ELİPTİK-PARABOLİK DİFERANSİYEL VE FARK DENKLEMLERİ İÇİN LOKAL OLMAYAN SINIR DEĞER PROBLEMLERİ

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ÖZ

H Hilbert uzayında self-adjoint pozitif tanımlı A operatörlü diferansiyel denklemleri için lokal olmayan sınır değer problemi

$$\begin{cases} -\dfrac{d^2u(t)}{dt^2}+Au(t)=g(t),(0\leq t\leq 1), \\ \dfrac{du(t)}{dt}-Au(t)=f(t),(-1\leq t\leq 0), \\ u(1)=u(-1)+\mu \end{cases}$$

ele alınmıştır. Bu sınır değer probleminin iyi konumlanmışlığı ağırlıklı Hölder uzaylarında doğruluğu ortaya konulmuştur. Eliptik-parabolik denklemlerin lokal olmayan sınır değer problemlerinin çözümü için koersatif eşitsizlikleri elde edilmiştir. Lokal olmayan sınır değer probleminin yaklaşık çözümü için birinci ve ikinci derecedeki yakınlaşması olan fark şemaları sunulmuştur. Bu fark şemalarının iyi konumlanmışlığı Hölder uzaylarında kanıtlanmıştır. Uygulamalarda, Elliptik-parabolik denklemlerin fark şemalarının çözümü için koersatif eşitsizlikleri sağlanmıştır. Elliptik-parabolik denklemler için fark şemalarının Matlab ile çözümleri elde edilmiştir.

Anahtar Kelimeler: Eliptik-Parabolik Denklem, Fark Şemaları, İyi konumlanmışlık, Koersatif Eşitsizlikleri, Kararlılık Kestirimleri, Sayısal Çözümler .

NONLOCAL BOUNDARY VALUE PROBLEMS FOR ELLIPTIC-PARABOLIC DIFFERENTIAL AND DIFFERENCE EQUATIONS

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ABSTRACT

The abstract nonlocal boundary value problem

$$\begin{cases} -\dfrac{d^2u(t)}{dt^2} + Au(t) = g(t), (0 \le t \le 1), \\ \dfrac{du(t)}{dt} - Au(t) = f(t), (-1 \le t \le 0), \\ u(1) = u(-1) + \mu \end{cases}$$

for differential equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. The well-posedness of this problem in Hölder spaces with a weight is established. The coercivity inequalities for the solutions of the boundary value problems for elliptic-parabolic equations are obtained. The first and the second order accuracy difference schemes for the approximate solutions of this nonlocal boundary value problem are presented. The well-posedness of these difference schemes in Hölder spaces is established. In applications, the coercitivity inequalities for the solutions of difference schemes for elliptic-parabolic equation are obtained. The Matlab implementation of these difference schemes for elliptic-parabolic equation is presented.

Keywords: Elliptic-Parabolic Equation, Difference Schemes, Well-posedness, Coercivity Inequalities, Stability Estimates, Numerical Solutions.

CHAPTER 1

INTRODUCTION

It is known that various problems in fluid mechanics and dynamics, elasticity and other areas of engineering, physics and biological systems lead to partial differential equations of variable type. Methods of solutions of the nonlocal boundary value problems for partial differential equations of variable type have been studied extensively by many researchers (see, e.g.,[1]-[4],[16]-[41] and the references given therein).

Our goal in this work is to investigate the stability of difference schemes of the approximate solutions of the nonlocal boundary value problems for differential equations of elliptic-parabolic type.

It is known that the mixed problem for elliptic-parabolic equations can be solved analytically by Fourier series method, by Fourier transform method and by Laplace transform method.

Now, let us illustrate these three different analytical methods by examples.

First, we consider the following simple nonlocal boundary value problem for elliptic-parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}=(1-t)\sin x, \quad -1<t\leq 0,\ 0<x<\pi, \\ \frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}=-t\sin x, \quad 0<t<1,\ 0<x<\pi, \\ u\left(1,x\right)=u\left(-1,x\right)+2\sin x,\ 0\leq x\leq \pi, \\ u\left(t,0\right)=u\left(t,\pi\right)=0, \quad -1\leq t\leq 1. \end{cases} \tag{1.1}$$

For the solution of the problem (1.1), we use the Fourier series method. In order to solve the problem we need to separate $u(t,x)$ into two parts

$$u(t,x)=v(t,x)+w(t,x) \tag{1.2}$$

where $v(t,x)$ and $w(t,x)$ are the solutions of the problems

$$\begin{cases} \frac{\partial v}{\partial t} = -\frac{\partial^2 v}{\partial x^2}, \ -1<t<0, \ 0<x<\pi, \\ \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial x^2} = 0, \ 0<t<1, \ 0<x<\pi, \\ v(1,x) = v(-1,x) + 2\sin x, \ 0 \le x \le \pi, \\ v(t,0) = v(t,\pi) = 0, \ -1 \le t \le 1. \end{cases} \tag{1.3}$$

and

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} = (1-t)\sin x, \ -1<t<0, \ 0<x<\pi, \\ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} = -t\sin x, \ 0<t<1, \ 0<x<\pi, \\ w(1,x) = w(-1,x), \ 0 \le x \le \pi, \\ w(t,0) = w(t,\pi) = 0, \ -1 \le t \le 1. \end{cases} \tag{1.4}$$

Now, let us obtain the solution of (1.3) when $-1 \le t \le 0$, by the method of separation of variables. To do this, a solution of the form

$$v(t,x) = T(t)X(x) \neq 0$$

is suggested. Taking the partial derivatives and substituting the result in (1.3), we obtain

$$\frac{T'(t)}{T(t)} + \frac{X''(x)}{X(x)} = 0$$

or

$$-\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda. \tag{1.5}$$

The boundary conditions presented in (1.3), require $X(0) = X(\pi) = 0$. Hence from (1.5) we have the ordinary differential equations

$$X''(x) = \lambda X(x), \ X(0) = X(\pi) = 0. \tag{1.6}$$

If $\lambda \ge 0$, then the boundary value problem (1.6) has only trivial solution $X(x) = 0$. For $\lambda < 0$, the nontrivial solutions of the boundary value problem (1.6) are

$$X_k(x) = \sin kx, \ \text{where } k = 1,2,3,\cdots, \ \lambda = \lambda_k = -k^2, k = 1,2,3,\cdots.$$

So, the nontrivial solutions of the boundary value problem (1.6) are

$$X_k(x) = \sin kx, \ \text{where } k = 1,2,3,\cdots. \tag{1.7}$$

The other ordinary differential equations presented in (1.5) is

$$T'(t) = -\lambda T(t),$$

with $\lambda = -k^2$, $k = 1, \cdots$. The solution of this ordinary differential equation is

$$T_k(t) = A_k e^{k^2 t}, \text{ where } k = 1, 2, 3, \cdots.$$

Thus,

$$v(t, x) = \sum_{k=1}^{\infty} v_k(t, x) = \sum_{k=1}^{\infty} A_k e^{k^2 t} \sin kx.$$

Now, we consider if $0 \le t \le 1$ by the same method of separation of variables. To do this a solution of the form

$$v(t, x) = T(t)X(x) \neq 0$$

is suggested. Taking the partial derivatives and substituting the result in (1.3), we obtain

$$\frac{T''(t)}{T(t)} + \frac{X''(x)}{X(x)} = 0$$

or

$$-\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda. \tag{1.8}$$

The boundary conditions presented in (1.3), require $X(0) = X(\pi) = 0$. Hence from (1.8) we have the ordinary differential equation

$$X''(x) = \lambda X(x), \; X(0) = X(\pi) = 0.$$

We have already solved this ordinary differential equation in the previous part.The solution is presented in (1.7). The other ordinary differential equation presented in (1.8) is

$$T''(t) = -\lambda T(t),$$

with $\lambda = -k^2$, $k = 1, \cdots$. The solution of this ordinary differential equation is

$$T_k(t) = (B_k e^{kt} + C_k e^{-kt}), \text{ where } k = 1, 2, 3, \cdots.$$

Thus,

$$v(t, x) = \sum_{k=1}^{\infty} v_k(t, x) = \sum_{k=1}^{\infty} \left(B_k e^{kt} + C_k e^{-kt}\right) \sin kx.$$

Using the nonlocal boundary conditions

$$\begin{cases} v(1, x) = v(-1, x) + 2\sin x, \\ v(0_+, x) = v(0_-, x), \\ v'(0_+, x) = v'(0_-, x), \end{cases}$$

we obtain

$$\begin{cases} B_k e^k + C_k e^{-k} = A_k e^{-k^2}, \\ B_k + C_k = A_k, \\ k(B_k - C_k) = k^2 A_k \end{cases}$$

for $k \neq 1$ and

$$\begin{cases} B_1 e + C_1 e^{-1} = A_1 e^{-1} + 2, \\ B_1 + C_1 = A_1, \\ B_1 - C_1 = A_1 \end{cases}$$

It is easy to see that $C_1 = 0, A_1 = B_1 = \frac{2}{e-e^{-1}} = \frac{4}{\sinh 1}, B_k = C_k = A_k = 0$ for all $k \neq 1$.Then the solution of (1.3) is

$$v(t,x) \equiv \frac{4}{\sinh 1} e^t \sin x.$$

Second, we obtain the solution of (1.4). We seek a solution of the form

$$w(t,x) = \sum_{k=1}^{\infty} D_k(t) \sin kx.$$

If $0 \leq t \leq 1$, then

$$w_{tt} + w_{xx} = \sum_{k=1}^{\infty} \left(D_k''(t) - k^2 D_k(t) \right) \sin kx = -t \sin x.$$

From that it follows that

$$D_k''(t) - k^2 D_k(t) = 0$$

for all $k \neq 1$ and $D_1''(t) - D_1(t) = -t$. Solving it, we can write

$$D_k(t) = C_k \cosh kt + B_k \sinh kt,$$

for all $k \neq 1$ and $D_1(t) = C_1 \cosh t + B_1 \sinh t + t$.Thus,

$$\begin{aligned} w(t,x) &= \sum_{k=1}^{\infty} v_k(t,x) = \sum_{k=2}^{\infty} (C_k \cosh kt + B_k \sinh kt) \sin kx \\ &\quad + (C_1 \cosh t + B_1 \sinh t + t) \sin x. \end{aligned}$$

If $-1 \leq t \leq 0$, then

$$w_t + w_{xx} = \sum_{k=1}^{\infty} \left(D_k'(t) - k^2 D_k(t) \right) \sin kx = (1-t) \sin x.$$

From that it follows that

$$D_k'(t) - k^2 D_k(t) = 0$$

for all $k \neq 1$ and $D_1'(t) - D_1(t) = 1 - t$. Solving it, we can write

$$D_k(t) = A_k e^{k^2 t},$$

for all $k \neq 1$ and $D_1(t) = A_1 e^t + t$.Thus,

$$w(t,x) = \sum_{k=1}^{\infty} v_k(t,x) = \sum_{k=1}^{\infty} A_k e^{k^2 t} \sin kx + (A_1 e^t + t) \sin x.$$

Using the nonlocal boundary conditions

$$\begin{cases} w\left(1,x\right)=w\left(-1,x\right), \\ w\left(0_{+},x\right)=w\left(0_{-},x\right), \\ w'\left(0_{+},x\right)=w'\left(0_{-},x\right), \end{cases}$$

we obtain

$$\begin{cases} C_k=A_k, \\ kB_k=k^2A_k, \\ B_k\sinh k+C_k\cosh k=A_ke^{-k^2} \end{cases}$$

for $k\neq 1$ and

$$\begin{cases} C_1=A_1, \\ B_1+1=A_1+1, \\ B_1\sinh 1+C_1\cosh 1+1=A_1e^{-1}-1. \end{cases}$$

It is easy to see that $C_1=A_1=B_1=\frac{2}{e^{-1}-\sinh 1-\cosh 1}=-\frac{4}{\sinh 1}$, $B_k=C_k=A_k=0$ for all $k\neq 1$.Then the solution of (1.4) is

$$w\left(t,x\right)\equiv(-\frac{4}{\sinh 1}e^t+t)\sin x.$$

Therefore

$$u\left(t,x\right)=v\left(t,x\right)+w\left(t,x\right),$$

and

$$u\left(t,x\right)=t\sin x.$$

Note that using the same manner, one obtains the solution of the following nonlocal boundary value problem for the multidimensional elliptic-parabolic equation

$$\begin{cases} \frac{\partial^2u(t,x)}{\partial t^2}+\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2u(t,x)}{\partial x_r^2}=g(t,x), \\ x=(x_1,\dots,x_n)\in\overline{\Omega},0\leq t\leq T, \\ \frac{\partial u(t,x)}{\partial t}+\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2u(t,x)}{\partial x_r^2}=f(t,x), \\ x=(x_1,\dots,x_n)\in\overline{\Omega},-T\leq t\leq 0, \\ u_t(0+,x)=u_t(0-,x),x\in\overline{\Omega} \\ u(T,x)=u\left(-T,x\right)+\varphi(x),\ x\in\overline{\Omega}, \\ u(t,x)=0,x\in S \end{cases}$$

where $\alpha_r>0$ and $f(t,x)$ $(t\in[0,T],\ x\in\overline{\Omega})$, $g(t,x)$ $(t\in[-T,0],\ x\in\overline{\Omega})$, $\varphi(x)$, $\psi(x)$ $(x\in\overline{\Omega})$ are given smooth functions. Here Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<1,1\leq k\leq n)$ with boundary

$$S,\ \ \overline{\Omega}=\Omega\cup S.$$

However, the method of separation of variables can be used for problems having constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Second, we will consider a mixed problem

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = (1+t)e^{-x}, \quad -1 < t < 0,\ 0 < x < \infty, \\ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = te^{-x},\ 0 < t < 1,\ 0 < x < \infty, \\ u\left(1, x\right) = u\left(-1, x\right) + 2e^{-x},\ 0 \leq x < \infty, \\ u\left(t, 0\right) = t, u_x\left(t, 0\right) = -t, \quad -1 \leq t \leq 1. \end{cases} \tag{1.9}$$

It can be solved by Laplace transformation method (in x). Let $0 \leq t \leq 1$. Then, taking the Laplace transform of both sides of the differential equation

$$u_{tt} + u_{xx} = te^{-x},$$

we can write

$$\mathbf{L}\left\{u_{tt}\right\} + \mathbf{L}\left\{u_{xx}\right\} = \mathbf{L}\left\{te^{-x}\right\}$$

or

$$\left(\mathbf{L}\left\{u\left(t,x\right)\right\}\right)_{tt} + s^2\mathbf{L}\left\{u\left(t,x\right)\right\} - su\left(t,0\right) - u_x\left(t,0\right) = \frac{t}{s+1}.$$

Let

$$\mathbf{L}\{u(t,x)\} = v(t,s).$$

So our problem becomes

$$v_{tt}\left(t,s\right) + s^2 v\left(t,s\right) - st + t = \frac{t}{s+1}$$

or

$$v_{tt}\left(t,s\right) + s^2 v\left(t,s\right) = \frac{s^2 t}{s+1}.$$

Now the complementary solution is

$$v_c\left(t,s\right) = c_1 \sin st + c_2 \cos st.$$

For the particular solution we can write

$$v_p\left(t,s\right) = \frac{t}{s+1}.$$

So

$$v\left(t,s\right) = c_1 \sin st + c_2 \cos st + \frac{t}{s+1}. \tag{1.10}$$

Now, let $-1 \leq t \leq 0$. Then,

$$u_t + u_{xx} = \left(1+t\right)e^{-x}.$$

By taking the Laplace transform of both sides of the last differential equation, we obtain

$$\mathbf{L}\left\{u_t\right\}+\mathbf{L}\left\{u_{xx}\right\}=\mathbf{L}\left\{(1+t)\,e^{-x}\right\}$$

or

$$\left(\mathbf{L}\left\{u\left(t,x\right)\right\}\right)_t+s^2\mathbf{L}\left\{u\left(t,x\right)\right\}-su\left(t,0\right)-u_x\left(t,0\right)=\frac{1+t}{s+1}.$$

Let

$$\mathbf{L}\left\{u\left(t,x\right)\right\}=v\left(t,s\right).$$

So our problem becomes

$$v_t\left(t,s\right)+s^2v\left(t,s\right)-st+t=\frac{1+t}{s+1}$$

or

$$v_t\left(t,s\right)+s^2v\left(t,s\right)=\frac{s^2t+1}{s+1}.$$

So

$$v\left(t,s\right)=c_3e^{-s^2t}+\frac{t}{s+1}. \tag{1.11}$$

Using the nonlocal boundary conditions

$$\begin{cases} u\left(1,x\right)=u\left(-1,x\right)+2e^{-x}, \\ u\left(0_+,x\right)=u\left(0_-,x\right), \\ u'\left(0_+,x\right)=u'\left(0_-,x\right), \end{cases}$$

we obtain

$$\begin{cases} v\left(1,s\right)=v\left(-1,s\right)+\frac{2}{1+s}, \\ v\left(0_+,s\right)=v\left(0_-,s\right), \\ v'\left(0_+,s\right)=v'\left(0_-,s\right). \end{cases}$$

Applying these conditions and using (1.10), (1.11), we get

$$\begin{cases} c_2=c_3, \\ sc_1+\frac{1}{s+1}=-s^2c_3+\frac{1}{s+1}, \\ c_1\sin s+c_2\cos s+\frac{1}{s+1}=c_3e^{s^2}-\frac{1}{s+1}+\frac{2}{1+s}. \end{cases}$$

Solving it, we can write $c_1=c_2=c_3=0$. Then

$$v\left(t,s\right)=\frac{t}{s+1}.$$

Hence taking the inverse of Laplace transform, we obtain

$$u\left(t,x\right)=\mathbf{L}^{-1}\left\{v\left(t,s\right)\right\}=\mathbf{L}^{-1}\left\{\frac{t}{s+1}\right\}=t\mathbf{L}^{-1}\left\{\frac{1}{s+1}\right\}=te^{-x}.$$

So

$$u\left(t,x\right)=te^{-x}$$

is the solution of the given nonlocal boundary value problem (1.9).

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the multidimensional elliptic-parabolic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = g(t,x), \\ x = (x_1, \ldots, x_n) \in \overline{\Omega}^+, 0 \le t \le T, \\ \frac{\partial u(t,x)}{\partial t} + \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, \ldots, x_n) \in \overline{\Omega}^+, -T \le t \le 0, \\ u(T,x) = u(-T,x) + \varphi(x), \\ u_t(0+,x) = u_t(0-,x) + \varphi(x),\ x \in \overline{\Omega}^+, \\ u(t,x) = 0,\ x \in S^+, \end{cases}$$

where $\alpha_r > 0$ and $f(t,x)$ $\left(t \in [0,T],\ x \in \overline{\Omega}^+\right)$, $g(t,x)$ $\left(t \in [-T,0],\ x \in \overline{\Omega}^+\right)$, $\varphi(x)$, $\psi(x)$ $\left(x \in \overline{\Omega}^+\right)$ are given smooth functions. Here Ω^+ is the open set in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n)$ with boundary

$$S^+,\ \ \overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However, Laplace transform method can be used only in the case of constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Third, we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = (1 + t(4x^2 - 2))\, e^{-x^2}, \ \ -1 \le t \le 0, \ \ -\infty < x < \infty, \\ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = t\,(4x^2 - 2)\, e^{-x^2},\ \ 0 < t < 1, \ \ -\infty < x < \infty, \\ u\,(1,x) = u\,(-1,x) + 2e^{-x^2}, \ \ -\infty \le x \le \infty. \end{cases} \tag{1.12}$$

It can be solved by using Fourier Transform method. We denote

$$v\,(t,s) = \mathbf{F}\left\{u\,(t,x)\right\}.$$

Then, taking the Fourier transform of both sides of the differential equation in (1.12) for $-1 \le t \le 0$, we obtain

$$v_t\,(t,s) - s^2 v\,(t,s) = \mathbf{F}\left\{\left(1 + t(4x^2 - 2)\right) e^{-x^2}\right\}.$$

Since $(e^{-x^2})'' = (4x^2-2)e^{-x^2}$, we have that

$$\mathbf{F}\left\{(4x^2-2)e^{-x^2}\right\} = \mathbf{F}\left\{(e^{-x^2})''\right\} = -s^2\mathbf{F}\left\{e^{-x^2}\right\}. \tag{1.13}$$

Then

$$v_t\left(t,s\right) - s^2 v\left(t,s\right) = (1-ts^2)\mathbf{F}\left\{e^{-x^2}\right\}.$$

Solving it we can write

$$v\left(t,s\right) = c_1 e^{s^2 t} + t\mathbf{F}\left\{e^{-x^2}\right\}. \tag{1.14}$$

Now, taking the Fourier transform of both sides of the differential equation in(1.12) for $0<t<1$, we obtain

$$v_{tt}\left(t,s\right) - s^2 v\left(t,s\right) = \mathbf{F}\left\{t\left(4x^2-2\right)e^{-x^2}\right\}.$$

Using (1.13), we get

$$v_{tt}\left(t,s\right) - s^2 v\left(t,s\right) = -ts^2\mathbf{F}\left\{e^{-x^2}\right\}.$$

Solving it we can write

$$v\left(t,s\right) = c_2\cosh st + c_3\sinh st + t\mathbf{F}\left\{e^{-x^2}\right\}. \tag{1.15}$$

Using the nonlocal boundary conditions

$$\begin{cases} u\left(1,x\right) = u\left(-1,x\right) + 2e^{-x^2}, \\ u\left(0_+,x\right) = u\left(0_-,x\right), \\ u'\left(0_+,x\right) = u'\left(0_-,x\right), \end{cases}$$

we obtain

$$\begin{cases} v\left(1,s\right) = v\left(-1,s\right) + 2\mathbf{F}\left\{e^{-x^2}\right\}, \\ v\left(0_+,s\right) = v\left(0_-,s\right), \\ v'\left(0_+,s\right) = v'\left(0_-,s\right). \end{cases}$$

Applying these conditions and using (1.14), (1.15), we get

$$v\left(t,s\right) = c_1 e^{s^2 t} + t\mathbf{F}\left\{e^{-x^2}\right\}, \tag{1.16}$$

$$\begin{cases} c_2 = c_1, \\ sc_3 + \mathbf{F}\left\{e^{-x^2}\right\} = s^2 + \mathbf{F}\left\{e^{-x^2}\right\}, \\ c_2\cosh s + c_3\sinh s + \mathbf{F}\left\{e^{-x^2}\right\} = c_1 e^{-s^2} - \mathbf{F}\left\{e^{-x^2}\right\} + 2\mathbf{F}\left\{e^{-x^2}\right\}. \end{cases}$$

It is easy to see that $c_1 = c_2 = c_3 = 0$. Then

$$v(t,s) = t\mathbf{F}\left\{e^{-x^2}\right\}.$$

Finally taking the inverse of Fourier transform we obtain the solution of the problem (1.12) as

$$u\left(t,x\right) = te^{-x^2}.$$

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the 2m-th order multidimensional elliptic-parabolic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \sum\limits_{|r|=2m} a_r \frac{\partial^{|\tau|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = g(t,x), \\ 0 \le t \le T, x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ \frac{\partial u}{\partial t} + \sum\limits_{|r|=2m} a_r \frac{\partial^{|\tau|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = f(t,x), \\ -T \le t \le 0, x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ u(T,x) = u\left(-T,x\right) + \varphi(x), x \in \mathbb{R}^n, \\ u_t(0+,x) = u_t(0-,x),\ x \in \mathbb{R}^n, \end{cases}$$

where $\alpha_r,\ f(t,x)\ (t \in [0,T],\ x \in R^n),\ g(t,x)\ (t \in [-T,0], x \in R^n), \varphi(x), \psi(x)\ (x \in R^n)$ are given smooth functions.

As in the previous two analytical methods mentioned above, the Fourier transform method can be used only in the case of constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

To sum up, all analytical methods described above, namely the Fourier series method, the Laplace transform method and the Fourier transform method can be used only when the differential equation has constant coefficients. It is well-known that the most general method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method, which is basically realized by digital computers and known to be numerical method. However the stability of different difference schemes used in numerical methods need to be proved or justified theoretically.

In the present work the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2 u(t)}{dt^2} + Au(t) = g(t), (0 \le t \le 1), \\ \frac{du(t)}{dt} - Au(t) = f(t), (-1 \le t \le 0), \\ u(1) = u(-1) + \mu \end{cases}$$

for differential equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. The well-posedness of this problem in Hölder spaces with a weight is established. The coercivity inequalities for the solutions of the boundary value problems for elliptic-parabolic equations are obtained. The first order of accuracy and second order of accuracy difference schemes for the approximate solutions of this nonlocal boundary value problem are presented. The well-posedness of these difference schemes in Hölder spaces is established. In applications, the coercivity inequalities for the solutions of difference schemes for elliptic-parabolic equation are obtained. Numerical examples are given. The Matlab implementation of these difference schemes for elliptic-parabolic equation is presented.

Let us briefly describe the contents of the various sections of the thesis. It consists of six chapters.

First chapter is the introduction.

Second chapter presents elementary statements in a Hilbert space that is needed for this work.

Third chapter consists the main theorem about well-posedness of the nonlocal boundary value problem for elliptic-parabolic equation in a Hilbert space. In applications this abstract result permits us to obtain the coercivity stability inequalities for the solution of the two nonlocal boundary value problems for elliptic-parabolic equations.

Fourth chapter presents the stable first order of accuracy difference scheme approximately solving the nonlocal boundary value problem for elliptic-parabolic equation in a Hilbert space H with self-adjoint positive definite operator A. The well-posedness of this difference scheme in Hölder spaces is established. In applications, the stability, almost coercivity inequalities, coercivity inequalities for the solutions of difference scheme for the approximate solution of the nonlocal boundary value problem for elliptic-parabolic equation are obtained.

Fifth chapter presents the stable second order of accuracy difference scheme approximately solving the nonlocal boundary value problem for elliptic-parabolic equation in a Hilbert space H with self-adjoint positive definite operator A. The well-posedness of this difference scheme in Hölder spaces is established. In applications, the stability, almost coercivity inequalities, coercivity inequalities for the solutions of difference scheme for the approximate solution of the nonlocal boundary value problem for elliptic-parabolic equation are obtained.

Sixth chapter is devoted to the applications. The method is considered by numerical examples. A matlab program is given to conclude that the second order of accuracy is more accurate. The figures and table are included for comparison.

Seventh chapter contains the conclusions.

CHAPTER 2

ELEMENTS OF HILBERT SPACE

This chapter covers selected concepts of the elementary Hilbert space theory as developed in [Krein, S. G., 1966]. It also includes the basis for the solution properties in an Hilbert space of the initial value problem considered in this thesis.

2.1 HILBERT SPACE

Definition 2.1. A complex linear space H is called an inner product space if there is a complex-valued function $\langle .,. \rangle : H \times H \to C$ with the properties

i. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Longleftrightarrow x = \sigma$,

ii. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in H$,

iii. $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$, for all $x, y \in H$ and $\alpha \in C$,

iv. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in H$.

The function $\langle x, y\rangle$ is called the inner product of x and y. A Hilbert space is a complete inner product space. An inner product on H defines a norm on H given by $\|x\| = \langle x, x\rangle^{1/2}$. Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

Example 2.1. *The space $C_2[-1, 1]$ of all defined and continuous functions on a given closed interval $[-1, 1]$ is an inner product space with the inner product given by*

$$\langle x, y\rangle = \int_{-1}^{1} x(t)\overline{y(t)}dt. \tag{2.1}$$

Note that the space $C_2[-1, 1]$ is not complete. So, $C_2[-1, 1]$ is not a Hilbert space.

Example 2.2. *The space $L_2[-1, 1] = \overline{C_2[-1, 1]}$ with the inner product (2.1) is a Hilbert space.*

Theorem 2.1. *Let x, y be any two vectors in a Hilbert space, then*

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \quad (\text{ Schwartz inequality}). \tag{2.2}$$

Note that the inner product is related to the norm by the following identity

$$\langle x, y\rangle = \frac{1}{4}\left[\left(\|x+y\|^2 - \|x-y\|^2\right) + i\left(\|x+iy\|^2 - \|x-iy\|^2\right)\right]. \tag{2.3}$$

A norm on an inner product space satisfies the important Parallelogram law

Theorem 2.2. *If H is a Hilbert space, then*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \ , \quad \forall \ x, y \in H. \ \ (\text{Parallelogram law}) \tag{2.4}$$

Conversely, H is a complex complete normed space with the norm $\|\cdot\|$ satisfying the equation (2.4) then H is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ satisfying $\|x\| = \langle x, x\rangle^{1/2}$.

Example 2.3. *The space l^p of all sequence, $x = (\xi_i) = (\xi_1, \xi_2, ...)$ such that $|\xi_1|^p + |\xi_2|^p + ...$ converges with $p \neq 2$ is not an inner product space,hence not a Hilbert space.*

Example 2.4. *The space $C[a, b]$ is not an inner product space,hence not a Hilbert space.*

2.2 BOUNDED LINEAR OPERATORS IN H

Definition 2.2. Let H_1 and H_2 be two Hilbert spaces. A linear operator A is an operator such that $A : H_1 \to H_2$

$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all $\alpha, \beta \in C$ and $x, y \in H_1$.

The domain of A $D(A) = \{x \in H_1, \exists Ax \in H_2\}$ is a vector space and

$R(A) = \{y = Ax, \forall x \in D(A)\}$ denotes the range of A.

A linear operator $A : H \to H$ is said to be bounded if there exist a real number $M > 0$ such that

$$\|Ax\|_H \leq M \ \|x\|_H \quad for\ all \ \ x \in H. \tag{2.5}$$

If a linear operator $A : H \to H$ is bounded with M, then

$$\|A\| = \inf M \tag{2.6}$$

is called the norm of operator A.

Example 2.5. *A bounded linear operator from $H = L_2[0, 1]$ into itself is defined by*

$$Ax = tx(t) \ , \ 0 \leq t \leq 1. \tag{2.7}$$

Example 2.6. *Another bounded linear operator $L_2[0, 1]$ into itself is defined by*

$$Ax(t) = \int_0^1 tsx(s)ds. \tag{2.8}$$

Theorem 2.3. *The norm of the bounded linear operator A is*

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{x\neq\sigma} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| . \tag{2.9}$$

Example 2.7. *A is an operator defined by $Ax = \alpha x(t)$, $A : L_2[0,1] \longrightarrow L_2[0,1]$. Then, $\|Ax\| = |\alpha|$.*

2.3 ADJOINT OF AN OPERATOR

Definition 2.3. Let $A : H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the Hilbert adjoint operator A^* of A is the operator

$$A^* : H_2 \to H_1,$$

such that for all $x \in H_1$ and $y \in H_2$

$$\langle Ax, y\rangle = \langle x, A^*y\rangle .$$

Theorem 2.4. *The Hilbert adjoint operator A^* of A is unique and bounded linear operator with the norm*

$$\|A^*\| = \|A\| . \tag{2.10}$$

Definition 2.4. A bounded linear operator $A : H \longrightarrow H$ on a Hilbert space H is said to be self-adjoint if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$.

Definition 2.5. A self-adjoint operator A is said to be positive if $A \geq 0$, that is, $(Ax, x) \geq 0$ for all $x \in H$.

Example 2.8. *A is an operator defined on the example 2.5. If $\alpha \in R^1$, then A is a self-adjoint operator.*

Example 2.9. *A is an operator defined on the example 2.7. Then, A is a self-adjoint positive operator.*

Definition 2.6. Let $A : D(A) \longrightarrow H$ be a linear operator with $\overline{D(A)} = H$. Then A is called symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D(A)$. If A is symmetric and $D(A) = D(A^*)$, then A is a self-adjoint operator.

Example 2.10. *Let $Au = -\dfrac{d^2u}{dx^2} + u$, $u(a) = u(b) = 0$ and $H = L_2[a,b]$. Then, A is a self-adjoint positive operator.*

2.4 SPECTRUM

Definition 2.7. Let H be a Hilbert space and $A : H \longrightarrow H$ be a linear operator with $D(A) \subset H$. We associate the operator $A_\lambda = A - \lambda I$, where $\lambda \in C$ and I is the identity operator on $D(A)$.

If A_λ has an inverse, we denote it by $R_\lambda(A)$ and we call it the resolvent operator of A, or simply, resolvent of A.

$$R_\lambda(A) = (A - \lambda I)^{-1} . \tag{2.11}$$

Definition 2.8. (Regular value, resolvent set, spectrum)

Let A be a linear operator with $D(A) \subset H$ and H is a Hilbert space. A regular value λ of A is a complex number such that

$(R1)$ $R_\lambda(A)$ exists.

$(R2)$ $R_\lambda(A)$ is bounded.

$(R3)$ $R_\lambda(A)$ is defined on a set which is dense in H.

The resolvent set $\rho(A)$ of A is the set of all regular values of A. Its complement $\sigma(A) = C - \rho(A)$ is called spectrum of A , and $\lambda \in \sigma(A)$ is called spectral value of A. Furthermore, the spectrum $\rho(A)$ is partitioned into three disjoint sets as follows.

The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_\lambda(A)$ does not exist. A $\lambda \in \sigma(A)$ is called an eigenvalue of A.

The continuous spectrum $\sigma_c(A)$ is the set such that $R_\lambda(A)$ exists and satisfies $(R3)$ but not $(R2)$, that is, $R_\lambda(A)$ is unbounded.

The residual spectrum $\sigma_r(A)$ is the set such that $R_\lambda(A)$ exists (and may be bounded or not) but does not satisfy $(R3)$, that is, the domain of $R_\lambda(A)$ is not dense in H.

If $A_\lambda x = (A - \lambda I)x = 0$ for some $x \neq 0$, then $\lambda \in \sigma_p(A)$, by definition, that is, λ is an eigenvalue of A.

The vector x is called an eigenvector of A corresponding to eigenvalue λ. The subspace of $D(A)$ consisting of 0 and all eigenvectors of A corresponding to an eigenvalue λ of A is called the eigenspace of A corresponding to that eigenvalue λ.

$$\sigma(A) = \sigma_c(A) \cup \sigma_p(A) \cup \sigma_r(A), \tag{2.12}$$

$$\sigma(A) \cup \rho(A) = C.$$

Definition 2.9. Let H be a Hilbert space over the field of real numbers and for any $x \in H$, let $\|x\|$ denote the norm of x. Let J be any interval of the real line R. A function $x : J \to H$ is called an abstract function. A function $x(t)$ is said to be continuous at the point $t_0 \in J$, if

$$\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0.$$

If $x : J \to H$ is continuous at each point of J, then we say that x is continuous on J and we write $x \in C[J, H]$.

Definition 2.10. The Stieltjes integral of a function x:$[a, b]$ →H with respect to a function y:$[a, b]$ →H_1. Let H, H_1 and H_2 be three Hilbert spaces. A bilinear operator $P : H \times H_1 \to H_2$ whose norm is less than or equal to 1, that is,

$$\|P(x, y)\| \leq \|x\| \|y\|, \tag{2.13}$$

is called a product operator. We shall agree to write $P(x, y) = xy$. Let $x : [a, b] \to H$ and $y : [a, b] \to H_1$ be two bounded functions such that the product $x(t)y(t) \in H_2$, for each $t \in [a, b]$ is linear in both x and y and

$$\|x(t)y(t)\| \leq \|x(t)\| \, \|y(t)\|$$

(for example, $x(t) = A(t)$ is an operator with domain $D[A(t)] \supset H_1$, or one of the function x,y is a scalar function). We denote the partition $(a = t_0 < t_1 < t_2 < ... < t_n = b)$ together with the points τ_i $(t_i < \tau_1 < t_{i+1}, i = 0, 1, 2, ..., n-1)$ by π and set $|\pi| = \max_i |t_{i+1} - t_i|$. We form the Stieltjes sum

$$S_\pi = \sum_{i=1}^{n-1} x(\tau_i)[y(t_{i+1}) - y(t_i)]. \tag{2.14}$$

If the $\lim S_\pi$ exist as $|\pi| \to 0$ and defines an element I in H_2 independent of π, then I is called the Stieltjes integral of the function $x(t)$ by the function $y(t)$, and is denoted by

$$\int_a^b x(t)\, dy(t). \tag{2.15}$$

Theorem 2.5. *If $x \in C[[a,b], H]$ and $y : [a,b] \to H_1$ are bounded variations on $[a,b]$, then the Stieltjes integral (2.15) exists.*

Consider the function $y : [a,b] \to H_1$ and the partition

$$\pi : a = t_0 < t_1 < t_2 < ... < t_n = b.$$

Form the sum

$$V = \sum_{i=1}^{n-1} \|y(t_{i+1}) - y(t_i)\|. \tag{2.16}$$

The least upper bound of the set of all possible sums V is called the (strong) total variation of the function $y(t)$ on the interval $[a,b]$ and is denoted by $V_a^b(y)$. If $V_a^b(y) < \infty$, then $y(t)$ is called an abstract function of bounded variation on $[a,b]$.

Example 2.11. *If $x \in C[[a,b], H]$ and $y : [a,b] \to H_1$ are bounded variations on $[a,b]$, then*

$$\left\| \int_a^b x(t)\, dy(t) \right\| \le \int_a^b \|x(t)\|\, dV_a^t[y(t)] \le \max_{t\in[a,b]} \|x(t)\|\, V_a^b[y(t)]. \tag{2.17}$$

2.5 PROJECTION OPERATOR, SPECTRAL FAMILY

Definition 2.11. A Hilbert space H is represented as the direct sum of a closed subspace Y and its orthogonal complement $Y^\perp$:

$$H = Y \oplus Y^\perp \tag{2.18}$$

$$x = y + z \quad , \quad \textit{where} \;\; y \in Y, z \in Y^\perp.$$

Since the sum is direct, y is unique for any given $x \in H$. Hence (2.18) defines a linear operator

$$P : H \longrightarrow H,$$

$$x \longrightarrow y = Px.$$

P is called an orthogonal projection or projection on H.

Theorem 2.6. *A bounded linear operator $P : H \longrightarrow H$ on a Hilbert space H is projection if and only if P is self-adjoint and idempotent that is, $P^2 = P$.*

Spectral family from dimensional case as follows: If matrix A has n different eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 ... < \lambda_n$. Then A has an orthogonal set of n vectors $x_1, x_2, x_3, ..., x_n,$where x_j corresponds to λ_j and we write these vectors as column vectors, for convenience. This basis for H, has a unique representation:

$$x = \sum_{j=1}^{n} \gamma_j x_j \ , \qquad \gamma_j = (x, x_j) = x^T \overline{x_j} \ , \tag{2.19}$$

x_j is an eigenvector of A, so that we have $Ax_j = \lambda_j x_j$.

$$Ax = \sum_{j=1}^{n} \lambda_j \gamma_j x_j. \tag{2.20}$$

We can define an operator

$$P_j : H \longrightarrow H, \tag{2.21}$$

$$x \longrightarrow \gamma_j x_j \ .$$

Obviously, P_j is the projection (orthogonal projection) of H onto the eigenspace of A corresponding to λ_j . From the equation (2.19) can be written

$$x = \sum_{j=1}^{n} P_j x \ \text{ hence } \ I = \sum_{j=1}^{n} P_j, \tag{2.22}$$

where I is the identity operator on H. Formula (2.20) becomes

$$Ax = \sum_{j=1}^{n} \lambda_j P_j x \ \text{ hence } \ A = \sum_{j=1}^{n} \lambda_j P_j. \tag{2.23}$$

This is a representation of A in terms of projections.

Theorem 2.7. *(Spectral Theorem) A family of an orthogonal projection operators E_λ $(-\infty < \lambda < \infty)$ is said to be spectral representation identity if:*

1) E_λ is strongly left-continuous in λ;

2) $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$ for $\lambda < \mu$;

3) $E_{-\infty} = \lim\limits_{\lambda \to -\infty} E_\lambda = 0$ and $E_{+\infty} = \lim\limits_{\lambda \to \infty} E_\lambda = I$, where the limits are understood in the sense of strong convergence. For every bounded function $F(\lambda)$ defined on the entire real axis, one can define the Stieltjes operator integral

$$\int_a^b F(\lambda)\, dE_\lambda. \tag{2.24}$$

This integral is defined as the limit in the norm of integral sums of the form

$$\sum_{k=0}^{N} F(\lambda_k) \left(E_{\lambda_{k+1}} - E_{\lambda_k} \right),$$

if the segment $[a, b]$ *is finite, and as an important integral if* $a = -\infty$ *or* $b = \infty$. *The integral (2.24) is a bounded operator with*

$$\left\| \int_a^b F(\lambda)\, dE_\lambda \right\| \leq \sup_{a \leq \lambda \leq b} |F(\lambda)| \, .$$

If the function $F(\lambda)$ *takes on only real values, the operator (2.24) is self-adjoint. If the function* $F(\lambda)$ *is real and bounded, then formula (2.24), after assigning an appropriate meaning to the integral, yields a self-adjoint and generally speaking bounded operator whose domain consists of only those elements* x *for which*

$$\int_{-\infty}^{\infty} |F(\lambda)|^2 \, d(E_\lambda x, x) < \infty.$$

It turns out that to every self-adjoint operator A *there corresponds some spectral representation* E_λ *of the identity with*

$$Ax = \int_{-\infty}^{\infty} \lambda dE_\lambda x$$

for $x \in D(A)$. *The operators* E_λ *commute with any operator commuting with* A.

If A *is bounded, and* m *and* M *are the greatest lower bound and least upper bound of its spectrum then* $E_\lambda = I$ *for* $\lambda > M$, *so that*

$$Ax = \int_m^{M+0} \lambda dE_\lambda x.$$

If the operator A *is positive definite, i.e.,* $\langle Ax, x\rangle \ \geq \delta \langle x, x\rangle$ *for some* $\delta \geq 0$, *then*

$$Ax = \int_a^{\infty} \lambda dE_\lambda x.$$

The real regular points of A *are characterized by the fact that in their neighborhoods the operator* E_λ *is constant. Thus, the points of the spectrum of* A *coincide with the points of growth of the operator function* E_λ.

By using the spectral representation one may bring into consideration a wide class of functions of an unbounded self-adjoint operator. Thus, for example, for any continuous function $F(\lambda)$ *it is natural to put*

$$F(A)\, x = \int_a^{\infty} F(\lambda)\, dE_\lambda x.$$

where E_λ *is the spectral resolution of the identity corresponding to operator* A. *From that, it follows*

$$\|F(A)\, x\|_H \leq \sup_{\delta \leq \mu < \infty} |F(\lambda)|.$$

Example 2.12. *A is a self-adjoint positive definite operator. Show the below inequalities;*

$$\| \ A^\alpha e^{-tA} ||_{H \to H} \leq t^{-\alpha}, t > 0, 0 \leq \alpha \leq 1. \tag{2.25}$$

$$\| \left(A^{\frac{1}{2}}\right)^{\alpha} e^{-tA^{\frac{1}{2}}} \|_{H\to H} \leq t^{-\alpha}, t>0, 0\leq \alpha \leq 1, \tag{2.26}$$

$$\| \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1} \|_{H\to H} \leq M, \tag{2.27}$$

$$\| \left(I + e^{-2A^{\frac{1}{2}}} + A^{\frac{1}{2}}(I - e^{-2A^{\frac{1}{2}}}) - 2e^{-(A^{\frac{1}{2}}+A)}\right)^{-1} \|_{H\to H} \leq M, \tag{2.28}$$

$$\| P^k \|_{H\to H} \leq M(1+\delta\tau)^{-k}, k\tau \| AP^k \|_{H\to H} \leq M, k\geq 1, \delta>0, P=P(\tau A)=(I+\tau A)^{-1} \tag{2.29}$$

$$\| A^{\beta}(P^{k+r}-P^k) \|_{H\to H} \leq M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+r\leq N,\ 0\leq \alpha,\ \beta\leq 1, \tag{2.30}$$

$$\| R^k \|_{H\to H} \leq M(1+\delta\tau)^{-k}, k\tau \| BR^k \|_{H\to H} \leq M, k\geq 1, \delta>0, R=(I+\tau B)^{-1}, \tag{2.31}$$

$$\| B^{\beta}(R^{k+r}-R^k) \|_{H\to H} \leq M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+r\leq N,\ 0\leq \alpha,\ \beta\leq 1, \tag{2.32}$$

$$||(I-R^{2N})^{-1}||_{H\to H} \leq M, \tag{2.33}$$

$$\begin{gathered}\left\|(I+(I+\tau A)\,(I+2\tau A)^{-1}\left(R^{2N-1}+B^{-1}A\left(I-R^{2N-1}\right)\right)-\right. \\ \left.-(2I+\tau B)\,(I+2\tau A)^{-1}\,R^{N}P^{N-1})^{-1}\right\|_{H\to H}\leq M.\end{gathered} \tag{2.34}$$

$$\| P^k \|_{H\to H} \leq 1, \| C \|_{H\to H} \leq 1, k\tau \| AP^kC^2 \|_{H\to H} \leq M, k\geq 1, \delta>0, \tag{2.35}$$

$$\| A^{\beta}(P^{k+r}-P^k)C^2 \|_{H\to H} \leq M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+r\leq N,\ 0\leq \alpha,\ \beta\leq 1, \tag{2.36}$$

where

$$P=P(\tau A)=(I-\frac{\tau A}{2})(I+\frac{\tau A}{2})^{-1}, C=C(\tau A)=(I+\frac{\tau A}{2})^{-1},$$

$$\begin{gathered}\left\|(I+B^{-1}A(I+\tau A+\tau C^2)D\left(I-R^{2N-1}\right)\right. \\ \left.+DP^2R^{2N-1}-(2I+\tau B)DR^{N}P^{N+2})^{-1}\right\|_{H\to H}\leq M,\end{gathered} \tag{2.37}$$

where

$$D=\left(I+2\tau A+\frac{5}{4}\left(\tau A\right)^2\right)^{-1}.$$

Solution. *Using the spectral representation of the self-adjoint positive defined operators we can write*

$$A^{\alpha}\exp(-At)\varphi=\int_{\delta}^{\infty}\mu^{\alpha}\exp(-\mu t)dE_{\mu}\varphi,$$

where (E_μ) *is the spectral family associated with* A. *Therefore, for any* $t \geq 0$ *we have that*

$$\|A^\alpha \exp(-At)\|_{H \to H} \leq \sup_{\delta \leq \mu < \infty} \mu^\alpha |\exp(-\mu t)| = t^{-\alpha}.$$

The estimate (2.25) is proved. The proof of estimates (2.26) - (2.37) follow the same scheme and relies on the spectral representation of the self-adjoint positive defined operators.

CHAPTER 3

A NONLOCAL BOUNDARY VALUE DIFFERENTIAL PROBLEM: WELL-POSEDNESS

We consider the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u(t)}{dt^2} + Au(t) = g(t), (0 \le t \le 1), \\ \frac{du(t)}{dt} - Au(t) = f(t), (-1 \le t \le 0), \\ u(1) = u(-1) + \mu \end{cases} \tag{3.1}$$

for differential equation in a Hilbert space H with the self-adjoint positive definite operator A.

Let us denote by $C^{\alpha}_{0,1}([-1,1],H)$, $C^{\alpha}_{0,1}([0,1],H), C^{\alpha}_{0}([-1,0],H), 0<\alpha<1$ the Banach spaces obtained by completion of the set of all smooth H-valued functions $\varphi(t)$ on $[a,b]$ in the norms

$$\| \varphi \|_{C^{\alpha}_{0,1}([-1,1],H)} = \|\varphi\|_{C([-1,1],H)} + \sup_{-1<t<t+\tau<0} \frac{(-t)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}}$$

$$+ \sup_{0<t<t+\tau<1} \frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}},$$

$$\| \varphi \|_{C^{\alpha}_{0}([-1,0],H)} = \|\varphi\|_{C([-1,0],H)} + \sup_{-1<t<t+\tau<0} \frac{(-t)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}},$$

$$\| \varphi \|_{C^{\alpha}_{0,1}([0,1],H)} = \|\varphi\|_{C([0,1],H)} + \sup_{0<t<t+\tau<1} \frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}},$$

where $C([a,b],H)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[a,b]$ with values in H equipped with the norm

$$||\varphi||_{C([a,b],H)} = \max_{a\le t\le b} \|\varphi(t)\|_H.$$

A function $u(t)$ is called a solution of problem (3.1) if the following conditions are satisfied:

i. $u(t)$ is a twice continuously differentiable on the segment $[0,1]$ and continuously differentiable on the segment $[-1,1]$.

ii. The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the function $Au(t)$ is continuous on $[-1,1]$.

iii. $u(t)$ satisfies the equation and nonlocal boundary condition (3.1).

A solution of problem (3.1) defined in this manner will from now on be referred to as a solution of problem (3.1) in the space $C(H) = C([-1,1],H)$.

We say that the problem (3.1) is well-posed in $C(H)$, if there exists the unique solution $u(t)$ in $C(H)$ of problem (3.1) for any $g(t) \in C([0,1],H)$, $f(t) \in C([-1,0],H)$ and $\mu \in D(A)$ and the following coercivity inequality is satisfied:

$$\|u''\|_{C([0,1],H)} + \|u'\|_{C([-1,0],H)} + \|Au\|_{C(H)} \tag{3.2}$$
$$\leq M[\|g\|_{C([0,1],H)} + \|f\|_{C([-1,0],H)} + \|A\mu\|_{H}] \ ,$$

where M does not depend on μ, $f(t)$ and $g(t)$.

The problem (3.1) is not well- posed in C(H)[4]. The well-posedness of the boundary value problem (3.1) can be established if one considers this problem in certain spaces $F(H)$ of smooth H-valued functions on $[-1,1]$.

A function $u(t)$ is said to be a solution of problem (3.1) in $F(H)$ if it is a solution of this problem in $C(H)$ and the functions $u''(t)$ $(t \in [0,1]), u'(t)(t \in [-1,1])$ and $Au(t)(t \in [-1,1])$ belong to $F(H)$.

As in the case of the space $C(H)$, we say that the problem (3.1) is well-posed in $F(H)$, if the following coercivity inequality is satisfied:

$$\|u''\|_{F([0,1],H)} + \|u'\|_{F([-1,0],H)} + \|Au\|_{F(H)} \tag{3.3}$$
$$\leq M[\|g\|_{F([0,1],H)} + \|f\|_{F([-1,0],H)} + \|A\mu\|_{H}] \ ,$$

where M does not depend on μ, $f(t)$ and $g(t)$.

We set $F(H)$ equal to $C^{\alpha}_{0,1}(H) = C^{\alpha}_{0,1}([-1,1],H), (0 < \alpha < 1)$ and we can establish the following coercivity inequality.

Theorem 3.1. *Suppose $\mu \in D(A)$. Then the boundary value problem (3.1) is well-posed in a Hölder space $C^{\alpha}_{0,1}(H)$ and the following coercivity inequality holds:*

$$\|u'\|_{C^{\alpha}_{0}([-1,0],H)} + ||Au||_{C^{\alpha}_{0,1}(H)} + \|u''\|_{C^{\alpha}_{0,1}([0,1],H)} \tag{3.4}$$

$$\leq \frac{M}{\alpha(1-\alpha)}\left[\|f\|_{C^{\alpha}_{0}([-1,0],H)} + \|g\|_{C^{\alpha}_{0,1}([0,1],H)}\right] \ + M\,\|A\mu\|_{H} \,.$$

Here M is independent of $f(t), g(t)$ and μ.

Proof. First, we will obtain the formula for solution of the problem (3.1). It is known that (see, e.g., [5]) for smooth data of the problems

$$\begin{cases} -u''\left(t\right) + Au\left(t\right) = g\left(t\right), \left(0 \leq t \leq 1\right), \\ u\left(0\right) = u_0,\ u\left(1\right) = u_1, \end{cases} \tag{3.5}$$

$$\begin{cases} u'(t) - Au(t) = f(t), (-1 \le t \le 0), \\ u(0) = u_0, \end{cases} \tag{3.6}$$

there are unique solutions of the problems (3.5), (3.6), and the following formulas hold:

$$u(t) = \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1} \left[\left(e^{-tA^{\frac{1}{2}}} - e^{-(-t+2)A^{\frac{1}{2}}}\right) u_0 \right. \tag{3.7}$$

$$\left. + \left(e^{-(1-t)A^{\frac{1}{2}}} - e^{-(t+1)A^{\frac{1}{2}}}\right) u_1 \right] + \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1}$$

$$\times \left(e^{-(1-t)A^{\frac{1}{2}}} - e^{-(t+1)A^{\frac{1}{2}}}\right) \int_0^1 A^{-\frac{1}{2}} 2^{-1} \left(e^{-(1-s)A^{\frac{1}{2}}} - e^{-(s+1)A^{\frac{1}{2}}}\right) g(s)ds$$

$$- \int_0^1 A^{-\frac{1}{2}} 2^{-1} \left(e^{-(t+s)A^{\frac{1}{2}}} - e^{-|t-s|A^{\frac{1}{2}}}\right) g(s)ds, \; 0 \le t \le 1,$$

and

$$u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} f(s)ds, \quad -1 \le t \le 0. \tag{3.8}$$

Using the condition $u(1) = u(-1) + \mu$ and formulas (3.7), (3.8), we can write

$$u(t) = \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1} \left[\left(e^{-tA^{\frac{1}{2}}} - e^{-(-t+2)A^{\frac{1}{2}}}\right) u_0 \right. \tag{3.9}$$

$$\left. + \left(e^{-(1-t)A^{\frac{1}{2}}} - e^{-(t+1)A^{\frac{1}{2}}}\right) \left(e^{-A} u_0 + \int_0^{-1} e^{-(1+s)A} f(s)ds + \mu\right)\right] + \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1}$$

$$\times \left(e^{-(1-t)A^{\frac{1}{2}}} - e^{-(t+1)A^{\frac{1}{2}}}\right) \int_0^1 A^{-\frac{1}{2}} 2^{-1} \left(e^{-(1-s)A^{\frac{1}{2}}} - e^{-(s+1)A^{\frac{1}{2}}}\right) g(s)ds$$

$$- \int_0^1 A^{-\frac{1}{2}} 2^{-1} \left(e^{-(t+s)A^{\frac{1}{2}}} - e^{-|t-s|A^{\frac{1}{2}}}\right) g(s)ds, \; 0 \le t \le 1.$$

For u_0, using the condition $u'(0+) = Au(0) + f(0)$ and formula (3.9), we obtain the operator equation

$$Au(0) + f(0) = \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1} \left[-A^{\frac{1}{2}} \left(I + e^{-2A^{\frac{1}{2}}}\right) u_0 \right. \tag{3.10}$$

$$\left. + 2A^{\frac{1}{2}} e^{-A^{\frac{1}{2}}} \left(e^{-A} u_0 + \int_0^{-1} e^{-(1+s)A} f(s)ds + \mu\right)\right] + \int_0^1 e^{-sA^{\frac{1}{2}}} g(s)ds$$

$$+\left(I-e^{-2A^{\frac{1}{2}}}\right)^{-1}2A^{\frac{1}{2}}e^{-A^{\frac{1}{2}}}\int_{0}^{1}A^{-\frac{1}{2}}2^{-1}\left(e^{-(1-s)A^{\frac{1}{2}}}-e^{-(s+1)A^{\frac{1}{2}}}\right)g(s)ds.$$

Since the operator

$$I+e^{-2A^{\frac{1}{2}}}+A^{\frac{1}{2}}(I-e^{-2A^{\frac{1}{2}}})-2e^{-(A^{\frac{1}{2}}+A)}$$

has an inverse

$$T=\left(I+e^{-2A^{\frac{1}{2}}}+A^{\frac{1}{2}}(I-e^{-2A^{\frac{1}{2}}})-2e^{-(A^{\frac{1}{2}}+A)}\right)^{-1},$$

for the solution of the operator equation (3.10) we have the formula

$$u_0=T\left[e^{-A^{\frac{1}{2}}}\left[2\int_{0}^{-1}e^{-(1+s)A}f(s)ds\right.\right.\tag{3.11}$$

$$\left.\left.+\int_{0}^{1}A^{-\frac{1}{2}}\left(e^{-(1-s)A^{\frac{1}{2}}}-e^{-(s+1)A^{\frac{1}{2}}}\right)g(s)ds\right]+2e^{-A^{\frac{1}{2}}}\mu\right]$$

$$+\left(I-e^{-2A^{\frac{1}{2}}}\right)T\left[-A^{-\frac{1}{2}}f(0)+\int_{0}^{1}A^{-\frac{1}{2}}e^{-sA^{\frac{1}{2}}}g(s)ds\right].$$

Hence, for the solution of the nonlocal boundary value problem (3.1), we have formulas (3.8), (3.9) and (3.11).

Second, we will establish estimate (3.4). It is based on the estimates

$$\|u'\|_{C_0^{\alpha}([-1,0],H)}+||Au||_{C_0^{\alpha}([-1,0],H)}\tag{3.12}$$

$$\leq\frac{M}{\alpha(1-\alpha)}\|f\|_{C_0^{\alpha}([-1,0],H)}+M\left\|Au_0\right\|_H$$

for the solution of an inverse Cauchy problem (3.6) and on the estimates

$$\|u''\|_{C_{0,1}^{\alpha}([0,1],H)}+||Au||_{C_{0,1}^{\alpha}([0,1],H)}\leq\frac{M}{\alpha(1-\alpha)}\|g\|_{C_{0,1}^{\alpha}([0,1],H)}\tag{3.13}$$

$$+M[\|Au_0\|_H+\|Au_1\|_H]$$

for the solution of the boundary value problem (3.5) and on the estimates

$$\|Au_0\|_H\leq\frac{M}{\alpha(1-\alpha)}[\|g\|_{C_{0,1}^{\alpha}([0,1],H)}+\|f\|_{C_0^{\alpha}([-1,0],H)}]+M\left\|A\mu\right\|_H,\tag{3.14}$$

$$\|Au_1\|_H\leq\frac{M}{\alpha(1-\alpha)}[\|f\|_{C_0^{\alpha}([-1,0],H)}+\|g\|_{C_{0,1}^{\alpha}([0,1],H)}]+M\left\|A\mu\right\|_H\tag{3.15}$$

for the solution of the boundary value problem (3.1). Estimates (3.12) and (3.13) were established in [6] and [7]. The proof of estimates (3.14)-(3.15) is based on the formulas

$$Au_0 = Te^{-A^{\frac{1}{2}}}\left[2\int_0^{-1} Ae^{-(1+s)A}\left(f(s)-f(-1)\right)ds\right.$$

$$\left.+\int_0^1 A^{\frac{1}{2}}e^{-(1-s)A^{\frac{1}{2}}}\left(g(s)-g(1)\right)ds-\int_0^1 A^{\frac{1}{2}}e^{-(s+1)A^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds+2A\mu\right]$$

$$+\left(I-e^{-2A^{\frac{1}{2}}}\right)T\int_0^1 A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds+2Te^{-A^{\frac{1}{2}}}\left(e^{-A}-I\right)f(-1)$$

$$+T\left(e^{-A^{\frac{1}{2}}}-e^{-2A^{\frac{1}{2}}}\right)g(1)+T\left(I+2e^{-3A^{\frac{1}{2}}}-2e^{-2A^{\frac{1}{2}}}-e^{-A^{\frac{1}{2}}}\right)g(0)$$

$$+T\left(e^{-2A^{\frac{1}{2}}}-2e^{-(A^{\frac{1}{2}}+A)}\right)f(0),$$

$$Au_1=-e^{-A}f(0)+A\mu$$

$$+e^{-A}\left\{Te^{-A^{\frac{1}{2}}}\left[2\int_0^{-1} Ae^{-(1+s)A}\left(f(s)-f(-1)\right)ds\right.\right.$$

$$\left.+\int_0^1 A^{\frac{1}{2}}e^{-(1-s)A^{\frac{1}{2}}}\left(g(s)-g(1)\right)ds-\int_0^1 A^{\frac{1}{2}}e^{-(s+1)A^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds+2A\mu\right]$$

$$+\left(I-e^{-2A^{\frac{1}{2}}}\right)T\int_0^1 A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds+2Te^{-A^{\frac{1}{2}}}\left(e^{-A}-I\right)f(-1)$$

$$+T\left(e^{-A^{\frac{1}{2}}}-e^{-2A^{\frac{1}{2}}}\right)g(1)+T\left(I+2e^{-3A^{\frac{1}{2}}}-2e^{-2A^{\frac{1}{2}}}-e^{-A^{\frac{1}{2}}}\right)g(0)$$

$$\left.+T\left(I+e^{-2A^{\frac{1}{2}}}-2e^{-(A^{\frac{1}{2}}+A)}\right)f(0)\right\}$$

$$+\int_0^{-1} Ae^{-(1+s)A}\left(f(s)-f(-1)\right)ds+\left(e^{-A}-I\right)f(-1)$$

for the solution of problem (3.1) and on the estimates

$$\parallel \left(I-e^{-2A^{\frac{1}{2}}}\right)^{-1}\parallel_{H\to H}\leq M, \tag{3.16}$$

$$\parallel \left(I+e^{-2A^{\frac{1}{2}}}+A^{\frac{1}{2}}(I-e^{-2A^{\frac{1}{2}}})-2e^{-(A^{\frac{1}{2}}+A)}\right)^{-1}\parallel_{H\to H}\leq M, \tag{3.17}$$

$$\parallel \left(A^{\frac{1}{2}}\right)^{\alpha}e^{-tA^{\frac{1}{2}}}||_{H\to H}\leq t^{-\alpha}, t>0, 0\leq\alpha\leq 1, \tag{3.18}$$

$$\| A^{\alpha}e^{-tA}||_{H\to H}\leq t^{-\alpha},t>0,0\leq\alpha\leq 1. \tag{3.19}$$

Theorem 3.1 is proved.

Now, we consider the applications of this abstract result. First, the mixed boundary value problem for elliptic-parabolic equation

$$\begin{cases} -u_{tt}-(a(x)u_x)_x+\delta u=g(t,x),0<t<1,0<x<1,\\ u_t+(a(x)u_x)_x-\delta u=f(t,x),-1<t<0,0<x<1,\\ u(t,0)=u(t,1),u_x(t,0)=u_x(t,1),-1\leq t\leq 1,\\ u(1,x)=u(-1,x)+\mu(x),0\leq x\leq 1,\\ u(0+,x)=u(0-,x),u_t(0+,x)=u_t(0-,x),0\leq x\leq 1 \end{cases} \tag{3.20}$$

is considered. Problem (3.20) has a unique smooth solution $u(t,x)$ for the smooth $a(x)\geqslant a>0(x\in(0,1))$, and $g(t,x)(t\in[0,1],\ x\in[0,1])$, $f(t,x)(t\in[-1,0],x\in[0,1])$ functions and $\delta=\text{const}>0$. This allows us to reduce the mixed problem(3.20) to the nonlocal boundary value problem (3.1) in a Hilbert space $H=L_2[0,1]$ with a self-adjoint positive definite operator A defined by (3.20).

Theorem 3.2. *The solutions of the nonlocal boundary value problem (3.20) satisfy the coercivity inequality*

$$\| u_{tt}\|_{C^{\alpha}_{0,1}([0,1],L_2[0,1])}+\| u_t\|_{C^{\alpha}_{0}([-1,0],L_2[0,1])}+\| u\|_{C^{\alpha}_{0,1}([-1,1],W_2^2[0,1])}$$
$$\leq\frac{M}{\alpha(1-\alpha)}\left[\| g\|_{C^{\alpha}_{0,1}([0,1],L_2[0,1])}+\| f\|_{C^{\alpha}_{0}([-1,0],L_2[0,1])}\right]+M\left\|\mu\right\|_{W_2^2[0,1]}.$$

Here M does not depend on $f(t,x)$, $g(t,x)$ and $\mu(x)$.

The proof of Theorem 3.2 is based on the abstract Theorem 3.1 and the symmetry properties of the space operator generated by the problem (3.20).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<1,\ \ 1\leq k\leq n)$ with boundary S, $\overline{\Omega}=\Omega\cup S$. In $[-1,1]\times\Omega$, the mixed boundary value problem for multi-dimensional mixed equation

$$\begin{cases} -u_{tt}-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}=g(t,x),0<t<1,x\in\Omega,\\ u_t+\sum\limits_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}=f(t,x),-1<t<0,x\in\Omega,\\ u(t,x)=0,x\in S,\ -1\leq t\leq 1;u(1,x)=u(-1,x)+\mu(x),x\in\ \overline{\Omega},\\ u(0+,x)=u(0-,x),u_t(0+,x)=u_t(0-,x),x\in\ \overline{\Omega} \end{cases} \tag{3.21}$$

is considered. The problem (3.21) has a unique smooth solution $u(t,x)$ for the smooth $a_r(x)\geqslant a>0$ $(x\in\Omega)$ and $g(t,x)$ $(t\in(0,1),\ x\in\ \overline{\Omega})$, $f(t,x)$ $(t\in(-1,0),\ x\in\ \overline{\Omega})$ functions. This allows us to reduce the mixed problem (3.21) to the nonlocal boundary value problem (3.1) in a Hilbert space $H=L_2(\overline{\Omega})$ of all the integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f\|_{L_2(\overline{\Omega})}=\{\int\cdots\int_{x\in\overline{\Omega}}|f(x)|^2dx_1\cdots dx_n\}^{\frac{1}{2}}$$

with a self- adjoint positive definite operator A defined by (3.21).

Theorem 3.3. *The solutions of the nonlocal boundary value problem (3.21) satisfy the coercivity inequality*

$$\| u_{tt} \|_{C^{\alpha}_{0,1}([0,1],L_2(\overline{\Omega}))} + \| u_t \|_{C^{\alpha}_{0}([-1,0],L_2(\overline{\Omega})])} + \| u \|_{C^{\alpha}_{0,1}([-1,1],W_2^2(\overline{\Omega}))}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \left[\| g \|_{C^{\alpha}_{0,1}([0,1],L_2(\overline{\Omega}))} + \| f \|_{C^{\alpha}_{0}([-1,0],L_2(\overline{\Omega}))} \right] + M \|\mu\|_{W_2^2(\overline{\Omega})} .$$

Here M does not depend on $f(t,x)$, $g(t,x)$ and $\mu(x)$.

The proof of Theorem 3.3 is based on the abstract Theorem 1.1 and the symmetry properties of the space operator generated by the problem (3.21) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3.4. *For the solutions of the elliptic differential problem*

$$\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = \omega(x), x \in \widetilde{\Omega}, \tag{3.22}$$

$$u(x) = 0, x \in S$$

the following coercivity inequalities are valid [8],

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M||\omega||_{L_2(\overline{\Omega})}.$$

CHAPTER 4

FIRST ORDER OF ACCURACY DIFFERENCE SCHEME: WELL-POSEDNESS

Let us associate the boundary-value problem (3.1) with the corresponding first order of accuracy difference scheme

$$\begin{cases} -\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_k=g_k, \\ g_k=g\left(t_k\right),\ t_k=k\tau, 1\leq k\leq N-1, \\ \tau^{-1}\left(u_k-u_{k-1}\right)-Au_{k-1}=f_k, \quad f_k=f(t_{k-1}), \\ t_{k-1}=(k-1)\tau,\ -(N-1)\leq k\leq 0, \\ u_N=u_{-N}+\ \mu, u_1-u_0=u_0-u_{-1}. \end{cases} \tag{4.1}$$

A study of discretization, over time only, of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A, is replaced by the difference operators A_h that act in the Hilbert spaces H_h and are uniformly self-adjoint positive definite in h for $0<h\leq h_0$.

Let $P=P(\tau A)=(I+\tau A)^{-1}$, then the following estimates are satisfied [8]:

$$\| P^k \|_{H\to H}\leq M(1+\delta\tau)^{-k}, k\tau \| AP^k \|_{H\to H}\leq M, k\geq 1, \delta>0, \tag{4.2}$$

$$\| A^{\beta}(P^{k+r}-P^k) \|_{H\to H} \leq M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+r\leq N,\ 0\leq\alpha,\ \beta\leq 1. \tag{4.3}$$

Furthermore, for a self-adjoint positive definite operator A it follows that the operator $R=(I+\tau B)^{-1}$ is defined on the whole space H and it is a bounded operator and the following estimates hold:

$$\| R^k \|_{H\to H}\leq M(1+\delta\tau)^{-k}, k\tau \| BR^k \|_{H\to H}\leq M, k\geq 1, \delta>0, \tag{4.4}$$

$$\| B^{\beta}(R^{k+r}-R^k) \|_{H\to H} \leq M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+r\leq N,\ 0\leq\alpha,\ \beta\leq 1. \tag{4.5}$$

Here $B=\frac{1}{2}(\tau A+\sqrt{A(4+\tau^2A)})$. From (4.2) and (4.4) it follows that

$$||(I-R^{2N})^{-1}||_{H\to H}\leq M, \tag{4.6}$$

$$\begin{gathered}\left\|(I+(I+\tau A)\,(I+2\tau A)^{-1}\,R^{2N-1}+B^{-1}A\,(I+2\tau A)^{-1}\left(I-R^{2N-1}\right)-\right.\\ \left.-(2I+\tau B)\,(I+2\tau A)^{-1}\,R^{N}P^{N-1})^{-1}\right\|_{H\to H}\leq M.\end{gathered} \tag{4.7}$$

Theorem 4.1. *For any g_k, $1\leq k\leq N-1$ and f_k, $-N+1\leq k\leq 0$ the solution of the problem (4.1) exists and the following formulas hold*

$$\begin{gathered}u_k=(I-R^{2N})^{-1}\left\{\left[R^k-R^{2N-k}\right]u_0\right.\\ +\left[R^{N-k}-R^{N+k}\right]\left[P^Nu_0-\tau\sum_{s=-N+1}^{0}P^{s+N}f_s+\mu\right]\\ \left.-\left[R^{N-k}-R^{N+k}\right](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s}-R^{N+s}\right]g_s\tau\right\}\\ +(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{|k-s|}-R^{k+s}\right]g_s\tau\ ,1\leq k\leq N,\end{gathered} \tag{4.8}$$

$$u_k=P^{-k}u_0-\tau\sum_{s=k+1}^{0}P^{s-k}f_s,-N\leq k\leq 0, \tag{4.9}$$

$$\begin{gathered}u_0=T_\tau\,(I+2\tau A)^{-1}\,(I+\tau A)\left\{\left\{(2+\tau B)\,R^N\left[-\tau\sum_{s=-N+1}^{0}P^{s+N}f_s+\mu\right]\right.\right.\\ \left.-R^{N-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s}-R^{N+s}\right]g_s\tau\right\}\\ \left.+(I-R^{2N})B^{-1}\sum_{s=1}^{N-1}R^{s-1}g_s\tau-(I-R^{2N})\,(I+\tau B)\,B^{-1}Pf_0\right\},\end{gathered} \tag{4.10}$$

where

$$\begin{gathered}T_\tau=(I+(I+\tau A)\,(I+2\tau A)^{-1}\,R^{2N-1}+B^{-1}A\,(I+2\tau A)^{-1}\left(I-R^{2N-1}\right)\\ -(2I+\tau B)\,(I+2\tau A)^{-1}\,R^NP^{N-1})^{-1}.\end{gathered}$$

Proof. By [8], [9],

$$\begin{gathered}u_k=(I-R^{2N})^{-1}\left\{\left[R^k-R^{2N-k}\right]\xi+\left[R^{N-k}-R^{N+k}\right]\psi\right.\\ \left.-\left[R^{N-k}-R^{N+k}\right](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s}-R^{N+s}\right]g_s\tau\right\}\end{gathered}$$

$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{|k-s|}-R^{k+s}\right]g_s\tau\ ,1\leq k\leq N \tag{4.11}$$

is the solution of the boundary value difference problem

$$\begin{cases} -\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_k=g_k, \\ g_k=g\left(t_k\right),\ t_k=k\tau,\ 1\leq k\leq N-1, \\ u_0=\xi,\ u_N=\psi \end{cases} \tag{4.12}$$

and

$$u_k=P^{-k}\xi-\tau\sum_{s=k+1}^{0}P^{s-k}f_s, -N\leq k\leq 0 \tag{4.13}$$

is the solution of the inverse Cauchy problem

$$\begin{cases} \tau^{-1}\left(u_k-u_{k-1}\right)-Au_{k-1}=f_k, \quad f_k=f(t_{k-1}), \\ t_{k-1}=(k-1)\tau,\ -(N-1)\leq k\leq 0,\ u_0\ =\xi. \end{cases} \tag{4.14}$$

Using (4.11), (4.13) and the formulas

$$\psi=u_{-N}+\ \mu, \xi=u_0,$$

we obtain formulas (4.8), (4.9). For u_0, using (4.8), (4.9) and the formula

$$u_1-u_0=u_0-u_{-1}$$

we obtain the operator equation

$$\left(I-R^{2N}\right)^{-1}\left\{\left[R-R^{2N-1}\right]u_0+\left[R^{N-1}-R^{N+1}\right]\right.$$

$$\times\left[P^Nu_0-\tau\sum_{s=-N+1}^{0}P^{s+N}f_s+\mu\right]$$

$$\left.-\left[R^{N-1}-R^{N+1}\right](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s}-R^{N+s}\right]g_s\tau\right\}$$

$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{s-1}-R^{1+s}\right]g_s\tau=2u_0-Pu_0+\tau Pf_0.$$

The operator

$$I+(I+\tau A)\left(I+2\tau A\right)^{-1}R^{2N-1}+B^{-1}A\left(I+2\tau A\right)^{-1}\left(I-R^{2N-1}\right)$$

$$-(2I+\tau B)\left(I+2\tau A\right)^{-1}R^NP^{N-1}$$

has an inverse

$$T_\tau=(I+(I+\tau A)\left(I+2\tau A\right)^{-1}R^{2N-1}+B^{-1}A\left(I+2\tau A\right)^{-1}\left(I-R^{2N-1}\right)$$

$$-(2I+\tau B)\left(I+2\tau A\right)^{-1}R^NP^{N-1})^{-1}.$$

and the following formula is satisfied:

$$u_0 = T_\tau (I+\tau A)\,(I+2\tau A)^{-1}\left\{\left\{(2+\tau B)\,R^N\left[-\tau\sum_{s=-N+1}^{0}P^{s+N}f_s+\mu\right]\right.\right.$$

$$\left.-R^{N-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s}-R^{N+s}\right]g_s\tau\right\}$$

$$\left.+(I-R^{2N})B^{-1}\sum_{s=1}^{N-1}R^{s-1}g_s\tau-(I-R^{2N})\,(I+\tau B)\,B^{-1}Pf_0\right\}.$$

Theorem 4.1 is proved.

Let $F_\tau(H)=F([a,b]_\tau,H)$ be the linear space of mesh functions $\varphi^\tau=\{\varphi_k\}_{N_a}^{N_b}$ with values in the Hilbert space H. Next on $F_\tau(H)$ we denote $C([a,b]_\tau,H)$ and $C^\alpha_{0,1}([-1,1]_\tau,H)$, $C^\alpha_{0,1}([0,1]_\tau,H), C^\alpha_0([-1,0]_\tau,H), 0<\alpha<1$ - Banach spaces with the norms

$$\|\ \varphi^\tau\|_{C([a,b]_\tau,H)}=\max_{N_a\le k\le N_b}\|\ \varphi_k\ \|_H,$$

$$\|\ \varphi^\tau\|_{C^\alpha_{0,1}([-1,1]_\tau,H)}=\|\ \varphi^\tau\|_{C([-1,1]_\tau,H)}+\sup_{-N\le k<k+r\le 0}\|\ \varphi_{k+r}-\varphi_k\ \|_E\ \frac{(-k)^\alpha}{r^\alpha}$$

$$+\sup_{1\le k<k+r\le N-1}\|\ \varphi_{k+r}-\varphi_k\ \|_E\ \frac{((k+r)\,\tau)^\alpha(N-k)^\alpha}{r^\alpha},$$

$$\|\ \varphi^\tau\|_{C^\alpha_0([-1,0]_\tau,H)}=\|\ \varphi^\tau\|_{C([-1,0]_\tau,H)}+\sup_{-N\le k<k+r\le 0}\|\ \varphi_{k+r}-\varphi_k\ \|_E\ \frac{(-k)^\alpha}{r^\alpha},$$

$$\|\ \varphi^\tau\|_{C^\alpha_{0,1}([0,1]_\tau,H)}=\|\ \varphi^\tau\|_{C([0,1]_\tau,H)}+\sup_{1\le k<k+r\le N-1}\|\ \varphi_{k+r}-\varphi_k\ \|_E\ \frac{((k+r)\tau)^\alpha\,(N-k)^\alpha}{r^\alpha}.$$

The nonlocal boundary value problem (4.1) is said to be stable in $F([-1,1]_\tau,H)$ if we have the inequality

$$\|\ u^\tau\ \|_{F([-1,1]_\tau,H)}\le M\left[\|\ f^\tau\ \|_{F([-1,0]_\tau,H)}+\|\ g^\tau\ \|_{F([0,1]_\tau,H)}+\|\ \mu\ \|_H\right],$$

where M is independent not only of f^τ,g^τ,μ but also of τ.

Theorem 4.2. *The nonlocal boundary value problem (4.1) is stable in $C([-1,1]_\tau,H)$ norm.*

Proof. By [8],

$$\left\|\{u_k\}_{-N}^0\right\|_{C([-1,0]_\tau,H)}\le M\left[\|\ f^\tau\ \|_{C([-1,0]_\tau,H)}+\|u_0\|_H\ \right] \tag{4.15}$$

for the solution of an inverse Cauchy difference problem (4.14) and

$$\left\|\{u_k\}_1^{N-1}\right\|_{C([0,1]_\tau,H)}\le M\left[\|\ g^\tau\ \|_{C([0,1]_\tau,H)}+\|u_0\|_H\ +\|u_N\|_H\right] \tag{4.16}$$

for the solution of the boundary value problem (4.12). The proof of Theorem 4.2 is based on the stability inequalities (4.15) -(4.16) and on the estimates

$$\| u_0 \|_H \leq M \left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)} + \| \mu \|_H \right], \tag{4.17}$$

$$\| u_N \|_H \leq M \left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)} + \| \mu \|_H \right] \tag{4.18}$$

for the solution of the boundary value problem (4.1). Estimates (4.17) and (4.18) are derived from formula (4.10) and estimates (4.2), (4.4), (4.7). Theorem 4.2 is proved.

The nonlocal boundary value problem (4.1) is said to be coercively stable (well posed) in $F([-1,1]_\tau, H)$ if we have the coercive inequality

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{F([0,1]_\tau,H)} + \| \{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0} \|_{F([-1,0]_\tau,H)}$$

$$+ \left\| \{Au_k\}_{-N}^{N-1} \right\|_{F([-1,1]_\tau,H)} \leq M \left[\| f^\tau \|_{F([-1,0]_\tau,H)} + \| g^\tau \|_{F([0,1]_\tau,H)} + \| A\mu \|_H \right],$$

where M is independent not only of f^τ, g^τ, μ but also of τ.

Since the nonlocal boundary value problem (3.1) in the space $C([0,1], H)$ of continuous functions defined on $[-1,1]$ and with values in H is not well-posed for the general positive unbounded operator A and space H, then the well-posedness of the difference nonlocal boundary value problem (4.1) in $C([-1,1]_\tau, H)$ norm does not take place uniformly with respect to $\tau > 0$. This means that the coercive norm

$$\| u^\tau \|_{K_\tau(E)} = \left\| \{Au_k\}_{-N}^{N-1} \right\|_{C([-1,1]_\tau,H)}$$

$$+ \| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau,H)} + \| \{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0} \|_{C([-1,0]_\tau,H)}$$

tends to ∞ as $\tau \to 0^+$. The investigation of the difference problem (4.1) permits to establish the order of growth of this norm to ∞.

Theorem 4.3. *Assume that $\mu \in D(A)$ and $f_0 \in D(I + \tau B)$. Then for the solution of the difference problem (4.1) we have almost coercivity inequality*

$$\| u^\tau \|_{K_\tau(E)} \leq M [\| A\mu \|_H + \| (I + \tau B) f_0 \|_H]$$

$$+ M \min \left\{ \ln \frac{1}{\tau}, 1 + |\ln \| A \|_{H \to H}| \right\} \left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)} \right],$$

where M is independent not only of f^τ, g^τ, μ but also of τ.

Proof. By [8],

$$\| \{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0} \|_{C([-1,0]_\tau,H)} + \left\| \{Au_k\}_{-N}^{0} \right\|_{C([-1,0]_\tau,H)} \tag{4.19}$$

$$\leq M \min \left\{ \ln \frac{1}{\tau}, 1 + |\ln \| A \|_{H \to H}| \right\} \| f^\tau \|_{C([-1,0]_\tau,H)} + M \| Au_0 \|_H$$

for the solution of an inverse Cauchy difference problem (4.14) and

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau,H)} + \left\| \{Au_k\}_1^{N-1} \right\|_{C([0,1]_\tau,H)} \tag{4.20}$$

$$\leq M \min\left\{ \ln\frac{1}{\tau}, 1+|\ln \| A \|_{H\to H}| \right\} \| g^\tau \|_{C([0,1]_\tau,H)} + M[\|Au_0\|_H + \|Au_N\|_H]$$

for the solution of the boundary value problem (4.12). Then the proof of Theorem 4.3 is based on the almost coercivity inequalities (4.19), (4.20) and on the estimates

$$\| Au_0\|_H \leq M\,[\| A\mu \|_H + \| (I+\tau B)\, f_0\|_H]$$

$$+M \min\left\{ \ln\frac{1}{\tau}, 1+|\ln \| A \|_{H\to H}| \right\} \left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)}\right],$$

$$\|Au_N\|_H \leq M\,[\| A\mu \|_H + \| (I+\tau B)\, f_0\|_H]$$

$$+M \min\left\{ \ln\frac{1}{\tau}, 1+|\ln \| A \|_{H\to H}| \right\} \left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)}\right]$$

for the solution of the boundary value problem (4.1) . The proof of these estimates follows the scheme of the papers [8] and [9] and relies on the formula (4.10) and on the estimates (4.2), (4.4) and (4.7). Theorem 4.3 is proved.

Theorem 4.4. *Let the assumptions of Theorem 4.3 be satisfied. Then the boundary value problem (4.1) is well-posed in a Hölder space $C_{0,1}^{\alpha}([-1,1]_\tau,H)$ and the following coercivity inequality holds:*

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_{0,1}^{\alpha}([0,1]_\tau,H)} + \left\| \{Au_k\}_{-N}^{N-1} \right\|_{C_{0,1}^{\alpha}([-1,1]_\tau,H)}$$

$$+\| \{\tau^{-1}(u_k-u_{k-1})\}_{-N+1}^{0} \|_{C_0^{\alpha}([-1,0]_\tau,H)} \leq M\,[\| A\mu \|_H + \| (I+\tau B)\, f_0\|_H]$$

$$+\frac{M}{\alpha(1-\alpha)}\left[\| f^\tau \|_{C_0^{\alpha}([-1,0]_\tau,H)} + \| g^\tau \|_{C_{0,1}^{\alpha}([0,1]_\tau,H)}\right],$$

where M is independent not only of f^τ, g^τ, μ but also of τ and α.

Proof. By [8] and [9],

$$\| \{\tau^{-1}(u_k-u_{k-1})\}_{-N+1}^{0} \|_{C_0^{\alpha}([-1,0]_\tau,H)} + \left\| \{Au_k\}_{-N}^{0} \right\|_{C_0^{\alpha}([-1,0]_\tau,H)} \tag{4.21}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \| f^\tau \|_{C_0^{\alpha}([-1,0]_\tau,H)} + M\,\|Au_0\|_H$$

for the solution of an inverse Cauchy difference problem (4.14) and

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_{0,1}^{\alpha}([0,1]_\tau,H)} + \left\| \{Au_k\}_1^{N-1} \right\|_{C_{0,1}^{\alpha}([0,1]_\tau,H)} \tag{4.22}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \| g^{\tau} \|_{C_{0,1}^{\alpha}([0,1]_{\tau},H)} + M[\|Au_0\|_H + \|Au_N\|_H]$$

for the solution of the boundary value problem (4.12). Then the proof of Theorem 4.4 is based on the coercivity inequalities (4.21) and (4.22) and on the estimates

$$\|Au_0\|_H \leq M\left[\|A\mu\|_H + \|\left(I+\tau B\right) f_0\|_H\right], \tag{4.23}$$

$$+\frac{M}{\alpha(1-\alpha)}\left[\| f^{\tau} \|_{C_0^{\alpha}([-1,0]_{\tau},H)} + \| g^{\tau} \|_{C_{0,1}^{\alpha}([0,1]_{\tau},H)}\right]$$

$$\|Au_N\|_H \leq M\left[\|A\mu\|_H + \|\left(I+\tau B\right) f_0\|_H\right] \tag{4.24}$$

$$+\frac{M}{\alpha(1-\alpha)}\left[\| f^{\tau} \|_{C_0^{\alpha}([-1,0]_{\tau},H)} + \| g^{\tau} \|_{C_{0,1}^{\alpha}([0,1]_{\tau},H)}\right]$$

for the solution of the boundary value problem (4.1). Estimates (4.23)-(4.24) are derived from the formulas

$$Au_0 = T_{\tau}\left(I+2\tau A\right)^{-1}\left(I+\tau A\right)$$

$$\times\left\{\left\{\left(2+\tau B\right)R^N\left[-\tau\sum_{s=-N+1}^{0} AP^{s+N}\left(f_s - f_{-N+1}\right) + A\mu\right]\right.\right.$$

$$\left.\left.-R^{N-1}AB^{-2}\left\{\sum_{s=1}^{N-1} BR^{N-s}\left(g_s - g_{N-1}\right)\tau + \sum_{s=1}^{N-1} BR^{N+s}\left(g_1 - g_s\right)\tau\right\}\right\}\right.$$

$$\left.+(I-R^{2N})AB^{-2}\sum_{s=1}^{N-1} BR^{s-1}\left(g_s - g_1\right)\tau\right\}$$

$$+T_{\tau}\left(I+2\tau A\right)^{-1}\left(I+\tau A\right)\left\{\left\{\left(2+\tau B\right)R^N\left(P^N - I\right)f_{-N+1}\right.\right.$$

$$\left.-R^{N-1}AB^{-2}\left\{\left(I-R^{N-1}\right)g_{N-1} - \left(R^{N-2}-R^{2N-1}\right)g_1\right\}\right\}$$

$$\left.+(I-R^{2N})AB^{-2}\left(I-R^{N-1}\right)g_1 - \left(I-R^{2N}\right)\left(I+\tau B\right)B^{-1}APf_0\right\},$$

$$Au_N = P^N\left\{T_{\tau}\left(I+2\tau A\right)^{-1}\left(I+\tau A\right)\right.$$

$$\times\left\{\left\{\left(2+\tau B\right)R^N\left[-\tau\sum_{s=-N+1}^{0} AP^{s+N}\left(f_s - f_{-N+1}\right) + A\mu\right]\right.\right.$$

$$\left.\left.-R^{N-1}AB^{-2}\left\{\sum_{s=1}^{N-1} BR^{N-s}\left(g_s - g_{N-1}\right)\tau + \sum_{s=1}^{N-1} BR^{N+s}\left(g_1 - g_s\right)\tau\right\}\right\}\right.$$

$$\left.\left.+(I-R^{2N})AB^{-2}\sum_{s=1}^{N-1} BR^{s-1}\left(g_s - g_1\right)\tau\right\}\right\}$$

$$-\tau\sum_{s=-N+1}^{0} AP^{s+N}\left(f_s - f_{-N+1}\right) + A\mu + \left(P^N - I\right)f_{-N+1}$$

$$+P^N\left\{T_{\tau}\left(I+2\tau A\right)^{-1}\left(I+\tau A\right)\left\{\left\{\left(2+\tau B\right)R^N\left(P^N - I\right)f_{-N+1}\right.\right.\right.$$

$$\left.-R^{N-1}AB^{-2}\left\{\left(I-R^{N-1}\right)g_{N-1} - \left(R^{N-2}-R^{2N-1}\right)g_1\right\}\right\}$$

$$+(I-R^{2N})AB^{-2}\left(I-R^{N-1}\right)g_1-(I-R^{2N})\left(I+\tau B\right)B^{-1}APf_0\}\}$$

for the solution of problem (4.1) and estimates (4.2), (4.4) and (4.7). Theorem 4.4 is proved.

Now, the applications of this abstract result to the approximate solution of the mixed boundary value problem for elliptic-parabolic equation (3.21) are considered. The discretization of problem (3.21) was carried out in two steps. In the first step the grid sets

$$\widetilde{\Omega}_h=\{x=x_m=(h_1m_1,\cdots,h_nm_n),m=(m_1,\cdots,m_n)\ ,$$
$$0\le m_r\le N_r,h_rN_r=L,r=1,\cdots,n\}\ ,$$
$$\Omega_h=\widetilde{\Omega}_h\cap\Omega,S_h=\widetilde{\Omega}_h\cap S$$

are defined. To the differential operator A generated by the problem (3.21) we assign the difference operator A_h^x by the formula

$$A_h^xu_x^h=-\sum_{r=1}^n\left(a_r(x)u_{\bar{x}_r}^h\right)_{x_r,j_r} \tag{4.25}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x)=0$ for all $x\in S_h$.With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\begin{cases}-\frac{d^2u^h(t,x)}{dt^2}+A_h^xu^h(t,x)=g^h(t,x),\ 0\le t\le 1,\ x\in\widetilde{\Omega}_h,\\ \frac{du^h(t,x)}{dt}-A_h^xu^h(t,x)=f^h(t,x),-1\le t\le 0,\ x\in\widetilde{\Omega}_h,\\ u^h(-1,x)=u^h(1,x)+\mu^h(x),x\in\widetilde{\Omega}_h,\\ u^h(0+,x)=u^h(0-,x),\frac{du^h(0+,x)}{dt}=\frac{du^h(0-,x)}{dt},x\in\widetilde{\Omega}_h\end{cases} \tag{4.26}$$

for an infinite system of ordinary differential equations.

In the second step problem (4.26) is replaced by the difference scheme (4.1)

$$\begin{cases}-\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^xu_k^h\left(x\right)=g_k^h(x),\\ g_k^h(x)=\{g(t_k,x_n)\}_1^{M-1},t_k=k\tau,1\le k\le N-1,\ N\tau=1,x\in\widetilde{\Omega}_h,\\ \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}-A_h^xu_{k-1}^h\left(x\right)=f_k^h(x),\\ f_k^h(x)=\{f(t_{k-1},x_n)\}_1^{M-1},t_{k-1}=(k-1)\,\tau,-N+1\le k\le -1,x\in\widetilde{\Omega}_h,\\ u_{-N}^h(x)=u_N^h(x)+\mu^h(x),x\in\widetilde{\Omega}_h,\\ u_1^h(x)-u_0^h(x)=u_0^h(x)-u_{-1}^h(x),x\in\widetilde{\Omega}_h.\end{cases} \tag{4.27}$$

Based on the number of corollaries of the abstract theorems given above, to formulate the result, one needs to introduce the space $L_{2h} = L_2(\widetilde{\Omega}_h)$ of all the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x\in\overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2}.$$

Theorem 4.5. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then the solutions of difference scheme (4.27) satisfy the following stability and almost coercivity estimates:*

$$\left\|\{u_k^h\}_{-N}^{N-1}\right\|_{C([-1,1]_\tau, L_{2h})} \leq M\left[\left\| \left(f^h\right)^\tau \right\|_{C([-1,0]_\tau, L_{2h})} + \left\| \left(g^h\right)^\tau \right\|_{C([0,1]_\tau, L_{2h})}\right] + M\left\|\mu^h\right\|_{L_{2h}},$$

$$\left\| \{\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\}_1^{N-1} \right\|_{C([0,1]_\tau, L_{2h})} + \left\| \{\tau^{-1}(u_k^h - u_{k-1}^h)\}_{-N+1}^0 \right\|_{C([-1,0]_\tau, L_{2h})}$$
$$+\left\|\{u_k^h\}_{-N}^{N-1}\right\|_{C([-1,1]_\tau, W_{2h}^2)} \leq M\left[\left\|\mu^h\right\|_{W_{2h}^2} + \tau\left\|f_0^h\right\|_{W_{2h}^1}\right]$$
$$+M\ln\frac{1}{\tau + |h|}\left[\left\| \left(f^h\right)^\tau \right\|_{C([-1,0]_\tau, L_{2h})} + \left\| \left(g^h\right)^\tau \right\|_{C([0,1]_\tau, L_{2h})}\right].$$

Here M does not depend on τ, h, $\mu^h(x)$ and $g_k^h(x), 1 \leq k \leq N-1, f_k^h, -N+1 \leq k \leq 0$.

The proof of Theorem 4.5 is based on the abstract Theorems 4.2-4.3, on the estimate

$$\min\left\{\ln\frac{1}{\tau}, 1 + \left|\ln \left\| A_h^x \right\|_{L_{2h}\to L_{2h}}\right|\right\} \leq M\ln\frac{1}{\tau + |h|} \tag{4.28}$$

as well as the symmetry properties of the difference operator A_h^x defined by the formula (4.25) in L_{2h}, along with the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 4.6. *For the solutions of the elliptic difference problem*

$$A_h^x u^h(x) = \omega^h(x), x \in \Omega_h, \tag{4.29}$$
$$u^h(x) = 0, x \in S_h$$

the following coercivity inequality holds[10]:

$$\sum_{r=1}^{n}\left\|\left(u^h\right)_{\bar{x}_r x_r, j_r}\right\|_{L_{2h}} \leq M\|\omega^h\|_{L_{2h}}.$$

Theorem 4.7. *Let τ and $|h|$ be sufficiently small numbers. Then the solutions of difference scheme (4.27) satisfy the following coercivity stability estimates:*

$$\left\| \{\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\}_1^{N-1} \right\|_{C_{0,1}^\alpha([0,1]_\tau, L_{2h})} + \left\| \{\tau^{-1}(u_k^h - u_{k-1}^h)\}_{-N+1}^0 \right\|_{C_0^\alpha([-1,0]_\tau, L_{2h})}$$
$$+\left\|\{u_k^h\}_{-N}^{N-1}\right\|_{C_{0,1}^\alpha([-1,1]_\tau, W_{2h}^2)} \leq M\left[\left\|\mu^h\right\|_{W_{2h}^2} + \tau\left\|f_0^h\right\|_{W_{2h}^1}\right]$$
$$+\frac{M}{\alpha(1-\alpha)}\left[\left\| \left(f^h\right)^\tau \right\|_{C_0^\alpha([-1,0]_\tau, L_{2h})} + \left\| \left(g^h\right)^\tau \right\|_{C_{0,1}^\alpha([0,1]_\tau, L_{2h})}\right].$$

Here M does not depend on τ, h, $\mu^h(x)$ and $g_k^h(x), 1 \leq k \leq N-1, f_k^h, -N+1 \leq k \leq 0$.

The proof of Theorem 4.7 is based on the abstract Theorem 4.4, and the symmetry properties of the difference operator A_h^x defined by the formula (4.25) and on the Theorem 4.6 on the coercivity inequality for the solution of the elliptic difference equation (4.29) in L_{2h}.

Note that in a similar manner can be constructed the difference schemes of the first order of accuracy with respect to one variable for approximate solutions of the boundary value problem (3.20). Abstract theorems given from above permit us to obtain the stability, the almost stability and the coercive stability estimates for the solutions of these difference schemes.

CHAPTER 5

SECOND ORDER OF ACCURACY DIFFERENCE SCHEME

Now, the second order of accuracy difference scheme

$$\begin{cases} -\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_k=g_k, \\ g_k=g\left(t_k\right),\ t_k=k\tau, 1\leq k\leq N-1, \\ \tau^{-1}\left(u_k-u_{k-1}\right)-\frac{1}{2}\left(Au_{k-1}+Au_k\right)=f_k, \quad f_k=f(t_{k-\frac{1}{2}}), \\ t_{k-\frac{1}{2}}=(k-\frac{1}{2})\tau,\ -(N-1)\leq k\leq 0, \\ u_N=u_{-N}+\ \mu, u_2-4u_1+3u_0=-3u_0+4u_{-1}-u_{-2} \end{cases} \tag{5.1}$$

for the approximate solution of problem (3.1) is considered.

Let $P=P(\tau A)=(I-\frac{\tau A}{2})(I+\frac{\tau A}{2})^{-1}, C=C(\tau A)=(I+\frac{\tau A}{2})^{-1}$, then the following estimates are valid [11],[12]:

$$\| P^k \|_{H\to H}\leq 1, \| C \|_{H\to H}\leq 1, k\tau \| AP^kC^2 \|_{H\to H}\leq M, k\geq 1, \delta>0, \tag{5.2}$$

$$\| A^{\beta}(P^{k+r}-P^k)C^3 \|_{H\to H}\leq M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+r\leq N,\ 0\leq \alpha,\ \beta\leq 1, \tag{5.3}$$

$$\| A^{\beta}(P^{k+2r}-P^k)C^2 \|_{H\to H}\leq M\frac{(2r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+2r\leq N,\ 0\leq \alpha,\ \beta\leq 1. \tag{5.4}$$

From these estimates and (4.4) it follows that

$$\begin{gathered} \left\|(I+B^{-1}A(I+\tau A+\tau C^2)D\left(I-R^{2N-1}\right) \right. \\ \left. +DP^2R^{2N-1}-(2I+\tau B)DR^NP^{N+2})^{-1}\right\|_{H\to H}\leq M, \end{gathered} \tag{5.5}$$

where

$$D=\left(I+2\tau A+\frac{5}{4}\left(\tau A\right)^2\right)^{-1}.$$

Theorem 5.1. *For any g_k, $1 \leq k \leq N-1$ and f_k, $-N+1 \leq k \leq 0$ the solution of the problem (5.1) exists and the following formula holds*

$$u_k = (I - R^{2N})^{-1} \left\{ \left[R^k - R^{2N-k} \right] u_0 + \right. \tag{5.6}$$

$$\left[R^{N-k} - R^{N+k} \right] \left[P^N u_0 - \tau \sum_{s=-N+1}^{0} P^{s+N-1} C f_s + \mu \right]$$

$$\left. - \left[R^{N-k} - R^{N+k} \right] (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s} \right] g_s \tau \right\}$$

$$+ (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{|k-s|} - R^{k+s} \right] g_s \tau \ , 1 \leq k \leq N,$$

$$u_k = P^{-k} u_0 - \tau \sum_{s=k+1}^{0} P^{s-k-1} C f_s, -N \leq k \leq 0, \tag{5.7}$$

$$u_0 = \frac{1}{2} T_\tau \left(2I + 4\tau A + \frac{5}{2} (\tau A)^2 \right)^{-1} C^{-2} \tag{5.8}$$

$$\times \left\{ \left(2I - \tau^2 A \right) \left\{ (2 + \tau B) R^N \left[-\tau \sum_{s=-N+1}^{0} P^{s+N-1} C f_s + \mu \right] \right. \right.$$

$$\left. - R^{N-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s} \right] g_s \tau \right\}$$

$$\left. \left. + (I - R^{2N}) B^{-1} \sum_{s=1}^{N-1} R^{s-1} g_s \tau \right\} \right\}$$

$$+ (I - R^{2N})(I + \tau B) \left(\tau B^{-1} g_1 - 4 C B^{-1} f_0 + P C B^{-1} f_0 + C B^{-1} f_{-1} \right) \} ,$$

where

$$T_\tau = \left(I + B^{-1} A (I + \tau A + \frac{\tau}{2} C^{-2}) D \left(I - R^{2N-1} \right) \right.$$

$$\left. + D(I - \frac{\tau^2 A}{2}) C^{-2} R^{2N-1} - D(I - \frac{\tau^2 A}{2}) C^{-2} (2I + \tau B) R^N P^N \right)^{-1} .$$

Proof. By [11],

$$u_k = P^{-k} \xi - \tau \sum_{s=k+1}^{0} P^{s-k-1} C f_s, -N \leq k \leq 0 \tag{5.9}$$

is the solution of the inverse Cauchy difference problem

$$\left\{ \begin{array}{c} \tau^{-1} (u_k - u_{k-1}) - \frac{1}{2} (A u_{k-1} + A u_k) = f_k, \\ -(N-1) \leq k \leq 0, \ u_0 \ = \xi. \end{array} \right. \tag{5.10}$$

Using (4.11), (5.9) and the formulas

$$\psi = u_{-N} + \ \mu, \xi = u_0,$$

we obtain the formulas (5.6), (5.7). For u_0, using (5.6), (5.7) and the formula

$$u_2 - 4u_1 + 3u_0 = -3u_0 + 4u_{-1} - u_{-2},$$

we obtain the operator equation

$$\begin{aligned}
&\left(2I - \tau^2 A\right) \left\{ (I - R^{2N})^{-1} \left\{ \left[R - R^{2N-1}\right] u_0 \right.\right.\\
&+ \left[R^{N-1} - R^{N+1}\right] \left[P^N u_0 - \tau \sum_{s=-N+1}^{0} P^{s+N-1} C f_s + \mu \right] \\
&\left. - \left[R^{N-1} - R^{N+1}\right] (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s}\right] g_s \tau \right\} \\
&\left. + (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{s-1} - R^{1+s}\right] g_s \tau \right\} \\
&= -\tau^2 g_1 + C^2 (2I + 4\tau A + \frac{5}{2} (\tau A)^2) u_0 + 4C\tau f_0 - PC\tau f_0 - C\tau f_{-1}.
\end{aligned}$$

The operator

$$\begin{aligned}
&I + B^{-1} A (I + \tau A + \frac{\tau}{2} C^{-2}) D \left(I - R^{2N-1}\right) \\
&+ D(I - \frac{\tau^2 A}{2}) C^{-2} R^{2N-1} - D(I - \frac{\tau^2 A}{2}) C^{-2} (2I + \tau B) R^N P^N
\end{aligned}$$

has an inverse

$$\begin{aligned}
T_\tau = &\left(I + B^{-1} A (I + \tau A + \frac{\tau}{2} C^{-2}) D \left(I - R^{2N-1}\right) \right.\\
&\left. + D(I - \frac{\tau^2 A}{2}) C^{-2} R^{2N-1} - D(I - \frac{\tau^2 A}{2}) C^{-2} (2I + \tau B) R^N P^N \right)^{-1}
\end{aligned}$$

it follows that

$$\begin{aligned}
u_0 = &\frac{1}{2} T_\tau \left(I + 2\tau A + \frac{5}{4} (\tau A)^2 \right)^{-1} C^{-2} \\
&\times \left\{ \left(2I - \tau^2 A\right) \left\{ (2 + \tau B) R^N \left[-\tau \sum_{s=-N+1}^{0} P^{s+N-1} C f_s + \mu \right] \right.\right.\\
&\left. - R^{N-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s}\right] g_s \tau \right\} \\
&\left. + (I - R^{2N}) B^{-1} \sum_{s=1}^{N-1} R^{s-1} g_s \tau \right\} \Bigg\} \\
&+ (I - R^{2N})(I + \tau B) \left(\tau B^{-1} g_1 - 4CB^{-1} f_0 + PCB^{-1} f_0 + CB^{-1} f_{-1} \right) \Big\}
\end{aligned}$$

for the solution of problem (5.1). Theorem 5.1 is proved.

Theorem 5.2. *The nonlocal boundary value problem (5.1) is stable in* $C([-1,1]_\tau, H)$ *norm.*

Proof. By [11],

$$\left\|\{u_k\}_{-N}^{0}\right\|_{C([-1,0]_\tau,H)} \leq M\left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \|u_0\|_H \quad\right] \tag{5.11}$$

for the solution of an inverse Cauchy difference problem (5.10). Then, the proof of Theorem 5.2 is based on the stability inequalities (5.11) and (4.16) and on the estimates

$$\| u_0 \|_H \leq M\left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)} + \| \mu \|_H\right], \tag{5.12}$$

$$\| u_N \|_H \leq M\left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)} + \| \mu \|_H\right] \tag{5.13}$$

for the solution of the boundary value problem (5.1). Estimates (5.12) and (5.13) follow from formula (5.8) and estimates (4.4), (5.2) and (5.5). Theorem 5.2 is proved.

Theorem 5.3. *Assume that* $\mu \in D(A)$ *and* $f_0, f_{-1}, g_1 \in D(I+\tau B)$. *Then for the solution of the difference problem (5.1) we have almost coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau,H)} + \| \{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0} \|_{C([-1,0]_\tau,H)}$$

$$+ \left\|\{Au_k\}_1^{N-1}\right\|_{C([0,1]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k + Au_{k-1})\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)}$$

$$\leq M\left[\| A\mu \|_H + \| (I+\tau B) f_0\|_H + \| (I+\tau B) g_1\|_H + \| (I+\tau B) f_{-1}\|_H\right]$$

$$+M \min\left\{\ln\frac{1}{\tau}, 1 + |\ln \| A \|_{H\to H}|\right\}\left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)}\right],$$

where M *is independent not only of* f^τ, g^τ, μ *but also of* τ.

Proof. By [13],

$$\| \{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0} \|_{C([-1,0]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k + Au_{k-1})\right\}_{-N+1}^{0}\right\|_{C([-1,0]_\tau,H)} \tag{5.14}$$

$$\leq M \min\left\{\ln\frac{1}{\tau}, 1 + |\ln \| A \|_{H\to H}|\right\} \| f^\tau \|_{C([-1,0]_\tau,H)} + M\|Au_0\|_H$$

for the solution of an inverse Cauchy difference problem (5.10). Then the proof of Theorem 5.3 is based on the almost coercivity inequalities (5.14), (4.20) and on the estimates

$$\| Au_0\|_H \leq M\left[\| A\mu \|_H + \| (I+\tau B) f_0\|_H\right]$$

$$+M \min\left\{\ln\frac{1}{\tau}, 1 + |\ln \| A \|_{H\to H}|\right\}\left[\| f^\tau \|_{C([-1,0]_\tau,H)} + \| g^\tau \|_{C([0,1]_\tau,H)}\right],$$

$$\|Au_N\|_H \leq M\left[\| A\mu \|_H + \| (I+\tau B) f_0\|_H\right]$$

$$+M\min\left\{\ln\frac{1}{\tau},1+|\ln\parallel A\parallel_{H\to H}|\right\}\left[\parallel f^{\tau}\parallel_{C([-1,0]_{\tau},H)}+\parallel g^{\tau}\parallel_{C([0,1]_{\tau},H)}\right]$$

for the solution of the boundary value problem (5.1). The proof of these estimates follow the scheme of the papers [11], [13] and relies on the formula (5.8) and on the estimates (4.4), (5.2) and (5.5). Theorem 5.3 is proved.

Let $\widetilde{C}^{\alpha}_{0,1}([-1,1]_{\tau},H)$, $\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H), 0<\alpha<1$ be the Banach spaces with the norms

$$\parallel\varphi^{\tau}\parallel_{\widetilde{C}^{\alpha}_{0,1}([-1,1]_{\tau},H)}=\parallel\varphi^{\tau}\parallel_{C([-1,1]_{\tau},H)}+\sup_{-N\le k<k+2r\le 0}\parallel\varphi_{2k+r}-\varphi_{k}\parallel_{E}\frac{(-k)^{\alpha}}{(2r)^{\alpha}}$$

$$+\sup_{1\le k<k+r\le N-1}\parallel\varphi_{k+r}-\varphi_{k}\parallel_{E}\frac{((k+r)\,\tau)^{\alpha}(N-k)^{\alpha}}{r^{\alpha}},$$

$$\parallel\varphi^{\tau}\parallel_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}=\parallel\varphi^{\tau}\parallel_{C([-1,0]_{\tau},H)}+\sup_{-N\le k<k+2r\le 0}\parallel\varphi_{k+2r}-\varphi_{k}\parallel_{E}\frac{(-k)^{\alpha}}{(2r)^{\alpha}}.$$

Theorem 5.4. *Let the assumptions of Theorem 5.3 be satisfied. Then the boundary value problem (5.1) is well-posed in Hölder spaces $C^{\alpha}_{0,1}([-1,1]_{\tau},H)$ and $\widetilde{C}^{\alpha}_{0,1}([-1,1]_{\tau},H)$ and the following coercivity inequalities hold:*

$$\parallel\{\tau^{-2}(u_{k+1}-2u_{k}+u_{k-1})\}_{1}^{N-1}\parallel_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}+\parallel\{\tau^{-1}(u_{k}-u_{k-1})\}_{-N+1}^{0}\parallel_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}$$

$$+\left\|\{Au_{k}\}_{1}^{N-1}\right\|_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}+\left\|\left\{\frac{1}{2}\left(Au_{k}+Au_{k-1}\right)\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}$$

$$\le\frac{M}{\alpha(1-\alpha)}\left[\parallel f^{\tau}\parallel_{C^{\alpha}_{0}([-1,0]_{\tau},H)}+\parallel g^{\tau}\parallel_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}\right]$$

$$+M\left[\parallel A\mu\parallel_{H}+\|(I+\tau B)f_{0}\|_{H}+\|(I+\tau B)g_{1}\|_{H}+\|(I+\tau B)f_{-1}\|_{H}\right],$$

$$\parallel\{\tau^{-2}(u_{k+1}-2u_{k}+u_{k-1})\}_{1}^{N-1}\parallel_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}+\parallel\{\tau^{-1}(u_{k}-u_{k-1})\}_{-N+1}^{0}\parallel_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}$$

$$+\left\|\{Au_{k}\}_{1}^{N-1}\right\|_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}+\left\|\left\{\frac{1}{2}\left(Au_{k}+Au_{k-1}\right)\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}$$

$$\le\frac{M}{\alpha(1-\alpha)}\left[\parallel f^{\tau}\parallel_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}+\parallel g^{\tau}\parallel_{C^{\alpha}_{0,1}([0,1]_{\tau},H)}\right]$$

$$+M\left[\parallel A\mu\parallel_{H}+\|(I+\tau B)f_{0}\|_{H}+\|(I+\tau B)g_{1}\|_{H}+\|(I+\tau B)f_{-1}\|_{H}\right],$$

where M is independent not only of f^{τ},g^{τ},μ but also of τ and α.

Proof. By [14], [15]

$$\parallel\{\tau^{-1}(u_{k}-u_{k-1})\}_{-N+1}^{0}\parallel_{C^{\alpha}_{0}([-1,0]_{\tau},H)}+\left\|\left\{\frac{1}{2}\left(Au_{k}+Au_{k-1}\right)\right\}_{-N+1}^{0}\right\|_{\widetilde{C}^{\alpha}_{0}([-1,0]_{\tau},H)}\tag{5.15}$$

$$\le\frac{M}{\alpha(1-\alpha)}\parallel f^{\tau}\parallel_{C^{\alpha}_{0}([-1,0]_{\tau},H)}+M\,\|Au_{0}\|_{H},$$

$$\| \{\tau^{-1}(u_k - u_{k-1})\}_{-N+1}^{0} \|_{C_0^\alpha([-1,0]_\tau,H)} + \left\| \left\{ \frac{1}{2}(Au_k + Au_{k-1}) \right\}_{-N+1}^{0} \right\|_{\widetilde{C}_0^\alpha([-1,0]_\tau,H)} \tag{5.16}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \| f^\tau \|_{\widetilde{C}_0^\alpha([-1,0]_\tau,H)} + M \|Au_0\|_H$$

for the solution of an inverse Cauchy difference problem (4.14). Then the proof of Theorem 5.4 is based on the coercivity inequalities (5.15), (5.16) and (4.22) and on the estimates

$$\|Au_0\|_H \leq \frac{M}{\alpha(1-\alpha)} \left[\| f^\tau \|_{\widetilde{C}_0^\alpha([-1,0]_\tau,H)} + \| g^\tau \|_{C_{0,1}^\alpha([0,1]_\tau,H)} \right] \tag{5.17}$$

$$+M \left[\| A\mu \|_H + \| (I+\tau B) f_0 \|_H + \| (I+\tau B) g_1 \|_H + \| (I+\tau B) f_{-1} \|_H \right],$$

$$\|Au_N\|_H \leq \frac{M}{\alpha(1-\alpha)} \left[\| f^\tau \|_{\widetilde{C}_0^\alpha([-1,0]_\tau,H)} + \| g^\tau \|_{C_{0,1}^\alpha([0,1]_\tau,H)} \right] \tag{5.18}$$

$$+M \left[\| A\mu \|_H + \| (I+\tau B) f_0 \|_H + \| (I+\tau B) g_1 \|_H + \| (I+\tau B) f_{-1} \|_H \right]$$

for the solution of the boundary value problem (5.1). Estimates (5.17)-(5.18) follow from the formulas

$$Au_0 = T_\tau \left(I + 2\tau A + \frac{5}{4}(\tau A)^2 \right)^{-1} C^{-2}$$

$$\times \left\{ (2I - \tau^2 A) \left\{ (2+\tau B) R^N \left[-\tau \sum_{s=-N+1}^{0} AP^{s+N-1} C (f_s - f_{-N+1}) + A\mu \right] \right.\right.$$

$$\left. -R^{N-1}AB^{-1} \sum_{s=1}^{N-1} R^{N-s} (g_s - g_{N-1}) \tau + R^{N-1}AB^{-1} \sum_{s=1}^{N-1} R^{N+s} (g_s - g_1) \tau \right\}$$

$$\left.\left. +(I - R^{2N})AB^{-1} \sum_{s=1}^{N-1} R^{s-1} (g_s - g_1) \tau \right\} \right\}$$

$$+(I - R^{2N})(I+\tau B) \left(\tau^2 A g_1 - 4C\tau A f_0 + PC\tau A f_0 + C\tau A f_{-1} \right) \}$$

$$+T_\tau \left(I + 2\tau A + \frac{5}{4}(\tau A)^2 \right)^{-1} C^{-2} \times \left\{ (2I - \tau^2 A) \left\{ (2+\tau B) R^N \left(P^N - I \right) f_{-N+1} \right.\right.$$

$$\left. +R^{N-1}AB^{-2} \left(R^{N-1} - I \right) g_{N-1} + R^{N-1}AB^{-2} \left(R^{N-2} - R^{2N-1} \right) g_1 \right\}$$

$$\left. +(I - R^{2N})AB^{-2} \left(I - R^{N-1} \right) g_1 \right\} \},$$

$$Au_N = P^N \left\{ T_\tau \left(I + \tau A + \frac{5}{4}(\tau A)^2 \right)^{-1} C^{-2} \right.$$

$$\times \left\{ (2I - \tau^2 A) \left\{ (2+\tau B) R^N \left[-\tau \sum_{s=-N+1}^{0} AP^{s+N-1} C (f_s - f_{-N+1}) + A\mu \right] \right.\right.$$

$$-R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N-s}\left(g_s-g_{N-1}\right)\tau+R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N+s}\left(g_s-g_1\right)\tau\Big\}$$

$$+(I-R^{2N})AB^{-2}\sum_{s=1}^{N-1}BR^{s-1}\left(g_s-g_1\right)\tau-(I-R^{2N})\left(I+\tau B\right)B^{-1}APf_0\Big\}\Big\}$$

$$+(I-R^{2N})(I+\tau B)\left(\tau^2Ag_1-4C\tau Af_0+PC\tau Af_0+C\tau Af_{-1}\right)\}$$

$$+P^NT_\tau\left(I+2\tau A+\frac{5}{4}\left(\tau A\right)^2\right)^{-1}C^{-2}\left\{\left(2I-\tau^2A\right)\left\{\left(2+\tau B\right)R^N\left(P^N-I\right)f_{-N+1}\right.\right.$$

$$+R^{N-1}AB^{-2}\left(R^{N-1}-I\right)g_{N-1}+R^{N-1}AB^{-2}\left(R^{N-2}-R^{2N-1}\right)g_1\}$$

$$+(I-R^{2N})AB^{-2}\left(I-R^{N-1}\right)g_1\}\}$$

$$-\tau\sum_{s=-N+1}^{0}AP^{s+N-1}C\left(f_s-f_{-N+1}\right)+A\mu+\left(P^N-I\right)f_{-N+1}$$

for the solution of problem (5.1) and estimates (4.4), (5.2), (5.3), (5.4) and (5.5). Theorem 5.4 is proved.

Now, the applications of this abstract result to the approximate solution of the mixed boundary value problem for elliptic-parabolic equation (3.21) is presented. Problem (4.26) is replaced by the difference scheme (5.1), one can obtain the second order of accuracy difference scheme

$$\begin{cases}
-\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^xu_k^h\left(x\right)=g_k^h(x),\\
g_k^h(x)=\{g(t_k,x_n)\}_1^{M-1},t_k=k\tau,1\leq k\leq N-1,\ N\tau=1,x\in\widetilde{\Omega}_h,\\
\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}-\frac{A_h^x}{2}\left(u_k^h\left(x\right)+u_{k-1}^h\left(x\right)\right)=f_k^h(x),\\
f_k^h(x)=\{f(t_{k-\frac{1}{2}},x_n)\}_1^{M-1},t_{k-\frac{1}{2}}=\left(k-\frac{1}{2}\right)\tau,-N+1\leq k\leq -1,x\in\widetilde{\Omega}_h,\\
u_{-N}^h(x)=u_N^h(x)+\mu^h(x),x\in\widetilde{\Omega}_h,\\
-u_2^h(x)+4u_1^h(x)-3u_0^h(x)=3u_0^h(x)-4u_{-1}^h(x)+u_{-2}^h(x),x\in\widetilde{\Omega}_h.
\end{cases}\tag{5.19}$$

Theorem 5.5. *Let τ and $|h|$ be a sufficiently small numbers.Then the solutions of difference scheme (5.19) satisfy the following stability and almost coercivity estimates:*

$$\left\|\left\{u_k^h\right\}_{-N}^{N-1}\right\|_{C([-1,1]_\tau,L_{2h})}\leq M\left[\|\ f^\tau\ \|_{C([-1,0]_\tau,L_{2h})}+\|\ g^\tau\ \|_{C([0,1]_\tau,L_{2h})}\right]+M\left\|\mu^h\right\|_{L_{2h}},$$

$$\|\ \{\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\}_1^{N-1}\ \|_{C([0,1]_\tau,L_{2h})}+\|\ \{\tau^{-1}(u_k^h-u_{k-1}^h)\}_{-N+1}^0\ \|_{C([-1,0]_\tau,L_{2h})}$$

$$+\left\|\left\{u_k^h\right\}_{-N}^{N-1}\right\|_{C([-1,1]_\tau,W_{2h}^2)}\leq M\left[\left\|\mu^h\right\|_{W_{2h}^2}+\tau\left\|f_0^h\right\|_{W_{2h}^1}\right]$$

$$+M\ln\frac{1}{\tau+|h|}\left[\| f^{\tau} \|_{C([-1,0]_{\tau},L_{2h})}+\| g^{\tau} \|_{C([0,1]_{\tau},L_{2h})}\right].$$

Here M does not depend on $\tau,\ h,\ \mu^h(x)$ *and* $g_k^h(x), 1\le k\le N-1, f_k^h, -N+1\le k\le 0.$

The proof of Theorem 5.5 is based on the abstract Theorems 5.2-5.3, on the estimate (4.28), as well as the symmetry properties of the difference operator A_h^x defined by the formula (4.25) in L_{2h} and on the Theorem 4.6 on the coercivity inequality for the solution of the elliptic difference equation (4.29) in L_{2h}.

Theorem 5.6. *Let τ and $|h|$ be sufficiently small numbers. Then the solutions of difference scheme (5.19) satisfy the following coercivity stability estimates:*

$$\| \{\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\}_1^{N-1} \|_{C_{0,1}^{\alpha}([0,1]_{\tau},L_{2h})}+\| \{\tau^{-1}(u_k^h-u_{k-1}^h)\}_{-N+1}^{0} \|_{\widetilde{C}_0^{\alpha}([-1,0]_{\tau},L_{2h})}$$

$$+\left\|\{u_k^h\}_1^{N-1}\right\|_{C_{0,1}^{\alpha}([0,1]_{\tau},W_{2h}^2)}+\left\|\left\{\frac{u_k^h+u_{k-1}^h}{2}\right\}_{-N+1}^{0}\right\|_{C_{0,1}^{\alpha}([-1,1]_{\tau},W_{2h}^2)}$$

$$\le M\left[\left\|\mu^h\right\|_{W_{2h}^2}+\tau\left\|f_0^h\right\|_{W_{2h}^1}+\tau\left\|f_{-1}^h\right\|_{W_{2h}^1}+\tau\left\|g_1^h\right\|_{W_{2h}^1}\right]$$

$$+\frac{M}{\alpha(1-\alpha)}\left[\| \left(f^h\right)^{\tau} \|_{C_0^{\alpha}([-1,0]_{\tau},L_{2h})}+\| \left(g^h\right)^{\tau} \|_{C_{0,1}^{\alpha}([0,1]_{\tau},L_{2h})}\right],$$

$$\| \{\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\}_1^{N-1} \|_{C_{0,1}^{\alpha}([0,1]_{\tau},L_{2h})}+\| \{\tau^{-1}(u_k^h-u_{k-1}^h)\}_{-N+1}^{0} \|_{\widetilde{C}_0^{\alpha}([-1,0]_{\tau},L_{2h})}$$

$$+\left\|\{u_k^h\}_1^{N-1}\right\|_{C_{0,1}^{\alpha}([0,1]_{\tau},W_{2h}^2)}+\left\|\left\{\frac{u_k^h+u_{k-1}^h}{2}\right\}_{-N+1}^{0}\right\|_{\widetilde{C}_0^{\alpha}([-1,0]_{\tau},L_{2h})}$$

$$\le M\left[\left\|\mu^h\right\|_{W_{2h}^2}+\tau\left\|f_0^h\right\|_{W_{2h}^1}+\tau\left\|f_{-1}^h\right\|_{W_{2h}^1}+\tau\left\|g_1^h\right\|_{W_{2h}^1}\right]$$

$$+\frac{M}{\alpha(1-\alpha)}\left[\| \left(f^h\right)^{\tau} \|_{\widetilde{C}_0^{\alpha}([-1,0]_{\tau},L_{2h})}+\| \left(g^h\right)^{\tau} \|_{C_{0,1}^{\alpha}([0,1]_{\tau},L_{2h})}\right].$$

Here M does not depend on $\tau,\ h,\ \mu^h(x)$ *and* $g_k^h(x), 1\le k\le N-1, f_k^h, -N+1\le k\le 0.$

The proof of Theorem 5.6 is based on the abstract Theorem 5.4, and the symmetry properties of the difference operator A_h^x defined by the formula (4.25) and on the Theorem 4.6 on the coercivity inequality for the solution of the elliptic difference equation (4.29) in L_{2h}.

Note that in a similar manner can be constructed the difference schemes of the second order of accuracy with respect to one variable for approximate solutions of the boundary value problem (3.20). Abstract theorems given from above permit to obtain the stability, the almost stability and the coercive stability estimates for the solutions of these difference schemes.

CHAPTER 6

NUMERICAL ANALYSIS

We consider the nonlocal boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}=(1-t)\sin x, \quad -1<t\leq 0,\ 0<x<\pi, \\ \frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}=-t\sin x, \quad 0<t<1,\ 0<x<\pi, \\ u(1,x)=u(-1,x)+2\sin x,\ 0\leq x\leq \pi, \\ u(t,0)=u(t,\pi)=0, \quad -1\leq t\leq 1. \end{cases} \tag{6.1}$$

for elliptic-parabolic equation.

Let

$$\begin{aligned} f(t,x) &= (1-t)\sin x, -1<t<0\ ,\ 0<x<\pi, \\ g(t,x) &= -t\sin x,\ 0<t<1,\ 0<x<\pi, \end{aligned}$$

and

$$\varphi(x)=2\sin x.$$

The exact solution of this problem is

$$u(t,x)=t\sin x.$$

For approximate solutions of the nonlocal boundary value problem (3.1), we will use the first order of accuracy and the second order of accuracy difference schemes. We have the second order difference equations with respect to n with matrix coefficients. To solve this difference equations, we have applied a procedure of modified Gauss elimination method for difference equations with respect to n with matrix coefficients. The results of numerical experiments permit us to show that the second order of accuracy difference schemes are more accurate comparing with the first order of accuracy difference scheme.

6.1 THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

We will consider the nonlocal boundary value problem (3.1) for elliptic-parabolic equation. For approximate solution of the nonlocal boundary-value problem (3.1), let's consider the set $[-1,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$\begin{aligned} [-1,1]_\tau \times [0,\pi]_h &= \{(t_k, x_n) : t_k = k\tau,\ -N \le k \le N,\ \ N\tau = 1, \\ x_n &= nh,\ \ 0 \le n \le M, Mh = \pi\}. \end{aligned}$$

Applying the formula

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O(h^2),$$

we present the following first order of accuracy difference scheme in t for the approximate solutions of the problem (4.1)

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} = f(t_{k-1}, x_n),\ -N+1 \le k \le 0, 1 \le n \le M-1, \\ \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = g(t_k, x_n),\ 1 \le k \le N-1, 1 \le n \le M-1, \\ \frac{u_n^1 - u_n^0}{\tau} = -\frac{u_{n+1}^{-1} - 2u_n^{-1} + u_{n-1}^{-1}}{h^2} + \sin x_n, x_n = nh, 1 \le n \le M-1, \\ u_n^N = u_n^{-N} + 2\sin x_n, x_n = nh, 0 \le n \le M, \\ u_0^k = u_M^k = 0, -N \le k \le N, \end{cases} \tag{6.2}$$

We have $(2N+1) \times (2N+1)$ system of linear equations in (4.2) and we will write them in the matrix form. We can rewrite this system as the following form

$$\begin{cases} \left(\frac{1}{h^2}\right) u_{n+1}^{k-1} + \frac{1}{\tau} u_n^k + \left(-\frac{1}{\tau} - \frac{2}{\tau^2}\right) u_n^{k-1} + \left(\frac{1}{h^2}\right) u_{n-1}^{k-1} = f(t_{k-1}, x_n), \\ -N+1 \le k \le 0,\ 1 \le n \le M-1, \\ \left(\frac{1}{h^2}\right) u_{n+1}^k + \left(\frac{1}{\tau^2}\right) u_n^{k+1} + \left(-\frac{2}{\tau^2} - \frac{2}{h^2}\right) u_n^k + \left(\frac{1}{\tau^2}\right) u_n^{k-1} + \left(\frac{1}{h^2}\right) u_{n-1}^k = g(t_k, x_n), \\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ \left(\frac{1}{h^2}\right) u_{n+1}^{-1} + \left(-\frac{2}{h^2}\right) u_n^{-1} + \left(-\frac{1}{\tau}\right) u_n^0 + \left(\frac{1}{\tau}\right) u_n^1 + \left(\frac{1}{h^2}\right) u_{n-1}^{-1} = \sin x_n, \\ x_n = nh, 1 \le n \le M-1, \\ u_n^N - u_n^{-N} = 2\sin x_n,\ 1 \le n \le M-1, \\ u_0^k = u_M^k = 0,\ \ -N \le k \le N, \end{cases} \tag{6.3}$$

We denote

$$\begin{aligned} a &= \frac{1}{h^2}, \quad b = -\frac{1}{\tau} - \frac{2}{h^2}, \quad c = \frac{1}{\tau}, \\ d &= \frac{1}{\tau^2}, \ e = -\frac{2}{\tau^2} - \frac{2}{h^2}, f = -\frac{2}{h^2}, g = -\frac{1}{\tau}. \end{aligned}$$

$$\varphi_n^k = \begin{cases} 2\sin x_n, \ k = -N, \\ f(t_{k-1}, x_n), \ \ -N+1 \le k \le 0, \\ g(t_k, x_n), \ 0 \le k \le N-1, \\ \sin x_n, \ k = N. \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{-N} \\ \varphi_n^{-N+1} \\ \ldots \\ \varphi_n^0 \\ \varphi_n^1 \\ \ldots \\ \varphi_n^N \end{bmatrix}_{(2N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ b & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots. & \ldots & \ldots \\ 0 & 0 & 0 & 0 & b & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & e & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & e & d & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & d & e & d \\ 0 & 0 & 0 & 0 & f & g & c & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

and

$$C = A,$$

$$D = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$U_s = \begin{bmatrix} U_s^{-N} \\ U_s^{-N+1} \\ \ldots \\ U_s^{0} \\ U_s^{1} \\ \ldots \\ U_s^{N-1} \\ U_s^{N} \end{bmatrix}_{(2N+1)\times(1)}, \quad s = \ n-1, \ n, n+1.$$

Then (6.3) can be written as

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ \\ U_0 = \widetilde{0}, \ \ U_M = \widetilde{0}. \end{cases} \tag{6.4}$$

So, we have the second order difference equation with respect to n with matrix coefficients. To solve this difference equation we have applied a procedure of modified Gauss elimination method for difference equation with respect to n with matrix coefficients. Hence, we seek a solution of the matrix equation in the following form

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, \cdot\cdot\cdot, 2, 1, 0, \tag{6.5}$$

where $\alpha_j \quad (j = 1, ..., M-1)$ are $(2N+1)\times(2N+1)$ square matrices and $\beta_j \quad (j = 1, ..., M-1)$ are $(2N+1)\times 1$ column matrices. Using the equality

$$U_s = \alpha_{s+1} U_{s+1} + \beta_{s+1}, \text{(for } s = n, \ n-1)$$

and the equality

$$A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n,$$

we can write

$$[A + B\alpha_{n+1} + C\alpha_n \alpha_{n+1}] U_{n+1} + \left[B\beta_{n+1} + C\alpha_n \beta_{n+1} + C\beta_n\right] = D\varphi_n.$$

The last equation is satisfied if we select

$$A + B\alpha_{n+1} + C\alpha_n \alpha_{n+1} = 0,$$

$$\left[B\beta_{n+1} + C\alpha_n \beta_{n+1} + C\beta_n\right] = D\varphi_n, \quad 1 \le n \le M-1.$$

From that it follows

$$\begin{aligned} \alpha_{n+1} &= -\left(B + C\alpha_n\right)^{-1} A, \\ \beta_{n+1} &= \left(B + C\alpha_n\right)^{-1} \left(D\varphi_n - C\beta_n\right), n = 1, 2, 3, \cdot\cdot\cdot, M-1. \end{aligned} \tag{6.6}$$

For the solution of difference equations we need to find α_1 and β_1. We can find them from $U_0 = \alpha_1 U_1 + \beta_1$. Thus, we have

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}, \tag{6.7}$$

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}_{(2N+1)\times 1}.$$

For the first step using formulas (6.6) and (6.7), we can compute α_{n+1} and β_{n+1}, $1 \leq n \leq M-1$. For the second step we will find u_n, $0 \leq n \leq M$. But, for this we need to find u_M. We can find u_M from $u_M = u_{M-1}$ and $u_{M-1} = \alpha_M u_M + \beta_M$. Namely

$$u_M = (I - \alpha_M)^{-1}\beta_M. \tag{6.8}$$

Thus using formulas (6.5) and (6.8), we can compute u_n, $0 \leq n \leq M$. We can summarize the computation procedure by the following algorithm.

Algorithm

1. **Step** Set input time step $\tau = \frac{1}{N}$ and space step $h = \frac{\pi}{M}$.
2. **Step** Use the first order of accuracy difference scheme and write it in matrix form

$$A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \leq n \leq M-1.$$

3. **Step** Determine the entries of the matrices A, B, C and D.
4. **Step** Find α_1, β_1 by the formula (6.7).
5. **Step** Compute α_{n+1}, $\beta_{n+1}, 1 \leq n \leq M-1$ by the formula (6.6).
6. **Step** Compute U_M by the formula (6.8).
7. **Step** Compute $U_n, n = M-1, \cdots, 1, 0$ by the formula (6.5).

Matlab Implementation of the First Order of Accuracy Difference Scheme

```
function [table,es,p]=rothermethod(N,M)
    % first order of accuracy rother method mixed type
    close; close;
    if nargin<1; N=30 ; M=30 ;end;
    tau=1/N; h=pi/M;
    A=zeros(2*N+1,2*N+1);
    for i=2:N+1; A(i,i-1)=1/(h^2); end;
    for i=N+2:2*N; A(i,i)=1/(h^2); end;
    A(2*N+1,N)=1/(h^2);
    B=zeros(2*N+1,2*N+1);
```

```
B(1,1)=-1;
B(1,2*N+1)=1;
for i=1:N;
B(i+1,i)=(-1/tau)-(2/h^2);
B(i+1,i+1)=1/tau;
end;
for i=N+2:2*N;
B(i,i)=(-2/(tau^2))-(2/(h^2));
B(i,i+1)=1/(tau^2);
end;
for i=N+1:2*N-1;B(i+1,i)=1/(tau^2);end;
B(2*N+1,N)=2/(h^2);
B(2*N+1,N+1)=-1/tau;
B(2*N+1,N+2)=1/tau;
for i=1:2*N+1;C(i,i)=1;end ;
C(2*N+1,2*N+1)=0;
alpha(2*N+1,2*N+1,1:1)= 0 ;
betha(2*N+1,1:1) = 0 ;
'fi(j) = fi(k,j) ' ;
for j=1:2*N+1;
x=j*h;
fii(1,j:j)=2*sin(x); for k=2:N+1; t=(-N+k-1)*tau ; fii( k, j:j ) = f(t,x); end;
for k=N+2:2*N; t=(-N+k-1)*tau+tau; fii( k, j:j ) = g(t,x) ; end;
end;
fii(2*N+1,j:j)=sin(x);
'alpha(N+1,N+1,j) ve betha(N+1,j) ' ;
for j=1:M-1;
alpha( :, :, j+1:j+1 ) = - inv(B+A*alpha(:, :, j:j))*A ;
betha( :, j+1:j+1 ) = inv(B+A*alpha(:, :, j:j ) )*(C*( fii(:, j:j ))-(A* betha(:,j:j) ));
end;
U( 2*N+1,1, M:M ) = 0;
for z = M-1:-1:1 ; U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 ) + betha(:,z+1:z+1);
end;
for z = 1:M ; p(:,z+1:z+1)=U(:,:,z:z); end;
'EXACT SOLUTION OF THIS PDE' ;
```

```
for j=1:M+1;
for k=1:2*N+1;
t=(-N+k-1)*tau;
x=(j-1)*h;
es(k,j) = exact(t,x);
end;
end;

'ERROR ANALYSIS' ;

maxerror=max(max(abs(es-p)))

%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%%%

[xler,tler]=meshgrid(0:h:pi,-1:tau:1);
table=[es;p];
table(1:2:end,:)=es; table(2:2:end,:)=p;
q=min(min(table)); w=max(max(table)); figure; surf(xler,tler,es); title('EXACT SOLUTION');
set(gca,'ZLim',[q w]);rotate3d;XLabel('x axis');YLabel('t axis');
figure; surf(xler,tler,p); title('EULER-ROTHER'); rotate3d ;
set(gca,'ZLim',[q w]);XLabel('x axis');YLabel('t axis');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function estx=exact(t,x);
estx=t*sin(x);
function ftx=f(t,x);
ftx=(1-t)*sin(x);
function gtx=g(t,x);
gtx=-t*sin(x);
```

6.2 THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME

We present the following second order of accuracy difference scheme in t for the approximate solutions of the problem (6.1)

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau}+\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{2h^2}=f(t_k-\frac{\tau}{2},x_n), \\ x_n=nh,\ t_k=k\tau, -N+\ 1\leq k\leq 0,\ 1\leq n\leq M-1, \\ \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}=g(t_k,x_n), \\ x_n=nh,\ t_k=k\tau,\ 1\leq k\leq N-1,\ 1\leq n\leq M-1, \\ -u_n^2+4u_n^1-3u_n^0=3u_n^0-4u_n^{-1}+u_n^{-2},\ x_n=nh,\ 1\leq n\leq M-1, \\ u_0^k=u_M^k=0, -N\leq k\leq N, \end{cases} \tag{6.9}$$

We have again the $(2N+1)\times(2N+1)$ system of linear equations and we will write them in the matrix form. We can rewrite this system in the following form

$$\begin{cases} \left(\frac{1}{2h^2}\right)u_{n+1}^{k-1}+\left(\frac{1}{2h^2}\right)u_{n+1}^k+\left(-\frac{1}{\tau}-\frac{1}{h^2}\right)u_n^{k-1}+\left(\frac{1}{\tau}-\frac{1}{h^2}\right)u_n^k+\left(\frac{1}{2h^2}\right)u_{n-1}^{k-1}+\left(\frac{1}{2h^2}\right)u_{n-1}^k \\ =f\left(t_k-\frac{\tau}{2},x_n\right), \\ -N+1\leq k\leq 0, 1\leq n\leq M-1, \\ \left(\frac{1}{h^2}\right)u_{n+1}^k+\left(\frac{1}{\tau^2}\right)u_n^{k+1}+\left(-\frac{2}{\tau^2}-\frac{2}{h^2}\right)u_n^k+\left(\frac{1}{\tau^2}\right)u_n^{k-1}+\left(\frac{1}{h^2}\right)u_{n-1}^k=g(t_k,x_n), \\ 1\leq k\leq N-1,\ 1\leq n\leq M-1, \\ u_n^{-2}-4u_n^{-1}+6u_n^0-4u_n^1+u_n^2=0, 0\leq n\leq M, \\ u_n^N-u_n^{-N}=2\sin x_n,\ 1\leq n\leq M-1, \\ u_0^k=u_M^k=0,\quad -N\leq k\leq N. \end{cases} \tag{6.10}$$

Denoting

$$\begin{aligned} x &= \frac{1}{2h^2},\ y=-\frac{1}{\tau}-\frac{1}{h^2},\ z=\frac{1}{\tau}-\frac{1}{h^2}, \\ a &= \frac{1}{h^2},\quad d=\frac{1}{\tau^2},\ e=-\frac{2}{\tau^2}-\frac{2}{h^2}, \end{aligned}$$

$$\varphi_n^k = \begin{cases} 2\sin x_n, k = -N, \\ f(t_k - \frac{\tau}{2}, x_n), -N+1 \le k \le 0, \\ g(t_k, x_n), 1 \le k \le N-1, \\ 0, k = N. \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{-N} \\ \varphi_n^{-N+1} \\ \varphi_n^{-N+2} \\ .. \\ \varphi_n^{N-2} \\ \varphi_n^{N-1} \\ \varphi_n^{N} \end{bmatrix}_{(2N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ x & x & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & x & x & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & x & x & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 1 \\ y & z & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & y & z & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & d & e & d & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & d & e & d & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & d & e & d \\ 0 & ... & 1 & -4 & 6 & -4 & 1 & ... & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

and

$$A = C,$$

$$D = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$U_s = \begin{bmatrix} U_s^{-N} \\ U_s^{-N+1} \\ U_s^{-N+2} \\ U_s^{-N+3} \\ .. \\ U_s^{N-1} \\ U_s^{N} \end{bmatrix}_{(2N+1)\times(1)}, \text{ where } s = n-1, n, n+1,$$

(6.10) can be written as

$$\begin{cases} A\,U_{n+1}+B\,U_n+C\,U_{n-1}=D\varphi_n, & 1\le n\le M-1, \\ \\ U_0=\widetilde{0}, \;\; U_M=\widetilde{0}. \end{cases} \tag{6.11}$$

So, we have the second order difference equation with respect to n with matrix coefficients. To solve this difference equation we have applied a procedure of modified Gauss elimination method for difference equation with respect to n with matrix coefficients. Hence, we seek a solution of the matrix equation in the following form

$$U_n=\alpha_{n+1}U_{n+1}+\beta_{n+1}, \quad n=M-1,\cdot\cdot\cdot,2,1,0, \tag{6.12}$$

where $\alpha_j \;\; (j=1,...,M-1)$ are $(2N+1)\times(2N+1)$ square matrices and $\beta_j \;\; (j=1,...,M-1)$ are $(2N+1)\times 1$ column matrices. Using the equality

$$U_s=\alpha_{s+1}U_{s+1}+\beta_{s+1}, (\text{for } s=n,\; n-1)$$

and the equality

$$A\,U_{n+1}+B\,U_n+C\,U_{n-1}=D\varphi_n,$$

we can write

$$[A+B\alpha_{n+1}+C\alpha_n\alpha_{n+1}]U_{n+1}+\left[B\beta_{n+1}+C\alpha_n\beta_{n+1}+C\beta_n\right]=D\varphi_n.$$

The last equation is satisfied if we select

$$A+B\alpha_{n+1}+C\alpha_n\alpha_{n+1}=0,$$

$$\left[B\beta_{n+1}+C\alpha_n\beta_{n+1}+C\beta_n\right]=D\varphi_n, \quad 1\le n\le M-1.$$

From that it follows

$$\begin{aligned} \alpha_{n+1} &= -\left(B+C\alpha_n\right)^{-1}A, \\ \beta_{n+1} &= \left(B+C\alpha_n\right)^{-1}\left(D\varphi_n-C\beta_n\right), n=1,2,3,\cdot\cdot\cdot,M-1. \end{aligned} \tag{6.13}$$

For the solution of difference equations we need to find α_1 and β_1. We can find them from $U_0=\alpha_1U_1+\beta_1$. Thus, we have

$$\alpha_1=\begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}, \tag{6.14}$$

$$\beta_1=\begin{bmatrix} 0 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}_{(2N+1)\times 1}.$$

For the first step using formulas (6.13) and (6.14), we can compute α_{n+1} and β_{n+1}, $1\le n\le M-1$. For the second step we will find u_n, $0\le n\le M$. But, for this we need to find u_M. We can find u_M from $u_M=u_{M-1}$ and $u_{M-1}=\alpha_Mu_M+\beta_M$. Namely

$$u_M=(I-\alpha_M)^{-1}\beta_M. \tag{6.15}$$

Thus using formulas (6.12) and (6.15), we can compute u_n, $0\le n\le M$. We can summarize the computation procedure by the following algorithm.

Algorithm

1. Step Set input time step $\tau = \frac{1}{N}$ and space step $h = \frac{\pi}{M}$.

2. Step Use the first order of accuracy difference scheme and write in matrix form

$$A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \le n \le M-1.$$

3. Step Determine the entries of the matrices $A,\ B,\ C$ and D.

4. Step Find α_1, β_1 by the formula (6.13).

5. Step Compute $\alpha_{n+1},\ \beta_{n+1}, 1 \le n \le M-1$ by the formula (6.14).

6. Step Compute U_M by the formula (6.12).

7. Step Compute $U_n, n = M-1, \cdots, 1, 0$ by the formula (6.15).

Matlab Implementation of Second Order of Accuracy Difference Scheme

```
function [table,es,p]=rothermethod(N,M)
    % second order accuracy rother method mixed type
    close; close;
    if nargin<1; N=30 ; M=30 ;end;
    tau=1/N; h=pi/M;
    A=zeros(2*N+1,2*N+1);
    for i=2:N+1; A(i,i-1)=1/(2*h^2); end;
    for i=2:N+1; A(i,i)=1/(2*h^2); end;
    for i=N+2:2*N; A(i,i)=1/(h^2); end;
    B=zeros(2*N+1,2*N+1);
    B(1,1)=-1;
    B(1,2*N+1)=1;
    for i=1:N; B(i+1,i)=(-1/tau)-(1/h^2); end;
    for i=2:N+1; B(i,i)=(1/tau)-(1/h^2); end;
    for i=N+2:2*N; B(i,i)=(-2/(tau^2))-(2/(h^2)); end;
    for i=N+2:2*N; B(i,i+1)=1/(tau^2); end;
    for i=N+1:2*N-1; B(i+1,i)=1/(tau^2); end;
    B(2*N+1,N-2)=1;
    B(2*N+1,N-1)=-4;
    B(2*N+1,N)=6;
    B(2*N+1,N+1)=-4;
```

```
B(2*N+1,N+2)=1;
for i=1:2*N+1; C(i,i)=1; end ;
C(2*N+1,2*N+1)=0;
alpha(2*N+1,2*N+1,1:1)= 0 ; betha(2*N+1,1:1) = 0 ;
'fi(j) = fi(k,j) ' ;
for j=1:2*N+1;
x=j*h;
fii(1,j:j)=2*sin(x);
for k=2:N+1;
x=j*h;
t=(-N+k-1)*tau-tau/2 ;
fii( k, j:j ) = f(t,x);
end;
for k=N+2:2*N;
t=(-N+k-1)*tau;
x=j*h; fii( k, j:j ) = g(t,x) ;
end;
end;
fii(2*N+1,j:j)=0;
'alpha(N+1,N+1,j) ve betha(N+1,j) ler' ;
for j=1:M-1;
alpha( :, :, j+1:j+1 ) = - inv(B+A*alpha(:, :, j:j))*A ;
betha( :, j+1:j+1 ) = inv(B+A*alpha(:, :, j:j ) )*(C*( fii(:, j:j ))-(A* betha(:,j:j) ));
end;
U( 2*N+1,1, M:M ) = 0;
for z = M-1:-1:1 ;
U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 ) + betha(:,z+1:z+1);
end;
for z = 1:M ; p(:,z+1:z+1)=U(:,:,z:z); end;

'EXACT SOLUTION OF THIS PDE' ;
for j=1:M+1;
for k=1:2*N+1;
t=(-N+k-1)*tau;
x=(j-1)*h;
```

```
es(k,j) = exact(t,x);
end;
end;

'ERROR ANALYSIS' ;

maxerror=max(max(abs(es-p)))
%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%%%

[xler,tler]=meshgrid(0:h:pi,-1:tau:1);
table=[es;p];
table(1:2:end,:)=es; table(2:2:end,:)=p;
q=min(min(table));
w=max(max(table));
figure; surf(xler,tler,es); title('EXACT SOLUTION'); set(gca,'ZLim',[q w]);
rotate3d;XLabel('x axis');YLabel('t axis');
figure; surf(xler,tler,p); title('APP. SOL.'); rotate3d ; set(gca,'ZLim',[q w]);
XLabel('x axis');YLabel('t axis');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function estx=exact(t,x)
estx=t*sin(x);
function ftx=f(t,x)
ftx=(1-t)*sin(x);
function gtx=g(t,x)
gtx=-t*sin(x);
```

6.3 COMPARISON OF THE RESULTS

Now, we will give the results of the numerical analysis. The exact and numerical solutions are given in the figures 6.1, 6.2 and 6.3 for $N = M = 30$.

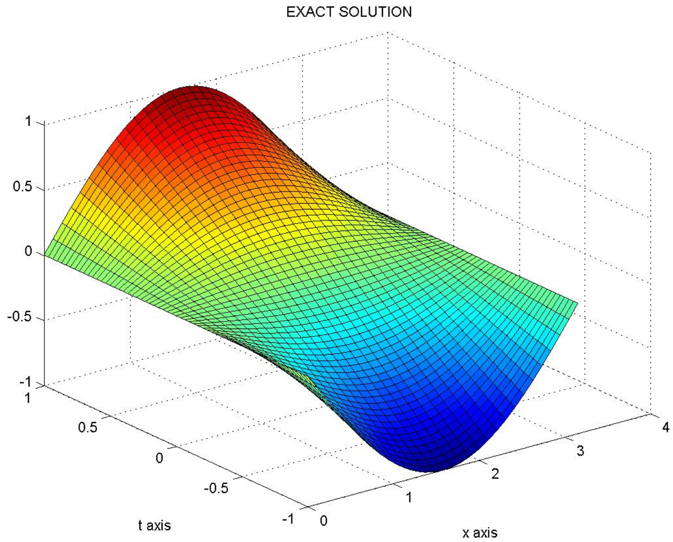

Figure 6.1:

Exact Solution

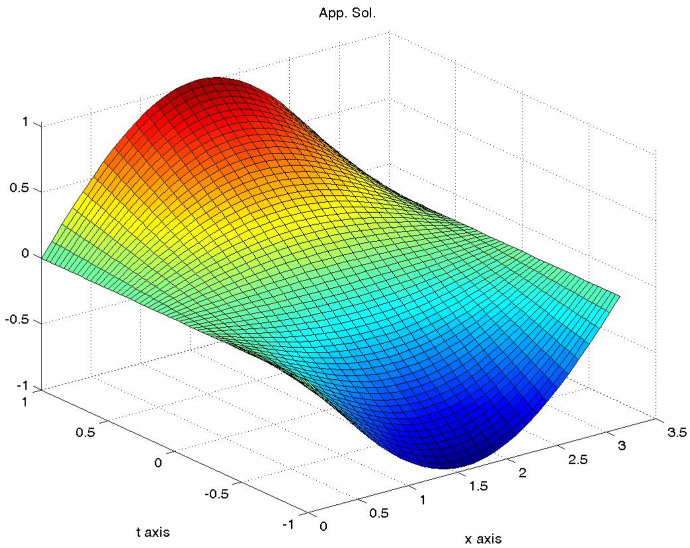

Figure 6.2:

Solution by the first order of accuracy difference scheme

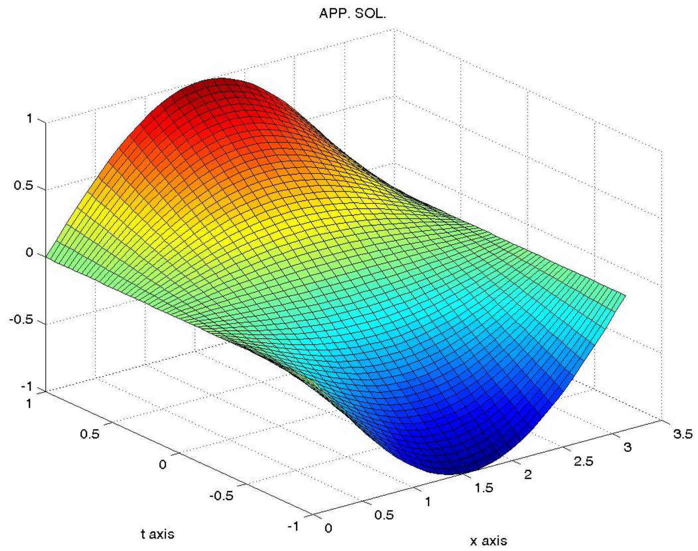

Figure 6.3:

Solution by the second order of accuracy difference schemes

The errors computed by

$$E_M^N = \max_{-N \leq k \leq N, 1 \leq n \leq M-1} \left| u(t_k, x_n) - u_n^k \right|$$

of the numerical solutions are given in the following table.

Table 6.1. Comparison of the errors

Difference schemes	N=M=20	N=M=30	N=M=60
The first order of accuracy difference scheme (6.2)	0.0468	0.0319	0.0163
The second order of accuracy difference scheme (6.9)	0.0008	O.00036	0.00009

Thus, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

CHAPTER 7

CONCLUSION

This work is devoted to the study of the well-posedness of the nonlocal boundary value problem for elliptic-parabolic differential and difference equations. The following original results are obtained:

- The abstract theorem on the well-posedness of the nonlocal boundary value problem for elliptic-parabolic equation in a Hilbert space is established.
- The coercivity stability inequalities for the solutions of the two nonlocal boundary value problems for elliptic-parabolic equations are obtained.
- The first and second order of accuracy difference schemes for the approximate solutions of the nonlocal boundary problem for elliptic-parabolic differential equations are presented.
- The abstract theorems on the well-posedness of the first and second order of accuracy difference schemes for the approximate solutions of the nonlocal boundary problem for elliptic-parabolic differential equation are established.
- The stability, almost coercivity inequalities, coercivity inequalities for the solutions of difference schemes for the approximate solution of the nonlocal boundary value problem for elliptic-parabolic equation are obtained.
- Numerical examples are presented. A Matlab program is given.

REFERENCES

[1] Salakhitdinov, M.S., *Equations of Mixed-Composite Type,* Fan:Tashkent, (Russian), 1974.

[2] Bazarov, D. and Soltanov, H., *Some Local and Nonlocal Boundary Value Problems for Equations of Mixed and Mixed-Composite Types,* Ylim:Ashgabat, (Russian), 1995.

[3] S.N. Glazatov, *Nonlocal boundary value problems for linear and nonlinear equations of variable type,* Sobolev Institute of Mathematics SB RAS, Preprint No. 46, 26p., 1998.

[4] Ashyralyev, A. and Soltanov, H., *On elliptic-parabolic equations in a Hilbert space,* in "Proceeding of the IMM and CS of Turkmenistan" Ashgabat, no. 4, 101-104. (Russian), 1995.

[5] Krein, S.G., *Linear Differential Equations in a Banach Space,* Nauka: Moskow, (Russian), 1966.

[6] Sobolevskii, P.E., "Coerciveness inequalities for abstract parabolic equations" *Dokl. Akad. Nauk SSSR*, Vol.157, No.1, pp. 52-55, (Russian), 1964.

[7] Sobolevskii, P.E., "On elliptic equations in a Banach space" *Differensialnyye Uravneniya*, Vol. 4, No.7, pp. 1346-1348, (Russian), 1969.

[8] Sobolevskii, P.E., " The coercive solvability of difference equations", *Dokl. Akad. Nauk SSSR* 201, no. 5, pp. 1063-1066. (Russian), 1971.

[9] Sobolevskii, P.E., "The theory of semigroups and the stability of difference schemes" *in:Operator Theory in Function Spaces (Proc. School, Novosibirsk, 1975)*, Nauka, Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, pp.304-337, (Russian), 1977.

[10] Sobolevskii, P.E., "On the Crank-Nicolson difference scheme for parabolic equations", *in: Nonlinear Oscillations and Control Theory*, pp.98-106, Izhevsk, (Russian), 1978.

[11] Sobolevskii, P.E., "On the stability and convergence of the Crank-Nicolson scheme" *in:Variational-Difference Methods in Mathematical Physics*, Vychisl.Tsentr Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, pp.146-151, (Russian),1974.

[12] A. Ashyralyev and P.E. Sobolevskii, “Coercive stability of a Crank-Nicholson difference scheme in $\widetilde{C}_0^{\alpha}$ spaces”, *Approximate Methods for Investigations of Differential Equations and Their Applications*, Kuybishev, pp.16-24, (Russian), 1982.

[13] A. Ashyralyev and P.E. Sobolevskii, “ The linear operator interpolation theory and the stability of difference schemes”, *Doklady Akademii Nauk SSSR* Vol. 275, No.6, pp.1289-1291, (Russian), 1984.

[14] A. Ashyralyev and P.E. Sobolevskii, “ Correct solvability of the Crank-Nicholson difference scheme for parabolic equations”, *Izv. Akad. Nauk Turkmen. SSR Ser. Fiz.-Tekhn. Khim. Geol. Nauk*, No.6, pp. 10-16, (Russian), 1981.

[15] A. Ashyralyev, and P.E.Sobolevskii, Well-Posedness of Parabolic Difference Equations, Birkhäuser Verlag, Basel, Boston, Berlin,Operator Theory and Appl., Vol.69, 349 p., 1994.

[16] Ashyralyev, A., Piskarev, S. and Wei, S., “On well-posedness of the difference schemes for abstract parabolic equations in $L_p([0,1],E)$ spaces”, *Numerical Functional Analysis and Optimization,* Vol. 23, No.7-8, pp. 669-693, 2002.

[17] Ashyralyev, A. and Soltanov, H., “On coercive stability of difference scheme for parabolic-elliptic equations” , *Labour of the IMM of AS of theTurkmenistan,* Ashgabat, No.3 pp. 63-66, (Russian) 1995.

[18] Ashyralyev, A. and Soltanov, H., “ On one difference schemes for an abstract nonlocal problem generated by the investigation of the motion of gas on the homogeneous space” , *in: Modeling Processes of Exploitation of Gas Places and Applied Problems of Theoretical Gasohydrodynamics,* Ilim, Ashgabat, pp. 147-154, (Russian), 1998.

[19] Ashyralyev, A. and Soltanov, H., “On the stability of the difference scheme for the parabolic-elliptic equations in a Hilbert Space” , *Labour of the IMM of AS of theTurkmenistan,* Ashgabat, pp. 53-57, 1994.

[20] Ashyralyev, A., “ Coercive solvability of parabolic equations in spaces of smooth functions”, *Izv. Akad. Nauk Turkmen. SSR Ser. Fiz. -Tekhn. Khim. Geol. Nauk*, No. 3, pp. 3-13, (Russian) 1989.

[21] Dzhuraev, T.D., *Boundary Value Problems for Equations of Mixed and Mixed-Composite Types,* Fan:Tashkent, (Russian), 1979.

[22] M.G. Karatopraklieva, “ On a nonlocal boundary value problem for an equation of mixed type” *Differensial’nye Uravneniya*,Vol. 27, No.1, pp. 68-79, (Russian), 1991.

[23] V.N. Vragov, *Boundary Value Problems for Nonclassical Equations of Mathematical Physics,* Textbook for Universities, NGU:Novosibirsk, (Russian), 1983.

[24] A.M. Nakhushev, *Equations of Mathematical Biology*, Textbook for Universities, Vysshaya Shkola:Moskow, (Russian), 1995.

[25] A. Ashyralyev and H.A. Yurtsever, " On a nonlocal boundary value problem for semilinear hyperbolic-parabolic equations*" Nonlinear Analysis-Theory, Methods and Applications*,Vol.47, No.5, pp.3585–3592, 2001.

[26] V.B. Shakhmurov, "*Maximal B-regular boundary value problems with parameters" Journal of Mathematical Analysis and Applications*,Vol.320, No.1, pp. 1-19, 2006.

[27] A. Ashyralyev, "A note on the nonlocal boundary value problem for elliptic-parabolic equations" *Nonlinear Studies*, 2004, 8p., accepted.

[28] A. Ashyralyev, " A note on the well-posedness of the nonlocal boundary value problem for elliptic-parabolic equation", *Journal of Physics A: Mathematical and General*, 2006, 9p., submitted.

[29] A. Ashyralyev and Y. Ozdemir, " Stability of difference schemes for hyperbolic-parabolic equations" *Computers and Mathematics with Applications*, Vol.50, No. 8-9,pp.1443-1476, 2005.

[30] Kubo ,M. and Yamazaki ,N., "Periodic solutions of elliptic-parabolic variational inequalities with time-dependent constraints", *Journal of evolution equations,Vol.* 6, No.1, pp. 71-93, 2006

[31] Escher, J., "Classical solutions to a moving boundary problem for an elliptic-parabolic system " ,*Interfaces and Free Boundaries,* Vol. 6, No. 2, pp. 175-193, 2004.

[32] Díaz, J. I., Lerena, B., Padial, J. F. and Rakotoson, J. M., "Global behaviour for bounded solutions of a porous media equation of elliptic-parabolic type" Comptes Rendus de l'Académie des Sciences - Series I - Mathematics, Vol.329, No. 9, pp.773-777, 1999

[33] Diaz, J., Lerena, M., Padial, J. and Rakotoson, J., "An elliptic-parabolic equation with a nonlocal term for the transient regime of a plasma in a Stellarator", *Journal of differential equations ,* Vol. 198,No. 2, pp. 321-355, 2004.

[34] Andreianov, B., Gutnic, M., and Wittbold, P., "Convergence of finite volume approximations for a nonlinear elliptic-parabolic problem: A continuous approach" *,Siam Journal on Numerical Analysis,* Vol. 42, No. 1, pp. 228-251, 2004.

[35] Andreianov, B. and Bouhsiss, F., "Uniqueness for an elliptic-parabolic problem with Neumann boundary condition", *Journal of Evolution Equations*, Vol. 4, No. 2, pp. 273-295, 2004.

[36] Benilan, P. and Wittbold, P., "On mild and weak solutions of elliptic-parabolic problems", *Adv. Differential Equations,* Vol. 16, pp.1053-1073, 1996.

[37] Pao,C.V.,"Dynamics of reaction-diffusion equations with nonlocal boundary conditions ", *Quart. Appl. Math.,*Vol. 53, pp.173-186, 1995.

[38] Chipot, M. and Lovat, B., "Some remarks on non local elliptic and parabolic problems", *Nonlinear Analysis*, Vol. 30, No. 7, pp. 4619-4627, 1997.

[39] Kröner, D. and Rodrigues, J.F., "Global behaviour for bounded solutions of a porous media equation of elliptic-parabolic type", *J. Math. Pures Appl.,*Vol. 64, pp.105-120, 1985.

[40] Lagnese, J., "Elliptic and parabolic boundary value problems of nonlocal type",*Journal of Mathematical Analysis and Applications,*Vol. 40,No. 1, pp.183-201, 1972.

[41] Hulshof, D. H., "An elliptic-parabolic problem in combustion theory: Convergence to travelling waves", *Nonlinear Analysis,* Vol. 17,No. 6, pp.519-546, 1991.