APPLICATION OF MATRIX ALGEBRA TO ECONOMICS

by

İbrahim KOCAOĞUL

June 2005

APPLICATION OF MATRIX ALGEBRA TO ECONOMICS

by

İbrahim KOCAOĞUL

A thesis submitted to

the Graduate Institute of Sciences and Engineering

of

Fatih University

in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

June 2005
Istanbul, Turkey

APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

____

Assit. Prof. Dr. Ali Şahin
Head of Department

This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

____

Assit. Prof. Dr. Afgan Aslanov
Supervisor

Examining Committee Members

Assit. Prof. Dr. Afgan Aslanov ______________________

Assit. Prof. Dr. Ali Şahin ______________________

Assit. Prof. Dr. Yüksel Köseoğlu ______________________

It is approved that this thesis has been written in compliance with the formatting rules laid down by the Graduate Institute of Sciences and Engineering.

____

Assit. Prof. Dr. Nurullah Arslan
Director

Date
2005

APPLICATION OF MATRIX ALGEBRA TO ECONOMICS

İbrahim KOCAOĞUL

M. S. Thesis - Mathematics
June 2005

Supervisor: Assist. Prof. Dr. Afgan ASLANOV

ABSTRACT

This paper discusses an improvement of some concepts of the Leontief's Input-Output Analysis, when some industries are expected to make a serious loss. If one industry, at the end of the estimation period, has serious losses so that could not even cover its expenses (say, due to unexpected exchange rates), the standard Leontief Model will not describe this process properly. The mathematical base established here would be useful to improve the applicability of input-output system for practical goals.

Firstly, application of that method has been tried to be shown basically by taking into account three industries in that study. It has been shown how an economy may be considered profitable even in the case of that one of those three industries has lost. Then, that model has been applied to the n industries by expanding those three industries. Eventually, it has been concluded that even if some industries have lost money in economy, that economy may be considered profitable if some industries covers the loss of other industries.

Keywords: Leontief model, input-output system.

MATRİS CEBİRİN EKONOMİ ÜZERİNE UYGULAMASI

İbrahim KOCAOĞUL

Yüksek Lisans Tezi – Matematik
Haziran 2005

Tez Yöneticisi: Yard. Doç. Dr. Afgan ASLANOV

ÖZET

Bu çalışma bazı endüstrilerin ciddi zarara uğraması beklendiği durumlarda Leontief input-output modeline ait bazı kavramların geliştirilmesini tartışıyor. Eğer bir endüstri, tahmin sürecinin sonunda masraflarını bile karşılayamayacak kadar ciddi zarara uğramışsa (mesela beklenmeyen kur oranlarından dolayı), standart Leontief modeli bu süreci tam olarak açıklayamayacaktır. Burada ortaya konan matematiksel temel input-output sisteminin uygulanabilirliğini, pratik amaçlar için, geliştirmede faydalı olacaktır.

İlk olarak, bu çalışmada üç endüstri ele alınarak basit bir şekilde bu metodun uygulaması gösterilmeye çalışıldı. Üç endüstriden birinin kaybetmesi durumunda bile bir ekonominin nasıl kar eden bir ekonomi olarak görülebileceği gösterildi. Daha sonra bu model n endüstriye genişletilerek uygulandı. Sonuç olarak şuna varıldı: Ekonomide bazı endüstrilerin zarar etmesine rağmen, bazı endüstrilerin kaybının diğerleri tarafından karşılandığı durumlarda bu ekonomi kar eden bir ekonomi olarak görülebilir.

Anahtar Kelimeler: Leontief modeli, input-output sistemi.

ACKNOWLEDGEMENT

First of all, I would like to thank my thesis supervisor Afgan ASLANOV. His motivation and will power kept me doing all the work .He has been a great model for me, who deserve to be followed academically and ethically.

I know that, that thesis is a product of my all instructors throughout all my education so I would like to present my deepest gratitude to all of them.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

This thesis is about a model of an economy with n industries. We will study a linear system which is called an open Leontief system after Wassily Leontief, who first studied this type of system in the 1930s and later won a Nobel Prize in economics for his work [1].

Linear models of production are perhaps the simplest production models to describe. Here, we will describe the simplest of linear models [2]. We will suppose that our economy has n goods. Each of goods 1 through n is produced by one production process. A production process is simply a list of amounts of goods: so much of good 1, so much of good 2, and so on. These quantities are the amounts of input needed to produce one unit of process's output. For example, the making of one car requires so much steel, so much plastic, so much electricity, and so forth. In fact, some production processes, such as those steel or automobiles, use some of their own output to aid in subsequent production.

The simplicity of the linear production model is due to two facts. First, in these models, the amounts of inputs needed to produce two automobiles are exactly twice those required for the production of one automobile. Three cars require 3 times as much of the inputs, and so on. In the jargon of microeconomics, each production process exhibits constant returns to scale. The production of 2, 3 or k cars requires 2, 3 or k times the amounts of inputs required for the production of 1 car. Second, in these models there is only one way to produce a car. Output cannot be increased by using more of any factor alone; more of all the factors are needed, and always in the same proportions. This simplifies the analysis of production problems, because the optimal input mix for the production of, say, 1000 cars, does not have to be computed. It is simply 1000 times the optimal input mix required for the production of 1 car.

After giving that information on linear model we can deal with Leontief model that we will study. Let's describe some notations we use. Denote c_i with the consumer demand for good *i*. We want each process to produce an output that is sufficient to meet both consumer demand and the input requirements of *n* industries. For our simple linear economy, this is the law of supply and demand: output produced must be used in production or in consumption. Let x_j denote the amount of output produced by process *j*. If process *j* produces x_j units of output, it will need $a_{ij}x_j$ units of good *i*. Adding these terms up over all the industries gives the demand for good *i*: $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + c_i$. The law of supply and demand then requires

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + c_i . \tag{1.1}$$

for $i = 1,2,...,n$. In matrix notation, this system of equations becomes

$$X = AX + C \tag{1.2}$$

which is more conveniently written as

$$(I - A)X = C. \tag{1.3}$$

The matrix $A = (a_{ik})$ for $i,k = 1,2,...,n$ of intermediate factor demands is sometimes called the technology matrix. The $(i, j)th$ entry a_{ij} of technology matrix *A* indicates how many millions of dollars of good *i* are needed to produce 1 million dollars of good *j*. It is convenient to study solutions of equation by working with the inverse:

$$X = (I - A)^{-1}C. \tag{1.4}$$

In addition to requiring that *I-A* be invertible, we also require that the solution to the equation be nonnegative whenever *C* is nonnegative. If we expect each industry to make a positive profit, the sum of entries in each column expected to be less than one. Now, we consider the case when sum of the entries in some columns may be greater than or equal to 1. Economically, it could be explained by this way: We expect some industries that may have serious losses and/or some of them may be expected to be non profitable (for instance, let one of them produces a product for military forces). Observations on the last period for the economy of Turkey shows that it really may be happened. At the beginning of the second quarter of 2003, the course of 1 dollar was around 1750000 TL (Turkish Lira) but at the end of the same quarter, 1 dollar was around 1350000 TL. It is clear that in this case, some industries really had serious losses (at least in terms of dollars) and no estimation could be made using standard Leontief's input-output model.

Thus, we will study on that problem. In that thesis we will approach to new assumptions on that model. That problem could be seen in developing countries whose economies are not stable.

CHAPTER 2

THE ANALYSIS OF THE STANDARD CASE

Before undertaking an abstract analysis, we will work out an example to illustrate the key features of the model. Consider the economy of an organic farm which produces two goods: corn and fertilizer. Corn is produced using corn (to plant) and using fertilizer. Fertilizer is made from old corn stalks (by feeding the corn to cows who then produce useful end products). Suppose that the production of 1 ton of corn requires as inputs 0.1 ton of corn and 0.8 ton of fertilizer. The production of 1 ton of fertilizer requires no fertilizer and 0.5 ton of corn.

We can describe each of the two production processes by pairs of numbers (a,b), where a represents the corn input and b represents the fertilizer input. The corn production process is described by the pair of numbers (0.1, 0.8). The fertilizer production process is described by the pair of numbers (0.5, 0).

The most important question to ask of this model is: What can be produced for consumption? Corn is used both in the production of fertilizer. Fertilizer is used in the production of corn. Is there any way of running both process so as leave some corn and some fertilizer for individual consumption? If so, what combinations of corn and fertilizer for consumption are feasible?

Answers to these questions can be found by examining a particular system of linear equations. Suppose the two production processes are run so as to produce x_c tons of corn and x_f tons of fertilizer. The amount of corn actually used in the production of corn is $0.1\,x_c$ - the amount of corn needed per ton of corn output times the number of tons to be produced. Similarly, the amount of corn used in the production of fertilizer is $0.5\,x_f$. The amount of corn left over for consumption will be the total amount produced minus the amounts used for production of corn and fertilizer: $x_c - 0.1x_c - 0.5x_f$, or $0.9x_c - 0.5x_f$ tons. The amount of fertilizer needed in production is $0.8x_c$ tons. Thus the amount left over for consumption is $x_f - 0.8x_c$ tons.

Suppose we want our farm to produce for consumption 4 tons of corn and 2 tons of fertilizer. How much total production of corn and fertilizer will be required? Put another way, how much total production of corn and fertilizer will the farm have to produce in order to have 4 tons of corn and 2 tons of fertilizer left over for consumers? We can answer this question by solving the pair of linear equations

$$0.9x_c - 0.5x_f = 4,$$

$$-0.8x_c + x_f = 2.$$

This system is easily solved. Solve the second equation for x_f in terms of x_c:

$$x_f = 2 + 0.8x_c \qquad (2.1)$$

Substitute this expression for x_f into the first equation:

$$0.9x_c - 0.5(0.8x_c + 2) = 4$$

and solve for x_c :

$$0.5\ x_c = 5, \text{ so } x_c = 10.$$

Finally, substitute $x_c = 10$ back into (2.1) to compute

$$x_f = 0.8 \cdot 10 + 2 = 10.$$

In the general case, the production process for good *j* can be described by a set of input-output coefficients $\{a_{1j}, a_{2j}, \ldots, a_{nj}\}$, where a_{ij} denotes the input of good *i* needed to output one unit of good *j*. Keep in mind that the first subscript stands for the input good and the second stands for the output good. The production of x_j units of good *j* requires $a_{1j}x_j$ units of good 1, $a_{2j}x_j$ units of good 2, and so on.

Total output of good i must be allocated between production activities and consumption. Denote by c_i the consumer demand for good i. This demand is given exogenously, which is to say that it is not solved for in the model. An n-tuple $(c_1, c_2, \ldots, c_n)$ is said to be an admissible n-tuple of consumer demands if all c_i's are nonnegative. We want each process to produce an output that is sufficient to meet both consumer demand and the input requirements of n industries. For our simple linear economy, this is the law of supply and demand: output produced must be used in production or in consumption. Let x_j denote the amount of output produced by process j. If process j produces x_j units of output, it will need $a_{ij}x_j$ units of good i. Adding these terms up over all the industries gives the demand for good i: $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + c_i$. The law of supply and demand then requires

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + c_i. \qquad (2.2)$$

It is convenient to rearrange this equation to say that consumer demand must equal gross output less the amount of the good needed as an input for the production processes. For good 1, this says

$$(1 - a_{11})x_1 - a_{12}x_2 - \ldots - a_{1n}x_n = c_i. \qquad (2.3)$$

The analogous equation for good i is

$$-a_{i1}x_1 - \ldots - a_{i,i-1}x_{i-1} - (1 - a_{ii})x_i - a_{i,i+1}x_{i+1} - \ldots - a_{in}x_n = c_i. \qquad (2.4)$$

This leads to the following system of n equations in n unknowns, which summarizes the equilibrium output levels for the entire n-industry economy:

$$\begin{aligned} (1 - a_{11})x_1 - a_{12}x_2 - \ldots - a_{1n}x_n &= c_1, \\ -a_{21}x_1 + (1 - a_{22})x_2 - \ldots - a_{2n}x_n &= c_2, \\ &\ldots \\ -a_{n1}x_1 - a_{n2}x_2 - \ldots - (1 - a_{nn})x_n &= c_n. \end{aligned} \qquad (2.5)$$

This linear system is called an open Leontief system. It is said to be open because the demand $c_1, c_2, \ldots, c_n$ is exogenously given, while the supply of goods is endogenously determined, that is, the demand is determined by the equations under study. In this system of equations, the a_{ij}'s and the c_i's are given and we must solve for the x_i's, the gross outputs of the industries.

In matrix notation, the system of equations (2.5) becomes

$$X = AX + C, \tag{2.6}$$

which is more conveniently written as

$$(I - A)X = C. \tag{2.7}$$

The matrix A of intermediate factor demands is sometimes called the technology matrix. Thus, it is convenient to study solutions to (2.7) by working with the inverse:

$$X = (I - A)^{-1}C. \tag{2.8}$$

Notice that in addition to requiring that $I - A$ be invertible, it is also required that the solution to (2.7) be nonnegative whenever C is nonnegative. This corresponds to the requirement that any solution to our economic system produces nonnegative amounts of each commodity. For this happen, all entries of the matrix $(I - A)^{-1}$ must be nonnegative. Furthermore, the study of this system is complicated by the fact that all economic data in the model are contained in the matrix A. It is not enough simply to assume that $I - A$ has a nonnegative inverse. We must find assumptions on A which will imply the desired behavior of $I - A$.

Since the factors of production have different natural units, it is convenient to express them all in monetary terms, say in millions of dollars, in an input-output analysis. In this case, the (i, j)th entry a_{ij} of technology matrix A indicates how many million dollars of good i are needed to produce 1 million dollars of good j. The sum of the entries in each column of A gives the total cost of producing 1 million dollars of the product that column represents.

Since we expect each industry to make a positive accounting profit, the sum of the entries in each column should be less than 1.

Theorem 2.1 Let A be a $n \times n$ matrix with the properties that each entry is nonnegative and the sum of the entries in each column is less than 1. Then, $(I - A)^{-1}$ exists and contains only nonnegative entries [3, 4].

Proof We conclude this section by proving Theorem 2.1. Let A be a technology matrix that satisfies the hypotheses of Theorem 2.1: nonnegative entries and each column sums less than 1. Then, -A has all its entries and its each column sums between 0 and -1 and $I-A$ satisfies the following three properties:

(a) each off-diagonal entry is less than or equal to zero,

(b) each diagonal entry is positive, and

(c) the sum of the entries in each column is positive.

Matrices which satisfy these three conditions are a special case of the class of dominant diagonal matrices. General definition of a dominant diagonal matrix requires that in each column the absolute value of the diagonal entry is at least as large as the sum of the absolute values of the other entries in that column. To prove Theorem 2.1, we need only prove the following result.

Theorem 2.2 Let B be square matrix which satisfies conditions *a,b*, and *c* above. Then, all entries of B^{-1} are nonnegative.

Proof To keep better track of the signs and sizes of the entries of the matrix B, we write it as

$$B=\begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & b_{nn} \end{pmatrix}, \tag{2.9}$$

where each $b_{ij}\geq 0$ and $0\leq \sum_{h\neq j} b_{hj} < b_{jj}$ for all j.

Let C be a vector with all positive entries and consider the system $BX=C$. To solve the system, we perform Gaussian elimination on the augmented matrix [B | C]. Add b_{j1}/b_{11} times row 1 to row j for all $j>1$. The result is the new augmented matrix

$$\begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} & | & c_1 \\ 0 & b_{22}-\frac{b_{21}}{b_{11}}b_{12} & \cdots & -b_{2n}-\frac{b_{21}}{b_{11}}b_{1n} & | & c_2+\frac{b_{21}}{b_{11}}c_1 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & -b_{n2}-\frac{b_{n1}}{b_{11}}b_{12} & \cdots & b_{nn}-\frac{b_{n1}}{b_{11}}b_{1n} & | & c_n+\frac{b_{n1}}{b_{11}}c_1 \end{pmatrix} \tag{2.10}$$

$$\equiv \begin{pmatrix} b_{11} & * & | & c_1 \\ 0 & \bar{B} & | & \bar{c} \end{pmatrix}. \tag{2.11}$$

The $(n-1)\times(n-1)$ matrix $\bar{B}$ is still dominant diagonal, since its off-diagonal entries are still nonpositive and the sum of the entries in its $(j-1)st$ column is

$$\left(b_{jj} - \frac{b_{j1}}{b_{11}} b_{1j}\right) + \sum_{h\neq 1,j}\left(-b_{hj} - \frac{b_{h1}}{b_{11}} b_{1j}\right) = b_{jj} - \left(\sum_{h\neq 1,j} b_{hj}\right) - b_{1j}\frac{b_{21}+\ldots+b_{n1}}{b_{11}}$$

$$> b_{jj} - \sum_{h\neq 1,j} b_{hj} - b_{1j}$$

$$> 0.$$

The new right hand side $\bar{c}$ has all entries positive. Continue applying Gaussian elimination; at each stage, the resulting submatrix still satisfies *a,b,* and *c*. We conclude that the row echelon form of $[B \mid C]$ has sign pattern

$$\begin{pmatrix} + & - & - & \cdots & - & | & + \\ 0 & + & - & \cdots & - & | & + \\ 0 & 0 & + & \cdots & - & | & + \\ \vdots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & 0 & 0 & + & | & + \end{pmatrix} \tag{2.12}$$

Back substitution from such a matrix yields a positive solution X to the system $BX = C$. If the nonzero right-hand side C had some zero entries and if A had some zero off-diagonal terms, the same argument yields a nonnegative solution of $BX = C$. The entries of B^{-1} are all nonnegative numbers.

CHAPTER 3

THE GENERALIZATION OF LEONTIEF MODEL IN CASE N=3

Now, we are coming the basic matter of our thesis. Until now, we have studied on the sum of the entries in each column is less than 1. From now on, we will study on a different situation that is some of the sum of the entries in each column is not less than 1. In other words, some industries will not be able to make profit. On the contrary, they will lose. We will try to determine that general economy may be profitable in spite of heavy losses of some industries. We will examine that in basic way in 3×3 matrix.

There is an economy which is constituted of 3 industries. Two of those industries are making profit and the other is losing. We will examine whether the profit of the industry 1 is covering the loss of the industry 3. If we are able to demonstrate that covering, we will prove that the economy which is constituted of 3 industries a profitable one.

Definition 3.1 Matrix *A* is a 3×3 matrix with the properties that each entry is nonnegative and sum of the entries in each column is less than1 except 3rd column. The sum of the entries in 3rd column may be greater than or equal to 1 or less than or equal to1.
We can write the matrix *A* as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{3.1}$$

Then, -*A* has all its entries and its column sums is less than 0 and $I - A$ satisfies the following properties:

a) each off-diagonal entry is less than or equal to zero,
b) each diagonal entry is positive ,
c) the sum of the entries in each column, except 3rd column is positive. The sum of the entries in 3rd column may be greater than or equal to 0 or less than or equal to 0.

We can write the matrix $I - A$ as

$$\begin{pmatrix} 1-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1-a_{33} \end{pmatrix}. \tag{3.2}$$

Let's rearrange the matrix $I - A$ as the matrix

$$B = \begin{pmatrix} b_{11} & -b_{12} & -b_{13} \\ -b_{21} & b_{22} & -b_{23} \\ -b_{31} & -b_{32} & b_{33} \end{pmatrix}. \tag{3.3}$$

where each $b_{ij} \geq 0$ for $i, j = 1,2,3$

Definition 3.2 The matrix *B* has the following properties:

a) each off-diagonal entry is less than or equal to zero,
b) each diagonal entry is positive ,
c) the sum of the entries in each column except 3rd column is positive. The sum of the entries in 3rd column may be greater than or equal to 0 or less than or equal to 0. It can be also expressed in the following way

$$d_1 = b_{11} - b_{21} - b_{31} > 0,$$

$$d_2 = b_{22} - b_{12} - b_{32} > 0,$$

$$d_3 = b_{33} - b_{13} - b_{23},$$

d) $\begin{vmatrix} b_{11} & -b_{13} \\ d_1 & d_3 \end{vmatrix} > 0.$

Theorem 3.1 Let B be a 3×3 matrix which satisfies conditions *a,b,c,* and *d* above. Then, all entries of B^{-1} are nonnegative.

Proof To solve this system, we perform Gaussian elimination on the augmented matrix [B | C]. We will apply that method until the under parts of the diagonal are 0.

$$\to\begin{pmatrix} b_{11} & -b_{12} & -b_{13} & | & c_1 \\ 0 & b_{22}-\frac{b_{21}}{b_{11}}b_{12} & -b_{23}-\frac{b_{21}}{b_{11}}b_{13} & | & c_2+\frac{b_{21}}{b_{11}}c_1 \\ 0 & -b_{32}-\frac{b_{31}}{b_{11}}b_{12} & b_{33}-\frac{b_{31}}{b_{11}}b_{13} & | & c_3+\frac{b_{31}}{b_{11}}c_1 \end{pmatrix} \tag{3.4}$$

$$\begin{pmatrix} b_{11} & -b_{12} & -b_{13} & | & c_1 \\ 0 & b_{22}-\frac{b_{21}}{b_{11}}b_{12} & -b_{23}-\frac{b_{21}}{b_{11}}b_{13} & | & c_2+\frac{b_{21}}{b_{11}}c_1 \\ 0 & 0 & b_{33}-\frac{b_{31}}{b_{11}}b_{13}+\frac{b_{32}+\frac{b_{31}}{b_{11}}b_{12}}{b_{22}-\frac{b_{21}}{b_{11}}b_{12}}(-b_{23}-\frac{b_{21}}{b_{11}}b_{13}) & | & c_3+\frac{b_{31}}{b_{11}}c_1+\frac{b_{32}+\frac{b_{31}}{b_{11}}b_{12}}{b_{22}-\frac{b_{21}}{b_{11}}b_{12}}(c_2+\frac{b_{21}}{b_{11}}c_1) \end{pmatrix} \tag{3.5}$$

If we are able to show that the following expression is greater than 0, than we will show that all of the entries of B^{-1} are nonnegative.

$$b_{33}-\frac{b_{31}}{b_{11}}b_{13}+\frac{b_{32}+\frac{b_{31}}{b_{11}}b_{12}}{b_{22}-\frac{b_{21}}{b_{11}}b_{12}}(-b_{23}-\frac{b_{21}}{b_{11}}b_{13})\,. \tag{3.6}$$

We arrange that expression in another way in order to show that it is greater than 0. For instance:

$$(b_{22}-\frac{b_{21}}{b_{11}}b_{12})^{-1}\bullet\begin{vmatrix} b_{22}-\frac{b_{21}}{b_{11}}b_{12} & b_{23}+\frac{b_{21}}{b_{11}}b_{13} \\ b_{32}+\frac{b_{31}}{b_{11}}b_{12} & b_{33}-\frac{b_{31}}{b_{11}}b_{13} \end{vmatrix} > 0. \tag{3.7}$$

Here, $(b_{22} - \frac{b_{21}}{b_{11}} b_{12})^{-1}$ is positive. To show that the expression above is greater than 0, it will be enough to prove the following expression:

$$\begin{vmatrix} b_{22} - \frac{b_{21}}{b_{11}} b_{12} & b_{23} + \frac{b_{21}}{b_{11}} b_{13} \\ b_{32} + \frac{b_{31}}{b_{11}} b_{12} & b_{33} - \frac{b_{31}}{b_{11}} b_{13} \end{vmatrix} > 0. \tag{3.8}$$

As we we will use some determinant rules in our following processes, we write that rule:

Lemma Let's consider such a 2×2 matrix**:** $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. If $a, d > 0$ and $a > |b|$ and $d > |c|$, we reach that the determinant of that matrix is greater than 0.

If we show that are $b_{22} - \frac{b_{21}}{b_{11}} b_{12} > b_{32} + \frac{b_{31}}{b_{11}} b_{12}$ and $b_{33} - \frac{b_{31}}{b_{11}} b_{13} > b_{23} + \frac{b_{21}}{b_{11}} b_{13}$, we obtain the determinant (3.8) that is positive (from lemma).

$$\left(b_{22} - \frac{b_{21}}{b_{11}} b_{12} \right) - \left(b_{32} + \frac{b_{31}}{b_{11}} b_{12} \right) = \frac{1}{b_{11}} \left(\begin{vmatrix} b_{11} & -b_{12} \\ -b_{21} & b_{22} \end{vmatrix} - \begin{vmatrix} b_{11} & -b_{12} \\ b_{31} & b_{32} \end{vmatrix} \right) \tag{3.9}$$

$$= \frac{1}{b_{11}} \left(\begin{vmatrix} b_{11} & -b_{12} \\ -b_{21} - b_{31} & b_{22} - b_{32} \end{vmatrix} \right) \tag{3.10}$$

Here, we clearly determine that $b_{11} > -b_{21} - b_{31}$ and $b_{22} - b_{32} > -b_{12}$, therefore (from lemma) we obtain

$$\left(b_{22} - \frac{b_{21}}{b_{11}} b_{12} \right) - \left(b_{32} + \frac{b_{31}}{b_{11}} b_{12} \right) > 0.$$

Similarly,

$$\left(b_{33} - \frac{b_{31}}{b_{11}} b_{13} \right) - \left(b_{23} + \frac{b_{21}}{b_{11}} b_{13} \right) = \frac{1}{b_{11}} \left(\begin{vmatrix} b_{11} & -b_{13} \\ -b_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & -b_{13} \\ -b_{21} & -b_{23} \end{vmatrix} \right)$$

$$= \frac{1}{b_{11}} \left(\begin{vmatrix} b_{11} & -b_{13} \\ b_{11} & -b_{13} \end{vmatrix} + \begin{vmatrix} b_{11} & -b_{13} \\ -b_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & -b_{13} \\ -b_{21} & -b_{23} \end{vmatrix} \right) \tag{3.11}$$

By the property of the determinant we get,

$$\left(b_{33}-\frac{b_{31}}{b_{11}}b_{13}\right)-\left(b_{23}+\frac{b_{21}}{b_{11}}b_{13}\right)=\frac{1}{b_{11}}\left(\begin{vmatrix} b_{11} & -b_{13} \\ b_{11}-b_{21}-b_{31} & b_{33}-b_{13}-b_{23}\end{vmatrix}\right)$$

$$=\frac{1}{b_{11}}\left(\begin{vmatrix} b_{11} & -b_{13} \\ d_1 & d_3\end{vmatrix}\right) \quad \text{(by condition d)}$$

$$>0.$$

Finally, we proved that

$$b_{33}-\frac{b_{31}}{b_{11}}b_{13}+\frac{b_{32}+\frac{b_{31}}{b_{11}}b_{12}}{b_{22}-\frac{b_{21}}{b_{11}}b_{12}}\left(-b_{23}-\frac{b_{21}}{b_{11}}b_{13}\right)>0, \tag{3.12}$$

therefore, we showed that all of the entries of B^{-1} are nonnegative.

We conclude that the row echelon form of $[B \mid C]$ has sign pattern

$$\left(\begin{vmatrix} + & - & - & \mid & + \\ 0 & + & - & \mid & + \\ 0 & 0 & + & \mid & + \end{vmatrix}\right) \tag{3.13}$$

Back substitution from such a matrix yields a positive solution X to the system $BX=C$.

We examined the equation $X=B^{-1}C$ and obtained that the components of X are positive and the components of C are already greater than or equal to zero. We want to show that the entries of B^{-1} are all nonnegative numbers. For simplicity, we write matrix B^{-1} another form that is

$$B^{-1}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \tag{3.14}$$

The $X=B^{-1}C$ has the sign pattern:

$$\begin{pmatrix} + \\ + \\ + \end{pmatrix}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} + \\ + \\ + \end{pmatrix} \tag{3.15}$$

Indeed, for example, if the term in the first row and the second column is negative $(b<0)$ in B^{-1}, then taking $c_1=c_3=0$ and $c_2>0$ large enough, we can make the product of the first row of B^{-1} and C, the result is negative, which is impossible. Therefore, we showed that the entries of B^{-1} are all nonnegative numbers.

In conclusion, we showed that first industry's profit covered third industry's damage and the economy is profitable.

CHAPTER 4

THE MAIN THEOREMS (n IS AN ARBITRARY)

In this thesis, we want to generalize this subject over n-industries. Except for nth industry, all the other industries make profit; however, we do not have any exact information about nth industry. If it makes profit, we have shown that process above. If it does not make profit, we will examine that.

Definition 4.1 The matrix B is a $n \times n$ matrix with the following properties:

$$B = \begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & b_{nn} \end{pmatrix} \tag{4.1}$$

where $b_{ij} \geq 0$ for $i, j = 1,2,\ldots,n$ and let $d_k = b_{kk} - \sum_{h \neq k} b_{hk}$ for $k = 1,2,\ldots,n$. Then,

a) each off-diagonal entry is less than or equal to zero,

b) each diagonal entry is positive,

c) the sum of the entries in each column is positive except nth column. There is no exact information on the sum of the entries in the nth column. We can write this such a way that

$$d_1 > 0, d_2 > 0, \ldots, d_{n-1} > 0,$$

d) At least one of the following determinants are positive,

$$\begin{vmatrix} b_{kk} & -b_{kn} \\ d_k & d_n \end{vmatrix}, \begin{vmatrix} b_{kk} & -b_{km} & -b_{kn} \\ -b_{mk} & b_{mm} & -b_{mn} \\ d_k & d_m & d_n \end{vmatrix}, \dots, \begin{vmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n-1,1} & -b_{n-1,2} & \cdots & -b_{n-1,n} \\ d_1 & d_2 & \cdots & d_n \end{vmatrix}$$

for $k < m < n$.

Theorem 4.1 Let B a square matrix which satisfies conditions *a, b, c* and *d* above. Then, all entries of B^{-1} are nonnegative.

Proof Let us consider the case when the first two determinants in condition *d* are positive and for simplicity consider the case $k = 1$ (we replace the first industry instead of *k*-th industry). Let C be a vector with all positive entries and consider the system $BX=C$. In order to solve the system, we apply Gaussian elimination on the augmented matrix
$[B \mid C]$.

$$\begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} & c_1 \\ -b_{21} & b_{22} & \cdots & -b_{2n} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & b_{nn} & c_n \end{pmatrix} \to \tag{4.2}$$

$$\begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} & c_1 \\ 0 & b_{22} - \frac{b_{21}}{b_{11}} b_{12} & \cdots & -b_{2n} - \frac{b_{21}}{b_{11}} b_{1n} & c_2 + \frac{b_{21}}{b_{11}} c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -b_{n2} - \frac{b_{n1}}{b_{11}} b_{12} & \cdots & b_{nn} - \frac{b_{n1}}{b_{11}} b_{1n} & c_n + \frac{b_{n1}}{b_{11}} c_1 \end{pmatrix} \equiv \tag{4.3}$$

$$\equiv \begin{pmatrix} b_{11} & * & c_1 \\ 0 & \bar{B} & \bar{c} \end{pmatrix} \tag{4.4}$$

The off-diagonal entries of $\bar{B}$ are still negative, all diagonal entries are positive, and the sum of the entries in its $(j-1)st$ column is

$$d'_j = \left(b_{jj} - \frac{b_{j1}}{b_{11}} b_{1j} \right) + \sum_{k \neq 1, j} \left(-b_{kj} - \frac{b_{k1}}{b_{11}} b_{1j} \right)$$

$$= b_{jj} - \sum_{k \neq 1, j} b_{kj} - b_{1j} \frac{b_{21} + b_{31} + \cdots + b_{n1}}{b_{11}}$$

$$= b_{jj} - \sum_{k \neq 1,j} b_{kj} - b_{1j} + b_{1j} - b_{1j} \frac{b_{21} + b_{31} + \cdots + b_{n1}}{b_{11}}$$

$$= d_j + b_{1j} \frac{b_{11} - b_{21} - b_{31} - \cdots - b_{n1}}{b_{11}}$$

$$d'_j = d_j + \frac{b_{1j}}{b_{11}} d_1. \tag{4.5}$$

Thus we obtain that $d'_j \geq d_j$, for all $j = 1,2,\ldots,n$.

Since

$$d'_n = \frac{1}{b_{11}} \begin{vmatrix} b_{11} & -b_{1n} \\ d_1 & d_n \end{vmatrix},$$

we conclude that $d'_n > 0$.

For the matrix [$\bar{B} \mid \bar{c}$], we apply the same procedure:

$$\begin{pmatrix} b'_{22} & -b'_{23} & \cdots & -b'_{2n} & c_2 + \dfrac{b_{21}}{b_{11}} c_1 \\ -b'_{32} & b'_{33} & \cdots & -b'_{3n} & c_3 + \dfrac{b_{31}}{b_{11}} c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -b'_{n2} & -b'_{n3} & \cdots & b'_{nn} & c_n + \dfrac{b_{n1}}{b_{11}} c_1 \end{pmatrix} \rightarrow \tag{4.6}$$

$$\begin{pmatrix} b'_{22} & -b'_{23} & \cdots & -b'_{2n} & c_2 + \dfrac{b_{21}}{b_{11}} c_1 \\ 0 & b'_{33} - \dfrac{b'_{32}}{b'_{22}} b'_{23} & \cdots & -b'_{3n} - \dfrac{b'_{32}}{b'_{22}} b'_{2n} & c_3 + \dfrac{b_{31}}{b_{11}} c_1 + \dfrac{b'_{32}}{b'_{22}} \left(c_2 + \dfrac{b_{21}}{b_{11}} c_1 \right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -b'_{n3} - \dfrac{b'_{n2}}{b'_{22}} b'_{23} & \cdots & b'_{nn} - \dfrac{b'_{n2}}{b'_{22}} b'_{2n} & c_n + \dfrac{b_{n1}}{b_{11}} c_1 + \dfrac{b'_{n2}}{b'_{22}} \left(c_2 + \dfrac{b_{21}}{b_{11}} c_1 \right) \end{pmatrix} \tag{4.7}$$

$$\equiv \begin{pmatrix} b'_{22} & * & c_2 + \dfrac{b_{21}}{b_{11}} c_1 \\ 0 & \bar{\bar{B}} & \bar{\bar{c}} \end{pmatrix} \tag{4.8}$$

As in the case of $\overline{B}$, the sum $d_k^{''}$ of column entries in $\overline{\overline{B}}$ will be greater than or equal to corresponding sum of the entries in $\overline{B}$, and if we apply (4.5) for the last column of $\overline{\overline{B}}$ we obtain

$$d_n^{''} = d_n^{'} + \frac{b_{2n}^{'}}{b_{22}^{'}} d_2^{'} \geq d_n^{'} \tag{4.9}$$

$$d_n^{''} = d_n + \frac{b_{1n}}{b_{11}} d_1 + \frac{b_{2n} + \frac{b_{21}}{b_{11}} b_{1n}}{b_{22} - \frac{b_{21}}{b_{11}} b_{12}} \left(d_2 + \frac{b_{12}}{b_{11}} d_1 \right)$$

$$= d_n + \left(\frac{b_{1n}}{b_{11}} + \frac{b_{2n}b_{11} + b_{21}b_{1n}}{b_{22}b_{11} - b_{21}b_{12}} \bullet \frac{b_{12}}{b_{11}} \right) d_1 + \frac{b_{2n}b_{11} + b_{21}b_{1n}}{b_{22}b_{11} - b_{21}b_{12}} d_2$$

$$= d_n + \frac{b_{1n}b_{22}b_{11} - b_{1n}b_{21}b_{12} + b_{12}b_{2n}b_{11} + b_{21}b_{1n}b_{12}}{b_{11}(b_{22}b_{11} - b_{21}b_{12})} d_1 + \frac{b_{2n}b_{11} + b_{21}b_{1n}}{b_{22}b_{11} - b_{21}b_{12}} d_2$$

$$= \begin{vmatrix} b_{11} & -b_{12} \\ -b_{21} & b_{22} \end{vmatrix}^{-1} \left(\begin{vmatrix} b_{11} & -b_{12} \\ -b_{21} & b_{22} \end{vmatrix} d_n + \begin{vmatrix} -b_{12} & -b_{1n} \\ b_{22} & -b_{2n} \end{vmatrix} d_1 - \begin{vmatrix} b_{11} & -b_{1n} \\ -b_{21} & -b_{2n} \end{vmatrix} d_2 \right)$$

$$= (b_{22}b_{11} - b_{21}b_{12})^{-1} \begin{vmatrix} b_{11} & -b_{12} & -b_{1n} \\ -b_{21} & b_{22} & -b_{2n} \\ d_1 & d_2 & d_n \end{vmatrix} \tag{4.10}$$

Let us consider the last entry of the matrix $\overline{\overline{B}}$:

$$b_{nn}^{'} - \frac{b_{n2}^{'}}{b_{22}^{'}} b_{2n}^{'} = b_{nn} - \frac{b_{n1}}{b_{11}} b_{1n} - \frac{-b_{n2} - \frac{b_{n1}}{b_{11}} b_{12}}{b_{22} - \frac{b_{21}}{b_{11}} b_{12}} \left(-b_{2n} - \frac{b_{21}}{b_{11}} b_{1n} \right)$$

$$= b_{nn} - \frac{b_{n1}}{b_{11}} b_{1n} + \frac{-b_{11}b_{n2} - b_{12}b_{n1}}{b_{11}b_{22} - b_{12}b_{21}} \bullet \frac{b_{11}b_{2n} + b_{1n}b_{21}}{b_{11}}$$

$$= b_{nn} + \frac{-b_{11}b_{n2}b_{11}b_{2n} - b_{11}b_{n2}b_{1n}b_{21} - b_{12}b_{n1}b_{11}b_{2n} - b_{12}b_{n1}b_{1n}b_{21} - b_{n1}b_{1n}b_{11}b_{22} + b_{n1}b_{1n}b_{12}b_{21}}{(b_{11}b_{22} - b_{12}b_{21})b_{11}}$$

$$= b_{nn} + (b_{11}b_{22} - b_{21}b_{12})^{-1} \left(-b_{n1} \begin{vmatrix} -b_{12} & -b_{1n} \\ b_{22} & -b_{2n} \end{vmatrix} + b_{2n} \begin{vmatrix} b_{11} & -b_{1n} \\ -b_{21} & -b_{2n} \end{vmatrix} \right)$$

$$= (b_{11}b_{22} - b_{21}b_{12})^{-1} \begin{vmatrix} b_{11} & -b_{12} & -b_{1n} \\ -b_{21} & b_{22} & -b_{2n} \\ -b_{n1} & -b_{n2} & b_{nn} \end{vmatrix}. \tag{4.11}$$

If we continue this process, we conclude that the row echelon form of [*B* | *C*] has the sign pattern

$$\begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} & c_1 \\ 0 & b_{22}^{'} & \cdots & -b_{2n}^{'} & c_2^{'} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d & c^{'} \end{pmatrix} \equiv \begin{pmatrix} + & - & \cdots & - & + \\ 0 & + & \cdots & - & + \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & + & + \end{pmatrix} \tag{4.12}$$

where

$$d = \begin{vmatrix} b_{11} & -b_{12} & \cdots & -b_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n-1,1} & -b_{n-1,2} & \cdots & b_{n-1,n-1} \end{vmatrix}^{-1} \bullet \sum_{k=1} B_k d_k \geq \ldots \geq d_n^{"} \geq d_n^{'} > 0 \tag{4.13}$$

where B_k is the algebraic cofactor of *k*-th entry in the *n*-th column of *B*.

Back substitution from such a matrix yields a positive solution *X* to the system *BX*=*C*. If we replace *C* by $e_k = (0,0,\ldots,0,1,0,\ldots,0)^T$, (*k*-th coordinate is 1), we obtain that all entries in the k-th column of B^{-1} are all nonnegative numbers. Since the columns of B^{-1} are the solution vectors of $BX = e_i$, for $i = 1,2,\ldots,n$, the entries of B^{-1} are all nonnegative numbers.

Thus, we obtain that 2×2 determinants in condition *d* are positive, then $(I - A)^{-1}$ exists and contains only nonnegative entries. Now, we assume that 2×2 determinant is ≤ 0 (for all $k = 1,2,\ldots,n-1$) and consider the case when 3×3 determinants in condition *d* are positive (for some *k* and *m*). Let us take basically $k = 1, m = 2$. If we repeat exactly the same procedure, we might get $d_n^{'} \leq 0$ but get $d_n^{"} > 0$ exactly. If we look at the above processions, we can easily see that result. Since each elimination procedure just may increase the sum of entries in the last column, we conclude that again $d > 0$ and B^{-1} which exists and contains only nonnegative entries. Similarly, if 3×3 determinants are not positive, then we continue this procedure with the same logic and so on. Thus, we conclude that if matrix *B* satisfies the condition *d* then B^{-1} exists and contains just nonnegative entries. This completes the proof of the Theorem 4.1.

Economically, the condition

$$\begin{vmatrix} b_{kk} & -b_{kn} \\ d_k & d_n \end{vmatrix} > 0 \tag{4.14}$$

can be explained by the way that, the loss of the industry n can be (say "conditionally") covered by the industry k.

Similarly, the condition

$$\begin{vmatrix} b_{kk} & -b_{km} & -b_{kn} \\ -b_{mk} & b_{mm} & -b_{mn} \\ d_k & d_m & d_n \end{vmatrix} > 0 \tag{4.15}$$

for some k and m, that is $k < m < n$, means that the loss of industry n can be covered by the industries k and m. It is clear that (4.15) is a weaker condition than (4.14), it means that if the loss of one industry can be covered by another industry, then the loss can be covered by two industries as well(!). The next theorem will be suitable for generalizing our study.

Theorem 4.2 Let the matrix B satisfies the following properties:

a) each off-diagonal entry is less than or equal to zero,
b) each diagonal entry is positive**,**
c) $d_1 > 0, d_2 > 0, \ldots, d_{n-1} > 0,$
d)

$$\begin{vmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n-1,1} & -b_{n-1,2} & \cdots & -b_{n-1,n} \\ d_1 & d_2 & \cdots & d_n \end{vmatrix} > 0.$$

Then, B^{-1} exists and contains only nonnegative entries**.**

In general, if the matrix B satisfies the condition *d* of the Theorem 4.2, then we define that economically the loss of the industry n can be covered by the industries *1,2,…,n-1.*

Now, let us suppose that the number of industries with loss is more than one. For simplicity, let first s industries have a positive surplus, but $s+1, s+2, \ldots, nth$ industries have no profit (more exactly the revenue is less than or equal to the cost). Suppose that the industries $s+1, s+2, \ldots, n$ are ordered so that the loss of industry $s+1$ can be covered by the industries $1,2,\ldots,s$ then the loss of the industry $s+2$ can be covered by the industries $1,2,\ldots,s,s+1$, and so on and the loss of the n-th industry can be covered by the industries $1,2,\ldots,n-1$.

Theorem 4.3 Let the matrix B satisfies the following properties**:**

a) each off-diagonal entry is less than or equal to zero,
b) each diagonal entry is positive**,**
c) the sum of each column $d_k > 0$ for $k = 1,2,\ldots,s$**,** and $d_k \leq 0$ for $k = s+1, s+2, \ldots, n$:
   $d_1 > 0, d_2 > 0, \ldots, d_s > 0$**,** $d_{s+1} \leq 0, d_{s+2} \leq 0, \ldots, d_n \leq 0.$
d) the losses of the industries $s+1, s+2, \ldots, n$ can be covered by the previous industries.

Then, B^{-1} exists and contains only nonnegative entries.

We just need to repeat the proof of Theorem 4.1 in order to make the positive of the sum of entries in the $(s+1)st$ column (we apply n replaced by $(s+1)$ on the proof of Theorem 4.1). Then we apply the same procedure to make the sum of entries positive in the $(s+2)nd$ column and so on. That continues to the n-th industry.

Thus, in spite of heavy losses of some industries or economies, in general, may be profitable.

CHAPTER 5

CONCLUSIONS

We showed that how an economy can be considered profitable even if one or more than one industries are expected to make a serious loss. Leontief model that discusses on stable economies is not sufficient in that cases. We showed that how Leontief model can be applied to cases in which some industries make loss, in order that economy to be considered profitable in spite of some of its industries' loss.

The next problems have been considered:

1) We generalized the Leontief Model in case of three industry fields and have taken into account how one economy which is consistituted of three industries can be seen as a profitable one.
2) The case of an arbitrary n analysied. We made a detailed analysis for the case when one of the industry fields may has losses.
3) We considered the case when few industry fields may have losses.
4) We established the relationship between the profitability of the economy (in general) and (mathematical) solvability of the problem.

In the future, it would be possible to consider the different generalizations of other linear models.

REFERENCES

[1] Wassily, W Leontief, “ The Structure of American Economy, 1919-1939 ”, *an Empirical Application of Equilibrium Analysis*, Oxford University, New York, 1951

[2] Dobson, A.J., *an Introduction to Generalized Linear Models*, Chapman and Hall/CRC, London, 2001

[3] Miller, R.E. and Blair, P.D., *Input-Output Analysis*, Foundations and Extensions Prentice-Hall Inc, Englewood Cliffs, New Jersey. 1985

[4] Simon, C.P. and Blume, L., *Mathematics for Economists*, W W Norton & Company, New York, London, 1994