau]-1}}{\tau} + \varphi,$$

we obtain the operator equation

$$u_{-N} = \alpha \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} R^{-s+1} g_s\right]$$

$$+\beta\tau^{-1}\Bigg(\Bigg\{\Bigg\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right) - R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\Bigg\}$$

$$\times \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} R^{-s+1} g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\tau^{2}A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]g_{0}$$

$$+\sum_{s=1}^{[\lambda/\tau]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\lambda/\tau]-s}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-s}\left(\tau A^{1/2}\right)\right]f_{s}$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^{2}A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times\left[R^{N}u_{-N}+\tau\sum_{s=-N+1}^{0}R^{-s+1}g_{s}\right]$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\tau^{2}A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]g_{0}$$

$$\left.-\sum_{s=1}^{[\lambda/\tau]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right]f_{s}\right)+\varphi.$$

The operator

$$I-\left[\alpha-\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^{2}A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.\left.\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^{2}A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}\right)\right]R^{N}$$

has an inverse

$$T_{\tau}=\left(I-\left[\alpha-\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^{2}A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\Big\}\Big)\Big]R^N\Big)^{-1}$$

and the formula holds

$$u_{-N}=T_\tau\left[\alpha\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s\right. \tag{4.18}$$

$$+\beta\tau^{-1}\Big(\Big\{\Big\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$\times\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]g_0$$

$$+\sum_{s=1}^{[\lambda/\tau]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\lambda/\tau]-s}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-s}\left(\tau A^{1/2}\right)\right]f_s$$

$$-\Big\{\Big\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$\times\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]g_0$$

$$\left.-\sum_{s=1}^{[\lambda/\tau]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right]f_s\Big)+\varphi\right].$$

If $2\tau<\mu$, $2\tau<\lambda$, then using the condition

$$u_{-N}=\alpha u_{[\mu/\tau]}+\beta\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+\varphi,$$

we obtain the operator equation

$$u_{-N}=\alpha\Big[\Big\{\Big\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$\times\left[R^N u_{-N}+\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]g_0$$

$$\left.+\sum_{s=1}^{[\mu/\tau]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right]f_s\right]$$

$$+\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$\times\left[R^N u_{-N}+\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]g_0$$

$$+\sum_{s=1}^{[\lambda/\tau]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\lambda/\tau]-s}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-s}\left(\tau A^{1/2}\right)\right]f_s$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\Big\}$$

$$\times\left[R^N u_{-N}+\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s\right]$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]g_0$$

$$\left.-\sum_{s=1}^{[\lambda/\tau]-2}\frac{\tau}{2i}A^{-1/2}[R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)]f_s\right)+\varphi.$$

The operator

$$I-\left[\alpha\left\{\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\Big[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\Big]\Big\}R^N$$
$$-\beta\tau^{-1}\Bigg(\Big\{\Big\{\frac{1}{2}\Big[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\Big]$$
$$+\frac{1}{2i}A^{1/2}\Big[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\Big]\Big\}$$
$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\Big[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\Big]\Big\}$$
$$-\Big\{\Big\{\frac{1}{2}\Big[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\Big]$$
$$+\frac{1}{2i}A^{1/2}\Big[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\Big]\Big\}$$
$$+\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\Big[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\Big]\Big\}\Bigg)R^N\Big]$$

has an inverse

$$T_\tau=\Bigg(I-\Big[\alpha\Big\{\Big\{\frac{1}{2}\Big[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\Big]$$
$$+\frac{1}{2i}A^{1/2}\Big[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\Big]\Big\}$$
$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\Big[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\Big]\Big\}R^N$$
$$-\beta\tau^{-1}\Bigg(\Big\{\Big\{\frac{1}{2}\Big[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\Big]$$
$$+\frac{1}{2i}A^{1/2}\Big[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\Big]\Big\}$$
$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\Big[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\Big]\Big\}$$
$$-\Big\{\Big\{\frac{1}{2}\Big[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\Big]$$
$$+\frac{1}{2i}A^{1/2}\Big[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\Big]\Big\}$$
$$+\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\Big[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\Big]\Big\}\Bigg)R^N\Big]\Bigg)^{-1}$$

and the formula holds

$$u_{-N}=T_\tau\Bigg(\alpha\Big[\Big\{\Big\{\frac{1}{2}\Big[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\Big] \tag{4.19}$$
$$+\frac{1}{2i}A^{1/2}\Big[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\Big]\Big\}$$
$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\Big[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\Big]\Big\}$$

$$\times\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]g_0$$

$$\left.+\sum_{s=1}^{[\mu/\tau]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right]f_s\right]$$

$$+\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]g_0$$

$$+\sum_{s=1}^{[\lambda/\tau]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\lambda/\tau]-s}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-s}\left(\tau A^{1/2}\right)\right]f_s$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times\tau\sum_{s=-N+1}^{0}R^{-s+1}g_s$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]g_0$$

$$-\sum_{s=1}^{[\lambda/\tau]-2}\frac{\tau}{2i}A^{-1/2}$$

$$\left.\times\left[R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right]f_s\right)+\varphi\Big).$$

Now, we will obtain estimates (4.8), (4.9) and (4.10). First, we obtain the estimate for $\|u_{-N}\|_H$.

If $\mu\le 2\tau$, $\lambda\le 2\tau$, then using the Abel's formula and formula (4.16), we obtain

$$u_{-N}=T_\tau\left[\left(\alpha-\beta A\right)A^{-1}\left\{-R^Ng_{-N+1}\right.\right. \tag{4.20}$$

$$\left.\left.+\sum_{s=-N+2}^{0}R^{-s+1}\left(g_{s-1}-g_s\right)\right\}+\alpha A^{-1}g_0+\varphi\right].$$

Using formula (4.20), estimates (4.10), (4.7), (4.2) and (4.3) we obtain

$$\|u_{-N}\|_H \le \|T_\tau\|_{H\to H} \Big[|\alpha| \, \|A^{-1}\|_{H\to H} \Big\{ \left\|A^{1/2}R^N\right\|_{H\to H} \left\|A^{-1/2}g_{-N+1}\right\|_H$$

$$+ \sum_{s=-N+2}^{0} \left\|A^{1/2}R^{-s+1}\right\|_{H\to H} \left\|A^{-1/2}(g_{s-1}-g_s)\right\|_H \Big\}$$

$$+|\alpha| \left\|A^{-1/2}\right\|_{H\to H} \left\|A^{-1/2}g_0\right\|_H \Big] + \Big[|\beta| \Big\{ \left\|A^{1/2}R^N\right\|_{H\to H} \left\|A^{-1/2}g_{-N+1}\right\|_H$$

$$+ \sum_{s=-N+2}^{0} \left\|A^{1/2}R^{-s+1}\right\|_{H\to H} \left\|A^{-1/2}(g_{s-1}-g_s)\right\|_H \Big\} + \|\varphi\|_H \Big]$$

$$\le M \left[\sum_{s=-(N-1)}^{-1} \frac{\tau}{\sqrt{(-s+1)}} \left\|A^{-1/2}(g_s - g_{s-1})\tau^{-1}\right\|_H + \|\varphi\|_H + \left\|A^{-1/2}g_0\right\|_H \right]$$

$$\le M \left[\max_{2\le s\le N-1} \left\|A^{-1/2}(g_s - g_{s-1})\tau^{-1}\right\|_H + \|\varphi\|_H + \left\|A^{-1/2}g_0\right\|_H \right]$$

$$\le M \Big[\|\varphi\|_H + \left\|A^{-1/2}f_1\right\|_H + \max_{2\le k\le N-1} \left\|A^{-1/2}(f_k - f_{k-1})\tau^{-1}\right\|_H$$

$$+ \left\|A^{-1/2}g_0\right\|_H + \max_{-(N-1)\le k\le 0} \left\|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\right\|_H \Big]. \tag{4.21}$$

If $2\tau < \mu$, $\lambda \le 2\tau$, then using the Abel's formula and formula (4.17), we obtain

$$u_{-N} = T_\tau \Big(\alpha \Big[\Big\{ \Big\{ \frac{1}{2} \left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right) \right] \tag{4.22}$$

$$+\frac{1}{2i} A^{1/2} \left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right) \right] \Big\}$$

$$-\frac{1}{2i} A^{1/2} (I+\tau^2 A) \left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right) \right] \Big\}$$

$$\times A^{-1} \left[g_0 - R^N g_{-N+1} + \sum_{s=-N+2}^{0} R^{-s+1}[g_{s-1} - g_s] \right]$$

$$+\frac{1}{2i} A^{-1/2} (I+\tau^2 A) \left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right) - R^{[\mu/\tau]}\left(\tau A^{1/2}\right) \right] g_0$$

$$+A^{-1} \Big[f_{[\mu/\tau]-1} - \frac{1}{2} \left[R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right) + R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right) \right] f_1$$

$$+ \sum_{s=2}^{[\mu/\tau]-1} \frac{1}{2} \left[R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right) + R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right) \right] [f_{s-1} - f_s] \Big]$$

$$-\beta \left[-R^N g_{-N+1} + \sum_{s=-N+2}^{0} R^{-s+1}[g_{s-1} - g_s] \right] + \varphi \Big).$$

Using formula (4.21), estimates (4.10), (4.13), (4.2) and (4.3) we obtain

$$\|u_{-N}\|_H \leq \|T_\tau\|_{H\to H}\left(|\alpha|\left[\left\{\left\{\left\|A^{-1/2}\right\|_{H\to H}\frac{1}{2}\left[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right.\right.\right.\right.\right.$$

$$\left.+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.+\frac{1}{2}\left[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\left.+\frac{1}{2}\left[\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\times\left[\left\|A^{-1/2}g_0\right\|_H+\left\|R^N\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_H\right.$$

$$\left.+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1}-g_s\right]\right\|_H\right]$$

$$+\frac{1}{2}\left[\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\times\left\|A^{-1/2}g_0\right\|_H+\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|A^{-1/2}f_{[\mu/\tau]-1}\right\|_H\right.$$

$$+\frac{1}{2}\left[\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\times\left\|A^{-1/2}f_1\right\|_H+\sum_{s=2}^{[\mu/\tau]-1}\frac{1}{2}\left[\left\|R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{-1/2}\left[f_{s-1}-f_s\right]\right\|_H\right]$$

$$+|\beta|\left[\left\|A^{1/2}R^N\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_H\right.$$

$$\left.\left.+\sum_{s=-N+2}^{0}\left\|A^{1/2}R^{-s+1}\right\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1}-g_s\right]\right\|_H\right]+\|\varphi\|_H\right)$$

$$\leq M\left[\sum_{s=-(N-1)}^{-1}\frac{\tau}{\sqrt{(-s+1)}}\left\|A^{-1/2}\left(g_s-g_{s-1}\right)\tau^{-1}\right\|_H+\|\varphi\|_H+\left\|A^{-1/2}g_0\right\|_H\right]$$

$$+\left[\left\|A^{-1/2}f_{[\mu/\tau]-1}\right\|_H+\left\|A^{-1/2}f_1\right\|_H+\sum_{s=2}^{[\mu/\tau]-1}\left\|A^{-1/2}\left[f_{s-1}-f_s\right]\right\|_H\right]$$

$$\leq M\left[\max_{2\leq s\leq N-1}\left\|A^{-1/2}\left(g_s-g_{s-1}\right)\tau^{-1}\right\|_H+\|\varphi\|_H+\left\|A^{-1/2}g_0\right\|_H\right]$$

$$\leq M\left[\|\varphi\|_H+\left\|A^{-1/2}f_1\right\|_H+\max_{2\leq k\leq N-1}\left\|A^{-1/2}\left(f_k-f_{k-1}\right)\tau^{-1}\right\|_H\right.$$

$$\left.+\left\|A^{-1/2}g_0\right\|_H+\max_{-(N-1)\leq k\leq 0}\left\|A^{-1/2}\left(g_k-g_{k-1}\right)\tau^{-1}\right\|_H\right]. \tag{4.23}$$

If $\mu \leq 2\tau$, $2\tau < \lambda$, then using the Abel's formula and formula (4.18), we obtain

$$u_{-N}=T_\tau\left[\alpha A^{-1}\left[g_0-R^N g_{-N+1}+\sum_{s=-N+2}^{0}R^{-s+1}\left[g_{s-1}-g_s\right]\right]\right. \tag{4.24}$$

$$+\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times A^{-1}\left[-R^N g_{-N+1}+\sum_{s=-N+2}^{0}R^{-s+1}\left[g_{s-1}-g_s\right]\right]$$

$$+A^{-1}\left[f_{[\lambda/\tau]-1}-\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]f_1\right.$$

$$\left.+\sum_{s=2}^{[\lambda/\tau]-1}\frac{1}{2}\left[R^{[\lambda/\tau]-s}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-s}\left(\tau A^{1/2}\right)\right]\left[f_{s-1}-f_s\right]\right]$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times A^{-1}\left[-R^N g_{-N+1}+\sum_{s=-N+2}^{0}R^{-s+1}\left[g_{s-1}-g_s\right]\right]$$

$$-A^{-1}\left[f_{[\lambda/\tau]-2}-\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right]f_1\right.$$

$$\left.\left.+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}\left[R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right]\left[f_{s-1}-f_s\right]\right]+\varphi\right].$$

Using formula (4.24), estimates (4.10), (4.13), (4.2) and (4.3) we obtain

$$\|u_{-N}\|_H\leq\|T_\tau\|_{H\to H}\left[|\alpha|\left\|A^{-1/2}\right\|_{H\to H}\right.$$

$$\times\left[\left\|A^{-1/2}g_0\right\|_H+\|R^N\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_H\right.$$

$$\left.+\sum_{s=-N+2}^{0}\|R^{-s+1}\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1}-g_s\right]\right\|_H\right]$$

$$+|\beta|\left(|i|\left\{\left\{\frac{1}{2}\left[\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right.\right.\right.$$

$$\left.+\frac{1}{2|i|}\left[\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$+\frac{1}{2\left|i\right|}\left[\left\|\left(I+\tau^{2}A\right)R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^{2}A\right)R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\Bigg\}$$

$$\times\left[\left\|R^{N}\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_{H}+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1}-g_{s}\right]\right\|_{H}\right]$$

$$+\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\left[f_{[\lambda/\tau]-1}-f_{[\lambda/\tau]-2}\right]\right\|_{H\to H}\right.$$

$$+\frac{1}{2}\left|i\right|\left[\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{-1/2}f_{1}\right\|$$

$$+\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|R\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\times\tau^{-1}\left\|A^{-1/2}\left[f_{[\lambda/\tau]-2}-f_{[\lambda/\tau]-1}\right]\right\|_{H}$$

$$+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}\left|i\right|\left[\left\|R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.\times\left\|A^{-1/2}\left[f_{s-1}-f_{s}\right]\right\|_{H}\right]\Bigg)+\left\|\varphi\right\|_{H}\Bigg].$$

$$\leq M\left[\sum_{s=-(N-1)}^{-1}\frac{\tau}{\sqrt{(-s+1)}}\left\|A^{-1/2}\left(g_{s}-g_{s-1}\right)\tau^{-1}\right\|_{H}+\left\|\varphi\right\|_{H}+\left\|A^{-1/2}g_{0}\right\|_{H}\right.$$

$$+\left\|A^{-1/2}f_{[\mu/\tau]-1}\right\|_{H}+\left\|A^{-1/2}f_{1}\right\|_{H}$$

$$\left.+\sum_{s=2}^{[\mu/\tau]-1}\left\|A^{-1/2}\left[f_{s-1}-f_{s}\right]\right\|_{H}\right]$$

$$\leq M\left[\left\|\varphi\right\|_{H}+\left\|A^{-1/2}f_{1}\right\|_{H}+\max_{2\leq k\leq N-1}\left\|A^{-1/2}\left(f_{k}-f_{k-1}\right)\tau^{-1}\right\|_{H}\right.$$

$$\left.+\left\|A^{-1/2}g_{0}\right\|_{H}+\max_{-(N-1)\leq k\leq 0}\left\|A^{-1/2}\left(g_{k}-g_{k-1}\right)\tau^{-1}\right\|_{H}\right]. \tag{4.25}$$

If $2\tau<\mu$, $2\tau<\lambda$, then using the Abel's formula and formula (4.19), we obtain

$$u_{-N}=T_{\tau}\left(\alpha\left[\left\{\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.\right. \tag{4.26}$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^{2}A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times A^{-1}\left[-R^{N}g_{-N+1}+\sum_{s=-N+2}^{0}R^{-s+1}\left[g_{s-1}-g_{s}\right]\right]$$

$$+A^{-1}\left[f_{[\mu/\tau]-1}-\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right]f_{1}\right.$$

$$+\sum_{s=2}^{[\mu/\tau]-1}\frac{1}{2}\left[R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)+R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right][f_{s-1}-f_s]\Bigg]$$

$$+\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times A^{-1}\left[-R^N g_{-N+1}+\sum_{s=-N+2}^{0}R^{-s+1}[g_{s-1}-g_s]\right]$$

$$+A^{-1}\Bigg[f_{[\lambda/\tau]-1}-\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]f_1$$

$$+\sum_{s=2}^{[\lambda/\tau]-1}\frac{1}{2}\left[R^{[\lambda/\tau]-s}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-s}\left(\tau A^{1/2}\right)\right][f_{s-1}-f_s]\Bigg]$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times A^{-1}\left[-R^N g_{-N+1}+\sum_{s=-N+2}^{0}R^{-s+1}[g_{s-1}-g_s]\right]$$

$$-A^{-1}\Bigg[f_{[\lambda/\tau]-2}-\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right]f_1$$

$$\left.+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}\left[R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right][f_{s-1}-f_s]\Bigg]+\varphi\right).$$

Using formula (4.26), estimates (4.10), (4.13), (4.2) and (4.3) we obtain

$$\|u_{-N}\|_H\le\|T_\tau\|_{H\to H}\left(|\alpha|\left[\left\{\left\{\frac{1}{2}\left[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right.\right.\right.\right.\right.$$

$$\left.+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.+\frac{1}{2|i|}\left[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\times\left[\left\|R^N\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_H+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\left\|A^{-1/2}[g_{s-1}-g_s]\right\|_H\right]$$

$$\frac{1}{2|i|}\left\|A^{1/2}\right\|_{H\to H}\left[\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.$$

$$
\begin{aligned}
&+\left\|\left(I+\tau^2 A\right) R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\Big\}\Big[\left\|A^{1/2}R^{N}\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_{H}\\
&\quad+\sum_{s=-N+2}^{0}\left\|A^{1/2}R^{-s+1}\right\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1}-g_{s}\right]\right\|_{H}\Big]\\
&+\left\|A^{-1/2}\right\|_{H\to H}\Big[\left\|A^{-1/2}f_{[\mu/\tau]-1}\right\|_{H}+\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\\
&\quad+\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\left\|A^{-1/2}f_{1}\right\|_{H}\\
&+\sum_{s=2}^{[\mu/\tau]-1}\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\\
&\quad\times\left\|A^{-1/2}\left[f_{s-1}-f_{s}\right]\right\|_{H}\Big]+|\beta|\Big(\Big\{\Big\{\frac{1}{2}|i|\left\|A^{-1/2}\right\|_{H\to H}\\
&\quad\times\Big[\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]\\
&\quad+\frac{1}{2}\Big[\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]\Big\}\\
&+\frac{1}{2}\Big[\left\|\left(I+\tau^2 A\right)R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2 A\right)R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\Big\}\\
&\quad\times\Big[\left\|A^{1/2}R^{N}\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_{H}\\
&\quad+\sum_{s=-N+2}^{0}\left\|A^{1/2}R^{-s+1}\right\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1}-g_{s}\right]\right\|_{H\to H}\Big]\\
&\quad+\Big[\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{-1/2}\tau^{-1}\left[f_{[\lambda/\tau]-1}-f_{[\lambda/\tau]-2}\right]\right\|_{H\to H}\\
&+\frac{1}{2}|i|\Big[\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\left\|A^{-1/2}f_{1}\right\|_{H\to H}\\
&+\Big[\left\|R\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\left\|A^{-1/2}\tau^{-1}\left[f_{[\lambda/\tau]-2}-f_{[\lambda/\tau]-1}\right]\right\|_{H}\\
&+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}|i|\Big[\left\|R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\\
&\quad\left\|A^{-1/2}\left[f_{s-1}-f_{s}\right]\right\|_{H\to H}\Big]+\|\varphi\|_{H}\Big)\\
&\leq M\Big[\sum_{s=-(N-1)}^{-1}\frac{\tau}{\sqrt{(-s+1)}}\left\|A^{-1/2}\left(g_{s}-g_{s-1}\right)\tau^{-1}\right\|_{H}+\|\varphi\|_{H}+\left\|A^{-1/2}g_{0}\right\|_{H}\\
&\quad+\left\|A^{-1/2}f_{[\mu/\tau]-1}\right\|_{H}+\left\|A^{-1/2}f_{1}\right\|_{H}+\sum_{s=2}^{[\mu/\tau]-1}\left\|A^{-1/2}\left[f_{s-1}-f_{s}\right]\right\|_{H}\Big]\\
&\leq M\Big[\|\varphi\|_{H}+\left\|A^{-1/2}f_{1}\right\|_{H}+\max_{2\leq k\leq N-1}\left\|A^{-1/2}\left(f_{k}-f_{k-1}\right)\tau^{-1}\right\|_{H}\\
&\quad+\left\|A^{-1/2}g_{0}\right\|_{H}+\max_{-(N-1)\leq k\leq 0}\left\|A^{-1/2}\left(g_{k}-g_{k-1}\right)\tau^{-1}\right\|_{H}\Big]. \qquad (4.27)
\end{aligned}
$$

Finally, applying estimates (4.21), (4.23), (4.25) and (4.27), we obtain

$$\|u_{-N}\|_H \leq M\left[\|\varphi\|_H + \left\|A^{-1/2}f_1\right\|_H + \max_{2\leq k\leq N-1}\left\|A^{-1/2}(f_k - f_{k-1})\tau^{-1}\right\|_H\right.$$

$$\left. + \left\|A^{-1/2}g_0\right\|_H + \max_{-(N-1)\leq k\leq 0}\left\|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\right\|_H\right]. \tag{4.28}$$

Now, we will obtain the estimate for $\|u_k\|_H$, $-N+1 \leq k \leq N$. Let $-(N-1) \leq k \leq 0$. Then using the Abel's formula and formula (4.14), we obtain

$$u_k = R^{N+k}u_{-N} \tag{4.29}$$

$$+A^{-1}\left[g_k - R^{N+k-1}g_{-N+1} + \sum_{s=-N+2}^{k} R^{k-s+1}\left[g_{s-1} - g_s\right]\right].$$

Using formula (4.29) and estimate (4.3) we obtain

$$\|u_k\|_H \leq \left\|R^{N+k}\right\|_{H\to H}\|u_{-N}\|_H$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|A^{-1/2}g_k\right\|_H + \left\|R^{N+k-1}\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_H\right.$$

$$\left. + \sum_{s=-N+2}^{k}\left\|R^{k-s+1}\right\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1} - g_s\right]\right\|_H\right]$$

$$\leq \|u_{-N}\|_H + 2\left\|A^{-1/2}g_0\right\|_H + \max_{-(N-1)\leq k\leq 0}\left\|A^{-1/2}(g_k - g_{k-1})\tau^{-1}\right\|_H. \tag{4.30}$$

Let $1 \leq k \leq N$. Then using the Abel's formula and formula (4.15), we obtain

$$\begin{cases} u_1 = (I-\tau A)\left(R^N u_{-N} + A^{-1}\left[g_0 - R^N g_{-N+1}\right.\right. \\ \left.\left. + \sum_{s=-N+2}^{0} R^{-s+1}\left[g_{s-1} - g_s\right]\right]\right) + \tau g_0, \\ u_k = \left\{\left\{\frac{1}{2}\left[R^{k-1}\left(\tau A^{1/2}\right) + R^{k-1}\left(-\tau A^{1/2}\right)\right]\right.\right. \\ \left. +\frac{1}{2i}A^{1/2}\left[R^{k-1}\left(\tau A^{1/2}\right) + R^{k-1}\left(-\tau A^{1/2}\right)\right]\right\} \\ \left. -\frac{1}{2i}A^{1/2}\left(I+\tau^2 A\right)\left[R^k\left(-\tau A^{1/2}\right) - R^k\left(\tau A^{1/2}\right)\right]\right\} \\ \times\left[R^N u_{-N} + A^{-1}\left[-R^N g_{-N+1} + \sum_{s=-N+2}^{0} R^{-s+1}\left[g_{s-1} - g_s\right]\right]\right] \\ +A^{-1}\left[f_{k-1} - \frac{1}{2}\left[R^{k-1}\left(-\tau A^{1/2}\right) + R^{k-1}\left(\tau A^{1/2}\right)\right]f_1\right. \\ \left. + \sum_{s=2}^{k-1}\frac{1}{2}\left[R^{k-s}\left(-\tau A^{1/2}\right) + R^{k-s}\left(\tau A^{1/2}\right)\right]\left[f_{s-1} - f_s\right]\right], 2 \leq k \leq N. \end{cases} \tag{4.31}$$

Using formula (4.29) and estimates (4.2), (4.3) we obtain

$$\|u_1\|_H \leq \left[\left\|(I-\tau A)R^N\right\|_{H\to H}\|u_{-N}\|_H + \left\|A^{-1/2}\right\|_{H\to H}\right.$$

$$\left[\left\|(I-\tau A)R^{N}\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_{H}\right.$$

$$\left.\left.+\sum_{s=-N+2}^{0}\left\|(I-\tau A)R^{-s+1}\right\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1}-g_{s}\right]\right\|_{H}\right]\right]$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{-1/2}g_{0}\right\|_{H}$$

$$\leq M\left[\|u_{-N}\|_{H}+\left\|A^{-1/2}g_{0}\right\|_{H}+\max_{-(N-1)\leq k\leq 0}\left\|A^{-1/2}\left(g_{k}-g_{k-1}\right)\tau^{-1}\right\|_{H}\right]. \tag{4.32}$$

$$\|u_{k}\|_{H}\leq\left\{\left\{\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right.\right.$$

$$\left.+\frac{1}{2|i|}\left[\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\left.+\frac{1}{2|i|}\left[\left\|\left(I+\tau^{2}A\right)R^{k}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^{2}A\right)R^{k}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\times\left[\left\|A^{1/2}R^{N}\right\|_{H\to H}\|u_{-N}\|_{H}+\left[\left\|R^{N}\right\|_{H\to H}\left\|A^{-1/2}g_{-N+1}\right\|_{H}\right.\right.$$

$$\left.\left.+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\left\|A^{-1/2}\left[g_{s-1}-g_{s}\right]\right\|_{H}\right]\right]$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|A^{-1/2}f_{k-1}\right\|_{H}+\frac{1}{2}\left[\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{-1/2}f_{1}\right\|_{H}$$

$$\left.+\sum_{s=2}^{k-1}\frac{1}{2}\left[\left\|R^{k-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{-1/2}\left[f_{s-1}-f_{s}\right]\right\|_{H}\right]$$

$$\leq M\left[\|u_{-N}\|_{H}+\left\|A^{-1/2}f_{1}\right\|_{H}+\max_{2\leq k\leq N-1}\left\|A^{-1/2}\left(f_{k}-f_{k-1}\right)\tau^{-1}\right\|_{H}\right.$$

$$\left.+\left\|A^{-1/2}g_{0}\right\|_{H}+\max_{-(N-1)\leq k\leq 0}\left\|A^{-1/2}\left(g_{k}-g_{k-1}\right)\tau^{-1}\right\|_{H}\right],2\leq k\leq N. \tag{4.33}$$

Combining the estimates (4.28), (4.30), (4.32), (4.33) and using the triangle inequality we obtain estimate (4.8).

Second, we obtain the estimate for $\left\|A^{1/2}u_{-N}\right\|_{H}$. If $\mu\leq 2\tau$, $\lambda\leq 2\tau$, then using formula (4.20), estimates (4.10), (4.7), (4.2), and (4.3) we obtain

$$\left\|A^{1/2}u_{-N}\right\|_{H}\leq\|T_{\tau}\|_{H\to H}\left[|\alpha|\left\|A^{-1}\right\|_{H\to H}\left\{\left\|A^{1/2}R^{N}\right\|_{H\to H}\|g_{-N+1}\|_{H}\right.\right.$$

$$\left.+\sum_{s=-N+2}^{0}\left\|A^{1/2}R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_{s}\|_{H}\right\}$$

$$\left.+|\alpha|\left\|A^{-1/2}\right\|_{H\to H}\|g_{0}\|_{H}\right]+\left[|\beta|\left\{\left\|A^{1/2}R^{N}\right\|_{H\to H}\|g_{-N+1}\|_{H}\right.\right.$$

$$+\sum_{s=-N+2}^{0}\left\|A^{1/2}R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\Bigg\}+\left\|A^{1/2}\varphi\right\|_H\Bigg]$$

$$\leq M\left[\sum_{s=-(N-1)}^{-1}\frac{\tau}{\sqrt{(-s+1)}}\left\|(g_s-g_{s-1})\,\tau^{-1}\right\|_H+\left\|A^{1/2}\varphi\right\|_H+\|g_0\|_H\right]$$

$$\leq M\left[\max_{2\leq s\leq N-1}\left\|(g_s-g_{s-1})\,\tau^{-1}\right\|_H+\left\|A^{1/2}\varphi\right\|_H+\|g_0\|_H\right]$$

$$\leq M\Bigg[\left\|A^{1/2}\varphi\right\|_H+\|f_1\|_H+\max_{2\leq k\leq N-1}\left\|(f_k-f_{k-1})\,\tau^{-1}\right\|_H$$

$$+\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|(g_k-g_{k-1})\,\tau^{-1}\right\|_H\Bigg]. \qquad (4.34)$$

If $2\tau<\mu$, $\lambda\leq 2\tau$, then using formula (4.21), estimates (4.10), (4.13), (4.2), and (4.3) we obtain

$$\left\|A^{1/2}u_{-N}\right\|_H\leq\|T_\tau\|_{H\to H}\Bigg(|\alpha|\Bigg[\Bigg\{\Bigg\{\left\|A^{-1/2}\right\|_{H\to H}\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}$$

$$+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]$$

$$+\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]\Bigg\}$$

$$+\frac{1}{2}\Big[\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]\Bigg\}$$

$$\times\Big[\|g_0\|_H+\left\|R^N\right\|_{H\to H}\|g_{-N+1}\|_H$$

$$+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\Big]$$

$$+\frac{1}{2}\Big[\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]$$

$$\times\|g_0\|_H+\left\|A^{-1/2}\right\|_{H\to H}\Big[\left\|f_{[\mu/\tau]-1}\right\|_H$$

$$+\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]$$

$$\times\|f_1\|_H+\sum_{s=2}^{[\mu/\tau]-1}\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}$$

$$+\left\|R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\|f_{s-1}-f_s\|_H\Big]$$

$$+|\beta|\Big[\left\|A^{1/2}R^N\right\|_{H\to H}\|g_{-N+1}\|_H$$

$$+\sum_{s=-N+2}^{0}\left\|A^{1/2}R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\Big]+\left\|A^{1/2}\varphi\right\|_H\Bigg)$$

$$\leq M\left[\sum_{s=-(N-1)}^{-1}\frac{\tau}{\sqrt{(-s+1)}}\left\|(g_s-g_{s-1})\tau^{-1}\right\|_H+\left\|A^{1/2}\varphi\right\|_H+\|g_0\|_H\right]$$

$$+\left[\left\|f_{[\mu/\tau]-1}\right\|_H+\|f_1\|_H+\sum_{s=2}^{[\mu/\tau]-1}\|f_{s-1}-f_s\|_H\right]$$

$$\leq M\left[\max_{2\leq s\leq N-1}\left\|(g_s-g_{s-1})\tau^{-1}\right\|_H+\left\|A^{1/2}\varphi\right\|_H+\|g_0\|_H\right]$$

$$\leq M\left[\left\|A^{1/2}\varphi\right\|_H+\|f_1\|_H+\max_{2\leq k\leq N-1}\left\|(f_k-f_{k-1})\tau^{-1}\right\|_H\right.$$

$$\left.+\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|(g_k-g_{k-1})\tau^{-1}\right\|_H\right]. \quad (4.35)$$

If $\mu\leq 2\tau$, $2\tau<\lambda$, then using formula (4.24), estimates (4.10), (4.13), (4.2) and (4.3) we obtain

$$\left\|A^{1/2}u_{-N}\right\|_H\leq\|T_\tau\|_{H\to H}\left[|\alpha|\left\|A^{-1/2}\right\|_{H\to H}\right.$$

$$\times\left[\|g_0\|_H+\left\|R^N\right\|_{H\to H}\|g_{-N+1}\|_H\right.$$

$$\left.+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\right]$$

$$+|\beta|\left(|i|\left\{\left\{\frac{1}{2}\left[\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right.\right.\right.$$

$$\left.+\frac{1}{2|i|}\left[\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\left.+\frac{1}{2|i|}\left[\left\|\left(I+\tau^2A\right)R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\times\left[\left\|R^N\right\|_{H\to H}\|g_{-N+1}\|_H+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\right]$$

$$+\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|\tau^{-1}\left[f_{[\lambda/\tau]-1}-f_{[\lambda/\tau]-2}\right]\right\|_{H\to H}\right.$$

$$+\frac{1}{2}|i|\left[\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\|f_1\|_H$$

$$+\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|R\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\times\tau^{-1}\left\|\left[f_{[\lambda/\tau]-2}-f_{[\lambda/\tau]-1}\right]\right\|_H$$

$$+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}|i|\left[\left\|R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.\times\|f_{s-1}-f_s\|_H]\right)+\left\|A^{1/2}\varphi\right\|_H\Big]$$

$$\leq M\left[\sum_{s=-(N-1)}^{-1}\frac{\tau}{\sqrt{(-s+1)}}\left\|(g_s-g_{s-1})\tau^{-1}\right\|_H+\left\|A^{1/2}\varphi\right\|_H+\|g_0\|_H\right.$$

$$+\left\|f_{[\mu/\tau]-1}\right\|_H+\|f_1\|_H+\sum_{s=2}^{[\mu/\tau]-1}\|f_{s-1}-f_s\|_H\Bigg]$$

$$\leq M\left[\max_{2\leq s\leq N-1}\left\|(g_s-g_{s-1})\,\tau^{-1}\right\|_H+\left\|A^{1/2}\varphi\right\|_H+\|g_0\|_H\right.$$

$$\left.+\|f_1\|_H+\max_{2\leq k\leq N-1}\left\|(f_k-f_{k-1})\,\tau^{-1}\right\|_H\right].\tag{4.36}$$

If $2\tau<\mu$, $2\tau<\lambda$, then using formula (4.26), estimates (4.10), (4.13), (4.2) and (4.3) we obtain

$$\left\|A^{1/2}u_{-N}\right\|_H\leq\|T_\tau\|_{H\to H}\left(|\alpha|\left[\left\{\left\{\frac{1}{2}\left[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right.\right.\right.\right.\right.$$

$$\left.+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.+\frac{1}{2\,|i|}\left[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\times\left[\left\|R^N\right\|_{H\to H}\|g_{-N+1}\|_H+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\right]$$

$$\frac{1}{2\,|i|}\left\|A^{1/2}\right\|_{H\to H}\left[\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}\left[\left\|A^{1/2}R^N\right\|_{H\to H}\|g_{-N+1}\|_H\right.$$

$$\left.+\sum_{s=-N+2}^{0}\left\|A^{1/2}R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\right]$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|f_{[\mu/\tau]-1}\right\|_H+\frac{1}{2}\left[\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\|f_1\|_H$$

$$+\sum_{s=2}^{[\mu/\tau]-1}\frac{1}{2}\left[\left\|R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.\times\|f_{s-1}-f_s\|_H\right]+|\beta|\left(\left\{\left\{\frac{1}{2}\,|i|\left\|A^{-1/2}\right\|_{H\to H}\right.\right.\right.$$

$$\times\left[\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.+\frac{1}{2}\left[\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\left.+\frac{1}{2}\left[\left\|\left(I+\tau^2A\right)R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\times\left[\left\|A^{1/2}R^N\right\|_{H\to H}\|g_{-N+1}\|_H\right.$$

$$\left.+\sum_{s=-N+2}^{0}\left\|A^{1/2}R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_{H\to H}\right]$$

$$+\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|\tau^{-1}\left[f_{[\lambda/\tau]-1}-f_{[\lambda/\tau]-2}\right]\right\|_{H\to H}\right.$$

$$+\frac{1}{2}|i|\left[\left\|R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\|f_1\|_{H\to H}$$

$$+\left[\left\|R\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|\tau^{-1}\left[f_{[\lambda/\tau]-2}-f_{[\lambda/\tau]-1}\right]\right\|_{H}$$

$$+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}|i|\left[\left\|R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.\|f_{s-1}-f_s\|_{H\to H}\right]+\left\|A^{1/2}\varphi\right\|_{H}\Bigg)$$

$$\leq M\left[\sum_{s=-(N-1)}^{-1}\frac{\tau}{\sqrt{(-s+1)}}\left\|(g_s-g_{s-1})\tau^{-1}\right\|_H+\left\|A^{1/2}\varphi\right\|_H+\|g_0\|_H\right.$$

$$\left.+\left\|f_{[\mu/\tau]-1}\right\|_H+\|f_1\|_H+\sum_{s=2}^{[\mu/\tau]-1}\|f_{s-1}-f_s\|_H\right]$$

$$\leq M\left[\left\|A^{1/2}\varphi\right\|_H+\|f_1\|_H+\max_{2\leq k\leq N-1}\left\|(f_k-f_{k-1})\tau^{-1}\right\|_H\right.$$

$$\left.+\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|(g_k-g_{k-1})\tau^{-1}\right\|_H\right]. \tag{4.37}$$

Finally, applying estimates (4.34), (4.35), (4.36) and (4.37), we obtain

$$\left\|A^{1/2}u_{-N}\right\|_H\leq M\left[\left\|A^{1/2}\varphi\right\|_H+\|f_1\|_H+\max_{2\leq k\leq N-1}\left\|(f_k-f_{k-1})\tau^{-1}\right\|_H\right.$$

$$\left.+\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|(g_k-g_{k-1})\tau^{-1}\right\|_H\right]. \tag{4.38}$$

Now, we will obtain the estimate for $\left\|A^{1/2}u_k\right\|_H$, $-N+1\leq k\leq N$. Let $-(N-1)\leq k\leq 0$. Then using the formula (4.29) and estimate (4.3) we obtain

$$\left\|A^{1/2}u_k\right\|_H\leq\left\|R^{N+k}\right\|_{H\to H}\left\|A^{1/2}u_{-N}\right\|_H$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\|g_k\|_H+\left\|R^{N+k-1}\right\|_{H\to H}\|g_{-N+1}\|_H\right.$$

$$\left.+\sum_{s=-N+2}^{k}\left\|R^{k-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\right]$$

$$\leq\left\|A^{1/2}u_{-N}\right\|_H+2\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|(g_k-g_{k-1})\tau^{-1}\right\|_H\Bigg]. \tag{4.39}$$

Let $1\leq k\leq N$. Then using the formula (4.29) and estimates (4.2), (4.3) we obtain

$$\left\|A^{1/2}u_1\right\|_H\leq\left[\left\|(I-\tau A)R^N\right\|_{H\to H}\left\|A^{1/2}u_{-N}\right\|_H+\left\|A^{-1/2}\right\|_{H\to H}\right.$$

$$\left[\left\|(I-\tau A)R^N\right\|_{H\to H}\|g_{-N+1}\|_H\right.$$

$$+\sum_{s=-N+2}^{0}\left\|(I-\tau A)R^{-s+1}\right\|_{H\to H}\|g_{s-1}-g_s\|_H\Big]\Big]$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\|g_0\|_H$$

$$\leq M\left[\left\|A^{1/2}u_{-N}\right\|_H+\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|(g_k-g_{k-1})\tau^{-1}\right\|_H\right]. \quad (4.40)$$

$$\left\|A^{1/2}u_k\right\|_H\leq\left\{\left\{\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right.\right.$$

$$\left.+\frac{1}{2|i|}\left[\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\left.+\frac{1}{2|i|}\left[\left\|(I+\tau^2A)R^k\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|(I+\tau^2A)R^k\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\times\Big[\left\|A^{1/2}R^N\right\|_{H\to H}\left\|A^{1/2}u_{-N}\right\|_H+\Big[\left\|R^N\right\|_{H\to H}\|g_{-N+1}\|_H$$

$$+\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\|[g_{s-1}-g_s]\|_H\Big]\Big]$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\|f_{k-1}\|_H+\frac{1}{2}\left[\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\|f_1\|_H$$

$$\left.+\sum_{s=2}^{k-1}\frac{1}{2}\left[\left\|R^{k-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\|f_{s-1}-f_s\|_H\right]$$

$$\leq M\left[\left\|A^{1/2}u_{-N}\right\|_H+\|f_1\|_H+\max_{2\leq k\leq N-1}\left\|(f_k-f_{k-1})\tau^{-1}\right\|_H\right.$$

$$\left.+\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|(g_k-g_{k-1})\tau^{-1}\right\|_H\right],\ 2\leq k\leq N. \quad (4.41)$$

Combining the estimates (4.38), (4.37), (4.39), (4.40), (4.41) and using the triangle inequality we obtain estimate (4.9).

Third, we obtain the estimate for $\|Au_{-N}\|_H$. If $\mu\leq 2\tau$, $\lambda\leq 2\tau$, then using the Abel's formula and formula (4.20), we obtain

$$Au_{-N}=T_\tau\Big[-(\alpha-\beta A)R^Ng_{-N+1}$$

$$-\beta\tau^{-1}\left[g_{-1}-g_0-R^{N-1}(g_{-N+1}-g_{-N+2})+\sum_{s=-N+2}^{-1}R^{-s}(g_{s+1}-2g_s+g_{s-1})\right]$$

$$\left.+\alpha\sum_{s=-N+2}^{0}R^{-s+1}(g_{s-1}-g_s)+\alpha g_0+A\varphi\right].$$

Using the last formula, estimates (4.10), (4.7), (4.2) and (4.3) we obtain

$$\|Au_{-N}\|_H\leq\|T_\tau\|_{H\to H}\Big[\left\|(\alpha-\beta A)R^N\right\|_{H\to H}\|g_{-N+1}\|_H$$

$$+\left|\alpha\right|\left[\sum_{s=-N+2}^{0}\left\|R^{-s+1}\right\|_{H\to H}\left\|g_{s-1}-g_{s}\right\|_{H}+\left\|g_{0}\right\|_{H}\right]$$

$$+|\beta|\tau^{-1}\left[\left\|R^{N-1}\right\|_{H\to H}\left\|g_{-N+1}-g_{-N+2}\right\|_{H}+\left\|g_{-1}-g_{0}\right\|_{H}\right]$$

$$+|\beta|\sum_{s=-N+2}^{-1}\left\|R^{-s+1}\right\|_{H\to H}\tau^{-1}\left\|(g_{s+1}-2g_{s}-g_{s-1})\right\|_{H}+\left\|A\varphi\right\|_{H}\Bigg]$$

$$\leq M\left[\left\|A\varphi\right\|_{H}+\left\|A^{1/2}g_{0}\right\|_{H}\right.$$

$$\left.+\left\|(g_{0}-g_{-1})\tau^{-1}\right\|_{H}+\max_{-(N-1)\leq k\leq -1}\left\|(g_{k+1}-2g_{k}+g_{k-1})\tau^{-2}\right\|_{H}\right]. \qquad (4.42)$$

If $2\tau<\mu$, $\lambda\leq 2\tau$, then using the Abel's formula and formula (4.17), we obtain

$$Au_{-N}=T_{\tau}\Bigg(\alpha\Bigg[\Bigg\{\Bigg\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^{2}A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Bigg\}$$

$$\times\Bigg[-R^{N}g_{-N+1}+(\tau A)^{-1}\Big[(g_{-1}-g_{0})-R^{N-1}\left(g_{-N+1}-g_{-N+2}\right)$$

$$+\sum_{s=-N+2}^{-1}R^{-s}\left(g_{s+1}-2g_{s}+g_{s-1}\right)\Big]\Bigg]$$

$$+\alpha\Bigg[\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$

$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg]g_{0}$$

$$+\Bigg[f_{[\mu/\tau]-1}-\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right]f_{1}$$

$$+\sum_{s=2}^{[\mu/\tau]-1}\frac{1}{2}\left[R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)+R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right]\left[f_{s-1}-f_{s}\right]\Bigg]$$

$$-\beta\Big\{-R^{N}Ag_{-N+1}+\tau^{-1}\Big[(g_{-1}-g_{0})-R^{N-1}\left(g_{-N+1}-g_{-N+2}\right)$$

$$+\sum_{s=-N+2}^{-1}R^{-s}\left(g_{s+1}-2g_{s}+g_{s-1}\right)\Big]\Big\}+A\varphi\Bigg).$$

Using formula (4.21), estimates (4.10), (4.13), (4.2) and (4.3) we obtain

$$\left\|Au_{-N}\right\|_{H}\leq\left\|T_{\tau}\right\|_{H\to H}\Bigg(|\alpha|\Bigg[\Bigg\{\Bigg\{\left\|A^{-1/2}\right\|_{H\to H}\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}$$

$$+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]$$

$$+\frac{1}{2}\left[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\Big\}$$

$$+\frac{1}{2}\left[\left\|\left(I+\tau^2 A\right)R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2 A\right)R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\Big\}$$

$$\times\left\|A^{1/2}R^N\right\|_{H\to H}\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g_0\right\|_H+\|g_{-N+1}-g_0\|_H\right]$$

$$+\left\|A^{-1}\right\|_{H\to H}\left[\left[1+\left\|R^{N-1}\right\|_{H\to H}\right]\left\|\tau^{-1}\left(g_{-1}-g_0\right)\right\|_H\right]$$

$$+\left\|R^{N-1}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{-1}-g_0-g_{-N+1}+g_{-N+2}\right)\right\|_H$$

$$\left.+\sum_{s=-N+2}^{-1}\left\|R^{-s}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{s+1}-2g_s+g_{s-1}\right)\right\|_H\right]$$

$$+|\alpha|\frac{1}{2}\left[\left[\left\|A^{-1/2}\right\|_{H\to H}+1\right]\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{1/2}g_0\right\|_H$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}f_1\right\|_H+\left\|f_{[\mu/\tau]-1}-f_1\right\|_H+\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}$$

$$\times\left[\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{1/2}f_1\right\|_H$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\tau^{-1}\left[\frac{1}{2}\left[\left\|R^{[\mu/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\right.$$

$$\times\|f_1-f_2\|_H+\sum_{s=2}^{[\mu/\tau]-2}\frac{1}{2}\left[\left\|R^{[\mu/\tau]-s-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{[\mu/\tau]-s-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\|f_{s+1}-2f_s+f_{s-1}\|_H\right]\Big]$$

$$+|\beta|\left[\left[1+\left\|R^{N-1}\right\|_{H\to H}\right]\left\|\tau^{-1}\left(g_{-1}-g_0\right)\right\|_H\right]$$

$$+\left\|R^{N-1}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{-1}-g_0-g_{-N+1}+g_{-N+2}\right)\right\|_H$$

$$\left.+\sum_{s=-N+2}^{-1}\left\|R^{-s}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{s+1}-2g_s+g_{s-1}\right)\right\|_H\right]$$

$$\left.+|\beta|\left[\left\|A^{1/2}R^N\right\|_{H\to H}\left\|A^{1/2}g_0\right\|_H+\left\|AR^N\right\|_{H\to H}\|g_0-g_{-N+1}\|_H\right]+\|A\varphi\|_H\right)$$

$$\leq M\left[\|A\varphi\|_H+\left\|A^{1/2}f_1\right\|_H+\left\|\left(f_1-f_2\right)\tau^{-1}\right\|_H\right.$$

$$+\max_{2\leq k\leq N-2}\left\|\left(f_{k+1}-2f_k+f_{k-1}\right)\tau^{-2}\right\|$$

$$\left.+\left\|A^{1/2}g_0\right\|_H+\left\|\left(g_0-g_{-1}\right)\tau^{-1}\right\|_H+\max_{-(N-1)\leq k\leq -1}\left\|\left(g_{k+1}-2g_k+g_{k-1}\right)\tau^{-2}\right\|_H\right]. \quad (4.43)$$

If $\mu \leq 2\tau$, $2\tau < \lambda$, then using the Abel's formula and formula (4.18), we obtain

$$Au_{-N}=T_\tau\left[\alpha\left[g_0-R^N g_{-N+1}\right.\right.$$

$$+(\tau A)^{-1}\left[(g_{-1}-g_0)-R^{N-1}(g_{-N+1}-g_{-N+2})\right.$$

$$\left.\left.+\sum_{s=-N+2}^{-1}R^{-s}(g_{s+1}-2g_s+g_{s-1})\right]\right]$$

$$+\beta\tau^{-1}\left(\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right.\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times\left[-R^Ng_{-N+1}+(\tau A)^{-1}\left[(g_{-1}-g_0)-R^{N-1}(g_{-N+1}-g_{-N+2})\right.\right.$$

$$\left.\left.+\sum_{s=-N+2}^{-1}R^{-s}(g_{s+1}-2g_s+g_{s-1})\right]\right]$$

$$+\left[f_{[\lambda/\tau]-1}-\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]f_1\right.$$

$$\left.+\sum_{s=2}^{[\lambda/\tau]-1}\frac{1}{2}\left[R^{[\lambda/\tau]-s}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-s}\left(\tau A^{1/2}\right)\right][f_{s-1}-f_s]\right]$$

$$-\left\{\left\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right.\right.$$

$$\left.+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\right\}$$

$$\left.+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\right\}$$

$$\times\left[-R^Ng_{-N+1}+(\tau A)^{-1}\left[(g_{-1}-g_0)-R^{N-1}(g_{-N+1}-g_{-N+2})\right.\right.$$

$$\left.\left.+\sum_{s=-N+2}^{-1}R^{-s}(g_{s+1}-2g_s+g_{s-1})\right]\right]$$

$$-\left[f_{[\lambda/\tau]-2}-\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right]f_1\right.$$

$$\left.\left.+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}\left[R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right][f_{s-1}-f_s]\right]+A\varphi\right].$$

Using formula (4.24), estimates (4.10), (4.13), (4.2) and (4.3) we obtain

$$\|Au_{-N}\|_H\le\|T_\tau\|_{H\to H}\left[|\alpha|\left[\left\|A^{-1/2}\right\|_{H\to H}\right.\right.$$

$$\times\left\|A^{1/2}g_0\right\|_H\left[1+\|R^N\|_{H\to H}\right]+\|R^N\|_{H\to H}\|g_{-N+1}-g_0\|_H$$

$$+\|A^{-1}\|_{H\to H}\left[\|\tau^{-1}(g_{-1}-g_0)\|_H\left[1+\|R^{N-1}\|_{H\to H}\right]\right.$$

$$\left.+\|R^{N-1}\|_{H\to H}\|\tau^{-1}(g_{-N+1}-g_{-N+2}-g_{-1}+g_0)\|_H\right]$$

$$+\sum_{s=-N+2}^{-1}\left\|R^{-s}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{s+1}-2g_s+g_{s-1}\right)\right\|_H\Bigg]$$

$$+|\beta|\left(\left\{\left\{\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right.\right.\right.\right.$$

$$\left.+\left\|R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$+\frac{1}{2}\left[\left\|R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\Bigg\}$$

$$+\frac{1}{2}\left[\left\|\left(I+\tau^2A\right)R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\right.$$

$$\left.\left\|\left(I+\tau^2A\right)R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\Bigg\}$$

$$\times\left\|AR^N\right\|_{H\to H}\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g_0\right\|_H+\left\|g_{-N+1}-g_0\right\|_H\right]$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|\tau^{-1}\left(g_{-1}-g_0\right)\right\|_H\left[1+\left\|R^{N-1}\right\|_{H\to H}\right]\right.$$

$$+\left\|R^{N-1}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{-N+1}-g_{-N+2}-g_{-1}+g_0\right)\right\|_H\Big]$$

$$+\sum_{s=-N+2}^{-1}\left\|R^{-s}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{s+1}-2g_s+g_{s-1}\right)\right\|_H\Bigg]$$

$$+\tau^{-1}\left[\frac{1}{2}\left[\left\|R^{[\mu/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|f_1-f_2\right\|_H\right.$$

$$+\sum_{s=2}^{[\mu/\tau]-2}\frac{1}{2}\left[\left\|R^{[\mu/\tau]-s-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{[\mu/\tau]-s-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|f_{s+1}-2f_s+f_{s-1}\right\|_H\right]\Bigg]$$

$$+\left[\left\|\tau^{-1}\left[f_1-f_2\right]\right\|_{H\to H}+\left\|\tau^{-1}\left[f_{[\lambda/\tau]-1}-f_{[\lambda/\tau]-2}-f_1+f_2\right]\right\|_{H\to H}\right.$$

$$+\frac{1}{2}\left[\left\|R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{1/2}f_1\right\|$$

$$+\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|R\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\times\tau^{-1}\left\|\left[f_{[\lambda/\tau]-2}-f_{[\lambda/\tau]-1}\right]\right\|_H$$

$$+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}|i|\left[\left\|R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.\times\left\|\left[f_{s-1}-f_s\right]\right\|_H\right)+\left\|A\varphi\right\|_H\Big].$$

$$\leq M\left[\left\|A\varphi\right\|_H+\left\|A^{1/2}f_1\right\|_H+\left\|\left(f_1-f_2\right)\tau^{-1}\right\|_H\right.$$

$$+\max_{2\leq k\leq N-2}\left\|\left(f_{k+1}-2f_k+f_{k-1}\right)\tau^{-2}\right\|_H$$

$$\left.+\left\|A^{1/2}g_0\right\|_H+\left\|\left(g_0-g_{-1}\right)\tau^{-1}\right\|_H+\max_{-(N-1)\leq k\leq -1}\left\|\left(g_{k+1}-2g_k+g_{k-1}\right)\tau^{-2}\right\|_H\right].\qquad(4.44)$$

If $2\tau<\mu$, $2\tau<\lambda$, then using the Abel's formula and formula (4.19), we obtain

$$Au_{-N}=T_\tau\Bigg(\alpha\Bigg[\Bigg\{\Bigg\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$
$$+\frac{1}{2i}A^{1/2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg\}$$
$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)-R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right]\Bigg\}$$
$$\times\Bigg[-R^Ng_{-N+1}+(\tau A)^{-1}\Big[(g_{-1}-g_0)-R^{N-1}\left(g_{-N+1}-g_{-N+2}\right)$$
$$+\sum_{s=-N+2}^{-1}R^{-s}\left(g_{s+1}-2g_s+g_{s-1}\right)\Big]\Bigg]$$
$$+\Bigg[f_{[\mu/\tau]-1}-\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)+R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right]f_1$$
$$+\sum_{s=2}^{[\mu/\tau]-1}\frac{1}{2}\left[R^{[\mu/\tau]-s}\left(-\tau A^{1/2}\right)+R^{[\mu/\tau]-s}\left(\tau A^{1/2}\right)\right]\left[f_{s-1}-f_s\right]\Bigg]$$
$$+\beta\tau^{-1}\Bigg(\Bigg\{\Bigg\{\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]$$
$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right]\Bigg\}$$
$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]}\left(\tau A^{1/2}\right)\right]\Bigg\}$$
$$\times\Bigg[-R^Ng_{-N+1}+(\tau A)^{-1}\Big[(g_{-1}-g_0)-R^{N-1}\left(g_{-N+1}-g_{-N+2}\right)$$
$$+\sum_{s=-N+2}^{-1}R^{-s}\left(g_{s+1}-2g_s+g_{s-1}\right)\Big]\Bigg]$$
$$+\Bigg[f_{[\lambda/\tau]-1}-\frac{1}{2}\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]f_1$$
$$+\sum_{s=2}^{[\lambda/\tau]-1}\frac{1}{2}\left[R^{[\lambda/\tau]-s}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-s}\left(\tau A^{1/2}\right)\right]\left[f_{s-1}-f_s\right]\Bigg]$$
$$-\Bigg\{\Bigg\{\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]$$
$$+\frac{1}{2i}A^{1/2}\left[R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right]\Bigg\}$$
$$+\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)-R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right]\Bigg\}$$
$$\times\Bigg[-R^Ng_{-N+1}+(\tau A)^{-1}\Big[(g_{-1}-g_0)-R^{N-1}\left(g_{-N+1}-g_{-N+2}\right)$$

$$+\sum_{s=-N+2}^{-1} R^{-s}\left(g_{s+1}-2g_s+g_{s-1}\right)\Big]\Big]$$

$$-\left[f_{[\lambda/\tau]-2}-\frac{1}{2}\left[R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right]f_1\right.$$

$$\left.+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}\left[R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)+R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right]\left[f_{s-1}-f_s\right]\right]+A\varphi\Big).$$

Using formula (4.26), estimates (4.10), (4.13), (4.2) and (4.3) we obtain

$$\|Au_{-N}\|_H \le \|T_\tau\|_{H\to H}\Big(|\alpha|\Big[\Big\{\Big\{\left\|A^{-1/2}\right\|_{H\to H}\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}$$

$$+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]$$

$$+\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]\Big\}$$

$$+\frac{1}{2}\Big[\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^{[\mu/\tau]}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]\Big\}$$

$$\times\left\|A^{1/2}R^N\right\|_{H\to H}\Big[\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g_0\right\|_H+\|g_{-N+1}-g_0\|_H\Big]$$

$$+\left\|A^{-1}\right\|_{H\to H}\left[\left[1+\left\|R^{N-1}\right\|_{H\to H}\right]\left\|\tau^{-1}\left(g_{-1}-g_0\right)\right\|_H\right]$$

$$+\left\|R^{N-1}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{-1}-g_0-g_{-N+1}+g_{-N+2}\right)\right\|_H$$

$$+\sum_{s=-N+2}^{-1}\left\|R^{-s}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{s+1}-2g_s+g_{s-1}\right)\right\|_H\Big]$$

$$+|\alpha|\frac{1}{2}\Big[\Big[\left\|A^{-1/2}\right\|_{H\to H}+1\Big]\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}$$

$$+\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]\left\|A^{1/2}g_0\right\|_H$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}f_1\right\|_H+\left\|f_{[\mu/\tau]-1}-f_1\right\|_H$$

$$+\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\Big[\left\|R^{[\mu/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\left\|A^{1/2}f_1\right\|_H$$

$$\left\|A^{-1/2}\right\|_{H\to H}\tau^{-1}\Big[\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\|f_1-f_2\|_H$$

$$+\sum_{s=2}^{[\mu/\tau]-2}\frac{1}{2}\Big[\left\|R^{[\mu/\tau]-s-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}$$

$$+\left\|R^{[\mu/\tau]-s-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\Big]\|f_{s+1}-2f_s+f_{s-1}\|_H\Big]\Big]$$

$$+|\beta|\Big(\Big\{\Big\{\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\Big[\left\|R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]$$

$$+\frac{1}{2}\Big[\left\|R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\Big]\Big\}$$

$$+\frac{1}{2}\left[\left\|\left(I+\tau^2A\right)R^{[\lambda/\tau]-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^{[\lambda/\tau]-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\Bigg\}$$

$$\times\left\|AR^N\right\|_{H\to H}\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g_0\right\|_H+\left\|g_{-N+1}-g_0\right\|_H\right]$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|\tau^{-1}\left(g_{-1}-g_0\right)\right\|_H\left[1+\left\|R^{N-1}\right\|_{H\to H}\right]\right.$$

$$+\left\|R^{N-1}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{-N+1}-g_{-N+2}-g_{-1}+g_0\right)\right\|_H\Big]$$

$$\left.+\sum_{s=-N+2}^{-1}\left\|R^{-s}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{s+1}-2g_s+g_{s-1}\right)\right\|_H\right]\Bigg]$$

$$+\tau^{-1}\left[\frac{1}{2}\left[\left\|R^{[\mu/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\mu/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\right.$$

$$\times\left\|f_1-f_2\right\|_H+\sum_{s=2}^{[\mu/\tau]-2}\frac{1}{2}\left[\left\|R^{[\mu/\tau]-s-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right.$$

$$\left.\left.+\left\|R^{[\mu/\tau]-s-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|f_{s+1}-2f_s+f_{s-1}\right\|_H\right]\Bigg]$$

$$+\left[\left\|\tau^{-1}\left[f_1-f_2\right]\right\|_{H\to H}+\left\|\tau^{-1}\left[f_{[\lambda/\tau]-1}-f_{[\lambda/\tau]-2}-f_1+f_2\right]\right\|_{H\to H}\right.$$

$$+\frac{1}{2}\left[\left\|R^{[\lambda/\tau]-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{1/2}f_1\right\|$$

$$+\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|R\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\times\tau^{-1}\left\|A^{-1/2}\left[f_{[\lambda/\tau]-2}-f_{[\lambda/\tau]-1}\right]\right\|_H$$

$$+\sum_{s=2}^{[\lambda/\tau]-2}\frac{1}{2}\left|i\right|\left[\left\|R^{[\lambda/\tau]-1-s}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{[\lambda/\tau]-1-s}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left.\times\left\|A^{-1/2}\left[f_{s-1}-f_s\right]\right\|_H\Big]\right)+\left\|A\varphi\right\|_H\Big].$$

$$\leq M\left[\left\|A\varphi\right\|_H+\left\|A^{1/2}f_1\right\|_H+\left\|\left(f_1-f_2\right)\tau^{-1}\right\|_H\right.$$

$$+\max_{2\leq k\leq N-2}\left\|\left(f_{k+1}-2f_k+f_{k-1}\right)\tau^{-2}\right\|$$

$$\left.+\left\|A^{1/2}g_0\right\|_H+\left\|\left(g_0-g_{-1}\right)\tau^{-1}\right\|_H+\max_{-(N-1)\leq k\leq -1}\left\|\left(g_{k+1}-2g_k+g_{k-1}\right)\tau^{-2}\right\|_H\right].\qquad(4.45)$$

Finally, applying estimates (4.42), (4.43), (4.44) and (4.45), we obtain

$$\left\|Au_{-N}\right\|_H\leq M\left[\left\|A\varphi\right\|_H+\left\|A^{1/2}f_1\right\|_H+\left\|\left(f_2-f_1\right)\tau^{-1}\right\|_H\right.$$

$$+\max_{2\leq k\leq N-2}\left\|\left(f_{k+1}-2f_k+f_{k-1}\right)\tau^{-2}\right\|_H+\left\|A^{1/2}g_0\right\|_H$$

$$\left.+\left\|\left(g_0-g_{-1}\right)\tau^{-1}\right\|_H+\max_{-(N-1)\leq k\leq -1}\left\|\left(g_{k+1}-2g_k+g_{k-1}\right)\tau^{-2}\right\|_H\right].\qquad(4.46)$$

Now, we will obtain the estimate for $\|Au_k\|_H$, $-N+1 \leq k \leq N$. Let $-(N-1) \leq k \leq 0$. Then using the Abel's formula and formula (4.14), we obtain

$$Au_k = R^{N+k}Au_{-N}$$
$$+\Big[g_k - R^{N+k-1}g_{-N+1} + (\tau A)^{-1}$$
$$\times\Big[(g_{k-1}-g_k) - R^{k+N-1}(g_{-N+1}-g_{-N+2})$$
$$+\sum_{s=-N+2}^{k-1} R^{k-s}(g_{s+1}-2g_s+g_{s-1})\Big]\Big].$$

Using formula (4.29) and estimate (4.3) we obtain

$$\|Au_k\|_H \leq \left\|R^{N+k}\right\|_{H\to H}\|Au_{-N}\|_H$$
$$+\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g_0\right\|_H + \|g_k-g_0\|_H$$
$$+\left\|R^{N+k-1}\right\|_{H\to H}\left[\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}g_0\right\|_H + \|g_{-N+1}-g_0\|_H\right]$$
$$+\|A^{-1}\|_{H\to H}\Big[\left\|\tau^{-1}(g_{-1}-g_0)\right\|_H\left(1+\left\|R^{N+k-1}\right\|_{H\to H}\right)$$
$$+\left\|\tau^{-1}(g_{k-1}-g_k-g_{-1}+g_0)\right\|_H + \left\|R^{N+k-1}\right\|_{H\to H}$$
$$\times\left\|\tau^{-1}(g_{-N+1}-g_{-N+2}-g_{-1}+g_0)\right\|_H$$
$$+\sum_{s=-N+2}^{k-1}\left\|R^{k-s}\right\|_{H\to H}\tau^{-1}\|g_{s+1}-2g_s+g_{s-1}\|_H\Big]$$
$$\leq M\Big[\|Au_{-N}\|_H + \left\|\tau^{-1}(g_{-1}-g_0)\right\|_H + \left\|A^{1/2}g_0\right\|_H$$
$$+\max_{-(N-2)\leq k\leq -1}\left\|\tau^{-2}(g_{s+1}-2g_s+g_{s-1})\right\|_H\Big]. \tag{4.47}$$

Let $1 \leq k \leq N$. Then using the Abel's formula and formula (4.15), we obtain

$$Au_1 = R^N Au_{-N} + \Big[g_0 - R^N g_{-N+1} + (\tau A)^{-1}\left[(g_{-1}-g_0)\right.$$
$$\left.-R^{N-1}(g_{-N+1}-g_{-N+2}) + \sum_{s=-N+2}^{-1} R^{-s}(g_{s+1}-2g_s+g_{s-1})\right]\Big]$$
$$-\tau A\Big(R^N Au_{-N} - R^N g_{-N+1} + (\tau A)^{-1}\left[(g_{-1}-g_0)\right.$$
$$\left.-R^{N-1}(g_{-N+1}-g_{-N+2}) + \sum_{s=-N+2}^{-1} R^{-s}(g_{s+1}-2g_s+g_{s-1})\right]\Big),$$
$$Au_k = \Big\{\Big\{\frac{1}{2}\left[R^{k-1}\left(\tau A^{1/2}\right) + R^{k-1}\left(-\tau A^{1/2}\right)\right]$$
$$+\frac{1}{2i}A^{1/2}\left[R^{k-1}\left(\tau A^{1/2}\right) + R^{k-1}\left(-\tau A^{1/2}\right)\right]\Big\}$$

$$-\frac{1}{2i}A^{1/2}\left(I+\tau^2A\right)\left[R^k\left(-\tau A^{1/2}\right)-R^k\left(\tau A^{1/2}\right)\right]\Bigg\}$$

$$\times\left[R^NAu_{-N}-R^Ng_{-N+1}+(\tau A)^{-1}\left[(g_{-1}-g_0)\right.\right.$$

$$\left.\left.-R^{N-1}\left(g_{-N+1}-g_{-N+2}\right)+\sum_{s=-N+2}^{-1}R^{-s}\left(g_{s+1}-2g_s+g_{s-1}\right)\right]\right]$$

$$+\left[f_{k-1}-\frac{1}{2}\left[R^{k-1}\left(-\tau A^{1/2}\right)+R^{k-1}\left(\tau A^{1/2}\right)\right]f_1\right.$$

$$+\left(i\tau A^{1/2}\right)^{-1}\left\{\frac{1}{2}\left[-R^{k-2}\left(-\tau A^{1/2}\right)+R^{k-2}\left(\tau A^{1/2}\right)\right][f_1-f_2]\right.$$

$$+\sum_{s=2}^{k-2}\frac{1}{2}\left[-R^{k-s-1}\left(-\tau A^{1/2}\right)+R^{k-s-1}\left(\tau A^{1/2}\right)\right][f_{s+1}-2f_s+f_{s-1}]\,,2\le k\le N.$$

Using formula (4.29) and estimates (4.2), (4.3) we obtain

$$\|Au_1\|_H=\left[\left\|(I-\tau A)\,R^N\right\|_{H\to H}\|Au_{-N}\|_H+\left\|A^{-1/2}\right\|_{H\to H}\right.$$

$$+\left[\left\|A^{-1/2}\right\|_{H\to H}\left[1+\left\|R^N\right\|_{H\to H}\right]+\left\|A^{1/2}R^N\right\|_{H\to H}\right]\left\|A^{1/2}g_0\right\|_H$$

$$+\left\|(I-\tau A)\,R^N\right\|_{H\to H}\|g_{-N+1}-g_0\|_H$$

$$+\left[\tau+\left\|A^{-1}\right\|_{H\to H}\right]\left[1+\left\|R^{N-1}\right\|_{H\to H}\right]\left\|\tau^{-1}\left(g_{-1}-g_0\right)\right\|_H$$

$$+\left[\tau+\left\|A^{-1}\right\|_{H\to H}\right]\left\|R^{N-1}\right\|_{H\to H}\left\|\tau^{-1}\left[g_{-N+1}-g_{-N+2}-g_{-1}+g_0\right]\right\|_H$$

$$\left.+\left[\tau+\left\|A^{-1}\right\|_{H\to H}\right]\sum_{s=-N+2}^{-1}\left\|R^{-s}\right\|_{H\to H}\left\|\tau^{-1}\left[g_{s+1}-2g_s+g_{s-1}\right]\right\|_H\right]\Bigg]$$

$$\le M\left[\|Au_{-N}\|_H+\left\|A^{1/2}g_0\right\|_H+\left\|\tau^{-1}\left(g_{-1}-g_0\right)\right\|_H\right.$$

$$\left.+\max_{-(N-1)\le k\le -1}\left\|\tau^{-1}\left[g_{s+1}-2g_s+g_{s-1}\right]\right\|_H\right],$$

$$\|Au_k\|_H=\left\{\left\{\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}\left[\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right.\right.$$

$$\left.+\frac{1}{2\,|i|}\left[\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\left.+\frac{1}{2\,|i|}\left[\left\|\left(I+\tau^2A\right)R^k\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|\left(I+\tau^2A\right)R^k\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\right\}$$

$$\times\left[\left\|A^{1/2}R^N\right\|_{H\to H}\|Au_{-N}\|_H+\left[\left\|R^N\right\|_{H\to H}\left\|A^{1/2}g_0\right\|_H\right.\right.$$

$$+\left\|A^{1/2}R^N\right\|_{H\to H}\|g_{-N+1}-g_0\|_H$$

$$+\left[\left\|R^{N-1}\right\|_{H\to H}+1\right]\left\|A^{-1}\right\|_{H\to H}\left\|\tau^{-1}\left(g_{-1}-g_0\right)\right\|_H$$

$$+\left\|A^{-1}\right\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}\left\|\tau^{-1}\left[g_{-N+1}-g_{-N+2}-g_{-1}+g_0\right]\right\|_H$$

$$+\left\|A^{-1}\right\|_{H\to H}\sum_{s=-N+2}^{-1}\left\|R^{-s}\right\|_{H\to H}\left\|\tau^{-1}\left[g_{s+1}-2g_s+g_{s-1}\right]\right\|_H\Big]\Big]$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\left\|A^{1/2}f_1\right\|_H+\left\|f_{k-1}-f_1\right\|_H+\frac{1}{2}\left\|A^{-1/2}\right\|_{H\to H}$$

$$\times\left[\left\|R^{k-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]\left\|A^{1/2}f_1\right\|_H$$

$$+\left\|A^{-1/2}\right\|_{H\to H}\frac{1}{2}\left[\left\|R^{k-2}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-2}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\times\left\|\tau^{-1}\left[f_1-f_2\right]\right\|_H+\left\|A^{-1/2}\right\|_{H\to H}$$

$$\times\sum_{s=2}^{k-2}\frac{1}{2}\left[\left\|R^{k-s-1}\left(-\tau A^{1/2}\right)\right\|_{H\to H}+\left\|R^{k-s-1}\left(\tau A^{1/2}\right)\right\|_{H\to H}\right]$$

$$\left\|\tau^{-1}\left[f_{s+1}-2f_s+f_{s-1}\right]\right\|_H\Big]$$

$$\leq M\Big[\left\|Au_{-N}\right\|_H+\left\|A^{1/2}f_1\right\|_H+\left\|\left(f_1-f_2\right)\tau^{-1}\right\|_H$$

$$+\max_{2\leq k\leq N-2}\left\|\left(f_{k+1}-2f_k+f_{k-1}\right)\tau^{-2}\right\|_H$$

$$+\left\|A^{1/2}g_0\right\|_H+\left\|\left(g_{-1}-g_0\right)\tau^{-1}\right\|_H$$

$$+\max_{-(N-2)\leq k\leq -1}\left\|\tau^{-2}\left(g_{s+1}-2g_s+g_{s-1}\right)\right\|_H\Big],\ 2\leq k\leq N.$$

Combining the estimates (4.28), (4.30), (4.32), (4.33) and using the triangle inequality we obtain estimate (4.10). Theorem 4.1 is proved.

Now we consider the applications of Theorem 4.1. First, we consider the mixed problem for hyperbolic-parabolic equation (3.21). The abstract Theorem 4.1 is applied in the investigation of difference scheme of the first order of accuracy with respect to one variable for approximate solutions of the mixed boundary value problem (3.21). The discretization of problem (3.21) is carried out in two steps. In the first step let us define the grid space

$$[0,1]_h=\{x:x_n=nh,0\leq n\leq M,Mh=1\}.$$

We introduce the Hilbert space $L_{2h}=L_2\left([0,1]_h\right)$ of the grid functions $\varphi^h(x)$ defined on $[0,1]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{n=1}^{M-1}\left|\varphi^h(x)\right|^2h\right)^{1/2}.$$

To the differential operator A generated by the problem (3.21) we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x)=\left\{-(a(x)\varphi_{\bar{x}})_{x,n}+\delta\varphi_n\right\}_1^{M-1}, \tag{4.48}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^n\}_0^M$ satisfying the conditions $\varphi^0 = \varphi^M$, $\varphi^1 - \varphi^0 = \varphi^M - \varphi^{M-1}$. It is known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\begin{cases} \frac{d^2v^h(t,x)}{dy^2} + A_h^x v^h(y,x) = f^h(y,x),\ 0 \le y \le 1,\ x \in [0,1]_h\,, \\ \frac{dv^h(t,x)}{dy} + A_h^x v^h(y,x) = f^h(y,x), -1 \le y \le 0,\ x \in [0,1]_h\,, \\ v^h(-1,x) = \frac{dv^h(1,x)}{dy} + \varphi^h(x), x \in [0,1]_h\,, \\ v^h(0+,x) = v^h(0-,x), \frac{dv^h(0+,x)}{dy} = \frac{dv^h(0-,x)}{dy}, x \in [0,1]_h \end{cases} \tag{4.49}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (4.49) by the difference scheme (4.1)

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h = f_k^h(x),\ x \in [0,1]_h\,, \\ f_{k+1}^h(x) = \{f(y_{k+1}, x_n)\}_1^{M-1}, y_{k+1} = (k+1)\tau, 1 \le k \le N-1,\ N\tau = 1, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h = g_k^h(x),\ x \in [0,1]_h\,, \\ g_k^h(x) = \{g(y_k, x_n)\}_1^{M-1}, y_k = k\tau, -N+1 \le k \le 0, \\ u_{-N}^h(x) = u_N^h(x) + \frac{u_N^h(x) - u_{N-1}^h(x)}{\tau} + \varphi^h(x),\ x \in [0,1]_h\,, \\ \frac{u_1^h(x) - u_0^h(x)}{\tau} = -A_h^x u_0^h(x) + g_0^h(x), g_0^h(x) = g^h(0,x),\ x \in [0,1]_h\,. \end{cases} \tag{4.50}$$

Theorem 4.2. *Let τ and h be a sufficiently small numbers. Then the solutions of difference scheme (4.50) satisfy the following stability estimates:*

$$\max_{-N \le k \le N} \left\| u_k^h \right\|_{L_{2h}} \le M_1 \left[\left\| f_1^h \right\|_{L_{2h}} + \max_{2 \le k \le N-1} \left\| \left(f_k^h - f_{k-1}^h \right) \tau^{-1} \right\|_{L_{2h}} \right.$$
$$\left. + \left\| g_0^h \right\|_{L_{2h}} + \max_{-N+1 \le k \le 0} \left\| \left(g_k^h - g_{k-1}^h \right) \tau^{-1} \right\|_{L_{2h}} + \left\| \varphi^h \right\|_{L_{2h}} \right],$$
$$\max_{-N+1 \le k \le N} \left\| \tau^{-1} \left(u_k^h - u_{k-1}^h \right) \right\|_{L_{2h}} + \max_{-N \le k \le N} \left\| \left(u_k^h \right)_x \right\|_{L_{2h}}$$
$$\le M_1 \left[\left\| f_1^h \right\|_{L_{2h}} + \max_{2 \le k \le N-1} \left\| \left(f_k^h - f_{k-1}^h \right) \tau^{-1} \right\|_{L_{2h}} \right.$$
$$\left. + \left\| g_0^h \right\|_{L_{2h}} + \max_{-N+1 \le k \le 0} \left\| \left(g_k^h - g_{k-1}^h \right) \tau^{-1} \right\|_{L_{2h}} + \left\| \varphi_{\bar{x}}^h \right\|_{L_{2h}} \right],$$
$$\max_{1 \le k \le N-1} \left\| \tau^{-2} \left(u_{k+1}^h - 2u_k^h + u_{k-1}^h \right) \right\|_{L_{2h}}$$
$$+ \max_{-N \le k \le N} \left\| \left(u_{\bar{x}}^k \right)_x \right\|_{L_{2h}} + \max_{-N+1 \le k \le 0} \left\| \tau^{-1} \left(u_k^h - u_{k-1}^h \right) \right\|_{L_{2h}}$$

$$\leq M_1 \left[\left\| f_{1x}^h \right\|_{L_{2h}} + \left\| \tau^{-1} \left(f_2^h - f_1^h \right) \right\|_{L_{2h}} \right.$$

$$+ \max_{2\leq k\leq N-1} \left\| \tau^{-2} \left(f_{k+1}^h - 2f_k^h + f_{k-1}^h \right) \right\|_{L_{2h}} + \left\| g_{0x}^h \right\|_{L_{2h}} + \left\| \tau^{-1} \left(g_0^h - g_{-1}^h \right) \right\|_{L_{2h}}$$

$$\left. + \max_{-N+1\leq k\leq -1} \left\| \tau^{-2} \left(g_{k+1}^h - 2g_k^h + g_{k-1}^h \right) \right\|_{L_{2h}} + \left\| \left(\varphi_{\bar{x}}^h \right)_x \right\|_{L_{2h}} \right].$$

Here M_1 does not depend on $\tau,\ h,\ \varphi^h(x)$ and $f_k^h(x), 1 \leq k \leq N-1, g_k^h, -N+1 \leq k \leq 0$.

The proof of Theorem 4.2 is based on the abstract Theorem 4.1, and the symmetry properties of the difference operator A_h^x defined by the formula (4.48).

Second, we consider the mixed boundary value problem for the multidimensional hyperbolic-parabolic equation (3.22).

Now, the abstract Theorem 4.1 is applied in the investigation of difference schemes of the first order of accuracy with respect to one variable for approximate solutions of the mixed boundary value problem (3.22). The discretization of problem (3.22) is carried out in two steps. In the first step let us define the grid sets

$$\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \cdots, h_n m_n), m = (m_1, \cdots, m_n)\ ,$$
$$0 \leq m_r \leq N_r, h_r N_r = L, r = 1, \cdots, n\}\ ,$$
$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.$$

We introduce the Hilbert space $L_{2h} = L_2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\| \varphi^h \right\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x\in\overline{\Omega}_h} \left| \varphi^h(x) \right|^2 h_1 \cdots h_n \right)^{1/2}.$$

To the differential operator A generated by the problem (3.22) we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{n} \left(a_r(x) u_{\bar{x}_r}^h \right)_{x_r, j_r} \tag{4.51}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\begin{cases} \frac{d^2 v^h(y,x)}{dy^2} + A_h^x v^h(y,x) = f^h(y,x),\ 0 \leq y \leq 1,\ x \in \widetilde{\Omega}_h, \\ \frac{dv^h(y,x)}{dy} + A_h^x v^h(y,x) = f^h(y,x), -1 \leq y \leq 0,\ x \in \widetilde{\Omega}_h, \\ v^h(-1,x) = v^h(1,x) + \frac{dv^h(1,x)}{dy} + \varphi^h(x), x \in \widetilde{\Omega}_h, \\ v^h(0+,x) = v^h(0-,x), \frac{dv^h(0+,x)}{dy} = \frac{dv^h(0-,x)}{dy}, x \in \widetilde{\Omega}_h \end{cases} \tag{4.52}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (4.52) by the difference scheme (4.1)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_{k+1}^h=f_k^h(x), x\in\widetilde{\Omega}_h, \\ f_{k+1}^h(x)=\{f(y_{k+1},x_n)\}_1^{M-1}, y_{k+1}=(k+1)\tau, 1\le k\le N-1,\ N\tau=1, \\ \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h=g_k^h(x), x\in\widetilde{\Omega}_h, \\ g_k^h(x)=\{g(y_k,x_n)\}_1^{M-1}, y_k=k\tau, -N+1\le k\le -1, \\ u_{-N}^h(x)=u_N^h(x)+\frac{u_N^h(x)-u_{N-1}^h(x)}{\tau}+\varphi^h(x), x\in\widetilde{\Omega}_h, \\ \frac{u_1^h(x)-u_0^h(x)}{\tau}=-A_h^x u_0^h(x)+g_0^h(x), g_0^h(x)=g^h(0,x), x\in\widetilde{\Omega}_h. \end{cases} \tag{4.53}$$

Theorem 4.3. *Let τ and $|h|$ be a sufficiently small numbers.Then the solutions of difference scheme (4.53) satisfy the following stability estimates:*

$$\max_{-N\le k\le N}\left\|u_k^h\right\|_{L_{2h}}\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\left(f_k^h-f_{k-1}^h\right)\tau^{-1}\right\|_{L_{2h}}\right.$$
$$\left.+\left\|g_0^h\right\|_{L_{2h}}+\max_{-N+1\le k\le 0}\left\|\left(g_k^h-g_{k-1}^h\right)\tau^{-1}\right\|_{L_{2h}}+\left\|\varphi^h\right\|_{L_{2h}}\right],$$

$$\max_{-N+1\le k\le N}\left\|\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{-N\le k\le N}\sum_{r=1}^n\left\|\left(u_k^h\right)_{x_r,j_r}\right\|_{L_{2h}}$$
$$\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\left(f_k^h-f_{k-1}^h\right)\tau^{-1}\right\|_{L_{2h}}\right.$$
$$\left.+\left\|g_0^h\right\|_{L_{2h}}+\max_{-N+1\le k\le 0}\left\|\left(g_k^h-g_{k-1}^h\right)\tau^{-1}\right\|_{L_{2h}}+\sum_{r=1}^n\left\|\left(\varphi^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}$$
$$+\max_{-N\le k\le N}\sum_{r=1}^n\left\|\left(u_k^h\right)_{\bar{x}_r x_r,j_r}\right\|_{L_{2h}}+\max_{-N+1\le k\le 0}\left\|\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\|_{L_{2h}}$$
$$\le M_1\left[\sum_{r=1}^n\left\|\left(f_1^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}+\left\|\tau^{-1}\left(f_2^h-f_1^h\right)\right\|_{L_{2h}}\right.$$
$$+\max_{2\le k\le N-1}\left\|\tau^{-2}\left(f_{k+1}^h-2f_k^h+f_{k-1}^h\right)\right\|_{L_{2h}}+\sum_{r=1}^n\left\|\left(g_0^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}+\left\|\tau^{-1}\left(g_0^h-g_{-1}^h\right)\right\|_{L_{2h}}$$
$$\left.+\max_{-N+1\le k\le -1}\left\|\tau^{-2}\left(g_{k+1}^h-2g_k^h+g_{k-1}^h\right)\right\|_{L_{2h}}+\sum_{r=1}^n\left\|\left(\varphi^h\right)_{\bar{x}_r x_r,j_r}\right\|_{L_{2h}}\right].$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$ and $f_k^h(x), 1\le k\le N-1, g_k^h, -N+1\le k\le 0$.

The proof of Theorem 4.3 is based on the abstract Theorem 4.1, and the symmetry properties of the difference operator A_h^x defined by the formula (4.51).

4.2 The Second Order of Accuracy Difference Schemes

In the present section the second order of accuracy difference schemes

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k, \\ f_k=f(t_k), \quad t_k=k\tau,\ 1\le k\le N-1, \\ \tau^{-1}(I+\tau^2A)(u_1-u_0)=Z_1, \\ Z_1=\frac{\tau}{2}(f(0)-Au_0)+(g(0)-Au_0), \\ \tau^{-1}(u_k-u_{k-1})+A(I+\frac{\tau}{2}A)u_k=(I+\frac{\tau}{2}A)g_k, \\ g_k=g(t_k-\frac{\tau}{2}),\ t_k=k\tau,\ -(N-1)\le k\le 0, \\ u_{-N}=\alpha\,(u_0+\mu\,(-Au_0+g_0))+\beta\,(-Au_0+g_0 \\ +\lambda\,(-Au_0+f_0))+\varphi,\ \mu\le 2\tau,\ \lambda\le 2\tau, \\ u_{-N}=\alpha\left(u_{[\mu/\tau]}+(\mu-[\frac{\mu}{\tau}]\tau)\frac{u_{[\mu/\tau]}-u_{[\mu/\tau]-1}}{\tau}\right) \\ +\beta\,(-Au_0+g_0+\lambda\,(-Au_0+f_0))+\varphi, 2\tau<\mu,\lambda\le 2\tau, \\ u_{-N}=\alpha\,(u_0+\mu\,(-Au_0+g_0)) \\ +\beta\left(\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})(f_{[\frac{\lambda}{\tau}]}-Au_{[\frac{\lambda}{\tau}]})\right)+\varphi,\mu\le 2\tau,2\tau<\lambda, \\ u_{-N}=\alpha\left(u_{[\mu/\tau]}+(\mu-[\frac{\mu}{\tau}]\tau)\frac{u_{[\mu/\tau]}-u_{[\mu/\tau]-1}}{\tau}\right) \\ +\beta\left(\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})(f_{[\frac{\lambda}{\tau}]}-Au_{[\frac{\lambda}{\tau}]})\right)+\varphi,\ 2\tau<\mu,\ 2\tau<\lambda. \end{cases} \tag{4.54}$$

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k, \\ f_k=f(t_k), \quad t_k=k\tau,\ 1\le k\le N-1, \\ \tau^{-1}(I+\tau^2A)(u_1-u_0)=Z_1, \\ Z_1=\frac{\tau}{2}(f(0)-Au_0)+(g(0)-Au_0), \\ \tau^{-1}(u_k-u_{k-1})+A(I+\frac{\tau}{2}A)u_k=(I+\frac{\tau}{2}A)g_k, \\ g_k=g(t_k-\frac{\tau}{2}),\ t_k=k\tau,\ -(N-1)\le k\le 0, \\ u_{-N}=\alpha\,(u_0+\mu\,(-Au_0+g_0))+\beta\,(-Au_0+g_0 \\ +\lambda\,(-Au_0+f_0))+\varphi,\ \mu\le 2\tau,\ \lambda\le 2\tau, \\ u_{-N}=\alpha\left(u_{[\mu/\tau]}+(\mu-[\frac{\mu}{\tau}]\tau)\frac{u_{[\mu/\tau]}-u_{[\mu/\tau]-1}}{\tau}\right) \\ +\beta\,(-Au_0+g_0+\lambda\,(-Au_0+f_0))+\varphi, 2\tau<\mu,\lambda\le 2\tau, \\ u_{-N}=\alpha\,(u_0+\mu\,(-Au_0+g_0)) \\ +\beta\left(\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})(f_{[\frac{\lambda}{\tau}]}-Au_{[\frac{\lambda}{\tau}]})\right)+\varphi,\mu\le 2\tau,2\tau<\lambda, \\ u_{-N}=\alpha\left(u_{[\mu/\tau]}+(\mu-[\frac{\mu}{\tau}]\tau)\frac{u_{[\mu/\tau]}-u_{[\mu/\tau]-1}}{\tau}\right) \\ +\beta\left(\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})(f_{[\frac{\lambda}{\tau}]}-Au_{[\frac{\lambda}{\tau}]})\right)+\varphi,\ 2\tau<\mu,\ 2\tau<\lambda \end{cases} \tag{4.55}$$

for the approximate solutions of the boundary-value problem (3.1) are presented. The stability estimates for the solutions of these difference schemes are established.

Theorem 4.4. *Suppose that $\varphi\in D(A)$, $g_0\in D(A^{1/2})$ and $f_0\in D(A^{1/2})$. Then for the solution of the difference scheme (4.54) the stability estimates hold:*

$$\max_{-N\le k\le N}\|u_k\|_H \tag{4.56}$$

$$\le M\left[\|\varphi\|_H+\left\|A^{-1/2}f_0\right\|_H+\max_{1\le k\le N-1}\left\|A^{-1/2}\,(f_k-f_{k-1})\,\tau^{-1}\right\|_H\right.$$

$$\left.+\left\|A^{-1/2}g_0\right\|_H+\max_{-(N-1)\le k\le 0}\left\|A^{-1/2}\,(g_k-g_{k-1})\,\tau^{-1}\right\|_H\right],$$

$$\max_{-N\le k\le N}\left\|A^{1/2}u_k\right\|_H \tag{4.57}$$

$$\leq M\left[\left\|A^{1/2}\varphi\right\|_H+\|f_0\|_H+\max_{1\leq k\leq N-1}\left\|(f_k-f_{k-1})\,\tau^{-1}\right\|_H\right.$$
$$\left.+\|g_0\|_H+\max_{-(N-1)\leq k\leq 0}\left\|(g_k-g_{k-1})\,\tau^{-1}\right\|_H\right],$$

$$\max_{1\leq k\leq N-1}\left\|\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)\right\|_H \tag{4.58}$$
$$+\max_{-(N-1)\leq k\leq 0}\left\|\tau^{-1}\left(u_k-u_{k-1}\right)\right\|_H+\max_{-N\leq k\leq N}\|Au_k\|_H$$
$$\leq M\left[\|A\varphi\|_H+\left\|A^{1/2}f_0\right\|_H+\left\|(f_1-f_0)\,\tau^{-1}\right\|_H\right.$$
$$+\max_{1\leq k\leq N-2}\left\|(f_{k+1}-2f_k+f_{k-1})\,\tau^{-2}\right\|_H+\left\|A^{1/2}g_0\right\|_H$$
$$\left.+\left\|(g_0-g_{-1})\,\tau^{-1}\right\|_H+\max_{-(N-1)\leq k\leq -1}\left\|(g_{k+1}-2g_k+g_{k-1})\,\tau^{-2}\right\|_H\right],$$

where M does not depend on τ, $f_k, 0\leq k\leq N-1$, $g_k, -N+1\leq k\leq 0$ and φ.

Proof. We will obtain the formula for the solution of the difference scheme (4.54). It is known that there are unique solutions of the initial value difference problems

$$\begin{cases}\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k,\\ f_k=f\left(t_k\right),\ t_k=k\tau,\ 1\leq k\leq N-1,\\ u_0=\xi,\ \tau^{-1}(u_1-u_0)=\psi,\end{cases} \tag{4.59}$$

$$\begin{cases}\tau^{-1}\left(u_k-u_{k-1}\right)+A\left(I+\frac{\tau}{2}A\right)u_k=\left(I+\frac{\tau}{2}A\right)g_k,\\ g_k=g\left(t_k-\frac{\tau}{2}\right),t_k=k\tau,-(N-1)\leq k\leq 0,\ u_{-N}\ \text{ is given}\end{cases} \tag{4.60}$$

and the following formulas hold:

$$\begin{cases}u_1=\xi+\tau\psi,\\ u_k=\frac{1}{2}\left[R^{k-1}(\tau A^{1/2})\left(\dot{I}-\frac{i\tau A^{1/2}}{2}\right)+R^{k-1}(-\tau A^{1/2})\left(\dot{I}+\frac{i\tau A^{1/2}}{2}\right)\right]\xi\\ +\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^k(-\tau A^{1/2})-R^k(\tau A^{1/2})\right]\psi\\ +\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}(-\tau A^{1/2})-R^{k-s}(\tau A^{1/2})\right]f_s,\ 2\leq k\leq N,\end{cases} \tag{4.61}$$

where $R^{k-1}(\pm\tau A^{1/2})=\left(I\pm i\tau A^{1/2}-\frac{\tau^2}{2}A\right)^{-1}$,

$$u_k=R^{N+k}u_{-N}+\tau\sum_{s=-N+1}^{k}\left(I+\frac{\tau}{2}A\right)R^{k-s+1}g_s,\ \ -(N-1)\leq k\leq 0, \tag{4.62}$$

where $R=\left(I+\tau A+\frac{\tau^2}{2}A^2\right)^{-1}$. Using (4.61) , (4.62) and the formulas

$$\begin{aligned}\tau^{-1}\left(u_1-u_0\right)&=\left(I+\tau^2A\right)^{-1}\left(-(\frac{\tau}{2}+1)Au_0+\frac{\tau}{2}f(0)+g_0\right),\\ u_0&=\xi,\tau^{-1}\left(u_1-u_0\right)=\psi,\end{aligned}$$

we obtain

$$\xi = u_0 = R^N u_{-N} + \tau \sum_{s=-N+1}^{0} (I + \frac{\tau}{2}A) R^{-s+1} g_s,$$

$$\psi = \tau^{-1}(u_1 - u_0) = (I + \tau^2 A)^{-1}$$

$$\times \left\{ -A(\frac{\tau}{2}+1) \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} (I + \frac{\tau}{2}A) R^{-s+1} g_s \right] + \frac{\tau}{2} f(0) + g_0 \right\}.$$

Therefore

$$u_1 = \left(I - \tau A(\frac{\tau}{2}+1)(I+\tau^2 A)^{-1} \right) \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} (I + \frac{\tau}{2}A) R^{-s+1} g_s \right]$$

$$u_k = \frac{1}{2} \left[R^{k-1}(\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{k-1}(-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right]$$

$$+ R^{k-1}(-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \Bigg] \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} \left(I + \frac{\tau}{2}A \right) R^{-s+1} g_s \right]$$

$$+ \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^k(-\tau A^{1/2}) - R^k(\tau A^{1/2}) \right]$$

$$\times (I + \tau^2 A)^{-1} \left\{ -A(\frac{\tau}{2}+1) \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} (I + \frac{\tau}{2}A) R^{-s+1} g_s \right] + \frac{\tau}{2} f(0) + g_0 \right\}$$

$$+ \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2} \left[R^{k-s}(-\tau A^{1/2}) - R^{k-s}(\tau A^{1/2}) \right] f_s, \ 2 \le k \le N. \tag{4.63}$$

If $\mu \le 2\tau$, $\lambda \le 2\tau$, then using the condition

$$u_{-N} = \alpha (u_0 + \mu (-A u_0 + g_0)) + \beta(-A u_0 + g_0 + \lambda (-A u_0 + f_0)) + \varphi,$$

we obtain the operator equation

$$u_{-N} = (\alpha - (\alpha\mu + \beta(1+\lambda))A) \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} (I + \frac{\tau}{2}A) R^{-s+1} g_s \right]$$

$$+ (\alpha\mu + \beta) g_0 + \lambda f_0 + \varphi.$$

Since the operator

$$I - (\alpha - (\alpha\mu + \beta(1+\lambda))A) R^N$$

has an inverse

$$T_\tau = \left(I - (\alpha - (\alpha\mu + \beta(1+\lambda))A) R^N \right)^{-1},$$

we have that

$$u_{-N} = T_\tau \Bigg[(\alpha - (\alpha\mu + \beta(1+\lambda))A)\tau \tag{4.64}$$

$$\sum_{s=-N+1}^{0} \left(I + \frac{\tau}{2}A \right) R^{-s+1} g_s + (\alpha\mu + \beta) g_0 + \lambda\beta f_0 + \varphi \Bigg].$$

If $2\tau < \mu$, $\lambda \le 2\tau$, then using the condition

$$u_{-N} = \alpha\{u_{[\mu/\tau]} + (\mu - [\frac{\mu}{\tau}]\tau)\frac{u_{[\mu/\tau]} - u_{[\mu/\tau]-1}}{\tau}\} + \beta\,(-Au_0 + g_0 + \lambda\,(-Au_0 + f_0)) + \varphi,$$

we obtain the operator equation

$$u_{-N} = \alpha\left(1 + \left(\frac{\mu}{\tau} - [\frac{\mu}{\tau}]\right)\right)\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left. + R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right]\left[R^N u_{-N} + \tau\sum_{s=-N+1}^{0}(I + \frac{\tau}{2}A)R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2}) - R^{[\mu/\tau]}(\tau A^{1/2})\right]$$

$$\times(I + \tau^2 A)^{-1}\left\{-A(\frac{\tau}{2} + 1)\left[R^N u_{-N} + \tau\sum_{s=-N+1}^{0}(I + \frac{\tau}{2}A)R^{-s+1}g_s\right] + \frac{\tau}{2}f(0) + g_0\right\}$$

$$\left. + \sum_{s=1}^{[\frac{\mu}{\tau}]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\mu}{\tau}]-s}\left(-\tau A^{1/2}\right) - R^{[\frac{\mu}{\tau}]-s}\left(\tau A^{1/2}\right)\right]f_s\right\}$$

$$-\alpha\left(\frac{\mu}{\tau} - [\frac{\mu}{\tau}]\right)\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left. + R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right]$$

$$\times\left[R^N u_{-N} + \tau\sum_{s=-N+1}^{0}\left(I + \frac{\tau}{2}A\right)R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2}) - R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\times(I + \tau^2 A)^{-1}\left\{-A(\frac{\tau}{2} + 1)\left[R^N u_{-N} + \tau\sum_{s=-N+1}^{0}(I + \frac{\tau}{2}A)R^{-s+1}g_s\right] + \frac{\tau}{2}f(0) + g_0\right\}$$

$$\left. + \sum_{s=1}^{[\frac{\mu}{\tau}]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\mu}{\tau}]-1-s}(-\tau A^{1/2}) - R^{[\frac{\mu}{\tau}]-1-s}(\tau A^{1/2})\right]f_s\right\}$$

$$-\beta(1 + \lambda)A\left[R^N u_{-N} + \tau\sum_{s=-N+1}^{0}\left(I + \frac{\tau}{2}A\right)R^{-s+1}g_s\right] + \beta g_0 + \lambda\beta f_0 + \varphi.$$

Since the operator

$$I - \alpha(1 + (\frac{\mu}{\tau} - [\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.\left. + R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right]R^N\right\}$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2})-R^{[\mu/\tau]}(\tau A^{1/2})\right]$$
$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N$$
$$-\alpha(\frac{\mu}{\tau}-[\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$
$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$
$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$
$$\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\right\}-\beta(1+\lambda)AR^N$$

has an inverse

$$T_\tau=\left(I-\alpha(1+(\frac{\mu}{\tau}-[\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.\right.$$
$$\left.\left.+R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N\right\}$$
$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2})-R^{[\mu/\tau]}(\tau A^{1/2})\right]$$
$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N$$
$$-\alpha(\frac{\mu}{\tau}-[\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}\left(\tau A^{1/2}\right)\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$
$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$
$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$
$$\left.\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\right\}-\beta(1+\lambda)AR^N\right)^{-1},$$

we have that

$$u_{-N}=T_\tau\left(\alpha(1+(\frac{\mu}{\tau}-[\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}\left(\tau A^{1/2}\right)\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.\right. \tag{4.65}$$
$$\left.+R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s$$
$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2})-R^{[\mu/\tau]}(\tau A^{1/2})\right]$$
$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\mu}{\tau}]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\mu}{\tau}]-s}(-\tau A^{1/2})-R^{[\frac{\mu}{\tau}]-s}(\tau A^{1/2})\right]f_s\right\}$$

$$-\alpha(\frac{\mu}{\tau}-[\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\mu}{\tau}]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\mu}{\tau}]-1-s}(-\tau A^{1/2})-R^{[\frac{\mu}{\tau}]-1-s}(\tau A^{1/2})\right]f_s\right\}$$

$$\left.-\beta(1+\lambda)A\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s+\beta g_0+\lambda\beta f_0+\varphi\right).$$

If $\mu\le 2\tau$, $2\tau<\lambda$, then using the condition

$$u_{-N}=\alpha\left(u_0+\mu\left(-Au_0+g_0\right)\right)+\beta\left(\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})(f_{[\frac{\lambda}{\tau}]}-Au_{[\frac{\lambda}{\tau}]})\right)+\varphi,$$

we obtain the operator equation

$$u_{-N}=\alpha(I-\mu A)\left[R^Nu_{-N}+\tau(I+\frac{\tau}{2}A)\sum_{s=-N+1}^{0}R^{-s+1}g_s\right]$$

$$+\frac{\beta}{\tau}(I-(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})A)$$

$$\times\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\right.$$

$$\times\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\lambda}{\tau}]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\lambda}{\tau}]-s}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-s}(\tau A^{1/2})\right]f_s\right\}$$

$$-\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\right.$$

$$\times \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} (I + \frac{\tau}{2}A) R^{-s+1} g_s \right]$$

$$+ \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2}) \right]$$

$$\times (I + \tau^2 A)^{-1} \left\{ -A(\frac{\tau}{2}+1) \left[R^N u_{-N} + \tau \sum_{s=-N+1}^{0} (I + \frac{\tau}{2}A) R^{-s+1} g_s \right] + \frac{\tau}{2} f(0) + g_0 \right\}$$

$$+ \sum_{s=1}^{[\frac{\lambda}{\tau}]-2} \frac{\tau}{2i} A^{-1/2} \left[R^{[\frac{\lambda}{\tau}]-1-s}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]-1-s}(\tau A^{1/2}) \right] f_s \Bigg\}$$

$$+ \beta(\lambda - [\frac{\lambda}{\tau}]\tau + \frac{\tau}{2}) f_{[\frac{\lambda}{\tau}]} + \alpha \mu g_0 + \varphi.$$

Since the operator

$$I - \alpha(I - \mu A) R^N - \frac{\beta}{\tau}(I - (\lambda - [\frac{\lambda}{\tau}]\tau + \frac{\tau}{2})A)$$

$$\times \left\{ \frac{1}{2} \left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right.$$

$$\left. - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right\}$$

$$+ \frac{\beta}{\tau} \left\{ \frac{1}{2} \left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right.$$

$$- \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2}) \right]$$

$$\left. (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right\}$$

has an inverse

$$T_\tau = I - \alpha(I - \mu A) R^N - \frac{\beta}{\tau}(I - (\lambda - [\frac{\lambda}{\tau}]\tau + \frac{\tau}{2})A)$$

$$\times \left\{ \frac{1}{2} \left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right.$$

$$\left. - \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2}) \right] (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right\}$$

$$+ \frac{\beta}{\tau} \left\{ \frac{1}{2} \left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2}) \left(I - \frac{i\tau A^{1/2}}{2} \right) + R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2}) \left(I + \frac{i\tau A^{1/2}}{2} \right) \right] R^N \right.$$

$$- \frac{1}{2i} A^{-1/2} \left(I + \frac{\tau^4 A^2}{4} \right) \left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2}) \right]$$

$$\left. \times (I + \tau^2 A)^{-1} A \left(\frac{\tau}{2} + 1 \right) R^N \right\} \Bigg)^{-1},$$

we have that

$$u_{-N} = T_\tau \left[\alpha(I - \mu A) \tau (I + \frac{\tau}{2}A) \sum_{s=-N+1}^{0} R^{-s+1} g_s \right. \tag{4.66}$$

$$+\frac{\beta}{\tau}(I-(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})A)\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s+\frac{\tau}{2}f(0)+g_0\right\}$$

$$\left.+\sum_{s=1}^{[\frac{\lambda}{\tau}]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\lambda}{\tau}]-s}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-s}(\tau A^{1/2})\right]f_s\right\}$$

$$-\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s+\frac{\tau}{2}f(0)+g_0\right\}$$

$$\left.+\sum_{s=1}^{[\frac{\lambda}{\tau}]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\lambda}{\tau}]-1-s}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1-s}(\tau A^{1/2})\right]f_s\right\}$$

$$\left.+\beta(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})f_{[\frac{\lambda}{\tau}]}+\alpha\mu g_0+\varphi\right].$$

If $2\tau<\mu$, $2\tau<\lambda$, then using the condition

$$u_{-N}=\alpha\left(u_{[\mu/\tau]}+(\mu-[\frac{\mu}{\tau}]\tau)\frac{u_{[\mu/\tau]}-u_{[\mu/\tau]-1}}{\tau}\right)$$

$$+\beta\left(\frac{u_{[\lambda/\tau]}-u_{[\lambda/\tau]-1}}{\tau}+(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})(f_{[\frac{\lambda}{\tau}]}-Au_{[\frac{\lambda}{\tau}]})\right)+\varphi,$$

we obtain the operator equation

$$u_{-N}=\alpha(1+(\frac{\mu}{\tau}-[\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]$$

$$\times\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2})-R^{[\mu/\tau]}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A(\frac{\tau}{2}+1)\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\mu}{\tau}]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\mu}{\tau}]-s}(-\tau A^{1/2})-R^{[\frac{\mu}{\tau}]-s}(\tau A^{1/2})\right]f_s\Bigg\}$$

$$-\alpha\left(\frac{\mu}{\tau}-[\frac{\mu}{\tau}]\right)\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]$$

$$\times\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A(\frac{\tau}{2}+1)\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\mu}{\tau}]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\mu}{\tau}]-1-s}(-\tau A^{1/2})-R^{[\frac{\mu}{\tau}]-1-s}(\tau A^{1/2})\right]f_s\Bigg\}$$

$$+\frac{\beta}{\tau}(I-(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})A)$$

$$\times\left\{\left[\frac{1}{2}R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\right.$$

$$\times\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\lambda}{\tau}]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\lambda}{\tau}]-s}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-s}(\tau A^{1/2})\right]f_s\Bigg\}$$

$$-\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\right.$$

$$\times\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\left[R^Nu_{-N}+\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s\right]+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\lambda}{\tau}]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\lambda}{\tau}]-1-s}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1-s}(\tau A^{1/2})\right]f_s\Bigg\}$$

$$+\beta(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})f_{[\frac{\lambda}{\tau}]}+\varphi.$$

Since the operator

$$I-\alpha(1+(\frac{\mu}{\tau}-[\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2})-R^{[\mu/\tau]}(\tau A^{1/2})\right]$$

$$\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\right\})$$

$$-\alpha(\frac{\mu}{\tau}-[\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N+\frac{\beta}{\tau}(I-(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})A)$$

$$\times\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N\right.$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right]$$

$$\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\right\}$$

$$-\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N\right.$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N$$

has an inverse

$$T_\tau = \Bigg(I - \alpha(1 + (\frac{\mu}{\tau} - [\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left. + R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right] R^N$$

$$- \frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2}) - R^{[\mu/\tau]}(\tau A^{1/2})\right]$$

$$\left.\times (I + \tau^2 A)^{-1} A\left(\frac{\tau}{2} + 1\right) R^N\right\}\Bigg)$$

$$-\alpha(\frac{\mu}{\tau} - [\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left. + R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right] R^N$$

$$- \frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2}) - R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\left.\times (I + \tau^2 A)^{-1} A\left(\frac{\tau}{2} + 1\right) R^N\right\} + \frac{\beta}{\tau}(I - (\lambda - [\frac{\lambda}{\tau}]\tau + \frac{\tau}{2})A)$$

$$\times\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right) + R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right] R^N\right.$$

$$- \frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right]$$

$$\left.\times (I + \tau^2 A)^{-1} A\left(\frac{\tau}{2} + 1\right) R^N\right\}$$

$$-\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right) + R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right] R^N\right.$$

$$- \frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right]$$

$$\left.\times (I + \tau^2 A)^{-1} A\left(\frac{\tau}{2} + 1\right) R^N\right\}\Bigg)^{-1}$$

we have that

$$u_{-N} = T_\tau\left[\alpha(1 + (\frac{\mu}{\tau} - [\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right)\right.\right.\right. \tag{4.67}$$

$$\left. + R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right] \tau \sum_{s=-N+1}^{0}(I + \frac{\tau}{2}A)R^{-s+1}g_s$$

$$+ \frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2}) - R^{[\mu/\tau]}(\tau A^{1/2})\right]$$

$$\times (I + \tau^2 A)^{-1}\left\{-A\left(\frac{\tau}{2} + 1\right)\tau \sum_{s=-N+1}^{0}(I + \frac{\tau}{2}A)R^{-s+1}g_s + \frac{\tau}{2}f(0) + g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\mu}{\tau}]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\mu}{\tau}]-s}(-\tau A^{1/2})-R^{[\frac{\mu}{\tau}]-s}(\tau A^{1/2})\right]f_s\Bigg\}$$

$$-\alpha(\frac{\mu}{\tau}-[\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\mu}{\tau}]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\mu}{\tau}]-1-s}(-\tau A^{1/2})-R^{[\frac{\mu}{\tau}]-1-s}(\tau A^{1/2})\right]f_s\Bigg\}$$

$$+\frac{\beta}{\tau}(I-(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})A)$$

$$\times\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\lambda}{\tau}]-1}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\lambda}{\tau}]-s}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-s}(\tau A^{1/2})\right]f_s\Bigg\}$$

$$-\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s$$

$$+\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}\left\{-A\left(\frac{\tau}{2}+1\right)\tau\sum_{s=-N+1}^{0}(I+\frac{\tau}{2}A)R^{-s+1}g_s+\frac{\tau}{2}f(0)+g_0\right\}$$

$$+\sum_{s=1}^{[\frac{\lambda}{\tau}]-2}\frac{\tau}{2i}A^{-1/2}\left[R^{[\frac{\lambda}{\tau}]-1-s}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1-s}(\tau A^{1/2})\right]f_s\Bigg\}$$

$$+\beta(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})f_{[\frac{\lambda}{\tau}]}+\varphi\Big].$$

The proof of estimates (4.56), (4.57), (4.58) follows the scheme of proof of Theorem 4.1 and relies on the formulas (4.62), (4.63), (4.64), (4.65), (4.67), (4.66) and on the estimates

$$\|T_\tau\|_{H\mapsto H}\leq M, \tag{4.68}$$

$$\left\|R(\pm\tau A^{1/2})\right\|_{H\mapsto H}\leq 1, \left\|\tau A^{1/2}R(\pm\tau A^{1/2})\right\|_{H\mapsto H}\leq 1, \left\|\tau^2 AR(\pm\tau A^{1/2})\right\|_{H\mapsto H}\leq 1, \tag{4.69}$$

$$(k\tau)^\beta\left\|A^\beta R^k\right\|_{H\mapsto H}\leq M, \quad k\geq 1, \quad 0\leq\beta\leq 1. \tag{4.70}$$

Note that if $\mu\leq 2\tau,\lambda\leq 2\tau$, then

$$T_\tau-\left(I-(\alpha-\beta A)\,e^{-A}\right)^{-1}$$

$$=T_\tau\left(I-(\alpha-\beta A)\,e^{-A}\right)^{-1}\left[(\alpha-(\alpha\mu+\beta\,(1+\lambda))\,A)\,R^N-(\alpha-\beta A)\,e^{-A}\right].$$

If $2\tau<\mu,\lambda\leq 2\tau$, then

$$T_\tau-\left(I-(\alpha\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta A)e^{-A}\right)^{-1}$$

$$=T_\tau\left(I-(\alpha\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta A)e^{-A}\right)^{-1}$$

$$\times\left(\alpha(1+(\frac{\mu}{\tau}-[\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.\right.$$

$$\left.+R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2})-R^{[\mu/\tau]}(\tau A^{1/2})\right]$$

$$\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\right\}$$

$$-\alpha(\frac{\mu}{\tau}-[\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N-\beta(1+\lambda)AR^N\right\}$$

$$\left.-(\alpha\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta A)e^{-A}\right).$$

If $\mu\leq 2\tau,2\tau<\lambda$, then

$$T_\tau-\left(I-(\alpha-\beta\left[s\left([\lambda/\tau]\,\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A)\,e^{-A}\right)^{-1}$$

$$= T_\tau \left(I - (\alpha - \beta\left[s\left([\lambda/\tau]\tau\right) + c\left([\lambda/\tau]\tau\right)\right] A)e^{-A}\right)^{-1}$$

$$\times \Bigg[\Bigg[\alpha(I-\mu A)R^N - \frac{\beta}{\tau}(I - (\lambda - [\frac{\lambda}{\tau}]\tau + \frac{\tau}{2})A)$$

$$\times \Bigg\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right) + R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right] R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right](I+\tau^2 A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\Bigg\}$$

$$+\frac{\beta}{\tau}\Bigg\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right) + R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right] R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right](I+\tau^2 A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\Bigg\}\Bigg]$$

$$-(\alpha - \beta\left[s\left([\lambda/\tau]\tau\right) + c\left([\lambda/\tau]\tau\right)\right] A)e^{-A}\Bigg].$$

If $2\tau < \mu$, $2\tau < \lambda$, then

$$T_\tau - \left(I - \alpha\left(\left[c\left([\mu/\tau]\tau\right) - As\left([\mu/\tau]\tau\right)\right] - \beta\left[s\left([\lambda/\tau]\tau\right) + c\left([\lambda/\tau]\tau\right)\right] A\right) e^{-A}\right)^{-1}$$

$$= T_\tau \left(I - \alpha\left(\left[c\left([\mu/\tau]\tau\right) - As\left([\mu/\tau]\tau\right)\right] - \beta\left[s\left([\lambda/\tau]\tau\right) + c\left([\lambda/\tau]\tau\right)\right] A\right) e^{-A}\right)^{-1}$$

$$\times \Bigg[\Bigg[\alpha(1 + (\frac{\mu}{\tau} - [\frac{\mu}{\tau}]))\Bigg\{\frac{1}{2}\Bigg[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right)$$

$$+R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\Bigg] R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2}) - R^{[\mu/\tau]}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2 A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\Bigg\}\Bigg)$$

$$-\alpha(\frac{\mu}{\tau} - [\frac{\mu}{\tau}])\Bigg\{\frac{1}{2}\Bigg[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right)$$

$$+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\Bigg] R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2}) - R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2 A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N + \frac{\beta}{\tau}(I - (\lambda - [\frac{\lambda}{\tau}]\tau + \frac{\tau}{2})A)$$

$$\times \Bigg\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I - \frac{i\tau A^{1/2}}{2}\right) + R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I + \frac{i\tau A^{1/2}}{2}\right)\right] R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I + \frac{\tau^4 A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2}) - R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right]$$

$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N$$

$$-\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N\right.$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right]$$

$$\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\right]$$

$$-\alpha\left([c([\mu/\tau]\tau)-As([\mu/\tau]\tau)]-\beta\left[s([\lambda/\tau]\tau)+c([\lambda/\tau]\tau)\right]A\right)e^{-A}\right]$$

Using the last formulas and the estimates

$$\left\|(\alpha-(\alpha\mu+\beta(1+\lambda))A)R^N-(\alpha-\beta A)e^{-A}\right\|_{H\to H}\le M\tau, \tag{4.71}$$

$$\left\|\left(\left\{\alpha(1+(\frac{\mu}{\tau}-[\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.\right.\right.\right.$$

$$\left.+R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2})-R^{[\mu/\tau]}(\tau A^{1/2})\right]$$

$$\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\right\}$$

$$-\alpha(\frac{\mu}{\tau}-[\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$

$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$

$$\left.\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N-\beta(1+\lambda)AR^N\right\}$$

$$-(\alpha\left[c([\mu/\tau]\tau)-As([\mu/\tau]\tau)\right]-\beta A)e^{-A})\Big\|_{H\to H}\le M\tau, \tag{4.72}$$

$$\left\|\left[\left[\alpha(I-\mu A)R^N-\frac{\beta}{\tau}(I-(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})A)\right.\right.\right.$$

$$\times\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N\right.$$

$$\left.-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right](I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\right\}$$

$$+\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N\right.$$

$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right](I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\Big\}\Big]$$

$$-(\alpha-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A)e^{-A}\Big]\Big\|_{H\to H}\leq M\tau, \tag{4.73}$$

$$\Bigg\|\Bigg[\Bigg[\alpha(1+(\frac{\mu}{\tau}-[\frac{\mu}{\tau}]))\left\{\frac{1}{2}\left[R^{[\mu/\tau]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$
$$\left.+R^{[\mu/\tau]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$
$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]}(-\tau A^{1/2})-R^{[\mu/\tau]}(\tau A^{1/2})\right]$$
$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\Big\}\Big)$$
$$-\alpha(\frac{\mu}{\tau}-[\frac{\mu}{\tau}])\left\{\frac{1}{2}\left[R^{[\mu/\tau]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)\right.\right.$$
$$\left.+R^{[\mu/\tau]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N$$
$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\mu/\tau]-1}(-\tau A^{1/2})-R^{[\mu/\tau]-1}(\tau A^{1/2})\right]$$
$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N+\frac{\beta}{\tau}(I-(\lambda-[\frac{\lambda}{\tau}]\tau+\frac{\tau}{2})A)$$
$$\times\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N\right.$$
$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]}(\tau A^{1/2})\right]$$
$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N$$
$$-\frac{\beta}{\tau}\left\{\frac{1}{2}\left[R^{[\frac{\lambda}{\tau}]-2}(\tau A^{1/2})\left(I-\frac{i\tau A^{1/2}}{2}\right)+R^{[\frac{\lambda}{\tau}]-2}(-\tau A^{1/2})\left(I+\frac{i\tau A^{1/2}}{2}\right)\right]R^N\right.$$
$$-\frac{1}{2i}A^{-1/2}\left(I+\frac{\tau^4A^2}{4}\right)\left[R^{[\frac{\lambda}{\tau}]-1}(-\tau A^{1/2})-R^{[\frac{\lambda}{\tau}]-1}(\tau A^{1/2})\right]$$
$$\times(I+\tau^2A)^{-1}A\left(\frac{\tau}{2}+1\right)R^N\Big]$$

$$-\alpha\left(\left[c\left([\mu/\tau]\tau\right)-As\left([\mu/\tau]\tau\right)\right]-\beta\left[s\left([\lambda/\tau]\tau\right)+c\left([\lambda/\tau]\tau\right)\right]A\right)e^{-A}\Big]\Big\|_{H\to H}\leq M\tau, \tag{4.74}$$

we can obtain the estimate (4.68). The estimates (4.71), (4.72), (4.73) and (4.74) are proved using the estimates (4.69), (4.70). Theorem 4.4 is proved.

In a similar manner one establishes the following theorem.

Theorem 4.5. *Suppose that $\varphi \in D(A)$, $g_0 \in D(A^{1/2})$ and $f_0 \in D(A^{1/2})$. Then for the solution of the difference scheme (4.55) the stability estimates hold:*

$$\max_{-N\leq k\leq N} \|u_k\|_H \leq M \left[\|\varphi\|_H + \left\|A^{-1/2} f_0\right\|_H + \max_{1\leq k\leq N-1} \left\|A^{-1/2}\left(f_k - f_{k-1}\right)\tau^{-1}\right\|_H \right.$$

$$\left. + \left\|A^{-1/2} g_0\right\|_H + \max_{-(N-1)\leq k\leq 0} \left\|A^{-1/2}\left(g_k - g_{k-1}\right)\tau^{-1}\right\|_H \right],$$

$$\max_{-N\leq k\leq N} \left\|A^{1/2} u_k\right\|_H \leq M \left[\left\|A^{1/2}\varphi\right\|_H + \|f_0\|_H + \max_{1\leq k\leq N-1} \left\|\left(f_k - f_{k-1}\right)\tau^{-1}\right\|_H \right.$$

$$\left. + \|g_0\|_H + \max_{-(N-1)\leq k\leq 0} \left\|\left(g_k - g_{k-1}\right)\tau^{-1}\right\|_H \right],$$

$$\max_{1\leq k\leq N-1} \left\|\tau^{-2}\left(u_{k+1} - 2u_k + u_{k-1}\right)\right\|_H$$

$$+ \max_{-(N-1)\leq k\leq 0} \left\|\tau^{-1}\left(u_k - u_{k-1}\right)\right\|_H + \max_{-N\leq k\leq N} \|Au_k\|_H$$

$$\leq M \left[\|A\varphi\|_H + \left\|A^{1/2} f_0\right\|_H + \left\|\left(f_1 - f_0\right)\tau^{-1}\right\|_H \right.$$

$$+ \max_{1\leq k\leq N-2} \left\|\left(f_{k+1} - 2f_k + f_{k-1}\right)\tau^{-2}\right\|_H + \left\|A^{1/2} g_0\right\|_H$$

$$\left. + \left\|\left(g_0 - g_{-1}\right)\tau^{-1}\right\|_H + \max_{-(N-1)\leq k\leq -1} \left\|\left(g_{k+1} - 2g_k + g_{k-1}\right)\tau^{-2}\right\|_H \right],$$

where M does not depend on τ, $f_k, 0 \leq k \leq N-1$, $g_k, -N+1 \leq k \leq 0$ and φ.

Note that applying the second order of accuracy difference schemes (4.54) and (4.55), we can construct the second order of accuracy difference schemes with respect to one variable for approximate solutions of the boundary-value problems (3.21) and (3.22). This approach permit us to obtain the stability estimates for the solutions of these difference schemes.

CHAPTER 5

NUMERICAL ANALYSIS

We consider the nonlocal boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \ 0 < t < 1, \ 0 < x < 1, \\ \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = g(t,x), \ -1 < t < 0, \ 0 < x < 1, \\ u(0+,x) = u(0-,x), u_t(0+,x) = u_t(0-,x), 0 \le x \le 1, \\ u(-1,x) = u(1,x) + u_t(1,x) + \varphi(x), \ 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0, \ -1 \le t \le 1, \end{cases} \tag{5.1}$$

for hyperbolic-parabolic equation.

Let

$$\begin{aligned} f(t,x) &= \left[-2 + \left(1 - t^2\right)\pi^2\right]\sin \pi x, \ 0 < t < 1, \ 0 < x < 1, \\ g(t,x) &= \left[-2t + \left(1 - t^2\right)\pi^2\right]\sin \pi x, \ -1 < t < 0, \ 0 < x < 1, \end{aligned}$$

and

$$\varphi(x) = 2 \sin \pi x.$$

The exact solution of this problem is

$$u(t,x) = \left(1 - t^2\right)\sin \pi x.$$

For approximate solutions of the nonlocal boundary value problem (5.1), we will use the first order of accuracy and a second order of accuracy difference schemes with $\tau = \frac{1}{30}$, $h = \frac{1}{30}$. We have the second order or fourth order difference equations with respect to n with matrix coefficients. To solve this difference equations we have applied a procedure of modified Gauss elimination method for difference equations with respect to n with matrix coefficients. The results of numerical experiments permit us to show that the second order of accuracy difference schemes are more accurate comparing with the first order of accuracy difference scheme.

5.1 The First Order of Accuracy Difference Scheme

Consider the nonlocal boundary value problem (5.1) for hyperbolic-parabolic equation. For approximate solution of the nonlocal boundary-value problem (5.1), consider the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$\begin{aligned} [0,1]_\tau \times [0,\pi]_h &= \{(t_k, x_n) : t_k = k\tau,\ \ 1 \le k \le N-1,\ \ N\tau = 1, \\ x_n &= nh,\ \ 1 \le n \le M-1, Mh = \pi\}. \end{aligned}$$

Applying the formulas

$$\begin{aligned} \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_{k+1}) &= O(\tau), \\ \frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) &= O(h^2), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \frac{u(1) - u(0)}{\tau} - u'(0) &= O(\tau), \\ \frac{u(1) - u(1-\tau)}{\tau} - u'(1) &= O(\tau) \end{aligned} \tag{5.3}$$

the first order of accuracy in t for the approximate solutions of the nonlocal boundary value problem

$$\begin{cases} \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} = f(t_{k+1}, x_n),\ 1 \le k \le N-1, 1 \le n \le M-1, \\ \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = g(t_k, x_n),\ -N+1 \le k \le 0, 1 \le n \le M-1, \\ u_n^1 - u_n^0 = u_n^0 - u_n^{-1}, 1 \le n \le M-1, \\ u_n^{-N} = u_n^N + \frac{u_n^N - u_n^{N-1}}{\tau} + 2\sin \pi x, 1 \le n \le M-1, \\ u_0^k = u_M^k = 0, 0 \le k \le N, \\ f(t,x) = (-2 + \pi^2(1-t^2)) \sin \pi x, \\ g(t,x) = (-2t + \pi^2(1-t^2)) \sin \pi x, \end{cases}$$

for hyperbolic-parabolic equation (5.1) are presented.

Here we have $(2N+1) \times (2N+1)$ system of linear equations and we will write them in the matrix form. By resorting the system

$$\begin{cases} \left(-\frac{1}{h^2}\right) u_{n+1}^k + \left(-\frac{1}{\tau}\right) u_n^{k-1} + \left(\frac{1}{\tau} + \frac{2}{h^2}\right) u_n^k + \left(-\frac{1}{h^2}\right) u_{n-1}^k = g(t_k, x_n), \\ -N+1 \le k \le 0,\ 1 \le n \le M-1, \\ \left(-\frac{1}{h^2}\right) u_{n+1}^{k+1} + \frac{1}{\tau^2} u_n^{k-1} + \left(-\frac{2}{\tau^2}\right) u_n^k + \left(\frac{2}{h^2}\right) u_n^{k+1} + \left(-\frac{1}{h^2}\right) u_{n-1}^{k+1} = f(t_{k+1}, x_n), \\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ u_n^1 - u_n^0 = u_n^0 - u_n^{-1},\ 1 \le n \le M-1, \\ u_n^{-N} = u_n^N + \frac{u_n^N - u_n^{N-1}}{\tau} + 2\sin \pi x,\ 1 \le n \le M-1, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N, \end{cases}$$

So,

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 0 \le n \le M, \\ U_0 = \widetilde{0},\ \ U_M = \widetilde{0}. \end{cases} \tag{5.4}$$

Denote

$$a = -\frac{1}{h^2}, \qquad b = \frac{1}{\tau}, \qquad c = \frac{1}{\tau} + \frac{2}{h^2}, \quad d = \frac{1}{\tau^2},\ e = -\frac{2}{\tau^2}, \quad s = \frac{1}{\tau^2} + \frac{2}{h^2}.$$

Then

$$\varphi_n^k = \begin{cases} 2\sin(\pi x_n), \\ g(t_k, x_n), \ \ -N+1 \le k \le 0 \\ f(t_{k+1}, x_n),\ 1 \le k \le N-1, \\ 0,\ k = N. \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{-N} \\ \varphi_n^{-N+1} \\ \cdots \\ \varphi_n^0 \\ \varphi_n^1 \\ \cdots \\ \varphi_n^N \end{bmatrix}_{(2N+1)\times 1},$$

$$A = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&\alpha&0&0&0&0&0&0&0&0 \\ 0&0&\alpha&0&0&0&0&0&0&0 \\ 0&0&0&\cdots&0&0&0&0&0&0 \\ 0&0&0&0&\alpha&0&0&0&0&0 \\ 0&0&0&0&0&0&\alpha&0&0&0 \\ 0&0&0&0&0&0&0&\alpha&0&0 \\ 0&0&0&0&0&0&0&0&\cdots&0 \\ 0&0&0&0&0&0&0&0&0&\alpha \\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau} & -\frac{1}{\tau}-1 \\ b & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & b & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & e & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & e & s & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & d & e & s \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

and $C=A$.

$$D=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$U_s=\begin{bmatrix} U_s^{-N} \\ U_s^{-N+1} \\ \dots \\ U_s^{0} \\ U_s^{1} \\ \dots \\ U_s^{N-1} \\ U_s^{N} \end{bmatrix}_{(2N+1)\times(1)}, \quad s=\ n-1,\ n, n+1.$$

For the solution of the last matrix equation, the modified variant Gauss elimination method is used. We seek a solution of the matrix equation by the following form

$$U_n=\alpha_{n+1}U_{n+1}+\beta_{n+1}, \quad n=M-1,...,2,1,0,$$

where α_j $(j=1,...,M-1)$ are $(2N+1)\times(2N+1)$ square matrices and β_j $(j=1,...,M-1)$ are $(2N+1)\times 1$ column matrices and $\alpha_1,\ \beta_1$:

$$\alpha_1=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$\beta_1=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}_{(2N+1)\times 1}.$$

Using the equality

$$U_s=\alpha_{s+1}U_{s+1}+\beta_{s+1}, (for\ s=n,\ n-1)$$

and the equality

$$AU_{n+1} + B\,U_n + CU_{n-1} = D\varphi_n$$

we can write

$$[A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1}]U_{n+1} + \left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n.$$

The last equation is satisfied if it is to be selected

$$A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1} = 0,$$

$$\left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n, \quad 1 \le n \le M-1.$$

Formulas for $\alpha_{n+1},\ \beta_{n+1}$:

$$\begin{aligned}\alpha_{n+1} &= -\left(B + C\alpha_n\right)^{-1} A,\\ \beta_{n+1} &= \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), n = 1,2,3,...M-1.\end{aligned}$$

So,

$$U_M = \widetilde{0},$$

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 2, 1, 0.$$

Algorithm

1. **Step** Input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$.
2. **Step** Use the first order of accuracy difference scheme and write in matrix form

$$A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 0 \le n \le M.$$

3. **Step** Determine the entries of the matrices A, B, C and D.
4. **Step** Find α_1, β_1.
5. **Step** Compute $\alpha_{n+1},\ \beta_{n+1}$.
6. **Step** Compute U_n's $(n = M-1, ..., 2, 1)$, $\left(U_M = \widetilde{0}\right)$ using the following formula

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}.$$

Matlab Implementation of the First Order of Accuracy Difference Scheme

```
function [table,es,p]=rothermethod(N,M)
    % first order accuracy rother method
    % mixed type
    close; close;
    if nargin<1; N=30 ; M=30 ;end; tau=1/N; h=1/M;
    A=zeros(2*N+1,2*N+1);
    for i=2:N+1;
```

```
A(i,i)=-1/(h^2);
end;
for i=N+2:2*N;
A(i,i+1)=-1/(h^2);
end;
B=zeros(2*N+1,2*N+1);
B(1,1)=1; B(1,2*N)=1/tau;
B(1,2*N+1)=-1/tau-1;
for i=1:N; B(i+1,i)=-1/tau;
end;
for i=2:N+1;
B(i,i)=1/tau+2/(h^2);
end;
for i=N+2:2*N;
B(i,i)=-2/(tau^2);
end;
for i=N+2:2*N-1;
B(i,i+1)=1/(tau^2)+2/(h^2);
end; for i=N+2:2*N;
B(i,i+1)=1/(tau^2)+2/(h^2);
end;
for i=N+1:2*N-1;
B(i+1,i)=1/(tau^2);
end;
B(2*N+1,N)=1;
B(2*N+1,N+1)=-2;
B(2*N+1,N+2)=1;
for i=1:2*N+1;
C(i,i)=1;
end ;
C(2*N+1,2*N+1)=0;
alpha(2*N+1,2*N+1,1:1)= 0 ;
betha(2*N+1,1:1) = 0 ;
'fi(j) = fi(k,j) hesaplanıyor ' ;
for j=1:M; x=j*h;
fii(1,j:j)=2*sin(pi*x); %%% alfa(xn)
for k=2:N+1;
```

```
x=j*h;
t=(-N+k-1)*tau ;
fii( k, j:j ) = g(t,x); %(pi^2)*sin(pi*x); %%g(t,x,aaa);
end;
for k=N+2:2*N+1;
t=(-N+k-1)*tau+tau ;
x=j*h;
fii( k, j:j ) = f(t,x) ; %(pi^2)*sin(pi*x); %%f(t,x,aaa);
end;
end;
'alpha(N+1,N+1,j) ve betha(N+1,j) ler hesaplanacak' ;
for j=1:M-1;
alpha( :, :, j+1:j+1 ) = - inv(B+A*alpha(:, :, j:j))*A ;
betha(:,j+1:j+1)=inv(B+A*alpha(:,:, j:j))*(C*(fii(:, j:j ))-(A* betha(:,j:j)));
end;
U( 2*N+1,1, M:M ) = 0;
for z = M-1:-1:1 ;
U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 ) + betha(:,z+1:z+1);
end;
for z = 1:M ;
p(:,z+1:z+1)=U(:,:,z:z);
end;
'EXACT SOLUTION OF THIS PDE' ;
for j=1:M+1;
for k=1:2*N+1;
t=(-N+k-1)*tau;
x=(j-1)*h; %exact solution on grid points, es(k,j) = exact(t,x);
end;
end;
'ERROR ANALYSIS' ;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxes,maxapp,maxerror,relativeerror]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
```

```
%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%
q=min(min(table));
w=max(max(table));
figure; surf(es); title('EXACT SOLUTION'); set(gca,'ZLim',[q w]);
rotate3d;
figure; surf(p); title('EULER-ROTHER'); rotate3d ;
set(gca,'ZLim',[q w]);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function estx=exact(t,x)
estx= (1 - t)*(1 + t)*sin(pi*x);
function ftx=f(t,x)
ftx=(-2 - pi^2*(-1 + t^2))*sin(pi*x);
function gtx=g(t,x)
gtx=(-2*t - pi^2*(-1 + t^2))*sin(pi*x);
```

5.2 Second Order Accuracy Difference Scheme

First, we consider again the nonlocal boundary value problem (5.1). The second order of accuracy in t for the approximate solutions of the nonlocal boundary value problem (5.1)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}+u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2} = f(t_k,x_n), \\ x_n = nh,\ t_k = k\tau,\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ \frac{u_n^k-u_n^{k-1}}{\tau} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \tau\left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{2h^4}\right) \\ = g(t_k-\frac{\tau}{2},x_n) - \frac{\tau}{2h^2}\left[g\left(t_k-\frac{\tau}{2},x_{n+1}\right) - 2g\left(t_k-\frac{\tau}{2},x_n\right) + g\left(t_k-\frac{\tau}{2},x_{n-1}\right)\right], \\ x_n = nh,\ t_k = k\tau,\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ \frac{u_n^1-u_n^0}{\tau} - \frac{\tau}{h^2}\left(u_{n+1}^1 - 2u_n^1 + u_{n-1}^1 - u_{n+1}^0 + 2u_n^0 - u_{n-1}^0\right) \\ = \frac{\tau}{2}\left(f(0,x_n) + \frac{1}{h^2}\left(u_{n+1}^0 - 2u_n^0 + u_{n-1}^0\right)\right) \\ + \left(g(0,x_n) + \frac{1}{h^2}\left(u_{n+1}^0 - 2u_n^0 + u_{n-1}^0\right)\right),\ x_n = nh,\ 2 \le n \le M-2, \\ u_n^{-N} = u_n^N + \frac{3u_n^N-4u_n^{N-1}+u_n^{N-2}}{2\tau} + \varphi(x_n),\ x_n = nh,\ 1 \le n \le M-1, \\ u_0^k = u_M^k = 0, 0 \le k \le N, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k,\ u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k,\ 0 \le k \le N, \\ f(t,x) = (-2+\pi^2(1-t^2))\sin\pi x, \\ g(t,x) = (-2t+\pi^2(1-t^2))\sin\pi x, \end{cases}$$

are presented.

We have again the $(2N+1)\times(2N+1)$ system of linear equations. We will write in the matrix form. We will resort the system

$$\begin{cases} \left(-\frac{1}{4h^2}\right) u_{n+1}^{k-1} + \left(-\frac{1}{2h^2}\right) u_{n+1}^{k} + \left(-\frac{1}{4h^2}\right) u_{n+1}^{k+1} + \left(\frac{1}{\tau^2} - \frac{1}{2h^2}\right) u_n^{k-1} + \left(-\frac{2}{\tau^2} + \frac{1}{h^2}\right) u_n^k \\ + \left(\frac{1}{\tau^2} + \frac{1}{h^2}\right) u_n^{k+1} + \left(-\frac{1}{4h^4}\right) u_{n-1}^{k-1} + \left(-\frac{1}{2h^2}\right) u_{n-1}^{k} + \left(-\frac{1}{4h^2}\right) u_{n-1}^{k+1} = f\left(t_k, x_n\right), \\ 1 \le k \le N-1, 2 \le n \le M-2, \\ \left(\frac{\tau}{2h^4}\right) u_{n+2}^k + \left(-\frac{2\tau}{h^4} - \frac{1}{h^2}\right) u_{n+1}^k + \left(-\frac{1}{\tau}\right) u_n^{k-1} + \left(\frac{3\tau}{h^4} + \frac{1}{\tau} + \frac{2}{h^2}\right) u_n^k \\ + \left(-\frac{2\tau}{h^4} - \frac{1}{h^2}\right) u_{n-1}^k + \left(\frac{\tau}{2h^4}\right) u_{n-2}^k \\ = g\left(t_k - \frac{\tau}{2}, x_n\right) - \frac{\tau}{2h^2}\left[g\left(t_k - \frac{\tau}{2}, x_{n+1}\right) - 2g\left(t_k - \frac{\tau}{2}, x_n\right) + g\left(t_k - \frac{\tau}{2}, x_{n-1}\right)\right], \\ -N+1 \le k \le 0,\ 2 \le n \le M-2, \\ \frac{u_n^1 - u_n^0}{\tau} - \frac{\tau}{h^2}\left(u_{n+1}^1 - 2u_n^1 + u_{n-1}^1 - u_{n+1}^0 + 2u_n^0 - u_{n-1}^0\right) \\ = \frac{\tau}{2}\left(f\left(0, x_n\right) + \frac{1}{h^2}\left(u_{n+1}^0 - 2u_n^0 + u_{n-1}^0\right)\right) \\ + \left(g\left(0, x_n\right) + \frac{1}{h^2}\left(u_{n+1}^0 - 2u_n^0 + u_{n-1}^0\right)\right),\ x_n = nh,\ 2 \le n \le M-2, \\ u_n^{-N} = u_n^N + \frac{3u_n^N - 4u_n^{N-1} + u_n^{N-2}}{2\tau} + \varphi\left(x_n\right),\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k,\ u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k,\ 0 \le k \le N. \\ f(t,x) = \left(-2 + \pi^2\left(1 - t^2\right)\right)\sin \pi x, \\ g(t,x) = (-2t + \pi^2(1 - t^2))\sin \pi x. \end{cases}$$

where

$$\varphi_n^k = \begin{cases} 2\sin(\pi x_n), \\ g(t_k - \frac{\tau}{2}, x_n) - \frac{\tau}{2h^2}\left[g(t_k - \frac{\tau}{2}, x_{n+1}) + g(t_k - \frac{\tau}{2}, x_n) + g(t_k - \frac{\tau}{2}, x_{n-1})\right], -N+1 \le k \le 0 \\ f(t_k, x_n), 1 \le k \le N-1, \\ 0, k = N \end{cases}$$

We have that

$$\begin{cases} A\, U_{n+2} + B\, U_{n+1} + C\, U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \quad 2 \le n \le M-2, \\ U_0 = 0,\ \ U_M, = 0 \\ U_1 = \frac{4}{5}U_2 - \frac{1}{5}U_3 \\ U_{M-1} = \frac{4}{5}U_{M-2} - \frac{1}{5}U_{M-3}, \end{cases}$$

where

$$\varphi_n = \begin{bmatrix} \varphi_n^{-N} \\ \varphi_n^{-N+1} \\ \varphi_n^{-N+2} \\ .. \\ \varphi_n^{N-2} \\ \varphi_n^{N-1} \\ \varphi_n^{N} \end{bmatrix}_{(2N+1)\times 1} .$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & e & f & e & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & e & f & e & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & e & f & e \\ 0 & 0 & 0 & ... & m & n & ... & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & -\frac{1}{2\tau} & \frac{2}{\tau} & -\frac{3}{2\tau} - 1 \\ z & w & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & z & w & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & z & w & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & c & d & c & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & c & d & c & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & c & d & c \\ 0 & 0 & ... & p & q & p & ... & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

where

$$\begin{aligned}
x &= \frac{\tau}{2h^4},\\
y &= -\frac{2\tau}{h^4} - \frac{1}{2h^2}, z = -\frac{1}{\tau},\\
w &= \frac{1}{\tau} + \frac{2}{h^2} + \frac{3\tau}{h^4},\\
e &= -\frac{1}{4h^2}, \ f = -\frac{1}{2h^2}\\
m &= \frac{\tau}{h^2} - \frac{\tau}{2h^2} - \frac{1}{h^2}, \ n = -\frac{\tau}{h^2},\\
c &= \frac{1}{\tau^2} + \frac{1}{2h^2}, \ d = -\frac{2}{\tau^2} + \frac{1}{2h^2}\\
p &= \frac{1}{\tau} + \frac{2\tau}{h^2}, \ q = -\frac{1}{\tau} - \frac{2\tau}{h^2} + \frac{\tau}{h^2} + \frac{2}{h^2},
\end{aligned}$$

and $D = B, \quad E = A$,

$$R = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}_{(2N+1)\times(2N+1)} .$$

$$U_s = \begin{bmatrix} U_s^{-N} \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ .. \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(2N+1)\times(1)}, \quad \text{where } s = n-2, \ n-1, \ n, \ n+1, \ n+2.$$

For the solution of the last matrix equation, we use the modified variant Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$\begin{cases} U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}, \quad n = M-2, ..., 2, 1, 0, \\ U_M = 0, \\ U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}], \\ U_{M-2} = [(4I - \alpha_{M-2})]^{-1}[(\beta_{M-2} + 5I)U_{M-1} + \gamma_{M-2}], \\ U_0 = 0. \end{cases}$$

Here α_j , β_j $(j = 1 : M-1)$ are $(2N+1) \times denk1(2N+1)$ square matrices and γ_j-s are $2(N+1) \times 1$ column matrices.

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}, \quad \beta_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$\gamma_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .. \\ 0 \end{bmatrix}_{(2N+1)\times(1)}, \quad \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .. \\ 0 \end{bmatrix}_{(2N+1)\times(1)}$$

$$\alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & 0 & \dots & 0 \\ 0 & \frac{4}{5} & 0 & \dots & 0 \\ 0 & 0 & \frac{4}{5} & \dots & 0 \\ & . & . & \dots & . \\ 0 & 0 & 0 & \dots & \frac{4}{5} \end{bmatrix}_{(2N+1)\times(2N+1)}, \quad \beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & 0 & \dots & 0 \\ 0 & -\frac{1}{5} & 0 & \dots & 0 \\ 0 & 0 & -\frac{1}{5} & \dots & 0 \\ & . & . & \dots & . \\ 0 & 0 & 0 & \dots & -\frac{1}{5} \end{bmatrix}_{(2N+1)\times(2N+1)}$$

Using the equality

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1} U_{n+2} + \gamma_{n+1},$$

and the equality

$$A\, U_{n+2} + B\, U_{n+1} + C\, U_n + DU_{n-1} + EU_{n-2} = R\varphi_n,$$

we can write

$$\begin{aligned} &[A + C\beta_{n+1} + +D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1}]U_{n+2} \\ &+[B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n + E\alpha_{n-1}\alpha_{n+1} + E\alpha_{n-1}\beta_n + E\beta_{n-1}\alpha_{n+1}]U_{n+1} \\ &+C\gamma_{n+1} + D\alpha_n\gamma_{n+1} + D\gamma_n + E\alpha_{n-1}\alpha_n\gamma_{n+1} + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\gamma_{n+1} + E\gamma_{n-1} = R\varphi_n. \end{aligned}$$

The last equation is satisfied if it is to be selected :

$$\begin{cases} A + C\beta_{n+1} + +D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1} = 0, \\ B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n + E\alpha_{n-1}\alpha_{n+1} + E\alpha_{n-1}\beta_n + E\beta_{n-1}\alpha_{n+1} = 0, \\ C\gamma_{n+1} + D\alpha_n\gamma_{n+1} + D\gamma_n + E\alpha_{n-1}\alpha_n\gamma_{n+1} + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\gamma_{n+1} + E\gamma_{n-1} = R\varphi_n. \end{cases}$$

Then we obtain the formulas for α_{n+1}, β_{n+1}, γ_{n+1},

$$\begin{aligned} \beta_{n+1} &= -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \\ \alpha_{n+1} &= -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \gamma_{n+1} &= +(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}), \end{aligned}$$

where $n = 2 : M - 2$.

After determining these values, we can use modified Gauss formula to compute U_n-s.

$$\{\ U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1} U_{n+2} + \gamma_{n+1}, \quad n = M - 3, ..., 2, 1,$$

where,

$$\begin{aligned} &U_M = 0, \\ &U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}], \\ &U_{M-2} = [(4I - \alpha_{M-2})]^{-1}[(\beta_{M-2} + 5I)U_{M-1} + \gamma_{M-2}], \\ &U_0 = 0. \end{aligned}$$

Algorithm

1. **Step;** Input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$.

2. **Step;** Substitute the second order difference approximations into the equations and write these equations in matrix form to obtain the equality

$$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \quad 2 \leq n \leq M-2.$$

3. **Step;** Determine the entries of the matrices A, B, C, D, E and R.

4. **Step;** $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are put.

5. **Step;** Compute α_{n+1}, β_{n+1}, γ_{n+1}, using the following formulas ($n = 2$ to $M-2$),

$$\beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A),$$

$$\alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n),$$

$$\gamma_{n+1} = (C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}),$$

where $n = 2 : M-2$.

6. **Step;** Compute

$$\begin{aligned} U_M &= 0, \\ U_{M-1} &= [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}], \\ U_{M-2} &= [(4I - \alpha_{M-2})]^{-1}\left[(\beta_{M-2} + 5I)\,U_{M-1} + (4I + \gamma_{M-2})\right]. \end{aligned}$$

7. **Step;** Compute U_n-s (n=M-3,...,2,1), using the following formula

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}.$$

Matlab Implementation on Second Order of Accuracy Difference Scheme

```
function [table,es,p]=ss(N,M) %p=parsecor %(sest,spt)=
   % mixed type
   % second order
   close;close; close;close;
   if nargin<1;N= 30 ; M= 30 ;end
   tau= 1/N ;
   h= 1 /M ; A=zeros(2*N+1,2*N+1);
   for i=2:N+1;
   A(i,i)=tau/(2*(h^4)); % diagonal
   end;
```

```
B=zeros(2*N+1,2*N+1);
for i=2:N+1;
B(i,i)=-1/(h^2)-2*tau/(h^4); %diagonal
end;
for i=N+2:2*N;
B(i,i)=-1/(2*h^2); %% 2. diagonal
end;
for i=N+2:2*N;
B(i,i-1)=-1/(4*h^2); % 2. diagonal altı
end;
for i=N+2:2*N;
B(i,i+1)=-1/(4*h^2) ; % 2. diagonal üstü
end;
%%%%%%%%%%%%%%
B(2*N+1,N+1)= tau/(h^2)-tau /(2*(h^2))-1/(h^2);
B(2*N+1,N+2)=-tau/(h^2);
%%%%%%%%%%%%%%
C=zeros(2*N+1,2*N+1);
C(1,1)=1;
C(1,2*N-1)=-1/(2*tau);
C(1,2*N)=4/(2*tau);
C(1,2*N+1)=-3/(2*tau)-1;
for i=2:N+1;
C(i,i-1)=-1/tau; %diagonal altı
end;
for i=2:N+1;
C(i,i)=2/(h^2)+1/tau+3*tau/(h^4); %diagonal
end;
for i=N+2:2*N;
C(i,i)=1/(h^2)-2/(tau^2); %% 2. diagonal
end;
for i=N+2:2*N;
C(i,i-1)=1/(2*h^2)+1/(tau^2); % 2. diagonal altı
end;
for i=N+2:2*N;
C(i,i+1)=1/(2*h^2)+1/(tau^2) ; % 2. diagonal üstü
end;
```

```
%C(2*N+1,N)=1/tau+2*tau/(h^2);
C(2*N+1,N+1)=-1/tau-2*tau/(h^2)+tau/(h^2)+2/(h^2);
C(2*N+1,N+2)=1/tau+2*tau/(h^2);
%%%%%%%%%%%%
E=A ;
D=B ;
for i=1:2*N+1; R(i,i)= 1 ; end;
%R(1,1)=0;
alpha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;
betha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;
gamma(2*N+1,1:1)= 0 ;
alpha(1:2*N+1,1:2*N+1,2:2) = (4/5)*eye(2*N+1) ; %%%%%%
betha(1:2*N+1,1:2*N+1,2:2) = (-1/5)*eye(2*N+1);
gamma(2*N+1,2:2)= 0 ;
'fi(j) = fi(k,j) hesaplanıyor ' ;
for j=2:M-1;
x=j*h;
fii(1,j:j)=2*sin(pi*x); %%% alfa(xn)
for k=2:N+1;
x=j*h;
t=(-N+k-1)*tau ;
t=t-tau/2;
fii( k, j:j ) = g(t,x)-(tau/(2*h^2))*(g(t,x+h)-2*g(t,x)+g(t,x-h));
end;
for k=N+2:2*N+1;
t=(-N+k-1)*tau ;
x=j*h;
fii( k, j:j ) = f(t,x) ; %(pi^2)*sin(pi*x); %%f(t,x,aaa);
end;
fii(2*N+1,j:j)=(tau/2)*f(0,x)+g(0,x);
end;
fii;
'alpha(N+1,N+1,j) ve betha(N+1,j) ler hesaplanacak' ;
for n = 2:M-2 ;
bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)...
+ E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;
betha(:,:,n+1:n+1 ) = - inv( bebek )*(A) ;
```

```
alpha(:,:,n+1:n+1) = - inv(bebek )*(B +D*betha(:,:,n:n) ...
+ E * alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*...
(R*fii(:,n:n) - D * gamma(:,n:n)...
-E * alpha(:,:,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
U(1:2*N+1,1:2*N+1)=nan;
U( 1:2*N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2) + 5*eye(2*N+1)) ...
- (4*eye(2*N+1)-alpha(:,:,M-2:M-2) )*alpha(:,:,M-1:M-1))...
*((4*eye(2*N+1)-alpha(:,:,M-2:M-2))*gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) = inv(4*eye(2*N+1)-alpha(:,:,M-2:M-2))*( (betha(:,:,M-2:M-2) +...
5* eye(2*N+1) )*U(:,M-1:M-1)+gamma(:,M-2:M-2) );
'INITIAL VALUEs OF U Is OBTAINED HERE' ;
for z = M-3:-1:1 ;
U(:,z:z )=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)+ ...
betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamma(:,z+1:z+1);
end;
for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;
'EXACT SOLUTION OF THIS PDE' ;
for j=1:M+1;
for k=1:2*N+1;
t=(-N+k-1)*tau;
x=(j-1)*h;
%exact solution on grid points,
es(k,j) = exact(t,x);
end;
end;
'ERROR ANALYSIS' ;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxes,maxapp,maxerror,relativeerror]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%
q=min(min(table)); w=max(max(table));
figure;
surf(es); title('EXACT SOLUTION');
set(gca,'ZLim',[q w]);
rotate3d;
figure;
surf(p); title('SECOND ORDER APPROXIMATE SOL.');
set(gca,'ZLim',[q w]);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function estx=exact(t,x)
estx= (1 - t)*(1 + t)*sin(pi*x);
function ftx=f(t,x)
ftx=(-2 - pi^2*(-1 + t^2))*sin(pi*x);
function gtx=g(t,x)
gtx=(-2*t - pi^2*(-1 + t^2))*sin(pi*x);
```

Second, we consider again the nonlocal boundary-value problem (5.1). The second order of accuracy in t for the approximate solutions of the nonlocal boundary value problem (5.1)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \tau^2\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{4h^4}\right) \\ = f(t_k,x_n),\ 1 \le k \le N-1, 2 \le n \le M-2, \\ \frac{u_n^k-u_n^{k-1}}{\tau} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \tau\left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{2h^4}\right) \\ = g(t_k-\frac{\tau}{2},x_n),\ -N+1 \le k \le 0, 2 \le n \le M-2, \\ \frac{u_n^1-u_n^0}{\tau} - \tau\left(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1-u_{n+1}^0+2u_n^0-u_{n-1}^0}{h^2}\right) = \frac{\tau}{2}\left(f(0,x_n)+\frac{1}{h^2}\left(u_{n+1}^0-2u_n^0+u_{n-1}^0\right)\right) \\ +\left(g(0,x_n)+\frac{1}{h^2}\left(u_{n+1}^0-2u_n^0+u_{n-1}^0\right)\right),\ x_n=nh,\ 1 \le n \le M-1, \\ u_n^{-N} = u_n^N + \frac{3u_n^N-4u_n^{N-1}+u_n^{N-2}}{2\tau} + \varphi(x_n),\ 0 \le n \le M, \\ u_0^k = u_M^k = 0, 0 \le k \le N, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k,\ u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k,\ 0 \le k \le N, \end{cases}$$

where

$$\begin{aligned} f(t,x) &= \left[-2+\left(1-t^2\right)\pi^2\right]\sin\pi x, \\ g(t,x) &= \left[-2t+\left(1-t^2\right)\pi^2\right]\sin\pi x, \\ \varphi(x) &= 2\sin\pi x \end{aligned}$$

are presented.

We have again the $(2N+1)\times(2N+1)$ system of linear equations. We will write in the matrix form. We will resort the system

$$\begin{cases} \left(\frac{\tau^2}{4h^4}\right)u_{n+2}^{k+1}+\left(-\frac{1}{h^2}\right)u_{n+1}^{k+1}+\left(-\frac{\tau^2}{h^2}\right)u_{n+1}^{k+1}+\left(-\frac{2}{\tau^2}+\frac{2}{h^2}\right)u_n^k+\left(\frac{1}{\tau^2}+\frac{3\tau^2}{2h^4}\right)u_n^{k+1} \\ +\left(-\frac{1}{h^2}\right)u_{n-1}^k+\left(-\frac{\tau^2}{h^4}\right)u_{n-1}^{k+1}+\left(\frac{\tau^2}{4h^4}\right)u_{n-2}^{k+1}=f\left(t_k,x_n\right),\ 1\le k\le N-1, 2\le n\le M-2, \\ \left(\frac{\tau}{2h^4}\right)u_{n+2}^k+\left(-\frac{2\tau}{h^4}-\frac{1}{h^2}\right)u_{n+1}^k+\left(-\frac{1}{\tau}\right)u_n^{k-1}+\left(\frac{3\tau}{h^4}+\frac{1}{\tau}+\frac{2}{h^2}\right)u_n^k \\ +\left(-\frac{2\tau}{h^4}-\frac{1}{h^2}\right)u_{n-1}^k+\left(\frac{\tau}{2h^4}\right)u_{n-2}^k=g\left(t_k-\frac{\tau}{2},x_n\right),\ \ -N+1\le k\le 0,\ 2\le n\le M-2, \\ u_n^{-N}=u_n^N+\frac{3u_n^N-4u_n^{N-1}+u_n^{N-2}}{2\tau}+\varphi\left(x_n\right),\ 0\le n\le M, \\ u_0^k=u_M^k=0, 0\le k\le N, \\ u_1^k=\frac{4}{5}u_2^k-\frac{1}{5}u_3^k,\ u_{M-1}^k=\frac{4}{5}u_{M-2}^k-\frac{1}{5}u_{M-3}^k,\ 0\le k\le N. \\ f(t,x)=\left[-2+\left(1-t^2\right)\pi^2\right]\sin\pi x, \\ g(t,x)=\left[-2t+\pi^2\left(1-t^2\right)\pi^2\right]\sin\pi x \end{cases}$$

where

$$\varphi_n^k=\begin{cases} 2\sin(\pi x_n), g(t_k-\frac{\tau}{2},x_n)-\frac{\tau}{2h^2}\left[g(t_k-\frac{\tau}{2},x_{n+1})+g(t_k-\frac{\tau}{2},x_n)+g(t_k-\frac{\tau}{2},x_{n-1})\right], \\ -N+1\le k\le 0, \\ f(t_k,x_n),\ 1\le k\le N-1, \\ 0,\ k=N. \end{cases}$$

We have that

$$\begin{cases} A\,U_{n+2}+B\,U_{n+1}+C\,U_n+DU_{n-1}+EU_{n-2}=R\varphi_n, \quad 2\le n\le M-2, \\ U_0=0,\ \ U_M,=0, \\ U_1=\frac{4}{5}U_2-\frac{1}{5}U_3, \\ U_{M-1}=\frac{4}{5}U_{M-2}-\frac{1}{5}U_{M-3}, \end{cases}$$

where

$$\varphi_n=\begin{bmatrix} \varphi_n^{-N} \\ \varphi_n^{-N+1} \\ \varphi_n^{-N+2} \\ .. \\ \varphi_n^{N-2} \\ \varphi_n^{N-1} \\ \varphi_n^{N} \end{bmatrix}_{(2N+1)\times 1}.$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & w & v & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & w & v & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & w & v \\ 0 & 0 & 0 & ... & m & n & ... & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & -\frac{1}{2\tau} & \frac{2}{\tau} & -\frac{3}{2\tau}-1 \\ 0 & a & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & b & c & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & b & c \\ 0 & 0 & ... & p & q & p & ... & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

where

$$\begin{aligned} x &= \frac{\tau}{2h^4},\ y=\frac{\tau^2}{4h^4}, \\ z &= -\frac{1}{h^2}-\frac{2\tau}{h^4},\ w=-\frac{1}{h^2},\ v=-\frac{\tau^2}{h^4}, \\ m &= \frac{\tau}{h^2}-\frac{\tau}{2h^2}-\frac{1}{h^2},\ n=\frac{1}{\tau}+\frac{2\tau}{h^2}, \\ a &= \frac{3\tau}{h^4}+\frac{1}{\tau}+\frac{2}{h^2},\ b=-\frac{2}{\tau^2}+\frac{2}{h^2},\ c=\frac{1}{\tau^2}+\frac{3\tau^2}{2h^4}, \\ p &= \frac{1}{\tau}+\frac{2\tau}{h^2},\ q=-\frac{1}{\tau}-\frac{2}{h^2}+\frac{\tau}{h^2}+\frac{2}{h^2}, \end{aligned}$$

and $D=B,\quad E=A,$

$$R=\begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$U_s = \begin{bmatrix} U_s^{-N} \\ U_s^{1} \\ U_s^{2} \\ U_s^{3} \\ .. \\ U_s^{N-1} \\ U_s^{N} \end{bmatrix}_{(2N+1)\times(1)}, \quad \text{where } s = n-2,\ n-1,\ n,\ n+1,\ n+2.$$

For the solution of the last matrix equation, we use the modified variant Gauss elimination method. We seek a solution of the matrix equation by the following form

$$\begin{cases} U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}, \quad n = M-2, ..., 2, 1, 0, \\ U_M = 0, \\ U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}], \\ U_{M-2} = [(4I - \alpha_{M-2})]^{-1}[(\beta_{M-2} + 5I)U_{M-1} + \gamma_{M-2}]. \end{cases}$$

Here α_j , β_j $(j = 1 : M-1)$ are $(2N+1)\times(2N+1)$ square matrices and γ_j-s are $(2N+1)\times 1$ column matrices.

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}, \quad \beta_1 = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$\gamma_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .. \\ 0 \end{bmatrix}_{(2N+1)\times(1)}, \quad \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .. \\ 0 \end{bmatrix}_{(2N+1)\times(1)}$$

$$\alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & 0 & ... & 0 \\ 0 & \frac{4}{5} & 0 & ... & 0 \\ 0 & 0 & \frac{4}{5} & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \frac{4}{5} \end{bmatrix}_{(2N+1)\times(2N+1)}, \quad \beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & 0 & ... & 0 \\ 0 & -\frac{1}{5} & 0 & ... & 0 \\ 0 & 0 & -\frac{1}{5} & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & -\frac{1}{5} \end{bmatrix}_{(2N+1)\times(2N+1)}$$

Using the equality

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}$$

and the equality

$$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n,$$

we can write

$$[A + C\beta_{n+1} + D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1}]U_{n+2}$$

$$+[B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n + E\alpha_{n-1}\alpha_{n+1} + E\alpha_{n-1}\beta_n + E\beta_{n-1}\alpha_{n+1}]U_{n+1}$$

$$+C\gamma_{n+1}+D\alpha_n\gamma_{n+1}+D\gamma_n+E\alpha_{n-1}\alpha_n\gamma_{n+1}+E\alpha_{n-1}\alpha_n+E\beta_{n-1}\gamma_{n+1}+E\gamma_{n-1}=R\varphi_n.$$

The last equation is satisfied if it is to be selected

$$\begin{cases} A+C\beta_{n+1}++D\alpha_n\beta_{n+1}+E\alpha_{n-1}\alpha_n\beta_{n+1}+E\beta_{n-1}\beta_{n+1}=0, \\ B+C\alpha_{n+1}+D\alpha_n\alpha_{n+1}+D\beta_n+E\alpha_{n-1}\alpha_{n+1}+E\alpha_{n-1}\beta_n+E\beta_{n-1}\alpha_{n+1}=0, \\ C\gamma_{n+1}+D\alpha_n\gamma_{n+1}+D\gamma_n+E\alpha_{n-1}\alpha_n\gamma_{n+1}+E\alpha_{n-1}\alpha_n+E\beta_{n-1}\gamma_{n+1}+E\gamma_{n-1}=R\varphi_n. \end{cases}$$

Then we obtain the formulas

$$\alpha_{n+1}=-(C+D\alpha_n+E\beta_{n-1}+E\alpha_{n-1}\alpha_n)^{-1}(B+D\beta_n+E\alpha_{n-1}\beta_n),$$

$$\beta_{n+1}=-(C+D\alpha_n+E\beta_{n-1}+E\alpha_{n-1}\alpha_n)^{-1}(A),$$

$$\gamma_{n+1}=(C+D\alpha_n+E\beta_{n-1}+E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n-D\gamma_n-E\alpha_{n-1}\gamma_n-E\gamma_{n-1}),$$

where $n=2:M-2$.

After determining these values, we can use modified Gauss formula to compute U_n-s.

$$\{\ U_n=\alpha_{n+1}U_{n+1}+\beta_{n+1}U_{n+2}+\gamma_{n+1},\quad n=M-3,...,2,1,$$

where,

$$\begin{aligned} &U_M=0, \\ &U_{M-1}=[(\beta_{M-2}+5I)-(4I-\alpha_{M-2})\alpha_{M-1}]^{-1}[(4I-\alpha_{M-2})\gamma_{M-1}-\gamma_{M-2}], \\ &U_{M-2}=[(4I-\alpha_{M-2})]^{-1}[(\beta_{M-2}+5I)U_{M-1}+\gamma_{M-2}], \\ &U_0=0. \end{aligned}$$

Algorithm

1. **Step;** Input time increment $\tau=\frac{1}{N}$ and space increment $h=\frac{1}{M}$.
2. **Step;** Substitute the second order difference approximations into the equations and write these equations in matrix form to obtain the equality

$$A\,U_{n+2}+B\,U_{n+1}+C\,U_n+DU_{n-1}+EU_{n-2}=R\varphi_n,\quad 2\le n\le M-2.$$

3. **Step;** Determine the entries of the matrices A, B, C, D, E and R.
4. **Step;** $\alpha_1,\beta_1,\gamma_1$ and $\alpha_2,\beta_2,\gamma_2$ are put.
5. **Step;** Compute α_{n+1}, β_{n+1}, γ_{n+1}, using the following formulas (n=2 to M-2)

$$\beta_{n+1}=-(C+D\alpha_n+E\beta_{n-1}+E\alpha_{n-1}\alpha_n)^{-1}(A),$$

$$\alpha_{n+1}=-(C+D\alpha_n+E\beta_{n-1}+E\alpha_{n-1}\alpha_n)^{-1}(B+D\beta_n+E\alpha_{n-1}\beta_n),$$

$$\gamma_{n+1}=(C+D\alpha_n+E\beta_{n-1}+E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n-D\gamma_n-E\alpha_{n-1}\gamma_n-E\gamma_{n-1}).$$

where $n = 2 : M - 2$.

6. Step; Compute

$$U_M = 0,$$

$$U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}],$$

$$U_{M-2} = [(4I - \alpha_{M-2})]^{-1}[(\beta_{M-2} + 5I)U_{M-1} + (4I + \gamma_{M-2}].$$

7. Step; Compute U_n-s (n=M-3,...,2,1), using the following formula

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}.$$

Matlab Implementation on Modified Second Order of Accuracy Difference Scheme

```
function [table,es,p]=ss(N,M)
  % mixed type
  % second order . % modified
  if nargin <1; N=30 ; M=30 ;end;
  tau= 1/N ;
  h= 1 /M ;
  A=zeros(2*N+1,2*N+1);
  for i=2:N+1;
  A(i,i)=tau/(2*(h^4)); % diagonal
  end;
  for i=N+2:2*N;
  A(i,i+1)=(tau^2)/(4*(h^4)); % 2. diagonal
  end;
  B=zeros(2*N+1,2*N+1);
  for i=2:N+1;
  B(i,i)=-1/(h^2)-2*tau/(h^4); %diagonal
  end;
  for i=N+2:2*N;
  B(i,i)=-1/(h^2); %% 2. diagonal
  end;
  for i=N+2:2*N;
  B(i,i+1)=-(tau^2)/(h^4) ; % 2. diagonal üstü
  end;
  %%%%%%%%%%%%%%
  B(2*N+1,N+1)= tau/(h^2)-tau /(2*(h^2))-1/(h^2);
  B(2*N+1,N+2)=-tau/(h^2);
```

```
%%%%%%%%%%%%%%
C=zeros(2*N+1,2*N+1);
C(1,1)=1;
C(1,2*N-1)=-1/(2*tau);
C(1,2*N)=4/(2*tau);
C(1,2*N+1)=-3/(2*tau);
for i=2:N+1;
C(i,i-1)=-1/tau; %diagonal altı
end;
for i=2:N+1;
C(i,i)=2/(h^2)+1/tau+3*tau/(h^4); %diagonal
end;
for i=N+2:2*N;
C(i,i)=2/(h^2)-2/(tau^2); %% 2. diagonal
end;
for i=N+2:2*N;
C(i,i-1)=1/(tau^2); % 2. diagonal altı
end;
for i=N+2:2*N;
C(i,i+1)=3*(tau^2)/(2*h^4)+1/(tau^2) ; % 2. diagonal üstü
end;
%C(2*N+1,N)=1/tau+2*tau/(h^2);
C(2*N+1,N+1)=-1/tau-2*tau/(h^2)+tau/(h^2)+2/(h^2);
C(2*N+1,N+2)=1/tau+2*tau/(h^2);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
E=A ;
D=B ;
for i=1:2*N+1; R(i,i)= 1 ; end;
%R(1,1)=0;
alpha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;
betha(1:2*N+1,1:2*N+1,1:1) = 0*eye(2*N+1) ;
gamma(2*N+1,1:1)= 0 ;
alpha(1:2*N+1,1:2*N+1,2:2) = (4/5)*eye(2*N+1) ; %%%%%%
betha(1:2*N+1,1:2*N+1,2:2) = (-1/5)*eye(2*N+1);
gamma(2*N+1,2:2)= 0 ;
'fi(j) = fi(k,j) hesaplanıyor ' ;
for j=2:M-1;
```

```
x=j*h;
fii(1,j:j)=2*sin(pi*x);   %%% alfa(xn)
for k=2:N+1;
x=j*h;
t=(-N+k-1)*tau ;
t=t-tau/2;
fii( k, j:j ) = g(t,x)-(tau/(2*h^2))*(g(t,x+h)-2*g(t,x)+g(t,x-h));
end;
for k=N+2:2*N+1;
t=(-N+k-1)*tau ;
x=j*h;
fii( k, j:j ) = f(t,x) ;   %(pi^2)*sin(pi*x);   %%f(t,x,aaa);
end;
fii(2*N+1,j:j)=(tau/2)*f(0,x)+g(0,x);
end;
fii;
'alpha(N+1,N+1,j) ve betha(N+1,j) ler hesaplanacak' ;
for n = 2:M-2 ;
bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)...
+ E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;
betha(:,:,n+1:n+1 ) = - inv( bebek )*(A) ;
alpha(:,:,n+1:n+1) = - inv(bebek )*(B +D*betha(:,:,n:n) ...
+ E * alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*...
(R*fii(:,n:n) - D * gamma(:,n:n)...
-E * alpha(:,:,n-1:n-1)* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
'U degerleri olushturuluyor. ve matrisine atılıyor.';
U(1:2*N+1,1:2*N+1)=nan;
U( 1:2*N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2) + 5*eye(2*N+1)) ...
- (4*eye(2*N+1)-alpha(:,:,M-2:M-2) )*alpha(:,:,M-1:M-1))...
*((4*eye(2*N+1)-alpha(:,:,M-2:M-2))*gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) = inv(4*eye(2*N+1)-alpha(:,:,M-2:M-2))*( (betha(:,:,M-2:M-2) +...
5* eye(2*N+1) )*U(:,M-1:M-1)+gamma(:,M-2:M-2) );
'INITIAL VALUEs OF U Is OBTAINED HERE' ;
for z = M-3:-1:1 ;
```

```
U(:,z:z )=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)+ ...
betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamma(:,z+1:z+1);
end;
for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;
'EXACT SOLUTION OF THIS PDE' ;
for j=1:M+1;
for k=1:2*N+1;
t=(-N+k-1)*tau;
x=(j-1)*h;
%exact solution on grid points,
es(k,j) = exact(t,x);
end;
end;
'ERROR ANALYSIS' ;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxes,maxapp,maxerror,relativeerror]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%
q=min(min(table)); w=max(max(table));
figure;
surf(es); title('EXACT SOLUTION'); set(gca,'ZLim',[q w]);
rotate3d;
figure;
surf(p); title('MODIFIED SECOND ORDER APP. SOL.'); rotate3d ;
set(gca,'ZLim',[q w]);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function estx=exact(t,x)
estx= (1 - t)*(1 + t)*sin(pi*x);
function ftx=f(t,x)
ftx=(-2 - pi^2*(-1 + t^2))*sin(pi*x);
```

```
function gtx=g(t,x)
gtx=(-2*t - pi^2*(-1 + t^2))*sin(pi*x);
```

5.3 Numerical Analysis

Consider the nonlocal boundary value problem for hyperbolic-parabolic equation (3.1). For the approximate solutions of the nonlocal boundary value problem (3.1), the first and the second order of accuracy difference schemes with $\tau = \frac{1}{30}$, $h = \frac{\pi}{30}$ will be used. The exact and numerical solutions are given in the figures 5.1, 5.2, 5.3 and 5.4.

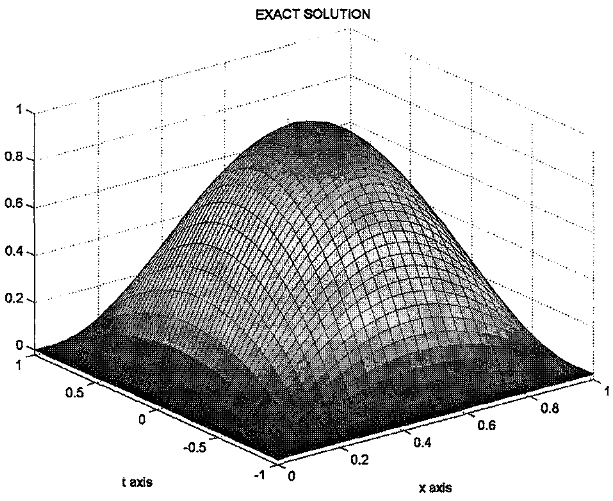

Figure 5.1: Exact Solution

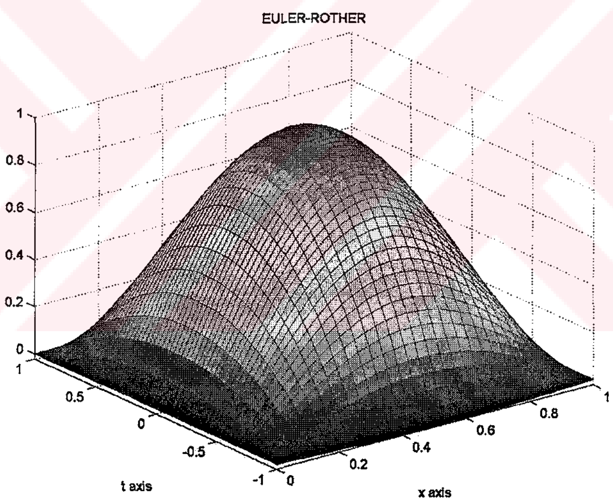

Figure 5.2: The first order of accuracy difference scheme

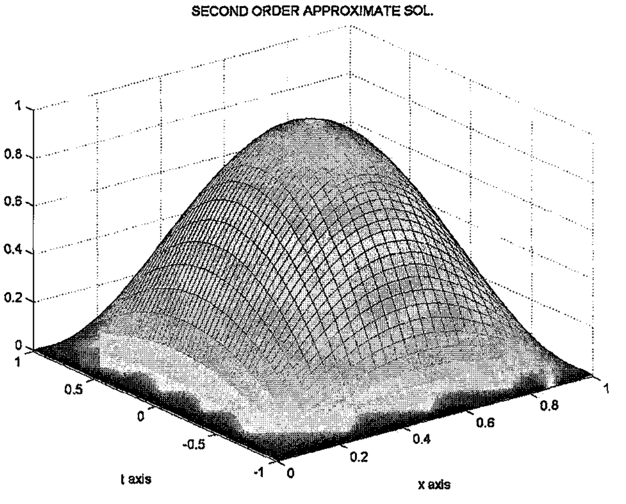

Figure 5.3: The second order of accuracy difference schemes (4.54)

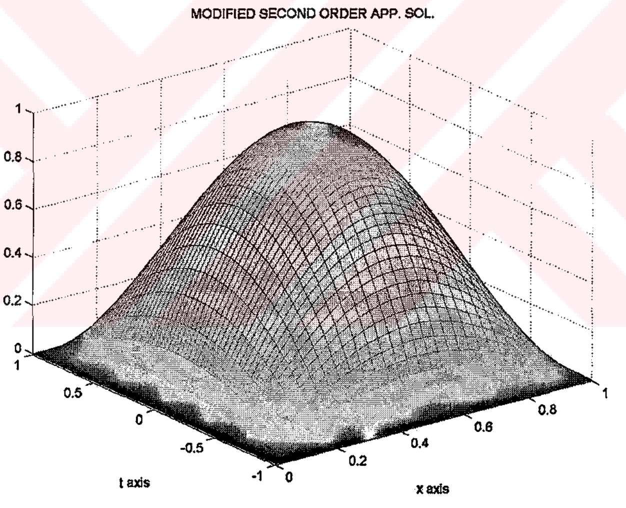

Figure 5.4: The second order of accuracy difference schemes (4.55)

The errors

$$E_M^N = \max_{-N \le k \le N, 1 \le n \le M-1} \left| u(t_k, x_n) - u_n^k \right|$$

of the numerical solutions are given in the following table.

Difference schemes	E_M^N
The first order of accuracy difference scheme (4.1)	0.0214
The second order of accuracy difference scheme (4.54)	0.0060
The second order of accuracy difference scheme (4.55)	0.0052

Thus, the second order of accuracy difference schemes are more accurate comparing with the first order of accuracy difference scheme.

CHAPTER 6

CONCLUSIONS

This work is devoted to the study of the stability of the nonlocal boundary value problem for hyperbolic-parabolic equations. The following original results are obtained:

- the abstract theorem on the stability estimates for the solution of the nonlocal boundary problem for hyperbolic-parabolic equation in a Hilbert space is proved,
- the theorems on the stability estimates for the solution of the nonlocal boundary problems for hyperbolic-parabolic equations are obtained,
- the first and second order of accuracy difference schemes for the approximate solutions of the nonlocal boundary problem for hyperbolic-parabolic differential equations are presented,
- the abstract theorems on the stability estimates for the solution of the first and second order of accuracy difference schemes for the approximate solutions of the nonlocal boundary problem for hyperbolic-parabolic differential equation in a Hilbert space are established,
- the theorems on the stability estimates for the solution of difference schemes for hyperbolic-parabolic equations are obtained,
- the theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments
- two papers from this work are submitted for publication in the journals.

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