METHODS FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

In this study, methods for analytical and numerical solutions of the Fractional Differential Equations are investigated. These methods are Green's Function Method, Adomian Decomposition Method, Power Series Method and Finite Difference Method. Also, they are illustrated with a special type of fractional differential equation

$$AD_t^2 x(t) + BD_t^{3/2} x(t) + Cg(x) = f(t),$$

where $A \neq 0$ and $B, C \in R$ which is known as Bagley-Torvik equation. The results are compared both numerically and graphically, computer programmes and algorithms are presented.

Keywords: Fractional differential equations, Bagley-Torvik equation, Green's Function method, power series method, ADM, FDM.

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ÖZ

Bu çalışmada Kesirli Mertebeden Diferansiyel Denklemlerin analitik ve sayısal çözüm yöntemleri araştırılmıştır. Bu metodlar Grenn's Fonksiyonu Metodu, Adomian Decomposition Metodu, Kuvvet Serisi Metodu ve Sonlu Fark Şemaları Metodlarıdır. Çözüm yöntemleri Bagley-Torvik diye bilinen

$$AD_t^2 x\left(t\right) + BD_t^{3/2} x\left(t\right) + Cg\left(x\right) = f\left(t\right),$$

$A \neq 0$ ve $B, C \in R$ özel bir kesirli mertebeden diferansiyel denklem ile örneklendirilmiştir. Sonuçlar sayısal ve grafiksel olarak karşılaştırılıp, bilgisayar programları ve algoritmaları yazılıp uygulanmıştır.

Anahtar Kelimeler: Kesirli mertebeden diferansiyel denklemler, Bagley-Torvik denklemi, Green's fonksiyon methodu, kuvvet serisi metodu, ADM, sonlu fark şemaları metodu.

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To my parents, my brother and sister.

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LIST OF SYMBOLS/ABBREVIATIONS

α	Order of a fractional operator
σ	shear stress
ρ	fluid density
μ	viscosity
ADM	Adomian decomposition method
FDE	Fractional differential equation
FDM	Finite difference method

INTRODUCTION
FRACTIONAL CALCULUS

Recently, the study of fractional calculus achieves a wide range of applications in many areas. Especially, in physics, chemistry and engineering it becomes a popular subject. The history of it began with a letter from L'Hospital to Leibniz in which is asked the meaning of the derivative of order 1/2 in 1695. In 1738, Euler did the first attempt with observing the result of the evaluation of the non-integer order derivative of a power function x^a has a meaning and right after in 1820, Lacroix repeated the Euler's idea and nearly found the exact formula for the evaluation of the half derivative of the power
function x^a. Then, first definition for the derivative of arbitrary positive order suitable for any sufficiently good function, not necessarily a power function was given by Fourier (1822) as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^\alpha d\lambda \int_{-\infty}^{\infty} f(t) \cos(\lambda x - t\lambda + \alpha\pi/2) dt. \tag{1.1}$$

Near all these studies, the first solution of a fractional order equation was maden by Abel in 1823 with the formulation of the tautochrone problem as an integral equation

$$\int_a^x \frac{\varphi(t)}{(x-t)^\mu} dt = f(x), \qquad x > a, 0 < \mu < 1. \tag{1.2}$$

After 1832, applications of the fractional calculus to the solution of some types of linear ordinary differential equations was seen in the papers of Liouville. His initial definition based on the formula for the differentiating an exponential function which may be expanded as the series $f(x) = \sum_{k=0}^{\infty} c_k e^{a_k x}$ is

$$D^{\alpha} f(x) = \sum_{k=0}^{\infty} c_k a_k^{\alpha} e^{a_k x}, \qquad \text{for any complex } \alpha. \tag{1.3}$$

Starting from the definition (1.3), he obtained the formula for the differentiation of a power function and fractional integration which is known Liouville's first formula

$$D^{-\alpha} f(x) = \frac{1}{(-1)^{\alpha}\Gamma(\alpha)} \int_0^{\infty} \varphi(x+t) t^{\alpha-1} dt, \quad -\infty < x < \infty, \operatorname{Re}\alpha > 0. \tag{1.4}$$

Next, Riemann's expression which was done when he was a student in 1847 has become one of the main formula with Liouville's construction. Riemann had lastly arrived the expression:

$$\frac{1}{\Gamma(\alpha)} \int_0^{x} \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \qquad x > 0. \tag{1.5}$$

Studies on fractional calculus achieved a significant and suitable level for modern mathematicians after 1880's. Being more applicable and veritable greatly enhanced the power of fractional calculus. Therefore, need of efficient and reliable techniques to solve the problems which are modeled with fractional integral and differential operators occur. Liouville was the first person who tried to solve fractional differential equations as mentioned above. Then, some books written by the authors Samko, Kilbas, Marichev [17], Podlubny [10], Miller and Ross [11], Oldham and Spainer [12] played a considerable role to understand the subject and gave the applications of fractional differential equations and methods for solutions.

The present study is starting with basic definitions for fractional calculus like as Gamma, Beta, Mittag-Leffler functions and definitions of methods that are used for solving ordinary-type fractional differential equations called extraordinary differential equations.

The methods are illustrated with a well-known equation:

$$AD_t^2 x(t) + BD_t^\alpha x(t) + Cg(x) = f(t), \tag{1.6}$$

where $A \neq 0$ and $B, C \in R, \alpha > 0$, which is known as Bagley-Torvik equation. It is one of the most popular fractional differential equation because of it's physical meaning and being easily modified to see the different analogies of fractional differential equations. Bagley-Torvik equation has a well-known application which is a rigid plate of m immersed into an infinite Newtonian fluid [46], [1].

Firstly, we will consider a motion of an infinite plate in a half-space Newtonian viscous fluid as shown in Fig. 1.1. to show the shear stress in the fluid can be expressed directly in terms of a fractional-order time derivative of the fluid velocity profile.

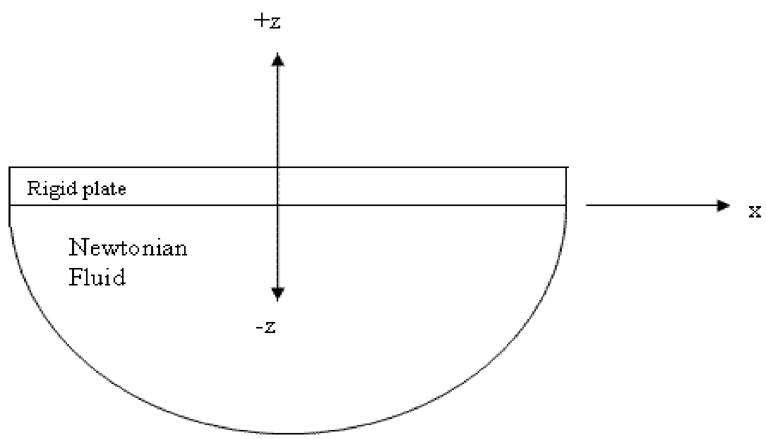

Figure 1.1 . Infinite plate in a half-space Newtonian viscous fluid

The equation of motion of the half-space fluid is the diffusion equation,

$$\rho \frac{\partial v(z,t)}{\partial t} = \mu \frac{\partial^2 v(z,t)}{\partial z^2}, \tag{1.7}$$

where ρ is the fluid density, μ is the viscosity and $v(z,t)$ describes the transverse fluid velocity as a function of z and t. Taking the Laplace transform of the equation (1.7) and using the following rule for the treatment of time-derivatives,

$$\mathcal{L}\left[\frac{\partial f(t)}{\partial t}\right] = s\mathcal{L}\left[f(t)\right] - f(t=0), \tag{1.8}$$

one obtains

$$\rho s\mathcal{L}\left[v(z,t)\right] - \rho v(z,t=0) = \mu \frac{\partial^2}{\partial z^2}\mathcal{L}\left[v(z,t)\right]. \tag{1.9}$$

Bagley and Torvik assumed the initial velocity profile in the fluid to be zero, thus the equation (1.9) reduces to

$$\rho s\mathcal{L}\left[v(z,t)\right] = \mu \frac{\partial^2}{\partial z^2}\mathcal{L}\left[v(z,t)\right]. \tag{1.10}$$

Since the Laplace Transformation is evaluated with respect to the time variable only, the following representation for the velocity profile with respect to the depth z can be used:

$$v(z,t) = v(t)e^{\lambda z} \tag{1.11}$$

$$\rightarrow \mathcal{L}\left[v(z,t)\right] = e^{\lambda z}\mathcal{L}\left[v(t)\right] \tag{1.12}$$

$$\rightarrow \frac{\partial^2}{\partial z^2}\mathcal{L}\left[v(z,t)\right] = \lambda^2 e^{\lambda z}\mathcal{L}\left[v(t)\right] \tag{1.13}$$

After insertion of (1.12) and (1.11) in (1.10) the following algebraic equation can be obtained for the unknown parameter λ.

$$\rho s e^{\lambda z}\mathcal{L}\left[v(t)\right] = \mu\lambda^2 e^{\lambda z}\mathcal{L}\left[v(t)\right] \rightarrow \lambda = \sqrt{\frac{s\rho}{\mu}} \tag{1.14}$$

From the boundary condition that the velocity of the fluid at $z = 0$ matches the prescribed velocity of the plate, $v_p(t)$, the complete velocity profile can be derived as

$$v(z=0,t) = v(t) = v_p(t) \rightarrow v(z,t) = v_p(t)e^{\sqrt{\frac{s\rho}{\mu}}z}. \tag{1.15}$$

In the next step, the shear stress relationship of the Newtonian fluid

$$\sigma(z,t) = \mu\frac{\partial v(z,t)}{\partial z} \tag{1.16}$$

is transformed into the Laplace domain using the above results.

$$\mathcal{L}\left[\sigma(z,t)\right] = \mu\sqrt{\frac{s\rho}{\mu}}e^{\sqrt{\frac{s\rho}{\mu}}z}\mathcal{L}\left[v(t)\right] = \sqrt{\mu\rho}\sqrt{s}\mathcal{L}\left[v(z,t)\right] \tag{1.17}$$

Equation (1.17) can be written as:

$$\mathcal{L}\left[\sigma(z,t)\right] = \sqrt{\mu\rho}\frac{s}{\sqrt{s}}\mathcal{L}\left[v(z,t)\right] \tag{1.18}$$

Now, the following two transforms can be identified in the equation (1.18):

$$s\mathcal{L}\left[v(z,t)\right] = \mathcal{L}\left[\frac{\partial v(z,t)}{\partial t}\right] \tag{1.19}$$

$$\frac{1}{\sqrt{s}} = \mathcal{L}\left[\frac{1}{\Gamma(1/2)\sqrt{t}}\right] \tag{1.20}$$

Using (1.19) and (1.20), it can be obtained as:

$$\mathcal{L}\left[\sigma(z,t)\right] = \sqrt{\mu\rho}\mathcal{L}\left[\frac{1}{\Gamma(1/2)\sqrt{t}}\right].\mathcal{L}\left[\dot{v}(z,t)\right]. \tag{1.21}$$

The product of two transforms in (1.21) corresponds to the following convolution when evaluating the inverse transformation:

$$\sigma(z,t) = \sqrt{\mu\rho}\frac{1}{\Gamma(1/2)}\int_0^t \frac{\dot{v}(\tau)}{(t-\tau)^{1/2}}d\tau. \tag{1.22}$$

Since the initial profile was assumed to be zero, the equation (1.22) can be written as:

$$\begin{aligned}\sigma(z,t) &= \sqrt{\mu\rho}\frac{1}{\Gamma(1/2)}\frac{d}{dt}\int_0^t \frac{v(z,t)}{(t-\tau)^{1/2}}d\tau \\ &= \sqrt{\mu\rho}_0 D_t^{1/2} v(z,t). \end{aligned} \tag{1.23}$$

In (1.23) a fractional derivative of degree $\alpha = 1/2$ can be identified within the shear stress-velocity relationship of a half-space Newtonian fluid. It is important because the fractional derivative is used to describe a real physical system, which was formulated in a conventional manner.

Now, we will consider a rigid plate of m immersed into an infinite Newtonian fluid as shown in the Fig. 1.2. The plate is held at a fixed point by means of a spring of stiffness k.

It is assumed that the motions of the spring do not influence the motion of the fluid and that the area A of the plate is very large, such that the stress-velocity relationship (1.22) is valid on both sides of the plate. Equilibrium of all forces acting on the plate gives

$$m\ddot{u}(t) + ku(t) + 2A\sigma(z=0,t). \tag{1.24}$$

Substituting the equation (1.23), it can be obtained as:

$$m\ddot{u}(t) + ku(t) + 2A\sqrt{\mu\rho}_0 D_t^{1/2} v(z=0,t) = 0 \tag{1.25}$$

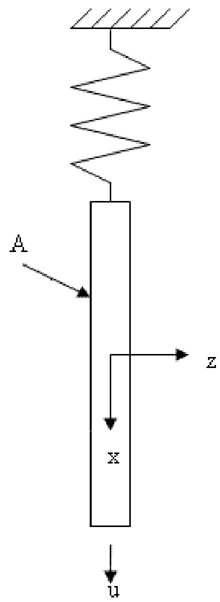

Figure 1.2 . A rigid plate of m immersed into an infinite Newtonian fluid

with

$$v(z=0,t)=\dot{u}(t), \tag{1.26}$$

a fractional differential equation of degree $\alpha = 3/2$ follows for the displacement of a rigid plate immersed into an infinite Newtonian fluid.

$$m\ddot{u}(t)+ku(t)+2A\sqrt{\mu\rho}_0 D_t^{3/2}u(t)=0. \tag{1.27}$$

Our examples are concerned with the differential equations of order 3/2 like as the equation (1.27).

In the third chapter, the stability estimates are done for both the differential and difference equation for Bagley-Torvik equation.

In the fourth chapter, the Green's Function method is defined to find the analytical solutions of Fractional Differential equations. It is investigated for one-term, two-term, three term and n-term equations. The analytical solution can be find with the help of Mittag-Leffler function and Laplace transform. Taking Laplace transform of the fractional differential equations is noted in the beginning of the chapter. Then, the method is applied to Bagley-Torvik equation, as an example of three-term equation and their algorithms, maple programs and graphical solutions are implemented.

In the fifth chapter, the Adomian Decomposition method is set to find the solutions of the Fractional Differential Equations. Relationship between finding the Adomian polynomials and Taylor series expansion of a function is investigated. Tools for the method is explained and method is applied to the Bagley-Torvik Equations. Algorithms, maple programs, and graphical solutions are implemented for the examples.

In the sixth chapter, the Finite Difference Method is defined for finding the numerical solutions of the Fractional Differential Equations. Fractional order finite difference is defined and first-order backward difference is used. Furthermore, to accelerate the computation Short Memory Principle is explained and applied graphically with the examples. Algorithms, maple programs, graphical solutions and error analysis are implemented.

In the seventh chapter, the Power Series Method is defined to find the series solution of the Fractional Differential Equations. First, equation is written in the form of a power series, then results are found. Algorithms, maple programs and graphical solutions of the example equations are implemented. Lastly, conclusions about the methods are explained in the conclusion part.

CHAPTER 2

SOME BASIC DEFINITIONS

2.1. PROPERTIES OF SOME SPECIAL FUNCTIONS

Here, some definitions and mathematical properties of special functions are given useful for numerical computation of fractional operators. More extensive information about special functions and their computations can be found in the books by Andrews [18], Zhang and Jin [19].

2.1.1. The Gamma Function

The Gamma function is very important function appearing by itself in physical applications, denoted by $\Gamma(z)$. The Gamma function has common definitions which are equal and most of them are related to Euler's works.

However the Gamma function first was defined by limit definition due to Gauss(1777-1855) :

Definition 1.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)(z+2)...(z+n)} \tag{2.1}$$

In the last equation there is an obvious restriction that (2.1) cannot be defined at negative integer values of z, but for all other values we can easily compute.

Although z is restricted in to only positive values, most popular definition of the gamma function is integral transform definition.

Definition 2. Integral representation of $\Gamma(x)$ is

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad \operatorname{Re} z > 0 \tag{2.2}$$

$\Gamma(x)$ is a continuous function for all $z \in C$, for which $\operatorname{Re} z > 0$. Also, the equation (2.2) can be extended by analytic continuation to the case $\operatorname{Re} z < 0$.

In addition to the definitions above, we can derive the following formula by setting $t = u^2$ and $dt = 2udu$ in the equation (2.2):

$$\Gamma(z) = \int_0^\infty e^{-u^2} u^{2z-2} 2udu = 2\int_0^\infty e^{-u^2} u^{2z-1} du, \qquad z > 0. \tag{2.3}$$

from equation (2.3)

$$\Gamma(x)\Gamma(y) = (2\int_0^\infty e^{-u^2} u^{2x-1})(2\int_0^\infty e^{-v^2} v^{2y-1}) dudv \qquad x > 0, y > 0. \tag{2.4}$$

Now let us write last equation in terms of polar coordinates variables (r, θ) :

$$u = r\cos\theta$$

$$v = r\sin\theta$$

equation (3.18) now becomes:

$$\Gamma(x)\Gamma(y) = 4\int_0^{\pi/2} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} \int_0^\infty e^{-r^2} r^{2(x+y)-1} drd\theta. \tag{2.5}$$

Since

$$2\int_0^\infty e^{-r^2} r^{2(x+y)-1} dr = \Gamma(x+y), \tag{2.6}$$

we have

$$\frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)} = \int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1} d\theta, \qquad x>0, y>0 \tag{2.7}$$

and if we write $x = 1/2$, $y = 1/2$ into the equation (2.7), we can get the value of $\Gamma(1/2)$ which is important for evaluating other values:

$$\frac{\Gamma(1/2)\Gamma(1/2)}{2\Gamma(1)} = \int_0^{\pi/2} 1 d\theta \tag{2.8}$$

$$\Gamma(1/2) = \sqrt{\pi}. \tag{2.9}$$

Properties of The Gamma Function.

1. Recurrence Formulas:

   When we write $z = 1$ in the equation (1), then we have

$$\Gamma(1) = \lim_{n\to\infty} \frac{n!n}{1.2.3....n(n+1)} = \lim_{n\to\infty} \frac{n}{n+1}$$

$$\Gamma(1) = 1, \tag{2.10}$$

other values of $\Gamma(z)$ can be easily obtained from the recurrence formula deduced by substituting $z+1$ into z:

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty} \frac{n!n^{z+1}}{(z+1)(z+2)...(z+n)(z+n+1)} \\ &= \lim_{n\to\infty} \frac{nz}{z+n+1}. \lim_{n\to\infty} \frac{n!n^z}{z(z+1)...(z+n)}\end{aligned}$$

$$\Gamma(z+1) = z\Gamma(z) \tag{2.11}$$

If we combine equations (2.11) and (2.10) we get

$$\Gamma(z+1) = z! \qquad z = 0, 1, 2... \tag{2.12}$$

From the equation (2.11), gamma function can be defined over the interval $z > -n$(except $z = 0, 1, 2, ..., -n+1$) as

$$\Gamma(z+1) = \frac{\Gamma(z+2)}{z+1} \qquad z > -1, \quad z \neq -1 \tag{2.13}$$

then one more we can combine equation (2.13) and equation (2.11):

$$\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)} \qquad z > -2, \quad z \neq 0, -1 \tag{2.14}$$

Using these expressions, we can find another recurrence formula for gamma function:

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)...(z+n-1)} \tag{2.15}$$

2. Binomial Formula:

   If z is not a positive integer we can write the binomial coefficient related to gamma function as:

$$\binom{-z}{r} = \frac{\Gamma(1-z)}{\Gamma(r+1)\Gamma(1-z-r)} \tag{2.16}$$

3. Reflection Formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad z \neq (0, \pm 1, \pm 2, ...) \tag{2.17}$$

4. Legendre Duplicated Formula:

$$\Gamma(2x)\sqrt{\pi} = 2^{2x-1}\Gamma(x+1/2)\Gamma(x) \tag{2.18}$$

2.1.2. The Beta Function:

Definition 3. The Beta function is a two variable function defined as:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \qquad x > 0, y > 0 \tag{2.19}$$

Properties of the Beta Function

1. Beta function is a symmetric function, i.e.,

$$\beta(x,y) = \beta(y,x) \tag{2.20}$$

2. Beta function can be evaluated in terms of the gamma function:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2.21}$$

2.1.3. Mittag-Leffer Function:

Definition 4. A two-parameter function of the Mittag-Leffer type is defined by the series expansion:

$$E_{\alpha,\beta} = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(\alpha j + \beta)}, \tag{2.22}$$

where $(\alpha > 0, \beta > 0)$ .And (k)th derivative of the function is:

$$E_{\alpha,\beta}^{(k)} = \sum_{j=0}^{\infty} \frac{(j+k)!y^j}{\Gamma(\alpha j + \alpha k + \beta)}, \quad (k = 0, 1, 2, ...) . \tag{2.23}$$

2.2. Riemann-Liouville Fractional Operator

The historical survey of the Riemann-Liouville definition is mentioned in the introduction part. In this section, most used definition and it's origin denoted.

First, let us consider a continuous function $y = f(t)$. According to the well-known definition, the first-order derivative of the function $f(t)$ is defined by

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}. \tag{2.24}$$

Applying this definition twice gives the second-order derivative:

$$\begin{aligned} f''(t) = \frac{d^2 f}{dt^2} &= \lim_{h\to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \left\{ \frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h} \right\} \\ &= \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}. \end{aligned} \tag{2.25}$$

Using (2.24) and (2.25), we obtain third-order derivative

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h\to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3}, \tag{2.26}$$

and, by induction, $n-th$ order derivative can be obtained as:

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh), \tag{2.27}$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \tag{2.28}$$

is the usual notation for the binomial coefficients.

Let us now consider the following expression generalizing the fractions in (2.24) to (2.27)

$$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh), \tag{2.29}$$

where p is an arbitrary integer number; n is also integer, as above.

Obviously, for $p \leq n$ we have

$$\lim_{h \to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p}, \tag{2.30}$$

because in such a case, as follows from (2.27), all the coefficients in the numerator after $\binom{p}{p}$ are equal to 0.

Let us consider negative values of p. For convenience, let us denote

$$\binom{p}{r} = \frac{p(p+1)\ldots(p+r-1)}{r!} \tag{2.31}$$

Then we have

$$\binom{-p}{r} = -\frac{p(-p-1)\ldots(-p-r+1)}{r!} = (-1)^r \binom{p}{r} \tag{2.32}$$

and replacing p in equation (2.29) with $-p$ we can write

$$f_h^{(-p)}(t) = \frac{1}{h^{-p}} \sum_{r=0}^{n} \binom{p}{r} f(t - rh), \tag{2.33}$$

where p is a positive integer number.

If n is fixed, then $f_h^{(-p)}(t)$ tends to the uninteresting limit 0 as $h \to 0$. To arrive at a nonzero limit, we have to suppose that $n \to \infty$ as $h \to 0$. We can take $h = \frac{t-a}{n}$, where a is a real constant, and consider the limit value, either finite or infinite, of $f_h^{(-p)}(t)$ which we denote as

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-p)}(t) = {}_a D_t^{-p} f(t). \tag{2.34}$$

Here ${}_aD_t^{-p}f(t)$ denotes, in fact, a certain operation performed on the function; a and t are the limits relating this operation.

Let us consider several particular cases:

For $p = 1$ we have:

$$f_h^{(-1)}(t) = h\sum_{r=0}^{n} f(t - rh). \tag{2.35}$$

Taking into account that $t - nh = a$ and the function $f(t)$ is assumed to be continuous, it can be concluded that

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-1)}(t) = {}_a D_t^{-1} f(t) = \int_0^{t-a} f(t-z)\,dz = \int_a^t f(\tau)\,d\tau. \tag{2.36}$$

Let us take $p = 2$. In this case

$$\binom{2}{r} = \frac{2\cdot 3\cdot ...(2+r+1)}{r!} = r+1$$

and we have:

$$f_h^{(-2)}(t) = h\sum_{r=0}^{n}(r+1)\,hf(t-rh). \tag{2.37}$$

Denoting $t + h = y$

$$f_h^{(-2)}(t) = h \sum_{r=1}^{n+1} (rh)\, f(y - rh) \tag{2.38}$$

and taking $h \to 0$ it will be

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-2)}(t) =_a D_t^{-2} f(t) = \int_0^{t-a} z f(t - z)\, dz = \int_a^t (t - \tau)\, f(\tau)\, d\tau, \tag{2.39}$$

because $y \to t$ as $h \to 0$.

The third particular case, namely $p = 3$, will show us the general expression for ${}_aD_t^{-2} f(t)$.

Taking into account that:

$$\binom{3}{r} = \frac{3 \cdot 4 \cdot ... (3 + r + 1)}{r!} = \frac{(r+1)(r+2)}{1 \cdot 2}, \tag{2.40}$$

we have

$$f_h^{(-3)}(t) = \frac{h}{1 \cdot 2} \sum_{r=0}^{n} (r+1)(r+2)\, h^2 f(t - rh). \tag{2.41}$$

Denoting as above, $t + h = y$ we write

$$f_h^{(-3)}(t) = \frac{h}{1 \cdot 2} \sum_{r=1}^{n+1} r(r+1)\, h^2 f(y - rh). \tag{2.42}$$

Expression (2.42) can be written as

$$f_h^{(-3)}(t) = \frac{h}{1 \cdot 2} \sum_{r=1}^{n+1} (rh)^2 f(y - rh) + \frac{h^2}{1 \cdot 2} \sum_{r=1}^{n+1} rhf(y - rh). \tag{2.43}$$

Taking $h \to 0$ we obtain

$$ {}_aD_t^{-3} f(t) = \frac{1}{2!} \int_0^{t-a} z^2 f(t-z)\, dz = \int_a^t (t-\tau)\, d\tau, \tag{2.44} $$

because $y \to t$ as $h \to 0$ and

$$ \lim_{\substack{h\to 0\\ nh=t-a}} \frac{h^2}{1\cdot 2} \sum_{r=1}^{n+1} rhf(y-rh) = \lim_{\substack{h\to 0\\ nh=t-a}} h \int_a^t (t-\tau) f(\tau)\, d\tau = 0. \tag{2.45} $$

Relationships (2.37)-(2.45) suggest the following general expression:

$$ {}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh) = \frac{1}{(p-1)!} \int_a^t (t-\tau)^{p-1} f(\tau)\, d\tau. \tag{2.46} $$

Riemann and Liouville continued this result with replacing the discrete factorial $(p-1)!$ with Euler's continuous gamma function

$$ {}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. \tag{2.47} $$

To obtain differentiation of fractional order, we can write

$$ \begin{aligned} {}_aD_t^{p} f(t) &= \frac{d^n}{dt^n}\, {}_aD_t^{-(n-p)} f(t) \\ &= \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-p-1} f(\tau) d\tau. \end{aligned} $$

If we take the lower limit 0, we can obtain the most used formula:

$$ {}_0D_t^{p} f(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-p-1} f(\tau) d\tau. $$

CHAPTER 3

STABILITY OF THE PROBLEM

3.1. The Stability of the Initial-Value Problem for Bagley-Torvik Equation

We consider the initial value problem for Bagley-Torvik equation

$$\begin{cases} D_t^2 u\left(t\right) + \frac{1}{2} D_t^{3/2} u\left(t\right) + \frac{1}{2} u(t) = f\left(t\right), \\ u(0) = 0, \ u'(0) = 0. \end{cases} \tag{3.1}$$

Note that $D_t u\left(t\right) = u'(t)$.

Then, we can rewrite it

$$\begin{cases} D_t^2 u\left(t\right) + \frac{1}{2} u(t) = f\left(t\right) - \frac{1}{2} D_t^{3/2} u\left(t\right), \\ u(0) = 0, \ u'(0) = 0 \end{cases} \tag{3.2}$$

Solving it, we get

$$\begin{aligned} u(t) = {} & \cos\frac{\sqrt{2}}{2} t u(0) + \sqrt{2} \sin\frac{\sqrt{2}}{2} t u'(0) + \\ & + \sqrt{2} \int_0^t \sin\frac{\sqrt{2}}{2}(t-s)\left(-\frac{1}{2} u^{3/2}(s) + f(s)\right) ds. \end{aligned} \tag{3.3}$$

Using the last formula, we can write

$$\begin{aligned} u'(t) = {} & -\frac{\sqrt{2}}{2} \sin\frac{\sqrt{2}}{2} t u(0) + \cos\frac{\sqrt{2}}{2} t u'(0) \\ & + \int_0^t \cos\frac{\sqrt{2}}{2}(t-s)\left(-\frac{1}{2} u^{3/2}(s) + f(s)\right) ds \end{aligned} \tag{3.4}$$

and

$$u^{1+1/2}(t) = \frac{d^{1/2}}{dt^{1/2}}\left(-\frac{\sqrt{2}}{2}\sin\frac{\sqrt{2}}{2}tu(0)\right) + \frac{d^{1/2}}{dt^{1/2}}\left(\cos\frac{\sqrt{2}}{2}tu'(0)\right) \tag{3.5}$$
$$+ \frac{1}{\sqrt{\pi}}\int_0^t \frac{\cos\frac{\sqrt{2}}{2}(t-s)}{(t-s)^{1/2}}\left(-\frac{1}{2}u^{3/2}(s) + f(s)\right) ds.$$

So, from the equation (3.5), an estimate for $u^{(3/2)}$ can be written as

$$t^{1/2}\left|u^{(3/2)}(t)\right| \leq \frac{t}{\sqrt{\pi}}|u(0)| + \frac{1}{\sqrt{\pi}}|u'(0)| + \int_0^t \frac{t^{1/2}}{\sqrt{\pi}}\frac{s^{1/2}\left|u^{(3/2)}(s)\right| ds}{s^{1/2}(t-s)^{1/2}} \tag{3.6}$$
$$+ \int_0^t \frac{t^{1/2}}{\sqrt{\pi}}\frac{|f(s)|\, ds}{(t-s)^{1/2}}$$

where $0 \leq t \leq T$.

Here, consider the function $z(t)$;

$$|z(t)| = \left|t^{1/2}u^{(3/2)}(t)\right|. \tag{3.7}$$

Then,

$$|z(t)| \leq \frac{t}{\sqrt{\pi}}|u(0)| + \frac{1}{\sqrt{\pi}}|u'(0)| + \int_0^t \frac{t^{1/2}}{\sqrt{\pi}}\frac{|z(s)|\, ds}{s^{1/2}(t-s)^{1/2}} + \int_0^t \frac{t^{1/2}}{\sqrt{\pi}}\frac{|f(s)|\, ds}{(t-s)^{1/2}} \tag{3.8}$$

Since, we have

$$\int_0^t \frac{t^{1/2}}{\sqrt{\pi}}\frac{|f(s)|\, ds}{(t-s)^{1/2}} \leq \max_{0\leq t\leq T}|f(t)|\frac{t^{1/2}}{\sqrt{\pi}}\int_0^t \frac{ds}{(t-s)^{1/2}} \tag{3.9}$$
$$\leq \frac{T}{\sqrt{\pi}}\max_{0\leq t\leq T}|f(t)|$$

and

$$\int_0^t \frac{ds}{s^{1/2}(t-s)^{1/2}} = B\left(\frac{1}{2},\frac{1}{2}\right), \tag{3.10}$$

we can obtain

$$\begin{aligned}
|z(t)| &= C_1 + \int_0^t \frac{T^{1/2} z(s)ds}{s^{1/2}(t-s)^{1/2}}, \\
C_1 &\leq \frac{T}{\sqrt{\pi}} \max_{0\leq t\leq T} |f(t)| + \frac{T}{\sqrt{\pi}} |u(0)| + \frac{1}{\sqrt{\pi}} |u'(0)| \\
&\leq C(T) \left[\max_{0\leq t\leq T} |f(t)| + |u(0)| + |u'(0)| \right] \qquad (3.11)
\end{aligned}$$

or

$$\begin{aligned}
|z(t)| &\leq C_1 e^{aT} \leq C(T) \left[\max_{0\leq t\leq T} |f(s)| + |u(0)| + |u'(0)| \right] \\
&\leq M \left[\max_{0\leq t\leq T} |f(t)| + |u(0)| + |u'(0)| \right]. \qquad (3.12)
\end{aligned}$$

Using the last estimate, we rewrite $\left| u^{(3/2)}(t) \right|$

$$\left| u^{(3/2)}(t) \right| \leq \frac{M \left[\max_{0\leq t\leq T} |f(t)| + |u(0)| + |u'(0)| \right]}{\sqrt{T}}. \qquad (3.13)$$

Next, we can obtain that the solution of the problem (3.2) satisfies the stability estimate:

$$\begin{aligned}
|u(t)| &\leq |u(0)| + \sqrt{2} |u'(0)| + \frac{M \left[\max_{0\leq t\leq T} |f(t)| + |u(0)| + |u'(0)| \right]}{\sqrt{T}} \\
&\quad + T \max_{0\leq t\leq T} |f(t)| \qquad (3.14) \\
&\leq \frac{\left[\max_{0\leq t\leq T} (M+T) |f(t)| + (M+1) |u(0)| + (M+\sqrt{2}) |u'(0)| \right]}{\sqrt{T}}.
\end{aligned}$$

Lastly, from the triangular inequality, we can obtain an estimate for $u''(t)$:

$$\begin{aligned}
|u''(t)| &\leq \frac{\left[\max_{0\leq t\leq T} (M+T) |f(t)| + (M+1) |u(0)| + (M+\sqrt{2}) |u'(0)| \right]}{\sqrt{T}} + \qquad (3.15) \\
&\quad + \frac{M \left[\max_{0\leq t\leq T} |f(t)| + |u(0)| + |u'(0)| \right]}{\sqrt{T}} + \max_{0\leq t\leq T} |f(t)|.
\end{aligned}$$

Using (3.13), (3.14) and (3.15), we get the following stability inequality for the solution of (3.2)

$$\left|u(t)\right|,\left|u^{(2)}(t)\right|,\left|u^{(3/2)}(t)\right|\leq A\left|u(0)\right|+B\left|u'(0)\right|+C\max_{0\leq t\leq T}\left|f(t)\right|, \tag{3.16}$$

where A, B and C are constants.

Now, we will consider the approximation formulas for the solution of (3.2). Using the fixed-point iteration, we can prove that (3.2) has a unique solution $u(t)$ and

$$u(t)=\lim_{n\to\infty}u_n(t), \tag{3.17}$$

where $u_n(t)$ defined by

$$\begin{aligned} u_n(t) &= \cos\frac{\sqrt{2}}{2}tu(0)+\sqrt{2}\sin\frac{\sqrt{2}}{2}tu'(0)+\\ &+\sqrt{2}\int_0^t\sin\frac{\sqrt{2}}{2}(t-s)\left(-\frac{1}{2}u_{n-1}^{3/2}(s)+2+s^2+\frac{4}{\sqrt{\pi}}s^{1/2}\right)ds, \end{aligned} \tag{3.18}$$

$n=1,2,3,...$ and $u_0(t)$ is given.

Here, we will consider (3.2) with $f(t)=2+t^2+\frac{4}{\sqrt{\pi}}t^{1/2}$. Using (3.18) and putting $u_0(t)=t^2$, we get

$$u_1(t)=t^2. \tag{3.19}$$

Let $u_{n-1}(t)=t^2$, so by induction

$$u_n(t)=t^2,\quad n=1,2,3,... \tag{3.20}$$

is obtained. Then,

$$u(t)=\lim_{n\to\infty}u_n(t)=\lim_{n\to\infty}t^2=t^2. \tag{3.21}$$

3.2. The Stability of the Difference Scheme for Bagley-Torvik Equation

Using the formula

$$u^{(1/2)} \approx \frac{1}{h^{1/2}} \frac{1}{\Gamma(1-1/2)} \sum_{n=1}^{k} \frac{1}{(k-n)!} \int_0^\infty t^{k-n-1/2} e^{-t} dt (u_n - u_{n-1}) \tag{3.22}$$

and approximate formulas for u'', we get a second order difference scheme for the numerical solution of (3.2):

$$\begin{cases} \frac{u_{n+1}-2u_n+u_{n-1}}{h^2} + \frac{1}{2}\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-1/2)} \sum\limits_{n=1}^{k} \frac{1}{(k-n)!} \int_0^\infty t^{k-n-1/2} e^{-t} dt (u_n - 2u_{n-1} + u_{n-2}) \\ \qquad +\frac{1}{2}u_{n+1} = f_n, \\ \qquad u_0 = 0, \ \frac{u_1-u_0}{h} = 0. \end{cases} \tag{3.23}$$

Then, we can rewrite it

$$\begin{cases} \frac{u_{n+1}-2u_n+u_{n-1}}{h^2} + \frac{1}{2}u_{n+1} = \varphi_n, \\ u_0 = 0, \ \frac{u_1-u_0}{h} = 0 \end{cases} \tag{3.24}$$

and

$$\begin{cases} \frac{u_{n+1}-2u_n+u_{n-1}}{h^2} + \frac{1}{2}u_{n+1} = \varphi_n, \\ u_0 = \zeta, \ \frac{u_1-u_0}{h} = \psi. \end{cases} \tag{3.25}$$

where $\varphi_n = f_n - \frac{1}{2}\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-1/2)} \sum\limits_{n=1}^{k} \frac{1}{(k-n)!} \int_0^\infty t^{k-n-1/2} e^{-t} dt (u_n - 2u_{n-1} + u_{n-2})$

Suppose that

$$u_n = v_n + w_n. \tag{3.26}$$

If we take

$$v_n = q^n \tag{3.27}$$

then we obtain

$$q^{n+1} - 2q^n + q^{n-1} + \frac{h^2}{2} q^{n+1} = 0, \tag{3.28}$$
$$q^{n-1}\left((1+\frac{h^2}{2})q^2 - 2q + 1\right) = 0. \tag{3.29}$$

So, the roots of the equation (3.28) are

$$q_{1,2} = \frac{1 \pm i\sqrt{h^2/2}}{1+h^2/2} = re^{\pm i\varphi} \tag{3.30}$$

where $r = \sqrt{\frac{1}{1+h^2/2}}, \quad \tan\varphi = \sqrt{h^2/2}.$

Since $\sqrt{-h^2/2} < 0$, the solution of the difference equation will be

$$u_n = \frac{\sin(1-n)\varphi}{\sin\varphi} r^n \zeta + \frac{\sin n\varphi}{\sin\varphi} r^n \psi + \left(\sum_{i=1}^{n} \frac{\sin(n-i)\varphi}{\sin\varphi} r^{n-i+1} \varphi_i\right) h^2. \tag{3.31}$$

Taking the *fractional* derivative of both sides of the last formula, we can write

$$\frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{k} \frac{1}{(k-n)!} \int_0^\infty t^{k-n-\alpha} e^{-t} dt \left(\frac{u_n - 2u_{n-1} + u_{n-2}}{h^{3/2}}\right) =$$
$$\frac{1}{h^{3/2}} \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{k} \frac{1}{(k-n)!} \int_0^\infty t^{k-n-\alpha} e^{-t} dt \frac{\sin(1-n)\varphi}{\sin\varphi} r^n \zeta$$
$$+\frac{1}{h^{3/2}} \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{k} \frac{1}{(k-n)!} \int_0^\infty t^{k-n-\alpha} e^{-t} dt \frac{\sin(n)\varphi}{\sin\varphi} r^n \psi$$
$$+\frac{1}{h^{3/2}} h^2 \frac{1}{\Gamma(1-\alpha)} \sum_{n=1}^{k} \frac{1}{(k-n)!} \int_0^\infty t^{k-n-\alpha} e^{-t} dt \left(\sum_{i=1}^{n} \frac{\sin(n-i)\varphi}{\sin\varphi} r^{n-i+1} fi\right)$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\\
&\left(\sum_{i=1}^{n}\frac{\sin(n-i)\varphi}{\sin\varphi}r^{n-i+1}\frac{1}{\Gamma(1-\alpha)}\sum_{z=1}^{i}\frac{1}{(i-z)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\frac{u_z-2u_{z-1}+u_{z-2}}{h^{3/2}}\right)\right)\\
&-2\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(2-n)\varphi}{\sin\varphi}r^{n-1}\zeta \qquad (3.32)\\
&-2\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(n)\varphi}{\sin\varphi}r^{n-1}\psi\\
&-2\frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n-1}\frac{\sin(n-1-i)\varphi}{\sin\varphi}r^{n-i}fi\right)\\
&+\frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\\
&\left(\sum_{i=1}^{n-1}\frac{\sin(n-1-i)\varphi}{\sin\varphi}r^{n-i}\frac{1}{\Gamma(1-\alpha)}\sum_{z=1}^{i}\frac{1}{(i-z)!}\int_0^\infty t^{k-z-\alpha}e^{-t}dt\left(\frac{u_z-2u_{z-1}+u_{z-2}}{h^{3/2}}\right)\right)\\
&\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(3-n)\varphi}{\sin\varphi}r^{n-2}\zeta\\
&+\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(n-2)\varphi}{\sin\varphi}r^{n-2}\psi\\
&+\frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n-2}\frac{\sin(n-2-i)\varphi}{\sin\varphi}r^{n-1-i}fi\right)\\
&-\frac{1}{2}\frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\\
&\left(\sum_{i=1}^{n-2}\frac{\sin(n-2-i)\varphi}{\sin\varphi}r^{n-1-i}\frac{1}{\Gamma(1-\alpha)}\sum_{z=1}^{i}\frac{1}{(i-z)!}\int_0^\infty t^{i-z-\alpha}e^{-t}dt\left(\frac{u_z-2u_{z-1}+u_{z-2}}{h^{3/2}}\right)\right)
\end{aligned}
$$

Denoting

$$
|z_k|=\left|\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\frac{u_n-2u_{n-1}+u_{n-2}}{h^{3/2}}\right)\right|
$$

the equation (3.32) will be

$$
\begin{aligned}
|z_k| &= \frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(1-n)\varphi}{\sin\varphi}r^n\zeta \\
&+ \frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(n)\varphi}{\sin\varphi}r^n\psi \\
&+ \frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n}\frac{\sin(n-i)\varphi}{\sin\varphi}r^{n-i+1}fi\right) \\
&- \frac{1}{2}\frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n}\frac{\sin(n-i)\varphi}{\sin\varphi}r^{n-i+1}\frac{1}{\Gamma(1-\alpha)}|z_i|\right) \\
&- 2\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(2-n)\varphi}{\sin\varphi}r^n\zeta \qquad (3.33) \\
&- 2\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(n-1)\varphi}{\sin\varphi}r^n\psi \\
&- 2\frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n-1}\frac{\sin(n-1-i)\varphi}{\sin\varphi}r^{n-i}fi\right) \\
&+ \frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n-1}\frac{\sin(n-1-i)\varphi}{\sin\varphi}r^{n-i}|z_i|\right) \\
&+ \frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(3-n)\varphi}{\sin\varphi}r^n\zeta \\
&+ \frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\frac{\sin(n-2)\varphi}{\sin\varphi}r^n\psi \\
&+ \frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n-2}\frac{\sin(n-2-i)\varphi}{\sin\varphi}r^{n-1-i}fi\right) \\
&- \frac{1}{2}\frac{1}{h^{3/2}}h^2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n-2}\frac{\sin(n-2-i)\varphi}{\sin\varphi}r^{n-1-i}|z_i|\right).
\end{aligned}
$$

Putting $\alpha = 1$, we have

$$\frac{1}{(k-n)!}\int_0^\infty t^{k-n-1}e^{-t}dt = \frac{(k-n-1)!}{(k-n)!} = \frac{1}{(k-n)}$$

and for $\alpha = 0$

$$\frac{1}{(k-n)!}\int_0^\infty t^{k-n}e^{-t}dt = \frac{(k-n)!}{(k-n)!} = 1. \qquad (3.34)$$

So, we obtain a formula for $\alpha = 1/2$ from interpolation theory

$$\frac{1}{(k-n)!}\int_0^\infty t^{k-n-1/2}e^{-t}dt \leq \frac{1}{\sqrt{k-n}}. \tag{3.35}$$

Then, we can write an estimate for z_k

$$\begin{aligned}
|z_k| \leq & \frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{\sqrt{k-n}}\frac{\sin(1-n)\varphi}{\sin\varphi}r^n\zeta \\
& + \frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{\sqrt{k-n}}\frac{\sin(n)\varphi}{\sin\varphi}r^n\psi \\
& + \frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{h}{\sqrt{h}}\frac{1}{\sqrt{k-n}}\left(\sum_{i=1}^{n}\frac{\sin(n-i)\varphi}{\sin\varphi}r^{n-i+1}fi\right) \\
& - \frac{1}{2}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{h}{\sqrt{h}}\frac{1}{\sqrt{k-n}}\left(\sum_{i=1}^{n}\frac{\sin(n-i)\varphi}{\sin\varphi}r^{n-i+1}\frac{1}{\Gamma(1-\alpha)}|z_i|\right) \\
& + 2\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{\sqrt{k-n}}\frac{\sin(2-n)\varphi}{\sin\varphi}r^n\zeta \\
& + 2\frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{\sqrt{k-n}}\frac{\sin(n-1)\varphi}{\sin\varphi}r^n\psi \\
& + 2\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{h}{\sqrt{h}}\frac{1}{(k-n)!}\int_0^\infty t^{k-n-\alpha}e^{-t}dt\left(\sum_{i=1}^{n-1}\frac{\sin(n-1-i)\varphi}{\sin\varphi}r^{n-i}fi\right) \\
& + \frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{h}{\sqrt{h}}\frac{1}{\sqrt{k-n}}\left(\sum_{i=1}^{n-1}\frac{\sin(n-1-i)\varphi}{\sin\varphi}r^{n-i}|z_i|\right) \\
& + \frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{\sqrt{k-n}}\frac{\sin(3-n)\varphi}{\sin\varphi}r^n\zeta \\
& + \frac{1}{h^{3/2}}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{1}{\sqrt{k-n}}\frac{\sin(n-2)\varphi}{\sin\varphi}r^n\psi \qquad (3.36) \\
& + \frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{h}{\sqrt{h}}\frac{1}{\sqrt{k-n}}\left(\sum_{i=1}^{n-2}\frac{\sin(n-2-i)\varphi}{\sin\varphi}r^{n-1-i}fi\right) \\
& - \frac{1}{2}\frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{k}\frac{h}{\sqrt{h}}\frac{1}{\sqrt{k-n}}\left(\sum_{i=1}^{n-2}\frac{\sin(n-2-i)\varphi}{\sin\varphi}r^{n-1-i}|z_i|\right)
\end{aligned}$$

Since

$$\sum_{n=1}^{k-1} \frac{h}{\sqrt{(k-n)h}} \leq \int_0^1 \frac{dt}{\sqrt{1-t}} = 2 \tag{3.37}$$

and from initial conditions are 0, then

$$z_k \leq \frac{8}{\pi} \max_{0\leq i\leq N} |f_i| + \sum_{n=1}^{k-1} \frac{h}{\sqrt{(k-n)h}} z_i. \tag{3.38}$$

From discrete analogies of integral inequalities, we obtain

$$|z_i| \leq \frac{8}{\pi} \max_{0\leq i\leq N} |f_i| \, e^{\sum_{n=1}^{k-1} \frac{h}{\sqrt{(k-n)h}}}, \tag{3.39}$$

$$|z_i| \leq \frac{8}{\pi} \max_{0\leq i\leq N} |f_i| \,. \tag{3.40}$$

Next, from the last estimate we can construct the following stability estimate for the solution of (3.2):

$$|u_n| \leq (1 + \frac{4}{\pi}) \max_{0\leq i\leq N} |f_i| \,. \tag{3.41}$$

Lastly, from the triangular inequality a stability estimate can be constructed as:

$$\left| \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} \right| \leq (\frac{1}{2} + \frac{6}{\pi}) \max_{0\leq i\leq N} |f_i| \,. \tag{3.42}$$

CHAPTER 4

GREEN'S FUNCTION METHOD

For the fractional differential equations, analytical solution can be found by using Green's function method with the help of Mittag-Leffler function and Laplace transform.

Now, we present Laplace transform of the fractional differential equations of the form

$$D_t^{\alpha} x(t) = f(t). \tag{4.1}$$

4.1. Laplace Transform

[12] For integer values of q, Laplace transform of a differintegrable function f is

$$\mathfrak{L}\left\{\frac{d^q f}{dx^q}\right\} \equiv \int_0^{\infty} \exp(-sx)\frac{d^q f}{dx^q}dx \tag{4.2}$$

First, recall the well-known property for integer order derivatives

$$\mathfrak{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathfrak{L}\{f\} - \sum_{k=0}^{\infty} s^{q-1-k}\frac{d^k f}{dx^k}(0), \qquad q = 1, 2, 3, ..., \tag{4.3}$$

and multiple integrals

$$\mathfrak{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathfrak{L}\{f\}, \qquad q = 0, -1, -2, ..., \tag{4.4}$$

and both formulas embraced by

$$\mathfrak{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathfrak{L}\{f\} - \sum_{k=0}^{q-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \qquad q = 0, \pm 1, \pm 2, \pm 3, ... \tag{4.5}$$

Now, we will try to generalize (4.5) formula for all q included non-integers with simple extension:

$$\mathfrak{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathfrak{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \qquad \text{all } q, \tag{4.6}$$

where n is the integer such that $n-1 < q \leq n$. The sum is empty and vanishes when $q \leq 0$.

Firstly, we consider $q < 0$, so that the Riemann-Liouville definition

$$\frac{d^q f}{dx^q} = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(y)dy}{(x-y)^{q+1}}, \qquad q < 0 \tag{4.7}$$

may be adopted. Direct application of the convolution theorem

$$\mathfrak{L}\left\{\int_0^x f_1(x-y) f_2(y) dy\right\} = \mathfrak{L}\{f_1\}\mathfrak{L}\{f_2\} \tag{4.8}$$

then gives

$$\mathfrak{L}\left\{\frac{d^q f}{dx^q}\right\} = \frac{1}{\Gamma(-q)} \mathfrak{L}\{x^{-1-q}\}\mathfrak{L}\{f\} = s^q \mathfrak{L}\{f\}, \qquad q < 0, \tag{4.9}$$

so that equation (4.9) generalizes unchanged for negative q.

For non-integer positive q, we use

$$\frac{d^q f}{dx^q} = \frac{d^n}{dx^n} \frac{d^{q-n} f}{dx^{q-n}} \tag{4.10}$$

where n is the integer such that $n-1 < q < n$.

Combining the last equation with (4.6), we obtain

$$\mathfrak{L}\left\{\frac{d^q f}{dx^q}\right\} = \mathfrak{L}\left\{\frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{dx^{q-n}}\right]\right\} = s^n \mathfrak{L}\left\{\frac{d^{q-n} f}{dx^{q-n}}\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k}}{dx^{q-1-k}}\left[\frac{d^{q-n} f}{dx^{q-n}}\right](0). \tag{4.11}$$

The difference $q - n$ being negative, the first right-hand term may be evaluated by use of (4.9).

Since $q - n < 0$, the composition rule may be applied to terms within the summation. The result

$$\mathfrak{L}\left\{\frac{d^q f}{dx^q}\right\} = s^q \mathfrak{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \qquad 0 < q \neq 1, 2, 3, ..., \tag{4.12}$$

follows from these two operations and is seen to be incorporated in (4.6)

The transformation (4.6) is a generalization of the classical formula for the Laplace transform of the derivative or integral of f.

4.2. Fractional Green's Function

Definition 5. [45] Function $G(t, \tau)$ satisfying the following conditions;

a) ${}_\tau\mathfrak{L}_t G(t, \tau) = 0$ for every $\tau \in (0, t)$;

b) $\lim_{\tau \to t-0}({}_\tau D_t^{\sigma_k} G(t, \tau)) = \delta_{k,n}$, $k = 0, 1, , ..., n$ ($\delta_{k,n}$ is Kronecker's delta) ;

c) $\lim_{\substack{\tau, t \to +0 \\ \tau < t}}({}_\tau D_t^{\sigma_k} G(t, \tau)) = 0$, $k = 0, 1, 2, ..., n-1$

is called the Green's function of the equation

${}_0\mathfrak{L}_t y(t) = f(t);$ $\qquad$ ${}_0D_t^{\sigma_{k-1}} y(t) \mid_{t=0} = 0$, $\qquad k = 1, ..., n.$

$$ {}_a\mathfrak{L}_t y(t) = {}_a D_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_k(t)\, {}_a D_t^{\sigma_{n-k}} y(t) + p_n(t) y(t), \tag{4.13}$$

where

${}_aD_t^{\sigma_k} =_a D_t^{\alpha_k} \ {}_aD_t^{\alpha_{k-1}} ... {}_aD_t^{\alpha_1}; \quad {}_aD_t^{\sigma_{k-1}} =_a D_t^{\alpha_{k-1}} \ {}_aD_t^{\alpha_{k-2}} ... {}_aD_t^{\alpha_1};$
$\sigma_k = \sum_{j=1}^{k} \alpha_j, \quad (k = 1, 2, .., n); \ \ 0 \leq \alpha_j \leq 1, \quad (j = 1, 2, ...,).$

4.2.1. Fractional Green's function for the one-term FDE

Fractional Green's function $G_1(t)$ for the one-term fractional-order differential equation with constant coefficients

$$a \ {}_0D_t^{\alpha} y(t) = f(t), \tag{4.14}$$

where the derivative can be either classic or sequential is found by the inverse Laplace transform of the following expression:

$$g_1(s) = \frac{1}{as^{\alpha}}. \tag{4.15}$$

The inverse Laplace transform then gives

$$G_1(t) = \frac{1}{a} \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{4.16}$$

The solution of equation (4.14) under homogeneous initial condition is

$$y(t) = \frac{1}{a\Gamma(\alpha)} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}} = \frac{1}{a} \ {}_0D_t^{-\alpha} f(t) \ , \qquad f(x) \text{ continuous in } \ [0, \infty) \tag{4.17}$$

4.2.2. Fractional Green's function for the two-term FDE

Fractional Green's function $G_2(t)$for the two-term fractional-order differential equation with constant coefficients

$$a {}_0D_t^{\alpha} y(t) + by(t) = f(t), \tag{4.18}$$

where the derivative can be either classic or sequential is found by inverse Laplace transform of the following expression :

$$g_2(s) = \frac{1}{as^\alpha + b} = \frac{1}{a}\frac{1}{s^\alpha + \frac{b}{a}}, \tag{4.19}$$

which leads to

$$G_1(t) = \frac{1}{a}t^{\alpha-1}E_{\alpha,\alpha}(-\frac{b}{a}t^\alpha). \tag{4.20}$$

4.2.3. Fractional Green's function for the three-term FDE

Fractional Green's function $G_3(t)$ for the 3-term fractional-order differential equation with constant coefficients

$$a_0 D_t^\beta y(t) + b_0 D_t^\alpha y(t) + cy(t) = f(t), \tag{4.21}$$

where the derivative can be either classic or sequential is found by inverse Laplace transform of the following expression :

$$g_3(s) = \frac{1}{as^\beta + bs^\alpha + c} \tag{4.22}$$

assuming $\beta > \alpha$, we can write $g_3(s)$ in the form

$$g_3(s) = \frac{1}{c}\frac{cs^{-\alpha}}{as^{\beta-\alpha} + b}\frac{1}{1 + \frac{cs^{-\alpha}}{as^{\beta-\alpha}+b}} = \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k \binom{c}{a}^{k+1} \frac{s^{-\alpha k-\alpha}}{(s^{\beta-\alpha} + \frac{b}{a})^{k+1}} \tag{4.23}$$

The term by term inversion ,

$$G_3(t) = \frac{1}{a}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{c}{a})^k t^{\beta(k+1)-1} E^{(k)}_{\beta-\alpha,\beta+\alpha k}(-\frac{b}{a}t^{\beta-\alpha}), \tag{4.24}$$

where $E_{\lambda,\mu}(z)$ is the Mittag-Leffler function in two parameters,

$$E_{\lambda,\mu}^{(k)}(y) = \frac{d^k}{dy^k} E_{\lambda,\kappa}(y) = \sum_{j=0}^{\infty} \frac{(j+k)!y^j}{j!\Gamma(\lambda j + \lambda k + \mu)}, \qquad (k = 0, 1, 2, ...) \tag{4.25}$$

Indeed, substituting (4.25) into (4.34) and changing the order of summation, we obtain:

$$\begin{aligned} G_3(t) &= \frac{1}{c} \sum_{k=0}^{\infty} (-1)^k \left(\frac{c}{a}\right)^{k+1} \sum_{j=0}^{\infty} (-1)^j \left(\frac{c}{a}\right)^j \frac{(j+k)! t^{\beta(j+k)+\beta-1-\alpha j}}{k! j! \Gamma(\beta(j+k+1) - \alpha j)} \\ &= \frac{1}{c} \sum_{j=0}^{\infty} \left(\frac{-b}{a}\right)^j \sum_{k=0}^{\infty} (-1)^k \left(\frac{c}{a}\right)^{k+1} \frac{(j+k)! t^{\beta(j+k)+\beta-1-\alpha j}}{k! j! \Gamma(\beta(j+k+1) - \alpha j)} \end{aligned} \tag{4.26}$$

4.2.4. Fractional Green's function for the general linear FDE

Fractional Green's function $G_n(t)$ for the n-term fractional-order differential equation with constant coefficients

$$a_n D^{\beta_n} y(t) + a_{n-1} D^{\beta_{n-1}} y(t) + ... + a_1 D^{\beta_1} y(t) + a_0 D^{\beta_0} y(t) = f(t), \tag{4.27}$$

where derivatives $D^{\alpha} = {}_0D_t^{\alpha}$ can be either classic or sequential is found by inverse Laplace transform of the following expression :

$$g_n(s) = \frac{1}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + ... + a_1 s^{\beta_1} + a_0 s^{\beta_0}} \tag{4.28}$$

Let us assume $\beta_n > \beta_{n-1} > ... > \beta_1 > \beta_0$ and write $g_n(s)$ in the form:

$$\begin{aligned} g_n(s) &= \frac{1}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}}} \frac{1}{1 + \frac{\sum_{k=0}^{n-2} a_k s^{\beta_k}}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}}}} \\ &= \frac{a_{-1} s^{-\beta_{n-1}}}{s^{\beta_n - \beta_{n-1}} + \frac{a_{n-1}}{a_n}} \frac{1}{1 + \frac{a_{-1} s^{-\beta_{n-1}} \sum_{k=0}^{n-2} a_k s^{\beta_k}}{s^{\beta_n - \beta_{n-1}} + \frac{a_{n-1}}{a_n}}} \end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m a_{-1} s^{-\beta_{n-1}}}{(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n})^{m+1}} \left(\sum_{k=0}^{n-2} (\frac{a_k}{a_n}) s^{\beta_k-\beta_{n-1}} \right)^m$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m a_{-1} s^{-\beta_{n-1}}}{(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n})^{m+1}} \qquad \text{(continued)}$$

$$\sum_{\substack{k_o+k_1+...+k_{n-2}=m \\ k_o\geq 0;...,k_{n-2}\geq 0}} (m; k_0, k_1, ..., k_{n-2}) \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n} \right)^{k_i} s^{(\beta_i-\beta_{n-1})k_i} \tag{4.29}$$

$$= \frac{1}{a_n} \sum_{m=0}^{\infty} (-1)^m \sum_{\substack{k_o+k_1+...+k_{n-2}=m \\ k_o\geq 0;...,k_{n-2}\geq 0}} (m; k_0, k_1, ..., k_{n-2}) \qquad \text{(continued)}$$

$$\prod_{i=0}^{n-2} \left(\frac{a_i}{a_n} \right)^{k_i} \frac{s^{-\beta_{n-1}+\sum_{i=0}^{n-2}\left(\beta_i-\beta_{n-1}\right)k_i}}{(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n})^{m+1}}$$

where $((m; k_0, k_1, ..., k_{n-2}))$ are the multinomial coefficients.

Term by term inversion, gives the final expression for the fractional Green's function for equation (4.27):

$$G_n(t) = \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left[\sum_{\substack{k_o+k_1+...+k_{n-2}=m \\ k_o\geq 0;...,k_{n-2}\geq 0}} (m; k_0, k_1, ..., k_{n-2}) \right. \qquad \text{(continued)}$$

$$\prod_{i=0}^{n-2} \left(\frac{a_i}{a_n} \right)^{k_i} t^{(\beta_n-\beta_{n-1})m+\beta_n+\sum_{j=0}^{n-2}\left(\beta_{n-1}-\beta_j\right)k_{j-1}} \qquad \text{(continued)}$$

$$\left. E^{(m)}_{\beta_n-\beta_{n-1},+\beta_n+\sum_{j=0}^{n-2}\left(\beta_{n-1}-\beta_j\right)k_j} \left(-\frac{a_{n-1}}{a_n} t^{\beta_n-\beta_{n-1}} \right) \right] \tag{4.30}$$

4.3. Solution of the General Bagley-Torvik Equation with Green's Function Method

Example 3.1

We will consider the Bagley-Torvik equation as an example of three-term fractional differential equation with constant coefficients

$$\begin{cases} AD^2x(t) + BD^{3/2}x(t) + Cx(t) = f(t), \\ x(0) = 0, x'(0) = 0, \end{cases} \tag{4.31}$$

where $A \neq 0, B, C \in R$.

Solution 3.1

First, taking Laplace transform of both sides, we obtain

$$g(s) = \frac{1}{As^2 + Bs^{3/2} + C}. \tag{4.32}$$

Then, $g(s)$ can be written in the form

$$g(s) = \frac{1}{C}\frac{Cs^{-3/2}}{As^{1/2}+B}\frac{1}{1+\frac{As^{-3/2}}{As^{1/2}+B}} = \frac{1}{C}\sum_{k=0}^{\infty}(-1)^k \binom{C}{A}^{k+1} \frac{s^{-3/2k-3/2}}{(s^{1/2}+\frac{B}{A})^{k+1}} \tag{4.33}$$

and term by term inversion

$$G(t) = \frac{1}{A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{C}{A}\right)^k t^{2(k+1)-1}E^{(k)}_{1/2,2+3/2k}(-\frac{B}{A}t^{1/2}) \tag{4.34}$$

where $E_{\lambda,\mu}(z)$ is the Mittag-Leffler function in two parameters

$$E^{(k)}_{\lambda,\mu}(y) = \frac{d^k}{dy^k}E_{\lambda,\kappa}(y) = \sum_{j=0}^{\infty}\frac{(j+k)!y^j}{j!\Gamma(\lambda j+\lambda k+\mu)}, \qquad (k=0,1,2,...). \tag{4.35}$$

Substituting (4.25) into (4.34) and changing the order of summation, we obtain

$$\begin{aligned} G(t) &= \frac{1}{C}\sum_{k=0}^{\infty}(-1)^k\left(\frac{C}{A}\right)^{k+1}\sum_{j=0}^{\infty}(-1)^j\left(\frac{C}{A}\right)^j\frac{(j+k)!t^{2(j+k)+1-3/2j}}{k!j!\Gamma(2(j+k+1)-3/2j)} \\ &= \frac{1}{C}\sum_{j=0}^{\infty}\left(\frac{-B}{A}\right)^j\sum_{k=0}^{\infty}(-1)^k\left(\frac{C}{A}\right)^{k+1}\frac{(j+k)!t^{2(j+k)+1-3/2j}}{k!j!\Gamma(2(j+k+1)-3/2j)}. \end{aligned} \tag{4.36}$$

Then, the generalized green's function for the equation (4.40) is:

$$G(t)=\frac{1}{A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{C}{A}\right)^r t^{2r+1}E^{(r)}_{\frac{1}{2},\frac{3}{2}r+2}\left(-\frac{B}{A}t^{\frac{1}{2}}\right). \tag{4.37}$$

Therefore, the solution of the problem under homogeneous initial conditions will be:

$$x\,(t)=\int_0^t G(t-\tau)f(\tau)d\tau. \tag{4.38}$$

In this problem, we obtain the general solution of the equation for all f, A, B, C above. Now, let us take examples with specific constants and the right side functions *named* first and second examples.

First Example

$$\begin{cases} D^2x(t)+\frac{1}{2}D^{3/2}x(t)+\frac{1}{2}x(t)=\left\{\begin{smallmatrix}8, & (0\le t\le 1) \\ 0, & (t>1),\end{smallmatrix}\right., \\ x(0)=0,\ x'(0)=0, \end{cases} \tag{4.39}$$

Second Example

$$\begin{cases} D^2x(t)+\frac{1}{2}D^{3/2}x(t)+\frac{1}{2}x(t)=f(t) \\ x(0)=0, \quad x'(0)=0, \end{cases} \tag{4.40}$$

where $f(t)=0,05t^4-0,03t^3+0,361t^{5/2}+0,145t^2-0,135t^{3/2}-0,36t+0,056t^{1/2}+0,1$

Here, we will write an algorithm to solve the Bagley-Torvik equation for the first and second examples. Moreover, with changing constants and the right side function more problems can be solved by using same algorithm.

4.4. Algorithm for the Green's Function Solution of the Bagley-Torvik Equation

1. Put A, B, C, h(step size), L(length of time interval), tk, Z, N and K (end points of the interval) as input values.
2. Give the Green's function $G(t)$ with two summation formulas from 0 to N.
3. Define $f(t)$ which is the right side function.
4. Integrate $G(t-s)f(s)$ with respect to s from 0 to t.
5. Set $tk = hr$ where r is from 0 to Z.
6. Evaluate solution for all tk.
7. Plot the graph from 0 to L.

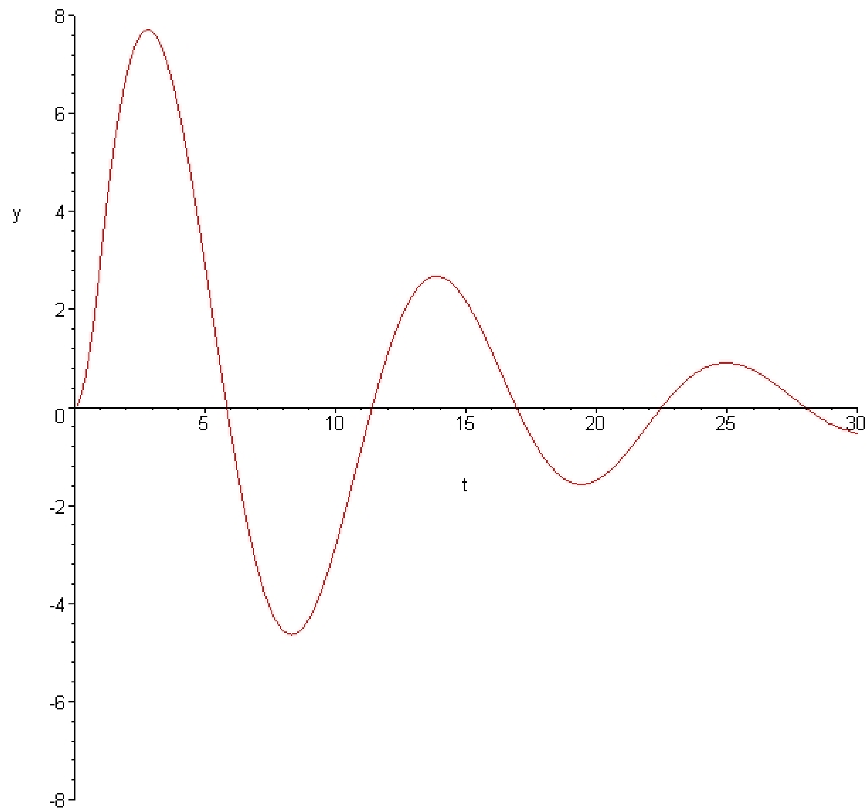

Figure 4.1 . Green's Function solution of the Bagley Torvik Equation for the first example.

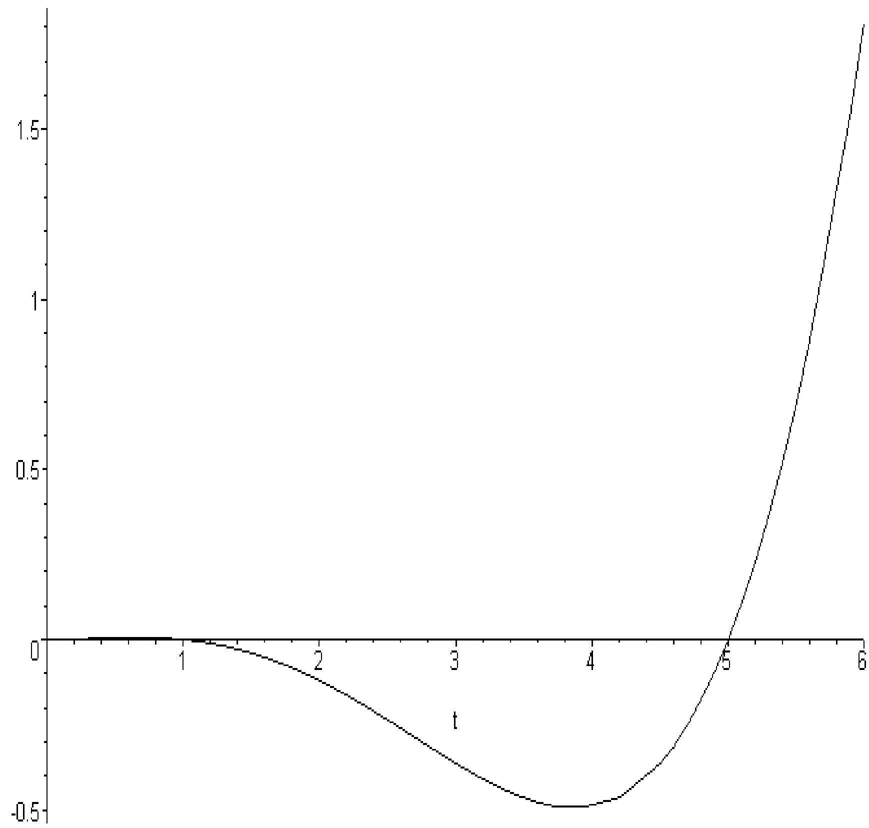

Figure 4.2 . Green's Function solution of the Bagley Torvik Equation for the second example.

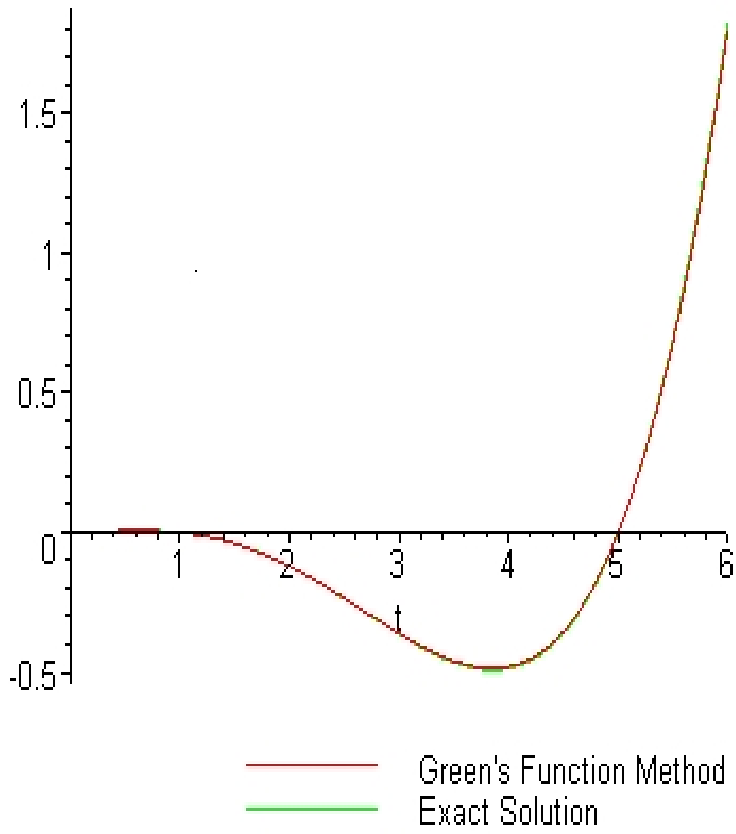

Figure 4.3 . Green's Function and Exact solution of the Bagley Torvik Equation for the first example.

CHAPTER 5

POWER SERIES METHOD

The power series method is a finite approach based on power-series expansion. The idea of this method is to look for the solution in the form of a power series, we need the power series expansions for all given functions in the fractional-order equation.

5.1. The Method

By definition, a power series is of the form

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + ...or \tag{5.1}$$

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + a_1(x-a) + C_2(x-a)^2 + ..., \tag{5.2}$$

where the coefficients C_n are constants. For each value of x (in the interval of convergence) the series has a finite sum whose value depends on the value of x. In the differential equations expanding the functions with the power series gives us a new equation including differential terms. But, if a power series converges for a particular range of x then the series obtained by differentiating every term and the series obtained by integrating every term also converge in this range. Then collecting all the terms with the same order of x and evaluating coefficients give us the solution of the differential equation of integer order.

The operation with the integer order differential equations is not very different if the equation is the fractional order differential equation. We can do the same operations with some changes, because now we have fractional order differential operators and our expansion should contain not only the integer order powers of x.

Consider the one-term equation

$$ {}_0D_t^{\alpha} x(t) = f(t) \tag{5.3} $$

and look for fractional power series form of $x(t)$ which is the solution of the equation (5.3):

$$ x(t) = \sum_{n=0}^{\infty} C_n t^{n\alpha} \tag{5.4} $$

and assume

$$ f(t) = \sum_{j=0}^{\infty} q_j t^{j\alpha}. $$

Substituting (5.4) into (5.3) we get:

$$ \sum_{n=0}^{\infty} C_n \frac{\Gamma(1+n\alpha)}{\Gamma(1+n\alpha-\alpha)} t^{n\alpha-\alpha} = f(t). \tag{5.5} $$

By the evaluation of every term of the $t's$ which have the same order with respect to the right hand side function we find a_n's. The procedure is also same for $n - term$ equations.

5.2. Solution of the General Bagley-Torvik Equation with Power Series Method

Example 4.1

We will consider the Bagley-Torvik equation as an example of three-term fractional differential equation with constant coefficients

$$\begin{cases} AD^2x(t) + BD^{3/2}x(t) + Cx(t) = f(t), \\ x(0) = 0, x'(0) = 0, \end{cases} \tag{5.6}$$

(where $A \neq 0$,$B, C \in R$)

Solution 4.1

Now, let us look for the solution of the Bagley-Torvik equation in the form of the fractional power series.

First, with the operations that we defined above, we define $x\,(t)$ in the form of the following power series:

$$x = \sum_{n=0}^{\infty} C_n t^{\frac{n}{2}} \tag{5.7}$$

$$x\,(t) = C_0 + C_1 t^{\frac{1}{2}} + C_2 t + ... + C_n t^{\frac{n}{2}} + ... \tag{5.8}$$

Let us take the first and second derivative

$$D_t^1 x(t) = \frac{1}{2}C_1 + C_2 + \frac{3}{2}C_3 t^{1/2} + ... + \frac{n}{2}C_n t^{\frac{n}{2}-1} + ... \tag{5.9}$$

$$D_t^2 x\,(t) = -\frac{1}{4}C_1 t^{-\frac{3}{2}} + \frac{3}{4}C_3 t^{-\frac{1}{2}} + C_4 + ... + \frac{n}{2}\frac{n-2}{2}C_n t^{\frac{n}{2}-2} + ... \tag{5.10}$$

Then, for the derivative of the fractional order of the equation, recall the rule of the Riemann-Liouville fractional differentiation for power functions:

$${}_oD_t^{\alpha} t^v = \frac{\Gamma\,(1+v)}{\Gamma\,(1+v-a)} t^{v-a}. \tag{5.11}$$

Substituting $\alpha = \frac{3}{2}$ into the equation (5.11) we get the $\frac{3}{2}th$ derivative of the (5.8):

$$ {}_oD_t^{3/2} t^{j/2} = \frac{\Gamma\left(1 + j/2\right)}{\Gamma\left(j/2 - 1/2\right)} t^{j/2-3/2}. \tag{5.12}$$

Collecting all the expressions (5.8), (5.10) and (5.12), we derive the power series expansion of the Bagley-Torvik equation

$$ A\sum_{j=0}^{\infty} C_j \frac{j}{2}\frac{j-2}{2} t^{j/2-2} + B\sum_{j=0}^{\infty} C_j \frac{\Gamma\left(j/2+1\right)}{\Gamma\left(j/2-1/2\right)} t^{j/2-3/2} + C\sum_{j=0}^{\infty} C_j t^{j/2} = f(t). \tag{5.13}$$

By the assumption, the series does not contain negative powers of t, we get

$$ C_0 = C_1 = C_2 = C_3 = 0 \tag{5.14}$$

and

$$ A(2C_4 + \frac{15}{4}C_5 t^{1/2} + 6C_6 t ... + \frac{n}{2}\frac{n-2}{2} C_n t^{\frac{n}{2}-2} + ...)+ \tag{5.15}$$

$$ + B(\frac{4}{\sqrt{\pi}} C_4 t^{1/2} + \frac{15}{8} C_5 \sqrt{\pi} t + \frac{8}{\sqrt{\pi}} C_6 t^{3/2} + ... + \frac{\Gamma(1+n/2)}{\Gamma(n/2-1/2)} t^{\frac{n}{2}-\frac{3}{2}}) \tag{5.16}$$

$$ + C(C_4 t^2 + C_5 t^{5/2} + C_6 t^3 + ... + C_n t^{\frac{n}{2}} + ...) \tag{5.17}$$

$$ = f(t). \tag{5.18}$$

Rearranging the last equation with respect to the same powers of $t^{1/2}$, we obtain

$$ (A2C_4)t^0 + (B\frac{4}{\sqrt{\pi}}C_4 + A\frac{15}{4}C_5)t^{1/2} + (B\frac{15}{8}C_5\sqrt{\pi} + A6C_6)t+ \tag{5.19}$$

$$ + (B\frac{8}{\sqrt{\pi}}C_6 + A\frac{35}{4}C_7)t^{3/2} + (CC_4 + B\frac{105}{32}\sqrt{\pi}C_7 + A12C_8)t^2+ \tag{5.20}$$

$$ + (CC_5 + B\frac{64}{5\sqrt{\pi}}C_8 + \frac{63}{5}C_9)t^{5/2} + ... + (CC_{n/2})t^{\frac{n}{2}} + ... \tag{5.21}$$

$$ = \sum_{j=0}^{\infty} q_j t^{j\alpha}. \tag{5.22}$$

By equating, the corresponding terms at the last equation, we obtain a recurrence formula to compute the coefficients in the solution series $x(t)$.

Here, we will write an algorithm to solve the Bagley-Torvik equation for the first and second examples. Moreover, with changing constants and the right side function more problems can be solved by using the same algorithm.

5.3. Algorithm for the Power Series Solution of the Bagley-Torvik Equation

1. Put A, B, C, h(stepsize), L(length of time interval), M as input values.
2. Define $f(t)$ which is the right side function.
3. Evaluate the coefficients of $f(t)$.
4. Give some initial values of $C[j]$.
5. Evaluate $C[j]s$ from the recurrence formula
   c[j+4]:=(q[j]-B*GAMMA((j+3)/2+1.) /
   GAMMA((j+3)/2.-1/2.)*c[j+3])
   /(A*(j+4)/2.*(j+2)/2)
   from 0 to 3.
6. Evaluate other $C[j]s$ from the recurrence formula
   c[j+4]:=(q[j]-B*GAMMA((j+3)/2.+1)/
   GAMMA((j+3)/2.-1/2)*c[j+3] -
   C*c[j]) /(A*(j+4)/2.*(j+2)/2)
   from 4 to $M - 4$.
7. Set R is the solution of the problem as c[k]*(t^(k/2) from 0 to M.
8. Plot the graph from 0 to L.

If the right side function is heaviside function:

1. Put A, B, C, h(stepsize), L(length of time interval), M as input values.
2. Give $f(t) = ut1$.
3. Set all the coefficients of $f(t)$ are 0.
4. Give the constant coefficient from the right side function.

5. Evaluate $C[j]s$ from the recurrence formula for (t)

   c[j+4]:=(q[j]-B*GAMMA((j+3)/2+1.) /

   GAMMA((j+3)/2.-1/2.)*c[j+3]) /

   (A*(j+4)/2.*(j+2)/2)

   from 0 to 3.

6. Evaluate other $C[j]s$ from the recurrence formula for (t)

   c[j+4]:=(q[j]-B*GAMMA((j+3)/2.+1)/GAMMA((j+3)/2.-1/2)*c[j+3] -C*c[j])

   /(A*(j+4)/2.*(j+2)/2)

   from 4 to $M-4$.

7. Give $f(t-1) = ut2$.

8. Evaluate $C[j]s$ from the recurrence formula for $(t-1)$

   d[j+4]:=(q[j]-B*GAMMA((j+3)/2+1.)

   / GAMMA((j+3)/2.-1/2.)*c[j+3]

)/(A*(j+4)/2.*(j+2)/2)

   from 0 to 3.

9. Evaluate other $C[j]s$ from the recurrence formula for $(t-1)$

   d[j+4]:=(q[j]-B*GAMMA((j+3)/2.+1)

   /GAMMA((j+3)/2.-1/2)*c[j+3] -

   C*c[j]) /(A*(j+4)/2.*(j+2)/2)

   from 4 to $M-4$.

10. Set $R1$ (c[k]*(t^(k/2)) from 0 to M.

11. Set $R2$ (d[k]*((t-1)^(k/2)) from 0 to M.

12. Set $R = R1 - R2$.

13. Give $ut1$ and $ut2$ as piecewise functions.

14. Plot the graph from 0 to L.

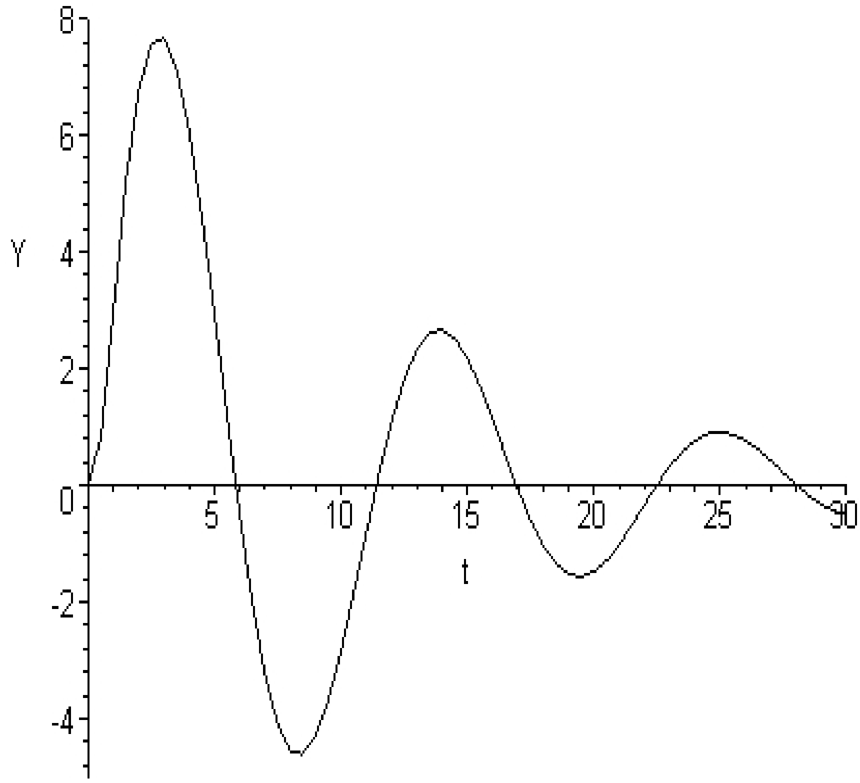

Figure 5.1 . Power Series solution of the Bagley-Torvik Equation for the first example.

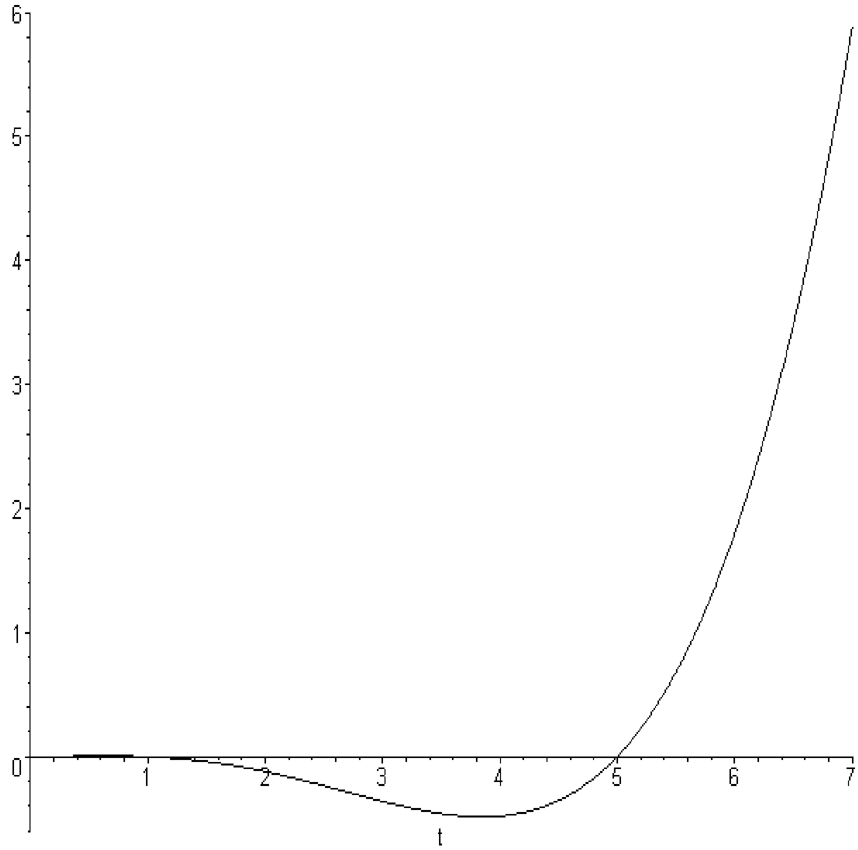

Figure 5.2 . Power Series solution of the Bagley-Torvik Equation for the second example

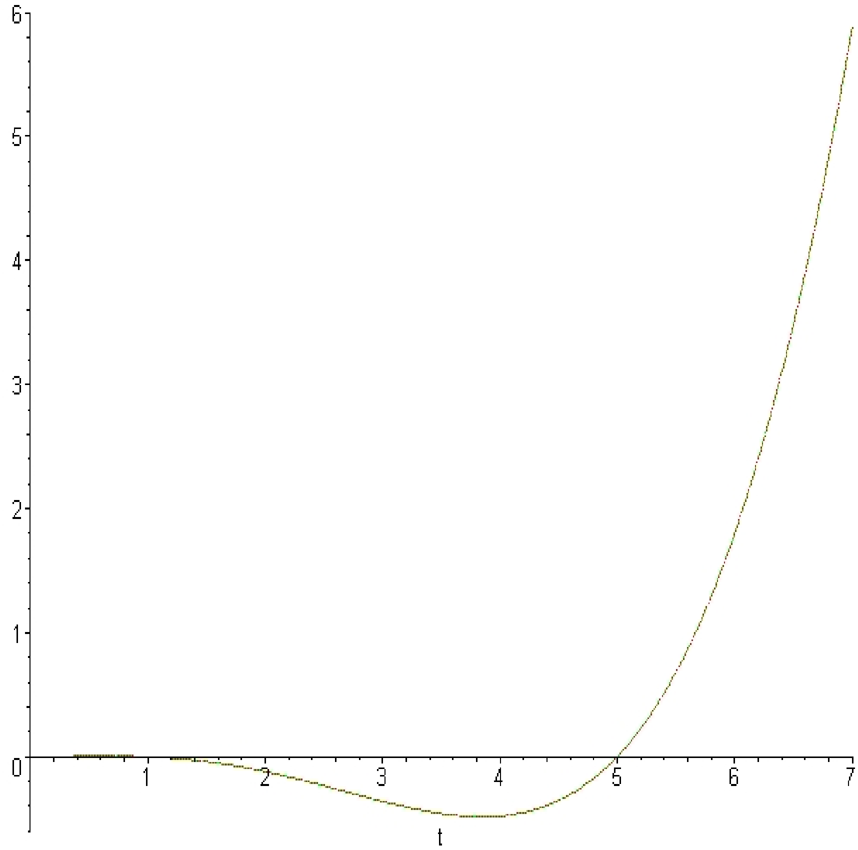

Figure 5.3 . Exact and Power Series solution of the Bagley-Torvik Equation for the second example.

CHAPTER 6

ADOMIAN DECOMPOSITION METHOD

The Adomian decomposition method is a method that gives us a series solution which was first defined by G. Adomian at the beginning of 1880s as a new iterative method, then was called ADM and now it has been used for solving a wide range of problems. G. Adomian explained this method in his book [7]. ADM is very easy to apply, has some advantages over standard numerical methods, eliminates cumbersome computational works, but we can face some difficulties when we choose the conditions because of the convergence of the method. Although it is shown that [24, 25] the method is rapidly convergent with some restrictions, we can see the improvements of the method to overcome these restrictions[40]. Recently, a new convergence proof based on properties of convergent series can be seen in the Cherruault and Adomian's paper [39]. Then, a reliable modification was done by Wazwaz [8] to accelerate the rapid convergence of the series solution that includes only two terms could be evaluated easily. He did it with removing the noise terms from first term of the series solution and had a solution in a closed form. Finally, with the fractional differential equations we can also use this method efficiently to have a series solution [3],[4]. ADM is a power-full tool for both linear and non-linear fractional differential equations especially in fractional dynamic models and their solutions.

6.1. The Method

First, we begin with an equation in the form of:

$$\begin{aligned} Lx + Rx + Nx &= g \\ Lx &= g - Rx - Nx \end{aligned} \tag{6.1}$$

where $Lx + Rx$ represents the linear term and L is the highest-ordered linear differential operator, R is the remainder of the linear operator. And, Nu represents the non-linear term. Applying L^{-1}, we obtain

$$L^{-1}Lx = L^{-1}g - L^{-1}Rx - L^{-1}Nx. \tag{6.2}$$

For initial-value problems, we conveniently define L^{-1} for $L = d^n/dt^n$ as the n-fold definite integration operator from 0 to t. For the operator $L = d^2/dt^2$, for example, we have

$$L^{-1}Lx = x - x(0) - tx'(0) \tag{6.3}$$

and therefore

$$x = x\,(0) + tx'\,(0) + L^{-1}g - L^{-1}Rx - L^{-1}Nx. \tag{6.4}$$

For the same operator equation but now considering a boundary value problem, we let L^{-1} be an indefinite integral and write $u = A + Bt$ for the first two terms and evaluate A, B from the given conditions. The first three terms are identified as x_0 in the assumed decomposition

$$x = \sum_{n=0}^{\infty} x_n. \tag{6.5}$$

Finally, assuming Nx is analytical, we write

$$Nx = \sum_{n=0} A_n \left(x_0, x_1, ..., x_n\right) \tag{6.6}$$

where the A_n are specially generated (Adomian) polynomials for the specific nonlinearity. They depend only on the x_0 to x_n components and form a rapidly convergent series. The A_n are given as:

$$\begin{aligned}
&A_0 = f\left(x_0\right) \\
&A_1 = x_1 \left(d/dx_0\right) f\left(x_0\right) \\
&A_2 = x_2 \left(d/dx_0\right) f\left(x_0\right) + \left(x_1^2/2!\right)\left(d^2/dx_0^2\right) f\left(x_0\right) \\
&A_3 = x_3 \left(d/dx_0\right) f\left(x_0\right) + x_1 x_2 \left(d^2/dx_0^2\right) f\left(x_0\right) + \left(x_1^3/3!\right)\left(d^3/dx_0^3\right) f\left(x_0\right) \\
&\quad ...
\end{aligned} \tag{6.7}$$

Also, we can see that the sum of the series $\sum_{n=0}^{\infty} A_n$ for Nx is equal to the sum of a generalized Taylor series about $x_0(t)$, $\sum_{n=0}^{\infty} x_n$ is equal to a generalized Taylor series about the function x_0.

Let's write the sum of Adomian polynomials as:

$$\begin{aligned}
f(x) &= A_0 + A_1 + A_2 + A_3 + ... \\
&= f(x_0) + x_1 f^{(1)}(x_0) + \frac{x_1^2}{2!} f^{(2)}(x_0) + \\
&+ x_2 f^{(1)}(x_0) + x_3 f^{(1)}(x_0) + x_1 x_2 f^{(2)}(x_0) + \frac{x_1^3}{3!} f^{(3)}(x_0) + ...
\end{aligned} \tag{6.8}$$

Then, let's write the generalized Taylor series form of $f(x)$ about x_0 :

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + f^{(2)}(x_0)\frac{(x - x_0)^2}{2!} + f^{(3)}(x_0)\frac{(x - x_0)^3}{3!} + ... \tag{6.9}$$

Here, the function is expanded about x_0 , then $(x - x_0) = (x_1 + x_2 + x_3 + x_4 + ...)$.

So $f(x)$ can be written as:

$$
\begin{aligned}
f(x) = & f(x_0) + f^{(1)}(x_0)(x_1 + x_2 + x_3 + x_4 + ...)+ \\
& + f^{(2)}(x_0)\frac{(x_1 + x_2 + x_3 + x_4 + ...)^2}{2!}+ \\
& f^{(3)}(x_0)\frac{(x_1 + x_2 + x_3 + x_4 + ...)^3}{3!} + ... \\
& = f(u_0) + f^{(1)}(x_0)(x_1 + x_2 + x_3 + x_4 + ...)+ \\
& + f^{(2)}(x_0)\frac{(x_1^2 + x_2^2 + ... + 2x_1x_2 + 2x_1x_3 + ...2x_2x_3 + 2x_2x_4 + ...)}{2!}+ \\
& + f^{(3)}(x_0)\frac{(x_1^3 + x_2^3 + ... + 3x_1^2x_2 + 3x_1^2x_3 + ... + 3x_2^2x_1 + 3x_2^2x_3 + ...)}{3!}+ \\
& ...
\end{aligned}
\tag{6.10}
$$

Then, the Adomian polynomials can be rearranged from the Taylor series:

$$
\begin{aligned}
A_0 &= f\left(x_0\right) \\
A_1 &= f^{(1)}\left(x_0\right)x_1 \\
A_2 &= f^{(1)}\left(x_0\right)x_2 + f^{(2)}\left(u_0\right)\frac{x_1^2}{2!} \\
A_3 &= f^{(1)}\left(x_0\right)x_3 + f^{(2)}\left(x_0\right)\frac{2x_1x_2}{2!} + f^{(3)}(x_0)\frac{x_1^3}{3!} \\
& ...
\end{aligned}
\tag{6.11}
$$

From that point, we can see that the form of Adomian polynomials is a rearranged form of the Taylor series. However, the complete method differs from the Taylor series method and has some advantages such as producing reliable results with few iterations, minimizing the computational difficulties, overcoming the deficiency of linearization of non-linear problems.

Let us apply the ADM to the general Bagley-Torvik equation

$$
\begin{cases}
AD_t^2x\left(t\right) + BD_t^{3/2}x\left(t\right) + Cg\left(x(t)\right) = f\left(t\right), \\
x\left(0\right) = 0, \qquad \text{and} \qquad D_tx(t)\mid_{t=0} = 0.
\end{cases}
\tag{6.12}
$$

where $A \neq 0$ and $B, C \in R$.

6.2. Solution of the general Bagley-Torvik equation with ADM

First, for linear equation $g(x) = x(t)$ we assume that $x(t) = x_0(t) + x_1(t) + x_2(t) + ...$ to be the solution of the equation (6.12)if we rearrange the equation

$$\frac{d^2x(t)}{dt^2} + \frac{B}{A}\frac{d^{\frac{3}{2}}x(t)}{dt^{\frac{3}{2}}} + \frac{C}{A}x(t) = \frac{f(t)}{A}. \tag{6.13}$$

It is seen that L should be the highest-ordered linear differential operator as d^2/dt^2, $d^{3/2}/dt^{3/2}$ is the remainder part and through $g(x) = x(t)$, N represents the linear part which is

$$Nx = z(x) = \sum_{n=0}^{\infty} A_n(x_0, x_1, ..., x_n) = \frac{C}{A}x. \tag{6.14}$$

So, the Adomian polynomials can be calculated as:

$$\begin{aligned}
A_0 &= z(x_0) = \frac{C}{A}x_{0,} & (6.15)\\
A_1 &= x_1 z^{(1)}(x_0) = \frac{C}{A}x_{1,} \\
A_2 &= x_2 z^{(1)}(x_0) + (x_1^2/2!)z^{(2)}(x_0) = \frac{C}{A}x_{2,} \\
A_3 &= x_3 z^{(1)}(x_0) + x_1 x_2 z^{(2)}(x_0) + (x_1^3/3!)z^{(3)}(x_0) = \frac{C}{A}x_{3,} \\
&\ldots & (6.16)
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
x(t) &= x(0) + tD_t x(t)\mid_{t=0} + \frac{1}{A}L^{-1}f(t) \\
&- L^{-1}\left(\frac{B}{A}D_t^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} x_n(t)\right)\right) - L^{-1}\sum_{n=0}^{\infty} A_n \qquad (6.17)\\
&= \frac{1}{A}L^{-1}f(t) - \frac{B}{A}L^{-1}\left(D_t^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} x_n(t)\right)\right) - L^{-1}\left(\sum_{n=0}^{\infty}\frac{C}{A}x_n(t)\right).
\end{aligned}$$

This implies that

$$x\left(t\right)=\frac{1}{A}\frac{d^{-2}}{dt^{-2}}f(t)-\frac{B}{A}\frac{d^{-\frac{1}{2}}}{dt^{-\frac{1}{2}}}\left(\sum_{n=0}^{\infty}x_n(t)\right)-\frac{C}{A}\frac{d^{-2}}{dt^{-2}}\left(\sum_{n=0}^{\infty}x_n(t)\right), \tag{6.18}$$

where

$$x_0(t)=\frac{1}{A}\frac{d^{-2}}{dt^{-2}}f\left(t\right), \tag{6.19}$$

$$x_1\left(t\right)=-\frac{B}{A}\frac{d^{-\frac{1}{2}}x_0\left(t\right)}{dt^{-\frac{1}{2}}}-\frac{C}{A}\frac{d^{-2}x_0(t)}{dt^{-2}}, \tag{6.20}$$

$$x_2(t)=-\frac{B}{A}\frac{d^{-\frac{1}{2}}x_1\left(t\right)}{dt^{-\frac{1}{2}}}-\frac{C}{A}\frac{d^{-2}x_1(t)}{dt^{-2}}, \tag{6.21}$$

$$x_3(t)=-\frac{B}{A}\frac{d^{-\frac{1}{2}}x_2\left(t\right)}{dt^{-\frac{1}{2}}}-\frac{C}{A}\frac{d^{-2}x_2(t)}{dt^{-2}} \tag{6.22}$$

and so on. Therefore, the general solution is:

$$\begin{aligned} x(t)=&\frac{1}{A}\frac{d^{-2}}{dt^{-2}}f\left(t\right)-\frac{B}{A}\left[\frac{d^{-\frac{1}{2}}x_0\left(t\right)}{dt^{-\frac{1}{2}}}+\frac{d^{-\frac{1}{2}}x_1\left(t\right)}{dt^{-\frac{1}{2}}}+\frac{d^{-\frac{1}{2}}x_2\left(t\right)}{dt^{-\frac{1}{2}}}+...\right]\\ &-\frac{C}{A}\left[\frac{d^{-2}x_0(t)}{dt^{-2}}+\frac{d^{-2}x_1(t)}{dt^{-2}}+\frac{d^{-2}x_2(t)}{dt^{-2}}+...\right]. \end{aligned} \tag{6.23}$$

Here, we give an algorithm to solve the Bagley-Torvik equation for the first and second examples. Moreover, with changing constants and the right side function more problems can be solved by using the same algorithm.

6.3. Algorithm for the ADM Solution of the Bagley-Torvik Equation

1. Put A, B, C, h(stepsize), b(length of time interval), P as input values.
2. Define $f(t)$ which is the right side function.
3. Evaluate x_0 from the right side function.
4. Evaluate the first part of the solution from the formula:
   $(1/GAMMA(1/2))*(-B/A)*int(x0/((-x+t)\hat{}(1/2)),x=0..t)$.
5. Evaluate the second part of the solution from the formula:
   $(1/GAMMA(2))*(-C/A)*int(x0*(-x+t),x=0..t)$.

6. Collect the part of the solutions.
7. Evaluate solution for all t from 0 to P.
8. Plot the graph from 0 to b.

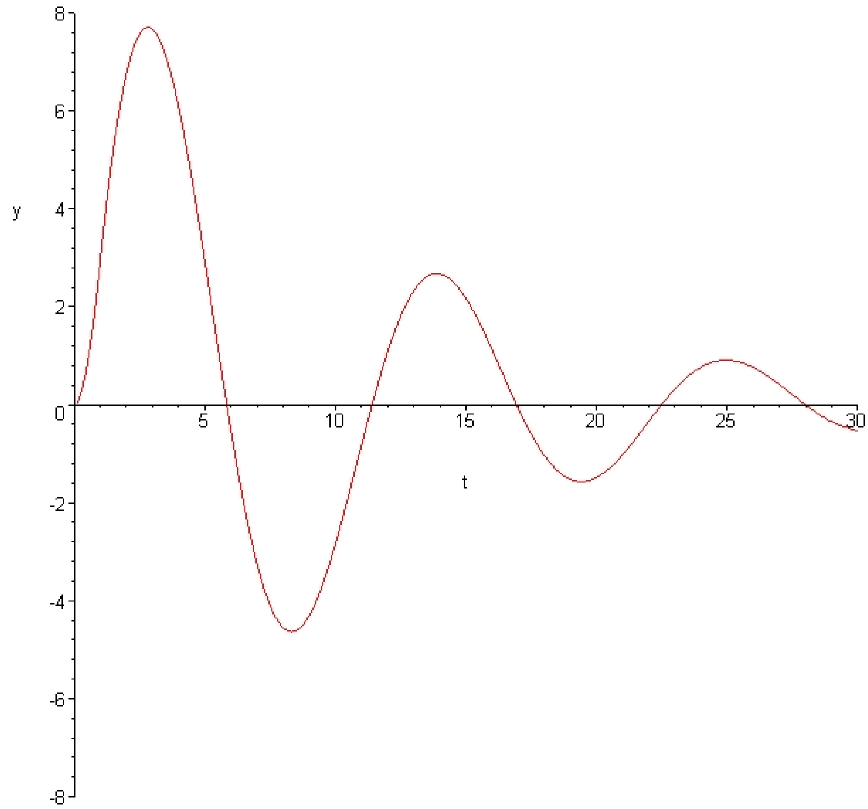

Figure 6.1 . Solution of the Bagley-Torvik equation for the first example with ADM.

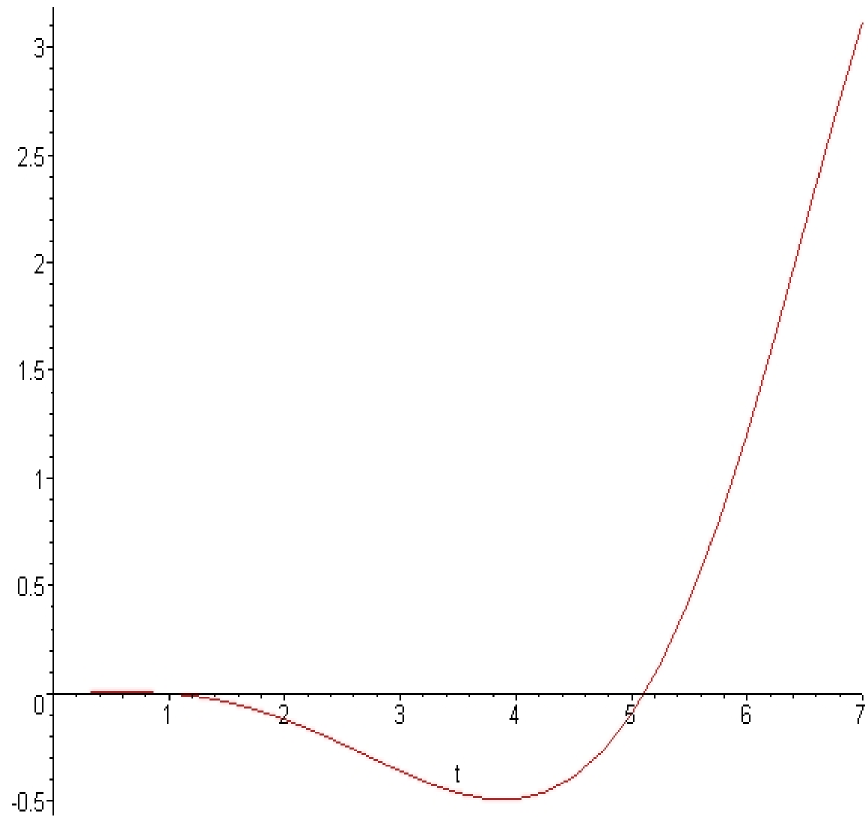

Figure 6.2 . Solution of the Bagley-Torvik equation for the second example with ADM.

CHAPTER 7

FINITE DIFFERENCE METHOD

FDM is a numerical method which is widely implemented on computers. It can be adopted to differential equations with non-integer order as easily as integer ones. The basic idea behind the finite difference method is to convert the differential equation into a system of algebraic equations by replacing each derivative with a finite difference.

7.1. The Method

The derivatives in the differential equation are substituted by finite divided differences approximations, for example the derivative of a function $f(x)$ at the point x_0 could be defined in any of the following three ways:

$$\frac{dx}{dt}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}, \tag{7.1}$$

$$\frac{dx}{dt}(t) = \lim_{h \to 0} \frac{x(t) - x(t-h)}{h}, \tag{7.2}$$

$$\frac{dx}{dt}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t-h)}{2h}. \tag{7.3}$$

If the derivative of $x(t)$ is continuous at t, all three expressions produce the same unique answer. Equation (7.1) introduces the forward difference, equation (7.2) introduces the backward difference and the last equation (7.3) introduces the central difference formulas. It can be also found higher order derivatives with the same way as :

$$\begin{aligned}\frac{d^2x}{dt^2}(t) &= \frac{d}{dt}(\frac{dx}{dt}(t)) = \frac{d}{dt}(\lim_{h\to 0}\frac{x(t)-x(t-h)}{h}) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{x(t)-x(t-h)}{h} - \frac{x(t-h)-x(t-h-h)}{h}\right) \\ &= \lim_{h\to 0}\frac{x(t)-2x(t-h)+x(t-2h)}{h^2}.\end{aligned} \tag{7.4}$$

Integer order finite differences are extensively known and used widely. To use finite difference method for fractional order derivatives first we should motivate to fractional difference. Like as integer order differences, fractional differencing has three definitions: forward, backward and central differences. Fractional forward difference is first defined by Osler (1984) [41] and Hosking (1981) [43] defined backward and lastly in the [44]is defined central difference with respect to the first two definitions. In our problems, we will use backward difference. If we omit the limit term from the definition of backward difference of first-order we get:

$$x'(t) = \Delta = \frac{x(t)-x(t-h)}{h} \tag{7.5}$$

replacing the Taylor series expansion form of $x(t-h)$

$$\begin{aligned}\Delta =& \frac{x(t)-x(t-h)}{h} = \frac{x'(t)h - \frac{x''(t)}{2}h^2 + ...}{h} \\ =& x'(t) - \frac{x''(t)}{2}h + ... = x'(t) + O(h),\end{aligned} \tag{7.6}$$

which means that

$$x'(t) - \Delta = O(h) \tag{7.7}$$

Theorem 7.1.1. *[10]First, let us show that*

$${}_aD_t^\alpha x(t) \approx_a \Delta_h^\alpha x(t) \tag{7.8}$$

where

$$ {}_aD_t^{\alpha}x(t) \approx \lim_{h\to 0}\ {}_a\Delta_h^{\alpha}x(t) \qquad \textit{and} \qquad {}_a\Delta_h^{\alpha}x(t) = \sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j}x(t-jh) \tag{7.9} $$

($[x]$ means the integer part of x.)

(7.10)

gives the first-order approximation for the $\alpha - th$ derivative.

Proof. For simplicity, it is convenient to assume that $a = 0$, and that the discretization step h and the number of nodes n are related by $t = nh$, where t is the point at which the derivative is evaluated. In this case we write the approximation of the $\alpha - th$ derivative as

$$ {}_0\Delta_h^{\alpha}x(t) = h^{-\alpha}\sum_{j=0}^{n}(-1)^j\binom{\alpha}{j}x(t-jh) \tag{7.11} $$

$$ = h^{-\alpha}\sum_{j=0}^{n}\binom{j-\alpha-1}{j}x(t-jh) \tag{7.12} $$

If we take $x_0(t) = 1$ $(t \geq 0)$, its exact $\alpha - th$ derivative is

$$ {}_0D_t^{\alpha}x_0(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{7.13} $$

On the other hand, the approximation (7.11)gives the approximate value

$$ {}_0\Delta_h^{\alpha}x_0(t) = h^{-\alpha}\sum_{j=0}^{n}\binom{j-\alpha-1}{j}. \tag{7.14} $$

Using the summation formula for the binomial coefficients

$$ \sum_{j=0}^{n}\binom{j-\alpha-1}{j} = \binom{n-\alpha}{n}, \tag{7.15} $$

and the asymptotic formula

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}=1+O\left(z^{-1}\right) \tag{7.16}$$

we have for fixed t

$$\begin{aligned}
{}_0\Delta_h^{\alpha}x_0(t) &= h^{-\alpha}\binom{n-\alpha}{n} && (7.17)\\
&= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\frac{n^{\alpha}\Gamma(n-\alpha+1)}{\Gamma(n+1)} && (7.18)\\
&= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\left(1+O(h)\right), && (7.19)
\end{aligned}$$

therefore for $x_0\ (t)=1\ (t\geq 0)$

$${}_0D_t^{\alpha}x_0(t)-_0\Delta_h^{\alpha}x_0(t)=O\left(h\right). \tag{7.20}$$

Now consider $x_m(t)=t^m,\ m=1,2,...$ In this case, the exact $\alpha-th$ derivative is

$${}_0D_t^{\alpha}x_m(t)=\frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)}t^{m-\alpha}, \tag{7.21}$$

and the approximation (7.11) of the exact derivative becomes

$${}_0\Delta_h^{\alpha}x_m(t)=t^{m-\alpha}n^{\alpha}\sum_{j=0}^{n}\binom{j-\alpha-1}{j}\left(1-\frac{j}{n}\right)^m \tag{7.22}$$

after some calculations[see [10], page 206-207],we obtain

$${}_0\Delta_h^{\alpha}x_m(t)=\frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)}t^{m-\alpha}+O\left(h\right) \tag{7.23}$$

and

$${}_0D_t^{\alpha}x_m(t)-_0\Delta_h^{\alpha}x_m(t)=O\left(h\right). \tag{7.24}$$

This means that if a function $x(t)$ can be written in the form of a power series

$$x(t) = \sum_{m=0}^{\infty} a_m t^m, \tag{7.25}$$

then (7.11) gives the first-order approximation for the fractional derivative order α at any point of the convergence region of the power series. The conditions on $f(t)$ also can be weakened. ∎

7.2. Solution of the General Bagley-Torvik Equation with FDM

Example 6.1

We consider the Bagley-Torvik equation as an example of three-term fractional differential equation with constant coefficient

$$\begin{cases} AD^2x(t) + BD^{3/2}x(t) + Cx(t) = f(t), \\ x(0) = 0, x'(0) = 0 \end{cases} \tag{7.26}$$

(where $A \neq 0, B, C \in R$).

Solution 6.1

First, recall the first order backward difference for second order differential operator with the time step h:

$$D_t^2 x\left(t\right) \approx_0 \Delta_h^2 x(t_n) = \frac{x_n - 2x_{n-1} + x_{n-2}}{h^2}, \quad (x_n = x(nh), n = 0, 1, 2, ...) \tag{7.27}$$

and for $\alpha-th$ order differential operator, we can write first order backward difference formula

$$D_t^{\alpha} x\left(t\right) \approx_0 \Delta_h^{\alpha} x(t_n) = h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} x_{n-k}. \tag{7.28}$$

Then, let us assume that

$$(-1)^k \binom{\alpha}{k} = w_k^\alpha, (k = 0, 1, 2, ...)\,, \tag{7.29}$$

since $w_0^\alpha = 1$, a recurrence formula for $w_k^\alpha, (k = 1, 2, 3, ...)$ can be obtained as

$$w_k^\alpha = \sum_{k=1}^{n} (1 - (\frac{\alpha+1}{k}))w_{k-1}^\alpha. \tag{7.30}$$

Combining (7.27), (7.28) and taking $g(x) = x(t)$, we get the first order backward difference formula for (7.26)

$$A\frac{(x_n - 2x_{n-1} + x_{n-2})}{h^2} + \frac{B\sum_{j=0}^{n} w_j^\alpha x_{n-j}}{h^\alpha} + Cx_n = fn, \quad (fn = f(nh)), \tag{7.31}$$

with the initial conditions

$$x_0 = 0, \qquad\qquad \frac{x_{1-}x_0}{h} = 0, \tag{7.32}$$

we have $x_0 = x_1 = 0$.

Arranging (7.31)

$$A\frac{(x_n - 2x_{n-1} + x_{n-2})}{h^2} + \frac{Bx_n}{h^\alpha} + \frac{B\sum_{j=1}^{n} w_j^\alpha x_{n-j}}{h^\alpha} + Cx_n = f_n$$

is obtained.

Then, if we take x_n from the last equation, we get the difference formula for the three-term equation:

$$x_n = \frac{h^2(f_n - Cx_n) + A(2x_{n-1} - x_{n-2}) - Bh^{2-\alpha}\sum_{j=1}^{n} w_j^\alpha x_{n-j}}{A + Bh^{2-\alpha} + Ch^2}, \quad (n = 2, 3, ...). \tag{7.33}$$

Therefore, the solution of the problem under homogeneous initial conditions will be:

$$x_n = \frac{h^2(f_n - Cx_n) + A(2x_{n-1} - x_{n-2}) - Bh^{1/2}\sum_{j=1}^{n} w_j^{3/2} x_{n-j}}{A + Bh^{1/2} + Ch^2}, \quad (n = 2, 3, ...).$$

In this problem, we write the solution of the equation for all f, A, B, C above. Now, let us choose some specific values for constants and the right side function. For instance, we will take constants such as $A = 1, B = 1/2, C = 1/2$ and the right side function $f(t) = \begin{cases} 8, & (0 \leq t \leq 1) \\ 0, & (t > 1), \end{cases}$ for the first example and $f(t) = 0,05t^4 - 0,03t^3 + 0,361t^{5/2} + 0,145t^2 - 0,135t^{3/2} - 0,36t + 0,056t^{1/2} + 0,1$ for the second example of our problem and solve both of them recursively from the difference formula (7.33.

Here, we will write an algorithm to solve the Bagley-Torvik equation for the first and second examples. Moreover, with changing constants and the right side function more problems can be solved by using same algorithm.

7.3. Algorithm for the Bagley-Torvik Equation with FDM

1. Put A, B, C, α, h(step size), b (length of time interval) as input values.
2. Put $M = b/h$.
3. Set W, x and tk as sequences from 0 to M.
4. Evaluate $tk[i]$ with the formula $t[k] = ih$.
5. Give the initial values $y[0] = 0$, $y[1] = 0$ and $W[0] = 1$.
6. Evaluate W with the recurrence formula $W[j] = (1 - (\alpha + 1)/j)W[j-1]$ from 1 to M.
7. Define $f(t)$ which is the right side function.
8. Evaluate $x[m]$ with the formula
   $x[m] = \frac{h^2(f(tk[m]) - Cx[m]) + A(2x[m-1] - x[m-2]) - Bh^{2-\alpha}\sum_{j=1}^{m} w_j^{\alpha} x[m-j]}{A + Bh^{2-\alpha} + Ch^2}$
   from 2 to M.

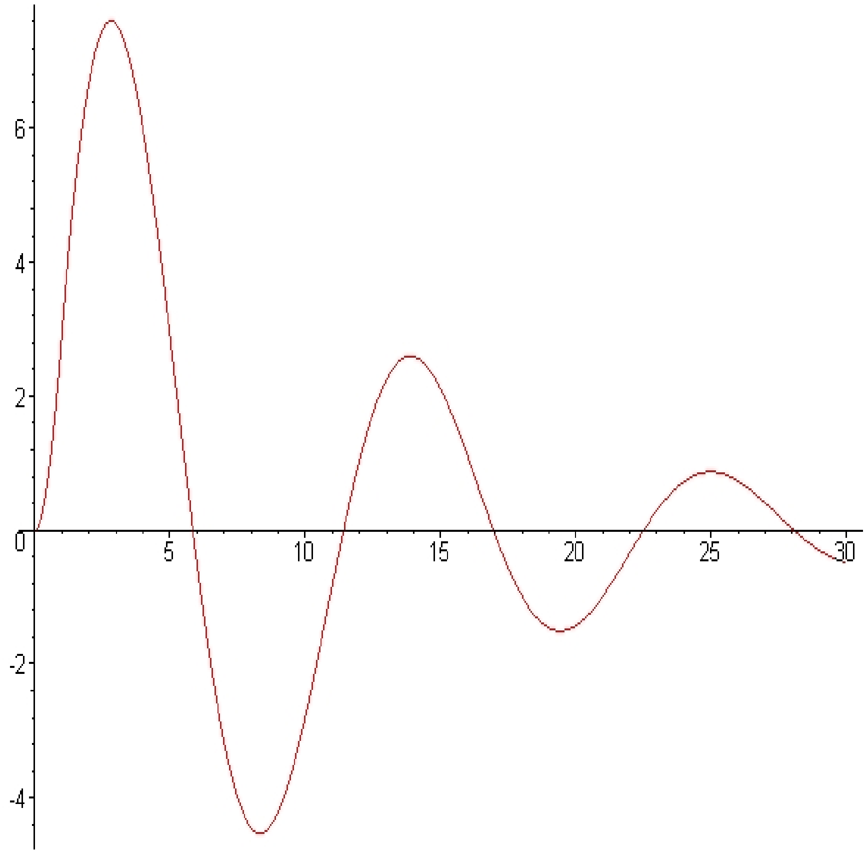

Figure 7.1 . Numerical solution of the Bagley-Torvik equation for the first example with FDM.

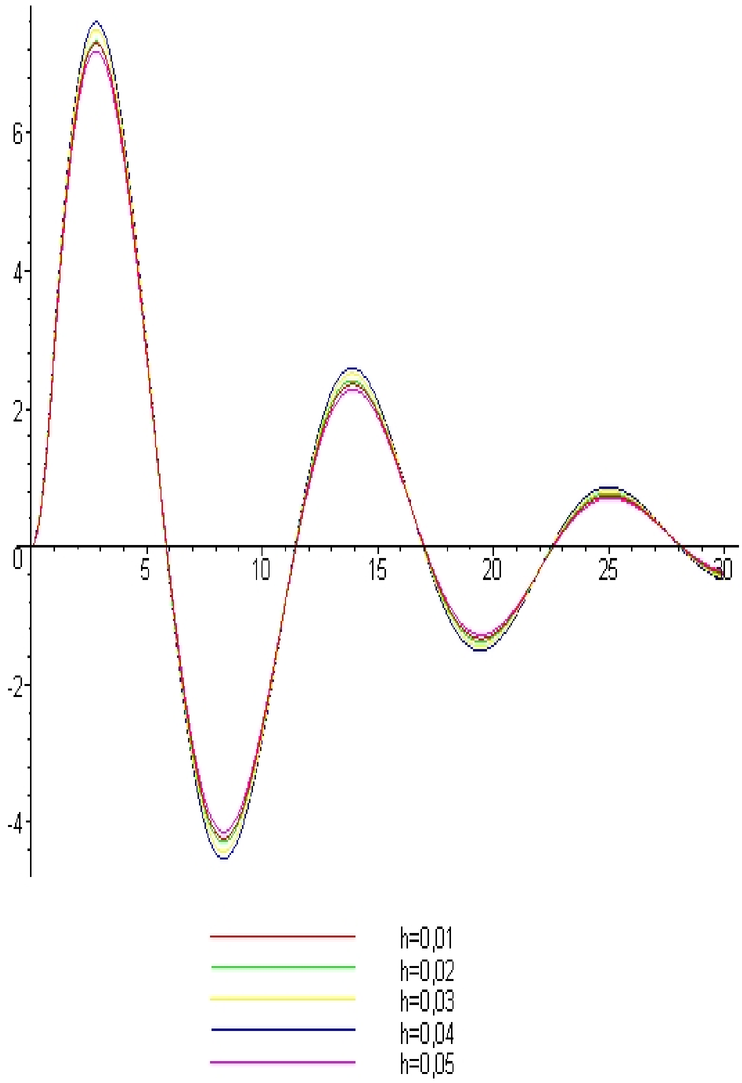

Figure 7.2 . Numerical solution of the Bagley-Torvik equation for the first example with FDM for different h values

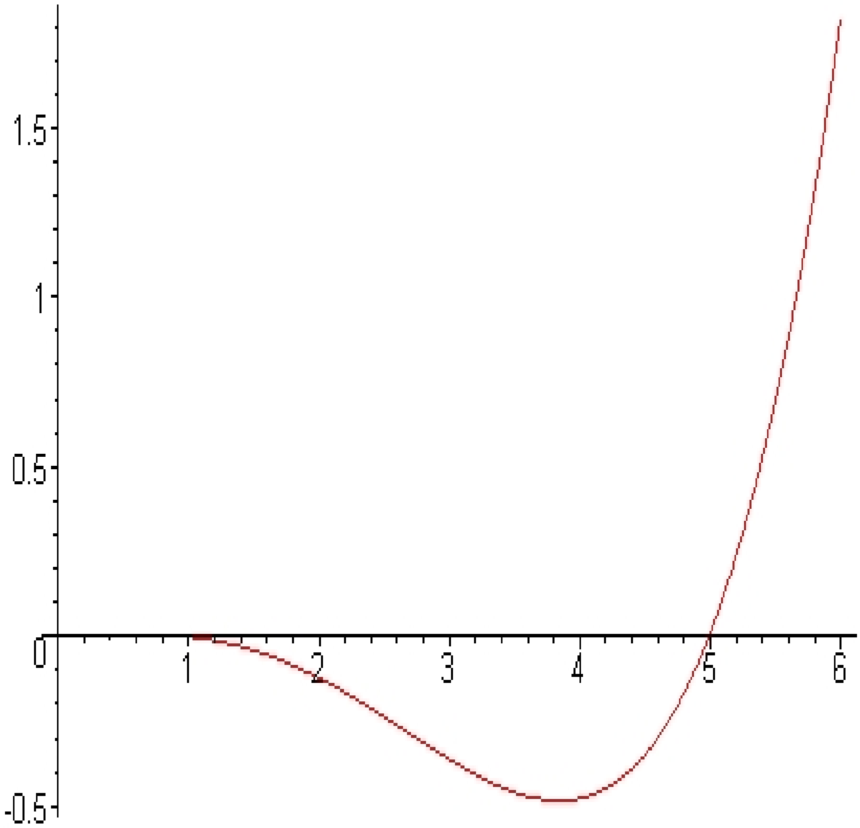

Figure 7.3 . Numerical solution of the Bagley-Torvik equation for the second example with FDM.

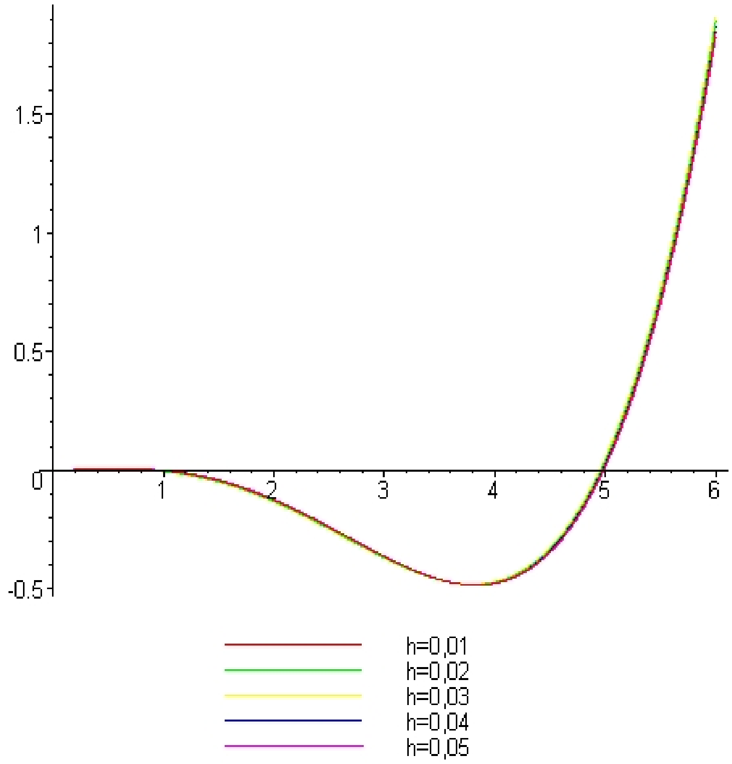

Figure 7.4 . Numerical solution of the Bagley-Torvik equation for the second example with FDM for different h values

Table 7.1 . Exact and Approximate values for some h values

tk	$h = 1/2^4$	$h = 1/2^5$	$h = 1/2^6$	$h = 1/2^7$	$h = 1/2^8$	$exact$
0	0	0	0	0	0	0
$0,625$	$0,0117749$	$0,0107524$	$0,0103242$	$0,0101665$	$0,0101254$	$0,0102539$
$1,25$	$-0,008603$	$-0,010621$	$-0,011477$	$-0,011806$	$-0,011905$	$-0,011718$
$1,875$	$-0,046109$	$-0,049257$	$-0,050641$	$-0,0512097$	$-0,051413$	$-0,051269$
$2,5$	$-0,086212$	$-0,090565$	$-0,092547$	$-0,093411$	$-0,093760$	$-0,09375$
$3,125$	$-0,114467$	$-0,120009$	$-0,122611$	$-0,123797$	$-0,124318$	$-0,1245117$
$3,75$	$-0,116447$	$-0,12306$	$-0,126258$	$-0,1277649$	$-0,128465$	$-0,1289062$
$4,375$	$-0,07770179$	$-0,0852210$	$-0,0889184$	$-0,0907136$	$-0,091583$	$-0,092285$
5	$0,0162713$	$0,0080860$	$0,0039924$	$0,0019612$	$0,0009464$	0
$5,625$	$0,1800369$	$0,1714335$	$0,1670737$	$0,1648734$	$0,1637501$	$0,1625976$
$6,25$	$0,42822725$	$0,419441$	$0,4149470$	$0,41264893$	$0,41145954$	$0,4101562$
$6,875$	$0,77553647$	$0,76676805$	$0,7622563$	$0,7599274$	$0,75871837$	$0,75732421$
$7,5$	$1,2367081$	$1,2281024$	$1,2236663$	$1,2213621$	$1,2201761$	$1,21875$
$8,125$	$1,82651839$	$1,8181582$	$1,8138581$	$1,8116188$	$1,8104789$	$1,8090820$
$8,75$	$2,5597566$	$2,5516619$	$2,5475227$	$2,5453678$	$2,54427182$	$2,5429687$
$9,375$	$3,4512079$	$3,44334259$	$3,4393559$	$3,4372839$	$3,4362629$	$3,4350585$
10	$4,515637$	$4,5079216$	$4,5040520$	$4,50204295$	$4,5011246$	$4,5$

Table 7.2 . Error analysis for $y_7 = 4,375$

$Stepsize, h$	$Approximation\ to\ y_7$	$error\ at\ t = 4,375:$	$O(h) = Ch,\ C = 0,256$
$1/2^4$	$0,0777017969$	$0,01458335935$	$0,016$
$1/2^5$	$0,08522104995$	$0,0070641063$	$0,008$
$1/2^6$	$0,08891840203$	$0,0033667542$	$0,004$
$1/2^7$	$0,09071366918$	$0,0015714871$	$0,002$
$1/2^8$	$0,09158346145$	$0,0007016948$	$0,001$

7.4. Short Memory Principle:

Numerical methods for solving fractional differential equations have been implemented extensively. For example, we used FDM for solving the Bagley-Torvik equation and while solving the equation by computer, it was seen that over long time intervals the computational effort to calculate the function values was too much. Then, we will make a modification to accelerate the computation and control the error for the solution simultaneously. Same observations were seen by some authors and Ford and Simpson [**?**] and [10] called this modification "short-memory principle" or "fixed memory principle". According to the short-memory principle, the fractional derivative with the lower limit a is approximated by the fractional derivative with moving lower limit $t-L$ where L is the memory length. With this approach, we do the calculations over a period of recent history and reduce the computational cost efficiently. However, because of omitting the calculations over a period, some errors occurs. Then, the following estimate can be established for the problem;

$$ {}_aD_t^{\alpha}x(t) = f(t). \tag{7.34} $$

Then,

$$ {}_aD_t^{\alpha}x(t) \approx_{t-L} D_t^{\alpha}x(t), \qquad (t > a + L), \tag{7.35} $$

$$\Delta(t) = \mid_a D_t^\alpha x(t) -_{t-L} D_t^\alpha x(t) \mid \leq \frac{ML^{-\alpha}}{\mid \Gamma(1-\alpha) \mid}, \qquad (a + L \leq t \leq b). \tag{7.36}$$

This inequality can be used for determining the "memory length" L providing the required accuracy ε :

$$\Delta(t) \leq \varepsilon, \quad (a + L \leq t \leq b), \;\; \text{if } L \geq \left(\frac{M}{\epsilon \mid \Gamma(1-\alpha) \mid} \right)^{1/\alpha}. \tag{7.37}$$

where $|f(t)| < M$ for $a \leq t \leq b$.

In our examples, we write an algorithm for the use of short memory principle and do a replacement of $\sum_{j=1}^{n}$ by $\sum_{j=1}^{N}$, where $N = \min\left\{n, \frac{L}{h}\right\}$ and L is the memory length that we choose. According to some calculations we saw that after an interval we reach the same value with the value that we had by the original algorithm and the computation worked fastener. However, the equations have two differential operators: $\alpha - th$ order and second order, so we can not evaluate a fixed L with this formula. Although ıt can be found for $\alpha - th$ order differential operator, there should be the effect of second order derivative.

Now, let us see the effect of the short memory principle in our second example:

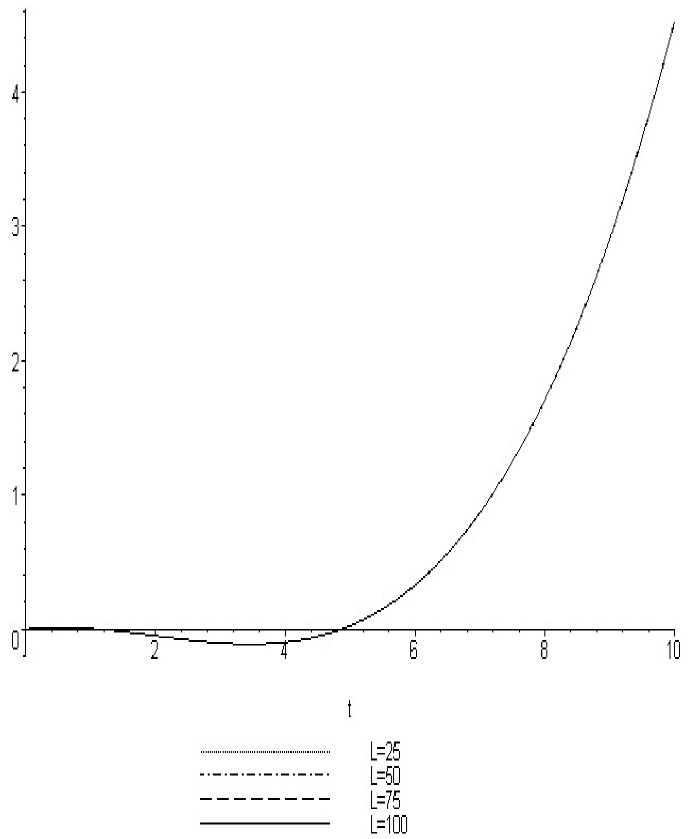

Figure 7.5 . Numerical solution of the Bagley-Torvik equation for the second example with FDM for different h values

7.5. Algorithm for the Short Memory Principle:

1. Put A, B, C, α, h(step size), b (length of time interval),L (memory), $M, W, i, j, f, m,$ as input values.
2. Set W, y and tk as sequences from 0 to M.
3. Evaluate $tk[i]$ with the formula $t[k] = ih$.
4. Give the initial values $y[0] = 0$, $y[1] = 0$ and $W[0] = 1$.
5. Evaluate W with the recurrence formula $W[j] = (1-(\alpha+1)/j)W[j-1]$ from 1 to M.
6. Give f as piecewise function f as piecewise function $f(t) = \begin{cases} 8, & (0 \le t \le 1) \\ 0, & (t>1), \end{cases}$ or $f(t) = 0,05t^4 - 0,03t^3 + 0,361t^{5/2} + 0,145t^2 - 0,135t^{3/2} - 0,36t + 0,056t^{1/2} + 0,1$.
7. Choose *memory* as the minimum of the (m, L),
8. Evaluate $y[m]$ with the formula from 2 to *memory* $y[m] = \frac{h^2(f(tk[m]) - Cy[m]) + A(2y[m-1] - y[m-2]) - Bh^{2-\alpha}\sum_{j=1}^{m} w_j^{\alpha} y[m-j]}{A + Bh^{2-\alpha} + Ch^2}$.

CHAPTER 8

CONCLUSION

In this thesis, methods for solving fractional differential equations are investigated. Although there are more methods, we choose four of them-Green's Function Method, Power Series Method, ADM and FDM- and express how they are used for fractional differential equations and formulated. Methods are also exemplified with Bagley-Torvik equation. First, the stability of the Bagley-Torvik equation is defined for both differential and difference equations for the reliability of the results. Then, the original equation is solved with all four methods. Also a test example for Bagley-Torvik equation which we know the exact solution is defined and solved to compare the methods, see error analysis easier. With the test example, more general algorithms are implemented to solve the different type of equations.

Some conclusions about the methods:

Table 8.1 . Error analysis for the methods

Method	max. error($N = 8$, $h = 0.01$, $b = 5$)
Green's Function	0.01136
Power Series	0
ADM	0.01222175
FDM	0.0105

1. Green's Function method gives us semi-analytical solution of the problems. However, in its formulation there are two summation together and. This means that if we truncate summations from 0 to N, green's function of the problem has N^2 terms. Because we should take the integral of the green's function with

right side function, it will take very long time periods to evaluate the integral and give results. When we increase N, results will be better.

2. In the Power Series Method, if the right side function is a polynomial function, it is enough to expand the series from 0 to $2m$ where m is the highest power of the right side function. Then error will be 0. But, if the right side function is not a polynomial function such as sin or cos functions, error will be smaller. It means also the computation effort will be more.
3. ADM method is also a series method which we expect results corresponds to Power Series method or Green's Function Method. Likely, in our examples our results are nearly same as the green's function method.
4. Lastly, FDM is the best way to solve fractional differential equations for all types. Also, it is not so important the right side function is polynomial or not like as the other methods. The errors will be smaller when we decrease the step size value(h). Also there is no need of so much computational effort. Results are generally better and computation is always faster than others. Furthermore, for long computations we can use short-memory principle with some very little errors, it is the fastest way to compute the results of the equation.

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APPENDIX

```
> restart:
  Digits:=14:
  N:=40: L:=30; A:=1.0:
  B:=0.5:  C:=0.5:

  G:=t-> (1/A)*sum(((((((-1)^k)/k!))*((C/A)^k)*t^(2*k + 1))*
         sum( (((j + k)!)*((-B/A)^j)*(t^(j/2)))/(j!)
         /GAMMA(j/2 + 2*k + 2),j=0..N),
          k=0..N):

  ft:=t-> piecewise(t>1,0,t<=1,8):

  Xt:= evalf(Int(simplify(G(t-S)*ft(S)),S=0..t)) :
  h:=0.1:
  tk:=0:
  Z:=L/h;

  for r from 1 to Z do
     tk:=tk+h:
     Xtk:=evalf(subs(t=tk,Xt)):
     dt[r]:=tk:
     y[r]:=Xtk:
     print(r,tk,y[r]);
  end do:

  fg:=[seq([dt[n],y[n]],n=1..r)]:
  plot(fg,t=0..L,y=-8..8);
```

Figure 8.1 . Maple programm for the solution of Bagley-Torvik equation with Green Function Method where $f(t)$ is piecewise function.

```
> restart:
  Digits:=5;
  N:=4:  A:=1:
  B:=1/2:   C:=1/2:

  G:=t-> (1/A)*sum((((((-1)^k)/k!))*((C/A)^k)*t^(2*k + 1))*
           sum( (((j + k)!)*((-B/A)^j)*(t^(j/2)))/(j!)
  /GAMMA(j/2 + 2*k + 2),j=0..N),
       k=0..N):

  ft:=t->.2000000000e-1*(t-1)*(t-5)+.4000000000e-1*t*(t-5)+
  .4000000000e-1*t*(t-1)+.2000000000e-1*t^2+
  .3610813334e-1*t^(5/2)-.1354055000*t^(3/2)+
  .5641895835e-1*sqrt(t)+.5000000000e-2*t^2*(t-1)*(t-5);
  Xt:=   int(G(t-S)*ft(S),S=0..t)    :
  print("integral alındı .... Grafik çiziliyor.....");
  K:=0;  L:=6;
  h:=0.1:
  tk:=K:
  Z:=L/h;
  for r from 0 to Z do
     tk:=h*r:
     Xtk:=evalf(subs(t=tk,Xt)):
     dt[r]:=tk:
     y[r]:=Xtk:
  end do:
  fg:=[seq([dt[n],y[n]],n=0..Z)];
  plot(fg,t=0..L);
```

Figure 8.2 . Maple programm for the solution of Bagley-Torvik equation with Green Function Method where
$f(t) = 0,05t^4 - 0,03t^3 + 0,361t^{5/2} + 0,145t^2 - 0,135t^{3/2} - 0,36t + 0,056t^{1/2} + 0,1.$

```
> restart;
  Digits:=20:
  M:=300:  L:=30;
  A:=1:  B:=1/2:   C:=1/2:

  f:=ut;
  s[0]:=subs(t=0,f):
  q[0]:=s[0]:
  for j from 1 by 1 while j<= M do
       q[j]:=0;
   end do:

  ##################################################
  ut1:=8*Heaviside(t):
  c[0]:=0: c[1]:=0: c[2]:=0:c[3]:=0: c[4]:= ut1*4:
  for j from 1 by 1 while j<=3 do
       c[j+4]:=(q[j]-B*GAMMA((j+3)/2+1.)
  / GAMMA((j+3)/2.-1/2.)*c[j+3] )/(A*(j+4)/2.*(j+2)/2);

  end do:

  for j from 4 by 1 while j<=M-4 do
       c[j+4]:=( q[j]-B*GAMMA((j+3)/2.+1)
  /GAMMA((j+3)/2.-1/2)*c[j+3] - C*c[j])
  /(A*(j+4)/2.*(j+2)/2 );
  end do:

  R1:=t-> sum(c[k]*(t^(k/2) ),k=0..M):
```

Figure 8.3 . Maple programm for the solution of Bagley-Torvik equation with Power Series Method where $f(t)$ is piecewise function.

```
ut2:=8*Heaviside(t-1):
d[0]:=0:d[1]:=0:d[2]:=0:d[3]:=0:d[4]:= ut2*4:
for j from 1 by 1 while j<=3 do
     d[j+4]:=(q[j]-B*GAMMA((j+3)/2+1.)
/ GAMMA((j+3)/2.-1/2.)*d[j+3] )/(A*(j+4)/2.*(j+2)/2);
end do:

for j from 4 by 1 while j<=M-4 do
     d[j+4]:=( q[j]-B*GAMMA((j+3)/2.+1)
/GAMMA((j+3)/2.-1/2)*d[j+3] - C*d[j])
/(A*(j+4)/2.*(j+2)/2 );
end do:
R2:=t-> sum(d[k]*((t-1)^(k/2) ),k=0..M):
####################################################
R:=t -> R1(t)-R2(t):
utt1:= t-> piecewise(t<=0,0,t>0,1):
utt2:= t-> piecewise(t<=1,0,t>1,1):
h:=0.5:
for j from 0 by 1 while j<301 do
  tt:=j*h:
  yytt:=simplify(subs(t=tt,ut1=utt1(tt)
,ut2=utt2(tt),R(t))):
  dt[j]:=tt:
  y[j]:=yytt:
  %print(j,tt,yytt):
end do:
fg[0]:=0:
fg:=seq([dt[n],y[n]],n=0..300):
plot([fg],t=0..30);
```

Figure 8.4 . Maple programm for the solution of Bagley-Torvik equation with Power Series Method where $f(t)$ is piecewise function.

```
> restart;
  M:=8:   L:=7;
  A:=1:   B:=1/2:    C:=1/2:
  f:=.1450000000*t^2-.3600000000*t+.1000000000+
  .3610813334e-1*t^(5/2)-.1354055000*t^(3/2)+
  .5641895835e-1*sqrt(t)+.5000000000e-2*t^4-.3000000000e-1*t^3;
  s[0]:=subs(t=0,f):
  q[0]:=s[0]:
  for j from 1 by 1 while j<= M do
     f:=f-s[0]:
     f:=expand(f/sqrt(t)):
     s[0]:=subs(t=0,f):
     q[j]:=s[0]:
  end do:
  c[0]:=0:   c[1]:=0:  c[2]:=0:c[3]:=0:   c[4]:= q[0]/(2*A):
  for j from 1 by 1 while j<=3 do
    c[j+4]:=(q[j]-B*GAMMA((j+3)/2+1.)
  /GAMMA((j+3)/2.-1/2.)*c[j+3])/(A*(j+4)/2.*(j+2)/2);
  end do:

  for j from 4 by 1 while j<=M-4 do
      c[j+4]:=( q[j]-B*GAMMA((j+3)/2.+1)
  /GAMMA((j+3)/2.-1/2)*c[j+3]-C*c[j])/( A*(j+4)/2.*(j+2)/2 );
  end do:

  R:=t-> sum(c[k]*(t^(k/2) ),k=0..M):
  h:=0.1;
  fg:=seq([p*h,R(p*h),abs(R(p*h)-x(p*h))],p=0..L/h);
  plot([R(t)],t=0..L);
```

Figure 8.5 . Maple programm for the solution of Bagley-Torvik equation with Power Series Method where
$f(t) = 0,05t^4 - 0,03t^3 + 0,361t^{5/2} + 0,145t^2 - 0,135t^{3/2} - 0,36t + 0,056t^{1/2} + 0,1.$

```
> restart:
A:=1:
B:=1/2:
C:=1/2:
h:=0.21:
x0:=simplify((1/GAMMA(2)/A)*int(8*(ut)*(x-t),t=0..x)):
ssum:=subs(x=t,x0):
%x0:=subs(t=x,x0):
for i from 0 by 1 while i<40 do
AA:=(1/GAMMA(1/2))*(-B/A)*int(x0/((-x+t)^(1/2)),x=0..t):
BB:=(1/GAMMA(2))*(-C/A)*int(x0*(-x+t),x=0..t):
xx:=AA+BB:
x0:=xx:
ssum:=simplify(x0+ssum):
x0:=subs(t=x,x0):
end do:
yt:=ssum:
yttt:=subs(t=(t-1),ut=utt,yt):
yyt:=ssum-yttt:
ut1:= t-> piecewise(t<=0,0,t>0,1):
utt1:= t-> piecewise(t<=1,0,t>1,1):
tt:=0.:
for j from 1 by 1 while j<81 do
tt:=tt+h:
yytt:=simplify(subs(t=tt,ut=ut1(tt),utt=utt1(tt),yyt)):
dt[j]:=tt:
y[j]:=yytt:
print(j,tt,yytt):
end do:
fg:=seq([dt[n],y[n]],n=1..80):
plot({fg},t=0..20,Y=-5..10,linestyle=1,style=point);
```

Figure 8.6 . Maple programm for the solution of Bagley-Torvik equation with ADM where $f(t)$ is piecewise function.

```
> restart;
  findiff:=proc(h,LL)
  local A,B,C,alfa,b,x,
    L,M,tk,y,W,i,j,f,m,memory,summm,result;
  A:=1:      B:=1/2.:  C:=1/2.:
  alfa:=1.5:
  b:=10:
  M:=b/h:
  seq(W[i],i=0..M):
  seq(y[i],i=0..M):
  seq(tk[i],i=0..M):
  for i from 0 to M do  tk[i]:= i*h;  end do:
  W[0]:=1:
  y[0]:=0:
  y[1]:=0:
  for j from 1 by 1 while j<=M do
     W[j]:= evalf( ((1-((alfa+1)/j))*W[j-1]) ):
  end do:

  ft:=t->.1450000000*t^2-.3600000000*t+.1000000000+
  .3610813334e-1*t^(5/2)-.1354055000*t^(3/2)+
  .5641895835e-1*sqrt(t)+.5000000000e-2*t^4-.3000000000e-1*t^3;
  L:=LL;
  for m from 2 by 1 while m<=M do
     memory:=min(m,L);
     summm:=sum(W[k]*y[m-k],k=1..memory):
     y[m]:=evalf( (A*(2*y[m-1]-y[m-2])-B*sqrt(h)*(summm)+
  f(tk[m])*(h^2))/(A+B*sqrt(h)+C*(h^2)) ):
  end do:
  result:=[seq([tk[m],y[m]], m=0..M )];
  end proc:
```

Figure 8.7 . Maple programm for the solution of Bagley-Torvik equation with ADM where

$f(t) = 0,05t^4 - 0,03t^3 + 0,361t^{5/2} + 0,145t^2 - 0,135t^{3/2} - 0,36t + 0,056t^{1/2} + 0,1.$

```
> findiff:=proc(h)
  local A, B,C,alfa,b,
        M,tk,y,W,i,j,f,m,summm,result;
  A:=1:      B:=1/2.:  C:=1/2.:
  alfa:=1.5:
  b:=30:
  M:=b/h:
  seq(W[i],i=0..M):
  seq(y[i],i=0..M):
  seq(tk[i],i=0..M):
  for i from 0 to M do ;
     tk[i]:= i*h;

  end do:
  W[0]:=1:
  y[0]:=0:
  y[1]:=0:
  for j from 1 by 1 while j<=M do
     W[j]:= evalf( ((1-((alfa+1)/j))*W[j-1])):
  end do:
  f:= t-> piecewise(t<=1,8,t>1,0):
  for m from 2 by 1 while m<=M do
     summm:=sum(W[k]*y[m-k],k=1..m):
     y[m]:=evalf( (A*(2*y[m-1]-y[m-2])-B*sqrt(h)*(summm)+
  f(tk[m])*(h^2))/(A+B*sqrt(h)+C*(h^2)) ):
  end do:
  result:=[seq([tk[m],y[m]], m=0..M )]:
  end proc:
```

Figure 8.8 . Maple programm for the solution of Bagley-Torvik equation with FDM where $f(t)$ is piecewise function.

```
> restart;
  findiff:=proc(h)
  local A, B,C,alfa,b,
       k1,M,tk,y,W,i,j,f,m,summm,result;
  A:=1:      B:=1/2.:   C:=1/2.:   alfa:=1.5:
  b:=6:     M:=b/h:
  seq(W[i],i=0..M):
  seq(y[i],i=0..M):
  seq(tk[i],i=0..M):
  for i from 0 to M do ;
     tk[i]:= i*h;
  end do:
  W[0]:=1:
  y[0]:=0:
  y[1]:=0:
  for j from 1 by 1 while j<=M do
     W[j]:= evalf( ((1-((alfa+1)/j))*W[j-1]) ):
  end do:

  ft:=t->.1450000000*t^2-.3600000000*t+.1000000000+
  .3610813334e-1*t^(5/2)-.1354055000*t^(3/2)+
  .5641895835e-1*sqrt(t)+.5000000000e-2*t^4-.3000000000e-1*t^3;
  for m from 2 by 1 while m<=M do
     summm:=sum(W[k]*y[m-k],k=1..m):
     y[m]:=evalf( (A*(2*y[m-1]-y[m-2])-B*sqrt(h)*(summm)+
  f(tk[m])*(h^2))/(A+B*sqrt(h)+C*(h^2)) ):
  end do:
  result:=[seq([tk[m],y[m]], m=0..M )]:
  end proc:
```

Figure 8.9 . Maple programm for the solution of Bagley-Torvik equation with FDM where

$f(t) = 0,05t^4 - 0,03t^3 + 0,361t^{5/2} + 0,145t^2 - 0,135t^{3/2} - 0,36t + 0,056t^{1/2} + 0,1.$

```
> restart;
findiff:=proc(h,LL)
local A,B,C,alfa,b,x,
  L,M,tk,y,W,i,j,f,m,memory,summm,result;
A:=1:     B:=1/2.:   C:=1/2.:
alfa:=1.5:
b:=10:
M:=b/h:
seq(W[i],i=0..M):
seq(y[i],i=0..M):
seq(tk[i],i=0..M):
for i from 0 to M do  tk[i]:= i*h;  end do:
W[0]:=1:
y[0]:=0:
y[1]:=0:
for j from 1 by 1 while j<=M do
   W[j]:= evalf( ((1-((alfa+1)/j))*W[j-1]) ):
end do:

ft:=t->.1450000000*t^2-.3600000000*t+.1000000000+
.3610813334e-1*t^(5/2)-.1354055000*t^(3/2)+
.5641895835e-1*sqrt(t)+.5000000000e-2*t^4-.3000000000e-1*t^3;
L:=LL;
for m from 2 by 1 while m<=M do
   memory:=min(m,L);
   summm:=sum(W[k]*y[m-k],k=1..memory):
   y[m]:=evalf( (A*(2*y[m-1]-y[m-2])-B*sqrt(h)*(summm)+
f(tk[m])*(h^2))/(A+B*sqrt(h)+C*(h^2)) ):
end do:
result:=[seq([tk[m],y[m]], m=0..M )];
end proc:
```

Figure 8.10 . Maple programm for the Short Memory Principle.