AN INVESTIGATION OF DIFFERENCE SCHEMES FOR HYPERBOLIC EQUATIONS

by

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ABSTRACT

The initial value problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + A(t)u(t) = f(t), \ 0 \le t \le 1, \\ u(0) = \psi, \ u'(0) = \varphi \end{cases}$$

for differential equation in a Hilbert space *H* with the self-adjoint positive definite operators *A(t)* is considered. The second order of accuracy difference schemes for the approximate solutions of this initial value problem are presented. The stability estimates for the solution of these difference schemes are established. The Matlab implementation of these difference schemes for hyperbolic equation is presented. The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

Keywords: Hyperbolic Equation, Difference Schemes, Stability Estimates, Numerical Solutions, Matlab Implementation.

HİPERBOLİK DENKLEMLERİN FARK ŞEMALARININ BİR TÜR ARAŞTIRMASI

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ÖZ

Hilbert uzayında self-adjoint pozitif tanımlı *A(t)* operatörlü diferansiyel denklemlerinin ilk değer problemi

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + A(t)u(t) = f(t), \; 0 \le t \le 1, \\ u(0) = \psi, \;\; u'(0) = \varphi, \end{cases}$$

ele alınmıştır. Bu ilk değer probleminin yaklaşık çözümü için ikinci derecedeki fark şemaları sunulmuştur. Bu fark şemalarının çözümü için kararlılık kestirimleri kurulmuştur. Hiperbolik denklemler için fark şemalarının Matlab ile çözümleri elde edilmiştir. Bu fark şemalarının çözümü için bulunan teorik sonuçlar, sayısal örneklerle desteklenmiştir.

Anahtar Kelimeler: Hiperbolik Denklem, Fark Şemaları, Kararlılık Kestirimleri, Sayısal Çözümler, Matlab Uygulamaları.

Contents

CHAPTER 1

INTRODUCTION

It is known that many problems in fluid mechanics, dynamics, elasticity and other areas of engineering, physics and biological systems lead to partial differential equations of hyperbolic type.

Methods of solutions of initial-boundary value problems for hyperbolic differential equations have been studied extensively by many researches (see [Sobolevskii, P. E. and Pogorelenko, V. A., 1967], [Sobolevskii, P. E. and Chebotaryeva, L. M., 1977], [Goldberg, M. and Tadmor, E., 1978], [Goldberg, M. and Tadmor, E., 1981], [Goldberg, M. and Tadmor, E., 1982], [Goldberg, M. and Tadmor, E., 1985], [Goldberg, M. and Tadmor, E., 1987], [Tadmor, E., 1987], [Renaut, R. A. and Petersen, J., 1989], [Duncan, D. B. and Lynch, M. A. M., 1991], [Ashyralyev, A. and Fattorini, H.O., 1992], [Ashyralyev, A. and Muradov, I., 1995], [Mohanty, R. K., Jain, M. K., and George, K., 1996], [Mohanty, R. K., Jain, M. K., and George, K., 1996], [Lasiecka, I., Triggianni, R., and Yao, P. F., 1997], [Krupa, V. G., 1997], [D'Ancona, P. and Racke, R., 1998], [Jeltsch, R., Renaut, R. A., and Smit, J. H., 1998], [Mohanty, R. K., Jain, M. K. and George, K., 1998], [Chen, Y. and Huang, Y., 1998], [Ashyralyev, A. and Muradov, I., 1998], [Cicognani, M. and Zanghirati, L., 1999], [Peiguang, W., 1999], [Peiguang, W., 1999], [Juras, M., 2000], [Kiguradze, T., 2000], [Chawla, M. M. and Al-Zanaidi, M. A., 2001], [Minamoto, T., 2001], [Ashyralyev, A. and Sobolevskii, P. E., 2001], [Kiguradze, T. and Lakshmikantham, V., 2002], [Kiguradze, T. and Lakshmikantham, V., 2002], [Cockburn, B., Luskin, M., Shu, C. and Suli, E., 2003], [Rasulov, M., Coskun, E. and Sinsoyasal, B., 2003], [Lotstedt, P., Ramage, A., Sydow, L. and Soderberg, S., 2004], [Mohanthy, R. K., 2004], [Mohanthy, R. K., 2004], [Xiao, T. and Liang, J., 2004], [Rasulov, M., Karaguler, T. and Sinsoyasal, B., 2004], [Ashyralyev, A. and Aggez, N., 2004], [Guidetti, D., Karasozen, B. and Piskarev, S., 2004], [Ashyralyev, A. and Sobolevskii, P. E., 2005], [Qikui, D. and Minxia, T., 2005], [Mohanty, R. K., 2005] and the references therein).

Our goal in this work, is to investigate the stability of difference schemes of approximate solutions of initial-boundary value problems for equations of hyperbolic type.

It is known that the mixed problem for hyperbolic equations can be solved analytically by Fourier series, Fourier transform and Laplace transform methods. Now, let us illustrate these three different analytical methods by examples.

Example 1.1. Consider the following simple initial-boundary value problem for hyperbolic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = [2 + 2(1-t)^2 - 8t(1-t) + 2t^2 + 2t^2(1-t)^2] \cos x, \\ 0 < t \le 1,\ 0 < x < \pi, \\ u\,(0,x) = \cos x,\ u_t\,(0,x) = 0, \quad 0 \le x \le \pi, \\ u_x\,(t,0) = u_x\,(t,\pi) = 0, \quad 0 \le t \le 1. \end{cases} \tag{1.1}$$

For the solution of the problem (1.1), we use the Fourier series method. In order to solve the problem we need to separate $u(t,x)$ into two parts

$$u(t,x) = v(t,x) + w(t,x) \tag{1.2}$$

where $v(t,x)$ is the solution of the problem

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + v = 0,\ 0 < t \le 1, \quad 0 < x < \pi, \\ v\,(0,x) = \cos x,\ v_t\,(0,x) = 0, \quad 0 \le x \le \pi, \\ v_x\,(t,0) = v_x\,(t,\pi) = 0, \quad 0 \le t \le 1. \end{cases} \tag{1.3}$$

and

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + w = [2 + 2(1-t)^2 - 8t(1-t) + 2t^2 + 2t^2(1-t)^2] \cos x, \\ 0 < t \le 1,\ 0 < x < \pi, \\ w\,(0,x) = w_t\,(0,x) = 0, \quad 0 \le x \le \pi, \\ w_x\,(t,0) = w_x\,(t,\pi) = 0, \quad 0 \le t \le 1. \end{cases} \tag{1.4}$$

Now, let us obtain the solution of (1.3), by the method of separation of variables. To do this a solution of the form

$$v(t,x) = T(t)X(x) \ne 0$$

is suggested. Taking the partial derivatives and substituting the result in (1.3), we obtain

$$\frac{T''(t) + T\,(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0,$$

or

$$\frac{T''(t) + T\,(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda. \tag{1.5}$$

The boundary conditions presented in (1.3), require $X'(0) = X'\,(\pi) = 0$. Hence, from (1.5) we have the ordinary differential equation

$$X''(x) = \lambda X(x),\ X'(0) = X'\,(\pi) = 0. \tag{1.6}$$

If $\lambda > 0$, then the boundary value problem (1.6) has only trivial solution $X(x) = 0$. For $\lambda < 0$, the nontrivial solutions of the boundary value problem (1.6) are

$$X_k(x) = \cos(kx), \quad \text{where } k = 1, 2, 3, \cdots, \quad \lambda = -k^2$$

For $\lambda = 0$, the nontrivial solution of the boundary value problem (1.6) is

$$X_0(x) = 1.$$

So, the nontrivial solutions of the boundary value problem (1.6) are

$$X_k(x) = \cos(kx), \quad \text{where } k = 0, 1, 2, 3, \cdots, \quad \text{and } \lambda = -k^2$$

The other ordinary differential equation presented in (1.5) is

$$T''(t) + T(t) = \lambda T(t),$$

with $\lambda = -k^2$, $k = 0, 1, 2, \cdots$. The solution of this ordinary differential equation is

$$T_k(t) = A_k \sin\sqrt{1+k^2}t + B_k \cos\sqrt{1+k^2}t, \quad \text{where } k = 0, 1, 2, 3, \cdots.$$

Thus,

$$v(t,x) = \sum_{k=1}^{\infty} v_k(t,x) = \sum_{k=1}^{\infty}\left(A_k \sin\sqrt{1+k^2}t + B_k \cos\sqrt{1+k^2}t\right)\cos kx.$$

Using the initial conditions

$$v(0,x) = \cos x, \;\; v_t(0,x) = 0,$$

we obtain

$$A_1 = 0, \;\; B_1 = 1, \;\; A_k = B_k = 0, \;\; k = 0, 2, 3, \cdots.$$

Hence, the solution of (1.3) is

$$v(t,x) = \cos\sqrt{2}t\cos x.$$

Second we obtain the solution of (1.4). We seek a solution of the form

$$w(t,x) = \sum_{k=0}^{\infty} E_k(t)\cos kx.$$

Then

$$w_{tt} - w_{xx} + w = \sum_{k=0}^{\infty}[E_k''(t) + \left(1+k^2\right)E_k(t)]\cos kx$$
$$= \left[2 + 2(1-t)^2 - 8t(1-t) + 2t^2 + 2t^2(1-t)^2\right]\cos x.$$

If $k \neq 1$,then

$$E_k''(t) + \left(1+k^2\right)E_k(t) = 0.$$

So we obtain

$$E_k(t) = c_1\cos\sqrt{1+k^2}t + c_2\sin\sqrt{1+k^2}t.$$

Using the initial conditions

$$E_k(0) = E_k'(0) = 0,$$

we get $c_1 = c_2 = 0$ and $E_k(t) = 0$.
$k = 1$, then

$$E_1''(t) + 2E_1(t) = \left[2 + 2(1-t)^2 - 8t(1-t) + 2t^2 + 2t^2(1-t)^2\right].$$

Using the initial conditions

$$E_1(0) = E_1'(0) = 0,$$

we get $c_1 = c_2 = 0$ and $E_1(t) = t^2(1-t)^2$. Thus the solution of (1.4) is

$$w(t,x) = [1 + t^2(1-t)^2] \cos x.$$

Finally, using (1.2) we obtain

$$u(t,x) = v(t,x) + w(t,x) = [\cos\sqrt{2}t + 1 + t^2(1-t)^2] \cos x.$$

Note that using similar procedure one can obtain the solution of the following initial-boundary value problem for the multidimensional hyperbolic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, \cdots, x_n) \in \overline{\Omega}, \quad 0 < t \leq T, \\ u(0,x) = \psi(x), \quad x \in \overline{\Omega} \\ u_t(0,x) = \varphi(x), \quad x \in \overline{\Omega}, \\ \frac{\partial u}{\partial n}(t,x) = 0, \quad x \in S \end{cases}$$

where $\alpha_r \geq a > 0$ and $f(t,x)$ $(t \in [0,T]$, $x \in \overline{\Omega})$, $\varphi(x), \psi(x)$ $(x \in \overline{\Omega})$ are given smooth functions. Here Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with the boundary

$$S, \quad \overline{\Omega} = \Omega \cup S.$$

Here $\frac{\partial}{\partial n}$ indicates differentiation in the direction of the exterior normal to S.

However, the method of separation of variables described in solving (1.3) can be used only in the case when (1.1) has constant coefficients.

Example 1.2. Now, we will consider the application of Laplace transformation method (in x) to the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = [2(1-t)^2 - 8t(1-t) + 2t^2]\, e^{-x}, \quad 0 < t \leq 1, \ 0 < x < \infty, \\ u(0,x) = u_t(0,x) = 0, \quad 0 < x < \infty, \\ u(t,0) = t^2(1-t)^2, \quad u_x(t,0) = -t^2(1-t)^2, \quad 0 \leq t \leq 1. \end{cases} \tag{1.7}$$

We denote

$$\mathbf{L}\{u(t,x)\} = v(t,s).$$

Then using the properties of the Laplace transform, we obtain

$$v_{tt}(t,s) + \left(1 - s^2\right) v(t,s) = \frac{12t^2 - 12t + 2}{s+1} + t^2(1-t)^2(1-s).$$

Solving it we can write

$$v(t,s) = c_1 e^{\sqrt{-1+s^2}t} + c_2 e^{-\sqrt{-1+s^2}t} + \frac{t^2(1-t)^2}{(s+1)}.$$

Now, using the initial conditions (1.7) which are transformed to

$$v\,(0,s)=0, v_t\,(0,s)=0,$$

we obtain

$$v\,(t,s)=\frac{t^2(1-t)^2}{(s+1)}.$$

Finally taking the inverse Laplace transform of this equation, we obtain

$$u\,(t,x)=\mathbf{L}^{-1}\left\{v\,(t,s)\right\}=\mathbf{L}^{-1}\left\{\frac{t^2(1-t)^2}{(s+1)}\right\}$$

$$=t^2(1-t)^2\mathbf{L}^{-1}\left\{\frac{1}{(s+1)}\right\}=t^2(1-t)^2e^{-x}.$$

Hence, the solution of (1.7) is

$$u\,(t,x)=t^2(1-t)^2e^{-x}.$$

Note that using similar procedure one can obtain the solution of the following initial-boundary value problem for the multidimensional hyperbolic equation

$$\begin{cases} \frac{\partial^2u(t,x)}{\partial t^2}-\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2u(t,x)}{\partial x_r^2}=f(t,x), \\ x=(x_1,\cdots,x_n)\in\overline{\Omega}^+,0\le t\le T, \\ u(0,x)=\varphi(x), \\ u_t(0,x)=\psi(x),\quad x\in\overline{\Omega}^+ \\ u(t,x)=0,\ x\in S^+, \end{cases}$$

where $\alpha_r\ \ge a>0$ and $f\,(t,x)\ \left(t\in[0,T]\,,\ x\in\overline{\Omega}^+\right),\ \varphi\,(x)\,,\psi(x)\ \left(x\in\overline{\Omega}^+\right)$ are given smooth functions. Here $\Omega\ ^+$ is the open set in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<\infty,\ 1\le k\le n)$ with boundary

$$S^+,\ \ \overline{\Omega}^+=\Omega^+\cup S^+.$$

However, as the method of separation of variables, Laplace transform method can be used only in the case when the differential equation has constant coefficients.

Example 1.3. The last example is a mixed problem solved by using Fourier transform method. Consider the problem

$$\begin{cases} \frac{\partial^2u}{\partial t^2}-\frac{\partial^2u}{\partial x^2}+u=\left[2(1-t)^2-8t(1-t)+2t^2+t^2(1-t)^2\left(3-4x^2\right)\right]e^{-x^2}, \\ 0<t\le 1,\quad -\infty<x<\infty, \\ u\,(0,x)=u_t\,(0,x)=0,\quad -\infty<x<\infty. \end{cases} \tag{1.8}$$

We denote

$$v\,(t,s)=\mathbf{F}\left\{u\,(t,x)\right\}.$$

Then, taking the Fourier transform of both sides of the differential equation in (1.8), we obtain

$$v_{tt}(t,s)+\left(s^2+1\right)v(t,s)$$
$$=\mathbf{F}\left\{\left[2(1-t)^2-8t(1-t)+2t^2+t^2(1-t)^2\left(1+s^2\right)\right]e^{-x^2}\right\}.$$

Solving it we can write

$$v(t,s)=c_1\cos\sqrt{s^2+1}t+c_2\sin\sqrt{s^2+1}t+t^2(1-t)^2\mathbf{F}\left\{e^{-x^2}\right\}.$$

Now using the initial conditions in (1.8) which are transformed to

$$v(0,s)=0, v_t(0,s)=0,$$

we get

$$v(t,s)=t^2(1-t)^2\mathbf{F}\left\{e^{-x^2}\right\}.$$

Finally taking the inverse of Fourier transformation we obtain the solution for the problem (1.8) as

$$u(t,x)=t^2(1-t)^2e^{-x^2}.$$

Note that using the same manner one obtains the solution of the following initial value problem for the 2m-th order multidimensional hyperbolic equation

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}-\sum\limits_{|r|=2m}a_r\frac{\partial^{|\tau|}u}{\partial x_1^{r_1}\dots\partial x_n^{r_n}}=f(t,x),\\ 0\le t\le T, x,r\in\mathbb{R}^n,\quad |r|=r_1+\dots+r_n,\\ u(0,x)=\varphi(x),\quad x\in\mathbb{R}^n,\\ u_t(0,x)=\psi(x),\quad x\in\mathbb{R}^n.\end{cases}$$

where $\alpha_r\ge a>o$, $f(t,x)$ $(t\in[0,T],\ x\in R^n)$, $\varphi(x),\psi(x)$ $(x\in R^n)$ are given smooth functions.

However, all analytical methods described above, namely the Fourier series method, the Laplace transform method and the Fourier transform method can be used only when the differential equation has constant coefficients. It is well-known that the most general method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method, which is basially realized by digital computers and known to be numerical method. However the stability of different difference schemes used in numerical methods need to be proved or justified theoretically.

It is known that various initial-boundary value problems for hyperbolic equations can be reduced to the initial value problem for differential equation in a Hilbert space H with self-adjoint positive definite operator A. In the present work the initial value problem

$$\begin{cases}\frac{d^2u(t)}{dt^2}+A(t)u(t)=f(t),\ 0\le t\le 1,\\ u(0)=\varphi, u'(0)=\psi\end{cases}$$

for differential equation in a Hilbert space H, with the self-adjoint positive definite operator $A(t)$ is considered. The second order of accuracy difference schemes are constructed.Applying the operator approach, the stability of difference schemes for the approximate solution of differential equations are obtained. The theoretical statements for the solution of this difference schemes are supported by the results of numerical examples.

Let us briefly describe the contents of the various sections of the thesis. It consists of five chapters.

First chapter is the introduction.

Second chapter presents elementary statements in a Hilbert space that is needed for this work.

Third chapter consists of two sections.The stable second order of accuracy difference schemes approximately solving the initial value problem for hyperbolic equation in a Hilbert space H with self-adjoint positive definite operator $A(t)$ are presented. The stability estimates for the solutions of the difference schemes of the initial value problems for hyperbolic equations are obtained.

Fourth chapter devoted to the applications. The first and second order of accuracy difference schemes are studied. A matlab program is given to conclude that the second order of accuracy is more accurate. Figures and tableS are included.

Fifth chapter containes conclusions.

CHAPTER 2

ELEMENTS OF HILBERT SPACE

This chapter covers selected concepts of the elementary Hilbert space theory as developed in [Krein, S. G., 1966]. It also includes tha basis for the solution proporties in an Hilbert space of the initial value problem considered in this thesis.

2.1 HILBERT SPACE

Definition 2.1. A complex linear space H is called an inner product space if there is a complex-valued function $\langle .,. \rangle : H \times H \to C$ with the properties

$i.$ $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = \sigma,$

$ii.$ $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in H$,

$iii.$ $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle,$ for all $x, y \in H$ and $\alpha \in C,$

$iv.$ $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in H.$

The function $\langle x, y \rangle$ is called the inner product of x and y, C is the set of complex numbers and the overline indicates the complex conjugate. A Hilbert space is a complete inner product space. An inner product on H defines a norm on H given by $\|x\| = \langle x, x \rangle^{1/2}$.

Example 2.1. *The space $C_2[-1, 1]$ of all defined and continuous functions on a given closed interval $[-1, 1]$ is an inner product space with the inner product given by*

$$\langle x, y \rangle = \int_{-1}^{1} x(t)\overline{y(t)}dt. \tag{2.1}$$

Note that the space $C_2[-1, 1]$ is not complete. So, $C_2[-1, 1]$ is not a Hilbert space (The overline denotes the closure).

Example 2.2. *The space $L_2[-1, 1] = \overline{C_2[-1, 1]}$ with the inner product (2.1) is a Hilbert space.*

Theorem 2.1. *Let x, y be any two vectors in a Hilbert space, then*

$$|\langle x, y \rangle| \leq \|x\| \, \|y\| \quad (\text{ Schwartz inequality}). \tag{2.2}$$

Note that the inner product is related to the norm by the following identity

$$\langle x, y \rangle = \frac{1}{4}\left[\left(\|x + y\|^2 - \|x - y\|^2\right) + i\left(\|x + iy\|^2 - \|x - iy\|^2\right)\right]. \tag{2.3}$$

Theorem 2.2. *If H is a Hilbert space, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \ , \quad \forall\ x, y \in H. \ \textit{(Parallelogram law)} \tag{2.4}$$

This theorem states that a norm on an inner product space satisfies the important Parallelogram law. Conversely, H is a complex complete normed space with the norm $\|\cdot\|$ satisfying the equation (2.4) then H is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ satisfying $\|x\| = \langle x, x\rangle^{1/2}$.

Example 2.3. *The space l^p of all sequence, $x = (\xi_i) = (\xi_1, \xi_2, \cdots)$ such that $|\xi_1|^p + |\xi_2|^p + \cdots$ converges with $p \neq 2$ is not an inner product space, hence not a Hilbert space.*

Example 2.4. *The space $C[a, b]$ is not an inner product space, hence not a Hilbert space.*

2.2 BOUNDED LINEAR OPERATORS IN H

Definition 2.2. Let H_1 and H_2 are two Hilbert spaces. An operator $A : H_1 \to H_2$ is said to be linear operator if

$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all $\alpha, \beta \in C$ and $x, y \in H_1$.

The domain of A $D(A) = \{x \in H_1, \exists Ax \in H_2\}$ is a vector space and

$R(A) = \{y = Ax, \forall x \in D(A)\}$ denotes the range of A.

A linear operator $A : H \to H$ is said to be bounded if there exist a real number $M > 0$ such that

$$\|Ax\|_H \leq M \ \|x\|_H \quad for\ all \ \ x \in H. \tag{2.5}$$

If a linear operator $A : H \to H$ is bounded with M, then

$$\|A\| = \inf M \tag{2.6}$$

is called the norm of operator A.

Theorem 2.3. *The norm of the bounded linear operator A is*

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| \ = \sup_{x\neq\sigma} \frac{\|Ax\|}{\|x\|} \ = \ \sup_{\|x\|=1} \|Ax\|. \tag{2.7}$$

Example 2.5. *A is an operator defined by $Ax = \alpha x(t)$, $\quad A : L_2[0,1] \longrightarrow L_2[0,1]$. Then, $\|Ax\| = |\alpha|$.*

2.3 ADJOINT OF AN OPERATOR

Definition 2.3. Let $A : H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the Hilbert adjoint operator A^* of A is the operator

$$A^* : H_2 \to H_1,$$

such that for all $x \in H_1$ and $y \in H_2$

$$\langle Ax, y\rangle = \langle x, A^* y\rangle.$$

Theorem 2.4. *The Hilbert adjoint operator A^* of A is unique and bounded linear operator with the norm*

$$\|A^*\| = \|A\| . \tag{2.8}$$

Definition 2.4. A bounded linear operator $A : H \longrightarrow H$ on a Hilbert space H is said to be self-adjoint if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$.

Definition 2.5. A self-adjoint operator A is said to be positive if $A \geq 0$, that is $(Ax, x) \geq 0$ for all $x \in H$.

Example 2.6. *A is an operator defined in the example 2.5. Then, if $\alpha \in R^1$, then A is a self-adjoint operator.*

Definition 2.6. Let $A : D(A) \longrightarrow H$ be a linear operator with $\overline{D(A)} = H$. Then A is called a symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D(A)$. If A is symmetric and $D(A) = D(A^*)$, then A is a self-adjoint operator.

Example 2.7. *Let* $Au = -\dfrac{d^2u}{dx^2} + u$, $u(a) = u(b) = 0$ *and* $H = L_2[a, b]$. *Then, A is a self-adjoint positive operator.*

2.4 SPECTRUM

Definition 2.7. Let H be a Hilbert space and $A : H \longrightarrow H$ be a linear operator with $D(A) \subset H$. We associate the operator $A_\lambda = A - \lambda I$, where $\lambda \in C$ and I is the identity operator on $D(A)$.

If A_λ has an inverse, we denote it by $R_\lambda(A)$ and we call it the resolvent operator of A, or simply, resolvent of A.

$$R_\lambda(A) = (A - \lambda I)^{-1} . \tag{2.9}$$

Definition 2.8. (Regular value, resolvent set, spectrum)

Let A be a linear operator with the $D(A) \subset H$ and H is a Hilbert space. A regular value λ of A is a complex number such that

- $(R1)$ $R_\lambda(A)$ exists.
- $(R2)$ $R_\lambda(A)$ is bounded.
- $(R3)$ $R_\lambda(A)$ is defined on a set which is dense in H.

The resolvent set $\rho(A)$ of A is the set of all regular values of A. Its complement $\sigma(A) = C \backslash \rho(A)$ is called the spectrum of A , and $\lambda \in \sigma(A)$ is called the spectral value of A. Furthermore, the spectrum $\rho(A)$ is partitioned into three disjoint sets as follows.

The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_\lambda(A)$ does not exist. Any $\lambda \in \sigma(A)$ is called an eigenvalue of A.

The continuous spectrum $\sigma_c(A)$ is the set such that $R_\lambda(A)$ exists and satisfies $(R3)$ but not $(R2)$, that is $R_\lambda(A)$ unbounded.

The residual spectrum $\sigma_r(A)$ is the set such that $R_\lambda(A)$ exists (and may be bounded or not) but does not satisfy $(R3)$, that is the domain of $R_\lambda(A)$ is not dense in H.

If $A_\lambda x = (A - \lambda I)x = 0$ for some $x \neq 0$, then $\lambda \in \sigma_p(A)$, by definition, that is, λ is an eigenvalue of A.

The vector x is called an eigenvector of A corresponding to the eigenvalue λ. The subspace of $D(A)$ consisting of 0 and all eigenvectors of A corresponding to an eigenvalue λ of A is called the eigenspace of A corresponding to that eigenvalue λ.

$$\sigma(A) = \sigma_c(A) \cup \sigma_p(A) \cup \sigma_r(A), \tag{2.10}$$
$$\sigma(A) \cup \rho(A) = C.$$

Definition 2.9. Let H be a Hilbert space over the field of real numbers and for any $x \in H$, let $\|x\|$ denote the norm of x. Let J be any interval of the real line R. A function $x : J \to H$ is called an abstract function. A function $x(t)$ is said to be continuous at the point $t_0 \in J$, if

$$\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0;$$

if $x : J \to H$ is continuous at each point of J, Then we say that x is continuous on J and we write $x \in C[J, H]$.

Definition 2.10. The Stieltjes integral of a function $x : [a, b] \to H$ with respect to a function $y : [a, b] \to H_1$. Let H, H_1 and H_2 be three Hilbert spaces. A bilinear operator $P : H \times H_1 \to H_2$ whose norm is less than or equal to 1, that is,

$$\|P(x, y)\| \leq \|x\| \|y\|, \tag{2.11}$$

is called a product operator. We shall agree to write $P(x, y) = xy$. Let $x : [a, b] \to H$ and $y : [a, b] \to H_1$ be two bounded functions such that the product $x(t)y(t) \in H_2$, for each $t \in [a, b]$ is linear in both x and y and

$$\|x(t)y(t)\| \leq \|x(t)\| \, \|y(t)\|$$

(for example, $x(t) = A(t)$ is an operator with domain $D[A(t)] \supset H_1$, or one of the function x,y is a scalar function). We denote the partition $(a = t_0 < t_1 < t_2 < ... < t_n = b)$ together with the points τ_i $(t_i < \tau_1 < t_{i+1}, i = 0, 1, 2, \cdots, n-1)$ by π and set $|\pi|$=$\max_i |t_{i+1} - t_i|$. We form the Stieltjes sum

$$S_\pi = \sum_{i=1}^{n-1} x(\tau_i) [y(t_{i+1}) - y(t_i)]. \tag{2.12}$$

If the $\lim S_\pi$ exist as $|\pi| \to 0$ and defines an element I in H_2 independent of π, then I is called the Stieltjes integral of the function $x(t)$ by the function $y(t)$, and is denoted by

$$\int_a^b x(t) \, dy(t). \tag{2.13}$$

Theorem 2.5. *If $x \in C[[a, b], H]$ and $y : [a, b] \to H_1$ is of bounded variation on $[a, b]$, then the Stieltjes integral (2.13) exists.*

Consider the function $y : [a, b] \to H_1$ and the partition

$$\pi : a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

Form the sum

$$V = \sum_{i=1}^{n-1} \|y(t_{i+1}) - y(t_i)\|. \tag{2.14}$$

The least upper bound of the set of all possible sums V is called the (strong) total variation of the function $y(t)$ on the interval $[a, b]$ and is denoted by $V_a^b(y)$. If $V_a^b(y) < \infty$, then $y(t)$ is called an abstract function of bounded variation on $[a, b]$.

Example 2.8. *If* $x \in C\left[[a,b], H\right]$ *and* $y : [a,b] \to H_1$ *is of bounded variation on* $[a,b]$, *then*

$$\left\| \int_a^b x(t)\, dy(t) \right\| \leq \int_a^b \|x(t)\|\, dV_a^t[y(t)] \leq \max_{t\in[a,b]} \|x(t)\|\, V_a^b[y(t)]. \tag{2.15}$$

2.5 PROJECTION OPERATOR, SPECTRAL FAMILY

Definition 2.11. A Hilbert space H is represented as the direct sum of a closed subspace Y and its orthogonal complement $Y^{\perp}$:

$$H = Y \oplus Y^{\perp}$$

$$x = y + z \quad , \quad \textit{where} \;\; y \in Y, z \in Y^{\perp}.$$

Since the sum is direct, y is unique for any given $x \in H$. Hence (**??**) defines a linear operator

$$P : H \longrightarrow H,$$

$$x \longrightarrow y = Px.$$

P is called an orthogonal projection or projection on H.

Theorem 2.6. *A bounded linear operator* $P : H \longrightarrow H$ *on a Hilbert space* H *is a projection if and only if* P *is self-adjoint and idempotent that is,* $P^2 = P$.

Spectral family for finite dimensional case as follows: If matrix A has n different eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n$. Then A has an orthogonal set of n vectors $x_1, x_2, x_3, \cdots, x_{n,}$where x_j corresponds to λ_j and we write these vectors as column vectors, for convenience. This basis for H, has a unique representation:

$$x = \sum_{j=1}^{n} \gamma_j x_j \ , \qquad \gamma_j = (x, x_j) = x^T \overline{x_j} \ , \tag{2.16}$$

x_j is an eigenvector of A, so that we have $Ax_j = \lambda_j x_j$.

$$Ax = \sum_{j=1}^{n} \lambda_i \gamma_j x_j. \tag{2.17}$$

We can define an operator

$$P_j : H \longrightarrow H, \tag{2.18}$$

$$x \longrightarrow \gamma_j x_j \ .$$

Obviously, P_j is the projection (orthogonal projection) of H onto the eigenspace of A corresponding to λ_j . From the equation (2.16) can be written

$$x = \sum_{j=1}^{n} P_i x \;\; \text{hence} \;\; I = \sum_{j=1}^{n} P_i, \tag{2.19}$$

where I is an identity operator on H. Formula (2.17) becomes

$$Ax = \sum_{j=1}^{n} \lambda_i P_i x \;\; \text{hence} \;\; A = \sum_{j=1}^{n} \lambda_i P_i. \tag{2.20}$$

This is a representation of A in terms of projections.

Theorem 2.7. *(Spectral Theorem) A family of an orthogonal projection operators E_λ $(-\infty < \lambda < \infty)$ is said to be spectral representation identity if:*

1) E_λ is strongly left-continuous in λ;

2) $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$ for $\lambda < \mu$;

3) $E_{-\infty} = \lim\limits_{\lambda\to-\infty} E_\lambda = 0$ and $E_{+\infty} = \lim\limits_{\lambda\to\infty} E_\lambda = I$, where the limits are understood in the sense of strong convergence. For every bounded function $F(\lambda)$ given in the entire real axis, one can define the Stieltjes operator integral

$$\int_a^b F(\lambda)\, dE_\lambda. \tag{2.21}$$

This integral is defined as the limit in the norm of integral sums of the form

$$\sum_{k=0}^{N} F(\lambda_k)\left(E_{\lambda_{k+1}} - E_\lambda\right),$$

if the segment $[a, b]$ is finite, and as an important integral if $a = -\infty$ or $b = \infty$. The integral (2.21) is a bounded operator with

$$\left\| \int_a^b F(\lambda)\, dE_\lambda \right\| \le \sup_{a\le\lambda\le b} |F(\lambda)|.$$

If the function $F(\lambda)$ takes on only real values, the operator (2.21) is self-adjoint. If the function $F(\lambda)$ is real and unbounded, then formula (2.21), after assigning an appropriate meaning to the integral, yields a self-adjoint and generally speaking unbounded operator whose domain consists of only those elements x for which

$$\int_{-\infty}^{\infty} |F(\lambda)|^2\, d(E_\lambda x, x) < \infty.$$

It turns out that to every self-adjoint operator A there corresponds some spectral representation E_λ of the identity with

$$Ax = \int_{-\infty}^{\infty} \lambda dE_\lambda x$$

for $x \in D(A)$. The operators E_λ commute with any operator commuting with A.

If A is bounded, and m and M are the greatest lower bound and least upper bound of its spectrum then $E_\lambda = I$ for $\lambda > M$, so that

$$Ax = \int_m^{M+0} \lambda dE_\lambda x.$$

If the operator A is positive definite and $\langle Ax, x\rangle \ge a\,\langle x, x\rangle$, then

$$Ax = \int_a^{\infty} \lambda dE_\lambda x.$$

The real regular points of A are characterized by the fact that in their neighborhoods the operator E_λ is constant. Thus, the points of the spectrum of A coincide with the points of growth of the operator function E_λ.

By using the spectral representation one may bring into consideration a wide class of functions of an unbounded self-adjoint operator. Thus, for example, for any continuous function $F(\lambda)$ it is natural to put

$$F(A)\,x = \int_a^\infty F(\lambda)\,dE_\lambda x,$$

where E_λ is the spectral resolution of the identity corresponding to operator A. Here, to the operations of addition and multiplication of the corresponding operators.

Lemma 2.1.

$$\left\| \left(I \mp i\tau A^{1/2}\right)^{-1} \right\|_{H\to H} \leq 1, \tag{2.22}$$

$$\left\| \left(I \mp \frac{i\tau}{2} A^{1/2}\right)^{-1} \left(I \pm \frac{i\tau}{2} A^{1/2}\right) \right\|_{H\to H} \leq 1, \tag{2.23}$$

$$\left\| \tau A^{1/2} \left(I \pm \frac{i\tau}{2} A^{1/2}\right)^{-1} \right\|_{H\to H} \leq 2. \tag{2.24}$$

$$\left\| A^{-\rho} \right\|_{H\to H} \leq M. \tag{2.25}$$

Proof. Using the spectral representation of the self-adjoint positive defined operators it can be written

$$\left(I \mp i\tau A^{1/2}\right)^{-1} \varphi = \int_\delta^\infty (1 \mp \tau i\sqrt{\mu})^{-1}\,dE_\mu \varphi.$$

Therefore, using the last theorem we get

$$\left\| \left(I \mp i\tau A^{1/2}\right)^{-1} \right\|_{H\to H} \leq \sup_{\delta \leq \mu < \infty} |1 \mp i\tau\sqrt{\mu}|^{-1} = \sup_{\delta \leq \mu < \infty} \frac{1}{\sqrt{1+\tau^2\mu}} \leq 1.$$

So, estimate (2.22) is proved. Now, using the same manner, we can write

$$\left(I \mp \frac{i\tau}{2} A^{1/2}\right)^{-1} \left(I \pm \frac{i\tau}{2} A^{1/2}\right) \varphi = \int_\delta^\infty \frac{1 \pm \frac{\tau i\sqrt{\mu}}{2}}{1 \mp \frac{\tau i\sqrt{\mu}}{2}} dE_\mu \varphi.$$

Therefore, using the last formula, we get

$$\left\| \left(I \mp \frac{i\tau}{2} A^{1/2}\right)^{-1} \left(I \pm \frac{i\tau}{2} A^{1/2}\right) \right\|_{H\to H} \leq \sup_{\delta \leq \mu < \infty} \left| \frac{1 \pm \frac{\tau i\sqrt{\mu}}{2}}{1 \mp \frac{\tau i\sqrt{\mu}}{2}} \right| = \sup_{\delta \leq \mu < \infty} \frac{\sqrt{1+\frac{\tau^2\mu}{4}}}{\sqrt{1+\frac{\tau^2\mu}{4}}} = 1.$$

Estimate (2.23) is proved. Now, using the same manner, we can write

$$\tau A^{1/2} \left(I \pm \frac{i\tau}{2} A^{1/2}\right)^{-1} \varphi = \int_\delta^\infty \frac{\tau\sqrt{\mu}}{\sqrt{1+\frac{\tau^2\mu}{4}}} dE_\mu \varphi.$$

Therefore, using the last formula, we get

$$\left\| \tau A^{1/2} \left(I \pm \frac{i\tau}{2} A^{1/2}\right)^{-1} \right\|_{H\to H} \leq \sup_{\delta \leq \mu < \infty} \left| \frac{\tau\sqrt{\mu}}{\sqrt{1+\frac{\tau^2\mu}{4}}} \right|$$

$$= \sup_{\delta \le \mu < \infty} \frac{\tau\sqrt{\mu}}{\sqrt{1+\frac{\tau^2\mu}{4}}} \le \sup_{\delta \le \mu < \infty} \frac{\tau\sqrt{\mu}}{\sqrt{\frac{\tau^2\mu}{4}}} = \frac{2\tau\sqrt{\delta}}{\tau\sqrt{\delta}} = 2.$$

Estimate (2.24) is proved. Now, using the same manner, we can write

$$\left(A^{-\rho}\right)\varphi = \int_{\delta}^{\infty} \frac{1}{\mu^{\rho}} dE_{\mu}\varphi.$$

Therefore, using the last formula, we get

$$\left\|A^{-\rho}\right\|_{H\to H} \le \sup_{\delta \le \mu < \infty} \left|\frac{1}{\mu^{\rho}}\right| = \frac{1}{\delta^{\rho}} \le M$$

Estimate (2.25) is proved. So, Lemma is established.

2.6 THE INITIAL VALUE PROBLEM FOR HYPERBOLIC EQUATION IN A HILBERT SPACE

We consider the initial-value problem

$$u'' + A(t)u(t) = f(t),\ 0 \le t \le T,\ u(0) = u_0,\ u'(0) = u_0' \tag{2.26}$$

in a Hilbert space H. Here $A(t)$ is a self-adjoint positive definite operator, having a region of definition $D = D(A(t))$ that does not depend on t, acting in H.

A function $u(t)$ is called the solution of the problem (2.26) if the following conditions are satisfied:

i. $u(t)$ is twice continuously differentiable on the segment $[0, T]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii. The element $u(t)$ belongs to D for all $t \in [0, T]$, and the function $A(t)u(t)$ is continuous on the segment $[0, T]$.

iii. $u(t)$ satisfies the equations and the initial conditions in (2.26).

In the paper [Sobolevskii, P. E. and Pogorelenko, V. A., 1967] the initial-value problem (2.26) in a Hilbert space H was investigated.

Let $A(t)$ for arbitrary $t \in [0, T]$ be positive-definite self-adjoint operator having a t

independent region of definition D. From [E. Heinz, 1951] it follows that $A^{\rho}(t)$ $(0 < \rho < 1)$ also has a constant region of definition D_{ρ}.

Let the operator-function $A(t)A^{-1}(0)$ be strongly continuously differentiable. In [Daletskii, Yu. L., 1958] it was proved that the operator-function $[A^{\rho}(t)]'A^{-\rho}(t)$ is bounded and strongly continuous with respect to t.

Lemma 2.2. Let the numbers α_i $(i = 1, 2, 3)$ be such that

$$\sum_{i=1}^{3} \alpha_i = 0, \quad |\alpha_i| \le 1.$$

Then the operator-function $B = A^{\alpha_1}(t)\left[A^{\alpha_2}(t)\right]' A^{\alpha_3}(t)$ is either strongly continuous or it admits a strongly continuous closure.

Proof. If $\alpha_2 = 0$, then $B = 0$, Furthermore, let $\alpha_2 = -\alpha$ where $\alpha > 0$. Then, as is well-known

$$\left[A^{-\alpha}(t)\right]' = -A^{-\alpha}(t)\left[A^{\alpha}(t)\right]' A^{-\alpha}(t)$$

and therefore the case $\alpha_2 < 0$ reduces to the case $\alpha_2 > 0$.

For $\alpha_2 > 0$ it is necessary to consider the following cases:

$$\begin{aligned} &1)\ \alpha_2 > 0, \quad \alpha_1 = 0, \quad \alpha_3 < 0; \\ &2)\ \alpha_2 > 0, \quad \alpha_1 > 0, \quad \alpha_3 < 0; \\ &3)\ \alpha_2 > 0, \quad \alpha_1 < 0, \quad \alpha_3 > 0; \\ &4)\ \alpha_2 > 0, \quad \alpha_1 < 0, \quad \alpha_3 < 0; \\ &5)\ \alpha_2 > 0, \quad \alpha_1 < 0, \quad \alpha_3 = 0. \end{aligned}$$

In case 1) a proof of the lemma is established in [Daletskii, Yu. L., 1958].

Let us show (case 5) that the operator $B = A^{-\rho}(t)\left[A^{\rho}(t)\right]'$ $(0 \le \rho \le 1)$ admits a strongly continuous closure.

Let $x \in D_\rho$, $y \in H$. Then, having used the symmetry of the operator $\left[A^{\rho}(t)\right]'$ on D_ρ, we have

$$(Bx,\ y) = (x,\ [A^{\rho}(t)]' A^{-\rho}(t) y).$$

Hence

$$|(Bx,\ y)| \le \left\| [A^{\rho}(t)]' A^{-\rho}(t) \right\| \|x\| \, \|y\| \, .$$

Having taken $y = Bx$, we obtain

$$\|Bx\| \le \left\| [A^{\rho}(t)]' A^{-\rho}(t) \right\| \|x\| \, .$$

Since D_ρ is dense in H, then from here it follows that the operator $\overline{B}$ is bounded.

If $x \in D_\rho$, then $x = A^{-\rho}(0) y$, where $y \in H$ and

$$Bx = A^{-\rho}(t)[A^{\rho}(t)]' A^{-\rho}(t) A^{\rho}(t) A^{-\rho}(0) y.$$

Since the operators $A^{-\rho}(t)$, $[A^{\rho}(t)]' A^{-\rho}(t)$ and $A^{\rho}(t) A^{-\rho}(0)$ are strongly continuous, then $\overline{B}$ is strongly continuous on D_ρ. Using the fact that $\overline{B}$ is bounded and D_ρ is dense in H, we conclude that $\overline{B}$ is strongly continuous.

For consideration of cases 2) and 4) it is convenient to change the notation: $\alpha_2 = \beta > 0$, $\alpha_3 = -\alpha < 0$, $\alpha_1 = \alpha - \beta$. In this connection for $\alpha > \beta$ we have case 4).

Let us consider case 2). The initial operator B in terms of the new notation appears as: $B = A^{\alpha-\beta}(t)[A^{\beta}(t)]' A^{-\alpha}(t)$ $(\alpha > \beta)$.

Let $x \in H$, $y \in D_\alpha$. We write down the identity

$$\begin{gathered} \left(A^{\alpha-\beta}(t+\Delta t) A^{\beta}(t) A^{-\alpha}(t) x,\ y\right) - \left(A^{\alpha-\beta}(t) A^{\beta}(t) A^{-\alpha}(t) x,\ y\right) \\ + \left(A^{\beta}(t+\Delta t) A^{-\alpha}(t) x, A^{\alpha-\beta}(t+\Delta t)\ y\right) - \left(A^{\beta}(t) A^{-\alpha}(t) x, A^{\alpha-\beta}(t+\Delta t)\ y\right) \\ + \left(A^{-\alpha}(t+\Delta t)\ x, A^{\alpha}(t+\Delta t)\ y\right) - \left(A^{-\alpha}(t) x, A^{\alpha}(t+\Delta t)\ y\right) \equiv 0. \end{gathered}$$

Having combined the terms in pairs and having divided by Δt, we obtain

$$\left(\frac{A^{\alpha-\beta}(t+\Delta t)-A^{\alpha-\beta}(t)}{\Delta t}A^{\beta-\alpha}(t)x, y\right)$$
$$+\left(\frac{A^{\beta}(t+\Delta t)-A^{\beta}(t)}{\Delta t}A^{-\alpha}(t)x, A^{\alpha-\beta}(t+\Delta t)y\ \right)$$
$$\left(\frac{A^{-\alpha}(t+\Delta t)-A^{-\alpha}(t)}{\Delta t}x, A^{\alpha}(t+\Delta t)y\right)\equiv 0 \tag{2.27}$$

Since $\alpha > \beta$, then on the basis of [Daletskii, Yu. L., 1958] as $\Delta t \to 0$ there exists a strong limit to the expression $\frac{A^{\alpha-\beta}(t+\Delta t)-A^{\alpha-\beta}(t)}{\Delta t}A^{\beta-\alpha}(t)$, this limit is equal to $[A^{\alpha-\beta}(t)]'A^{\beta-\alpha}(t)$. In similar fashion $\frac{A^{\beta}(t+\Delta t)-A^{-\beta}(t)}{\Delta t}A^{-\alpha}(t)$ strongly converges to $[A^{\beta}(t)]'A^{-\alpha}(t)$. It is obvious that

$\frac{A^{-\alpha}(t+\Delta t)-A^{-\alpha}(t)}{\Delta t}$ has a limit $-A^{-\alpha}(t)[A^{\alpha}(t)]'A^{-\alpha}(t)$. Finally, by taking into consideration that the operators $A^{\alpha-\beta}(t)$ and $A^{\alpha}(t)$ are strongly continuous on D_{α}, we conclude that in the identity (2.27) we can go over to the limit as $\Delta t \to 0$. After passing to the limit we obtain

$$\left([A^{\alpha-\beta}(t)]'A^{\beta-\alpha}(t)x,\ y\right)+\left([A^{\beta}(t)]'A^{-\alpha}(t)x,\ A^{\alpha-\beta}(t)y\right)$$
$$-\left(A^{-\alpha}(t)[A^{\alpha}(t)]'A^{-\alpha}(t)x,\ A^{\alpha}(t)y\right)\equiv 0$$

or

$$\left([A^{\beta}(t)]'A^{-\alpha}(t)x,\ A^{\alpha-\beta}(t)y\right)\equiv\left(\left\{[A^{\alpha}(t)]'A^{-\alpha}(t)-[A^{\alpha-\beta}(t)]'A^{\beta-\alpha}(t)\right\}x,\ y\right).$$

Now let y be any element of $D_{\alpha-\beta}$ and let $z = A^{\alpha-\beta}y$. Since D_{β} is dense in H, then there exists a sequence of elements $z_n \in D_{\beta}$ such that $z_n \to A^{\alpha-\beta}y$ as $n \to \infty$. The elements are

$$y_n = A^{-(\alpha-\beta)}z_n \in D_{\alpha},\ y_n \to y \text{ and } A^{\alpha-\beta}y_n \to A^{\alpha-\beta}y.$$

The last identity is valid for any y_n. Its validity for arbitrary $y \in D_{\alpha-\beta}$ can be established by taking the limit.

Since the operator appearing inside the curcly brackets is bounded, then $[A^{\beta}(t)]'A^{-\alpha}(t)x \in D_{\alpha-\beta}$ and

$$B = A^{\alpha-\beta}(t)[A^{\beta}(t)]'A^{-\alpha}(t)\equiv[A^{\alpha}(t)]'A^{-\alpha}(t)-[A^{\alpha-\beta}(t)]'A^{\beta-\alpha}(t).$$

Strong continuity of the operator B follows from here.

Now let us consider case 4) ($\alpha < \beta$). Let $x \in D_{\beta-\alpha}$, $y \in D_{\alpha}$. For these x and y in analogy to the previous case one can justify the feasibility of taking the limit as $\Delta t \to 0$ in the identity (2.27).

After taking the limit and simplifying, we obtain

$$(Bx,\ y)\equiv\left(\left\{A^{\alpha-\beta}(t)[A^{\beta-\alpha}(t)]'+[A^{\alpha}(t)]'A^{-\alpha}(t)\right\}x,\ y\right).$$

Since D_{α} is dense in H, then

$$B\equiv A^{\alpha-\beta}(t)[A^{\beta-\alpha}(t)]'+[A^{\alpha}(t)]'A^{-\alpha}(t).$$

The operator standing to the right admits a strongly continuous closure (case 5). Consequaently there exists a strongly continuous operator $\overline{B}$.

Finally, we consider case 3). Here it is convenient to introduce the notation $\alpha_2 = \beta$, $\alpha_1 = -\alpha$, $\alpha_3 = \alpha - \beta$ where $\alpha > \beta$. Changing from the initial operator B to its conjugate B^*, we obtain the operator considered in case 2). Therefore the initial operator admits a bounded closure which is strongly continuous with respect to t, since the initial operator is strongly continuous over the everywhere dense set D_α. The lemma is proved.

Let $u(t)$ be a classical solution of the problem (2.26). Then it is obvious that the pair of functions $v(t)$ and $w(t) = -iA^{-1/2}(t)dv/dt$ will satisfy the system of equations

$$\begin{cases} \frac{du}{dt} = iA^{1/2}(t)w, \\ \frac{dw}{dt} = iA^{1/2}(t)u - A^{-1/2}(t)\left[A^{1/2}(t)\right]' w - iA^{-1/2}(t)f(t). \end{cases} \tag{2.28}$$

equation (2.28) it follows that

$$\begin{cases} \frac{d(u+w)}{dt} = iA^{1/2}(t)(u+w) - A^{-1/2}(t)\left[A^{1/2}(t)\right]' w - iA^{-1/2}(t)f(t), \\ \frac{d(u-w)}{dt} = -iA^{1/2}(t)(u-w) + A^{-1/2}(t)\left[A^{1/2}(t)\right]' w + iA^{-1/2}(t)f(t). \end{cases} \tag{2.29}$$

this it follows that

$$\begin{cases} u(t) = \frac{U^+(t,0)+U^-(t,0)}{2}u_0 + \frac{U^+(t,0)-U^-(t,0)}{2}\left(-iA^{-1/2}(0)u_0'\right) \\ -\int\limits_0^t \frac{U^+(t,s)-U^-(t,s)}{2}\left[A^{-1/2}(s)\left[A^{1/2}(s)\right]' w(s) + iA^{-1/2}(s)f(s)\right]ds, \\ w(t) = \frac{U^+(t,0)+U^-(t,0)}{2}u_0 + \frac{U^+(t,0)-U^-(t,0)}{2}\left(-iA^{-1/2}(0)u_0'\right) \\ -\int\limits_0^t \frac{U^+(t,s)+U^-(t,s)}{2}\left[A^{-1/2}(s)\left[A^{1/2}(s)\right]' w(s) + iA^{-1/2}(s)f(s)\right]ds. \end{cases} \tag{2.30}$$

Here $U^{\pm}(t,s)$ $(0\leq s \leq t \leq T)$ is a certain operator-function which is strongly continuous with respect to the set of variables and the semigroup identity

$$U^{\pm}(t,s) = U^{\pm}(t,p)U^{\pm}(p,s),\ (0 \leq s \leq p \leq t \leq T),\ U^{\pm}(t,t) = I$$

is valid. Hence, in particular it follows that for any $x_0 \in D$ the function $U^{\pm}(t,s)x_0$ is continuously differentiable with respect to s and t and the following relations

$$\frac{\partial}{\partial t}(U^{\pm}(t,s)x_0) = (\pm iA^{1/2}(t))U^{\pm}(t,s)x_0,$$

$$\frac{\partial}{\partial s}(U^{\pm}(t,s)x_0) = -U^{\pm}(t,s)(\pm iA^{1/2}(s))x_0$$

are valid. Then, consequently $u(t)$ satisfies the equation

$$u(t) = \frac{U^+(t,0)+U^-(t,0)}{2}u_0 + \frac{U^+(t,0)-U^-(t,0)}{2}\left(-iA^{-1/2}(0)\right)u_0'$$

$$-\int_0^t \frac{U^+(t,s)-U^-(t,s)}{2}A^{-1/2}(s)\left[-i\left[A^{1/2}(s)\right]' A^{-1/2}(s)\frac{du(s)}{ds}+if(s)\right]ds.$$

In the paper [Sobolevskii, P. E. and Chebotaryeva, L. M., 1977] the first order of accuracy difference scheme

$$\begin{cases} \tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+A_k u_{k+1}=f_k,\ A_k=A(t_k), \\ f_k=f(t_k),\quad t_k=k\tau,\ 1\le k\le N-1,\ N\tau=T, \\ u_0=u(0),\quad u_1-u_0=\tau u'(0) \end{cases} \tag{2.31}$$

was investigated. The following theorems summarise Sobolevskii and Chebotaryeva's results.

Theorem 2.8. *Assume that the operator-function* $A^{\frac{1}{2}}(t)A^{-\frac{1}{2}}(0)$ *has a finite variation on the segment* $[0,T]$ *and*

$$\left\|A^{\frac{1}{2}}(0)A^{-\frac{1}{2}}(t)\right\|\le M_{1/2}$$

for any $t\in[0,T]$, $u(0)\in D(A(0)), u'(0)\in D(A^{1/2}(0))$ *. Then the following estimate holds:*

$$||\{\frac{u_k-u_{k-1}}{\tau}\}_1^{N-1}||_{C_\tau}+||\{A^{\frac{1}{2}}(0)u_k\}_1^{N-1}||_{C_\tau}+||u^\tau||_{C_\tau}$$

$$\le M_1[||A^{\frac{1}{2}}(0)u_0||_H+||u_0'||_H+\sum_{s=1}^{N-1}||f_s||_H\tau].$$

Here C_τ *is the normed space of the mesh functions* $u^\tau=\{u_k\}_1^{N-1}$ *with the norm*

$$||u^\tau||_{C_\tau}=\max_{1\le k\le N-1}||u_k||_H.$$

Theorem 2.9. *Assume that operator-function* $A(t)A^{-1}(0)$ *has a finite variation on the segment* $[0,T]$ *and*

$$\left\|A(0)A^{-1}(t)\right\|\le M_1$$

for any $t\in[0,T]$, $u(0)\in D(A(0)), u'(0)\in D(A^{1/2}(0))$ *. Then the following estimate holds:*

$$||\{A^{1/2}(0)\frac{u_k-u_{k-1}}{\tau}\}_1^{N-1}||_{C_\tau}+||\{A(0)u_k\}_1^{N-1}||_{C_\tau}$$

$$+||\{\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)\}_1^{N-1}||_{C_\tau}$$

$$\le M_2[||A(0)u_0||_H+||A^{\frac{1}{2}}(0)u_0'||_H+\max_{0\le s\le N-1}||f_s||_H+\sum_{s=1}^{N-1}||f_s-f_{s-1}||_H].$$

In the following Chapter we consider the stable second order of accuracy difference scheme approximately solving the initial value problem (2.26) .

CHAPTER 3

THE DIFFERENCE SCHEMES OF INITIAL VALUE PROBLEMS FOR HYPERBOLIC DIFFERENTIAL EQUATIONS

3.1 THE CONSTRUCTION OF DIFFERENCE SCHEME

We will consider the initial value problem (2.26). By (2.28), we have the equivalent initial value problem for a system of the first order linear differential equations

$$\begin{cases} \frac{du(t)}{dt} = iA^{1/2}(t)v(t),\ 0 < t < T, \quad u(0) = u_0, \quad u'(0) = u_0', \\ \frac{dv(t)}{dt} = iA^{1/2}(t)u(t) - A^{-1/2}(t)\left[A^{1/2}(t)\right]' v(t) - iA^{-1/2}(t)f(t). \end{cases} \tag{3.1}$$

For construction of a two-step difference scheme, we consider the uniform grid space

$$[0,T]_\tau = \{t_k = k\tau,\ 0 \le k \le N,\ N\tau = T\}.$$

Using the central difference formula for the derivative, we have that

$$\begin{cases} \tau^{-1}(u(t_{k+1}) - u(t_k)) = iA^{1/2}_{k+1/2}v(t_{k+1/2}) + o(\tau^2), \quad 0 \le k \le N-1, \\ \tau^{-1}(v(t_{k+1}) - v(t_k)) = iA^{1/2}_{k+1/2}u(t_{k+1/2}) - A^{-1/2}_{k+1/2}[A^{1/2}_{k+1/2}]'v(t_{k+1/2}) \\ -iA^{-1/2}_{k+1/2}f(t_{k+1/2}) + o(\tau^2), \quad 0 \le k \le N-1, \quad v_0 = -iA_0^{-1/2}u_0', \end{cases}$$

where

$$A^{1/2}_{k+1/2} = A^{1/2}(t_{k+1/2}),\ [A^{1/2}_{k+1/2}]' = (A')^{1/2}(t_{k+1/2}),\ t_{k+1/2} = (t_k + \frac{\tau}{2}),\ A_0 = A(0).$$

Then, replacing the functional values at $t_{k+1/2}$ by their means at t_k and t_{k+1}

$$\begin{cases} \tau^{-1}(u(t_{k+1}) - u(t_k)) = \frac{1}{2}iA^{1/2}_{k+1/2}(v(t_{k+1}) + v(t_k)) + o(\tau^2), \quad 0 \le k \le N-1, \\ \tau^{-1}(v(t_{k+1}) - v(t_k)) = \frac{1}{2}iA^{1/2}_{k+1/2}(u(t_{k+1}) + u(t_k)) \\ -\frac{1}{2}A^{-1/2}_{k+1/2}[A^{1/2}_{k+1/2}]'(v(t_{k+1}) + v(t_k)) - iA^{-1/2}_{k+1/2}f(t_{k+1/2}) + o(\tau^2), \quad 0 \le k \le N-1, \\ v_0 = -iA_0^{-1/2}u_0'. \end{cases}$$

Neglecting the last small terms of $o(\tau^2)$ and using (3.1), we obtain the following difference scheme for the approximate solution of the initial value problem (2.26) :

$$\begin{cases} \tau^{-1}(u_{k+1}-u_k)=iA_{k+1/2}^{1/2}2^{-1}\left(v_{k+1}+v_k\right), \\ u_0=u(0), \quad 0\le k\le N-1, \\ \tau^{-1}(v_{k+1}-v_k)=iA_{k+1/2}^{1/2}2^{-1}(u_{k+1}+u_k)-\frac{1}{2}A_{k+1/2}^{-1/2}[A_{k+1/2}^{1/2}]'(v_{k+1}+v_k) \\ -iA_{k+1/2}^{-1/2}f_{k+1/2}, \quad 0\le k\le N-1, \\ v_0=-iA_0^{-1/2}u_0'. \end{cases} \tag{3.2}$$

Eliminating v_k, collecting u_k on the left side and t_k on the right sides of the equation and rearranging the terms, we obtain

$$\begin{cases} \tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+A_{k+1/2}4^{-1}\left(u_{k+1}+u_k\right)+A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}4^{-1}\left(u_k+u_{k-1}\right) \\ +\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\tau^{-1}\left(u_k-u_{k-1}\right) \\ +2^{-1}\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\tau^{-1}\left(u_{k+1}-u_k\right) \\ +A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}2^{-1}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\tau^{-1}\left(u_k-u_{k-1}\right) \\ =2^{-1}\left(f_{k-1/2}+f_{k+1/2}\right)+\left(A_{k+1/2}^{1/2}-A_{k-1/2}^{1/2}\right)A_{k-1/2}^{-1/2}2^{-1}f_{k-1/2}, \\ 1\le k\le N-1, \quad u_0=u(0), \\ \tau^{-1}(u_1-u_0)+\frac{\tau}{2}A_{1/2}2^{-1}(u_1+u_0)+\frac{\tau}{2}\left(A_{1/2}^{1/2}\right)'A_{1/2}^{-1/2}\tau^{-1}(u_1-u_0) \\ =\frac{\tau}{2}f_{1/2}+A_{1/2}^{1/2}A_{1/2}^{-1/2}u_0'. \end{cases} \tag{3.3}$$

Now, we will obtain the solution of the difference scheme (3.3). We can rewrite equation (3.2) as follows

$$\begin{cases} \tau^{-1}\left(u_{k+1}-u_k\right)=iA_{k+1/2}^{1/2}2^{-1}\left(v_{k+1}+v_k\right), \ 0\le k\le N-1, \quad u_0=u(0), \\ \tau^{-1}(v_{k+1}-v_k)=iA_{k+1/2}^{1/2}2^{-1}(u_{k+1}+u_k)+\varphi_{k+1}, \\ \varphi_{k+1}=A_{k+1/2}^{-1/2}\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)iA_{k+1/2}^{-1/2}\tau^{-1}\left(u_{k+1}-u_k\right)-iA_{k+1/2}^{-1/2}f_{k+1/2}, \quad 0\le k\le N-1, \\ v_0=-iA_0^{-1/2}u_0'. \end{cases}$$

By making the transformation $\eta_k=u_k+v_k$ and $\mu_k=u_k-v_k$ in the above equations, we obtain the following system of the difference equations

$$\begin{cases} \tau^{-1}(\eta_{k+1}-\eta_k)=iA_{k+1/2}^{1/2}2^{-1}(\eta_{k+1}+\eta_k)+\varphi_{k+1}, \quad 0\le k\le N-1, \quad \eta_0=u_0-iA_0^{-1/2}u_0', \\ \tau^{-1}(\mu_{k+1}-\mu_k)=-iA_{k+1/2}^{1/2}2^{-1}\left(\mu_{k+1}+\mu_k\right)-\varphi_{k+1}, \\ \varphi_{k+1}=A_{k+1/2}^{-1/2}\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)iA_{k+1/2}^{-1/2}\tau^{-1}\left(u_{k+1}-u_k\right)-iA_{k+1/2}^{-1/2}f_{k+1/2}, \\ 0\le k\le N-1, \quad \mu_0=u_0+iA_0^{-1/2}u_0'. \end{cases}$$

Now, we will obtain η_k and μ_k for all $k=0,\cdots,N-1$. First, the first two equations are solved for η_{k+1} and μ_{k+1} respectively. The reslut are

$$\begin{cases} \eta_{k+1}=\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)\eta_k+\tau\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\varphi_{k+1}, \\ \mu_{k+1}=\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)\mu_k-\tau\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\varphi_{k+1}. \end{cases}$$

Then, using the induction method, we get

$$\eta_{k+1}=P_k^+(k)(u_0-iA_0^{-1/2}u_0')$$

$$+\sum_{m=1}^{k+1}R_m^+(k)\tau\left(I-\frac{i\tau}{2}A_{m-1/2}^{1/2}\right)^{-1}\left[A_{m-1/2}^{-1/2}\tau^{-1}(A_m^{1/2}-A_{m-1}^{1/2})\right.$$

$$\left.\times iA_{m-1/2}^{-1/2}\tau^{-1}(u_m-u_{m-1})-iA_{m-1/2}^{-1/2}f_{m-1/2}\right],$$

$$\mu_{k+1}=P_k^-(k)(u_0+iA_0^{-1/2}u_0')$$

$$-\sum_{m=1}^{k+1}R_m^-(k)\tau\left(I+\frac{i\tau}{2}A_{m-1/2}^{1/2}\right)^{-1}\left[A_{m-1/2}^{-1/2}\tau^{-1}(A_m^{1/2}-A_{m-1}^{1/2})\right.$$

$$\left.\times iA_{m-1/2}^{-1/2}\tau^{-1}(u_m-u_{m-1})-iA_{m-1/2}^{-1/2}f_{m-1/2}\right],$$

Here

$$P_k^{\pm}(k)=X_{k+1}^{\pm}X_k^{\pm}\cdots X_1^{\pm}, \ R_m^{\pm}(k)=X_{k+1}^{\pm}X_k^{\pm}\cdots X_{m+1}^{\pm},$$

where

$$X_{k+1}^{\pm}=\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\left(I\pm\frac{i\tau}{2}A_{k+1/2}^{1/2}\right).$$

Then, using the formula $u_k=\frac{1}{2}(\eta_k+\mu_k)$, we obtain

$$u_{k+1}=\frac{1}{2}\left\{[P_k^+(k)+P_k^-(k)]u_0-[P_k^+(k)-P_k^-(k)]iA_0^{-1/2}u_0'\right.$$

$$+\sum_{m=1}^{k+1}[R_m^+(k)\tau\left(I-\frac{i\tau}{2}A_{m-1/2}^{1/2}\right)^{-1}-R_m^-(k)\tau\left(I+\frac{i\tau}{2}A_{m-1/2}^{1/2}\right)^{-1}]\left[A_{m-1/2}^{-1/2}\tau^{-1}\left(A_m^{1/2}-A_{m-1}^{1/2}\right)\right.$$

$$\left.\left.\times iA_{m-1/2}^{-1/2}\tau^{-1}(u_m-u_{m-1})-iA_{m-1/2}^{-1/2}f_{m-1/2}\right]\right\}.$$

By making the transformation $k-m+1=s$, we obtain

$$u_{k+1}=\frac{1}{2}\left\{[P_k^+(k)+P_k^-(k)]u_0-[P_k^+(k)-P_k^-(k)]iA_0^{-1/2}u_0'\right. \tag{3.4}$$

$$+\sum_{s=0}^{k}[B_s^+(k)\tau\left(I-\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}-B_s^-(k)\tau\left(I+\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}]$$

$$\times\left[A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)iA_{k-s+1/2}^{-1/2}\tau^{-1}(u_{k-s+1}-u_{k-s})-iA_{k-s+1/2}^{-1/2}f_{k-s+1/2}\right]\Bigg\}.$$

where

$$B_s^{\pm}(k)=X_{k+1}^{\pm}X_k^{\pm}\cdots X_{k-s+2}^{\pm}$$

Finally, from the last formula it follows that

$$\tau^{-1}(u_{k+1}-u_k)=\frac{1}{4}\Big\{\left[iA_{k+1/2}^{1/2}\left[P_k^+(k)+P_{k-1}^+(k-1)\right]-iA_{k+1/2}^{1/2}\left[P_k^-(k)+P_{k-1}^-(k-1)\right]\right]u_0$$

$$+\left[A_{k+1/2}^{1/2}\left[P_k^+(k)+P_{k-1}^+(k-1)\right]\left(A_0^{-1/2}\right)+A_{k+1/2}^{1/2}\left[P_k^-(k)+P_{k-1}^-(k-1)\right]\left(A_0^{-1/2}\right)\right]u_0'$$

$$-A_{k+1/2}^{1/2}\left[B_0^+(k)\tau\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}+B_0^-(k)\tau\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right]$$

$$\times\left[A_{k+1/2}^{-1/2}\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\tau^{-1}(u_{k+1}-u_k)\right]$$

$$+A_{k+1/2}^{1/2}\left[B_0^+(k)\tau\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}+B_0^-(k)\tau\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right]\left(A_{k+1/2}^{-1/2}f_{k+1/2}\right)$$

$$-\sum_{s=1}^{k}A_{k+1/2}^{1/2}\left[[B_s^+(k)+B_{s-1}^+(k-1)]\tau\left(I-\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right.$$

$$\left.+[B_s^-(k)+B_{s-1}^-(k-1)]\tau\left(I+\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right]$$

$$\times\left[A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\tau^{-1}(u_{k-s+1}-u_{k-s})-A_{k-s+1/2}^{-1/2}f_{k-s+1/2}\right]\Big\} \quad (3.5)$$

where

$$B_0^{\pm}(k)=1$$

This formula will be used to establish the stability inequality of the difference schemes (3.3).

3.2 STABILITY OF DIFFERENCE SCHEME

First, we consider the subsidiary conditions.

1. Let $A(t)$ for arbitrary $t\in[0,T]$ be operator having a t-independent region of definition $D(A(t))$. Then $A(t)A^{-1}(s)$ is the bounded operator.

Actually, Let H be Hilbert space. Then $D(A(t))$ is the Hilbert space with norm

$$\|u\|_{D(A(t))}=\|A(t)u\|_H+\|u\|_H\,.$$

Let $D(A(t))=D(A(s))$ for any $t,s\in[0,T]$. This means that there exists $m>0$ and M such that

$$m\,\|u\|_{D(A(s))}\le\|u\|_{D(A(t))}\le M\,\|u\|_{D(A(s))}\,.$$

For a self-adjoint positive definite operator $A(s)$ there exists $m_1 > 0$ such that

$$\|u\|_H = \left\|A^{-1}(s)A(s)u\right\|_H \le \left\|A^{-1}(s)\right\|_{H\to H} \|A(s)u\|_H$$
$$\le m_1 \|A(s)u\|_H .$$

Then

$$\|A(t)u\|_H \le M \|A(s)u\|_H + M \|u\|_H$$
$$\le M \|A(s)u\|_H + Mm_1 \|A(s)u\|_H \le M(1+m_1) \|A(s)u\|_H \quad \text{for any } u \in H.$$

Denoting $A(s)u = v$, we get

$$\left\|A(t)A^{-1}(s)v\right\|_H \le M(1+m_1) \|v\|_H .$$

This means that

$$||A(t)A^{-1}(s)||_{H\to H} \le M(1+m_1).$$

Note that from (3.13) it follows that $D(A^\rho(t)) = D(A^\rho(s))$ for any $0<\rho<1$, $t,s \in [0,T]$ and $A^\rho(t)A^{-\rho}(s)$ is uniformly in $t,s\in[0,T]$ bounded. Thus

$$\left\|A^\rho(t)A^{-\rho}(s)\right\|_{H\to H} \le \left\|A(t)A^{-1}(s)\right\|^\rho_{H\to H} \le M_\rho. \tag{3.6}$$

2. We say that the operator-function $A^\rho(t)A^{-\rho}(s))$, $0<\rho<1$, have a finite variation on $[0,T]$, if there exists a number P_ρ such that

$$\sum_{k=1}^{N} \left\|\left(A_k^\rho - A_{k-1}^\rho\right) A_{k-1/2}^{-\rho}\right\|_{H\to H} \le P_\rho \tag{3.7}$$

for any $0 = t_0 < t_1 < \cdots < t_N = T$.

If the operator-function $A(t)A^{-1}(0)$ has a finite variation on $[0,T]$, then operator-function $A^\rho(t)A^{-\rho}(0)$, has a finite variation on $[0,T]$ for any $0<\rho<1$ [see Daletskii, Yu. L., 1962].

3. Let the operator-function $A(t)A^{-1}(0)$, $0<\rho<1$ have a finite variation on $[0,T]$ and $\|A(0)A^{-1}(t)\| \le M_1$ for any $t \in [0,T]$. Then

$$\sum_{k=0}^{N-1} \left\|A_{k+1/2}^{1/2}\left(A_{k+1}^{1/2} - A_k^{1/2}\right) A_{k+1/2}^{-1}\right\|_{H\to H} \le M_{1/2}M_1P_1 + M_1P_1 \tag{3.8}$$

for any $0 = t_0 < t_1 < \cdots < t_N = T$.

Actually, we have that

$$A_{k+1/2}^{1/2}\left(A_{k+1}^{1/2} - A_k^{1/2}\right) A_{k+1/2}^{-1} = A_{k+1/2}^{1/2}A_{k+1}^{-1/2}\left(A_{k+1} - A_{k+1}^{1/2}A_k^{1/2}\right) A_{k+1/2}^{-1}$$
$$= A_{k+1/2}^{1/2}A_{k+1}^{-1/2}\left(A_{k+1} - A_k\right) A_{k+1/2}^{-1} + A_{k+1/2}^{1/2}A_{k+1}^{-1/2}\left(A_k - A_{k+1}^{1/2}A_k^{1/2}\right) A_{k+1/2}^{-1}$$
$$= A_{k+1/2}^{1/2}A_{k+1}^{-1/2}\left(A_{k+1} - A_k\right) A_{k+1/2}^{-1} + A_{k+1/2}^{1/2}A_{k+1}^{-1/2}\left(A_k^{1/2} - A_{k+1}^{1/2}\right) A_k^{-1/2}A_kA_{k+1/2}^{-1}.$$

Then using the triangle inequality, we obtain

$$\sum_{k=0}^{N-1} \left\|A_{k+1}\left(A_{k+1}^{-1/2} - A_k^{-1/2}\right) A_k^{-1/2}\right\|_{H\to H}$$

$$\leq \sum_{k=0}^{N-1}\left\|A_{k+1/2}^{1/2}A_{k+1}^{-1/2}\right\|\left\|\left(A_{k+1}-A_k\right)A_{k+1/2}^{-1}\right\|_{H\to H}$$

$$+\sum_{k=0}^{N-1}\left\|A_{k+1/2}^{1/2}A_{k+1}^{-1/2}\right\|\|(A_k-A_{k+1})A_k^{-1}\|\left\|A_kA_{k+1/2}^{-1}\right\|_{H\to H}\leq M_{1/2}M_1P_1+M_1P_1.$$

4. Let $B_s^{\pm}(k)=X_{k+1}^{\pm}X_k^{\pm}\cdots X_{k-s+2}^{\pm}$ such that $X_{k+1}^{\pm}=\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\left(I\pm\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)$. Assume that $A^{\rho}(t)A^{-\rho}(0)$ has a finite variation on $[0,T]$ and $\|A^{\rho}(0)A^{-\rho}(t)\|\leq M_{\rho}$ for any $t\in[0,T]$. Then

$$\left\|A_{k+1/2}^{\rho}2^{-1}[P_k^{\pm}(k)+P_{k-1}^{\pm}(k-1)]A_{1/2}^{-\rho}\right\|_{H\to H}\leq e^{M_{\rho}\sum\limits_{i=1}^{k}\left\|\left(A_{i+1/2}^{\rho}-A_{i-1/2}^{\rho}\right)A_0^{-\rho}\right\|_{H\to H}},\tag{3.9}$$

$$\left\|A_{k+1/2}^{\rho}2^{-1}\left[B_s^{\pm}(k)+B_{s-1}^{\pm}(k-1)\right]A_{k-s+3/2}^{-\rho}\right\|_{H\to H}\leq e^{M_{\rho}\sum\limits_{i=1}^{k}\left\|\left(A_{i+1/2}^{\rho}-A_{i-1/2}^{\rho}\right)A_0^{-\rho}\right\|_{H\to H}}.\tag{3.10}$$

Actually, using the definition of $P_k^{\pm}(k)$ and the formula

$$I+X_k^{\pm}=2\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1},$$

we obtain

$$2^{-1}\left[P_k^{\pm}(k)+P_{k-1}^{\pm}(k-1)\right]=2^{-1}\left(X_{k+1}^{\pm}X_k^{\pm}\cdots X_1^{\pm}+X_k^{\pm}\cdots X_1^{\pm}\right)$$

$$=2^{-1}\left(I+X_{k+1}^{\pm}\right)\left(X_k^{\pm}\cdots X_2^{\pm}\right)=\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}(X_k^{\pm}\cdots X_1^{\pm}).$$

Applying the estimates (2.23) and (3.6), we get

$$\left\|A_{k+1/2}^{\rho}2^{-1}[P_k^{\pm}(k)+P_{k-1}^{\pm}(k-1)]A_{1/2}^{-\rho}\right\|\leq\left\|A_{k+1/2}^{\rho}X_k^{\pm}\cdots X_2^{\pm}A_{1/2}^{-\rho}\right\|$$

$$\leq\left\|A_{k+1/2}^{\rho}A_{k-1/2}^{-\rho}\right\|\|X_k^{\pm}\|\left\|A_{k-1/2}^{\rho}A_{k-3/2}^{-\rho}\right\|\|X_{k-1}^{\pm}\|\cdots\left\|A_{3/2}^{\rho}A_{1/2}^{-\rho}\right\|\|X_1^{\pm}\|$$

$$\leq e^{M_{\rho}\left\|\left(A_{i+1/2}^{\rho}-A_{i-1/2}^{\rho}\right)A_0^{-\rho}\right\|}e^{M_{\rho}\left\|\left(A_{i-1/2}^{\rho}-A_{i-3/2}^{\rho}\right)A_0^{-\rho}\right\|}\cdots e^{M_{\rho}\left\|\left(A_{3/2}^{\rho}-A_{1/2}^{\rho}\right)A_0^{-\rho}\right\|}$$

$$\leq e^{M_{\rho}\sum\limits_{i=1}^{k}\left\|\left(A_{i+1/2}^{\rho}-A_{i-1/2}^{\rho}\right)A_0^{-\rho}\right\|_{H\to H}}\leq e^{M_{\rho}P_{\rho}}.$$

The estimate (3.9) is proved. In the similar manner, we can write

$$2^{-1}[B_s^{\pm}(k)+B_{s-1}^{\pm}(k-1)]=2^{-1}(X_{k+1}^{\pm}X_k^{\pm}...X_{k-s+2}^{\pm}+X_k^{\pm}...X_{k-s+2}^{\pm})$$

$$=2^{-1}(I+X_{k+1}^{\pm})(X_k^{\pm}...X_{k-s+2}^{\pm})=\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}(X_k^{\pm}...X_{k-s+2}^{\pm}).$$

Applying the estimates (2.23) and (3.6), we get

$$\left\|A_{k+1/2}^{\rho}2^{-1}\left[B_s^{\pm}(k)+B_{s-1}^{\pm}(k-1)\right]A_{k-s+1/2}^{-\rho}\right\|\leq\left\|A_{k+1/2}^{\rho}X_k^{\pm}\cdots X_{k-s+2}^{\pm}A_{k-s+3/2}^{-\rho}\right\|$$

$$\leq\left\|A_{k+1/2}^{\rho}A_{k-1/2}^{-\rho}\right\|\|X_k^{\pm}\|\left\|A_{k-1/2}^{\rho}A_{k-3/2}^{-\rho}\right\|\|X_{k-1}^{\pm}\|\cdots\left\|A_{k-s+5/2}^{\rho}A_{k-s+3/2}^{-\rho}\right\|\|X_{k-s+2}^{\pm}\|$$

$$\leq e^{M_\rho \left\| \left(A^\rho_{i+1/2} - A^\rho_{i-1/2}\right) A_0^{-\rho}\right\|} e^{M_\rho \left\| \left(A^\rho_{i-1/2} - A^\rho_{i-3/2}\right) A_0^{-\rho}\right\|} \dots e^{M_\rho \left\| \left(A^\rho_{k-s+5/2} - A^\rho_{k-s+3/2}\right) A_0^{-\rho}\right\|}$$

$$\leq e^{M_\rho \sum\limits_{i=k-s+2}^{k} \left\| \left(A^\rho_{i+1/2} - A^\rho_{i-1/2}\right) A_0^{-\rho}\right\|} \leq e^{M_\rho \sum\limits_{i=1}^{k} \left\| \left(A^\rho_{i+1/2} - A^\rho_{i-1/2}\right) A_0^{-\rho}\right\|} \leq e^{M_\rho P_\rho}.$$

That is the proof of the estimate (3.10).

5. Finally, let the operator-function $A^\rho(t)A^{-\rho}(s)$ satisfies the Holder condition with respect to t

$$\left\| \left(A^\rho(t) - A^\rho(s)\right) A^{-\rho}(z)\right\|_{H\to H} \leq M_\rho \tau,\ t, s, z \in [0, T], \tag{3.11}$$

$$\left\| A^{1/2}(z) \left(A^{1/2}(t) - A^{1/2}(s)\right) A^{-1}(z)\right\|_{H\to H} \leq M_{1/2} \tau,\ t, s, z \in [0, T]. \tag{3.12}$$

Under the light of the estimates obtained in items 1-5, we have the following results.

Theorem 3.1. *Let we have the above etimates. Assume that the operator-function $A^{\frac{1}{2}}(t)A^{-\frac{1}{2}}(0)$ has a finite variation on the segment $[0,T]$ and*

$$\left\| A^{\frac{1}{2}}(0) A^{-\frac{1}{2}}(t)\right\| \leq M_{1/2} \tag{3.13}$$

for any $t \in [0,T]$, $u(0) \in D(A(0)), u'(0) \in D(A^{1/2}(0))$. Then for the solution of the difference scheme (3.3) the following estimate holds:

$$||\{\frac{u_k - u_{k-1}}{\tau}\}_1^{N-1}||_{C_\tau} + ||u^\tau||_{C_\tau}$$

$$\leq M_2[||A^{\frac{1}{2}}(0)u_0||_H + ||u_0'||_H + \sum_{s=0}^{N-1} ||f_{s+1/2}||_H \tau].$$

Theorem 3.2. *Let we have the above etimates. Assume that the operator-function $A(t)A^{-1}(0)$ has a finite variation on the segment $[0,T]$ and*

$$\left\| A(0) A^{-1}(t)\right\| \leq M_1 \tag{3.14}$$

for any $t \in [0,T]$, $u(0) \in D(A(0)), u'(0) \in D(A^{1/2}(0))$. Then for the solution of the difference scheme (3.3) the following estimate holds:

$$||\{A^{1/2}(0)\frac{u_k - u_{k-1}}{\tau}\}_1^{N-1}||_{C_\tau} +$$

$$||A_{k+1/2}4^{-1}(u_{k+1} + u_k) + A^{1/2}_{k+1/2}A^{1/2}_{k-1/2}4^{-1}(u_k + u_{k-1})$$

$$+\tau^{-1}\left(A^{1/2}_{k-1/2} - A^{1/2}_{k+1/2}\right) A^{-1/2}_{k-1/2}\tau^{-1}(u_k - u_{k-1})$$

$$+2^{-1}\tau^{-1}\left(A^{1/2}_{k+1} - A^{1/2}_{k}\right) A^{-1/2}_{k+1/2}\tau^{-1}(u_{k+1} - u_k)$$

$$+A^{1/2}_{k+1/2}A^{-1/2}_{k-1/2}2^{-1}\tau^{-1}\left(A^{1/2}_{k} - A^{1/2}_{k-1}\right) A^{-1/2}_{k-1/2}\tau^{-1}(u_k - u_{k-1})||_H,$$

$$+||\{\tau^{-2}\left(u_{k+1} - 2u_k + u_{k-1}\right)\}_1^{N-1}||_{C_\tau}$$

$$\leq M_2[||A(0)u_0||_H + ||A^{\frac{1}{2}}(0)u_0'||_H + \max_{0\leq s\leq k}\left\|f_{s+1/2}\right\|_H + \sum_{s=0}^{N-2} ||f_{s+1/2} - f_{s-1/2}||_H].$$

Proof of Theorem 3.1. We will estimate $\|\tau^{-1}(u_{k+1}-u_k)\|$. Applying formula (3.5), we can write

$$\tau^{-1}(u_{k+1}-u_k)=J_{1k}+J_{2k}+J_{3k}+J_{4k}+J_{5k}+J_{6k}, \tag{3.15}$$

where

$$J_{1k}=\frac{1}{4}\left[iA_{k+1/2}^{1/2}\left[P_k^{+}(k)+P_{k-1}^{+}(k-1)\right]-iA_{k+1/2}^{1/2}\left[P_k^{-}(k)+P_{k-1}^{-}(k-1)\right]\right]u_0,$$

$$J_{2k}=\frac{1}{4}\left[A_{k+1/2}^{1/2}\left[P_k^{+}(k)+P_{k-1}^{+}(k-1)\right]\left(A_0^{-1/2}\right)\right.$$
$$\left.+A_{k+1/2}^{1/2}\left[P_k^{-}(k)+P_{k-1}^{-}(k-1)\right]\left(A_0^{-1/2}\right)\right]u_0',$$

$$J_{3k}=-\frac{1}{4}A_{k+1/2}^{1/2}\left[B_0^{+}(k)\tau\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}+B_0^{-}(k)\tau\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right]$$
$$\times\left[A_{k+1/2}^{-1/2}\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\tau^{-1}(u_{k+1}-u_k)\right],$$

$$J_{4k}=\frac{1}{4}A_{k+1/2}^{1/2}\left[B_0^{+}(k)\tau\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}+B_0^{-}(k)\tau\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right]A_{k+1/2}^{-1/2}f_{k+1/2}$$

$$J_{5k}=-\frac{1}{4}\sum_{s=1}^{k}A_{k+1/2}^{1/2}\left[[B_s^{+}(k)+B_{s-1}^{+}(k-1)]\tau\left(I-\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right.$$
$$\left.+[B_s^{-}(k)+B_{s-1}^{-}(k-1)]\tau\left(I+\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right]$$
$$\times\left[A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\tau^{-1}(u_{k-s+1}-u_{k-s})\right],$$

$$J_{6k}=\frac{1}{4}\sum_{s=1}^{k}A_{k+1/2}^{1/2}\left[[B_s^{+}(k)+B_{s-1}^{+}(k-1)]\tau\left(I-\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right.$$
$$\left.+[B_s^{-}(k)+B_{s-1}^{-}(k-1)]\tau\left(I+\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right]A_{k-s+1/2}^{-1/2}f_{k-s+1/2}.$$

We will estimate $||J_{mk}||_H$, $m=\overline{1,6}$. Let $m=1$. Then applying the estimates (3.6) and (3.9), we get

$$||J_{1k}||_H\leq\left\|A_{k+1/2}^{1/2}2^{-1}[P_k^{\pm}(k)+P_{k-1}^{\pm}(k-1)]u_0\right\|$$
$$\leq\left\|A_{k+1/2}^{1/2}2^{-1}[P_k^{\pm}(k)+P_{k-1}^{\pm}(k-1)]A_{1/2}^{-1/2}\right\|\left\|A_{1/2}^{1/2}A_0^{-1/2}\right\|\left\|A_0^{1/2}u_0\right\|_H$$
$$\leq C_1\left\|A_0^{1/2}u_0\right\|_H,$$

where

$$C_1=M_{1/2}e^{M_{1/2}P_{1/2}}.$$

Let $m=2$. Then applying the estimates (3.6) and (3.9), we get

$$||J_{2k}||_H\leq\left\|A_{k+1/2}^{1/2}2^{-1}[P_k^{\pm}(k)+P_{k-1}^{\pm}(k-1)]A_0^{-1/2}u_0'\right\|_H$$

$$\leq \left\| A_{k+1/2}^{1/2} 2^{-1}[P_k^{\pm}(k) + P_{k-1}^{\pm}(k-1)] A_{1/2}^{-1/2} \right\| \left\| A_{1/2}^{1/2} A_0^{-1/2} \right\| \|u_0'\|_H \leq C_1 \|u_0'\|_H .$$

Let $m = 3$. Then applying the estimates (2.22) and (3.11), we get

$$||J_{3k}||_H \leq \left\| 4^{-1} A_{k+1/2}^{1/2} 2\tau B_0^{\pm}(k) \left(I \mp \frac{i\tau}{2} A_{k+1/2}^{1/2} \right)^{-1} \right.$$

$$\left. \times \left[A_{k+1/2}^{-1/2} \tau^{-1} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} \tau^{-1} (u_{k+1} - u_k) \right] \right\|_H$$

$$\leq 2^{-1} \left\| \left(I \mp \frac{i\tau}{2} A_{k+1/2}^{1/2} \right)^{-1} \right\| \left\| \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} \right\| \left\| \tau^{-1} (u_{k+1} - u_k) \right\|_H$$

$$\leq M_{1/2} \frac{\tau}{2} \left\| \tau^{-1} (u_{k+1} - u_k) \right\|_H .$$

Let $m = 4$. Then applying the estimate (2.22), we get

$$||J_{4k}||_H \leq \left\| 4^{-1} A_{k+1/2}^{1/2} 2\tau B_0^{\pm}(k) \left(I \mp \frac{i\tau}{2} A_{k+1/2}^{1/2} \right)^{-1} A_{k+1/2}^{-1/2} f_{k+1/2} \right\|_H$$

$$\leq 2^{-1} \left\| \left(I \mp \frac{i\tau}{2} A_{k+1/2}^{1/2} \right)^{-1} \right\| \tau \left\| f_{k+1/2} \right\|_H \leq \frac{\tau}{2} \left\| f_{k+1/2} \right\|_H .$$

Let $m = 5$. Then applying the estimates (2.22), (3.6) and (3.10), we get

$$||J_{5k}||_H \leq \left\| \sum_{s=1}^{k} A_{k+1/2}^{1/2} 2^{-1} \left[B_s^{\pm}(k) + B_{s-1}^{\pm}(k-1) \right] \tau \left(I \mp \frac{i\tau}{2} A_{k-s+1/2}^{1/2} \right)^{-1} \right.$$

$$\left. \times A_{k-s+1/2}^{-1/2} \tau^{-1} \left(A_{k-s+1}^{1/2} - A_{k-s}^{1/2} \right) A_{k-s+1/2}^{-1/2} \tau^{-1} (u_{k-s+1} - u_{k-s}) \right\|_H$$

$$\leq \sum_{s=1}^{k} \left\| A_{k+1/2}^{1/2} 2^{-1} \left[B_s^{\pm}(k) + B_{s-1}^{\pm}(k-1) \right] A_{k-s+3/2}^{-1/2} \right\| \left\| A_{k-s+3/2}^{1/2} A_{k-s+1/2}^{-1/2} \right\|$$

$$\times \left\| \left(I \mp \frac{i\tau}{2} A_{k-s+1/2}^{1/2} \right)^{-1} \right\| \left\| \left(A_{k-s+1}^{1/2} - A_{k-s}^{1/2} \right) A_{k-s+1/2}^{-1/2} \right\| \left\| \tau^{-1} (u_{k-s+1} - u_{k-s}) \right\|_H$$

$$\leq C_1 \tau \sum_{s=0}^{k-1} \left\| \left(A_{s+1}^{1/2} - A_s^{1/2} \right) A_{s+1/2}^{-1/2} \right\| \left\| \tau^{-1} (u_{s+1} - u_s) \right\|_H$$

Let $m = 6$. Then applying the estimates (2.22), (3.6) and (3.10), we get

$$||J_{6k}||_H = \left\| \sum_{s=1}^{k} A_{k+1/2}^{1/2} 2^{-1} \left[B_s^{\pm}(k) + B_{s-1}^{\pm}(k-1) \right] \tau \left(I \mp \frac{i\tau}{2} A_{k-s+1/2}^{1/2} \right)^{-1} A_{k-s+1/2}^{-1/2} f_{k-s+1/2} \right\|_H$$

$$\leq \sum_{s=1}^{k} \tau \left\| A_{k+1/2}^{1/2} 2^{-1} \left[B_s^{\pm}(k) + B_{s-1}^{\pm}(k-1) \right] A_{k-s+3/2}^{-1/2} \right\| \left\| A_{k-s+3/2}^{1/2} A_{k-s+1/2}^{-1/2} \right\|$$

$$\times \left\| \left(I \mp \frac{i\tau}{2} A_{k-s+1/2}^{1/2} \right)^{-1} \right\| \left\| f_{k-s+1/2} \right\|_H \leq C_1 \tau \sum_{s=1}^{k} \left\| f_{k-s+1/2} \right\|_H$$

Using the formula (3.15), the triangle inequality and the last six estimates, we obtain

$$\left\| \tau^{-1} (u_{k+1} - u_k) \right\|_H \leq C_1 \left\| A_0^{1/2} u_0 \right\|_H + C_1 \|u_0'\|_H$$

$$+M_{1/2}\frac{\tau}{2}\left\|\tau^{-1}(u_{k+1}-u_k)\right\|_H+\frac{\tau}{2}\left\|f_{k+1/2}\right\|_H$$

$$+C_1\tau\sum_{s=0}^{k-1}\left\|\left(A_{s+1}^{1/2}-A_s^{1/2}\right)A_{s+1/2}^{-1/2}\right\|\left\|\tau^{-1}(u_{s+1}-u_s)\right\|_H$$

$$+C_1\tau\sum_{s=1}^{k}\left\|f_{k-s+1/2}\right\|_H \tag{3.16}$$

$$\leq C_1\left\|A_0^{1/2}u_0\right\|_H+C_1\left\|u_0'\right\|_H+M_{1/2}\frac{\tau}{2}\left\|\tau^{-1}(u_{k+1}-u_k)\right\|_H$$

$$+C_1\tau\sum_{s=0}^{k-1}\left\|\left(A_{s+1}^{1/2}-A_s^{1/2}\right)A_{s+1/2}^{-1/2}\right\|\left\|\tau^{-1}(u_{s+1}-u_s)\right\|_H$$

$$+C_2\tau\sum_{s=0}^{k}\left\|f_{k-s+1/2}\right\|_H,$$

where

$$C_2=\max\{\frac{1}{2},\ C_1\}.$$

From the above result it follows that

$$\left\|\tau^{-1}(u_{k+1}-u_k)\right\|\leq C_3\left[\left\|A_0^{1/2}u_0\right\|_H+\left\|u_0'\right\|_H\right.$$

$$\left.+\sum_{s=0}^{k-1}\left\|(A_{s+1}^{1/2}-A_s^{1/2})A_{s+1/2}^{-1/2}\right\|\left\|\tau^{-1}(u_{s+1}-u_s)\right\|_H+\sum_{s=0}^{k}\left\|f_{s+1/2}\right\|_H\tau\right], \tag{3.17}$$

where

$$C_3=\max\left\{\left(I-\frac{1}{2}M_{1/2}\tau\right)^{-1}C_1,\ \left(I-\frac{1}{2}M_{1/2}\tau\right)^{-1}C_1\tau,\ \left(I-\frac{1}{2}M_{1/2}\tau\right)^{-1}C_2\tau\right\}.$$

Defining

$$\beta=\left\|A_0^{1/2}u_0\right\|_H+\left\|u_0'\right\|_H,\ \alpha_s=\left\|\left(A_{s+1}^{1/2}-A_s^{1/2}\right)A_{s+1/2}^{-1/2}\right\|_H,$$

$$z_{k+1}=\left\|\tau^{-1}(u_{k+1}-u_k)\right\|_H,\ z_{s+1}=\left\|\tau^{-1}(u_{s+1}-u_s)\right\|_H,\ \beta_s=\left\|f_{s+1/2}\right\|_H,\ \alpha=C_3,$$

the inequality (3.17) can be written as

$$z_{k+1}\leq\alpha\left(\beta+\sum_{s=0}^{k-1}\alpha_s z_{s+1}+\sum_{s=0}^{k}\beta_s\tau\right),\ k=0,1,\cdots,N-1.$$

We denote that

$$\nu_k=\alpha\left(\beta+\sum_{s=0}^{k-1}\alpha_s z_{s+1}+\sum_{s=0}^{k}\beta_s\tau\right).$$

Then

$$z_{k+1}\leq v_k$$

and

$$\nu_{k+1}-\nu_k=\alpha\alpha_k z_{k+1}+\alpha\beta_{k+1}\tau\leq\alpha\alpha_k\nu_k+\alpha\beta_{k+1}\tau$$

or

$$\nu_{k+1}\leq(1+\alpha\alpha_k)\nu_k+\alpha\beta_{k+1}\tau.$$

Solving it, we can write

$$\nu_k \leq (1+\alpha\alpha_{k-1})...(1+\alpha\alpha_0)\nu_0 + \sum_{i=1}^{k}[(1+\alpha\alpha_{k-1})...(1+\alpha\alpha_i)]\alpha\beta_i\tau.$$

Using the Bernolli inequality $1+\alpha\alpha_{k-1} \leq e^{\alpha\alpha_{k-1}}$, we can write

$$v_k \leq e^{\alpha\alpha_{k-1}}e^{\alpha\alpha_{k-2}}...e^{\alpha\alpha_0}\nu_0 + \sum_{i=1}^{k}\left(e^{\alpha\alpha_{k-1}}e^{\alpha\alpha_{k-2}}...e^{\alpha\alpha_i}\right)\alpha\beta_i\tau$$

$$= \left[e^{\alpha(\alpha_0+\alpha_1+...\alpha_{k-1})}\alpha(\beta+\beta_0\tau)\right] + \sum_{i=1}^{k}\left[e^{\alpha(\alpha_i+\alpha_{i+12}+...+\alpha_{k-1})}\alpha\beta_i\tau\right].$$

Since

$$\alpha_0+\alpha_1+\cdots+\alpha_{k-1}$$

$$= \left\|\left(A_1^{1/2}-A_0^{1/2}\right)A_{1/2}^{-1/2}\right\|_H \left\|\left(A_2^{1/2}-A_1^{1/2}\right)A_{3/2}^{-1/2}\right\|_H \cdots \left\|\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|_H$$

$$= \sum_{i=0}^{k-1}\left\|\left(A_{i+1}^{1/2}-A_i^{1/2}\right)A_{i+1/2}^{-1/2}\right\|_H$$

and

$$\sum_{i=0}^{k-1}\left\|\left(A_{i+1}^{1/2}-A_i^{1/2}\right)A_{i+1/2}^{-1/2}\right\|_H \leq P_{1/2},$$

we have that

$$e^{\alpha(\alpha_0+\alpha_1+...\alpha_{k-1})} \leq e^{\alpha P_{1/2}}.$$

Therefore

$$v_k \leq e^{\alpha P_{1/2}}\left[\alpha(\beta+\beta_0\tau)+\alpha\sum_{i=1}^{k}\beta_i\tau\right] = \alpha e^{\alpha P_{1/2}}\left[\beta+\sum_{i=0}^{k}\beta_i\tau\right]$$

$$= C_4\left[\left\|A_0^{1/2}u_0\right\|_H + \|u_0'\|_H + \sum_{i=0}^{k}\beta_i\tau\right]$$

or

$$z_{k+1} \leq C_4\left[\left\|A_0^{1/2}u_0\right\|_H + \|u_0'\|_H + \sum_{i=0}^{k}\beta_i\tau\right].$$

So,

$$\left\|\tau^{-1}(u_{k+1}-u_k)\right\|_H \leq C_4\left[\left\|A_0^{1/2}u_0\right\|_H + \|u_0'\|_H + \sum_{s=0}^{k}\left\|f_{s+1/2}\right\|_H\tau\right], \tag{3.18}$$

where

$$C_4 = \alpha e^{\alpha P_{1/2}}.$$

Now, we will obtain the estimate for $||u_k||_H$, $k=1,\cdots,N$. It is easy to show that

$$u_k = u_0 + \sum_{s=1}^{k}\tau^{-1}(u_s-u_{s-1})\tau.$$

Using the last formula, the estimate (3.18) and the triangular inequality, we obtain

$$\|u_k\|_H \leq \|u_0\|_H + \sum_{s=1}^{k}\left\|\tau^{-1}(u_s-u_{s-1})\right\|_H\tau = \left\|A_0^{-1/2}A_0^{1/2}u_0\right\|_H$$

$$\leq \left\|A_0^{-1/2}\right\| \left\|A_0^{1/2}u_0\right\|_H + C_4 k\tau \left[\left\|A_0^{1/2}u_0\right\|_H + \|u_0'\|_H + \sum_{s=0}^{k-1} \left\|f_{s+1/2}\right\|_H \tau\right].$$

Applying the estimate (2.25), we get

$$\left\|A_0^{-1/2}\right\|_{H\to H} \leq \sqrt{\delta}^{-1}.$$

Then for any $k = 1, \cdots N$ we have that

$$\|u_k\|_H \leq \left[\sqrt{\delta}^{-1} + TC_4\right] \left\|A_0^{1/2}u_0\right\|_H + TC_4 \|u_0'\|_H$$

$$+TC_4 \sum_{s=0}^{k-1} \left\|f_{s+1/2}\right\|_H \tau$$

$$\leq C_5 \left[\left\|A_0^{1/2}u_0\right\|_H + \|u_0'\|_H + \sum_{s=0}^{k-1} \left\|f_{s+1/2}\right\|_H \tau\right]$$

where

$$C_5 = \max\left\{\sqrt{\delta}^{-1} + TC_4,\ TC_4,\ C_4\right\}.$$

Combining the last estimate and (3.18), we obtain the conclusion of Theorem 3.1.

$$||\{\frac{u_k - u_{k-1}}{\tau}\}_1^{N-1}||_{C_\tau} + ||u^\tau||_{C_\tau}$$

$$\leq M_2[||A^{\frac{1}{2}}(0)u_0||_H + ||u_0'||_H + \sum_{s=0}^{N-1} ||f_{s+1/2}||_H\tau].$$

where

$$M_2 = \max\left(C_4 + C_5,\ 2\right).$$

Proof of Theorem 3.2. We will estimate $\left\|A_{k+1/2}^{1/2}\tau^{-1}(u_{k+1} - u_k)\right\|$. Applying formula (3.5), we can write

$$A_{k+1/2}^{1/2}\tau^{-1}(u_{k+1} - u_k) = A_{k+1/2}^{1/2}\left[J_{1k} + J_{2k} + J_{3k} + J_{4k} + J_{5k} + J_{6k}\right]. \tag{3.19}$$

We will estimate $||A_{k+1/2}^{1/2}J_{mk}||_H$, $m = \overline{1,6}$. Let $m = 1$. Then applying the estimates (3.6) and (3.9), we get

$$||A_{k+1/2}^{1/2}J_{1k}||_H \leq \left\|A_{k+1/2}2^{-1}[P_k^{\pm}(k) + P_{k-1}^{\pm}(k-1)]u_0\right\|_H$$

$$\leq \left\|A_{k+1/2}2^{-1}[P_k^{\pm}(k) + P_{k-1}^{\pm}(k-1)]A_{1/2}^{-1}\right\| \left\|A_{1/2}A_0^{-1}\right\| \left\|A_0u_0\right\|_H$$

$$\leq C_6 \left\|A_0u_0\right\|_H,$$

where

$$C_6 = M_1 e^{M_1 P_1}.$$

Let $m = 2$. Then applying the estimates (3.6) and (3.9), we get

$$||A_{k+1/2}^{1/2}J_{2k}||_H \leq \left\|A_{k+1/2}2^{-1}[P_k^{\pm}(k) + P_{k-1}^{\pm}(k-1)]A_0^{-1/2}u_0'\right\|_H$$

$$\leq \left\| A_{k+1/2} 2^{-1} [P_k^{\pm}(k) + P_{k-1}^{\pm}(k-1)] A_{1/2}^{-1} \right\| \left\| A_{1/2} A_0^{-1} \right\| \left\| A_0^{1/2} u_0' \right\|_H$$

$$\leq C_6 \left\| A_0^{1/2} u_0' \right\|_H .$$

Let $m = 3$. Then applying the estimates (2.22) and (3.12), we get

$$||A_{k+1/2}^{1/2} J_{3k}||_H \leq \Bigg\| 4^{-1} A_{k+1/2} 2\tau B_0^{\pm}(k) \left(I \mp \frac{i\tau}{2} A_{k+1/2}^{1/2} \right)^{-1}$$

$$\times A_{k+1/2}^{-1/2} \tau^{-1} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} \tau^{-1} \left(u_{k+1} - u_k \right) \Bigg\|_H$$

$$\leq 2^{-1} \left\| \left(I \mp \frac{i\tau}{2} A_{k+1/2}^{1/2} \right)^{-1} \right\| \left\| A_{k+1/2}^{1/2} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1} \right\|$$

$$\times \left\| A_{k+1/2}^{1/2} \tau^{-1} \left(u_{k+1} - u_k \right) \right\|_H \leq M_{1/2} \frac{\tau}{2} \left\| A_{k+1/2}^{1/2} \tau^{-1} \left(u_{k+1} - u_k \right) \right\|_H .$$

Let $m = 4$. Then applying the estimate (2.24), we get

$$||A_{k+1/2}^{1/2} J_{4k}||_H = \left\| 4^{-1} A_{k+1/2} 2\tau B_0^{\pm}(k) \left(I \mp \frac{i\tau}{2} A_{k+1/2}^{1/2} \right)^{-1} A_{k+1/2}^{-1/2} f_{k+1/2} \right\|_H$$

$$\leq \left\| \left(\frac{\tau}{2} A_{k+1/2}^{1/2} \right) \left(I \mp \frac{i\tau}{2} A_{k+1/2}^{1/2} \right)^{-1} \right\| \left\| f_{k+1/2} \right\|_H \leq \left\| f_{k+1/2} \right\|_H .$$

Let $m = 5$. Then applying the estimate (2.22), (3.6) and (3.10), we get

$$||A_{k+1/2}^{1/2} J_{5k}||_H \leq \Bigg\| \sum_{s=1}^{k} A_{k+1/2} 2^{-1} \left[B_s^{\pm}(k) + B_{s-1}^{\pm}(k-1) \right] \tau \left(I \mp \frac{i\tau}{2} A_{k-s+1/2}^{1/2} \right)^{-1}$$

$$\times A_{k-s+1/2}^{-1/2} \tau^{-1} \left(A_{k-s+1}^{1/2} - A_{k-s}^{1/2} \right) A_{k-s+1/2}^{-1/2} \tau^{-1} (u_{k-s+1} - u_{k-s}) \Bigg\|_H$$

$$\leq \sum_{s=1}^{k} \left\| A_{k+1/2} 2^{-1} \left[B_s^{\pm}(k) + B_{s-1}^{\pm}(k-1) \right] A_{k-s+3/2}^{-1} \right\|$$

$$\times \left\| A_{k-s+3/2} A_{k-s+1/2}^{-1} \right\| \left\| \left(I \mp \frac{i\tau}{2} A_{k-s+1/2}^{1/2} \right)^{-1} \right\|_H$$

$$\times \left\| A_{k-s+1/2}^{1/2} \left(A_{k-s+1}^{1/2} - A_{k-s}^{1/2} \right) A_{k-s+1/2}^{-1} \right\| \left\| A_{k-s+1/2}^{1/2} \tau^{-1} (u_{k-s+1} - u_{k-s}) \right\|_H$$

$$\leq \sum_{s=0}^{k-1} C_6 \left\| A_{s+1/2}^{1/2} \left(A_{s+1}^{1/2} - A_s^{1/2} \right) A_{s+1/2}^{-1} \right\| \left\| A_{s+1/2}^{1/2} \tau^{-1} (u_{s+1} - u_s) \right\|_H .$$

Finally, Let $m = 6$. We have that

$$\tau \left(I \pm \frac{i\tau}{2} A_{k-s+1/2}^{1/2} \right)^{-1} A_{k-s+1/2}^{-1/2} = i A_{k-s+1/2}^{-1} (I - X_{k-s+1}^{\pm}). \tag{3.20}$$

Using the Abel formula, we get

$$A_{k+1/2}^{1/2} J_{6k} = A_{k+1/2} 4^{-1} \left\{ \left[B_1^{+}(k) + B_0^{+}(k-1) \right] + \left[B_1^{-}(k) + B_0^{-}(k-1) \right] \right\} A_{k-1/2}^{-1} f_{k-1/2}$$

$$+ A_{k+1/2} 4^{-1} \left\{ \left[B_1^{+}(k) + B_0^{+}(k-1) \right] + \left[B_1^{-}(k) + B_0^{-}(k-1) \right] \right\} A_{1/2}^{-1} f_{1/2}$$

$$+\sum_{s=1}^{k-1} A_{k+1/2}4^{-1}\left\{\left[B_1^{+}(k)+B_0^{+}(k-1)\right]+\left[B_1^{-}(k)+B_0^{-}(k-1)\right]\right\}.$$

$$\times\left(A_{k-s-1/2}^{-1}f_{k-s-1/2}-A_{k-s+1/2}^{-1}f_{k-s+1/2}\right)$$

Then applying the triangular inequality, we get

$$||A_{k+1/2}^{1/2}J_{6k}||_H \leq \left\|A_{k+1/2}2^{-1}\left[B_1^{\pm}(k)+B_0^{\pm}(k-1)\right]A_{k-1/2}^{-1}f_{k-1/2}\right\|_H$$

$$+\left\|A_{k+1/2}2^{-1}\left[B_{k+1}^{\pm}(k)+B_k^{\pm}(k-1)\right]A_{1/2}^{-1}f_{1/2}\right\|_H$$

$$+\sum_{s=1}^{k-1}\left\|A_{k+1/2}2^{-1}\left[B_{s+1}^{\pm}(k)+B_s^{\pm}(k-1)\right]\left(A_{k-s-1/2}^{-1}f_{k-s-1/2}-A_{k-s+1/2}^{-1}f_{k-s+1/2}\right)\right\|_H.$$

It is easy to show that

$$\left(A_{k-s-1/2}^{-1}f_{k-s-1/2}-A_{k-s+1/2}^{-1}f_{k-s+1/2}\right)$$

$$=\left(A_{k-s-1/2}^{-1}-A_{k-s+1/2}^{-1}\right)f_{k-s-1/2}+A_{k-s+1/2}^{-1}\left(f_{k-s-1/2}-f_{k-s+1/2}\right)$$

$$=A_{k-s-1/2}^{-1}\left(A_{k-s+1/2}-A_{k-s-1/2}\right)A_{k-s+1/2}^{-1}f_{k-s-1/2}$$

$$+A_{k-s+1/2}^{-1}\left(f_{k-s-1/2}-f_{k-s+1/2}\right). \tag{3.21}$$

Using the above equation and triangular inequality, and then applying the estimates (2.22), (3.6), (3.10) and (3.11), we get

$$||A_{k+1/2}^{1/2}J_{6k}||_H \leq \left\|\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\left\|A_{k+1/2}A_{k-1/2}^{-1}\right\|\left\|f_{k-1/2}\right\|_H$$

$$+\left\|A_{k+1/2}2^{-1}\left[B_{k+2}^{\pm}(k)+B_{k+1}^{\pm}(k-1)\right]A_{1/2}^{-1}\right\|\left\|f_{1/2}\right\|_H$$

$$+\sum_{s=1}^{k-1}\left\|A_{k+1/2}2^{-1}\left[B_{s+1}^{\pm}(k)+B_s^{\pm}(k-1)\right]A_{k-s+1/2}^{-1}\right\|$$

$$\times\left\|\left(A_{k-s+1/2}-A_{k-s-1/2}\right)A_{k-s+1/2}^{-1}\right\|\left\|f_{k-s-1/2}\right\|_H$$

$$+\sum_{s=1}^{k-1}\left\|A_{k+1/2}2^{-1}\left[B_{s+1}^{\pm}(k)+B_s^{\pm}(k-1)\right]A_{k-s+1/2}^{-1}\right\|\left\|f_{k-s-1/2}-f_{k-s+1/2}\right\|_H$$

$$\leq M_1\left\|f_{k-1/2}\right\|_H+C_6\left\|f_{1/2}\right\|_H+C_6\tau\sum_{s=1}^{k-1}\left\|f_{k-s-1/2}\right\|_H+C_7\sum_{s=1}^{k-1}\left\|f_{k-s-1/2}-f_{k-s+1/2}\right\|_H$$

$$=M_1\left\|f_{k-1/2}\right\|_H+C_6\left\|f_{1/2}\right\|_H+C_6\tau\sum_{s=1}^{k-1}\left\|f_{s-1/2}\right\|_H+C_7\sum_{s=1}^{k-1}\left\|f_{s-1/2}-f_{s+1/2}\right\|_H,$$

where

$$C_7=e^{M_1P_1}.$$

Using the formula (3.19), the triangular inequality, and the last six estimates, we obtain

$$\left\|A_{k+1/2}^{1/2}\tau^{-1}(u_{k+1}-u_k)\right\|_H \leq C_6\left\|A_0u_0\right\|_H+C_6\left\|A_0^{1/2}u_0'\right\|_H$$

$$+M_{1/2}\frac{\tau}{2}\left\|A_{k+1/2}^{1/2}\tau^{-1}(u_{k+1}-u_k)\right\|_H$$

$$+C_6\sum_{s=0}^{k-1}\left\|A_{s+1/2}^{1/2}\left(A_{s+1}^{1/2}-A_s^{1/2}\right)A_{s+1/2}^{-1}\right\|\left\|A_{s+1/2}^{1/2}\tau^{-1}(u_{s+1}-u_s)\right\|_H$$

$$+\left\|f_{k+1/2}\right\|_H+C_6\left\|f_{1/2}\right\|_H+M_1\left\|f_{k-1/2}\right\|_H$$

$$\left.+C_6\tau\sum_{s=1}^{k-1}\left\|f_{s-1/2}\right\|_H+C_2\sum_{s=1}^{k-1}\left\|f_{s-1/2}-f_{s+1/2}\right\|_H\right]$$

$$\leq C_8\left[\left\|A_0u_0\right\|_H+\left\|A_0^{1/2}u_0'\right\|_H+\max_{0\leq s\leq k}\left\|f_s\right\|_H\right]$$

$$+\sum_{s=0}^{k-1}\left\|A_{s+1/2}^{1/2}\left(A_{s+1}^{1/2}-A_s^{1/2}\right)A_{s+1/2}^{-1}\right\|\left\|A_{s+1/2}^{1/2}\tau^{-1}(u_{s+1}-u_s)\right\|_H$$

$$+\sum_{s=1}^{k-1}\left\|f_{s-1/2}-f_{s+1/2}\right\|_H, \tag{3.22}$$

where

$$C_8=\max\left\{(I-M_{1/2}\frac{\tau}{2})^{-1}C_6,\ (I-M_{1/2}\frac{\tau}{2})^{-1}C_2,\ (I-M_{1/2}\frac{\tau}{2})^{-1},\right.$$
$$\left.(I-M_{1/2}\frac{\tau}{2})^{-1}M_1,\ (I-M_{1/2}\frac{\tau}{2})^{-1}\ C_6\tau\right\}.$$

Definig

$$\beta=\left\|A_0u_0\right\|_H+\left\|A_0^{1/2}u_0'\right\|_H+\max_{0\leq s\leq k}\left\|f_{s+1/2}\right\|_H,\alpha_s=\left\|A_{s+1/2}^{1/2}\left(A_{s+1}^{1/2}-A_s^{1/2}\right)A_{s+1/2}^{-1}\right\|_H$$

$$z_{k+1}=\left\|A_{k+1/2}^{1/2}\tau^{-1}(u_{k+1}-u_k)\right\|_H,\ \beta_{s+1}=\left\|f_{s-1/2}-f_{s+1/2}\right\|_H,\ \alpha=C_8,$$

the inequality (3.22) can be written as

$$z_{k+1}\leq\alpha\left(\beta+\sum_{s=0}^{k-1}\alpha_sz_{s+1}+\sum_{s=1}^{k-1}\beta_{s+1}\right),\ k=0,1,\cdot\cdot\cdot,N-1.$$

We denote that

$$\nu_k=\alpha\left(\beta+\sum_{s=0}^{k-1}\alpha_sz_{s+1}+\sum_{s=1}^{k-1}\beta_{s+1}\right).$$

Then

$$z_{k+1}\leq\nu_k,$$

and

$$\nu_{k+1}-\nu_k=\alpha\alpha_kz_{k+1}+\alpha\beta_{k+1}\leq\alpha\alpha_k\nu_k+\alpha\beta_{k+1},$$

or

$$\nu_{k+1}\leq(1+\alpha\alpha_k)\nu_k+\alpha\beta_{k+1}.$$

Solving it, we can write

$$\nu_k\leq(1+\alpha\alpha_{k-1})...(1+\alpha\alpha_0)\nu_0+\sum_{i=1}^{k}[(1+\alpha\alpha_{k-1})...(1+\alpha\alpha_i)]\alpha\beta_i.$$

Using the Bernolli inequality $1+\alpha\alpha_{k-1}\leq e^{\alpha\alpha_{k-1}}$, we can write

$$\nu_k \leq e^{\alpha\alpha_{k-1}}e^{\alpha\alpha_{k-2}}...e^{\alpha\alpha_0}\nu_0+\sum_{i=1}^{k}\left(e^{\alpha\alpha_{k-1}}e^{\alpha\alpha_{k-2}}...e^{\alpha\alpha_i}\right)\alpha\beta_i$$

$$=\left[e^{\alpha(\alpha_0+\alpha_1+...\alpha_{k-1})}\alpha\beta\right]+\sum_{i=1}^{k}\left[e^{\alpha(\alpha_0+\alpha_1+...\alpha_{k-1})}\alpha\beta_i\right].$$

Since

$$\alpha_0+\alpha_1+\cdots+\alpha_{k-1}$$
$$=\left\|A_{1/2}^{1/2}\left(A_1^{1/2}-A_0^{1/2}\right)A_{1/2}^{-1}\right\|\left\|A_{3/2}^{1/2}\left(A_2^{1/2}-A_1^{1/2}\right)A_{3/2}^{-1}\right\|$$
$$\cdots\left\|A_{k-1/2}^{1/2}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1}\right\|=\sum_{i=0}^{k-1}\left\|A_{i+1/2}^{1/2}\left(A_{i+1}^{1/2}-A_i^{1/2}\right)A_{i+1/2}^{-1}\right\|,$$

and

$$\sum_{i=0}^{k-1}\left\|A_{i+1/2}^{1/2}\left(A_{i+1}^{1/2}-A_i^{1/2}\right)A_{i+1/2}^{-1}\right\|\leq\left(M_{1/2}+I\right)M_1P_1,$$

we have that

$$e^{\alpha(\alpha_0+\alpha_1+...\alpha_{k-1})}\leq e^{\alpha\left(M_{1/2}+I\right)M_1P_1}.$$

Therefore

$$v_k\leq e^{\alpha\left(M_{1/2}+I\right)M_1P_1}\left[\alpha\beta+\alpha\sum_{i=1}^{k}\beta_i\right]=\alpha e^{\alpha\left(M_{1/2}+I\right)M_1P_1}\left[\beta+\sum_{i=1}^{k}\beta_i\right]$$

$$=C_9\left[\|A_0u_0\|_H+\left\|A_0^{1/2}u_0'\right\|_H+\max_{0\leq s\leq k}\|f_s\|_H+\sum_{s=1}^{k}\left\|f\left(t_{s-3/2}\right)-f\left(t_{s-1/2}\right)\right\|_H\right],$$

or

$$z_{k+1}\leq C_9\left[\|A_0u_0\|_H+\left\|A_0^{1/2}u_0'\right\|_H+\max_{0\leq s\leq k}\|f_s\|_H+\sum_{s=1}^{k}\left\|f\left(t_{s-3/2}\right)-f\left(t_{s-1/2}\right)\right\|_H\right].$$

So,

$$\left\|A_{k+1/2}^{1/2}\tau^{-1}(u_{k+1}-u_k)\right\|_H\leq C_9\left[\|A_0u_0\|_H+\left\|A_0^{1/2}u_0'\right\|_H\right.$$

$$\left.+\max_{0\leq s\leq k}\left\|f_{s+1/2}\right\|_H+\sum_{s=0}^{k-1}\left\|f\left(t_{s+1/2}\right)-f\left(t_{s-1/2}\right)\right\|_H\right],\tag{3.23}$$

where

$$C_9=\alpha e^{\alpha\left(M_{1/2}+I\right)M_1P_1}.$$

Now, we consider the equation (3.3). If we put the expression, for u_k, in the equation (3.3), we obtain the following formula

$$A_{k+1/2}4^{-1}(u_{k+1}+u_k)+A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}4^{-1}(u_k+u_{k-1})$$

$$+\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\tau^{-1}(u_k-u_{k-1})$$

$$+2^{-1}\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\tau^{-1}(u_{k+1}-u_k)$$

$$+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}2^{-1}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\tau^{-1}(u_k-u_{k-1})=\sum_{m=1}^{8}S_{mk},$$

where

$$S_{1k}=4^{-1}\left\{\left[A_{k+1/2}+i\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]2^{-1}\left[P_k^{+}(k)+P_{k-1}^{+}(k-1)\right]\right.$$

$$+\left[A_{k+1/2}-i\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]2^{-1}\left[P_k^{-}(k)+P_{k-1}^{-}(k-1)\right]$$

$$+\left[A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2i\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.+iA_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]2^{-1}\left[P_{k-1}^{+}(k-1)+P_{k-2}^{+}(k-2)\right]$$

$$+\left[A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}-2i\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.\left.-iA_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]2^{-1}\left[P_{k-1}^{-}(k-1)+P_{k-2}^{-}(k-2)\right]\right\}u_0,$$

$$S_{2k}=4^{-1}\left\{\left[A_{k+1/2}+i\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]2^{-1}\left[P_k^{+}(k)+P_{k-1}^{+}(k-1)\right]\right.$$

$$+\left[A_{k+1/2}-i\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]2^{-1}\left[P_k^{-}(k)+P_{k-1}^{-}(k-1)\right]$$

$$+\left[A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2i\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.+iA_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[P_{k-1}^{+}(k-1)+P_{k-2}^{+}(k-2)\right]\left(-iA_0^{-1/2}\right)$$

$$+\left[A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}-2i\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.-iA_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]$$

$$\left.\times 2^{-1}\left[P_{k-1}^{-}(k-1)+P_{k-2}^{-}(k-2)\right]\left(iA_0^{-1/2}\right)\right\}u_0',$$

$$S_{3k}=4^{-1}\left\{\left[\frac{i}{2}A_{k+1/2}-2^{-1}\tau\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]B_0^{+}(k)\,\tau\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right.$$

$$\left.+\left[-\frac{i}{2}A_{k+1/2}-2^{-1}\tau\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]B_0^{-}(k)\,\tau\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\}$$

$$\times\left[A_{k+1/2}^{-1/2}\tau\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\tau\left(u_{k+1}-u_k\right)\right],$$

$$S_{4k}=4^{-1}\left\{\left[\frac{i}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}-\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.\right.$$

$$\left.-\frac{1}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]B_0^{+}(k-1)\,\tau\left(I-\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}$$

$$+\left[-\frac{i}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}-\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$-\frac{1}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\Big]B_{0}^{-}\left(k-1\right)\tau\left(I+\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}$$

$$+\left[iA_{k+1/2}^{1/2}-\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_{1}^{+}\left(k\right)+B_{0}^{+}\left(k-1\right)\right]\tau\left(I-\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}$$

$$-\left[iA_{k+1/2}-\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_{1}^{-}\left(k\right)+B_{0}^{-}\left(k-1\right)\right]\tau\left(I+\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\Bigg\}$$

$$\times\left[A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1}^{-1/2}\tau^{-1}\left(u_{k}-u_{k-1}\right)\right],$$

$$S_{5k}=4^{-1}\Bigg\{\left[-\frac{i}{2}A_{k+1/2}+2^{-1}\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]B_{0}^{+}\left(k\right)\tau\left(I-\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}$$

$$+\left[\frac{i}{2}A_{k+1/2}+2^{-1}\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]B_{0}^{-}\left(k\right)\tau\left(I+\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\Bigg\}$$

$$\times\left(A_{k+1/2}^{-1/2}f_{k+1/2}\right),$$

$$S_{6k}=4^{-1}\Bigg\{\Big[-\frac{i}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}$$

$$+2^{-1}A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\Big]B_{0}^{+}\left(k-1\right)\tau\left(I-\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}$$

$$+\Big[\frac{i}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}$$

$$+2^{-1}A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\Big]B_{0}^{-}\left(k-1\right)\tau\left(I+\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}$$

$$+\left[-iA_{k+1/2}^{1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_{1}^{+}\left(k\right)+B_{0}^{+}\left(k-1\right)\right]\tau\left(I-\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}$$

$$+\left[iA_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_{1}^{-}\left(k\right)+B_{0}^{-}\left(k-1\right)\right]\tau\left(I+\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\Bigg\}$$

$$\times\left(A_{k-1/2}^{-1/2}f_{k-1/2}\right),$$

$$S_{7k}=4^{-1}\sum_{s=2}^{k}\Bigg\{\left[iA_{k+1/2}-\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_{s}^{+}\left(k\right)+B_{s-1}^{+}\left(k-1\right)\right]\tau\left(I-\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}$$

$$+\left[-iA_{k+1/2}-\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_s^{-}(k)+B_{s-1}^{-}(k-1)\right]\tau\left(I+\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}$$

$$+\left[iA_{k+1/2}^{1/2}A_{k-1/2}^{1/2}-2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.-A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_{s-1}^{+}(k-1)+B_{s-2}^{+}(k-2)\right]\tau\left(I-\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}$$

$$+\left[-iA_{k+1/2}^{1/2}A_{k-1/2}^{1/2}-2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.-A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]$$

$$\left.\times 2^{-1}\left[B_{s-1}^{-}(k-1)+B_{s-2}^{-}(k-2)\right]\tau\left(I+\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right\}$$

$$\times\left[A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\tau^{-1}\left(u_{k-s+1}-u_{k-s}\right)\right],$$

$$S_{8k}=4^{-1}\sum_{s=2}^{k}\left\{\left[-iA_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]\right.$$

$$\times 2^{-1}\left[B_s^{+}(k)+B_{s-1}^{+}(k-1)\right]\tau\left(I-\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}$$

$$+\left[iA_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_s^{-}(k)+B_{s-1}^{-}(k-1)\right]\tau\left(I+\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}$$

$$+\left[iA_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]$$

$$\times 2^{-1}\left[B_{s-1}^{+}(k-1)+B_{s-2}^{+}(k-2)\right]\tau\left(I-\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}$$

$$+\left[-iA_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]$$

$$\left.\times 2^{-1}\left[B_{s-1}^{-}(k-1)+B_{s-2}^{-}(k-2)\right]\tau\left(I+\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right\}$$

$$\times\left(A_{k-s+1/2}^{-1/2}f_{k-s+1/2}\right).$$

We will estimate $||S_{mk}||_H$, $m=\overline{1,8}$. Let $m=1$. Then applying the estimates (2.25), (3.6), (3.9) and (3.11), we get

$$||S_{1k}||_H\le 2^{-1}\left\{\left\|A_{k+1/2}2^{-1}\left[P_k^{\pm}(k)+P_{k-1}^{\pm}(k-1)\right]u_0\right\|_H\right.$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}2^{-1}\left[P_k^{\pm}(k)+P_{k-1}^{\pm}(k-1)\right]u_0\right\|_H$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]u_{0}\right\|_{H}$$

$$+\left\|2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]u_{0}\right\|_{H}$$

$$\left.+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]u_{0}\right\|_{H}\right\}$$

$$\leq 2^{-1}\left\{\left\|A_{k+1/2}2^{-1}\left[P_{k}^{\pm}(k)+P_{k-1}^{\pm}(k-1)\right]A_{1/2}^{-1}\right\|\left\|A_{1/2}A_{0}^{-1}\right\|\right.$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\left\|A_{k+1/2}^{1/2}2^{-1}\left[P_{k}^{\pm}\left(k\right)+P_{k-1}^{\pm}\left(k-1\right)\right]A_{1/2}^{-1/2}\right\|$$

$$\times\left\|A_{1/2}^{1/2}A_{0}^{-1/2}\right\|\left\|A_{0}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|A_{k-1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{1/2}^{-1}\right\|$$

$$\times\left\|A_{1/2}A_{0}^{-1}\right\|+2\left\|\tau^{-1}\left(A_{k+1/2}^{1/2}-A_{k-1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|$$

$$\times\left\|A_{k-1/2}^{1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{1/2}^{-1/2}\right\|\left\|A_{1/2}^{1/2}A_{0}^{-1/2}\right\|\left\|A_{0}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\tau^{-1}\left(A_{k}^{1/2}-A_{k-1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|$$

$$\left.\times\left\|A_{k-1/2}^{1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{1/2}^{-1/2}\right\|\left\|A_{1/2}^{1/2}A_{0}^{-1/2}\right\|\left\|A_{0}^{-1/2}\right\|\right\}\left\|A_{0}u_{0}\right\|_{H}$$

$$\leq 2^{-1}\left(C_{6}+3C_{10}+C_{11}+C_{12}\right)\left\|A_{0}u_{0}\right\|_{H},$$

where

$$C_{10}=\sqrt{\delta}^{-1}M_{1/2}^{2}e^{M_{1/2}P_{1/2}},\ C_{11}=M_{1/2}M_{1}e^{M_{1}P_{1}},\ C_{12}=\sqrt{\delta}^{-1}M_{1/2}^{3}e^{M_{1/2}P_{1/2}}.$$

Let $m=2$. Then applying the estimates (2.25), (3.6), (3.9) and (3.11), we get

$$||S_{2k}||_{H}\leq 2^{-1}\left\{\left\|A_{k+1/2}2^{-1}\left[P_{k}^{\pm}\left(k\right)+P_{k-1}^{\pm}\left(k-1\right)\right]A_{0}^{-1/2}u_{0}'\right\|_{H}\right.$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}2^{-1}\left[P_{k}^{\pm}\left(k\right)+P_{k-1}^{\pm}\left(k-1\right)\right]A_{0}^{-1/2}u_{0}'\right\|_{H}$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{0}^{-1/2}u_{0}'\right\|_{H}$$

$$+\left\|2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.\times 2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{0}^{-1/2}u_{0}'\right\|_{H}$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$

$$\left.\left.\times 2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{0}^{-1/2}u_{0}'\right\|_{H}\right\}$$

$$\leq 2^{-1}\left\{\left\|A_{k+1/2}2^{-1}\left[P_{k}^{\pm}\left(k\right)+P_{k-1}^{\pm}\left(k-1\right)\right]A_{1/2}^{-1}\right\|\left\|A_{1/2}A_{0}^{-1}\right\|\right.$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\left\|A_{k+1/2}^{1/2}2^{-1}\left[P_{k}^{\pm}\left(k\right)+P_{k-1}^{\pm}\left(k-1\right)\right]A_{1/2}^{-1/2}\right\|$$

$$\times\left\|A_{1/2}^{1/2}A_{0}^{-1/2}\right\|\left\|A_{0}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|A_{k-1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{1/2}^{-1}\right\|\left\|A_{1/2}A_{0}^{-1}\right\|$$

$$+2\left\|\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|$$

$$\times\left\|A_{k-1/2}^{1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{1/2}^{-1/2}\right\|\left\|A_{1/2}^{1/2}A_{0}^{-1/2}\right\|\left\|A_{0}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|$$

$$\times\left\|A_{k-1/2}^{1/2}2^{-1}\left[P_{k-1}^{\pm}\left(k-1\right)+P_{k-2}^{\pm}\left(k-2\right)\right]A_{1/2}^{-1/2}\right\|\left\|A_{1/2}^{1/2}A_{0}^{-1/2}\right\|\left\|A_{0}^{-1/2}\right\|\Big\}\left\|A_{0}^{1/2}u_{0}'\right\|_{H}$$

$$\leq 2^{-1}\left(C_{6}+3C_{10}+C_{11}+C_{12}\right)\left\|A_{0}^{1/2}u_{0}'\right\|_{H}.$$

Let $m=3$. Then applying the estimates (2.22), (2.25), (3.11) and (3.12), we get

$$||S_{3k}||_{H}\leq 4^{-1}\left\{\left\|\frac{1}{2}A_{k+1/2}B_{0}^{\pm}(k)2\tau\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}A_{k+1/2}^{-1/2}\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right.\right.$$

$$\left.\times\frac{u_{k+1}-u_{k}}{\tau}\right\|_{H}+\left\|\frac{1}{2}\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}B_{0}^{\pm}(k)2\tau\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\tau\right.$$

$$\left.\left.\times A_{k+1/2}^{-1/2}\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\frac{u_{k+1}-u_{k}}{\tau}\right\|_{H}\right\}$$

$$\leq 4^{-1}\left[\left\|\left(I\pm\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\left\|A_{k+1/2}^{1/2}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1}\right\|\right.$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right\|$$

$$\left.\times\left\|A_{k+1/2}^{-1/2}\right\|\right]\left\|A_{k+1/2}^{1/2}\frac{u_{k+1}-u_{k}}{\tau}\right\|_{H}$$

$$\leq 4^{-1}\left(M_{1/2}\tau+C_{13}\right)\left\|A_{k+1/2}^{1/2}\frac{u_{k+1}-u_{k}}{\tau}\right\|_{H},$$

where

$$C_{13}=\sqrt{\delta}^{-1}\tau M_{1/2}^{2}.$$

Let $m=4$. Then applying the estimates (2.22), (2.25), (3.6), (3.11) and (3.12), we get

$$||S_{4k}||_{H}\leq 4^{-1}\left\{\left\|\frac{1}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}B_{0}^{\pm}(k-1)2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right.\right.$$

$$\left.\times A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\frac{u_{k}-u_{k-1}}{\tau}\right\|_{H}$$

$$+\left\|\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}B_{0}^{\pm}(k-1)2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right.$$

$$\left.\times A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\frac{u_{k}-u_{k-1}}{\tau}\right\|_{H}$$

$$+\left\|\frac{1}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}B_{0}^{\pm}(k-1)2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right.$$

$$\times A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\frac{u_k-u_{k-1}}{\tau}\Bigg\|_H$$

$$+\left\|A_{k+1/2}2^{-1}\left[B_1^{\pm}(k)+B_0^{\pm}(k)\right]2\tau\left(1\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right.$$

$$\times A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\frac{u_k-u_{k-1}}{\tau}\Bigg\|_H$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}2^{-1}\left[B_1^{\pm}(k)+B_0^{\pm}(k)\right]2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right.$$

$$\left.\times A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\frac{u_k-u_{k-1}}{\tau}\Bigg\|_H\right\}$$

$$\leq 4^{-1}\left\{\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{-1/2}\right)^{-1}\right\|\left\|A_{k-1/2}^{1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1}\right\|\right.$$

$$+\left\|\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|2\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|\left\|\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|$$

$$\times\left\|A_{k-1/2}^{-1/2}\right\|+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|$$

$$\times\left\|\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|\left\|A_{k-1/2}^{-1/2}\right\|+2\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\left\|A_{k+1/2}A_{k-1/2}^{-1}\right\|$$

$$\times\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|\left\|A_{k-1/2}^{1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1}\right\|$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\right\|2\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|$$

$$\left.\times\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|\left\|\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|\left\|A_{k-1/2}^{-1/2}\right\|\right\}\left\|A_{k-1/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H$$

$$\leq 4^{-1}\left(C_{14}+2C_{13}+3C_{15}+2C_{16}\right)\left\|A_{k-1/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H,$$

where

$$C_{14}=\tau M_{1/2}^2,\ C_{15}=\sqrt{\delta}^{-1}\tau M_{1/2}^3,\ C_{16}=\tau M_1M_{1/2}.$$

Let $m=5$. Then applying the estimates (2.22), (2.24) and (3.11), we get

$$||S_{5k}||_H\leq 4^{-1}\left\{\left\|\frac{1}{2}A_{k+1/2}B_0^{\pm}(k)2\tau\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}A_{k+1/2}^{-1/2}f_{k+1/2}\right\|_H\right.$$

$$\left.+\left\|\frac{1}{2}\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}B_0^{\pm}(k)2\tau\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}A_{k+1/2}^{-1/2}f_{k+1/2}\right\|_H\right\}$$

$$\leq 4^{-1}\left\{2\left\|\left(\frac{\tau}{2}A_{k+1/2}^{1/2}\right)\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\right.$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\right\}\left\|f_{k+1/2}\right\|_{H}$$

$$\leq 4^{-1}\left(2I+\tau M_{1/2}\right)\left\|f_{k+1/2}\right\|_{H}.$$

Let $m=6$. Then applying the estimates (2.22), (2.24), (3.6) and (3.11), we get

$$||S_{6k}||_H \leq 4^{-1}\left\{\left\|\frac{1}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}B_0^{\pm}(k-1)2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}A_{k-1/2}^{-1/2}f\left(t_{k-1/2}\right)\right\|_H\right.$$

$$+\left\|\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}B_0^{\pm}(k-1)2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}A_{k-1/2}^{-1/2}f_{k-1/2}\right\|_H$$

$$+\left\|\frac{1}{2}A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right.$$

$$\left.\times A_{k-1/2}^{1/2}B_0^{\pm}(k-1)2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}A_{k-1/2}^{-1/2}f_{k-1/2}\right\|_H$$

$$+\left\|A_{k+1/2}2^{-1}\left[B_1^{\pm}(k)+B_0^{\pm}(k)\right]2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}A_{k-1/2}^{-1/2}f_{k-1/2}\right\|_H$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}2^{-1}\left[B_1^{\pm}(k)+B_0^{\pm}(k)\right]\right.$$

$$\left.\left.\times 2\tau\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}A_{k-1/2}^{-1/2}f_{k-1/2}\right\|_H\right\}$$

$$\leq 4^{-1}\left\{2\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\left(\frac{\tau}{2}A_{k-1/2}^{1/2}\right)\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|\right.$$

$$+2\tau\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|$$

$$+2\left\|\left(\tau A_{k+1/2}^{1/2}\right)\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|$$

$$+2\tau\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\left\|\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\right\|\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|$$

$$\left.\times\left\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\right\|\right\}\left\|f_{k-1/2}\right\|_H$$

$$\leq 4^{-1}\left(6M_{1/2}+2\tau M_{1/2}+3C_{14}\right)\left\|f_{k-1/2}\right\|_H.$$

Let $m=7$. Then applying the estimates (2.22), (2.25), (3.6), (3.10), (3.11) and (3.12), we get

$$||S_{7k}||_H = 4^{-1}\sum_{s=2}^{k}\left\{\left\|A_{k+1/2}2^{-1}\left[B_s^{\pm}(k)+B_s^{\pm}(k-1)\right]2\tau\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right.\right.$$

$$\times A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\frac{u_{k-s+1}-u_{k-s}}{\tau}\Bigg\|_H$$

$$+\Bigg\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}2^{-1}\left[B_s^{\pm}(k)+B_s^{\pm}(k-1)\right]$$

$$\times 2\tau\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\frac{u_{k-s+1}-u_{k-s}}{\tau}\Bigg\|_H\Bigg\}$$

$$+\Bigg\|A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}2^{-1}\left[B_{s-1}^{\pm}(k-1)+B_{s-2}^{\pm}(k-2)\right]2\tau\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}$$

$$\times A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\frac{u_{k-s+1}-u_{k-s}}{\tau}\Bigg\|_H$$

$$+\Bigg\|2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}2^{-1}\left[B_{s-1}^{\pm}(k-1)+B_{s-2}^{\pm}(k-2)\right]2\tau$$

$$\times\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\frac{u_{k-s+1}-u_{k-s}}{\tau}\Bigg\|_H$$

$$+\Bigg\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}2^{-1}\left[B_{s-1}^{\pm}(k-1)+B_{s-2}^{\pm}(k-2)\right]2\tau$$

$$\times\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}A_{k-s+1/2}^{-1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\frac{u_{k-s+1}-u_{k-s}}{\tau}\Bigg\|_H\Bigg\}$$

$$\leq 4^{-1}\sum_{s=2}^{k}\Bigg\{\left\|A_{k+1/2}2^{-1}\left[B_s^{\pm}(k)+B_s^{\pm}(k-1)\right]A_{k-s+3/2}^{-1}\right\|\left\|A_{k-s+3/2}A_{k-s+1/2}^{-1}\right\|$$

$$\times\left\|\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{-1/2}\right)^{-1}\right\|\left\|A_{k-s+1/2}^{1/2}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1}\right\|$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\left\|A_{k+1/2}^{1/2}2^{-1}\left[B_s^{\pm}(k)+B_s^{\pm}(k-1)\right]A_{k-s+3/2}^{-1/2}\right\|$$

$$\times\left\|A_{k-s+3/2}^{1/2}A_{k-s+1/2}^{-1/2}\right\|2\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right\|$$

$$\times\left\|\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\right\|\left\|A_{k-s+1/2}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|A_{k-1/2}2^{-1}\left[B_{s-1}^{\pm}(k-1)+B_{s-2}^{\pm}(k-2)\right]A_{k-s+3/2}^{-1}\right\|$$

$$\times\left\|A_{k-s+3/2}A_{k-s+1/2}^{-1}\right\|2\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right\|\left\|A_{k-s+1/2}^{1/2}\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1}\right\|$$

$$+2\left\|\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|\left\|A_{k-1/2}^{1/2}2^{-1}\left[B_{s-1}^{\pm}(k-1)+B_{s-2}^{\pm}(k-2)\right]A_{k-s+3/2}^{-1/2}\right\|$$

$$\times\left\|A_{k-s+3/2}^{1/2}A_{k-s+1/2}^{-1/2}\right\|2\tau\left\|\left(I\mp\frac{i\tau}{2}A_{k-s+1/2}^{1/2}\right)^{-1}\right\|$$

$$\times\left\|\tau^{-1}\left(A_{k-s+1}^{1/2}-A_{k-s}^{1/2}\right)A_{k-s+1/2}^{-1/2}\right\|\left\|A_{k-s+1/2}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|$$

$$\times \left\| A_{k-1/2}^{1/2} 2^{-1} \left[B_{s-1}^{\pm}(k-1) + B_{s-2}^{\pm}(k-2) \right] A_{k-s+3/2}^{-1/2} \right\| 2\tau \left\| A_{k-s+3/2}^{1/2} A_{k-s+1/2}^{-1/2} \right\|$$

$$\times \left\| \left(I \mp \frac{i\tau}{2} A_{k-s+1/2}^{1/2} \right)^{-1} \right\| \left\| \tau^{-1} \left(A_{k-s+1}^{1/2} - A_{k-s}^{1/2} \right) A_{k-s+1/2}^{-1/2} \right\| \left\| A_{k-s+1/2}^{-1/2} \right\| \Bigg\}$$

$$\times \left\| A_{k-s+1/2}^{1/2} \frac{u_{k-s+1} - u_{k-s}}{\tau} \right\|_H$$

$$\leq \sum_{s=2}^{k} 4^{-1} \left(2C_{17} + 6C_{18} + 2C_{19} + 2C_{20} \right) \left\| A_{k-s+1/2}^{1/2} \frac{u_{k-s+1} - u_{k-s}}{\tau} \right\|_H,$$

where

$$C_{17} = \tau M_{1/2} M_1 e^{M_1 P_1}, \; C_{18} = \sqrt{\delta}^{-1} \tau M_{1/2}^3 e^{M_{1/2} P_{1/2}}, \; C_{19} = \tau M_{1/2}^2 M_1 e^{M_1 P_1},$$

$$C_{20} = \sqrt{\delta}^{-1} \tau M_{1/2}^2 e^{M_{1/2} P_{1/2}}.$$

Let $m = 8$. Using the equation (3.20) and (3.21), we obtain

$$S_{8k} = 4^{-1} \Bigg\{ \left[-iA_{k+1/2} + \tau^{-1} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} A_{k+1/2}^{1/2} \right] 2^{-1} \left[B_2^{+}(k) + B_1^{+}(k-1) \right]$$

$$+ \left[iA_{k+1/2} + \tau^{-1} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} A_{k+1/2}^{1/2} \right] 2^{-1} \left[B_2^{-}(k) + B_1^{-}(k-1) \right]$$

$$+ \left[iA_{k+1/2}^{1/2} A_{k-1/2}^{1/2} + 2\tau^{-1} \left(A_{k-1/2}^{1/2} - A_{k+1/2}^{1/2} \right) A_{k-1/2}^{-1/2} A_{k-1/2}^{1/2} \right.$$

$$\left. + A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} \tau^{-1} \left(A_k^{1/2} - A_{k-1}^{1/2} \right) A_{k-1/2}^{-1/2} A_{k-1/2}^{1/2} \right] 2^{-1} \left[B_1^{+}(k-1) + B_0^{+}(k-2) \right]$$

$$+ \left[-iA_{k+1/2}^{1/2} A_{k-1/2}^{1/2} + 2\tau^{-1} \left(A_{k-1/2}^{1/2} - A_{k+1/2}^{1/2} \right) A_{k-1/2}^{-1/2} A_{k-1/2}^{1/2} \right.$$

$$\left. + A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} \tau^{-1} \left(A_k^{1/2} - A_{k-1}^{1/2} \right) A_{k-1/2}^{-1/2} A_{k-1/2}^{1/2} \right] 2^{-1} \left[B_1^{-}(k-1) + B_0^{-}(k-2) \right] \Bigg\}$$

$$\times \left(A_{k-3/2}^{-1} f_{k-3/2} \right)$$

$$+ 4^{-1} \Bigg\{ \left[iA_{k+1/2} + \tau^{-1} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} A_{k+1/2}^{1/2} \right] 2^{-1} \left[B_{k+1}^{+}(k) + B_k^{+}(k-1) \right]$$

$$+ \left[-iA_{k+1/2} + \tau^{-1} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} A_{k+1/2}^{1/2} \right] 2^{-1} \left[B_{k+1}^{-}(k) + B_k^{-}(k-1) \right]$$

$$+ \left[-iA_{k+1/2}^{1/2} A_{k-1/2}^{1/2} + 2\tau^{-1} \left(A_{k-1/2}^{1/2} - A_{k+1/2}^{1/2} \right) A_{k-1/2}^{-1/2} A_{k-1/2}^{1/2} \right.$$

$$\left. + A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} \tau^{-1} \left(A_k^{1/2} - A_{k-1}^{1/2} \right) A_{k-1/2}^{-1/2} A_{k-1/2}^{1/2} \right] 2^{-1} \left[B_k^{+}(k-1) + B_{k-1}^{+}(k-2) \right]$$

$$+ \left[-iA_{k+1/2}^{1/2} A_{k-1/2}^{1/2} + 2\tau^{-1} \left(A_{k-1/2}^{1/2} - A_{k+1/2}^{1/2} \right) A_{k-1/2}^{-1/2} A_{k-1/2}^{1/2} \right.$$

$$\left. + A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} \tau^{-1} \left(A_k^{1/2} - A_{k-1}^{1/2} \right) A_{k-1/2}^{-1/2} A_{k-1/2}^{1/2} \right] 2^{-1} \left[B_k^{-}(k-1) + B_{k-1}^{-}(k-2) \right] \Bigg\}$$

$$\times i \left(A_{1/2}^{-1} f_{1/2} \right)$$

$$+ \sum_{s=2}^{k-1} 4^{-1} \Bigg\{ \left[-iA_{k+1/2} + \tau^{-1} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} A_{k+1/2}^{1/2} \right] 2^{-1} \left[B_{s+1}^{+}(k) + B_s^{+}(k-1) \right]$$

$$+ \left[iA_{k+1/2} + \tau^{-1} \left(A_{k+1}^{1/2} - A_k^{1/2} \right) A_{k+1/2}^{-1/2} A_{k+1/2}^{1/2} \right] 2^{-1} \left[B_{s+1}^{-}(k) + B_s^{-}(k-1) \right]$$

$$+\left[iA_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$
$$\left.+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]2^{-1}\left[B_{s}^{+}(k-1)+B_{s-1}^{+}(k-2)\right]$$
$$+\left[-iA_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$
$$\left.+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]$$
$$\left.\times 2^{-1}\left[B_{s}^{-}(k-1)+B_{s-1}^{-}(k-2)\right]\right\}iA_{k-s-1/2}^{-1}\left(A_{k-s+1/2}-A_{k-s-1/2}\right)A_{k-s+1/2}^{-1}f_{k-s-1/2}$$
$$+\sum_{s=2}^{k-1}4^{-1}\left\{\left[-iA_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]2^{-1}\left[B_{s+1}^{+}(k)+B_{s}^{+}(k-1)\right]\right.$$
$$+\left[iA_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]2^{-1}\left[B_{s+1}^{-}(k)+B_{s}^{-}(k-1)\right]$$

$$+\left[iA_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$
$$\left.\left.+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]2^{-1}\left[B_{s}^{+}(k-1)+B_{s-1}^{+}(k-2)\right]\right\}$$
$$+\left[-iA_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.$$
$$\left.\left.+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]2^{-1}\left[B_{s}^{-}(k-1)+B_{s-1}^{-}(k-2)\right]\right\}$$
$$\times iA_{k-s+1/2}^{-1}\left(f_{k-s-1/2}-f_{k-s+1/2}\right).$$

Then applying the triangular inequality, we get

$$\|S_{8k}\|_H\leq 2^{-1}\left\{\left\|2\left[A_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]\right.\right.$$
$$\left.\times 2^{-1}\left[B_{2}^{\pm}(k)+B_{1}^{\pm}(k-1)\right]A_{k-3/2}^{-1}\right\|$$
$$+\left\|\left[A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.\right.$$
$$\left.+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]$$
$$\left.\left.\times 2^{-1}\left[B_{1}^{\pm}(k-1)+B_{0}^{\pm}(k-2)\right]A_{k-3/2}^{-1}\right\|_H\right\}\left\|f_{k-3/2}\right\|_H$$
$$+2^{-1}\left\{\left\|\left[A_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]2^{-1}\left[B_{k+1}^{\pm}(k)+B_{k}^{\pm}(k-1)\right]A_{1/2}^{-1}\right\|\right.$$
$$+\left\|\left[A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.\right.$$
$$\left.\left.\left.+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_{k}^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right]2^{-1}\left[B_{k}^{\pm}(k-1)+B_{k-1}^{\pm}(k-2)\right]A_{1/2}^{-1}\right\|\right\}$$
$$\times\left\|f_{1/2}\right\|_H$$
$$+\sum_{s=2}^{k-1}2^{-1}\left\{\left\|\left[A_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_{k}^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\right]\right.\right.$$
$$\left.\times 2^{-1}\left[B_{s+1}^{\pm}(k)+B_{s}^{\pm}(k-1)\right]A_{k-s-1/2}^{-1}\left(A_{k-s+1/2}-A_{k-s-1/2}\right)A_{k-s+1/2}^{-1}\right\|$$
$$+\left\|\left[A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\right.\right.$$

$$+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\Big]$$

$$2^{-1}\left[B_s^{\pm}(k-1)+B_{s-1}^{\pm}(k-2)\right]A_{k-s-1/2}^{-1}\left(A_{k-s+1/2}-A_{k-s-1/2}\right)A_{k-s+1/2}^{-1}\Big\|_H\Big\}\left\|f_{k-s-1/2}\right\|_H$$

$$+\sum_{s=2}^{k-1}2^{-1}\Big\{\Big\|\Big[A_{k+1/2}+\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}A_{k+1/2}^{1/2}\Big]$$

$$\times 2^{-1}\left[B_{s+1}^{\pm}(k)+B_s^{\pm}(k-1)\right]A_{k-s+1/2}^{-1}\Big\|+\Big\|\Big[A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}$$

$$+2\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}A_{k-1/2}^{1/2}\Big]$$

$$\times 2^{-1}\left[B_s^{\pm}(k-1)+B_{s-1}^{\pm}(k-2)\right]A_{k-s+1/2}^{-1}\Big\|\Big\}\left\|f_{k-s-1/2}-f_{k-s+1/2}\right\|_H.$$

Applying the estimates (2.22), (2.23), (2.25), (3.6), (3.10) and (3.11), we get

$$\|S_{8k}\|_H\le 2^{-1}\Bigg[\Bigg\|\left(1\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\Bigg\|\left\|A_{k+1/2}A_{k-1/2}^{-1}\right\|\left\|X_k^{\pm}\right\|\left\|A_{k-1/2}A_{k-3/2}^{-1}\right\|$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\Bigg\|\left(I\mp\frac{i\tau}{2}A_{k+1/2}^{1/2}\right)^{-1}\Bigg\|$$

$$\times\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|X_k^{\pm}\right\|\left\|A_{k-1/2}^{1/2}A_{k-3/2}^{-1/2}\right\|\left\|A_{k-3/2}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\Bigg\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\Bigg\|\left\|A_{k-1/2}^{1/2}A_{k-3/2}^{-1/2}\right\|\left\|A_{k-3/2}^{-1/2}\right\|$$

$$+2\left\|\tau^{-1}\left(A_{k+1/2}^{1/2}-A_{k-1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|\Bigg\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\Bigg\|\left\|A_{k-1/2}^{1/2}A_{k-3/2}^{-1/2}\right\|\left\|A_{k-3/2}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|$$

$$\times\Bigg\|\left(I\mp\frac{i\tau}{2}A_{k-1/2}^{1/2}\right)^{-1}\Bigg\|\left\|A_{k-1/2}^{1/2}A_{k-3/2}^{-1/2}\right\|\left\|A_{k-3/2}^{-1/2}\right\|\Bigg]\left\|f_{k-3/2}\right\|_H$$

$$+2^{-1}\Big[\Big\|A_{k+1/2}2^{-1}\left[B_{k+1}^{\pm}(k)+B_k^{\pm}(k-1)\right]A_{1/2}^{-1}\Big\|$$

$$+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\right\|\left\|A_{k+1/2}^{1/2}2^{-1}\left[B_{k+1}^{\pm}(k)+B_k^{\pm}(k-1)\right]A_{1/2}^{-1/2}\right\|\left\|A_{1/2}^{-1/2}\right\|$$

$$+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|A_{k-1/2}2^{-1}\left[B_k^{\pm}(k-1)+B_{k-1}^{\pm}(k-2)\right]A_{1/2}^{-1}\right\|$$

$$+2\left\|\tau^{-1}\left(A_{k+1/2}^{1/2}-A_{k-1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|\left\|A_{k-1/2}^{1/2}2^{-1}\left[B_k^{\pm}(k-1)+B_{k-1}^{\pm}(k-2)\right]A_{1/2}^{-1/2}\right\|$$

$$\times\left\|A_{k-1/2}^{-1/2}\right\|+\left\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\right\|\left\|\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|$$

$$\times\left\|A_{k-1/2}^{1/2}2^{-1}\left[B_k^{\pm}(k-1)+B_{k-1}^{\pm}(k-2)\right]A_{1/2}^{-1/2}\right\|\left\|A_{1/2}^{-1/2}\right\|\Big]\left\|f_{1/2}\right\|_H$$

$$+\sum_{s=2}^{k-1}2^{-1}\Big\{\Big[\Big\|A_{k+1/2}2^{-1}\left[B_{s+1}^{\pm}(k)+B_s^{\pm}(k-1)\right]A_{k-s+1/2}^{-1}\Big\|\left\|A_{k-s+1/2}A_{k-s-1/2}^{-1}\right\|$$

$$\times\left\|\left(A_{k-s+1/2}-A_{k-s-1/2}\right)A_{k-s+1/2}^{-1}\right\|+\left\|\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\right\|$$

$$\times \left\| A_{k+1/2}^{1/2} 2^{-1} \left[B_{s+1}^{\pm}(k) + B_{s}^{\pm}(k-1) \right] A_{k-s+1/2}^{-1/2} \right\| \left\| A_{k-s+1/2}^{1/2} A_{k-s-1/2}^{-1/2} \right\| \left\| A_{k-s-1/2}^{-1/2} \right\|$$

$$\times \left\| \left(A_{k-s+1/2} - A_{k-s-1/2} \right) A_{k-s+1/2}^{-1} \right\| + \left\| A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} \right\|$$

$$\times \left\| A_{k-1/2} 2^{-1} \left[B_{s}^{\pm}(k) + B_{s-1}^{\pm}(k-1) \right] A_{k-s+1/2}^{-1} \right\| \left\| A_{k-s+1/2} A_{k-s-1/2}^{-1} \right\|$$

$$\times \left\| \left(A_{k-s+1/2} - A_{k-s-1/2} \right) A_{k-s+1/2}^{-1} \right\| + 2 \left\| \tau^{-1} \left(A_{k-1/2}^{1/2} - A_{k+1/2}^{1/2} \right) A_{k-1/2}^{-1/2} \right\|$$

$$\times \left\| A_{k-1/2}^{1/2} 2^{-1} \left[B_{s}^{\pm}(k-1) + B_{s-1}^{\pm}(k-2) \right] A_{k-s+1/2}^{-1/2} \right\| \left\| A_{k-s+1/2}^{1/2} A_{k-s-1/2}^{-1/2} \right\|$$

$$\times \left\| A_{k-s-1/2}^{-1/2} \right\| \left\| \left(A_{k-s+1/2} - A_{k-s-1/2} \right) A_{k-s+1/2}^{-1} \right\| + \left\| A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} \right\|$$

$$\times \left\| \tau^{-1} \left(A_{k}^{1/2} - A_{k-1}^{1/2} \right) A_{k-1/2}^{-1/2} \right\| \left\| A_{k-1/2}^{1/2} 2^{-1} \left[B_{s}^{\pm}(k-1) + B_{s-1}^{\pm}(k-2) \right] A_{k-s+1/2}^{-1/2} \right\|$$

$$\times \left\| A_{k-s+1/2}^{1/2} A_{k-s-1/2}^{-1/2} \right\| \left\| A_{k-s-1/2}^{-1/2} \right\| \left\| \left(A_{k-s+1/2} - A_{k-s-1/2} \right) A_{k-s+1/2}^{-1} \right\| \Big] \left\| f_{k-s-1/2} \right\|_H$$

$$+ \Big[\left\| A_{k+1/2} 2^{-1} \left[B_{s+1}^{\pm}(k) + B_{s}^{\pm}(k-1) \right] A_{k-s+1/2}^{-1} \right\|$$

$$+ \left\| \tau^{-1} \left(A_{k+1}^{1/2} - A_{k}^{1/2} \right) A_{k+1/2}^{-1/2} \right\| \left\| A_{k+1/2}^{1/2} 2^{-1} \left[B_{s+1}^{\pm}(k) + B_{s}^{\pm}(k-1) \right] A_{k-s+1/2}^{-1/2} \right\|$$

$$\times \left\| A_{k-s+1/2}^{-1/2} \right\| + \left\| A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} \right\| \left\| A_{k-1/2} 2^{-1} \left[B_{s}^{\pm}(k-1) + B_{s-1}^{\pm}(k-2) \right] A_{k-s+1/2}^{-1} \right\|$$

$$+2 \left\| \tau^{-1} \left(A_{k-1/2}^{1/2} - A_{k+1/2}^{1/2} \right) A_{k-1/2}^{-1/2} \right\| \left\| A_{k-1/2}^{1/2} 2^{-1} \left[B_{s}^{\pm}(k-1) + B_{s-1}^{\pm}(k-2) \right] A_{k-s+1/2}^{-1/2} \right\|$$

$$\times \left\| A_{k-s+1/2}^{-1/2} \right\| + \left\| A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} \right\| \left\| \tau^{-1} \left(A_{k}^{1/2} - A_{k-1}^{1/2} \right) A_{k-1/2}^{-1/2} \right\|$$

$$\times \left\| A_{k-1/2}^{1/2} 2^{-1} \left[B_{s}^{\pm}(k-1) + B_{s-1}^{\pm}(k-2) \right] A_{k-s+1/2}^{-1/2} \right\| \left\| A_{k-s+1/2}^{-1/2} \right\| \Big] \left\| f_{k-s-1/2} - f_{k-s+1/2} \right\|_H \Big\}$$

$$\leq 2^{-1} \left(M_1^2 + 3\sqrt{\delta}^{-1} M_{1/2}^2 + 2\sqrt{\delta}^{-1} M_{1/2}^3 \right) \left\| f_{k-3/2} \right\|_H + 2^{-1} \left(C_7 + 3C_{21} + C_{22} + C_{10} \right) \left\| f_{1/2} \right\|_H$$

$$+2^{-1} \left(C_{23} + 3C_{24} + C_{25} + C_{26} \right) \sum_{s=2}^{k-1} \left\| f_{k-s-1/2} \right\|_H$$

$$+2^{-1} \left(C_7 + 3C_{21} + C_{22} + C_{10} \right) \sum_{s=2}^{k-1} \left\| f_{k-s+1/2} - f_{k-s-1/2} \right\|_H ,$$

where

$$C_{21} = \sqrt{\delta}^{-1} M_{1/2} e^{M_{1/2} P_{1/2}}, \ C_{22} = M_{1/2} e^{M_1 P_1}, \ C_{23} = \tau M_1^2 e^{M_1 P_1}$$

$$C_{24} = \sqrt{\delta}^{-1} \tau M_{1/2}^2 M_1 e^{M_{1/2} P_{1/2}}, \ C_{25} = \tau M_{1/2} M_1^2 e^{M_1 P_1}, \ C_{26} = \sqrt{\delta}^{-1} \tau M_{1/2}^3 M_1 e^{M_{1/2} P_{1/2}}$$

Combining the last estimates, we obtain

$$||A_{k+1/2} 4^{-1} (u_{k+1} + u_k) + A_{k+1/2}^{1/2} A_{k-1/2}^{1/2} 4^{-1} (u_k + u_{k-1})$$

$$+\tau^{-1} \left(A_{k-1/2}^{1/2} - A_{k+1/2}^{1/2} \right) A_{k-1/2}^{-1/2} \tau^{-1} (u_k - u_{k-1})$$

$$+2^{-1} \tau^{-1} \left(A_{k+1}^{1/2} - A_{k}^{1/2} \right) A_{k+1/2}^{-1/2} \tau^{-1} (u_{k+1} - u_k)$$

$$+A_{k+1/2}^{1/2} A_{k-1/2}^{-1/2} 2^{-1} \tau^{-1} \left(A_{k}^{1/2} - A_{k-1}^{1/2} \right) A_{k-1/2}^{-1/2} \tau^{-1} (u_k - u_{k-1}) ||_H$$

$$\leq C_{27} \left[\left\| A_0 u_0 \right\|_H + \left\| A_0^{1/2} u_0' \right\|_H + \max_{0 \leq s \leq k} \left\| f_{s+1/2} \right\|_H + \sum_{s=0}^{k-1} \left\| f_{s+1/2} - f_{s-1/2} \right\|_H \right].$$

where

$$C_{27}=\max\left[2^{-1}\left(C_6+3C_{10}+C_{11}+C_{12}\right),\ 4^{-1}\left(\tau M_{1/2}+C_{13}\right),\right.$$
$$4^{-1}(C_{14}+2C_{13}+3C_{15}+2C_{16}),\ 4^{-1}(2I+\tau M_{1/2}),\ 4^{-1}(6M_{1/2}+2\tau M_{1/2}+3C_{14}),$$
$$4^{-1}(2C_{17}+6C_{18}+2C_{19}+2C_{20}),\ 2^{-1}(M_{1/2}^2+3\sqrt{\delta}^{-1}M_{1/2}^2+2\sqrt{\delta}^{-1}M_{1/2}^3),$$
$$\left.2^{-1}\left(C_{23}+3C_{24}+C_{25}+C_{26}\right),\ 2^{-1}\left(C_7+3C_{21}+C_{22}+C_{10}\right)\right]$$
$$\times\max\left[C_9(k+1)+1,\ 2\right].$$

Now, we consider right side system of difference equation (3.3). Applying the estimate (3.11), we get

$$\left\|2^{-1}\left(f_{k-1/2}+f_{k+1/2}\right)+2^{-1}\left(A_{k+1/2}^{1/2}-A_{k-1/2}^{1/2}\right)A_{k-1/2}^{-1/2}f_{k-1/2}\right\|$$
$$\leq 2^{-1}\left\|\left(A_{k+1/2}^{1/2}-A_{k-1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\right\|\left\|f_{k-1/2}\right\|_H+2^{-1}\left\|f_{k-1/2}+f_{k+1/2}\right\|_H$$
$$\leq 2^{-1}\tau\left\|f_{k-1/2}\right\|_H+2^{-1}(\|f_{k-1/2}\|_H+||f_{k+1/2}\|_H).$$

Applying the last two estimates and the triangular inequality, we get

$$\left\|\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)\right\|_H$$
$$\leq C_{28}\left[\|A_0u_0\|+\left\|A_0^{1/2}u_0'\right\|+\max_{0\leq s\leq k}\left\|f_{s+1/2}\right\|+\sum_{s=0}^{k-1}\left\|f_{s+1/2}-f_{s-1/2}\right\|_H\right].$$

where

$$C_{28}=\max_{0\leq s\leq k}\left(C_{27},\ 2^{-1}\tau,\ 2^{-1}\right).$$

Combining the estimates for $||A^{1/2}(0)\tau^{-1}\left(u_k-u_{k-1}\right)||_H$, $\|\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)\|_H$ and

$$||A_{k+1/2}4^{-1}(u_{k+1}+u_k)+A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}4^{-1}(u_k+u_{k-1})$$
$$+\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\tau^{-1}(u_k-u_{k-1})$$
$$+2^{-1}\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\tau^{-1}(u_{k+1}-u_k)$$
$$+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}2^{-1}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\tau^{-1}(u_k-u_{k-1})||_H,$$

we obtain the conclusion of Theorem 3.2.

$$||\{A^{1/2}(0)\frac{u_k-u_{k-1}}{\tau}\}_1^{N-1}||_{C_\tau}+$$
$$||A_{k+1/2}4^{-1}(u_{k+1}+u_k)+A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}4^{-1}(u_k+u_{k-1})$$
$$+\tau^{-1}\left(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2}\right)A_{k-1/2}^{-1/2}\tau^{-1}(u_k-u_{k-1})$$
$$+2^{-1}\tau^{-1}\left(A_{k+1}^{1/2}-A_k^{1/2}\right)A_{k+1/2}^{-1/2}\tau^{-1}(u_{k+1}-u_k)$$
$$+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}2^{-1}\tau^{-1}\left(A_k^{1/2}-A_{k-1}^{1/2}\right)A_{k-1/2}^{-1/2}\tau^{-1}(u_k-u_{k-1})||_H,$$
$$+||\{\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)\}_1^{N-1}||_{C_\tau}$$
$$\leq M_2[||A(0)u_0||_H+||A^{\frac{1}{2}}(0)u_0'||_H+\max_{0\leq s\leq k}\left\|f_{s+1/2}\right\|_H+\sum_{s=0}^{N-2}||f_{s+1/2}-f_{s-1/2}||_H]$$

where

$$M_2=\left(C_9+C_{27}+C_{28},\ 3\right).$$

3.3 THE DIFFERENCE SCHEME GENERATED BY THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME

We will consider again the initial value problem (2.26). Applying the system of the first order linear differential equations

$$\begin{cases} \frac{du(t)}{dt} = iA^{1/2}(t)v(t),\ 0<t<T,\ u(0)=u_0,\ u'(0)=u'_0, \\ \frac{dv(t)}{dt} = iA^{1/2}(t)u(t) - A^{-1/2}(t)\left[A^{1/2}(t)\right]'v(t) - iA^{-1/2}(t)f(t) \end{cases}$$

and the Taylor's formula, we get

$$\begin{cases} \tau^{-1}(u(t_{k+1})-u(t_k)) = iA^{1/2}_{k+1/2}v(t_{k+1/2})+o(\tau^2), 0\le k\le N-1, \\ \tau^{-1}(v(t_{k+1})-v(t_k)) = iA^{1/2}_{k+1/2}u(t_{k+1/2}) - A^{-1/2}_{k+1/2}[A^{1/2}_{k+1/2}]'v(t_{k+1/2}) \\ -iA^{-1/2}_{k+1/2}f(t_{k+1/2})+o(\tau^2), 0\le k\le N-1, v_0=-iA_0^{-1/2}u'_0, \end{cases}$$

where

$$A^{1/2}_{k+1/2}=A^{1/2}(t_{k+1/2}),\ [A^{1/2}_{k+1/2}]'=(A')^{1/2}(t_{k+1/2}),\ t_{k+1/2}=(t_k+\frac{\tau}{2}),\ A_0=A(0).$$

Then using the Taylor's formula, we can write

$$\begin{cases} \tau^{-1}(u(t_{k+1})-u(t_k)) = iA^{1/2}_{k+1/2}\left(v(t_{k+1})-\frac{\tau}{2}v'(t_{k+1})\right)+o(\tau^2), 0\le k\le N-1, \\ \tau^{-1}(v(t_{k+1})-v(t_k)) = iA^{1/2}_{k+1/2}\left(u(t_{k+1})-\frac{\tau}{2}u'(t_{k+1})\right) \\ -A^{-1/2}_{k+1/2}\left(A^{1/2}_{k+1/2}\right)'\left(v(t_{k+1})-\frac{\tau}{2}v'(t_{k+1})\right)-iA^{-1/2}_{k+1/2}f(t_{k+1/2})+o(\tau^2), 0\le k\le N-1, \\ v_0=-iA_0^{-1/2}u'_0. \end{cases}$$

Using (2.28) and putting $A^{1/2}_{k+1/2}$ instead of $A^{1/2}_{k+1}$, we obtain the following equation

$$\begin{cases} \tau^{-1}(u(t_{k+1})-u(t_k)) = \frac{\tau}{2}A_{k+1/2}u(t_{k+1}) \\ +\left[iA^{1/2}_{k+1/2}+i\frac{\tau}{2}\left(A^{1/2}_{k+1/2}\right)'\right]v(t_{k+1})-\frac{\tau}{2}f(t_{k+1/2})+o(\tau^2),\ 0\le k\le N-1, \\ \tau^{-1}(v(t_{k+1})-v(t_k)) = \left[iA^{1/2}_{k+1/2}+i\frac{\tau}{2}\left(A^{1/2}_{k+1/2}\right)'\right]u(t_{k+1}) \\ +\left[\frac{\tau}{2}A_{k+1/2}-A^{-1/2}_{k+1/2}\left(A^{1/2}_{k+1/2}\right)'-\frac{\tau}{2}A^{-1}_{k+1/2}\left(A^{1/2}_{k+1/2}\right)'\left(A^{1/2}_{k+1}\right)'\right]v(t_{k+1}) \\ +\left[-iA^{-1/2}_{k+1/2}-i\frac{\tau}{2}A^{-1}_{k+1/2}\left(A^{1/2}_{k+1/2}\right)'\right]f(t_{k+1/2})+o(\tau^2), 0\le k\le N-1, \\ v_0=-iA_0^{-1/2}u'_0. \end{cases}$$

Neglecting the last small terms and using (3.1), we obtain the following difference scheme for the approximate solution of the initial value problem (2.26) :

$$\begin{cases} \tau^{-1}(u_{k+1}-u_k)=\frac{\tau}{2}A_{k+1/2}u_{k+1} \\ +\left[iA_{k+1/2}^{1/2}+i\frac{\tau}{2}\left(A_{k+1/2}^{1/2}\right)'\right]v_{k+1}-\frac{\tau}{2}f_{k+1/2}, 0\le k\le N-1, \\ u_0=u(0), \\ \tau^{-1}(v_{k+1}-v_k)=\left[iA_{k+1/2}^{1/2}+i\frac{\tau}{2}\left(A_{k+1/2}^{1/2}\right)'\right]u_{k+1} \\ +\left[\frac{\tau}{2}A_{k+1/2}-A_{k+1/2}^{-1/2}\left(A_{k+1/2}^{1/2}\right)'-\frac{\tau}{2}A_{k+1/2}^{-1}\left(A_{k+1/2}^{1/2}\right)'\left(A_{k+1}^{1/2}\right)'\right]v_{k+1} \\ \left[-iA_{k+1/2}^{-1/2}-i\frac{\tau}{2}A_{k+1/2}^{-1}\left(A_{k+1/2}^{1/2}\right)'\right]f_{k+1/2},\ 0\le k\le N-1, \\ v_0=-iA_0^{-1/2}u_0'. \end{cases}$$

or

$$
\begin{cases}
\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+\Big\{\frac{i}{\tau}A_{k+1/2}^{1/2}\Big[I-\frac{\tau^2}{2}A_{k+1/2}+\tau A_{k+1/2}^{-1/2}\left(A_{k+1/2}^{1/2}\right)' \\
+\frac{\tau^2}{2}A_{k+1/2}^{-1}\left(A_{k+1/2}^{1/2}\right)'\left(A_{k+1}^{1/2}\right)'\Big]\left[i\tau A_{k+1/2}^{1/2}+i\frac{\tau^2}{2}\left(A_{k+1}^{1/2}\right)'\right]^{-1}\left(I-\frac{\tau^2}{2}A_{k+1/2}\right) \\
-\frac{i}{\tau}A_{k+1/2}^{1/2}\left[i\tau A_{k+1/2}^{1/2}+i\frac{\tau^2}{2}\left(A_{k+1/2}^{1/2}\right)'\right]-\frac{1}{\tau^2}\Big\}u_{k+1}+ \\
\Big\{\frac{i}{\tau}A_{k+1/2}^{1/2}\Big[I-\frac{\tau^2}{2}A_{k+1/2}-\tau A_{k+1/2}^{-1/2}\left(A_{k+1/2}^{1/2}\right)'+\frac{\tau^2}{2}A_{k+1/2}^{-1}\left(A_{k+1/2}^{1/2}\right)'\left(A_{k+1}^{1/2}\right)'\Big] \\
\times\left[i\tau A_{k+1/2}+i\frac{\tau^2}{2}\left(A_{k+1}^{1/2}\right)'\right]^{-1}-\frac{i}{\tau}A_{k+1/2}^{1/2}\left[i\tau A_{k-1/2}^{1/2}+i\frac{\tau^2}{2}\left(A_k^{1/2}\right)'\right]^{-1} \\
\times\left(I-\frac{\tau^2}{2}A_{k-1/2}\right)+\frac{2}{\tau^2}\Big\}u_k+\Big\{\frac{i}{\tau}A_{k+1/2}^{1/2}\left[i\tau A_{k-1/2}^{1/2}+i\frac{\tau^2}{2}\left(A_k^{1/2}\right)'\right]^{-1}-\frac{1}{\tau^2}\Big\}u_{k-1} \\
=\Big\{-\frac{i\tau}{2}A_{k+1/2}^{1/2}\Big[I-\frac{\tau^2}{2}A_{k+1/2}+\tau A_{k+1/2}^{-1/2}\left(A_{k+1/2}^{1/2}\right)'+\frac{\tau^2}{2}A_{k+1/2}^{-1}\left(A_{k+1/2}^{1/2}\right)'\left(A_{k+1}^{1/2}\right)'\Big] \\
\times\left[i\tau A_{k+1/2}^{1/2}+i\frac{\tau^2}{2}\left(A_{k+1}^{1/2}\right)'\right]^{-1}+I+\frac{\tau}{2}A_{k+1/2}^{-1/2}\left(A_{k+1/2}^{1/2}\right)'\Big\}f_{k+1/2}+ \\
\Big\{\frac{i\tau}{2}A_{k+1/2}^{1/2}\left[i\tau A_{k-1/2}^{1/2}+i\frac{\tau^2}{2}\left(A_k^{1/2}\right)'\right]^{-1}\Big\}f_{k-1/2} \\
1\le k\le N-1,\quad u_0=u(0), \\
\Big\{I-\Big[\frac{\tau^2}{2}A_{1/2}-\tau A_{1/2}^{-1/2}\left(A_{1/2}^{1/2}\right)'-\frac{\tau^2}{2}A_{1/2}^{-1}\left(A_{1/2}^{1/2}\right)'\left(A_1^{1/2}\right)'\Big]\Big\} \\
\times\left[iA_{1/2}^{1/2}+i\frac{\tau}{2}\left(A_{1/2}^{1/2}\right)'\right]^{-1}\tau^{-1}(u_1-u_0) \\
+\left[-\frac{\tau}{2}A_{1/2}-i\tau A_{1/2}^{1/2}-i\frac{\tau^2}{2}\left(A_{1/2}^{1/2}\right)'\right]u_1+\left(iA_0^{-1/2}\right)u_0' \\
=\Big\{-\frac{\tau}{2}\Big\{I-\Big[\frac{\tau^2}{2}A_{1/2}-\tau A_{1/2}^{-1/2}\left(A_{1/2}^{1/2}\right)'-\frac{\tau^2}{2}A_{1/2}^{-1}\left(A_{1/2}^{1/2}\right)'\left(A_1^{1/2}\right)'\Big]\Big\} \\
\times\left[iA_{1/2}^{1/2}+i\frac{\tau}{2}\left(A_{1/2}^{1/2}\right)'\right]^{-1}-i\tau A_{1/2}^{-1/2}-i\frac{\tau^2}{2}A_{1/2}^{-1}\left(A_{1/2}^{1/2}\right)'\Big\}f_{1/2}
\end{cases}
\tag{3.24}
$$

In the similar manner we can obtain the stability estimates for the solution of the difference scheme (3.24). This is left as a future work.

CHAPTER 4

NUMERICAL ANALYSIS

We have not been able to obtain a sharp estimate for the constants figuring in the stability inequality. Therefore we will give the following results of numerical experiments of the initial-boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - g(t)\frac{\partial^2 u(t,x)}{\partial x^2} = 2\exp(-t)\cos x, \\ 0 < t < 1,\ 0 < x < \pi, \\ u(0,x) = \cos x,\ u_t(0,x) = -\cos x, \quad 0 \le x \le \pi, \\ u_x(t,0) = u_x(t,\pi) = 0,\ 0 \le t \le 1, \end{cases} \tag{4.1}$$

for hyperbolic equation. The exact solution of this problem for $g(t) = 1$ is

$$u\,(t,x) = \exp(-t)\cos x.$$

In the present chapter for the approximate solutions of the initial-boundary value problem (4.1), we will use the first and second order of accuracy difference schemes with grid intervals $\tau = \frac{1}{20}$, $h = \frac{1}{20}$ for t and x, respectively. We have the second order or fourth order difference equations with respect to n with matrix coefficients. To solve these difference equations we have applied a procedure of modified Gauss elimination method for difference equations with respect to n with matrix coefficients. The results of numerical experiments permit us to show that the second order of accuracy difference schemes are more accurate compared with the first order of accuracy difference scheme.

4.1 THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

For the approximate solution of the initial-boundary value problem (4.1) we consider the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$\begin{aligned} [0,1]_\tau \times [0,\pi]_h &= \{(t_k, x_n) : t_k = k\tau, \quad 1 \le k \le N-1, \quad N\tau = 1, \\ x_n &= nh, \quad 1 \le n \le M-1, Mh = \pi\}. \end{aligned}$$

Applying the formulas

$$\begin{aligned} \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_{k+1}) &= O(\tau), \\ \frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) &= O(h^2), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \frac{u(h)-u(0)}{h}-u'(0) &= O(h), \\ \frac{u(\pi)-u(\pi-h)}{h}-u'(\pi) &= O(h) \end{aligned} \tag{4.3}$$

we present the following first order of accuracy in t difference scheme for the approximate solutions of the problem (4.1)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-g(t_k)\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}=f(t_k,x_n),\ 1\le k\le N-1, 1\le n\le M-1, \\ u_n^0=\varphi(x_n),\ 1\le n\le M-1, \\ \frac{u_n^1-u_n^0}{\tau}=-\varphi(x_n),\ 1\le n\le M-1, \\ u_0^k=u_1^k,\ u_{M-1}^k=u_M^k,\ 0\le k\le N, \\ f(t,x)=2\exp(-t)\cos x, \\ \varphi(x)=\cos(x), \\ g(t)=1. \end{cases} \tag{4.4}$$

Note that in the equations (4.2), (4.3) and (4.4), $u(t_k)$, $u(x_n)$, u_n^k represent $u(t_k,x_n)$.

We have $(N+1)\times(N+1)$ system of linear equations in (4.4) and we will write them in the matrix form. We can rewrite this system as the following form

$$\begin{cases} \left(-\frac{g(t_k)}{h^2}\right)u_{n+1}^{k+1}+\left(\frac{1}{\tau^2}+\frac{2g(t_k)}{h^2}\right)u_n^{k+1}+\left(-\frac{2}{\tau^2}\right)u_n^k+\left(\frac{1}{\tau^2}\right)u_n^{k-1}+\left(-\frac{g(t_k)}{h^2}\right)u_{n-1}^{k+1}=f(t_k,x_n), \\ 1\le k\le N-1,\ 1\le n\le M-1, \\ u_n^0=\varphi(x_n),\ 1\le n\le M-1, \\ \frac{u_n^1-u_n^0}{\tau}=-\varphi(x_n),\ 1\le n\le M-1, \\ u_0^k=u_1^k,\ u_{M-1}^k=u_M^k,\ 0\le k\le N, \end{cases} \tag{4.5}$$

We denote

$$a=-\frac{g(t_k)}{h^2},\quad b=\frac{1}{\tau^2}+\frac{2g(t_k)}{h^2},\quad c=-\frac{2}{\tau^2},\quad d=\frac{1}{\tau^2},$$

$$\varphi_n^k=\begin{cases} \cos(x_n),\quad k=0, \\ f(t_k,x_n),\ 1\le k\le N-1, \\ \cos(x_n),\ k=N. \end{cases}$$

$$\varphi_n=\begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & ... & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ d & c & b & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & d & c & b & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & d & c & b & 0 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & c & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & d & c & b \\ \frac{1}{\tau} & -\frac{1}{\tau} & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

and

$$C = A,$$

$$D = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ ... \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s = n-1, \ n, n+1.$$

Then (4.5) can be written as

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 0 \le n \le M, \\ U_0 = U_1, \ \ U_{M-1} = U_M. \end{cases} \tag{4.6}$$

So, we have the second order difference equation with respect to n with matrix coefficients. To solve this difference equation we have applied a procedure of modified Gauss elimination method for difference equation with respect to n with matrix coefficients. Hence, we seek a solution of the matrix equation in the following form

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, \cdots, 2, 1, 0, \tag{4.7}$$

where $\alpha_j \ (j=1,\cdots,M)$ are $(N+1)\times(N+1)$ square matrices and $\beta_j \ (j=1,\cdots,M)$ are $(N+1)\times 1$ column matrices. Using the equality

$$U_s = \alpha_{s+1}U_{s+1} + \beta_{s+1}, (for\ s=n,\ n-1)$$

and the equality

$$AU_{n+1} + B\,U_n + CU_{n-1} = D\varphi_n,$$

we can write

$$[A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1}]U_{n+1} + \left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n.$$

The last equation is satisfied if we select

$$A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1} = 0,$$

$$\left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n, \quad 1 \le n \le M-1.$$

From that it follows

$$\begin{aligned} \alpha_{n+1} &= -\left(B + C\alpha_n\right)^{-1} A, \\ \beta_{n+1} &= \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), n = 1,2,3,\cdots,M-1. \end{aligned} \tag{4.8}$$

For the solution of difference equations we need to find α_1 and β_1. We can find them from $U_0 = \alpha_1 U_1 + \beta_1$. Thus, we have

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{4.9}$$

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}_{(N+1)\times 1}.$$

For the first step using formulas (4.8) and (4.9), we can compute α_{n+1} and β_{n+1}, $1 \le n \le M-1$. For the second step we will find u_n, $0 \le n \le M$. But, for this we need to find u_M. We can find u_M from $u_M = u_{M-1}$ and $u_{M-1} = \alpha_M u_M + \beta_M$. Namely

$$u_M = (I - \alpha_M)^{-1}\beta_M. \tag{4.10}$$

Thus using formulas (4.7) and (4.10), we can compute u_n, $0 \le n \le M$. We can summarise the computation procedure by the following algorithm.

Algorithm

1. **Step** Set input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$.
2. **Step** Use the first order of accuracy difference scheme and write in matrix form

$$A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \le n \le M-1.$$

3. Step Determine the entries of the matrices A, B, C and D.

4. Step Find α_1, β_1 by the formula (4.9).

5. Step Compute $\alpha_{n+1},\ \beta_{n+1}, 1 \leq n \leq M-1$ by the formula (4.8).

6. Step Compute U_M by the formula (4.10).

7. Step Compute $U_n, n = M-1, \cdots, 1, 0$ by the formula (4.7).

Matlab Implementation of the First Order of Accuracy Difference Scheme .

```
function firstorder(N,M)
    time1=cputime;
    if nargin<1; close; close; close; close; end;
    N= 20 ; M= 20 ;
    h=pi/M;
    tau=1/N;
    c=-2/(tau^2);
    d=1/(tau^2);
    for k=2:N; t(k)=(k-1)*tau;
    a(k)=(-1/(h^2)); A(k,k+1)=a(k); A(N+1,N+1)=0; A;
    end;
    for k=2:N; t(k)=(k-1)*tau;
    b(k)=(((2)/(h^2))+(1/(tau^2))); B(k,k+1)=b(k);
    B(k,k-1)=d; B(k,k)=c; B;
    end;
    B(N+1,1)=1/tau; B(N+1,2)=-1/tau;
    B(1,N+1)=0; B(N+1,N)=0; B(N+1,N+1)=0; B(1,1)=1;
    C=A;
    for i=1:N+1; D(i,i)=1; end; D;
    for j=1:M-1;
    for k=1:N+1;
    t=(k-1)*tau; x=(j)*h;
    fii(k,j:j)=2*(exp(-t))*cos(x);
    end;
    end;
    for j=1:M-1;
    x=(j)*h;
    fii(1,j:j)=cos(x); fii(N+1,j:j)=(1-tau)*cos(x);
```

```
end;
I=eye(N+1,N+1);
alpha{1}=I;
betha{1}=zeros(N+1,1);
for j=1:M-1;
alpha{j+1}=inv(B+C*alpha{j})*(-A);
betha{j+1}=inv(B+C*alpha{j})*(D*fii(:,j:j)-C*betha{j});
end;
U{M}=inv(I-alpha{M})*betha{M};
for Z=M-1:-1:1;
U{Z}=alpha{Z+1}*U{Z+1}+betha{Z+1};
end;
for Z=1:M;
p(:,Z+1)=U{Z};
end;
p(:,1)=U{1};
time2=cputime;
time2-time1
%%%%%%%'EXACT SOLUTION OF THIS PDE' %%%%%%%%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=exp(-t)*cos(x);
end;
end;
%%%%%%%%% END EXACT SOLUTION %%%%%%%%%%%
%%%%%%%%%%%%% ERROR ANALYSIS %%%%%%%%%%%
abs(es-p)
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=maxerror/maxapp;
cevap = [maxes,maxapp,maxerror,relativeerror]
p; es;
[xler,tler]=meshgrid(0:h:pi,0:tau:1);
table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%
```

```
q=min(min(table)); w=max(max(table)); figure;
surf(xler,tler,es);
title('EXACT SOLUTION');
set(gca,'ZLim',[q w]); rotate3d;
XLabel('x axis');YLabel('t axis');
figure; surf(xler,tler,p);
title('EULER-ROTHER');
rotate3d ; set(gca,'ZLim',[q w]);
XLabel('x axis');YLabel('t axis');
%%%%%%%%%%%% END GRAPH %%%%%%%%%%%%%%%%%
```

4.2 THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME GENERATED BY THREE POINTS

Applying the formulas

$$\begin{aligned} \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_k) &= O(\tau^2), \\ \frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) &= O(h^2) \end{aligned}$$

and

$$\begin{aligned} \frac{2u(0) - 5u(\tau) + 4u(2\tau) - u(3\tau)}{\tau^2} - u''(0) &= O(\tau^2), \\ \frac{2u(1) - 5u(1-\tau) + 4u(1-2\tau) - u(1-3\tau)}{\tau^2} - u''(1) &= O(\tau^2). \end{aligned}$$

We present the following second order of accuracy in t difference scheme for the approximate solutions of the problem (4.1)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - g(t_k)\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} - g(t_{k+1})\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{4h^2} \; - g(t_{k-1})\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2} \\ = f(t_k,x_n), x_n = nh, \; t_k = k\tau, \; 1 \le k \le N-1, \; 1 \le n \le M-1, \\ u_n^0 = \varphi(x_n), \quad x_n = nh, \; 1 \le n \le M-1, \\ \frac{u_n^1-u_n^0}{\tau} = \frac{\tau}{2}\left(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2} + f(0,x_n)\right) - \varphi(x_n), \; x_n = nh, \; 1 \le n \le M-1, \\ \frac{u_1^k-u_1^k}{h} = \lambda_k\left(u_1^{k+1} - 2u_1^k + u_1^{k-1}\right) + \lambda_k\tau^2 f(t_k,h), \quad 1 \le k \le N-1, \\ \frac{u_1^0-u_0^0}{h} = \lambda_0\left(2u_0^0 - 5u_0^1 + 4u_0^2 - u_0^3\right) + \lambda_0\tau^2 f(0,h), \\ \frac{u_1^N-u_0^N}{h} = \lambda_N\left(2u_0^N - 5u_0^{N-1} + 4u_0^{N-2} - u_0^{N-3}\right) + \lambda_N\tau^2 f(1,h), \\ 3u_M^k = 4u_{M-1}^k - u_{M-2}^k, \; 0 \le k \le N, \\ f(t,x) = 2\exp(-t)\cos x, \\ \lambda_k = \frac{h}{2g(t_k)\tau^2}, \quad 0 \le k \le N, \\ g(t) = 1. \end{cases} \tag{4.11}$$

We have again $(N+1) \times (N+1)$ system of linear equations and we will write them in the matrix form. We can rewrite this system in the following form

$$\begin{cases} \left(-\frac{g(t_{k+1})}{4h^2}\right) u_{n+1}^{k-1} + \left(-\frac{g(t_k)}{2h^2}\right) u_{n+1}^{k} + \left(-\frac{g(t_{k-1})}{4h^2}\right) u_{n+1}^{k-1} + \left(\frac{1}{\tau^2} + \frac{g(t_{k+1})}{2h^2}\right) u_n^{k+1} \\ + \left(-\frac{2}{\tau^2} + \frac{g(t_k)}{h^2}\right) u_n^{k} + \left(\frac{1}{\tau^2} + \frac{g(t_{k-1})}{2h^2}\right) u_n^{k-1} + \left(-\frac{g(t_{k+1})}{4h^2}\right) u_{n-1}^{k+1} \\ + \left(-\frac{g(t_k)}{2h^2}\right) u_{n-1}^{k} + \left(-\frac{g(t_{k-1})}{4h^2}\right) u_{n-1}^{k-1} = f\left(t_k, x_n\right), \\ 1 \le k \le N-1, 1 \le n \le M-1, \\ u_n^0 = \varphi(x_n), \quad x_n = nh,\ 1 \le n \le M-1, \\ \frac{u_n^1 - u_n^0}{\tau} = \frac{\tau}{2}\left(\frac{u_{n+1}^1 - 2u_n^1 + u_{n-1}^1}{h^2} + f(0, x_n)\right) - \varphi(x_n),\ x_n = nh,\ 1 \le n \le M-1, \\ \frac{u_1^k - u_1^k}{h} = \lambda_k \left(u_1^{k+1} - 2u_1^k + u_1^{k-1}\right) + \lambda_k \tau^2 f(t_k, h), \quad 1 \le k \le N-1, \\ \frac{u_1^0 - u_0^0}{h} = \lambda_0 \left(2u_0^0 - 5u_0^1 + 4u_0^2 - u_0^3\right) + \lambda_0 \tau^2 f(0, h), \\ \frac{u_1^N - u_0^N}{h} = \lambda_N \left(2u_0^N - 5u_0^{N-1} + 4u_0^{N-2} - u_0^{N-3}\right) + \lambda_N \tau^2 f(1, h), \\ 3u_M^k = 4u_{M-1}^k - u_{M-2}^k,\ 0 \le k \le N, \\ f(t,x) = 2\exp(-t)\cos x, \\ \lambda_k = \frac{h}{2g(t_k)\tau^2}, \quad 0 \le k \le N, \\ g(t) = 1. \end{cases}$$

Denoting

$$\begin{aligned} a &= -\frac{g(t_{k+1})}{4h^2},\ b = -\frac{g(t_k)}{2h^2},\ c = -\frac{g(t_{k-1})}{4h^2}, \\ d &= \frac{1}{\tau^2} + \frac{g(t_{k+1})}{2h^2},\ e = -\frac{2}{\tau^2} + \frac{g(t_k)}{h^2},\ f = \frac{1}{\tau^2} + \frac{g(t_{k-1})}{2h^2}, \end{aligned}$$

$$\varphi_n^k = \begin{cases} \cos(x_n), \quad k = 0, \\ f(t_k, x_n),\ 1 \le k \le N-1, \\ \cos(x_n),\ k = N, \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ .. \\ \varphi_n^{N-2} \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ c & b & a & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & c & b & a & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & c & b & a & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & c & b & a \\ 0 & \frac{-\tau}{2h^2} & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ f & e & d & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & f & e & d & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & f & e & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & f & e & d & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & f & e & d \\ \frac{-1}{\tau} & \left(\frac{1}{\tau} + \frac{\tau}{h^2}\right) & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

and

$$C = A,$$

$$D = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ .. \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad \text{where } s = \ n-1, \ n, \ n+1,$$

we obtain the following second order difference equation with respect to n with matrix coefficients

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ 3U_M = 4U_{M-1} - U_{M-2}, \ \gamma_0 U_0 = \theta_1 U_1 + T_1, \end{cases}$$

where

$$\gamma_0 = \begin{bmatrix} 1+2\lambda_0 & -5\lambda_0 & 4\lambda_0 & -\lambda_0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & -\lambda_N & 4\lambda_N & -5\lambda_N & 1+2\lambda_N \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\theta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ l & s & l & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & l & s & l & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & l & s & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & s & l & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & l & s & l & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & l & s & l \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

and

$$l = \frac{h^2}{2g(t_k)\tau^2}, \quad s = \frac{h^2}{g(t_k)\tau^2},$$

$$T_1 = \begin{bmatrix} F_0^0 \\ F_1^1 \\ F_1^2 \\ .. \\ F_1^{N-1} \\ F_0^N \end{bmatrix}_{(N+1)\times(1)},$$

$$F_0^0 = \tau^2 h^2, \quad F_0^N = \frac{3}{4}\tau^2 \exp(-1),$$

$$F_1^k = \tau^2 \lambda_k f(t_k, h), \ 1 \le k \le N-1 \ .$$

To solve this difference equation we have applied the modified Gauss elimination method for difference equation with respect to n with matrix coefficients. Hence, we seek a solution $U_n, n = M-1, \cdots, 2, 1, 0$ of the matrix equation by the formula (4.7), where $\alpha_{n+1}, \beta_{n+1}, n = 1, \cdots, M-1$, are obtained by (4.10). Note that for obtaing $\alpha_{n+1}, \beta_{n+1}, n = 1, \cdots, M-1$, we need to find α_1 and β_1. We can find them from $U_0 = \alpha_1 U_1 + \beta_1$. Using the formula $\gamma_0 U_0 = \theta_1 U_1 + T_1$, we obtain

$$U_0 = inv(\gamma_0)\theta_1 U_1 + inv(\gamma_0) T_1,$$

where

$$\alpha_1 = inv(\gamma_0)\theta_1, \ \beta_1 = inv(\gamma_0) T_1. \tag{4.12}$$

Now we will find u_n, $0 \le n \le M$, by the formula (4.7), but, for this we need to find u_M. We can find u_M from $3U_M = 4U_{M-1} - U_{M-2}$ and $u_{M-1} = \alpha_M u_M + \beta_M, u_{M-2} = \alpha_{M-1} u_{M-1} + \beta_{M-1}$. Namely

$$u_M = [3I - 4\alpha_M + \alpha_{M-1}\alpha_M]^{-1}[4\beta_M - \alpha_{M-1}\beta_M - \beta_{M-1}]. \tag{4.13}$$

Thus using formulas (4.7) and (4.13), we can compute u_n, $0 \leq n \leq M$. We can summarize the computation procedure by the following algorithm.

Algorithm

1. **Step** Set input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$.
2. **Step** Use the second order of accuracy difference scheme and write in matrix form

$$A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \leq n \leq M-1.$$

3. **Step** Determine the entries of the matrices A, B, C and D.
4. **Step** Find α_1, β_1 by the formula (4.12).
5. **Step** Compute α_{n+1}, $\beta_{n+1}, 1 \leq n \leq M-1$ by the formula (4.8).
6. **Step** Compute U_M by the formula (4.13).
7. **Step** Compute $U_n, n = M-1, \cdots, 1, 0$ by the formula (4.7).

Matlab Implementation on Second Order of Accuracy Difference Scheme Generated by Three Points

```
function secondorder(N,M)
    time1=cputime;
    if nargin<1;N= 20 ; M= 20 ; end;
    close; close; close; close;
    h=pi/M; tau=1/N;
    t=0:tau:1;
    for k=1:N-1;
    c(k)=-g( tau*(k-1) ) /(4*(h^2));
    b(k)=-g( tau*(k) ) /(2*(h^2)) ;
    a(k)=-g( tau*(1+k) ) /(4*(h^2));
    A(k+1,k)=c(k); A(k+1,k+1)=b(k); A(k+1,k+2)=a(k);
    end;
    A(N+1,1)=0;
    A(N+1,2)=(-tau)/(2*(h^2));
    A(N+1,N)=0; A(N+1,N+1)=0;
    for k=1:N-1;
    f(k)=(1/(tau^2))+ g( tau*(k-1) )/(2*(h^2));
    e(k)=(-2/(tau^2))+ g( tau* k )/(h^2);
    d(k)=(1/(tau^2))+ g( tau*(k+1) )/(2*(h^2));
```

```
B(k+1,k) = f(k);
B(k+1,k+1)= e(k);
B(k+1,k+2)= d(k);
end;
B(1,1)=1; B(1,N+1)=0;
B(N+1,1)=(-1/tau);
B(N+1,2)=(1/tau)+((tau)/(h^2));
B(N+1,N)=0; B(N+1,N+1)=0;
C=A;
for i=1:N+1; D(i,i)=1; end; D;
for j=1:M-1;
for k=1:N-1;
t=k*tau; x=j*h;
fii(k+1,j:j)=2*(exp(-t))*cos(x);
end;
end;
for j=1:M-1;
x=j*h;
fii(1,j:j)= cos(x); % right side for k=0
fii(N+1,j:j)=(-1+tau)*cos(x);% right side for k=N
end;
lamda0=(h^2)/(g(0)*2*(tau^2));
lamdaN=(h^2)/(g(1)*2*(tau^2));
for k=1:N-1;
tk=k*tau;
lamda(k)=(h^2)/(2*(tau^2)*g(tk));
end;
for k=2:N;
fgama(k,k)=1;
end;
fgama(1,1)=1+2*lamda0;
fgama(1,2)=-5*lamda0;
fgama(1,3)=4*lamda0;
fgama(1,4)=-lamda0;
fgama(N+1,N+1)=1+2*lamdaN;
fgama(N+1,N)=-5*lamdaN;
fgama(N+1,N-1)=4*lamdaN;
fgama(N+1,N-2)=-lamdaN;
```

```
T(N+1,1)=(tau^2)*lamdaN * ( 2*exp(-1) );
%%t0=0;
T(1,1)=(tau^2)*lamda0*2; %(2*exp(-t0));
for k=1:N-1;
tk=k*tau;
T(k+1,1)=+(tau^2)*lamda(k)*(2*exp(-tk)*cos(h));
end;
teta(1,1)=1;
teta(N+1,N+1)=1;
for k=1:N-1;
tk=k*tau;
teta(k+1,k)=-lamda(k);
teta(k+1,k+1)=1+2*lamda(k);
teta(k+1,k+2)=-lamda(k);
end;
zalpha=inv(fgama)*teta;
zbetha=inv(fgama)*T;
alpha{1}=zalpha;
betha{1}=zbetha;
for j=1:M-1;
alpha{j+1}=inv(B+C*alpha{j})*(-A);
betha{j+1}=inv(B+C*alpha{j})*(D*fii(:,j:j)-C*betha{j});
end;
I=eye(N+1,N+1);
U{M}=inv(3*I-4*alpha{M}+alpha{M-1}*alpha{M})*...
(4*betha{M}-betha{M-1}-alpha{M-1}*betha{M});
for Z=M-1:-1:1;
U{Z}=alpha{Z+1}*U{Z+1}+betha{Z+1};
end;
U0= alpha{1}*U{1}+betha{1};
for Z=1:M;
p(:,Z+1)=U{Z};
end;
p(:,1)=U0;
time2=cputime;
time2-time1
%%%%%%%%%% EXACT SOLUTION OF THIS PDE %%%%%%%%%%%
for j=1:M+1;
```

```
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=exp(-t)*cos(x);
end;
end;
%%%%%%%%%%%  END EXACT SOLUTION  %%%%%%%%%%%%%
%%%%%%%%%%%%%%  EROR ANALYSIS %%%%%%%%%%%%%%%
abs(es-p)
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxertetar=max(max(abs(es-p)));
relativeertetar=maxertetar/maxapp;
cevap = [maxes,maxapp,maxertetar,relativeertetar]

%%%%%%%%%%  GRAPH OF THE SOLUTION  %%%%%%%%%%%%
[xler,tler]=meshgrid(0:h:pi,0:tau:1);
table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
q=min(min(table)); w=max(max(table));
figure;
surf(xler,tler,es); title('EXACT SOLUTION');
set(gca,'ZLim',[q w]);
rotate3d;XLabel('x axis');YLabel('t axis');
figure; surf(xler,tler,p);
title('SEC.ORD. APP. SOL. GEN. BY 3 POINTS');
rotate3d ; set(gca,'ZLim',[q w]);
XLabel('x axis');YLabel('t axis');

%%%%%%%%%%%%%%  END GRAPH   %%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%  SUB FUNCTIONS  %%%%%%%%%%%%%%%
function gt=g(t)
gt=1;
%function ftx=f(t,x)
%ftx=2*(exp(-t))*cos(x);
```

4.3 THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME GENERATED BY FIVE POINTS

Applying the formulas

$$\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-u''(t_k)=O(\tau^2),$$

$$\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2}-u''(x_n)=O(h^2),$$

$$\frac{u(x_{n+2})-4u(x_{n+1})+6u(x_n)-4u(x_{n-1})+u(x_{n-2})}{h^4}-u^{iv}(x_n)=O(h^2),$$

and

$$\begin{aligned}\frac{10u(0)-15u(h)+6u(2h)-u(3h)}{h^3}-u'''(0) &= O(h^2),\\ \frac{-10u(\pi)+15u(\pi-h)-6u(\pi-2h)+u(\pi-3h)}{h^3}-u'''(\pi) &= O(h^2).\end{aligned}$$

we present the following second order of accuracy in t difference scheme for the approximate solutions of the problem (4.1)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-g(t_k)\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+g(t_{k+1})\tau^2\left(\frac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{4h^4}\right)\\ =f(t_k,x_n),\ x_n=nh,\quad t_k=k\tau,\ 1\le k\le N-1,\ 2\le n\le M-2,\\ u_n^0=\varphi\left(x_n\right),\ x_n=nh,\quad 0\le n\le M,\\ \frac{u_n^1-u_n^0}{\tau}=\frac{\tau}{2}\left(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+f(0,x_n)\right)-\varphi(x_n),\ x_n=nh,\ 1\le n\le M-1,\\ 3u_M^k=4u_{M-1}^k-u_{M-2}^k,\ 0\le k\le N,\\ 10u_M^k=15u_{M-1}^k-6u_{M-2}^k+u_{M-3}^k,\ 0\le k\le N,\\ 10u_0^k=15u_1^k-6u_2^k+u_3^k,\ 0\le k\le N,\end{cases} \tag{4.14}$$

We have again $(N+1)\times(N+1)$ system of linear equations and we will write them in the matrix form. We can rewrite this system in the following form

$$\begin{cases} \left(g(t_{k+1})\frac{\tau^2}{4h^4}\right)u_{n+2}^{k+1} + \left(-g(t_{k+1})\frac{\tau^2}{h^4}\right)u_{n+1}^{k+1} + \left(-g(t_k)\frac{1}{h^2}\right)u_{n+1}^{k} \\ + \left(g(t_{k+1})\frac{6\tau^2}{4h^4} + \frac{1}{\tau^2}\right)u_n^{k+1} + \left(g(t_k)\frac{2}{h^2} - \frac{2}{\tau^2}\right)u_n^k + \left(\frac{1}{\tau^2}\right)u_n^{k-1} \\ + \left(-g(t_{k-1})\frac{\tau^2}{h^4}\right)u_{n-1}^{k+1} + \left(-g(t_k)\frac{1}{h^2}\right)u_{n-1}^{k} \\ + \left(g(t_{k+1})\frac{\tau^2}{4h^4}\right)u_{n-2}^{k+1} = f\left(t_k, x_n\right),\ 1 \le k \le N-1, 2 \le n \le M-2, \\ u_n^0 = \varphi\left(x_n\right),\ x_n = nh, \quad 0 \le n \le M, \\ \frac{u_n^1 - u_n^0}{\tau} = \frac{\tau}{2}\left(\frac{u_{n+1}^1 - 2u_n^1 + u_{n-1}^1}{h^2} + f\left(0, x_n\right)\right) - \varphi\left(x_n\right), \\ 3u_M^k = 4u_{M-1}^k - u_{M-2}^k,\ 0 \le k \le N, \\ 10u_M^k = 15u_{M-1}^k - 6u_{M-2}^k + u_{M-3}^k,\ 0 \le k \le N, \\ 10u_0^k = 15u_1^k - 6u_2^k + u_3^k,\ 0 \le k \le N, \\ f\left(t, x\right) = 2\exp(-t)\cos x \\ g(t) = 1. \end{cases}$$

We denote that

$$\varphi_n^k = \begin{cases} \cos(x_n),\ k = 0, \\ f(t_k, x_n),\ 1 \le k \le N-1, \\ (-1+\tau)\cos(x_n),\ k = N, \end{cases}$$

and

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ .. \\ \varphi_n^{N-2} \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & c & b & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & c & b & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & c & b & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & c & b \\ 0 & -\frac{\tau}{2h^2} & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ f & e & d & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & f & e & d & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & f & e & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & e & d & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & f & e & d & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & f & e & d \\ -\frac{1}{\tau} & \left(\frac{1}{\tau} + \frac{\tau}{h^2}\right) & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$\begin{aligned} a &= g(t_{k+1})\frac{\tau^2}{4h^4}, \\ b &= -g(t_{k+1})\frac{\tau^2}{h^4},\ c = -g(t_k)\frac{1}{h^2}, \\ d &= g(t_{k+1})\frac{6\tau^2}{4h^4} + \frac{1}{\tau^2},\ e = g(t_{k+1})\frac{2}{h^2} - \frac{2}{\tau^2},\ f = \frac{1}{\tau^2}, \end{aligned}$$

and

$$D = B,\ E = A,$$

$$R = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ .. \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad \text{where } s = n-2,\ n-1,\ n,\ n+1,\ n+2.$$

Then

$$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n,$$

we can write

$$\begin{cases} A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \quad 2 \le n \le M-2, \\ 10U_0 = 15U_1 - 6U_2 + U_3,\ \gamma_0 U_0 = \theta_1 U_1 + T_1, \\ 3U_M = 4U_{M-1} - U_{M-2}, 10U_M = 15U_{M-1} - 6U_{M-2} + U_{M-3}. \end{cases}$$

So, we have the fourth order difference equation with respect to n with matrix coefficients. To solve this difference equation we have applied the modified Gauss elimination method for difference equation with respect to n with matrix coefficients. Hence, we seek a solution of the matrix equation in the following form

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}, \quad n = M-2, \cdots, 2, 1, 0. \tag{4.15}$$

Using this formula and the equality

$$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, n = M-2, \cdots, 2,$$

we can write

$$[A + C\beta_{n+1} + D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1}]U_{n+2}$$

$$+[B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n + E\alpha_{n-1}\alpha_n\alpha_{n+1} + E\alpha_{n-1}\beta_n + E\beta_{n-1}\alpha_{n+1}]U_{n+1}$$

$$+C\gamma_{n+1} + D\alpha_n\gamma_{n+1} + D\gamma_n + E\alpha_{n-1}\alpha_n\gamma_{n+1} + E\alpha_{n-1}\gamma_n + E\beta_{n-1}\gamma_{n+1} + E\gamma_{n-1} = R\varphi_n.$$

The last equation is satisfied if we select

$$\begin{cases} A + C\beta_{n+1} + +D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1} = 0, \\ B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n + E\alpha_{n-1}\alpha_n\alpha_{n+1} + E\alpha_{n-1}\beta_n + E\beta_{n-1}\alpha_{n+1} = 0, \\ C\gamma_{n+1} + D\alpha_n\gamma_{n+1} + D\gamma_n + E\alpha_{n-1}\alpha_n\gamma_{n+1} + E\alpha_{n-1}\gamma_n + E\beta_{n-1}\gamma_{n+1} + E\gamma_{n-1} = R\varphi_n. \end{cases}$$

From that it follows

$$\begin{cases} \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \\ \gamma_{n+1} = (C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}), \end{cases} \tag{4.16}$$

where $n = 2, \cdots, M-2$. Note that for obtaining $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$, $n = 1, \cdots, M-1$, we need to find α_1, β_1, γ_1 and α_2, β_2, γ_2. We can find them from $U_0 = \alpha_1 U_1 + \beta_1$. Using the formula $\gamma_0 U_0 = \theta_1 U_1 + T_1$, we obtain

$$U_0 = inv(\gamma_0)\theta_1 U_1 + inv(\gamma_0)T_1,$$

where

$$\alpha_1 = inv(\gamma_0)\theta_1,\ \beta_1 = \sigma, \gamma_1 = inv(\gamma_0)T_1. \tag{4.17}$$

Using the formulas

$$10U_0 = 15U_1 - 6U_2 + U_3,$$

$$U_0 = \alpha_1 U_1 + \beta_1 U_2 + \gamma_1,$$

$$U_1 = \alpha_2 U_2 + \beta_2 U_3 + \gamma_2,$$

we obtain

$$\begin{cases} \alpha_2 = (10\alpha_1 - 15I)^{-1}(-6I - 10\beta_1), \\ \beta_2 = (10\alpha_1 - 15I)^{-1}, \\ \gamma_2 = (10\alpha_1 - 15I)^{-1}(-10\gamma_1). \end{cases} \tag{4.18}$$

Now we will find u_n, $0 \le n \le M$, by the formula (4.7), But, for this we need to find u_M and u_{M-1}.We can find u_M and u_{M-1} from

$$\begin{cases} 3U_M = 4U_{M-1} - U_{M-2}, \\ 10U_M = 15U_{M-1} - 6U_{M-2} + U_{M-3}, \\ U_{M-2} = \alpha_{M-1}U_{M-1} + \beta_{M-1}U_M + \gamma_{M-1}, \\ U_{M-3} = \alpha_{M-2}U_{M-2} + \beta_{M-2}U_{M-1} + \gamma_{M-2}. \end{cases}$$

Solving this system, we obtain

$$U_M = \left[(4I - \alpha_{M-1})^{-1}(3I + \beta_{M-1}) - (9I - \beta_{M-2} - 4\alpha_{M-2})^{-1}(8I - 3\alpha_{M-2})\right]^{-1}$$

$$\times[-(4I - \alpha_{M-1})^{-1}\gamma_{M-1} + (9I - \beta_{M-2} - 4\alpha_{M-2})^{-1}\gamma_{M-2}], \tag{4.20}$$

$$U_{M-1} = \left[(3I + \beta_{M-1})^{-1}(-4I + \alpha_{M-1}) + (8I - 3\alpha_{M-2})^{-1}(9I - \beta_{M-2} - 4\alpha_{M-2})\right]^{-1}$$

$$\times[-(3I + \beta_{M-1})^{-1}\gamma_{M-1} + (8I - 3\alpha_{M-2})^{-1}\gamma_{M-2}] \tag{4.21}$$

Thus using formulas (4.15) and (**??**), (**??**), we can compute u_n, $0 \le n \le M$. We can summarize the computation procedure by the following algorithm

Algorithm

1. **Step** Set input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$.
2. **Step** Use the second order of accuracy difference scheme and write in matrix form

$$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \quad 2 \le n \le M-2.$$

3. **Step** Determine the entries of the matrices A, B, C, D, E and R.
4. **Step** Find α_1, β_1, γ_1 by the formula (4.17).
5. **Step** Find α_2, β_2, γ_2 by the formula (4.18).
6. **Step** Compute $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$, $n = 2, \cdots, M-1$ by the formula (4.16).
7. **Step** Compute U_M by the formula (**??**).
8. **Step** Compute U_{M-1} by the formula (**??**).
9. **Step** Compute $U_n, n = M-2, \cdots, 1, 0$ by the formula (4.15).

Matlab Implementation on the Second Order of Accuracy Difference Scheme Generated by Five Points

```
function ssemir(N,M)
    time1=cputime;
    if nargin<1; N=20; M=20; end;
    close; close; close; close;
    h=pi/M; tau=1/N;
    t=0:tau:1;
    A(N+1,N+1)=0;
    for k=1:N-1;
    a(k)=(g(tau*(k+1))*(tau^2))/(4*(h^4));
    A(k+1,k+2)=a(k);
    end;
    B(N+1,N+1)=0;
    for k=1:N-1;
    c(k)=-g(tau*(k))/((h^2));
    b(k)=-(g(tau*(k+1))*(tau^2))/(h^4) ;
    B(k+1,k+1)=c(k); B(k+1,k+2)=b(k);
    end;
    B(N+1,2)=(-tau)/(2*(h^2));
    C(N+1,N+1)=0;
    for k=1:N-1;
    f(k)= 1/(tau^2);
    e(k)=(-2/(tau^2))+2*g(tau*k)/(h^2);
    d(k)=(1/(tau^2))+(g(tau*(k+1) ) *(6*(tau^2)))/(4*(h^4));
    C(k+1,k) = f(k); C(k+1,k+1)= e(k); C(k+1,k+2)= d(k);
    end;
    C(1,1)=1; C(N+1,1)=(-1/tau); C(N+1,2)=(1/tau)+((tau)/(h^2));
    D=B; E=A;
    for i=1:N+1;
    F(i,i)=1;
    end;
    for j=1:M-1;
    for k=1:N-1;
    t=k*tau; x=j*h;
    fii(k+1,j:j)=2*(exp(-t))*cos(x);
    end;
```

```
end;
for j=1:M-1;
x=j*h;
fii(1,j:j)= cos(x); % right side for k=0
fii(N+1,j:j)=(-1+tau)*cos(x);% right side for k=N
end;
lamda0=(h^2)/(g(0)*2*(tau^2));
lamdaN=(h^2)/(g(1)*2*(tau^2));
for k=1:N-1;
tk=k*tau;
lamda(k)=(h^2)/(2*(tau^2)*g(tk));
end;
for k=2:N;
gama(k,k)=1;
end;
gama(1,1)=1+2*lamda0;
gama(1,2)=-5*lamda0;
gama(1,3)=4*lamda0;
gama(1,4)=-lamda0;
gama(N+1,N+1)=1+2*lamdaN;
gama(N+1,N)=-5*lamdaN;
gama(N+1,N-1)=4*lamdaN;
gama(N+1,N-2)=-lamdaN;
T(N+1,1)=(tau^2)*lamdaN * ( 2*exp(-1) );
T(1,1)=(tau^2)*lamda0*2;
for k=1:N-1;
tk=k*tau;
T(k+1,1)=+(tau^2)*lamda(k)*(2*exp(-tk)*cos(h));
end;
ro(1,1)=1;
ro(N+1,N+1)=1;
for k=1:N-1;
tk=k*tau;
ro(k+1,k)=-lamda(k);
ro(k+1,k+1)=1+2*lamda(k);
ro(k+1,k+2)=-lamda(k);
end;
zalpha=inv(gama)*ro;
```

```
zbetha=inv(gama)*T;
eskialpha{1}=zalpha;
eskibetha{1}=zbetha;
I=eye(N+1,N+1);
alpha{1}=eskialpha{1};
betha{1}=zeros(N+1,N+1);
dama{1}=eskibetha{1};
alpha{2}=inv( 10*alpha{1}-15*I ) * ( -6*I-10*betha{1} );
betha{2}=inv( 10*alpha{1}-15*I ) ;
dama{2}= inv( 10*alpha{1}-15*I )*(-10*dama{1}) ;
for n = 2:M-2 ; mat = inv(C + D*alpha{n} +...
E*betha{n-1}+ E*alpha{n-1}*alpha{n}) ;
alpha{n+1} = - mat *(B +D*betha{n}+ E *   alpha{n-1}*betha{n} ) ;
betha{n+1} = - mat*A ; dama{n+1} = -mat*(-F*fii(:,n)+D*dama{n}+...
E*alpha{n-1}* dama{n}+E*dama{n-1} ) ;
end;
I=eye(N+1,N+1);
U{M}=inv( inv(4*I-alpha{M-1})*(3*I+betha{M-1})-...
inv(9*I-betha{M-2}-4*alpha{M-2})*(8*I-3*alpha{M-2}))*...
(-inv(4*I-alpha{M-1})*(dama{M-1}) + ...
inv(9*I-betha{M-2}-4*alpha{M-2})*dama{M-2});
U{M-1}= inv( inv( 3*I+betha{M-1})*(-4*I + alpha{M-1} )+...
inv( 8*I-3*alpha{M-2} )* ( 9*I-betha{M-2}-4*alpha{M-2} ))*...
( -inv( 3*I+betha{M-1})*(dama{M-1} )+...
inv( 8*I-3*alpha{M-2} )* ( dama{M-2} ) );
for n = M-2:-1:1 ; U{n}= alpha{n+1}*U{n+1}+...
betha{n+1}*U{n+2}+dama{n+1}; end;
U0= alpha{1}*U{1}+ betha{1}*U{2}+dama{1};
for z = 1 : M ; p(:,z+1)=U{z}; end; p(:,1)=U0;
time2=cputime;
time2-time1
%%%%%%%%  EXACT SOLUTION OF THIS PDE  %%%%%%%%
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j:j)=exp(-t)*cos(x);
end;
```

```
end;
%%%%%%%%%% END EXACT SOLUTION %%%%%%%%%%%
%%%%%%%%%%% ERROR ANALYSIS %%%%%%%%%%%%%
abs(es-p)
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=maxerror/maxapp;
cevap = [maxes,maxapp,maxerror,relativeerror]
%%%%%%%%%%% GRAPH OF THE SOLUTION %%%%%%%%%%
[xler,tler]=meshgrid(0:h:pi,0:tau:1);
table=[es;p];
table(1:2:end,:)=es;
table(2:2:end,:)=p;
q=min(min(table)); w=max(max(table));
figure;
surf(xler,tler,es);
title('EXACT SOLUTION');
set(gca,'ZLim',[q w]);
rotate3d;XLabel('x axis');
YLabel('t axis'); figure; surf(xler,tler,p);
title('SEC.ORD. APP. SOL. GEN. BY 5 POINTS');
rotate3d ; set(gca,'ZLim',[q w]);
XLabel('x axis');YLabel('t axis');
%%%%%%%%%%%%%%% END GRAPH %%%%%%%%%%%%%%%
%%%%%%%%%%%%%% SUB FUNCTIONS %%%%%%%%%%%%%
function gt=g(t)
gt=1;
%function ftx=f(t,x)
%ftx=2*(exp(-t))*cos(x);
%function exact=exf(t,x)
%ftx=2*(exp(-t))*cos(x);
```

4.4 COMPARISON OF THE RESULTS

Now, we will give the results of the numerical analysis. The exact and numerical solutions are given in the figures 4.1, 4.2, 4.3 and 4.4 for $N = M = 20$ time and space intervals.

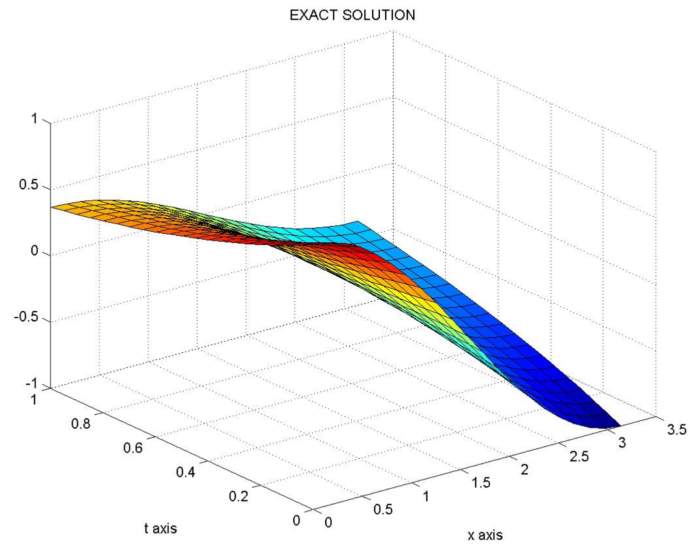

Figure 4.1:

Exact Solution

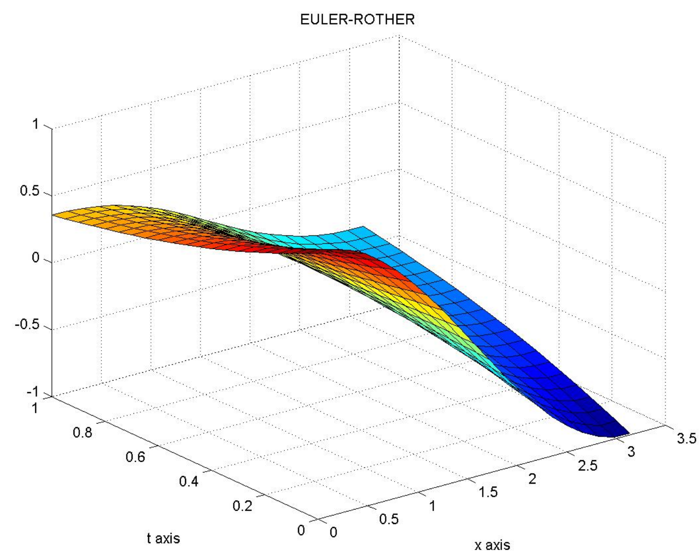

Figure 4.2:

Solution by the first order of accuracy difference scheme.

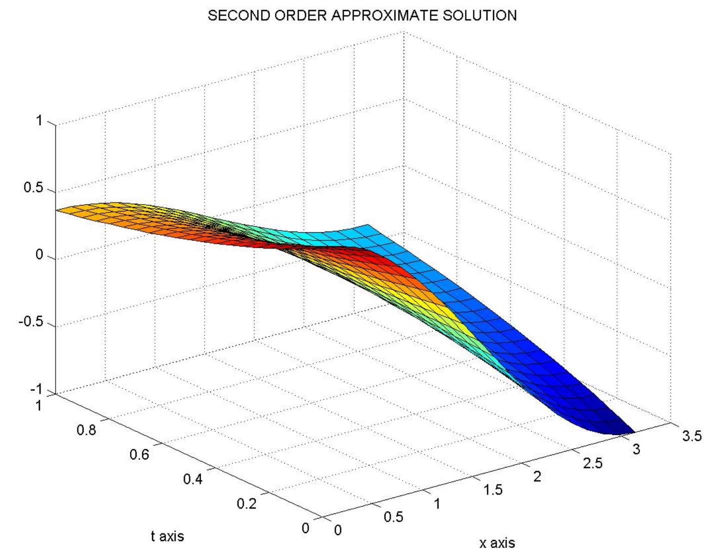

Figure 4.3:

Solution by the second order of accuracy difference schemes (4.11).

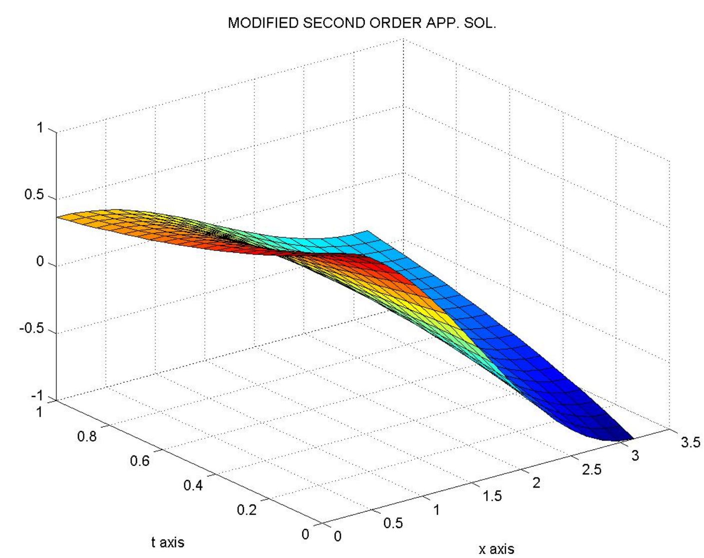

Figure 4.4:

Solution by the second order of accuracy difference scheme (4.14).

Appearntly all the waveforms seem equivalent. For their comparison, the errors computed by

$$E_M^N = \max_{1 \le k \le N-1, 1 \le n \le M-1} \left| u(t_k, x_n) - u_n^k \right|$$

of the numerical solutions and the computer cpu time are recorded for different values of $M = N$, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). The result are shown in Tables 4.1, 4.2, 4.3 and 4.4 for $N = M = 20$, 93, 200 and 300 respectively. The simulation result are obtained by a PC Pentium (R) 4CPV, 3.00 6Hz, 2.99 6Hz, 512 MB of RAM.

Table 4.1. Comparison of the errors and cpu times of different difference schemes for $N = M = 20$.

Difference schemes	E_M^N	$N\ CPUtimes(s)$
The first order of accuracy difference scheme (4.4)	0.0143	0.27
The second order of accuracy difference scheme (4.11)	0.0013	0.23
The second order of accuracy difference scheme (4.14)	0.0039	0.26

Table 4.2. Comparison of the errors and cpu times of different difference schemes for $N = M = 93$.

Difference schemes	E_M^N	N *CPU times* (s)
The first order of accuracy difference scheme (4.4)	0.0039	0.8
The second order of accuracy difference scheme (4.11)	0.0001	0.8
The second order of accuracy difference scheme (4.14)	0.0017	1.1

Table 4.3. Comparison of the errors and cpu times of different difference schemes for $N = M = 200$.

Difference schemes	E_M^N	N *CPU times* (s)
The first order of accuracy difference scheme (4.4)	0.0019	14
The second order of accuracy difference scheme (4.11)	0.0000	11
The second order of accuracy difference scheme (4.14)	0.0005	13

Table 4.4. Comparison of the errors and cpu times of different difference schemes for $N = M = 300$.

Difference schemes	E_M^N	N *CPU times* (s)
The first order of accuracy difference scheme (4.4)	0.0013	245
The second order of accuracy difference scheme (4.11)	0.0000	34
The second order of accuracy difference scheme (4.14)	0.0003	62

All recorded cpu times are on the arrange base with an error less than ∓ 15 %, which does not effect the conclusions stated below basically.

Comparison of the reslut in these tables reveal the following factors.

i) For $N = M = 20$, although the cpu times of all three difference schemes are more or less equal the second order of accuracy difference schemes produces $0.0143/0.0013 \cong 11$ times smaller error than the first order of accuracy difference scheme. For the second order of accuracy difference scheme generated by 5 points this ratio reduces to $0.0143/0.0039 \cong 3.6$ times.

ii) To have the same accuracy of computation of the second order of accuracy difference scheme generated by 5 points and $N = M = 20$, the first order of accuracy difference scheme needs $N = M = 93$ intervals and $0.8/0.26 \cong 3.07$ times larger cpu time.

iii) To have the same accuracy of computation of the second order of accuracy difference scheme generated by 3 points with $N = M = 20$, the first order of accuracy difference scheme needs $N = M = 300$ intervals and $245/0.23 \cong 1065$ times larger cpu time.

iv) All cpu times exceed 1s after aporximately $N = M = 100$.

v) CPU time for the first order of accuracy difference scheme increases drantically for large values of $N = M$ is much larger that needed for the second order of accuracy difference schemes.

CHAPTER 5

CONCLUSIONS

This work is devoted to study the stability of the difference schemes for the approximate solutions of the initial value problem for hyperbolic equations with t dependent coefficients. The following original results are obtained:

- The second order of accuracy difference schemes for the approximate solutions of the initial value problems for hyperbolic differential equations in a Hilbert space are presented.
- The theorems on the stability estimates for the solution of these difference schemes and its diference derivatives are established.
- The Matlab implementation of these difference schemes generated by three and five points are presented.
- The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

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