MULTIPOINT NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

by

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HİPERBOLİK DENKLEMLER İÇİN ÇOK NOKTALI LOKAL OLMAYAN SINIR DEĞER PROBLEMLERİ

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ÖZ

Bilindiği gibi çeşitli local olmayan hiperbolik tip sınır-değer denklemleri, Hilbert uzayındaki kendi kendine eş positive operatör A ile yerel olmayan sınır-değer problemine dönüştürülebilir.Operator metod kullanarak, bu yerel olmayan problemin kararlılığı elde edilmiştir.Yapılan soyut uygulamalar bize yerel olmayan iki hiperbolik tip sınır-değer problemlerinin kararlılığını elde etmemizi sağlamıştır. Bu yerel olmayan hiperbolik tip sınır-değer problemleri A-nın tamsayı değerli üslerinin oluşturduğu birinci ve ikinci mertebeden yaklaşımlı sonlu farklar metodlarıyla kurulmuştur. Bu sonlu farklar metodları ile çözümün kararlı olup olmadığı incelenmiştir ve yapılan nümerik denemelerle, elde edilen teorik sonuçların doğruluğu desteklenmiştir.

Anahtar Kelimeler: Hiperbolik denklemler, Fark şemaları, Yakınsaklık, Kararlılık, Sayısal analiz.

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ABSTRACT

It is known that various nonlocal boundary value problem for the hyperbolic equations can be reduced to the nonlocal boundary problem for differential equation in a Hilbert space H with self - adjoint positive operator A.Applying the operator approach we obtain the stability estimates for solution of this nonlocal boundary problem. In applications this abstract result permit us to obtain the stability estimates for the solution of nonlocal boundary value problem for hyperbolic equations. The first and second order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem are presented. The stability estimates for the solution of these difference schemes are obtained. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

Keywords: Hyperbolic equation, Difference schemes, Convergence, Stability, Numerical analysis.

CHAPTER 1

INTRODUCTION

It is known that most problems in fluid mechanics (dynamics,elasticity)and other areas of physics lead to partial differential equations of the hyperbolic type. These equations can be derived as models of physical systems and are considered as the methods for solving boundary value problems.

It is known that the hyperbolic problem can be solved by Fourier series method, by Fourier transform method and by Laplace transform method.

Now let us give some examples.

First let us consider the simple nonlocal boundary value problem for hyperbolic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = t \sin x, \ 0 < t < 0 \ , 0 < x < \pi, \\ \\ u(0,x) = \frac{1}{4}u(1,x) - \frac{1}{4}u(\frac{1}{2},x) - \frac{1}{8}\sin x, \ 0 \leq x \leq \pi, \\ \\ u_t(0,x) = \frac{1}{4}u_t(1,x) - \frac{1}{4}u_t(\frac{1}{2},x) + \sin x, \ 0 \leq x \leq \pi, \\ \\ u(t,0) = u(t,\pi) = 0, \ 0 \leq t \leq 1. \end{cases} \tag{1.1}$$

For the solution of the problem (1.1), we use the method of separation of variables or so called Fourier series method. In order to solve the problem we need to separate $u(t,x)$ into two parts

$$u(t,x) = v(t,x) + w(t,x),$$

where $v(t,x)$ is the solution of the problem

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0, \ 0 \le t \le 1, \ 0 < x < \pi, \\ v(0,x) - \frac{1}{4}v(1,x) + \frac{1}{4}v(\frac{1}{2},x) = -\frac{1}{8}\sin x \ , \ 0 \le x \le \pi, \\ v_t(0,x) - \frac{1}{4}v_t(1,x) + \frac{1}{4}v_t(\frac{1}{2},x) = \sin x, \ 0 \le x \le \pi, \\ v\left(t,0\right) = v\left(t,\pi\right) = 0, \ \ -1 \le t \le 1. \end{cases} \tag{1.2}$$

And $w(t,x)$ is the solution of problem

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = t\sin x \ , \ 0 \le t \le 1, \ 0 < x < \pi, \\ w(0,x) = \frac{1}{4}w(1,x) - \frac{1}{4}w(\frac{1}{2},x) \ , 0 \le x \le \pi, \\ w_t(0,x) = \frac{1}{4}w_t(1,x) - \frac{1}{4}w_t(\frac{1}{2},x), 0 \le x \le \pi, \\ w(t,0) = w(t,\pi) = 0 \quad , 0 \le t \le 1. \end{cases} \tag{1.3}$$

First we will obtain the solution of the problem (1.2). By the method of separation of variables, we obtain

$$v(t,x) = T(t)X(x) \neq 0. \tag{1.4}$$

We have that

$$T''(t)X(x) - T\left(t\right)X''\left(x\right) = 0$$

or

$$\frac{T''(t)}{T\left(t\right)} = \frac{X''(x)}{X(x)} = -k^2 = \lambda. \tag{1.5}$$

The boundary conditions presented in (1.2), require $X(0) = X\left(\pi\right) = 0$. Hence, from (1.5) we have the ordinary differential equation

$$X''(x) = \lambda X(x), \ X(0) = X\left(\pi\right) = 0. \tag{1.6}$$

If $\lambda \ge 0$, then the boundary value problem (1.6) has only trivial solution $X(x) = 0$. For $\lambda < 0$, the nontrivial solutions of the boundary value problem (1.6) are

$$X_k(x) = \sin kx, \quad \text{where } k = 1, 2, \cdots, \quad \lambda = -k^2.$$

So, the nontrivial solutions of the boundary value problem (1.6) are

$$X_k(x) = \sin kx, \quad \text{where } k = 1, 2, \cdots, \quad \text{and } \lambda = -k^2.$$

The other ordinary differential equation presented in (1.5) is

$$T''(t) = \lambda T(t),$$

with $\lambda = -k^2, \ \ k = 1, 2, \cdots$. The solution of this ordinary differential equation is

$$T_k(t) = c_{1k} \sin kt + c_{2k} \cos kt, \quad \text{where } k = 1, 2, \cdots.$$

Thus,

$$v(t,x) = \sum_{k=1}^{\infty} v_k(t,x) = \sum_{k=1}^{\infty} (c_{1k} \cos kt + c_{2k} \sin kt) \sin kx.$$

Using the nonlocal boundary condition

$$v(0,x) = \frac{1}{4} v(1,x) - \frac{1}{4} v(\frac{1}{2}, x) - \frac{1}{8} \sin x, \ 0 \le x \le \pi,$$

we obtain the following equation

$$\sum_{k=1}^{\infty} c_{1k} \sin kx = \frac{1}{4} \sum_{K=1}^{\infty} (c_{1k} \cos k + c_{2k} \sin k) \sin kx$$

$$-\frac{1}{4} \sum_{k=1}^{\infty} (c_{1k} \cos \frac{k}{2} + c_{2k} \sin \frac{k}{2}) \sin kx - \frac{1}{8} \sin x$$

or

$$\sum_{k=1}^{\infty} \left[c_{1k} - \frac{1}{4}(c_{1k} \cos k + c_{2k} \sin k) + \frac{1}{4}(c_{1k} \cos \frac{k}{2} + c_{2k} \sin \frac{k}{2}) \right]$$

$$\times \sin kx = -\frac{1}{8} \sin x.$$

Equating the coefficients of sin kx,we get

$$c_{1k} - \frac{1}{4}(c_{1k} \cos k + c_{2k} \sin k) + \frac{1}{4}(c_{1k} \cos \frac{k}{2} + c_{2k} \sin \frac{k}{2}) = 0,$$

$$2 \le k < \infty,$$

$$c_{11}(1 + \frac{1}{4}(\cos \frac{1}{2} - \cos 1)) + c_{21}(\frac{\sin \frac{1}{2}}{4} - \frac{\sin 1}{4}) = -\frac{1}{8} \ . \tag{1.7}$$

Using the nonlocal boundary condition

$$v_t(0,x) = \frac{1}{4} v_t(1,x) - \frac{1}{4} v_t(\frac{1}{2}, x) + \sin x, \ 0 \le x \le \pi,$$

we get the following equation

$$\sum_{k=1}^{\infty}\left[c_{2k}k-\frac{1}{4}(-c_{1k}k\sin k+c_{2k}k\cos k)+\frac{1}{4}(-c_{1k}k\sin\frac{k}{2}+c_{2k}k\sin\frac{k}{2})\right]$$
$$\times\sin kx=\sin x.$$

Equating the coefficients of sin kx,we get

$$c_{2k}k-\frac{1}{4}(-c_{1k}k\sin k+c_{2k}k\cos k)+\frac{1}{4}(-c_{1k}k\sin\frac{k}{2}+c_{2k}k\sin\frac{k}{2})=0,$$
$$2\le k<\infty,$$

$$c_{11}(\frac{\sin 1}{4}-\frac{\sin\frac{1}{2}}{4})+c_{21}(1-\frac{\cos 1}{4}+\frac{\cos\frac{1}{2}}{4})=1. \qquad (1.8)$$

So, we have that

$$\begin{cases} c_{11}(1+\frac{1}{4}(\cos\frac{1}{2}-\cos 1))+c_{21}(\frac{\sin\frac{1}{2}}{4}-\frac{\sin 1}{4})=-\frac{1}{8},\\ c_{11}(\frac{\sin 1}{4}-\frac{\sin\frac{1}{2}}{4})+c_{21}(1-\frac{\cos 1}{4}+\frac{\cos\frac{1}{2}}{4})=1\end{cases}$$

for $k=1$ and

$$\begin{cases} c_{1k}-\frac{1}{4}(c_{1k}\cos k+c_{2k}\sin k)+\frac{1}{4}(c_{1k}\cos\frac{k}{2}+c_{2k}\sin\frac{k}{2})=0,\\ c_{2k}k-\frac{1}{4}(-c_{1k}k\sin k+c_{2k}k\cos k)+\frac{1}{4}(-c_{1k}k\sin\frac{k}{2}+c_{2k}k\sin\frac{k}{2})=0\end{cases}$$

for $k\ge 2$.

Using the Crammer's rule

$$\triangle=\begin{vmatrix} 1-\frac{\cos 1}{4}+\frac{\cos\frac{1}{2}}{4} & \frac{\sin\frac{1}{2}}{4}-\frac{\sin 1}{4}\\ \frac{\sin 1}{4}-\frac{\sin\frac{1}{2}}{4} & 1-\frac{\cos 1}{4}+\frac{\cos\frac{1}{2}}{4}\end{vmatrix}=$$
$$(1-\frac{\cos 1}{4}+\frac{\cos\frac{1}{2}}{4})^2+(\frac{\sin 1}{4}-\frac{\sin\frac{1}{2}}{4})^2\neq 0,$$

and

$$c_{11}=\frac{\begin{vmatrix} -\frac{1}{8} & \frac{\sin\frac{1}{2}}{4}-\frac{\sin 1}{4}\\ 1 & 1-\frac{\cos 1}{4}+\frac{\cos\frac{1}{2}}{4}\end{vmatrix}}{\Delta},$$

$$c_{21} = \frac{\begin{vmatrix} 1 - \frac{\cos 1}{4} + \frac{\cos\frac{1}{2}}{4} & -\frac{1}{8} \\ \frac{\sin 1}{4} - \frac{\sin\frac{1}{2}}{4} & 1 \end{vmatrix}}{\Delta}$$

are the coefficients.

Putting c_{11} and c_{21} in $v(t,x)$, the following equation can be written

$$v_1(t,x) = \Bigg[\Big(\frac{\frac{\cos 1}{32} - \frac{1}{8} - \frac{\cos\frac{1}{2}}{32} - \frac{\sin\frac{1}{2}}{4} + \frac{\sin 1}{4}}{\Delta} \Big) cost$$

$$+ \Big(\frac{1 + \frac{\cos\frac{1}{2}}{4} - \frac{\cos 1}{4} + \frac{\sin 1}{32} - \frac{\sin\frac{1}{2}}{32}}{\Delta} \Big) \sin t \Bigg] \sin x.$$

For the $w(t,x)$ the solution is

$$w(t,x) = \sum_{k=1}^{\infty} A_k(t) \sin kx$$

$$\sum_{k=1}^{\infty} A_k''(t) \sin kx + \sum_{k=1}^{\infty} A_k(t) k^2 \sin kx = t \sin x$$

$$+ \sum_{k=2}^{\infty} (A_k''(t) + A_k(t) k^2) \sin kx.$$

The complimentary solution of the equation is

$$A_k''(t) + A_k(t)k^2 = t, \qquad A_k^c(t) : m^2 + k^2 = 0 \Longrightarrow m = \pm ik,$$

$$A_k^c(t) : \sum_{k=1}^{\infty} (d_{1k} \cos kt + d_{2k} \sin kt) \quad A_p(t) = at + b,$$

$$ak^2 t + bk^2 = t \Longrightarrow ak^2 = 1 \Longrightarrow a = \frac{1}{k^2},$$

$$ak^2 t + bk^2 = t \Longrightarrow ak^2 = 1 \Longrightarrow a = \frac{1}{k^2},$$

$$(d_{11} \cos t + d_{21} \sin t + t) \sin x,$$

$$w_k(t,x) = (d_{11} \cos t + d_{21} \sin t + t) \sin x$$

$$+\sum_{k=2}^{\infty}(d_{1k}\cos kt + d_{2k}\sin kt + \frac{1}{k^2}t)\sin kx.$$

The equation can be written as

$$d_{11}\sin x = \frac{1}{4}(d_{11}\cos 1 + d_{21}\sin 1 + 1)\sin x$$
$$-\frac{1}{4}(d_{11}\cos\frac{1}{2} + d_{21}\sin\frac{1}{2} + \frac{1}{2})\sin x,$$

$$d_{21} \ +1 \ = \ \frac{1}{4}(-d_{11}\sin 1 + d_{21}\cos 1 + 1)$$
$$-\frac{1}{4}(-d_{11}\sin\frac{1}{2} + d_{21}\cos\frac{1}{2} + 1) \quad ,$$

for $k = 1$

$$d_{11}(1 - \frac{\cos 1}{4} + \frac{\cos\frac{1}{2}}{4}) + d_{21}(\frac{\sin\frac{1}{2}}{4} - \frac{\sin 1}{4}) = \frac{1}{8}, \tag{1.9}$$

$$d_{11}(\frac{\sin 1}{4} - \frac{\sin\frac{1}{2}}{4}) + d_{21}(1 - \frac{\cos 1}{4} + \frac{\cos\frac{1}{2}}{4}) \ = -1, \tag{1.10}$$

for $k \geq 2$

$$d_{1k}(1 - \frac{\cos 1}{4} + \frac{\cos\frac{1}{2}}{4}) + d_{2k}(\frac{\sin\frac{1}{2}}{4} - \frac{\sin 1}{4}) = 0$$
$$d_{1k}(\frac{\sin 1}{4} - \frac{\sin\frac{1}{2}}{4}) + d_{2k}(1 - \frac{\cos 1}{4} + \frac{\cos\frac{1}{2}}{4}) \ = 0.$$

Using the Crammer's rule

$$\Delta = \begin{vmatrix} 1 - \frac{\cos 1}{4} + \frac{\cos\frac{1}{2}}{4} & \frac{\sin 1}{4} - \frac{\sin\frac{1}{2}}{4} \\ \frac{\sin 1}{4} - \frac{\sin\frac{1}{2}}{4} & 1 - \frac{\cos 1}{4} + \frac{\cos\frac{1}{2}}{4} \end{vmatrix} =$$

$$(1 + \frac{\cos 1}{4} - \frac{\cos\frac{1}{2}}{4})^2 + (\frac{\sin 1}{4} - \frac{\sin\frac{1}{2}}{4})^2 \neq 0$$

$$d_{11} = \frac{\begin{vmatrix} \frac{1}{8} & \frac{\sin 1}{4} - \frac{\sin\frac{1}{2}}{4} \\ -1 & 1 - \frac{\cos 1}{4} + \frac{\cos\frac{1}{2}}{4} \end{vmatrix}}{\Delta} =$$
$$\frac{\frac{1}{8} - \frac{\cos 1}{32} + \frac{\cos\frac{1}{2}}{32} + \frac{\sin\frac{1}{2}}{4} - \frac{\sin 1}{4}}{\Delta}$$

$$d_{21} = \frac{\begin{vmatrix} 1 - \frac{\cos 1}{4} + \frac{\cos \frac{1}{2}}{4} & \frac{1}{8} \\ \frac{\sin 1}{4} - \frac{\sin \frac{1}{2}}{4} & -1 \end{vmatrix}}{\Delta} =$$

$$\frac{\frac{\cos 1}{4} - \frac{\cos \frac{1}{2}}{4} - 1 + \frac{\sin \frac{1}{2}}{32} - \frac{\sin 1}{32}}{\Delta}$$

d_{11} and d_{21} are two coefficients of $w(t,x)$.Putting d_{11} and d_{21} in the $w(t,x)$ the following equation can be written

$$w(t,x) = \Bigg[(\frac{\frac{1}{8} - \frac{\cos 1}{32} + \frac{\cos \frac{1}{2}}{32} + \frac{\sin \frac{1}{2}}{4} - \frac{\sin 1}{4}}{\Delta}) cost +$$

$$(\frac{\frac{\cos 1}{4} - \frac{\cos \frac{1}{2}}{4} - 1 + \frac{\sin \frac{1}{2}}{32} - \frac{\sin 1}{32}}{\Delta}) \sin t + t \Bigg] \sin x.$$

Putting $v(t,x)$ and $w(t,x)$ in the $u(t,x) = v(t,x) + w(t,x)$,we obtain

$$u(t,x) = \Bigg(\frac{\frac{1}{8} + \frac{\cos 1}{32} - \frac{\cos \frac{1}{2}}{32} - \frac{\sin \frac{1}{2}}{4} + \frac{\sin 1}{4}}{\Delta} \tag{1.11}$$

$$- \frac{\frac{1}{8} + \frac{\cos 1}{32} - \frac{\cos \frac{1}{2}}{32} - \frac{\sin \frac{1}{2}}{4} + \frac{\sin 1}{4}}{\Delta} \Bigg) \cos t$$

$$+ (\frac{\frac{\cos 1}{4} + \frac{\cos \frac{1}{2}}{4} + 1 - \frac{\sin \frac{1}{2}}{32} + \frac{\sin 1}{32} - \frac{\cos 1}{4} - \frac{\cos \frac{1}{2}}{4} - 1 + \frac{\sin \frac{1}{2}}{32} - \frac{\sin 1}{32}}{\Delta}) \sin t$$

$$+ t \sin t.$$

$$u(t,x) = 0 + t \sin x.$$

$u(t,x) = t \sin x$ is the solution of the equation by the Fourier series method.

However, the method of separation of variables can be used only in the case when it has constant coefficients.

It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Now another example for a hyperbolic equation is given below.It can be solved by Laplace transformation method

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 2e^{-x} \quad ,0 < t < 1, 0 < x < \infty, \\ \\ u(0,x) = \frac{1}{4}u(1,x) - \frac{1}{4}u(\frac{1}{2},x) - \frac{3}{16}e^{-x} \quad , \\ -1 \leq t \leq 0, \ 0 < x < \infty, \\ \\ u_t(0,x) = \frac{1}{4}u_t(1,x) - \frac{1}{4}u_t(\frac{1}{2},x) - \frac{1}{4}e^{-x}, \\ -1 \leq t \leq 0, \ 0 < x < \infty, \\ \\ u_t(t,0) = t^2 \qquad u_x(t,0) = -t^2. \end{cases}$$

Taking the Laplace transform of both sides of the differential equation

$$\mathbf{L}\{u(t,x)\} = u(t,s).$$

It can be written as

$$L\{u_{tt}\} - L\{u_{xx}\} + L\{u\} = 2L\{e^{-x}\}$$

After transformation the differential equation becomes

$$u_{tt}(t,s) - s^2 L\{u\} + su(t,0) + u_x(t,0) + u(t,s) = \frac{2}{1+s}$$

$$u_{tt}(t,s) - s^2 u(t,s) + st^2 - t^2 + u(t,s) = \frac{2}{1+s}.$$

Using the UC method,where $u_c(t,s)$ is the complementary and $u_p(t,s)$ is the particular solution where

$$u(t,s) = u_c(t,s) + u_p(t,s).$$

We get

$$u_{tt}(t,s) + (1-s^2)u(t,s) = \frac{2}{1+s} + (1-s)t^2 \ .$$

Auxiliary equation is

$$u_c(t,s): m^2 = s^2 - 1 \Longrightarrow m = \pm\sqrt{s^2-1}\ ,$$

$$m^2 + \left(1 - s^2\right) = 0,$$

$$m_{1,2} = \pm\sqrt{-1+s^2}.$$

Now the complementary solution is

$$u_c(t,s) = c_1 e^{\sqrt{s^2-1}t} + c_2 e^{-\sqrt{s^2-1}t}.$$

Finding the particular solution we can write

$$u_p(t,s) = a(s)t^2 + b(s)t + c(s) \quad 2a(s) + (1-s^2)\left[a(s)t^2 + b(s)t + c(s)\right] =$$

$$\frac{2}{1+s} + (1-s)t^2,$$

$$a(s) = \frac{1}{1+s}, \; b(s) = 0 \quad , c(s) = 0$$

so the particular solution is

$$u_p(t,s) = \frac{t^2}{1+s}.$$

Thus we get $u(t,s)$ as

$$u(t,s) = c_1 e^{\sqrt{s^2-1}t} + c_2 e^{-\sqrt{s^2-1}t} + \frac{t^2}{1+s}.$$

Using the nonlocal boundary condition

$$u(0,x) = \frac{1}{4}u(1,x) - \frac{1}{4}u(\frac{1}{2},x) - \frac{3}{16}e^{-x} \quad , -1 \le t \le 0, \; 0 < x < \infty,$$

$$c_1 + c_2 = \frac{1}{4}(c_1 e^{\sqrt{s^2-1}} + c_2 e^{-\sqrt{s^2-1}t} + \frac{1}{1+s})$$
$$-\frac{1}{4}(c_1 e^{\frac{\sqrt{s^2-1}}{2}} + c_2 e^{-\frac{\sqrt{s^2-1}}{2}} + \frac{1}{4(1+s)}) - \frac{3}{16(1+s)}$$

we get

$$c_1 + c_2 = \frac{1}{4}(c_1 e^{\sqrt{s^2-1}} + c_2 e^{-\sqrt{s^2-1}}) - \frac{1}{4}(c_1 e^{\frac{\sqrt{s^2-1}}{2}} + c_2 e^{-\frac{\sqrt{s^2-1}}{2}}) \tag{1.12}$$

and using the second nonlocal boundary condition the following equation can be written as

$$u_t(0,x) = \frac{1}{4}u_t(1,x) - \frac{1}{4}u_t(\frac{1}{2},x) - \frac{1}{4}e^{-x} \ , -1 \le t \le 0 \ , \ 0 < x < \infty \ ,$$

$$c_1\sqrt{s^2-1} - c_2\sqrt{s^2-1} = \frac{\sqrt{s^2-1}}{4}(c_1 e^{\sqrt{s^2-1}} - c_2 e^{-\sqrt{s^2-1}} + \frac{2}{1+s})$$

$$-\frac{\sqrt{s^2-1}}{4}(c_1 e^{\frac{\sqrt{s^2-1}}{2}} - c_2 e^{-\frac{\sqrt{s^2-1}}{2}} + \frac{1}{1+s}) - \frac{1}{4(1+s)}$$

$$\sqrt{s^2-1}(c_1 - c_2) = \frac{\sqrt{s^2-1}}{4}(c_1 e^{\sqrt{s^2-1}} - c_2 e^{-\sqrt{s^2-1}})$$
$$-\frac{\sqrt{s^2-1}}{4}(c_1 e^{\frac{\sqrt{s^2-1}}{2}} - c_2 e^{-\frac{\sqrt{s^2-1}}{2}})$$

we get

$$(c_1 - c_2) = \frac{1}{4}(c_1 e^{\sqrt{s^2-1}} - c_2 e^{-\sqrt{s^2-1}}) - \frac{1}{4}(c_1 e^{\frac{\sqrt{s^2-1}}{2}} - c_2 e^{-\frac{\sqrt{s^2-1}}{2}}) \tag{1.13}$$

By(1.12) and (1.13)

$$c_1 + c_2 = \frac{1}{4}(c_1 e^{\sqrt{s^2-1}} + c_2 e^{-\sqrt{s^2-1}}) - \frac{1}{4}(c_1 e^{\frac{\sqrt{s^2-1}}{2}} + c_2 e^{-\frac{\sqrt{s^2-1}}{2}}) \ ,$$

$$(c_1 - c_2) = \frac{1}{4}(c_1 e^{\sqrt{s^2-1}} - c_2 e^{-\sqrt{s^2-1}}) - \frac{1}{4}(c_1 e^{\frac{\sqrt{s^2-1}}{2}} - c_2 e^{-\frac{\sqrt{s^2-1}}{2}}),$$

$$c_1 = 0 \ , \ c_2 = 0,$$

$u(t,s)$ is

$$u(t,s) = \frac{t^2}{1+s}.$$

By taking the inverse of the Laplace operator

$$L^{-1}\left\{u(t,s)\right\} = t^2 L^{-1}\left\{\frac{1}{1+s}\right\}.$$

we get

$u(t,x) = t^2 e^{-x}$ is the solution of the given nonlocal boundary value problem.

However, Laplace transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

And the last example is a problem solved by using Fourier Transform method.

Consider a nonlocal boundary value problem for hyperbolic equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = (3-4x^2)e^{-x^2} \quad ,0<t<1 \quad ,0<x<\infty, \\ \\ u(0,x) = \frac{1}{4}u(1,x) - \frac{1}{4}u(\frac{1}{2},x) + e^{-x} \ ,0<x<\infty \ ,0<t<1, \\ \\ u_t(0,x) = \frac{1}{4}u_t(1,x) - \frac{1}{4}u_t(\frac{1}{2},x) \ ,0<t<1 \quad ,0<x<\infty. \end{cases}$$

Applying the Fourier transform method to the both sides of the equation

$$F\{u_{tt}(t,x)\} - F\{u_{xx}(t,x)\} + F\{u\} = F\left\{(3-4x^2)e^{-x^2}\right\}.$$

Then, we have

$$F\{u_{tt}(t,x)\} - F\{u_{xx}(t,x)\} + F\{u\} = F\left\{(e^{-x^2})'' + e^{-x^2}\right\},$$

and the equation becomes

$$u_{tt}(t,s) + s^2 u(t,s) + u(t,s) = (s^2+1)F\left\{e^{-x^2}\right\},$$

$$u_{tt}(t,s) + (s^2+1)u(t,s) = (s^2+1)F\left\{e^{-x^2}\right\}. \tag{1.14}$$

We have that

$$u(t,s) = u_c(t,s) + u_p(t,s),$$

The auxiliary equation is

$$u_c(t,s) : m^2 + (s^2+1) = 0 \ ,$$

$$m = \pm i\sqrt{(s^2+1)},$$

$$m^2 + \left(s^2+1\right) = 0,$$

$$m_{1,2} = \pm i\sqrt{s^2+1}.$$

Now the complementary solution is

$$u_c(t,s) = c_1 \cos\sqrt{(s^2+1)}t + c_2 \sin\sqrt{(s^2+1)}t \ .$$

Using the UC method, we obtain

$$u_p(t,s) = F\left\{e^{-x^2}\right\}.$$

Using the nonlocal boundary condition

$$u(0,x) = \frac{1}{4}u(1,x) - \frac{1}{4}u(\frac{1}{2},x) + e^{-x} \ ,0 < x < \infty \ ,0 < t < 1$$

we get

$$c_1 - F\left\{e^{-x^2}\right\} = \frac{1}{4}\left[c_1 \cos\sqrt{(s^2+1)} + c_2 \sin\sqrt{(s^2+1)} + F\left\{e^{-x^2}\right\}\right]$$

$$-\frac{1}{4}\left[c_1 \cos\frac{\sqrt{(s^2+1)}}{2} + c_2 \sin\frac{\sqrt{(s^2+1)}}{2} + F\left\{e^{-x^2}\right\}\right] + F\left\{e^{-x^2}\right\}$$

$$c_1 = \frac{1}{4}\left[c_1(\cos\sqrt{(s^2+1)} - \cos\frac{\sqrt{(s^2+1)}}{2}) + c_2(\sin\sqrt{(s^2+1)} - \sin\frac{\sqrt{(s^2+1)}}{2})\right]. \tag{1.15}$$

Using the other nonlocal boundary condition

$$u_t(0,x) = \frac{1}{4}u_t(1,x) - \frac{1}{4}u_t(\frac{1}{2},x) \ ,0 < t < 1 \ ,0 < x < \infty,$$

$$c_2\sqrt{(s^2+1)} = \frac{1}{4}\left[-c_1(s^2+1)\sin\sqrt{(s^2+1)} + c_2\sqrt{(s^2+1)}\cos\sqrt{(s^2+1)}\right]$$

$$-\frac{1}{4}\left[-c_1\sqrt{(s^2+1)}\sin\frac{\sqrt{(s^2+1)}}{2}\right.$$

$$+c_2\sqrt{(s^2+1)}\cos\frac{\sqrt{(s^2+1)}}{2}\Bigg],$$

we get

$$c_2=\frac{1}{4}\Bigg[c_2(\cos\sqrt{(s^2+1)}-\cos\frac{\sqrt{(s^2+1)}}{2}) \tag{1.16}$$

$$+c_1(\sin\sqrt{(s^2+1)}-\sin\frac{\sqrt{(s^2+1)}}{2})\Bigg].$$

Solving (1.15) and (1.16) the coefficients $c_1=0$, $c_2=0$ and the complementary solution $u_c(t,s)=0$.And particular solution is

$$u_p(t,s)=\mathbf{F}\left\{e^{-x^2}\right\}$$

Putting $u_c(t,s)$ and $u_p(t,s)$ in $u(t,s)$ where

$$u(t,s)=u_c(t,s)+u_p(t,s),$$

we get

$$u(t,s)=0+\mathbf{F}\left\{e^{-x^2}\right\}.$$

Finally,taking the inverse of the Fourier transform

$$u(t,x)=\mathbf{F}^{-1}\left\{\mathbf{F}\left\{e^{-x^2}\right\}\right\},$$

$$u(t,x)=e^{-x^2}.$$

is the solution of the nonlocal boundary value problem by the Fourier transform method.

However, the Fourier transform method can be used only in the case when it has constant coefficients,it is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

In the present work we obtain the stability estimates for solution of this nonlocal boundary problem. In applications this abstract result permit us to obtain the stability estimates for the solution of nonlocal boundary value problem for the hyperbolic equations. The first and second order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem are presented. The stability estimates for the solution of these difference schemes are obtained. The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

We briefly describe the contents of the various chapters.

First chapter is the introduction.

Second chapter presents all the elementary Hilbert space theory that is needed for this work.

Third chapter consists of four sections. A brief survey of all investigations in this area can be found in the first section.Second section is devoted to the study of the stability of this nonlocal boundary value problem.We obtain the stability estimates for solution of this nonlocal boundary problem. In applications this abstract result permit us to obtain the stability estimates for the solution of nonlocal boundary value problem for the hyperbolic equations. In last two sections we describe the first and second order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem. The stability estimates for the solution of these difference schemes are obtained

Fourth chapter is the applications. The first and second order of accuracy difference schemes are studied. A Matlab program is given to conclude that the second order of accuracy is more accurate. The tables and figures are included.

Fifth chapter is the conclusions.

CHAPTER 2

ELEMENTS OF HILBERT SPACE

2.1 Hilbert Space

Definition 2.1. A complex linear space H is called an inner product space if there is a complex-valued function $\langle .,. \rangle : H \times H \to C$ with the properties

i. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = \sigma$,

ii. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in H$,

iii. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$, for all $x, y \in H$ and $\alpha \in C$,

iv. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in H$.

The function $\langle x, y \rangle$ is called the inner product of x and y. A Hilbert space is a complete inner product space. An inner product on H defines a norm on H given by $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

Example 2.1. *The space $C_2 [-1, 1]$ of all defined and continuous functions on a given closed interval $[-1, 1]$ is an inner product space with the inner product given by*

$$\langle x, y \rangle = \int_{-1}^{1} x(t)\overline{y(t)}dt.$$

Example 2.2. *The space $L_2 [-1, 1] = \overline{C_2 [-1, 1]}$ with the inner product*

$$\langle x, y \rangle = \int_{-1}^{1} x(t)\overline{y(t)}dt$$

is a Hilbert space.

2.2 Bounded Linear Operators in H

Definition 2.2. Let H_1 and H_2 are two Hilbert space. A linear operator A is an operator such that $A : H_1 \to H_2$

$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all $\alpha, \beta \in C$ and $x, y \in H_1$.

The domain of A $D(A) = \{x \in H_1, \exists Ax \in H_2\}$ is a vector space and

$R(A) = \{y = Ax, \forall x \in D(A)\}$ denotes the range of A.

A linear operator $A : H \to H$ is said to be bounded if there exist a real number $M > 0$ such that

$$\|Ax\|_H \leq M \ \|x\|_H \quad for\ all \ \ x \in H.$$

If A linear operator $A : H \to H$ is bounded with M, then

$$\|A\| = \inf M$$

is called the norm of operator A.

Example 2.3. *A bounded linear operator from $H = L_2[0,1]$ into itself is defined by*

$$Ax = tx(t) \ , \ 0 \leq t \leq 1.$$

Theorem 2.1. *The norm of the bounded linear operator A is*

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{x \neq \sigma} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| .$$

2.3 Adjoint of an Operator

Definition 2.3. Let $A : H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert space. Then the Hilbert adjoint operator A^* of A is the operator

$$A^* : H_2 \to H_1,$$

such that for all $x \in H_1$ and $y \in H_2$

$$\langle Ax, y \rangle = \langle x, A^* y \rangle .$$

Theorem 2.2. *The Hilbert adjoint operator A^* of A is unique and bounded linear operator with the norm*

$$\|A^*\| = \|A\| .$$

Definition 2.4. A bounded linear operator $A : H \longrightarrow H$ on a Hilbert space H is said to be self-adjoint if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$.

Definition 2.5. A self-adjoint operator A is said to be positive if $A \geq 0$, that is $(Ax, x) \geq 0$ for all $x \in H$.

Definition 2.6. Let $A : D(A) \longrightarrow H$ be a linear operator with $\overline{D(A)} = H$. Then

A is called a symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D(A)$.

If A is symmetric and $D(A) = D(A^*)$, then A is a self-adjoint operator.

Example 2.4. *Let* $Au = -\dfrac{d^2u}{dx^2} + u$, $u(a) = u(b) = 0$ *and* $H = L_2[a, b]$. *A is a self-adjoint positive operator.*

2.4 Spectrum

Definition 2.7. Let H be a Hilbert space and $A : H \longrightarrow H$ be a linear operator with $D(A) \subset H$. We associate the operator $A_\lambda = A - \lambda I$, where $\lambda \in C$ and I is the identity operator on $D(A)$.

If A_λ has an inverse, we denote it by $R_\lambda(A)$ and we call it the resolvent operator of A, or simply, resolvent of A.

$$R_\lambda(A) = (A - \lambda I)^{-1}.$$

Definition 2.8. (Regular value, resolvent set, spectrum)

Let A be a linear operator with the $D(A) \subset H$ and H is a Hilbert space. A regular value λ of A is a complex number such that

$(R1)$ $R_\lambda(A)$ exists.

$(R2)$ $R_\lambda(A)$ is bounded.

$(R3)$ $R_\lambda(A)$ is defined on a set which is dense in H.

The resolvent set $\rho(A)$ of A is the set of all regular values of A. Its complement $\sigma(A) = C - \rho(A)$ is called spectrum of A , and a $\lambda \in \sigma(A)$ is called spectral value of A. Furthermore, the spectrum $\rho(A)$ is partitioned into three disjoint sets as follows.

The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_\lambda(A)$ does not exist. A $\lambda \in \sigma(A)$ is called an eigenvalue of A.

The continuous spectrum $\sigma_c(A)$ is the set such that $R_\lambda(A)$ exists and satisfies $(R3)$ but not $(R2)$, that is $R_\lambda(T)$ unbounded.

The residual spectrum $\sigma_r(A)$ is the set such that $R_\lambda(A)$ exists (and may be bounded or not) but does not satisfy $(R3)$, that is the domain of $R_\lambda(A)$ is not dense in H.

If $A_\lambda x = (A - \lambda I)x = 0$ for some $x \neq 0$, then $\lambda \in \sigma_p(A)$, by definition, that is, λ is an eigenvalue of A.

The vector x is called an eigenvector of A corresponding to eigenvalue λ. The subspace of $D(A)$ consisting of 0 and all eigenvectors of A corresponding to an eigenvalue λ of A is called the eigenspace of A corresponding to that eigenvalue λ.

$$\sigma(A) = \sigma_c(A) \cup \sigma_p(A) \cup \sigma_r(A),$$

$$\sigma(A) \cup \rho(A) = C.$$

Definition 2.9. Let H be a Hilbert space over the field of real numbers and for any $x \in H$, let $\|x\|$ denote the norm of x. Let J be any interval of the real line R. A function $x : J \to H$ is called an abstract function.

A function $x(t)$ is said to be continuous at the point $t_0 \in J$,

if

$$\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0;$$

if $x : J \to H$ is continuous at each point of J, Then we say that x is continuous on J and we write $x \in C[J, H]$.

Definition 2.10. The Stieltjes integral of a function $x : [a, b] \to H$ with respect to a function $y : [a, b] \to H_1$. Let H, H_1, H_2 be three Hilbert space. A bilinear operator $P : H \times H_1 \to H_2$ whose norm is less than or equal to 1, that is,

$$\|P(x, y)\| \leq \|x\| \| y\|,$$

is called a product operator. We shall agree to write $P(x, y) = xy$. Let $x : [a, b] \to H$ and $y : [a, b] \to H_1$ be two bounded functions such that the product $x(t)y(t) \in H_2$, for each $t \in [a, b]$ is linear in both x and y and

$$\|x(t)y(t)\| \leq \|x(t)\| \, \|y(t)\|$$

(for example, $x(t) = A(t)$ is an operator with domain $D[A(t)] \supset H_1$, or one of the function x,y is a scalar function).

We denote the partition $(a = t_0 < t_1 < t_2 < ... < t_n = b)$ together with the points

τ_i $(t_i < \tau_1 < t_{i+1}, i = 0, 1, 2, ..., n-1)$ by π and set $|\pi|$=$\max_i |t_{i+1} - t_i|$. We form the Stieltjes sum

$$S_\pi = \sum_{i=1}^{n-1} x(\tau_i)[y(t_{i+1}) - y(t_i)].$$

If the $\lim S_\pi$ exist as $|\pi| \to 0$ and defines an element I in H_2 independent of π, then I is called the Stieltjes integral of the function $x(t)$ by the function $y(t)$, and is denoted by

$$\int_a^b x(t)\, dy(t).$$

Theorem 2.3. *If $x \in C[[a,b], H]$ and $y : [a,b] \to H_1$ is of bounded variation on $[a,b]$, then the Stieltjes integral*

$$\int_a^b x(t)\, dy(t) \quad exists.$$

Consider the function $y : [a,b] \to H_1$ and the partition

$$\pi : a = t_0 < t_1 < t_2 < ... < t_n = b.$$

Form the sum

$$V = \sum_{i=1}^{n-1} \|y(t_{i+1}) - y(t_i)\|.$$

The least upper bound of the set of all possible sums V is called the (strong) total variation of the function $y(t)$ on the interval $[a,b]$ and is denoted by $V_a^b(y)$. If $V_a^b(y) < \infty$, then $y(t)$ is called an abstract function of bounded variation on $[a,b]$.

Example 2.5. *If $x \in C[[a,b], H]$ and $y : [a,b] \to H_1$ is of bounded variation on $[a,b]$, then*

$$\left\| \int_a^b x(t)\, dy(t) \right\| \le \int_a^b \|x(t)\|\, dV_a^t[y(t)] \le \max_{t \in [a,b]} \|x(t)\|\, V_a^b[y(t)].$$

2.5 Projection Operator. Spectral Family

Definition 2.11. A Hilbert space H is represented as the direct sum of a closed subspace Y and its orthogonal complement $Y^{\perp}$:

$$H = Y \oplus Y^{\perp} \tag{2.1}$$

$$x = y + z \quad , \qquad where \;\; y \in Y, z \in Y^{\perp}.$$

Since the sum is direct, y is unique for any given $x \in H$. Hencethe spectral representation of unit matrix defines a linear operator

$$P : H \longrightarrow H,$$

$$x \longrightarrow y = Px.$$

P is called an orthogonal projection or projection on H.

Theorem 2.4. *A bounded linear operator $P : H \longrightarrow H$ on a Hilbert space H is projection if and only if P is self-adjoint and idempotent that is, $P^2 = P$.*

Spectral family from dimensional case as follows: If matrix A has n different eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 ... < \lambda_n$. then A has an orthogonal set of n vectors $x_1, x_2, x_3, ..., x_{n,}$where x_j corresponds to λ_j and we write these vectors as column vectors, for convenience. This basis for H, has a unique representation:

$$x = \sum_{j=1}^{n} \gamma_j x_j \ , \qquad \gamma_j = (x, x_j) = x^T \overline{x_j} \ , \tag{2.2}$$

x_j is an eigenvector of A, so that we have $Ax_j = \lambda_j x_j$.

$$Ax = \sum_{j=1}^{n} \lambda_i \gamma_j x_j. \tag{2.3}$$

We can define an operator

$$P_j : H \longrightarrow H,$$

$$x \longrightarrow \gamma_j x_j \ .$$

Obviously, P_j is the projection (orthogonal projection) of H onto the eigenspace of A corresponding to λ_j .

Theorem 2.5. ***Spectral Theorem:*** *Let $A : H \longrightarrow H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then there exists a family of orthogonal projection $\{E(\lambda)\}, \lambda \in R$ such that*

$$\lambda_1 \leq \lambda_2 \text{ implies that } E(\lambda_1)E(\lambda_1) = E(\lambda_2)E(\lambda_1) = E(\lambda_1);$$

$$E(\lambda+\varepsilon) \to E(\lambda) \quad (\text{strongly}) \text{ as } \varepsilon \to 0^+;$$

$$E(\lambda) \to 0 \quad (\text{strongly}) \text{ as } \lambda \to -\infty,$$

and

$$E(\lambda) \to I \quad (\text{strongly}) \text{ as } \lambda \to +\infty;$$

a) A has the spectral representation

$$A = \int_{m-0}^{M} \lambda dE_\lambda,$$

where E_λ is the spectral family associate with A; the integral is to be understood in the sense of uniform operator convergence, and for all $x, y \in H$.

$$\langle Ax, y\rangle = \int_{m-0}^{M} \lambda dw(\lambda) \qquad w(\lambda) = \langle E_\lambda x, y\rangle$$

where the integral is an ordinary Riemann-Stieltjes integral.

b) If P is a polynomial in λ with real coefficients,

$$P(\lambda) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + ... + \alpha_0$$

then the operator $P(A)$ defined by

$$P(A) = \alpha_n A^n + \alpha_{n-1}A^{n-1} + ... + \alpha_0 I$$

has the spectral representation

$$P(A) = \int_{m-0}^{M} P(\lambda)\, dE_\lambda$$

and for all $x, y \in H$.

Theorem 2.6. *Let $A : D(A) \longrightarrow H$ be a self-adjoint linear operator, where H is a complex Hilbert space and $D(A)$ is dense in H. Then A has the spectral representation*

$$A = \int_{m}^{\infty} \lambda dE_\lambda \qquad \text{and} \qquad I = \int_{m}^{\infty} dE_\lambda.$$

If F is the continuously bounded function on $[m, \infty]$, then

$$F(A) = \int_m^M F(\lambda)\, dE_\lambda.$$

Note that, from theorem 2.6 and property of E_λ and Stieltjes integral it follows

$$\|F(A)x\| \leq \int_m^\infty |f(\lambda)|\, d\|E_\lambda x\| \leq \int_m^\infty |f(\lambda)|\, dE_\lambda \|x\|$$

$$\leq \sup_{m \leq \lambda < \infty} |f(\lambda)| \int_m^\infty dE_\lambda \|x\|$$

$$\|F(A)x\| \leq \sup_{m \leq \lambda < \infty} |f(\lambda)|\, \|x\|$$

$$\|F(A)\| \leq \sup_{m \leq \lambda < \infty} |f(\lambda)|.$$

Example 2.6. *A is an operator defined on the example 2.5. Show that*

$$\|\exp(-At)\| \leq e^{-t}, \tag{2.4}$$

$$\left\|\cos\left(A^{1/2}t\right)\right\| \leq 1, \quad \left\|A^{1/2}\sin\left(A^{1/2}t\right)\right\| \leq 1.$$

Solution. *Using the spectral representation of the self-adjoint positive defined operators we can write*

$$\exp(-At)\varphi = \int_1^\infty \exp(-\mu t) dE_\mu \varphi,$$

where (E_μ) is the spectral family associated with A. Therefore, for any $t \geq 0$ we have that

$$\|\exp(-At)\|_{H \to H} \leq \sup_{1 \leq \mu < \infty} |\exp(-\mu t)| = \exp(-t).$$

The estimate (2.4) is proved. Using the spectral representation of the self-adjoint positive defined operators we can write

$$e^{\pm i A^{1/2} t}\varphi = \int_1^\infty e^{\pm i t \mu^{1/2}} dE_\mu \varphi.$$

Therefore, using the last theorem we obtain

$$\left\|e^{\pm i A^{1/2} t}\right\| \leq \sup_{1 \leq \mu < \infty} |e^{\pm i t \mu^{1/2}}| = 1.$$

So,

$$\left\|\cos\left(A^{1/2}t\right)\right\| = \left\|\frac{e^{iA^{1/2}t} + e^{-iA^{1/2}t}}{2}\right\|$$

$$\leq \frac{1}{2}\left[\left\|e^{iA^{1/2}t}\right\| + \left\|e^{-iA^{1/2}t}\right\|\right] \leq 1$$

and

$$\left\|A^{1/2}\sin\left(A^{1/2}t\right)\right\| = \left\|\frac{e^{iA^{1/2}t} - e^{-iA^{1/2}t}}{2i}\right\|$$

$$\leq \frac{1}{2\left|i\right|}\left[\left\|e^{iA^{1/2}t}\right\| + \left\|e^{-iA^{1/2}t}\right\|\right] \leq 1.$$

CHAPTER 3

DIFFERENCE SCHEMES OF NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

3.1 The Problem

In the papers [Ashyralyev, A. and Aggez, N., 2004] - [Ashyralyev, A. and Aggez, N., 2003] the first and second order difference schemes generated by the integer power of A for approximately solving the following nonlocal boundary-value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \\ \\ u(0) = \alpha u(1) + \varphi, \ u'(0) = \beta u'(1) + \psi \end{cases} \tag{3.1}$$

was presented. The stability estimates for the solutions of the nonlocal boundary-value problem (3.1) and of difference schemes were established. In applications, the stability estimates for the solutions of the difference schemes of the mixed type nonlocal boundary-value problems for hyperbolic equations were obtained.

We are interested in studying the stability of solutions of the problem (3.3) under the assumption

$$\sum_{k=1}^{n} |\alpha_k + \beta_k| + \sum_{m=1}^{n} |\alpha_m| \sum_{\substack{k=1 \\ k \neq m}}^{n} |\beta_k| < |1 + \sum_{k=1}^{n} \alpha_k \beta_k| \ . \tag{3.2}$$

It is known that various nonlocal boundary value problems for the hyperbolic equations can be reduced to the boundary value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \\ u(0) = \sum\limits_{r=1}^{n} \alpha_r u(\lambda_r) + \varphi, u_t(0) = \sum\limits_{r=1}^{n} \beta_r u_t(\lambda_r) + \psi, \\ \sum\limits_{k=1}^{n} |\alpha_k + \beta_k| + \sum\limits_{k=1}^{n} |\alpha_k| \sum\limits_{\substack{m=1 \\ m \neq k}}^{n} |\beta_m| < |1 + \sum\limits_{k=1}^{n} \alpha_k \beta_k|, \\ 0 < \lambda_1 < \lambda_2 < ... < \lambda_k \leq 1, \end{cases} \tag{3.3}$$

for differential equation in a Hilbert space H with self -adjoint positive definite operator A.

In this work the stability estimates for the solution of the problem (3.3) are obtained. The first order of accuracy difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_{k+1}=f_k,\ \ f_k=f(t_k), \\ t_k=k\tau,\ 1\le k\le N-1,\ N\tau=1, \\ u_0=\sum\limits_{r=1}^{n}\alpha_r u_{\left[\frac{\lambda_r}{\tau}\right]}+\varphi, \\ \tau^{-1}(u_1-u_0)=\sum\limits_{r=1}^{n}\beta_r\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]}\right)\frac{1}{\tau}+\psi. \end{cases} \tag{3.4}$$

and the second order of accuracy of the following difference schemes

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k, \\ f_k=f(t_k),\quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ u_0=\sum\limits_{\frac{\lambda_r}{\tau}\in Z}\alpha_r u_{\frac{\lambda_r}{\tau}}+\sum\limits_{\frac{\lambda_r}{\tau}\notin Z}\alpha_r\left(u_{\left[\frac{\lambda_r}{\tau}\right]}+\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]}\right)\left(\frac{\lambda_r}{\tau}-\left[\frac{\lambda_r}{\tau}\right]\right)\right)+\varphi, \\ (I+\frac{\tau^2A}{2})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0) \\ =\sum\limits_{r\neq n,\frac{\lambda_r}{\tau}\notin Z}\alpha_r\left(\frac{1}{2\tau}\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]-1}\right)+\left(f_{\left[\frac{\lambda_r}{\tau}\right]}-Au_{\left[\frac{\lambda_r}{\tau}\right]}\right)\left(\frac{\lambda_r}{\tau}-\left[\frac{\lambda_r}{\tau}\right]\right)\right) \\ +\alpha_n\left(\frac{1}{2\tau}\left(3u_{\left[\frac{\lambda_n}{\tau}\right]}-4u_{\left[\frac{\lambda_n}{\tau}\right]-1}+u_{\left[\frac{\lambda_n}{\tau}\right]-2}\right)+\left(f_{\left[\frac{\lambda_n}{\tau}\right]}-Au_{\left[\frac{\lambda_n}{\tau}\right]}\right)\left(\frac{\lambda_n}{\tau}-\left[\frac{\lambda_n}{\tau}\right]\right)\right) \\ +\sum\limits_{r\neq n,\frac{\lambda_r}{\tau}\in Z}\alpha_r\frac{1}{2\tau}\left(u_{\frac{\lambda_r}{\tau}+1}-u_{\frac{\lambda_r}{\tau}-1}\right)++\alpha_n\frac{1}{2\tau}\left(u_{\frac{\lambda_n}{\tau}}-4u_{\frac{\lambda_n}{\tau}-1}+3u_{\frac{\lambda_n}{\tau}-2}\right)+\psi, \\ f_0=f(0), f_N=f(1), \end{cases} \tag{3.5}$$

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k, \\ f_k=f(t_k),\quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ u_0=\sum\limits_{\frac{\lambda_r}{\tau}\in Z}\alpha_r u_{\frac{\lambda_r}{\tau}}+\sum\limits_{\frac{\lambda_r}{\tau}\notin Z}\alpha_r\left(u_{\left[\frac{\lambda_r}{\tau}\right]}+\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]}\right)\left(\frac{\lambda_r}{\tau}-\left[\frac{\lambda_r}{\tau}\right]\right)\right)+\varphi, \\ (I+\frac{\tau^2A}{4})[(I+\frac{\tau^2A}{4})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)] \\ =\sum\limits_{r\neq n,\frac{\lambda_r}{\tau}\notin Z}\alpha_r\left(\frac{1}{2\tau}\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]-1}\right)+\left(f_{\left[\frac{\lambda_r}{\tau}\right]}-Au_{\left[\frac{\lambda_r}{\tau}\right]}\right)\left(\frac{\lambda_r}{\tau}-\left[\frac{\lambda_r}{\tau}\right]\right)\right) \\ +\alpha_n\left(\frac{1}{2\tau}\left(3u_{\left[\frac{\lambda_n}{\tau}\right]}-4u_{\left[\frac{\lambda_n}{\tau}\right]-1}+u_{\left[\frac{\lambda_n}{\tau}\right]-2}\right)+\left(f_{\left[\frac{\lambda_n}{\tau}\right]}-Au_{\left[\frac{\lambda_n}{\tau}\right]}\right)\left(\frac{\lambda_n}{\tau}-\left[\frac{\lambda_n}{\tau}\right]\right)\right) \\ +\sum\limits_{r\neq n,\frac{\lambda_r}{\tau}\in Z}\alpha_r\frac{1}{2\tau}\left(u_{\frac{\lambda_r}{\tau}+1}-u_{\frac{\lambda_r}{\tau}-1}\right)++\alpha_n\frac{1}{2\tau}\left(3u_{\frac{\lambda_n}{\tau}}-4u_{\frac{\lambda_n}{\tau}-1}+u_{\frac{\lambda_n}{\tau}-2}\right)+\psi, \\ f_0=f(0), f_N=f(1). \end{cases} \tag{3.6}$$

for approximately solving the problem (3.3) are presented . The stability estimates for the solution of these difference schemes were obtained.

The well-posedness of the nonlocal boundary value problems for parabolic ,elliptic equations and equations of mixed types have been studied extensively, see for instance

[Salahatdinov M. S., 1974] - [Ashyralyev A. and Kendirli B., 2001] and the references therein.

3.2 The Differential Hyperbolic Equation

A function $u(t)$ is called a *solution* of the problem (3.3) if the following conditions are satisfied:

i) $u(t)$ is twice continuously differentiable on the segment $[0,1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$ and the function $Au(t)$ is continuous on the segment $[0,1]$.

iii) $u(t)$ satisfies the equation and the nonlocal boundary conditions (3.3).

In the paper [AshyralyevA. Martinez M., Paster J. and Piskarev S., 2007] the nonlocal boundary value problem (3.3) in the cases $\alpha_j = 0,\ j = 2, \cdots, n$ and $\beta_j = 0,\ j = 2, \cdots, n$, $\lambda_1 = 1$ was considered. The following theorem on the stability was proved.

Let H be a Hilbert space, A be a positive definite self-adjoint operator with $A \geq \delta I$, where $\delta > \delta_0 > 0$. Throughout this paper, $\{c(t), t \geq 0\}$ is a strongly continuous cosine operator-function defined by the formula

$$c\left(t\right) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}.$$

Then, from the definition of the sine operator-function $s\left(t\right)$

$$s(t)u = \int_0^t c(s)u \, ds$$

it follows that

$$s\left(t\right) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.$$

For the theory of cosine operator-function we refer to [Fattorini H.O., 1985] and [Piskarevs. and Shaw Y., 1997] .

Now, let us give some lemmas that will be needed below.

Lemma 2.1. The estimates hold:

$$\|c(t)\|_{H\to H} \leq 1, \left\|A^{1/2}s(t)\right\|_{H\to H} \leq 1. \tag{3.7}$$

Lemma 2.2. Suppose that the assumption (3.2) holds. Then, the operator

$$I-\sum_{k=1}^{n}\beta_k c(\lambda_k)-\sum_{m=1}^{n}\alpha_m c(\lambda_m)+\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(c(\lambda_m)c(\lambda_k)+As(\lambda_m)s(\lambda_k)\right)$$

has an inverse

$$T=\left(I-\sum_{k=1}^{n}\beta_k c(\lambda_k)-\sum_{m=1}^{n}\alpha_m c(\lambda_m)+\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(c(\lambda_m)c(\lambda_k)+As(\lambda_m)s(\lambda_k)\right)\right)^{-1}$$

and the following estimate is satisfied:

$$||T||_{H\to H}\leq\frac{1}{|1+\sum\limits_{k=1}^{n}\alpha_k\beta_k|-\sum\limits_{k=1}^{n}|\alpha_k+\beta_k|-\sum\limits_{m=1}^{n}|\alpha_m|\sum\limits_{\substack{k=1\\k\neq m}}^{n}|\beta_k|}. \tag{3.8}$$

Proof. The proof of estimate (3.8) is based on the estimate

$$\left\|I-\sum_{k=1}^{n}\beta_k c(\lambda_k)-\sum_{m=1}^{n}\alpha_m c(\lambda_m)+\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(c(\lambda_m)c(\lambda_k)+As(\lambda_m)s(\lambda_k)\right)\right\|_{H\to H}$$

$$\geq|1+\sum_{k=1}^{n}\alpha_k\beta_k|-\sum_{k=1}^{n}|\alpha_k+\beta_k|-\sum_{m=1}^{n}|\alpha_m|\sum_{\substack{k=1\\k\neq m}}^{n}|\beta_k|.$$

From the definitions of $c(\lambda_j)$ and $s(\lambda_j)$ $(\lambda_j, j=1,\cdot\cdot\cdot,n)$ it follows that

$$I-\sum_{k=1}^{n}\beta_k c(\lambda_k)-\sum_{m=1}^{n}\alpha_m c(\lambda_m)+\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(c(\lambda_m)c(\lambda_k)+As(\lambda_m)s(\lambda_k)\right)$$

$$I+\sum_{k=1}^{n}\alpha_k\beta_k I-\sum_{k=1}^{n}(\alpha_k+\beta_k)c(\lambda_k)+\sum_{m=1}^{n}\sum_{\substack{k=1\\k\neq m}}^{n}\alpha_m\beta_k c(\lambda_m-\lambda_k).$$

Then, using the triangle inequality and estimate (3.7), we obtain

$$\left\|I-\sum_{k=1}^{n}\beta_k c(\lambda_k)-\sum_{m=1}^{n}\alpha_m c(\lambda_m)+\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(c(\lambda_m)c(\lambda_k)+As(\lambda_m)s(\lambda_k)\right)\right\|_{H\to H}$$

$$\geq|1+\sum_{k=1}^{n}\alpha_k\beta_k|-\sum_{k=1}^{n}|\alpha_k+\beta_k|\,\|c(\lambda_k)\|_{H\to H}-\sum_{m=1}^{n}\sum_{\substack{k=1\\k\neq m}}^{n}|\alpha_m||\beta_k|\,\|c(\lambda_m-\lambda_k)\|_{H\to H}$$

$$\geq|1+\sum_{k=1}^{n}\alpha_k\beta_k|-\sum_{k=1}^{n}|\alpha_k+\beta_k|-\sum_{m=1}^{n}|\alpha_m|\sum_{\substack{k=1\\k\neq m}}^{n}|\beta_k|.$$

Lemma 2.2 is proved.

Now, we will obtain the formula for solution of the problem (3.3). It is clear that [Fattorini H.O.] the initial value problem

$$\frac{d^2u}{dt^2}+Au(t)=f(t), 0<t<1, u(0)=u_0, u'(0)=u_0'$$

has a unique solution

$$u(t)=c(t)u_0+s(t)u_0'+\int_0^t s(t-s)f(s)ds. \tag{3.9}$$

Using (3.9) and the nonlocal boundary conditions

$$u(0)=\sum_{m=1}^n \alpha_m u(\lambda_m)+\varphi, \quad u^{'}(0)=\sum_{k=1}^n \beta_k u^{'}(\lambda_k)+\psi,$$

it can be written as follows

$$\begin{cases} u(0)=\sum\limits_{m=1}^n \alpha_m \left\{c(\lambda_m)u(0)+s(\lambda_m)u^{'}(0)+\int\limits_0^{\lambda_m} s\left(\lambda_m-s\right)f(s)ds\right\}+\varphi, \\ u^{'}(0)=\sum\limits_{k=1}^n \beta_k \left\{-As(\lambda_k)u(0)+c(\lambda_k)u^{'}(0)+\int\limits_0^{\lambda_k} c\left(\lambda_k-s\right)f(s)ds\right\}+\psi. \end{cases} \tag{3.10}$$

Denoting

$$\Delta=\begin{vmatrix} I-\sum\limits_{m=1}^n \alpha_m c(\lambda_m) & -\sum\limits_{m=1}^n \alpha_m s(\lambda_m) \\ \sum\limits_{k=1}^n \beta_k As(\lambda_k) & I-\sum\limits_{k=1}^n \beta_k c(\lambda_k) \end{vmatrix},$$

and using the definitions of $c\left(\lambda_j\right)$ and $s\left(\lambda_j\right)$ $\left(\lambda_j, j=1,\cdots,n\right)$, we can write

$$\Delta=\left(I-\sum_{m=1}^n \alpha_m c(\lambda_m)\right)\left(I-\sum_{k=1}^n \beta_k c(\lambda_k)\right)+A\sum_{k=1}^n\sum_{k=1}^n \alpha_m\beta_k s(\lambda_m)s(\lambda_k)$$

$$=I-\sum_{k=1}^n \beta_k c(\lambda_k)-\sum_{m=1}^n \alpha_m c(\lambda_m)+\sum_{m=1}^n\sum_{k=1}^n \alpha_m\beta_k c(\lambda_m)c(\lambda_k)$$

$$+A\sum_{m=1}^n\sum_{k=1}^n \alpha_m\beta_k s(\lambda_m)s(\lambda_k),$$

$$=I-\sum_{k=1}^n \beta_k c(\lambda_k)-\sum_{m=1}^n \alpha_m c(\lambda_m)$$

$$+\sum_{m=1}^n\sum_{k=1}^n \alpha_m\beta_k\left(c(\lambda_m)c(\lambda_k)+As(\lambda_m)s(\lambda_k)\right).$$

Then, using the definition of the operator T, we obtain

$$T = \Delta^{-1}.$$

Solving system (3.10), we get

$$u(0) = T \begin{vmatrix} \sum\limits_{m=1}^{n} \alpha_m \int\limits_0^{\lambda_m} s(\lambda_m - s) f(s) ds + \varphi & -\sum\limits_{m=1}^{n} \alpha_m s(\lambda_m) \\ \psi + \sum\limits_{k=1}^{n} \beta_k \int\limits_0^{\lambda_k} c(\lambda_k - s) f(s) ds & I - \sum\limits_{k=1}^{n} \beta_k c(\lambda_k) \end{vmatrix} \tag{3.11}$$

$$= T \left\{ \left(I - \sum_{k=1}^{n} \beta_k c(\lambda_k) \right) \left(\sum_{m=1}^{n} \alpha_m \int_0^{\lambda_m} s(\lambda_m - s) f(s) ds + \varphi \right) \right.$$

$$\left. + \sum_{m=1}^{n} \alpha_m s(\lambda_m) \left(\psi + \sum_{k=1}^{n} \beta_k \int_0^{\lambda_k} c(\lambda_k - s) f(s) ds \right) \right\},$$

$$u'(0) = T \begin{vmatrix} I - \sum\limits_{m=1}^{n} \alpha_m c(\lambda_m) & \sum\limits_{m=1}^{n} \alpha_m \int\limits_0^{\lambda_m} s(\lambda_m - s) f(s) ds + \varphi \\ \sum\limits_{k=1}^{n} \beta_k A s(\lambda_k) & \sum\limits_{k=1}^{n} \beta_k \int\limits_0^{\lambda_k} c(\lambda_k - s) f(s) ds + \psi \end{vmatrix} \tag{3.12}$$

$$= T \left\{ \left(I - \sum_{m=1}^{n} \alpha_m c(\lambda_m) \right) \left(\sum_{k=1}^{n} \beta_k \int_0^{\lambda_k} c(\lambda_k - s) f(s) ds + \psi \right) \right.$$

$$\left. - A \sum_{k=1}^{n} \beta_k s(\lambda_k) \left(\sum_{m=1}^{n} \alpha_m \int_0^{\lambda_m} s(\lambda_m - s) f(s) ds + \varphi \right) \right\}.$$

Consequently, if the function $f(t)$ is not only continuous, but also continuously differentiable on [0,1], $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$ and formulas (3.9), (3.11), (3.12) give a solution of the problem (3.3).

Theorem 3.7. *Suppose that* $\varphi \in D(A)$,$\psi \in D(A^{\frac{1}{2}})$ *and* $f(t)$ *are continuously differentiable on* $[0,1]$ *function and* $\sum\limits_{k=1}^{n} |\alpha_k + \beta_k| + \sum\limits_{k=1}^{n} |\alpha_k| \sum\limits_{\substack{m=1 \\ m \neq k}}^{n} |\beta_m| < |1 + \sum\limits_{k=1}^{n} \alpha_k \beta_k|$. *Then there is a unique solution of the problem (3.3) and the stability inequalities*

$$\max_{0 \leq t \leq 1} \| u(t) \|_H \leq M \left[\| \varphi \|_H + \| A^{-1/2} \psi \|_H + \max_{0 \leq t \leq 1} \| A^{-1/2} f(t) \|_H \right], \tag{3.13}$$

$$\max_{0 \leq t \leq 1} \| A^{1/2} u(t) \|_H \leq M \left[\| A^{1/2} \varphi \|_H + \| \psi \|_H + \max_{0 \leq t \leq 1} \| f(t) \|_H \right], \tag{3.14}$$

$$\max_{0\le t\le 1}\parallel \frac{d^2u(t)}{dt^2}\parallel_H + \max_{0\le t\le 1}\parallel Au(t)\parallel_H \le M[\parallel A\varphi\parallel_H + \parallel A^{1/2}\psi\parallel_H \tag{3.15}$$

$$+\parallel f(0)\parallel_H + \int_0^1 \parallel f'(t)\parallel_H dt]$$

hold, where M does not depend on $f(t)$, $t\in[0,1]$, and φ,ψ.

Proof. Using formula (3.3) and estimates (3.7), (3.8), we obtain

$$\|u(t)\|_H \le \|c(t)\|_{H\to H}\|T\|_{H\to H}\left\{\left(1+\sum_{k=1}^n|\beta_k|\,\|c(\lambda_k)\|_{H\to H}\right)\right.$$

$$\times\sum_{m=1}^n|\alpha_m|\left(\int_0^{\lambda_m}\left\|A^{\frac{1}{2}}s\left(\lambda_m-s\right)\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds+\|\varphi\|_H\right)$$

$$+\sum_{m=1}^n|\alpha_m|\left\|A^{\frac{1}{2}}s(\lambda_m)\right\|_{H\to H}$$

$$\left.\times\left(\|A^{-\frac{1}{2}}\psi\|_H+\sum_{k=1}^n|\beta_k|\int_0^{\lambda_k}\|c\left(\lambda_k-s\right)\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds\right)\right\}$$

$$+\left\|A^{\frac{1}{2}}s(t)\right\|_{H\to H}\|T\|_{H\to H}\left\{\left(1+\sum_{m=1}^n|\alpha_m|\,\|c(\lambda_m)\|_{H\to H}\right)\right.$$

$$\times\left(\sum_{k=1}^n|\beta_k|\int_0^{\lambda_k}\|c\left(\lambda_k-s\right)\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds+\|A^{-\frac{1}{2}}\psi\|_H\right)$$

$$+\left(\sum_{k=1}^n|\beta_k|\left\|A^{\frac{1}{2}}s\left(\lambda_k\right)\right\|_{H\to H}\right)$$

$$\left.\times\left(\sum_{m=1}^n|\alpha_m|\int_0^{\lambda_m}\left\|A^{\frac{1}{2}}s\left(\lambda_m-s\right)\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds+\|\varphi\|_H\right)\right\}$$

$$+\int_0^t\left\|A^{\frac{1}{2}}s\left(t-s\right)\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds$$

$$\le M\left[\parallel\varphi\parallel_H+\parallel A^{-1/2}\psi\parallel_H+\max_{0\le t\le 1}\parallel A^{-1/2}f(t)\parallel_H\right].$$

Therefore, estimate (3.13) is proved.

Applying $A^{\frac{1}{2}}$ to formula (3.3) and using estimates (3.7) and (3.8), we obtain

$$\|A^{\frac{1}{2}}u(t)\|_H \le \|c(t)\|_{H\to H}\|T\|_{H\to H}\left\{\left(1+\sum_{k=1}^n|\beta_k|\,\|c(\lambda_k)\|_{H\to H}\right)\right.$$

$$\times\left(\sum_{m=1}^{n}|\alpha_m|\int_0^{\lambda_m}\left\|A^{\frac{1}{2}}s\left(\lambda_m-s\right)\right\|_{H\to H}\|f(s)\|_H\,ds+\|A^{\frac{1}{2}}\varphi\|_H\right)+\sum_{m=1}^{n}|\alpha_m|\left\|A^{\frac{1}{2}}s(\lambda_m)\right\|_{H\to H}$$

$$\times\left(\|\psi\|_H+\sum_{k=1}^{n}|\beta_k|\int_0^{\lambda_k}\|c\left(\lambda_k-s\right)\|_{H\to H}\|f(s)\|_H\,ds\right)\Bigg\}$$

$$+\left\|A^{\frac{1}{2}}s(t)\right\|_{H\to H}\|T\|_{H\to H}\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\,\|c(\lambda_m)\|_{H\to H}\right)\right.$$

$$\times\left(\sum_{k=1}^{n}|\beta_k|\int_0^{\lambda_k}\|c\left(\lambda_k-s\right)\|_{H\to H}\|f(s)\|_H\,ds+\|\psi\|_H\right)+\left(\sum_{k=1}^{n}|\beta_k|\left\|A^{\frac{1}{2}}s\left(\lambda_k\right)\right\|_{H\to H}\right)$$

$$\left.\times\left(\sum_{m=1}^{n}|\alpha_m|\int_0^{\lambda_m}\left\|A^{\frac{1}{2}}s\left(\lambda_m-s\right)\right\|_{H\to H}\|f(s)\|_H\,ds+\|A^{\frac{1}{2}}\varphi\|_H\right)\right\}$$

$$+\int_0^{t}\left\|A^{\frac{1}{2}}s\left(t-s\right)\right\|_{H\to H}\|f(s)\|_H\,ds\le M\left[\|\,A^{1/2}\varphi\,\|_H+\|\,\psi\,\|_H+\max_{0\le t\le 1}\|\,f(t)\,\|_H\right].$$

Thus, estimate (3.14) is proved.

Now, we obtain the estimate for $\|\,Au(t)\,\|_H$.Using the integration by parts and applying A to formula (3.3), we can write the formula

$$Au(t)=c(t)T\left\{\left(I-\sum_{k=1}^{n}\beta_k c(\lambda_k)\right)\right. \tag{3.16}$$

$$\times\left(\sum_{m=1}^{n}\alpha_m\left(f\left(\lambda_m\right)-c(\lambda_m)f\left(0\right)-\int_0^{\lambda_m}c\left(\lambda_m-s\right)f'(s)ds\right)+A\varphi\right)$$

$$\left.+\sum_{m=1}^{n}\alpha_m As(\lambda_m)\left(\sum_{k=1}^{n}\beta_k\left(s(\lambda_k)f\left(0\right)+\int_0^{\lambda_k}s\left(\lambda_k-s\right)f'(s)ds\right)+\psi\right)\right\}$$

$$+As(t)T\left\{\left(I-\sum_{m=1}^{n}\alpha_m c(\lambda_m)\right)\right.$$

$$\times\left(\sum_{k=1}^{n}\beta_k\left(s(\lambda_k)f\left(0\right)+\int_0^{\lambda_k}s\left(\lambda_k-s\right)f'(s)ds\right)+\psi\right)-\left(\sum_{k=1}^{n}\beta_k s\left(\lambda_k\right)\right)$$

$$\left.\times\left(\sum_{m=1}^{n}\alpha_m\left(f\left(\lambda_m\right)-c(\lambda_m)f\left(0\right)-\int_0^{\lambda_m}c\left(\lambda_m-s\right)f'(s)ds\right)+A\varphi\right)\right\}$$

$$+f\left(t\right)-c(t)f\left(0\right)-\int_0^{t}c\left(t-s\right)f'(s)ds.$$

Using formula (3.16) and estimates (3.7) and (3.8), we get

$$\|Au(t)\|_H \leq \|c(t)\|_{H\to H}\|T\|_{H\to H}\left\{\left(1+\sum_{k=1}^{n}|\beta_k|\,\|c(\lambda_k)\|_{H\to H}\right)\right.$$

$$\times\left(\sum_{m=1}^{n}|\alpha_m|\left(\|f(\lambda_m)\|_H+\|c(\lambda_m)\|_{H\to H}\|f(0)\|_H\right.\right.$$

$$\left.\left.+\int_0^{\lambda_m}\|c(\lambda_m-s)\|_{H\to H}\left\|f'(s)\right\|_H ds\right)+\|A\varphi\|_H\right)$$

$$+\sum_{m=1}^{n}|\alpha_m|\left\|A^{\frac{1}{2}}s(\lambda_m)\right\|_{H\to H}\left(\sum_{k=1}^{n}|\beta_k|\left(\left\|A^{\frac{1}{2}}s(\lambda_k)\right\|_{H\to H}\|f(0)\|_H\right.\right.$$

$$\left.\left.+\int_0^{\lambda_k}\left\|A^{\frac{1}{2}}s(\lambda_k-s)\right\|_{H\to H}\left\|f'(s)\right\|_H ds\right)\right)$$

$$\left.+\|A^{\frac{1}{2}}\psi\|_H\right)\Bigg\}+\left\|A^{\frac{1}{2}}s(t)\right\|_{H\to H}\|T\|_{H\to H}\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\,\|c(\lambda_m)\|_{H\to H}\right)\right.$$

$$\times\left(\sum_{k=1}^{n}|\beta_k|\left(\left\|A^{\frac{1}{2}}s(\lambda_k)\right\|_{H\to H}\|f(0)\|_H\right.\right.$$

$$\left.\left.+\int_0^{\lambda_k}\left\|A^{\frac{1}{2}}s(\lambda_k-s)\right\|_{H\to H}\left\|f'(s)\right\|_H ds\right)+\|A^{\frac{1}{2}}\psi\|_H\right)+\left(\sum_{k=1}^{n}|\beta_k|\left\|A^{\frac{1}{2}}s(\lambda_k)\right\|_{H\to H}\right)$$

$$\times\left(\sum_{m=1}^{n}|\alpha_m|\left(\|f(\lambda_m)\|_H+\|c(\lambda_m)\|_{H\to H}\|f(0)\|_H\right.\right.$$

$$\left.\left.\left.+\int_0^{\lambda_m}\|c(\lambda_m-s)\|_{H\to H}\left\|f'(s)\right\|_H ds\right)+\|A\varphi\|_H\right)\right\}$$

$$+\|f(t)\|_H+\|c(t)\|_{H\to H}\|f(0)\|_H+\int_0^{t}\|c(t-s)\|_{H\to H}\left\|f'(s)\right\|_H ds$$

$$\leq M\left[\|A\varphi\|_H+\|A^{\frac{1}{2}}\psi\|_H+\|f(0)\|_H+\int_0^{t}\|f'(s)\|_H ds\right]$$

for every t, $0\leq t\leq 1$. This shows that

$$\max_{0\leq t\leq 1}\|Au(t)\|_H\leq M\left[\|A\varphi\|_H+\|A^{1/2}\psi\|_H+\|f(0)\|_H+\max_{0\leq t\leq 1}\|f'(t)\|_H\right]. \quad (3.17)$$

From estimate (3.17) and the triangle inequality it follows estimate (3.15).Theorem 3.7 is proved.

Now, we will consider the application of Theorem 3.7.

First, the mixed problem for hyperbolic equation

$$\begin{cases} u_{tt} - (a(x)u_x)_x + \delta u = f(t,x),\ 0 < t < 1,\ 0 < x < 1, \\ u(0,x) = \sum_{m=1}^{n} \alpha_m u(\lambda_m, x) + \varphi(x), u_t(0,x) = \sum_{k=1}^{n} \beta_k u_t(\lambda_k, x) + \psi(x), \\ 0 \le x \le 1, \\ u(t,0) = u(t,1),\ u_x(t,0) = u_x(t,1),\ 0 \le t \le 1 \end{cases} \tag{3.18}$$

under assumption (3.2) is considered. The problem (3.18) has a unique smooth solution $u(t,x)$ for (3.2), $\delta > 0$ and the smooth functions $a(x) \ge a > 0 \quad (x \in (0,1))$, $\varphi(x), \psi(x) \quad (x \in [0,1])$ and $f(t,x) \quad (t, x \in [0,1])$. This allows us to reduce the mixed problem (3.18) to the nonlocal boundary value problem (3.3) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by (3.18).Let us give a number of corollaries of the abstract Theorem 3.7.

Theorem 3.8. *For solutions of the mixed problem (3.18), we have the following stability inequalities*

$$\max_{0\le t\le 1} \| u_x(t,\cdot) \|_{L_2[0,1]} \le M \left[\max_{0\le t\le 1} \| f(t,\cdot) \|_{L_2[0,1]} + \| \varphi_x \|_{L_2[0,1]} + \| \psi \|_{L_2[0,1]} \right],$$

$$\max_{0\le t\le 1} \| u_{xx}(t,\cdot) \|_{L_2[0,1]} + \max_{0\le t\le 1} \| u_{tt}(t,\cdot) \|_{L_2[0,1]}$$

$$\le M \left[\max_{0\le t\le 1} \| f_t(t,\cdot) \|_{L_2[0,1]} + \| f(0,\cdot) \|_{L_2[0,1]} + \| \varphi_{xx} \|_{L_2[0,1]} + \| \psi_x \|_{L_2[0,1]} \right],$$

where M does not depend on $\varphi(x), \psi(x)$ and $f(t,x)$.

The proof of this theorem is based on the abstract Theorem 3.7 and the symmetry properties of the space operator generated by the problem (3.18).Second, let Ω be the unit open cube in the m-dimensional Euclidean space$\mathbb{R}^m$ $\{x = (x_1, \cdots, x_m) : 0 < x_j < 1, 1 \le j \le m\}$

with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, the mixed boundary value problem for the multi-dimensional hyperbolic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} = f(t,x), \\ x = (x_1, \ldots, x_m) \in \Omega, \quad 0 < t < 1, \\ u(0,x) = \sum_{j=1}^{n} \alpha_j u(\lambda_j, x) + \varphi(x), x \in \overline{\Omega}, \\ u_t(0,x) = \sum_{k=1}^{n} \beta_k u_t(\lambda_k, x) + \psi(x), 0 \le x \le 1, x \in \overline{\Omega}, \\ u(t,x) = 0, \quad x \in S, \quad 0 \le r \le m \end{cases} \tag{3.19}$$

under assumption (3.2) is considered. Here $a_r(x)$, $(x \in \Omega)$, $\varphi(x), \psi(x)$ $(x \in \overline{\Omega})$ and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions and $a_r(x) \geq a > 0$.

We introduce the Hilbert space $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_m \}^{\frac{1}{2}}.$$

The problem (3.19) has a unique smooth solution $u(t,x)$ for (3.2) and the smooth functions $\varphi(x)$, $\psi(x)$, $a_r(x)$ and $f(t,x)$. This allows us to reduce the mixed problem (3.19) to the nonlocal boundary value problem (3.3) in a Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by (3.19).

Theorem 3.9. *For the solutions of the mixed problem (3.19), the following stability inequalities*

$$\max_{0 \leq t \leq 1} \sum_{r=1}^{m} \| u_{x_r}(t,\cdot) \|_{L_2(\overline{\Omega})} \leq M \left[\max_{0 \leq t \leq 1} \| f(t,\cdot) \|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \| \varphi_{x_r} \|_{L_2(\overline{\Omega})} + \| \psi \|_{L_2(\overline{\Omega})} \right],$$

$$\max_{0 \leq t \leq 1} \sum_{r=1}^{m} \| u_{x_r x_r}(t,\cdot) \|_{L_2(\overline{\Omega})} + \max_{0 \leq t \leq 1} \| u_{tt}(t,\cdot) \|_{L_2(\overline{\Omega})}$$

$$\leq M \left[\max_{0 \leq t \leq 1} \| f_t(t,\cdot) \|_{L_2(\overline{\Omega})} + \| f(0,\cdot) \|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \| \varphi_{x_r x_r} \|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \| \psi_{x_r} \|_{L_2(\overline{\Omega})} \right]$$

hold, where M does not depend on $\varphi(x)$, $\psi(x)$ and $f(t,x)$.

The proof of this theorem is based on the abstract Theorem 3.7, the symmetry properties of the operator A^x defined by formula (3.19) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3.10. *For the solutions of the elliptic differential problem*

$$A^x u(x) = \omega(x), x \in \Omega, \tag{3.20}$$

$$u(x) = 0, x \in S,$$

the following coercivity inequality holds [SobolevskiiP.E.1975]:

$$\sum_{r=1}^{m} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M \|\omega\|_{L_2(\overline{\Omega})}.$$

Note that the stability estimate (3.13)is not satisfied for the general α_k and β_k. the corresponding counterexampleve was given in the paper [Ashyralyev, A. and Aggez, N., 2004].

3.3 The First Order of Accuracy Difference Schemes

Throughout this research for simplicity $\lambda_1 > 2\tau$ and $\lambda_n < 1$ will be considered. Let us associate the boundary value problem (3.3) with the corresponding first order of accuracy difference scheme (3.4)

A study of discretization, over time only, of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A, is replaced by the difference operators A_h that act in the Hilbert spaces and are uniformly self-adjoint positive definite in h for $0 < h \leq h_0$.

In general, we have not been able to obtain the stability estimates for the solution of difference scheme (3.4) under assumption (3.2). Note that the stability of solutions of the difference scheme (3.4) will be obtained under the strong assumption

$$\sum_{k=1}^{n} |\alpha_k| + \sum_{k=1}^{n} |\beta_k| + \sum_{k=1}^{n} |\alpha_k| \sum_{k=1}^{n} |\beta_k| < 1. \tag{3.21}$$

Now, let us give some lemmas that will be needed below.

Lemma 3.1. The following estimates hold:

$$\begin{cases} \|R\|_{H\mapsto H} \leq 1, \ \|\tilde{R}\|_{H\mapsto H} \leq 1, \\ \\ \|\tilde{R}^{-1}R\|_{H\mapsto H} \leq 1, \ \|R^{-1}\tilde{R}\|_{H\mapsto H} \leq 1, \\ \\ \|\tau A^{1/2}R\|_{H\mapsto H} \leq 1, \|\tau A^{1/2}\tilde{R}\|_{H\mapsto H} \leq 1. \end{cases} \tag{3.22}$$

Here and in future $R = \left(I + i\tau A^{1/2}\right)^{-1}$, $\tilde{R} = \left(I - i\tau A^{1/2}\right)^{-1}$.

Lemma 3.2. Suppose that the assumption (3.21) holds. Then, the operator

$$I - \sum_{k=1}^{n} \beta_k \frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}\right) - \sum_{m=1}^{n} \alpha_m \frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)$$

$$+ \frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n} \alpha_m\beta_k \left(R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + R^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda k}{\tau}\right]+1}\right)$$
$$+ \frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n} \alpha_m\beta_k \left(-R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda k}{\tau}\right]}\right).$$

has an inverse

$$T_\tau = \Bigg\{ I - \sum_{k=1}^{n} \beta_k \frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}\right) - \sum_{m=1}^{n} \alpha_m \frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)$$

$$+\frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + R^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda k}{\tau}\right]+1}\right)$$

$$+\frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(-R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda k}{\tau}\right]}\right)\Bigg\}^{-1}$$

and the following estimate is satisfied:

$$\| T_\tau \|_{H\mapsto H} \leq \frac{1}{1 - \sum_{k=1}^{n}|\alpha_k| - \sum_{k=1}^{n}|\beta_k| - \sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|}. \tag{3.23}$$

Proof. The proof of estimate (3.23) is based on the estimate

$$\Bigg\| I - \sum_{k=1}^{n} \beta_k \frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}\right) - \sum_{m=1}^{n} \alpha_m \frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)$$

$$+\frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + R^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda k}{\tau}\right]+1}\right)$$

$$+\frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\Bigg(-R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]}$$

$$+R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda k}{\tau}\right]}\Bigg)\Bigg\|_{H\mapsto H} \geq 1 - \sum_{k=1}^{n}|\alpha_k| - \sum_{k=1}^{n}|\beta_k| - \sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|.$$

Using the triangle inequality, the definitions of R, $\tilde{R}$ and estimate (3.22), we obtain

$$\Bigg\| I - \sum_{k=1}^{n} \beta_k \frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}\right) - \sum_{m=1}^{n} \alpha_m \frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)$$

$$+\frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + R^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda k}{\tau}\right]+1}\right)$$

$$+\frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(-R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]} + R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda k}{\tau}\right]}\right)\Bigg\|_{H\mapsto H}$$

$$\geq 1 - \sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]+1}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+1}\|_{H\to H}\right)$$

$$-\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)$$

$$+\frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n}|\alpha_m||\beta_k|\left(\left\|R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\right\|_{H\to H}\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]+1}\right\|_{H\to H}\right)$$

$$+\frac{1}{4}\sum_{m=1}^{n}\sum_{k=1}^{n}|\alpha_m||\beta_k|\left(\left\|R^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]+\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right.$$

$$\left.+\left\|R^{\left[\frac{\lambda_m}{\tau}\right]}\right\|_{H\to H}\left\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}+\left\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right\|_{H\to H}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\right)$$

$$\geq 1-\sum_{m=1}^{n}|\alpha_m|-\sum_{k=1}^{n}|\beta_k|-\sum_{m=1}^{n}|\alpha_m|\sum_{k=1}^{n}|\beta_k|$$

$$=1-\sum_{k=1}^{n}|\alpha_k|-\sum_{k=1}^{n}|\beta_k|-\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|.$$

Lemma 3.2 is proved.

Now, we will obtain the formula for the solution of problem (3.4). It is clear that the first order of accuracy difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_{k+1}=f_k, \\ \quad f_k=f(t_{k+1}),\ t_{k+1}=(k+1)\tau,\ 1\leq k\leq N-1,\ N\tau=1, \\ u_0=\mu,\ \tau^{-1}(u_1-u_0)=\omega \end{cases} \tag{3.24}$$

has a solution and the following formula holds:

$$u_0=\mu, u_1 \ = \ \mu+\tau\omega,$$

$$u_k=\frac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]\mu+(R-\tilde{R})^{-1}\tau(R^k-\tilde{R}^k)\omega$$
$$-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}-\tilde{R}^{k-s}\right]f_s,\ 2\leq k\leq N. \tag{3.25}$$

Applying formula (3.25) and the nonlocal boundary conditions

$$u_0=\sum_{m=1}^{n}\alpha_m u_{\left[\frac{\lambda_m}{\tau}\right]}+\varphi, \tau^{-1}(u_1-u_0)=\sum_{k=1}^{n}\tau^{-1}\beta_k\left(u_{\left[\frac{\lambda_k}{\tau}\right]+1}-u_{\left[\frac{\lambda_k}{\tau}\right]}\right)+\psi,$$

we can write

$$\mu=\sum_{m=1}^{n}\alpha_m\left\{\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)\mu+(R-\tilde{R})^{-1}\tau\left(R^{\left[\frac{\lambda_m}{\tau}\right]}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right)\omega\right. \tag{3.26}$$

$$\left.-\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right)f_s\right\}+\varphi,$$

$$\omega=\sum_{k=1}^{n}\tau^{-1}\beta_k\left\{\frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]-1}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right)\mu\right. \tag{3.27}$$

$$+(R-\tilde{R})^{-1}\tau\left(R^{\left[\frac{\lambda_k}{\tau}\right]+1}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}-R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\omega-\frac{\tau}{2i}A^{-1/2}\left[R-\tilde{R}\right]f_{\left[\frac{\lambda_k}{\tau}\right]}$$

$$\left.-\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}-R^{\left[\frac{\lambda_k}{\tau}\right]-s}+\tilde{R}^{\left[\frac{\lambda_i}{\tau}\right]-s}\right)f_s\right\}+\psi.$$

Using formulas (3.26) and (3.27), we obtain

$$\mu=T_\tau\left\{\left(I-\sum_{k=1}^{n}\tau^{-1}\beta_k\left((R-\tilde{R})^{-1}\tau(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1})\right)\right)\right. \tag{3.28}$$

$$\times\left(\varphi-\sum_{m=1}^{n}\alpha_m\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right)f_s\right)$$

$$+\left(\sum_{m=1}^{n}\alpha_m(R-\tilde{R})^{-1}\tau\left(R^{\left[\frac{\lambda_m}{\tau}\right]}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right)\right)\left(\psi-\sum_{k=1}^{n}\tau^{-1}\beta_k\right.$$

$$\left.\left.\times\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s})f_s-\tau^2R\tilde{R}f_{\left[\frac{\lambda_k}{\tau}\right]}\right)\right)\right\},$$

$$\omega=T_\tau\left\{\left(I-\sum_{m=1}^{n}\alpha_m\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)\right)\left(\psi-\sum_{k=1}^{n}\tau^{-1}\beta_k\right.\right. \tag{3.29}$$

$$\left.\times\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(-i\tau A^{1/2}\right)(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s})f_s-\tau^2R\tilde{R}f_{\left[\frac{\lambda_k}{\tau}\right]}\right)\right)$$

$$+\left(\sum_{k=1}^{n}\tau^{-1}\beta_k\frac{1}{2}\left(i\tau A^{1/2}\right)\left(-R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\right)$$

$$\left.\times\left(\varphi-\sum_{m=1}^{n}\alpha_m\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s})f_s\right)\right\}.$$

So, formulas (3.25), (3.28) and (3.29) give a solution of problem (3.4).

Theorem 3.11. *Suppose that the assumption (3.21) holds and $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$. Then, for the solution of the difference scheme (3.4) satisfy the following stability estimates*

$$\|u_k\|_H\le M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\parallel A^{-1/2}\psi\parallel_H+\parallel\varphi\parallel_H\},k=0,2,\cdots,N, \tag{3.30}$$

$$\|u_1\|_H \le M[\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+ \parallel \varphi \parallel_H + \parallel (I+i\tau A^{1/2})A^{-1/2}\psi \parallel_H],$$

$$\|A^{1/2}u_k\|_H \le M\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+ \parallel \psi \parallel_H + \parallel A^{1/2}\varphi \parallel_H\}, k=0,2,\cdot\cdot\cdot,N, \quad (3.31)$$

$$\|A^{1/2}u_1\|_H \le M[\sum_{s=1}^{N-1}\|f_s\|_H\tau+ \parallel A^{1/2}\varphi \parallel_H + \parallel (I+i\tau A^{1/2})\psi \parallel_H],$$

$$\|Au_k\|_H \le M\{\sum_{s=2}^{N-1} \parallel f_s - f_{s-1} \parallel_H]+ \parallel f_1 \parallel_H + \parallel A^{1/2}\psi \parallel_H + \parallel A\varphi \parallel_H\}, k=0,2,\cdot\cdot\cdot,N, \quad (3.32)$$

$$\|Au_1\|_H \le M[\sum_{s=2}^{N-1} \parallel f_s - f_{s-1} \parallel_H]+ \parallel f_1 \parallel_H + \parallel A\varphi \parallel_H + \parallel (I+i\tau A^{1/2})A^{1/2}\psi \parallel_H]$$

hold, where M *does not depend on* τ, φ, ψ *and* f_s, $1 \le s \le N-1$.

Proof. Using formulas(3.28), (3.29) and estimates (3.22), (3.23), we obtain

$$\|\mu\|_H \le \|T_\tau\|_{H\to H}\left\{\left(1+\sum_{k=1}^{n}|\beta_k|\frac{1}{2}(\|\tilde{R}^{-1}R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{-1}\|_{H\to H})\right)(\|\varphi\|_H\right. \quad (3.33)$$

$$\left.+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\right)\right\}$$

$$+\left(\sum_{m=1}^{n}|\alpha_m|\left(\|\tilde{R}^{-1}R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}\right)\right)$$

$$\times\left(\sum_{k=1}^{n}|\beta_k|\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\right.\right.$$

$$\left.\left.+\left\|R\tilde{R}\right\|_{H\to H}\|A^{-1/2}f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau+\|A^{-1/2}\psi\|_H\right)\right\}$$

$$\le M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+ \parallel A^{-1/2}\psi \parallel_H + \parallel \varphi \parallel_H\right\}.$$

$$\|A^{-\frac{1}{2}}\omega\|_H \le \|T_\tau\|_{H\to H}\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\right)\right. \quad (3.34)$$

$$\times\left(\|A^{-1/2}\psi\|_H+\sum_{k=1}^{n}|\beta_k|\left(\left\|R\tilde{R}\right\|_{H\to H}\|A^{-1/2}f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau\right.\right.$$

$$\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\right)$$

$$+\left(\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)(\|\varphi\|_H$$

$$+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\Bigg)\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|A^{-1/2}\psi\|_H+\|\varphi\|_H\right\}.$$

Applying $A^{\frac{1}{2}}$ to formulas (3.28),(3.29) and using estimates (3.22) and (3.23) in a similar manner, we obtain

$$\|A^{1/2}\mu\|_H\leq\|T_\tau\|_{H\to H}\left\{\left(1+\sum_{k=1}^{n}|\beta_k|\frac{1}{2}(\|\tilde{R}^{-1}R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{-1}\|_{H\to H})\right)\right. \tag{3.35}$$

$$\times\left(\left\|A^{1/2}\varphi\right\|_H+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|f_s\|_H\tau\right)\Bigg\}$$

$$+\left(\sum_{m=1}^{n}|\alpha_m|\left(\|\tilde{R}^{-1}R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}\right)\right)$$

$$\times\left(\sum_{k=1}^{n}|\beta_k|\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|f_s\|_H\tau\right.\right.$$

$$\left.\left.+\left\|R\tilde{R}\right\|_{H\to H}\|f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau+\|\psi\|_H\right)\right\}$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\psi\|_H+\|A^{1/2}\varphi\|_H\right\},$$

$$\|\omega\|_H\leq\|T_\tau\|_{H\to H}\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\right)\right. \tag{3.36}$$

$$\times\left(\|\psi\|_H+\sum_{k=1}^{n}|\beta_k|\left(\left\|R\tilde{R}\right\|_{H\to H}\|f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau\right.\right.$$

$$\left.\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|f_s\|_H\tau\right)\right)$$

$$+\left(\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)(\left\|A^{1/2}\varphi\right\|_H$$

$$+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|f_s\|_H\tau\Bigg)\Bigg\}$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\parallel\psi\parallel_H+\parallel A^{1/2}\varphi\parallel_H\right\}.$$

Now, we obtain the estimates for $\parallel A\mu\parallel_H$ and $\|A^{\frac{1}{2}}\omega\|_H$. Applying A to formulas (3.28), (3.29) and using Abel's formula, we can write

$$A\mu=T_\tau\left\{\left(I-\sum_{k=1}^{n}\tau^{-1}\beta_k\left((R-\tilde{R})^{-1}\tau(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1})\right)\right)\right. \tag{3.37}$$

$$\times\left(A\varphi-\sum_{m=1}^{n}\alpha_m\left[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right)(f_{s-1}-f_s)\right.\right.$$

$$\left.\left.+\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)f_1-\left(R+\tilde{R}\right)f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right]\right)$$

$$+\left(\sum_{m=1}^{n}\alpha_m A^{1/2}\tau\left(R-\tilde{R}\right)^{-1}\left(R^{\left[\frac{\lambda_m}{\tau}\right]}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right)\right)$$

$$\times\left(A^{1/2}\psi+\sum_{k=1}^{n}\beta_k i\left[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)(f_{s+1}-f_s)\right.\right.$$

$$\left.\left.\left.+\left(R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)f_1-\left(R+\tilde{R}\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}-A\tau^2R\tilde{R}f_{\left[\frac{\lambda_k}{\tau}\right]}\right]\right)\right\},$$

$$A^{1/2}\omega=T_\tau\left\{\left(I-\sum_{m=1}^{n}\alpha_m\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)\right.\right. \tag{3.38}$$

$$\times\left(A^{1/2}\psi+\sum_{k=1}^{n}\beta_k i\left[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)(f_{s+1}-f_s)\right.\right.$$

$$\left.\left.+\left(R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)f_1-\left(R+\tilde{R}\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}+A^{1/2}\tau^2R\tilde{R}f_{\left[\frac{\lambda_k}{\tau}\right]}\right]\right)$$

$$+\left(\sum_{k=1}^{n}\tau^{-1}\beta_k\frac{1}{2}i\tau\left(-R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\right)$$

$$\times\left(A\varphi-\sum_{m=1}^{n}\alpha_m\left[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right)(f_{s-1}-f_s)\right.\right.$$

$$\left.\left.\left.+\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)f_1-\left(R+\tilde{R}\right)f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right]\right)\right\}.$$

Using formulas (3.37), (3.38) and estimates (3.22), (3.23), we obtain

$$\|A\mu\|_H \le \|T_\tau\|_{H\to H}\left\{\left(1+\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|\tilde{R}^{-1}R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}\|_{H\to H}\right)\right)\right. \tag{3.39}$$
$$\times\left(\|A\varphi\|_H+\sum_{m=1}^{n}|\alpha_m|\left[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|f_s-f_{s-1}\|_H\right.\right.$$
$$\left.\left.+\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\|f_1\|_H+\|R+\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right\|_H\right]\right)$$
$$+\left(\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}\left(\|\tilde{R}^{-1}R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\|_{H\to H}\right)\right)\left(\left\|A^{1/2}\psi\right\|_H+\sum_{k=1}^{n}|\beta_k|\right.$$
$$\times\left[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}\right)\|f_{s+1}-f_s\|_H\right.$$
$$+\|R+\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H$$
$$\left.\left.\left.+\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\|f_1\|_H+\|A\tau^2R\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\right]\right)\right\}$$
$$\le M\left\{\sum_{s=2}^{N-1}\|\ f_s-f_{s-1}\ \|_H+\|\ f_1\ \|_H+\|\ A^{1/2}\psi\ \|_H+\|\ A\varphi\ \|_H\right\},$$

$$\left\|A^{1/2}\omega\right\|_H \le \|T_\tau\|_{H\to H}\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\right)\right. \tag{3.40}$$
$$\times\left(\left\|A^{1/2}\psi\right\|_H+\sum_{k=1}^{n}|\beta_k|\right.$$
$$\times\left[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}\right)\|f_{s+1}-f_s\|_H+\|R+\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\right.$$
$$\left.\left.+\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\|f_1\|_H+\|A^{1/2}\tau^2R\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\right]\right)$$
$$+\left(\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)\left(\|A\varphi\|_H+\sum_{m=1}^{n}|\alpha_m|\right.$$
$$\times\left[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|f_s-f_{s-1}\|_H\right.$$

$$+\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\|f_1\|_H+\|R+\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right\|_H\Bigg]\Bigg)$$

$$\leq M\left\{\sum_{s=2}^{N-1}\| f_s-f_{s-1}\|_H+\| f_1\|_H+\| A^{1/2}\psi\|_H+\| A\varphi\|_H\right\}.$$

Now, we will prove estimates (3.30), (3.31) and (3.32). Let $k\geq 2$. Then using formula (3.25) and estimates (3.22), (3.33), (3.34), (3.35) and (3.36), we obtain

$$\|u_k\|_H\leq\frac{1}{2}\left[\|R^{k-1}\|_H+\|\tilde{R}^{k-1}\|_H\right]\|\mu\|_H+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1}\|_{H\to H}$$

$$+\|R^{-1}\tilde{R}^{k-1}\|_{H\to H})\|A^{-\frac{1}{2}}\omega\|_H+\sum_{s=1}^{k-1}\frac{\tau}{2}\left[\|R^{k-s}\|_H+\|\tilde{R}^{k-s}\|_H\right]\|A^{-\frac{1}{2}}f_s\|_H$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\| A^{-1/2}\psi\|_H+\|\varphi\|_H\right\}.$$

$$\|A^{\frac{1}{2}}u_k\|_H\leq\frac{1}{2}\left[\|R^{k-1}\|_H+\|\tilde{R}^{k-1}\|_H\right]\|A^{\frac{1}{2}}\mu\|_H+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1}\|_{H\to H}$$

$$+\|R^{-1}\tilde{R}^{k-1}\|_{H\to H})\|\omega\|_H+\sum_{s=1}^{k-1}\frac{\tau}{2}\left[\|R^{k-s}\|_H+\|\tilde{R}^{k-s}\|_H\right]\|f_s\|_H$$

$$\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\psi\|_H+\| A^{\frac{1}{2}}\varphi\|_H\right\}.$$

Now, we obtain the estimates for $\| Au_k\|_H$ for $k\geq 2$. Applying A to formula (3.25) and using Abel's formula, we can write

$$Au_k=\frac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]A\mu+(R-\tilde{R})^{-1}\tau(R^k-\tilde{R}^k)A\omega \tag{3.41}$$

$$+\frac{1}{2}\left(\sum_{s=2}^{k-1}(\left[R^{k-s}+\tilde{R}^{k-s}\right](f_{s-1}-f_s)+2f_{k-1}-\left[R^{k-1}+\tilde{R}^{k-1}\right]f_1\right).$$

Using formula (3.41) and estimates (3.22), (3.39), (3.40), we obtain

$$\|Au_k\|_H\leq\frac{1}{2}\left[\|R^{k-1}\|_{H\to H}+\|\tilde{R}^{k-1}\|_{H\to H}\right]\|A\mu\|_H+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1}+R^{-1}\tilde{R}^{k-1})\|A^{\frac{1}{2}}\omega\|_H$$

$$+\frac{1}{2}(\sum_{s=2}^{k-1}(\left[\|R^{k-s}\|_{H\to H}+\|\tilde{R}^{k-s}\|_{H\to H}\right]\|f_{s-1}-f_s\|_H$$

$$+2\|f_{k-1}\|_H+\Big[\|R^{k-1}\|_{H\to H}+\|\tilde{R}^{k-1}\|_{H\to H}\|f_1\|_H)$$

$$\leq M\left\{\sum_{s=2}^{N-1}\| f_s-f_{s-1}\|_H+\| f_1\|_H+\| A^{1/2}\psi\|_H+\| A\varphi\|_H\right\}.$$

Thus, estimates (3.30), (3.31), (3.32) for any $k \geq 2$ are obtained. From $u_0 = \mu$ and (3.33), (3.35), (3.39) it follows estimates (3.30), (3.31) and (3.32) for $k = 0$. Note that in a similar manner with estimates (3.34), (3.36), (3.40), (3.22) and (3.23), we obtain

$$\|\tau\omega\|_H \leq \|\tau A^{1/2}R\|_{H\to H}\|T_\tau\|_{H\to H} \tag{3.42}$$
$$\times\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H})\right)\right.$$
$$\times\left(\|A^{-1/2}(I+i\tau A^{-1/2})\psi\|_H+\sum_{k=1}^{n}|\beta_k|\left(\|\tilde{R}\|_{H\to H}\|A^{-1/2}f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau\right.\right.$$
$$\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\right)$$
$$+\left(\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)(\|\varphi\|_H$$
$$\left.\left.+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\right)\right\}$$
$$\leq M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\ A^{-1/2}(I+i\tau A^{-1/2})\psi\ \|_H+\|\ \varphi\ \|_H\right\},$$

$$\|\tau A^{1/2}\omega\|_H \leq \|\tau A^{1/2}R\|_{H\to H}\|T_\tau\|_{H\to H} \tag{3.43}$$
$$\times\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H})\right)\right.$$
$$\times\left(\|A(I+i\tau A^{-1/2})\psi\|_H+\sum_{k=1}^{n}|\beta_k|\left(\|\tilde{R}\|_{H\to H}\|f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau\right.\right.$$
$$\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|f_s\|_H\tau\right)$$
$$+\left(\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)(\|A^{1/2}\varphi\|_H$$
$$\left.\left.+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|f_s\|_H\tau\right)\right\}$$
$$\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\ (I+i\tau A^{-1/2})\psi\ \|_H+\|\ A^{\frac{1}{2}}\varphi\ \|_H\right\}.$$

Finally, in the same manner we get

$$\|\tau A\omega\|_H \leq M\left\{\sum_{s=2}^{N-1} \| f_s - f_{s-1} \|_H + \| f_1 \|_H + \| A^{1/2}(I + i\tau A^{-1/2})\psi \|_H + \| A\varphi \|_H\right\}. \tag{3.44}$$

Using the formula $u_1 = \mu + \tau\omega$ and the triangle inequality and estimates (3.33), (3.35), (3.39), (3.42) and (3.43), we obtain estimates(3.30), (3.31), (3.32) for $k = 1$. Theorem 3.1 is proved.

Remark 1. Note that stability estimates (3.30), (3.31) and (3.32) in the case $k = 1$ are weaker than respective estimates in the cases $k = 0, 2, \cdots, N$. However, obtaining this type of estimate is important for applications.We denote by $a^\tau = \{a_k\}_{k=0}^N$ the mesh function of approximation. Then $\|(I + i\tau A^{-1/2})a_1\|_H \sim \|a_1\|_H = o(\tau)$ if we assume that $\tau\|Aa_1\|_H$ tends to 0 as $\tau \to 0$ not slower than $\|a_1\|_H$. It takes place in applications by supplementary restriction of the smooth property of the data of space variables. It is clear that the uniformity in τ estimate

$$\|u_1\|_H \leq M\left[\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau + \| A^{-1/2}\psi \|_H + \| \varphi \|_H\right]$$

is absent. However, estimates for the solution of first order of accuracy modified difference scheme for approximately solving the boundary value problem (3.3)

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k,\ 1 \leq k \leq N-1, \\ u_0 = \sum_{m=1}^{n} \alpha_m u_{\left[\frac{\lambda_m}{\tau}\right]} + \varphi, \\ (I + \tau^2 A)\tau^{-1}(u_1 - u_0) = \sum_{k=1}^{n} \tau^{-1}\beta_k\left(u_{\left[\frac{\lambda_k}{\tau}\right]+1} - u_{\left[\frac{\lambda_k}{\tau}\right]}\right) + \psi \end{cases} \tag{3.45}$$

are better than the estimates for the solution of difference scheme (3.4).

Theorem 3.12. *Suppose that the assumption (3.21) holds and $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$. Then, for the solution of the difference scheme (3.45) the stability inequalities*

$$\max_{0\leq k\leq N} \|u_k\|_H \leq M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau + \| A^{-1/2}\psi \|_H + \| \varphi \|_H\},$$

$$\max_{0\leq k\leq N} \|A^{1/2}u_k\|_H \leq M\{\sum_{s=1}^{N-1} \| f_s \|_H \tau + \| A^{1/2}\varphi\|_H + \| \psi\|_H\},$$

$$\max_{1\leq k\leq N-1} \|\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\|_H + \max_{0\leq k\leq N} \|Au_k \|_H$$

$$\leq M\{\sum_{s=2}^{N-1} \| f_s - f_{s-1} \|_H + \| f_1 \|_H + \| A^{1/2}\psi \|_H + \| A\varphi \|_H\}$$

hold, where M does not depend on τ, φ, ψ and $f_s, 1 \leq s \leq N-1$.

The proof of this theorem follows the scheme of the proof of theorem 3.11 and it is based on the following formulas

$$u_0 = \mu, u_1 = \mu + \tau R\tilde{R}\omega,$$

$$u_k = \frac{1}{2}\left[R^{k-1} + \tilde{R}^{k-1}\right]\mu + (R-\tilde{R})^{-1}\tau(R^k - \tilde{R}^k)R\tilde{R}\omega - \sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s} - \tilde{R}^{k-s}\right]f_s$$

$$= \frac{1}{2}\left[R^{k-1} + \tilde{R}^{k-1}\right]\mu + (R-\tilde{R})^{-1}\tau(R^k - \tilde{R}^k)R\tilde{R}\omega + A^{-1}\frac{1}{2}\left(2f_{k-1} - \left[R^{k-1} + \tilde{R}^{k-1}\right]f_1\right)$$

$$+A^{-1}\frac{1}{2}(\sum_{s=2}^{k-1}(\left[R^{k-s} + \tilde{R}^{k-s}\right](f_{s-1} - f_s)), 2 \leq k \leq N,$$

$$\mu = T_\tau\left\{\left(I - R\tilde{R}\sum_{k=1}^{n}\tau^{-1}\beta_k\left((R-\tilde{R})^{-1}\tau(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1})\right)\right)\right.$$

$$\times\left(\varphi - \sum_{m=1}^{n}\alpha_m\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right)f_s\right)$$

$$+R\tilde{R}\left(\sum_{m=1}^{n}\alpha_m(R-\tilde{R})^{-1}\tau\left(R^{\left[\frac{\lambda_m}{\tau}\right]} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right)\right)$$

$$\left.\times\left(\psi - \sum_{k=1}^{n}\tau^{-1}\beta_k\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s})f_s\right)\right\},$$

$$\omega = T_\tau R\tilde{R}\left\{\left(I - \sum_{m=1}^{n}\alpha_m\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)\right)\right.$$

$$\times\left(\psi - \sum_{k=1}^{n}\tau^{-1}\beta_k\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(-i\tau A^{1/2}\right)(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s})f_s\right)$$

$$+\left(\sum_{k=1}^{n}\tau^{-1}\beta_k\frac{1}{2}\left(i\tau A^{1/2}\right)\left(-R^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\right)$$

$$\times\left(\varphi-\sum_{m=1}^{n}\alpha_m\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s})f_s\right)\Bigg\}$$

and on estimates (3.23) and (3.22).

Now, we consider the application of theorem 3.11.

First, the nonlocal boundary value problem (3.18) for one dimensional hyperbolic equation under assumption (3.21) is considered. The discretization of problem (3.18) is carried out in two steps. In the first step, let us define the grid space

$$[0,1]_h=\{x:x_r=rh,0\leq r\leq K,Kh=1\}.$$

We introduce the Hilbert space $L_{2h}=L_2\left([0,1]_h\right)$ of the grid functions $\varphi^h(x)=\{\varphi^r\}_1^{K-1}$ defined on $[0,1]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{r=1}^{K-1}\left|\varphi^h(x)\right|^2h\right)^{1/2}.$$

To the differential operator A generated by the problem (3.18), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x)=\left\{-(a(x)\varphi_{\bar{x}})_{x,r}+\delta\varphi^r\right\}_1^{K-1}, \tag{3.46}$$

acting in the space of grid functions $\varphi^h(x)=\{\varphi^r\}_0^K$ satisfying the conditions $\varphi^0=\varphi^K$, $\varphi^1-\varphi^0=\varphi^K-\varphi^{K-1}$. With the help of A_h^x we arrive at the nonlocal boundary value problem

$$\begin{cases}\frac{d^2v^h(t,x)}{dt^2}+A_h^xv^h(t,x)=f^h(t,x),\ 0\leq t\leq 1,\ x\in[0,1]_h,\\ v^h(0,x)=\sum\limits_{j=1}^{n}\alpha_jv^h\left(\lambda_j,x\right)+\varphi^h(x),x\in[0,1]_h,\\ v_t^h(0,x)=\sum\limits_{j=1}^{n}\beta_jv_t^h(\lambda_j,x)+\psi^h(x),x\in[0,1]_h\end{cases} \tag{3.47}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (3.47) by the difference scheme (3.48)

$$\begin{cases}\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^xu_{k+1}^h=f_k^h(x),\ x\in[0,1]_h,\\ f_{k+1}^h(x)=\{f(t_{k+1},x_n)\}_1^{K-1},t_{k+1}=(k+1)\tau,1\leq k\leq N-1,\ N\tau=1,\\ (I+\tau^2A_h^x)\frac{u_1^h(x)-u_0^h(x)}{\tau}=\sum\limits_{j=1}^{n}\beta_j\frac{u_{[\lambda_j/\tau]+1}^h(x)-u_{[\lambda_j/\tau]}^h(x)}{\tau}+\psi^h(x),\\ u_0^h(x)=\sum\limits_{j=1}^{n}\alpha_ju_{[\lambda_j/\tau]}^h(x)+\varphi^h(x),\ x\in[0,1]_h.\end{cases} \tag{3.48}$$

Theorem 3.13. *Let τ and h be sufficiently small numbers. Suppose that the assumption (3.21) holds. Then, the solutions of the difference scheme (3.48) satisfy the following stability estimates:*

$$\max_{0\leq k\leq N}\left\|u_k^h\right\|_{L_{2h}}+\max_{0\leq k\leq N}\left\|\left(u_k^h\right)_x\right\|_{L_{2h}}\leq M_1\left[\max_{1\leq k\leq N-1}\left\|f_k^h\right\|_{L_{2h}}\right.$$

$$\left.+\left\|\psi^h\right\|_{L_{2h}}+\left\|\varphi_{\bar{x}}^h\right\|_{L_{2h}}\right],$$

$$\max_{1\leq k\leq N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{0\leq k\leq N}\left\|\left(u_k^h\right)_{\bar{x}x}\right\|_{L_{2h}}$$

$$\leq M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\leq k\leq N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi_{\bar{x}}^h\right\|_{L_{2h}}+\left\|\left(\varphi_{\bar{x}}^h\right)_x\right\|_{L_{2h}}\right].$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x)$ and $f_k^h, 1\leq k<N$.

The proof of this theorem is based on the abstract Theorem 3.11 and the symmetry properties of the operator A_h^x defined by the formula (3.46).

Second, the nonlocal boundary value problem (3.19) for the m-dimensional hyperbolic equation under assumption (3.21) is considered. The discretization of problem (3.47) is carried out in two steps.

In the first step, let us define the grid sets

$$\widetilde{\Omega}_h=\{x=x_r=(h_1r_1,\cdots,h_mr_m),r=(r_1,\cdots,r_m),$$

$$0\leq r_j\leq N_j,h_jN_j=1,j=1,\cdots,m\},\Omega_h=\widetilde{\Omega}_h\cap\Omega,S_h=\widetilde{\Omega}_h\cap S.$$

We introduce the Banach space $L_{2h}=L_2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x)=\{\varphi(h_1r_1,\cdots,h_mr_m)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)}=\left(\sum_{x\in\overline{\Omega_h}}\left|\varphi^h(x)\right|^2h_1\cdots h_m\right)^{1/2}.$$

To the differential operator A generated by problem (3.47), we assign the difference operator A_h^x by the formula

$$A_h^xu_x^h=-\sum_{r=1}^m\left(a_r(x)u_{\bar{x}_r}^h\right)_{x_r,j_r} \tag{3.49}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x)=0$ for all $x\in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of A_h^x we arrive at the nonlocal boundary value problem

$$\begin{cases} \frac{d^2v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x),\ 0 \le t \le 1,\ x \in \widetilde{\Omega}_h, \\ v^h(0,x) = \sum\limits_{l=1}^{n} \alpha_l v^h(\lambda_l, x) + \varphi^h(x), x \in \widetilde{\Omega}_h, \\ \frac{dv^h(0,x)}{dt} = \sum\limits_{l=1}^{n} \beta_l v_t^h(\lambda_l, x) + \psi^h(x), x \in \widetilde{\Omega} \end{cases} \tag{3.50}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (3.50) by the difference scheme (3.51)

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h = f_k^h(x),\ x \in \widetilde{\Omega}_h, \\ f_{k+1}^h(x) = \{f(t_{k+1}, x_n)\}_1^{K-1}, t_{k+1} = (k+1)\tau, 1 \le k \le N-1,\ N\tau = 1, \\ (I + \tau^2 A_h^x) \frac{u_1^h(x) - u_0^h(x)}{\tau} = \sum\limits_{l=1}^{n} \beta_l \frac{u_{[\lambda_l/\tau]+1}^h(x) - u_{[\lambda_l/\tau]}^h(x)}{\tau} + \psi^h(x), \\ u_0^h(x) = \sum\limits_{l=1}^{n} \alpha_l u_{[\lambda_l/\tau]}^h(x) + \varphi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{3.51}$$

Theorem 3.14. *Let τ and $|h|$ be sufficiently small numbers. Suppose that the assumption (3.21) holds. Then, the solutions of the difference scheme (3.51) satisfy the following stability estimates:*

$$\max_{0 \le k \le N} \left\| u_k^h \right\|_{L_{2h}} + \max_{0 \le k \le N} \sum_{r=1}^{m} \left\| \left(u_k^h\right)_{x_r, j_r} \right\|_{L_{2h}} \le M_1 \left[\max_{1 \le k \le N-1} \left\| f_k^h \right\|_{L_{2h}} + \left\| \psi^h \right\|_{L_{2h}} + \sum_{r=1}^{m} \left\| \varphi^h_{\bar{x}_r, j_r} \right\|_{L_{2h}} \right],$$

$$\max_{1 \le k \le N-1} \left\| \tau^{-2} \left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right) \right\|_{L_{2h}} + \max_{0 \le k \le N} \sum_{r=1}^{m} \left\| \left(u_k^h\right)_{\bar{x}_r x_r, j_r} \right\|_{L_{2h}}$$

$$\le M_1 \left[\left\| f_1^h \right\|_{L_{2h}} + \max_{2 \le k \le N-1} \left\| \tau^{-1} \left(f_k^h - f_{k-1}^h\right) \right\|_{L_{2h}} + \sum_{r=1}^{m} \left\| \psi^h_{\bar{x}_r, j_r} \right\|_{L_{2h}} + \sum_{r=1}^{m} \left\| \varphi^h_{\bar{x}_r x_r, j_r} \right\|_{L_{2h}} \right].$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x)$ and $f_k^h, 1 \le k < N$.

The proof of this theorem is based on the abstract Theorem 3.11 and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 3.15. *For the solutions of the elliptic difference problem*

$$A_h^x u^h(x) = \omega^h(x), x \in \Omega_h, \tag{3.52}$$

$$u^h(x) = 0, x \in S_h$$

the following coercivity inequality holds [SobolevskiiP.E.1975]:

$$\sum_{r=1}^{m} \left\| u^h_{x_r \bar{x}_r, j_r} \right\|_{L_{2h}} \le M ||\omega^h||_{L_{2h}}.$$

3.4 The Second Order of Accuracy Difference Schemes

Now, we consider the second order accuracy difference schemes for approximately solving the boundary value problem (3.3)

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k, \\ f_k=f(t_k), \quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ (I+\frac{\tau^2A}{2})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0) \\ =\sum\limits_{k=1}^{n}\beta_k\left\{\tau^{-1}(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1})+\frac{\tau}{2}(f_{\left[\frac{\lambda_k}{\tau}\right]}-Au_{\left[\frac{\lambda_k}{\tau}\right]})\right\}+\psi, \\ u_0=\sum\limits_{m=1}^{n}\alpha_m\left\{u_{\left[\frac{\lambda_m}{\tau}\right]}+\tau^{-1}(u_{\left[\frac{\lambda_m}{\tau}\right]}-u_{\left[\frac{\lambda_m}{\tau}\right]-1})\left(\lambda_m-\left[\frac{\lambda_m}{\tau}\right]\tau\right)\right\}+\varphi, \\ f_0=f(0), \end{cases} \tag{3.53}$$

and

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k, \\ f_k=f(t_k), \quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ (I+\frac{\tau^2A}{4})[(I+\frac{\tau^2A}{4})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)] \\ =\sum\limits_{k=1}^{n}\beta_k\left\{\tau^{-1}(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1})+\frac{\tau}{2}(f_{\left[\frac{\lambda_k}{\tau}\right]}-Au_{\left[\frac{\lambda_k}{\tau}\right]})\right\}+\psi, \\ u_0=\sum\limits_{m=1}^{n}\alpha_m\left\{u_{\left[\frac{\lambda_m}{\tau}\right]}+\tau^{-1}(u_{\left[\frac{\lambda_m}{\tau}\right]}-u_{\left[\frac{\lambda_m}{\tau}\right]-1})\left(\lambda_m-\left[\frac{\lambda_m}{\tau}\right]\tau\right)\right\}+\varphi, \\ f_0=f(0). \end{cases} \tag{3.54}$$

Theorem 3.16. *Let* $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$ *and*

$$\left(\sum_{m=1}^{n}|\alpha_m|\left(1+2\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)\right)\right)+\sum_{k=1}^{n}|\beta_k|+$$

$$+\sum_{m=1}^{n}\sum_{k=1}^{n}|\alpha_m|\,|\beta_k|\left(1+3\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)+\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)^2\right)<1.$$

Then for the solution of the difference scheme(3.53) the stability inequalities

$$\max_{0\le k\le N}\|u_k\|_H\le M\{\sum_{s=0}^{N}\|A^{-1/2}f_s\|_H\tau+\|A^{-1/2}\psi\|_H+\|\varphi\|_H\}, \tag{3.55}$$

$$\max_{0\le k\le N}\|A^{1/2}u_k\|_H\le M\{\sum_{s=0}^{N}\|f_s\|_H\tau+\|A^{1/2}\varphi\|_H+\|\psi\|_H\}, \tag{3.56}$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\|_H+\max_{0\le k\le N}\|Au_k\|_H \tag{3.57}$$

$$\le M\{\sum_{s=1}^{N}\| f_s-f_{s-1}\|_H+\| f_0\|_H+\| A^{1/2}\psi\|_H+\| A\varphi\|_H\}$$

hold ,where M does not depend on $f_s, 0\le s\le N$ and φ,ψ.

Proof. We will write the formula for the solution of the difference scheme (3.53). It is easy to show that [Ashyralyev A. and Sobolevskii P.E. 2001] there are unique solution of the problem

$$\begin{cases}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k,\\ f_k=f(t_k),\quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1,\\ (I+\frac{\tau^2A}{2})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)=\omega, f_0=f(0), u_0=\mu\end{cases} \tag{3.58}$$

and for the solutions of these problems the following formulas hold:

$$u_0=\mu, u_1=\left(I+\frac{\tau^2A}{2}\right)^{-1}\left(\mu+\tau\omega+\frac{\tau^2}{2}f_0\right],$$

$$u_k=\left(I-i\tau A^{\frac12}\right)^{-1}\frac{R^k(I-i\tau A^{\frac12}+\frac{\tau^2A}{2})+\tilde R^{k-1}}{2}\mu$$

$$+\left(I+\tau^2A\right)^{-1}(iA^{\frac12})^{-1}\frac{\tilde R^{k-1}-R^k\tilde R^{-1}}{2}(\omega+\frac{\tau}{2}f_0)$$

$$-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}-\tilde R^{k-s}\right]f_s, 2\le k\le N, \tag{3.59}$$

where

$$R=\left(I+i\tau A^{1/2}-\frac{\tau^2}{2}A\right)^{-1}, \tilde R=\left(I-i\tau A^{1/2}-\frac{\tau^2}{2}A\right)^{-1}.$$

Applying the last formula and the nonlocal boundary conditions

$$u_0=\sum_{m=1}^{n}\alpha_m\left\{u_{\left[\frac{\lambda_m}{\tau}\right]}+\tau^{-1}(u_{\left[\frac{\lambda_m}{\tau}\right]}-u_{\left[\frac{\lambda_m}{\tau}\right]-1})\left(\lambda_m-\left[\frac{\lambda_m}{\tau}\right]\tau\right)\right\}+\varphi,$$

$$(I+\frac{\tau^2A}{2})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)$$

$$=\sum_{k=1}^{n}\beta_k\left\{\tau^{-1}(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1})+\frac{\tau}{2}(f_{\left[\frac{\lambda_k}{\tau}\right]}-Au_{\left[\frac{\lambda_k}{\tau}\right]})\right\}+\psi,$$

we can write

$$\mu=\sum_{m=1}^{n}\alpha_m\left\{\left(I-i\tau A^{\frac12}\right)^{-1}\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}(I-i\tau A^{\frac12}+\frac{\tau^2A}{2})+\tilde R^{\left[\frac{\lambda_m}{\tau}\right]-1}}{2}\mu\right.$$

$$+\left(I+\tau^2A\right)^{-1}(iA^{1/2})^{-1}\frac{\tilde R^{\left[\frac{\lambda_m}{\tau}\right]-1}-R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde R^{-1}}{2}(\omega+\frac{\tau}{2}f_0)$$

$$
-\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1} \frac{\tau}{2i} A^{-\frac{1}{2}} \left[R^{\left[\frac{\lambda_m}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right] f_s
$$

$$
\left(\left(I - i\tau A^{\frac{1}{2}}\right)^{-1} (i\tau A^{1/2}) \times \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) - R^{\left[\frac{\lambda_m}{\tau}\right]}\left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I - i\tau A^{1/2} + \frac{\tau^2}{2}A\right)}{2}\mu\right.
$$

$$
+ \left(I + \tau^2 A\right)^{-1} \tau \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) + R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{-1}\left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)}{2}(\omega + \frac{\tau}{2} f_0)
$$

$$
+ \sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-2} \frac{\tau^2}{2} \left[R^{\left[\frac{\lambda_m}{\tau}\right]-s}\left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right) + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right)\right] f_s
$$

$$
\left. +\tau^2 \left(I + \frac{\tau^4}{4}A^2\right)^{-1} f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right)\left(\frac{\lambda_m}{\tau} - \left[\frac{\lambda_m}{\tau}\right]\right)\Bigg\} + \varphi
$$

and

$$
\omega = \sum_{k=1}^{n} \beta_k \Bigg\{ \left(\left(I - i\tau A^{\frac{1}{2}}\right)^{-1} (iA^{1/2})\right.
$$

$$
\times \frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) - R^{\left[\frac{\lambda_k}{\tau}\right]}\left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I - i\tau A^{1/2} + \frac{\tau^2}{2}A\right)}{2}\mu
$$

$$
+ \left(I + \tau^2 A\right)^{-1} \frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) + R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}\left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)}{2}(\omega + \frac{\tau}{2} f_0)
$$

$$
+ \sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2} \frac{\tau}{2} \left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}\left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right) + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right)\right] f_s
$$

$$
-\frac{\tau}{2} A \left(\left(I - i\tau A^{\frac{1}{2}}\right)^{-1} \frac{R^{\left[\frac{\lambda_k}{\tau}\right]}(I - i\tau A^{\frac{1}{2}} + \frac{\tau^2 A}{2}) + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}}{2}\mu\right.
$$

$$
+ \left(I + \tau^2 A\right)^{-1} (iA^{1/2})^{-1} \frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1} - R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}}{2}(\omega + \frac{\tau}{2} f_0)
$$

$$
\left. - \sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1} \frac{\tau}{2i} A^{-\frac{1}{2}} \left[R^{\left[\frac{\lambda_k}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right] f_s\right) + \tau\left(I + \frac{\tau^4}{4}A^2\right)^{-1} f_{\left[\frac{\lambda_k}{\tau}\right]-1} + \frac{\tau}{2} f_{\left[\frac{\lambda_k}{\tau}\right]}\Bigg\} + \psi
$$

Using the last two formulas, we obtain

$$
\mu = T_\tau \left\{\sum_{m=1}^{n} \alpha_m \left[\left(I + \tau^2 A\right)^{-1} (iA^{1/2})^{-1} \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1} - R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{-1}}{2}\right.\right. \tag{3.60}
$$

$$
\left. + \left(I + \tau^2 A\right)^{-1} (i\tau A^{1/2}) \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right) - R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{-1}\left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)}{2} \times \left(\frac{\lambda_m}{\tau} - \left[\frac{\lambda_m}{\tau}\right]\right)\right)
$$

$$
-\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right]f_s
$$

$$
+\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-2}\frac{\tau^2}{2}\left[R^{\left[\frac{\lambda_m}{\tau}\right]-s}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right]f_s\times\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)
$$

$$
+\tau^2\left(I+\frac{\tau^4A^2}{4}\right)^{-1}\left(\lambda_m-\left[\frac{\lambda_m}{\tau}\right]\tau\right)f_{\left[\frac{\lambda_m}{\tau}\right]-1}\Bigg\}\Bigg]+\varphi\Bigg\}
$$

$$
\times\left(I+\sum_{k=1}^{n}\beta_k\left(I+\tau^2A\right)^{-1}(iA^{1/2})^{-1}\left(\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\left(\frac{\tau}{2}A+A-i\frac{\tau}{2}A^{\frac{3}{2}}\right)}{2}\right.\right.
$$

$$
\left.\left.-\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{-1}\left(\frac{\tau}{2}A+A+i\frac{\tau}{2}A^{\frac{3}{2}}\right)}{2}\right)\right)
$$

$$
+T_\tau\left\{\sum_{m=1}^{n}\alpha_m\left[\left(I+\tau^2A\right)^{-1}(iA^{1/2})^{-1}\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}-R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{-1}}{2}\right.\right.
$$

$$
+\left(I+\tau^2A\right)^{-1}(i\tau A^{1/2})\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)-R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{-1}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)}{2}
$$

$$
\left.\times\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)\right)
$$

$$
\times\left[\sum_{k=1}^{n}\beta_k\left(-\frac{\tau}{2}f_0\left(I+\tau^2A\right)^{-1}(iA^{1/2})^{-1}\left(\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(\frac{\tau}{2}A+A-i\frac{\tau}{2}A^{\frac{3}{2}}\right)}{2}\right.\right.\right.
$$

$$
\left.-\frac{R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}\left(\frac{\tau}{2}A+A+i\frac{\tau}{2}A^{\frac{3}{2}}\right)}{2}\right)
$$

$$
+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{\tau}{2}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right]f_s
$$

$$
-\frac{\tau}{2}A\left(\left(I-i\tau A^{\frac{1}{2}}\right)^{-1}\frac{R^{\left[\frac{\lambda_k}{\tau}\right]}(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}}{2}\right.
$$

$$
\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right]f_s\right)
$$

$$
\left.\left.\left.+\tau^2\left(I+\frac{\tau^4A^2}{4}\right)^{-1}f_{\left[\frac{\lambda_k}{\tau}\right]-1}+\frac{\tau}{2}f_{\left[\frac{\lambda_k}{\tau}\right]}\right]\right)+\psi\right\}
$$

$$
\begin{aligned}
\omega = T_\tau \Bigg\{ \Bigg(I - \sum_{m=1}^{n} \alpha_m \Bigg(\left(\left(I - i\tau A^{\frac{1}{2}} \right)^{-1} (i\tau A^{1/2}) \left(\frac{\lambda_m}{\tau} - \left[\frac{\lambda_m}{\tau} \right] \right) \right. \\
\times \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2} \right) - R^{\left[\frac{\lambda_m}{\tau}\right]} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2} \right) \left(I - i\tau A^{1/2} + \frac{\tau^2}{2} A \right)}{2} \\
- \left(I - i\tau A^{\frac{1}{2}} \right)^{-1} \frac{R^{\left[\frac{\lambda_m}{\tau}\right]}(I - i\tau A^{\frac{1}{2}} + \frac{\tau^2 A}{2}) + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}}{2} \Bigg) \Bigg) \\
\times \Bigg[\sum_{k=1}^{n} \beta_k \Bigg(-\frac{\tau}{2} f_0 \left(I + \tau^2 A \right)^{-1} (iA^{1/2})^{-1} \Bigg(\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1} \left(\frac{\tau}{2} A + A - i\frac{\tau}{2} A^{\frac{3}{2}} \right)}{2} \\
- \frac{R^{\left[\frac{\lambda_k}{\tau}\right]} \tilde{R}^{-1} \left(\frac{\tau}{2} A + A + i\frac{\tau}{2} A^{\frac{3}{2}} \right)}{2} \Bigg) \\
+ \sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2} \frac{\tau}{2} \left[R^{\left[\frac{\lambda_k}{\tau}\right]-s} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2} \right)^{-1} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2} \right)^{-1} \right] f_s \\
- \frac{\tau}{2} A \Bigg(\left(I - i\tau A^{\frac{1}{2}} \right)^{-1} \frac{R^{\left[\frac{\lambda_k}{\tau}\right]}(I - i\tau A^{\frac{1}{2}} + \frac{\tau^2 A}{2}) + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}}{2} \\
+ \sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1} \frac{\tau}{2i} A^{-\frac{1}{2}} \left[R^{\left[\frac{\lambda_k}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s} \right] f_s \Bigg) \\
+ \tau^2 \left(I + \frac{\tau^4 A^2}{4} \right)^{-1} f_{\left[\frac{\lambda_k}{\tau}\right]-1} + \frac{\tau}{2} f_{\left[\frac{\lambda_k}{\tau}\right]} \Bigg] \Bigg) + \psi \Bigg\} \\
- T_\tau \Bigg\{ \sum_{m=1}^{n} \alpha_m \Bigg[\left(I + \tau^2 A \right)^{-1} (iA^{1/2})^{-1} \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1} - R^{\left[\frac{\lambda_m}{\tau}\right]} \tilde{R}^{-1}}{2} \\
+ \left(I + \tau^2 A \right)^{-1} (i\tau A^{1/2}) \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2} \right) - R^{\left[\frac{\lambda_m}{\tau}\right]} \tilde{R}^{-1} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2} \right)}{2} \\
\times \left(\frac{\lambda_m}{\tau} - \left[\frac{\lambda_m}{\tau} \right] \right) \Bigg) \\
- \sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1} \frac{\tau}{2i} A^{-\frac{1}{2}} \left[R^{\left[\frac{\lambda_m}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s} \right] f_s \\
+ \sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-2} \frac{\tau^2}{2} \left[R^{\left[\frac{\lambda_m}{\tau}\right]-s} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2} \right)^{-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2} \right)^{-1} \right] f_s \\
\times \left(\frac{\lambda_m}{\tau} - \left[\frac{\lambda_m}{\tau} \right] \right)
\end{aligned}
\tag{3.61}
$$

$$+\tau^2\left(I+\frac{\tau^4A^2}{4}\right)^{-1}\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)f_{\left[\frac{\lambda_m}{\tau}\right]-1}\Bigg\}\Bigg]+\varphi\Bigg\}$$

$$\times\sum_{k=1}^{n}\beta_k\left(I-i\tau A^{\frac{1}{2}}\right)^{-1}\left(\frac{R^{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\tau}{2}A+\frac{\tau^3}{4}A^2-\frac{\tau}{2}A-i\frac{\tau^2}{2}A^{\frac{3}{2}}-iA^{1/2}\right)}{2}\right.$$

$$\left.-\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(\frac{\tau}{2}A-iA^{\frac{1}{2}}-\tau A+\frac{i\tau}{2}A-i\tau^2A^{\frac{3}{2}}+i\frac{\tau^3}{4}A^2\right)}{2}\right)$$

where

$$T_\tau=\left(I-\sum_{m=1}^{n}\alpha_m\left(\left(I-i\tau A^{\frac{1}{2}}\right)^{-1}\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)(i\tau A^{1/2})\right.\right.\tag{3.62}$$

$$\times\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)-R^{\left[\frac{\lambda_m}{\tau}\right]}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I-i\tau A^{1/2}+\frac{\tau^2}{2}A\right)}{2}$$

$$\left.+\left(I-i\tau A^{\frac{1}{2}}\right)^{-1}\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}}{2}\right)$$

$$-\sum_{k=1}^{n}\beta_k\left(I+\tau^2A\right)^{-1}\left(\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(\frac{i\tau A^{\frac{1}{2}}}{2}+\tau-\frac{i\tau^2A^{\frac{1}{2}}}{2}\right)}{2}\right.$$

$$\left.-\frac{R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}\left(\frac{i\tau A^{\frac{1}{2}}}{2}-\tau-\frac{i\tau^2A^{\frac{1}{2}}}{2}\right)}{2}\right)$$

$$+\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left\{\left(I-i\tau A^{\frac{1}{2}}\right)^{-1}\left(I+\tau^2A\right)^{-1}\right.$$

$$\times\left(\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})\left(\frac{i\tau A^{\frac{1}{2}}}{2}+\tau-\frac{i\tau^2A^{\frac{1}{2}}}{2}\right)}{4}\right.$$

$$-\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})\left(\frac{i\tau A^{\frac{1}{2}}}{2}-\tau-\frac{i\tau^2A^{\frac{1}{2}}}{2}\right)}{4}$$

$$+\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(\frac{i\tau A^{\frac{1}{2}}}{2}+\tau-\frac{i\tau^2A^{\frac{1}{2}}}{2}\right)-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}\left(\frac{i\tau A^{\frac{1}{2}}}{2}-\tau-\frac{i\tau^2A^{\frac{1}{2}}}{2}\right)}{4}$$

$$+\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda_k}{\tau}\right]}(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}}{4}$$

$$\left.-\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})+R^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}^{-1}}{4}\right)$$

$$+\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)\left(I-i\tau A^{\frac{1}{2}}\right)^{-1}\left(I+\tau^2A\right)^{-1}\left(i\tau A^{\frac{1}{2}}\right)$$

$$\times\left(\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)}{4}+\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)}{4}\right.$$

$$-\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})}{4}$$

$$+\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})}{4}$$

$$+\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda_k}{\tau}\right]}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})}{4}-\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)}{4}$$

$$\left.+\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\tilde{R}^{-1}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)(I-i\tau A^{\frac{1}{2}}+\frac{\tau^2A}{2})}{4}-\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\tilde{R}^{-1}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)}{4}\right)\Bigg\}.$$

Hence, for the formal solution of the difference scheme (3.53) we have the formula (3.59),(3.60),(3.61),(3.62). For substantiation of these formulas we need to obtain the stability estimates(3.55),(3.56),(3.57) for solutions of the difference scheme (3.53).From the symmetry and positivity properties of the operator A it follows that

$$\| T_\tau \|_{H\mapsto H} \le \left(1-\sum_{m=1}^{n}|\alpha_m|\left(1+2\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)\right)-\sum_{k=1}^{n}|\beta_k|+\right.$$
$$\left.-\sum_{m=1}^{n}\sum_{k=1}^{n}|\alpha_m|\,|\beta_k|\left(1+3\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)+\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)^2\right)\right)^{-1} \tag{3.63}$$

and the estimates

$$\begin{cases} \|R\|_{H\mapsto H}\le 1, & \|\tilde{R}\|_{H\mapsto H}\le 1, \\ \|R\tilde{R}^{-1}\|_{H\mapsto H}\le 1, & \|\tilde{R}R^{-1}\|_{H\mapsto H}\le 1, \\ \|\tau A^{1/2}R\|_{H\mapsto H}\le 1, & \|\tau A^{1/2}\tilde{R}\|_{H\mapsto H}\le 1. \end{cases} \tag{3.64}$$

The proof of estimates(3.55),(3.56),(3.57) uses the outline of the proof Theorem 3.10 and is based on the formulas(3.55),(3.56), (3.57)and estimates (3.59) and

(3.60). Theorem 3.12 is proved.

Theorem 3.17. *Let* $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$ *and*

$$\left(\sum_{m=1}^{n}|\alpha_m|\left(1+2\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)\right)+\sum_{k=1}^{n}|\beta_k|+\right.$$

$$+\sum_{m=1}^{n}\sum_{k=1}^{n}|\alpha_m|\,|\beta_k|\left(1+3\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)+\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)^2\right)\Bigg)<1.$$

Then for the solution of the difference scheme (3.54) the stability inequalities

$$\max_{0\le k\le N}\|u_k\|_H\le M\{\sum_{s=0}^{N}\|A^{-1/2}f_s\|_H\tau+\|\ A^{-1/2}\psi\ \|_H+\|\ \varphi\ \|_H\}, \tag{3.65}$$

$$\max_{0\le k\le N}\|A^{1/2}u_k\|_H\le M\{\sum_{s=0}^{N}\|\ f_s\ \|_H\ \tau+\|\ A^{1/2}\varphi\|_H+\|\ \psi\|_H\}, \tag{3.66}$$

$$\max_{1\le k\le N-1}\|\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\|_H+\max_{0\le k\le N}\|Au_k\ \|_H \tag{3.67}$$

$$\le M\{\sum_{s=1}^{N}\|\ f_s-f_{s-1}\ \|_H+\|\ f_0\ \|_H+\|\ A^{1/2}\psi\ \|_H+\|\ A\varphi\ \|_H\}$$

hold ,where M does not depend on $f_s, 0\le s\le N$ and φ,ψ.

Proof. We will write the formula for the solution of the difference scheme (3.54). It is easy to show that [Ashyralyev A. and Sobolevskii P.E. 2001] there are unique solution of the problem

$$\begin{cases}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k,\\ f_k=f(t_k),\quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1,\\ (I+\frac{\tau^2A}{4})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)=\omega, f_0=f(0), u_0=\mu\end{cases} \tag{3.68}$$

and for the solutions of these problems the following formulas hold:

$$u_0=\mu, u_1=\left(I+\frac{\tau^2A}{4}\right)^{-1}\left[(I-\frac{\tau^2A}{4})\mu+\tau\omega+\frac{\tau^2}{2}f_0\right],$$

$$u_k=\frac{R^k+\tilde{R}^k}{2}\mu+(iA^{\frac{1}{2}})^{-1}\frac{\tilde{R}^k-R^k}{2}(\omega+\frac{\tau}{2}f_0)$$

$$-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}-\tilde{R}^{k-s}\right]f_s, 2\le k\le N, \tag{3.69}$$

where

$$R=\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(I+\frac{i\tau A^{1/2}}{2}\right)^{-1},$$

$$\tilde{R}=\left(I+\frac{i\tau A^{1/2}}{2}\right)\left(I-\frac{i\tau A^{1/2}}{2}\right)^{-1}.$$

Applying the last formula and the nonlocal boundary conditions

$$u_0=\sum_{m=1}^{n}\alpha_m\left\{u_{[\frac{\lambda_m}{\tau}]}+\tau^{-1}(u_{[\frac{\lambda_m}{\tau}]}-u_{[\frac{\lambda_m}{\tau}]-1})\left(\lambda_m-\left[\frac{\lambda_m}{\tau}\right]\tau\right)\right\}+\varphi,$$

$$(I+\frac{\tau^2 A}{4})[(I+\frac{\tau^2 A}{4})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)]$$
$$=\sum_{k=1}^{n}\beta_k\left\{\tau^{-1}(u_{\left[\frac{\lambda_k}{\tau}\right]}-u_{\left[\frac{\lambda_k}{\tau}\right]-1})+\frac{\tau}{2}(f_{\left[\frac{\lambda_k}{\tau}\right]}-Au_{\left[\frac{\lambda_k}{\tau}\right]})\right\}+\psi,$$

we can write

$$\mu=\sum_{m=1}^{n}\alpha_m\left\{\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}}{2}\mu+(iA^{\frac{1}{2}})^{-1}\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}-R^{\left[\frac{\lambda_m}{\tau}\right]}}{2}(\omega+\frac{\tau}{2}f_0)\right.$$
$$-\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-\frac{1}{2}}\left[R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right]f_s$$
$$+\left[(-iA^{\frac{1}{2}})\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}\mu\right.$$
$$+\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}+R^{\left[\frac{\lambda_m}{\tau}\right]}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}(\omega+\frac{\tau}{2}f_0)$$
$$+\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-2}\frac{\tau}{2}\left[R^{\left[\frac{\lambda_m}{\tau}\right]-s}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right]f_s$$
$$\left.\left.+2\tau\left(I+\frac{\tau^2 A}{4}\right)^{-1}f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right]\times\left(\lambda_m-\left[\frac{\lambda_m}{\tau}\right]\tau\right)\right\}+\varphi$$

and

$$\omega=\left(I+\frac{\tau^2 A}{4}\right)^{-1}\sum_{k=1}^{n}\beta_k$$
$$\times\left\{(-iA^{\frac{1}{2}})\frac{R^{\left[\frac{\lambda_k}{\tau}\right]}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}\mu\right.$$
$$+\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}+R^{\left[\frac{\lambda_k}{\tau}\right]}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}(\omega+\frac{\tau}{2}f_0)$$
$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{\tau}{2}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right]f_s$$
$$+\tau\left(I+\frac{\tau^2 A}{4}\right)^{-1}f_{\left[\frac{\lambda_k}{\tau}\right]-1}$$
$$-A\frac{\tau}{2}\left[\frac{R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}}{2}\mu+(iA^{\frac{1}{2}})^{-1}\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}-R^{\left[\frac{\lambda_k}{\tau}\right]}}{2}(\omega+\frac{\tau}{2}f_0)\right.$$

$$- \sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1} \frac{\tau}{2i} A^{-\frac{1}{2}} \left[R^{\left[\frac{\lambda_k}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s} \right] f_s \Bigg] + \frac{\tau}{2} f_{\left[\frac{\lambda_k}{\tau}\right]} \Bigg\} + \left(I + \frac{\tau^2 A}{4}\right)^{-1} \psi$$

Using the last two formulas, we obtain

$$\mu = T_\tau \Bigg\{ \sum_{m=1}^{n} \alpha_m \Bigg[\frac{\tau}{2} f_0 \Bigg((iA^{\frac{1}{2}})^{-1} \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]} - R^{\left[\frac{\lambda_m}{\tau}\right]}}{2} \tag{3.70}$$

$$+ \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1} + R^{\left[\frac{\lambda_m}{\tau}\right]} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2} \left(\lambda_m - \left[\frac{\lambda_m}{\tau}\right] \tau \right) \Bigg)$$

$$- \sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1} \frac{\tau}{2i} A^{-\frac{1}{2}} \left[R^{\left[\frac{\lambda_m}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s} \right] f_s$$

$$+ \sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-2} \frac{\tau}{2} \left(R^{\left[\frac{\lambda_m}{\tau}\right]-s} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1} \right) f_s \times \left(\lambda_m - \left[\frac{\lambda_m}{\tau}\right] \tau \right)$$

$$+ 2\tau \left(I + \frac{\tau^2 A}{4}\right)^{-1} \left(\lambda_m - \left[\frac{\lambda_m}{\tau}\right] \tau \right) f_{\left[\frac{\lambda_m}{\tau}\right]-1} \Bigg\} \Bigg] + \varphi \Bigg\}$$

$$\times \Bigg(I + \left(I + \frac{\tau^2 A}{4}\right)^{-1} \sum_{k=1}^{n} \beta_k \Bigg(\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} \left(\frac{\tau^2 A}{4} - \frac{i\tau A^{\frac{1}{2}}}{2} - 1 \right) \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}$$

$$+ \frac{R^{\left[\frac{\lambda_k}{\tau}\right]} \left(\frac{\tau^2 A}{4} + \frac{i\tau A^{\frac{1}{2}}}{2} - 1 \right) \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2} \Bigg)$$

$$+ T_\tau \Bigg\{ \sum_{m=1}^{n} \alpha_m \Bigg[\frac{\tau}{2} f_0 \Bigg((iA^{\frac{1}{2}})^{-1} \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]} - R^{\left[\frac{\lambda_m}{\tau}\right]}}{2}$$

$$+ \frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]} \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1} + R^{\left[\frac{\lambda_m}{\tau}\right]} \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2} \left(\lambda_m - \left[\frac{\lambda_m}{\tau}\right] \tau \right) \Bigg]$$

$$\times \Bigg[- \left(I + \frac{\tau^2 A}{4}\right)^{-1} \frac{\tau}{2} f_0 \sum_{k=1}^{n} \beta_k \Bigg(\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} \left(\frac{\tau^2 A}{4} - \frac{i\tau A^{\frac{1}{2}}}{2} - 1 \right) \left(I + \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}$$

$$+ \frac{R^{\left[\frac{\lambda_k}{\tau}\right]} \left(\frac{\tau^2 A}{4} + \frac{i\tau A^{\frac{1}{2}}}{2} - 1 \right) \left(I - \frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2} \Bigg)$$

$$+\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-2}\frac{\tau}{2}\left[R^{\left[\frac{\lambda_m}{\tau}\right]-s}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right]f_s$$

$$+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau^2}{4i}A^{\frac{1}{2}}\left[R^{\left[\frac{\lambda_k}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right]f_s$$

$$+2\tau f_{\left[\frac{\lambda_k}{\tau}\right]-1}+\frac{\tau}{2}f_{\left[\frac{\lambda_k}{\tau}\right]}\Bigg]+\psi\left(I+\frac{\tau^2A}{4}\right)^{-1}\Bigg\}$$

where

$$T_\tau=\Bigg(I+\sum_{m=1}^{n}\alpha_m\Bigg(\left(\lambda_m-\left[\frac{\lambda_m}{\tau}\right]\tau\right)(iA^{\frac{1}{2}})$$

$$\times\frac{R^{\left[\frac{\lambda_k}{\tau}\right]}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}+\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}}{2}\Bigg)$$

$$+\left(I+\frac{\tau^2A}{4}\right)^{-1}\sum_{k=1}^{n}\beta_k\Bigg(\frac{\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{\tau^2A}{4}-\frac{i\tau A^{\frac{1}{2}}}{2}-1\right)\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}$$

$$+\frac{R^{\left[\frac{\lambda_k}{\tau}\right]}\left(I-\frac{\tau^2A}{4}-\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}\Bigg)$$

$$+\left(I+\frac{\tau^2A}{4}\right)^{-1}\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\Bigg\{\left(\lambda_m-\left[\frac{\lambda_m}{\tau}\right]\tau\right)$$

$$\Bigg((iA^{\frac{1}{2}})\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\left(2-\frac{\tau^2A}{4}-\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I+\frac{\tau^2A}{4}\right)^{-1}}{4}$$

$$+(iA^{\frac{1}{2}})\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\left(2-\frac{\tau^2A}{4}-\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-2}}{4}$$

$$-\frac{\tau A}{2}\Bigg(\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}+\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{2}$$

$$+\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{4}+\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}}{4}\Bigg)\Bigg)$$

$$+\frac{R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{i\tau A^{\frac{1}{2}}}{2}+\left(2-\frac{\tau^2A}{4}+\frac{i\tau A^{\frac{1}{2}}}{2}\right)\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right)}{4}$$

$$-\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{i\tau A^{\frac{1}{2}}}{2}+\left(\frac{\tau^2A}{4}-\frac{i\tau A^{\frac{1}{2}}}{2}-2\right)\left(I+\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right)}{4}$$

$$-\frac{\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\left(\frac{i\tau A^{\frac{1}{2}}}{2}+\left(\frac{\tau^2 A}{4}+\frac{i\tau A^{\frac{1}{2}}}{2}-2\right)\left(I-\frac{i\tau A^{\frac{1}{2}}}{2}\right)^{-1}\right)}{4}\Bigg\}.$$

Hence, for the formal solution of the difference scheme(3.54) we have the formula(3.39),(3.70),(3.71). For substantiation of these formulas we need to obtain the stability estimates(3.65) ,(3.66),(3.67) for solutions of the difference scheme (3.54).From the symmetry and positivity properties of the operator A it follows that

$$\| T_\tau \|_{H\mapsto H} \leq \left(1-\sum_{m=1}^{n}|\alpha_m|\left(1+2\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)\right)-\sum_{k=1}^{n}|\beta_k|+\right. \tag{3.71}$$

$$\left.-\sum_{m=1}^{n}\sum_{k=1}^{n}|\alpha_m|\,|\beta_k|\left(1+3\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)+\left(\frac{\lambda_m}{\tau}-\left[\frac{\lambda_m}{\tau}\right]\right)^2\right)\right)^{-1} \tag{3.72}$$

and the estimates

$$\begin{cases} \|R\|_{H\mapsto H}\leq 1, \qquad \|\tilde{R}\|_{H\mapsto H}\leq 1, \\ \\ \|\left(I+\frac{\tau^2 A}{4}\right)^{-1}\|_{H\mapsto H}\leq 1. \end{cases} \tag{3.73}$$

The proof of estimates (3.65),(3.66),(3.67) uses the outline of the proof Theorem 3.10 and is based on the formulas (3.69),(3.70),(3.71)and estimates (3.64),(3.65),(3.66),(3.67). Theorem 3.13 is proved.

We consider the application of Theorem 3.12 by the same way as it is shown in the

end of theorem 3.11.

CHAPTER 4

APPLICATIONS

4.1 The First Order of Accuracy in Time Difference Scheme

We consider the nonlocal boundary-value problem for wave equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = 6t\sin\pi x + \pi^2 t^3 \sin\pi x, \\ 0 < t < 1,\ 0 < x < 1, \\ u(0,x) = \frac{1}{4}u(1,x) - \frac{1}{4}u(\frac{1}{2},x) + \psi(x), \\ \psi(x) = -\frac{7}{32}\sin\pi x, 0 \le x \le 1, \\ u_t(0,x) = \frac{1}{4}u_t(1,x) - \frac{1}{4}u_t(\frac{1}{2},x) + \varphi(x), \\ \varphi(x) = -\frac{9}{16}\sin\pi x,\ 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0, \ \ -1 \le t \le 1. \end{cases} \tag{4.1}$$

The exact solution is:

$$u(t,x) = t^3 \sin\pi x.$$

For approximate solution of the nonlocal boundary-value problem (4.1), we consider the set $[0,1]_\tau \times [0,1]_h$ of a family of grid points depending on the 'small' parameters τ and h:

$[0,1]_\tau \times [0,1]_h = \{(t_k, x_n) : t_k = k\tau,\ \ 0 \le k \le N,\ \ N\tau = 1,$
$x_n = nh,\ \ 0 \le n \le M,\ Mh = 1\}.$

Applying the formulas

$$\frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_{k+1}) = O(\tau^2),$$

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O(h^2),$$

and

$$\frac{u(1) - u(0)}{\tau} - u'(0) = O(\tau),$$
$$\frac{u(1) - u(1-\tau)}{\tau} - u'(1) = O(\tau)$$

and using the first order of accuracy in t implicit difference scheme for wave equation, we obtain the difference scheme first order of accuracy in t and second order of accuracy in x for approximate solutions of the nonlocal boundary value problem (4.1)

$$\begin{cases} \frac{U_n^{k+1}-2U_n^k+u_n^{k-1}}{\tau^2} - \frac{U_{n+1}^{k+1}-2U_n^{k+1}+U_{n-1}^{k+1}}{h^2} = f(t_{k+1}, x_n),\ t_{k+1} = (k+1)\tau, \\ 0 \le k \le N,\ N\tau = 1,\ x_n = nh,\ 1 \le n \le M-1,\ Mh = 1, \\ U_n^0 = \frac{1}{4}U_n^N - \frac{1}{4}U_n^{(N/2)+1} - \frac{7}{32}\sin \pi x, 1 \le n \le M-1, \\ \tau^{-1}(U_n^1 - U_n^0) \ = \frac{1}{4}\,\tau^{-1}\left(U_n^N - U_n^{N-1}\right) - \frac{1}{4}\tau^{-1}\left(U_n^{(N/2)+1} - U_n^{N/2}\right) \\ -\frac{9}{16}\sin \pi x,\ x_n = nh,\ 1 \le n \le M-1, \\ U_0^k = U_M^k = 0, 0 \le k \le N. \end{cases} \tag{4.2}$$

We have the $(N+1)\times(M+1)$ system of linear equations. We will write it in the matrix form. We will resort the system

$$\begin{cases} \left(\frac{-1}{h^2}\right)U_{n+1}^{k+1} + \left(\frac{1}{\tau^2} + \frac{2}{h^2} + 1\right)U_n^{k+1} + \left[-\frac{2}{\tau^2}\right]U_n^k + \left(\frac{1}{\tau^2}\right)U_n^{k-1} + \left[-\frac{1}{h^2}\right]U_{n-1}^{k+1} = \varphi_n^k, \\ \varphi_n^k = \left[6(k+1)\tau + \pi^2\left((k+1)\tau\right)^3\right]\sin(nh) \quad, \\ \ 1 \le k \le N-1,\ 1 \le n \le M-1 \\ U_n^0 = \frac{1}{4}U_n^N - \frac{1}{4}U_n^{(N/2)+1} - \frac{7}{32}\sin \pi x, 1 \le n \le M-1, \\ \tau^{-1}(U_n^1 - U_n^0) \ = \frac{1}{4}\,\tau^{-1}\left(U_n^N - U_n^{N-1}\right) - \frac{1}{4}\tau^{-1}\left(U_n^{(N/2)+1} - U_n^{N/2}\right) \\ -\frac{9}{16}\sin \pi x,\ x_n = nh,\ 1 \le n \le M-1, \\ U_0^k = U_M^k = 0, 0 \le k \le N. \end{cases}$$

We have that

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ U_0 = 0,\ \ U_M = 0. \end{cases} \tag{4.3}$$

Denote

$$a = \left(-\frac{1}{h^2}\right), \qquad b = \left(\frac{1}{\tau^2}\right),$$
$$c = \left(\frac{-2}{\tau^2}\right), \qquad d = \left(\frac{1}{\tau^2} + \frac{2}{h^2}\right),$$

$$
\varphi_n^k = \begin{cases} 6t_{k+1}\sin(\pi x_n) + \pi^2 t_{k+1}^3 \sin(\pi x_n), \\ \ 1 \le k \le N-1, \\ f(t_{k+1}, x_n),\ 1 \le k \le N-1, \\ 0,\ k = N, \end{cases} ,
$$

$$
\varphi_n^0 = -\frac{7}{32}\sin(\pi x_n), \varphi_n^N = -\frac{9}{16}\sin(\pi x_n),
$$

$$
\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \ldots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1} ,
$$

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & .. & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0
\end{bmatrix}_{(N+1)\times(N+1)} ,
$$

$$
B = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \ldots & 1/4 & 0 & \ldots & 0 & 0 & 0 & -1/4 \\
b & c & d & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & b & c & d & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & b & c & d & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & & \ldots & 0 & 0 & \ldots & b & c & d & 0 \\
0 & 0 & 0 & 0 & & \ldots & 0 & 0 & \ldots & & b & c & d \\
-1 & 1 & 0 & 0 & & \ldots & -1/4 & 1/4 & \ldots & & 0 & 1/4 & -1/4
\end{bmatrix}_{(N+1)\times(N+1)} ,
$$

and $C = A$.

$$C = A,\ D = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ ... \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s = \ n-1,\ n,\ n+1.$$

For the solution of the last matrix equation, we will use the modified variant Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, ..., 2, 1, 0,$$

where α_j , β_j, $(j = 1, ..., M-1)$ are $(N+1)\times(N+1)$ square matrices . And $\alpha_1,\ \beta_1$:

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}_{(N+1)\times 1}, \text{ where } U_0 = \widetilde{0},\ U_M = \widetilde{0}.$$

Using the equality $U_s = \alpha_{s+1} U_{s+1} + \beta_{s+1}$, (for $s = n,\ n-1$) and the equality
$AU_{n+1} + B\,U_n + CU_{n-1} = D\varphi_n$,
we can write

$$[A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1}]U_{n+1} + \left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n.$$

The last equation is satisfied if it is to be selected:

$$\begin{cases} A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1} = 0, \\ \left[B\beta_{n+1} + C\alpha_n\beta_{n+1} + C\beta_n\right] = D\varphi_n, \\ 1 \le n \le M - 1. \end{cases}$$

Then we obtain the formulas for $\alpha_{n+1},\ \beta_{n+1}$:

$$\alpha_{n+1} = -\left(B + C\alpha_n\right)^{-1} A,$$
$$\beta_{n+1} = \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), n = 1, 2, 3, ...M - 1.$$

So,

$$U_M = \sigma,$$
$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}, \quad n = M - 1, ..., 2, 1, 0.$$

Algorithm

1. **Step** Input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$.

2. **Step** Use the second order of accuracy difference scheme and write in matrix form:

$$A\, U_{n+1} + B\, U_n + C\, U_{n-1} = D\varphi_n, \quad 1 \le n \le M - 1.$$

3. **Step** Determine the entries of the matrices $A,\ B,\ C$ and D.

4. **Step** Find α_1, β_1.

5. **Step** Compute $\alpha_{n+1},\ \beta_{n+1}$.

6. **Step;** Compute U_n-s $(n = M - 1, ..., 2, 1), (U_M = 0)$ using the following formula:

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}.$$

Matlab Implementation of the First Order of Accuracy Difference Scheme

```
function firstord(N,M)

close; close;

if nargin<1; N=10; M=10; end;
```

```
tau=1/N;

h=1/M;

al=1/4;

a = -1/(h^2);

b = 1/(tau^2);

c = -2/(tau^2);

d = (1/(tau^2)) + (2/(h^2));

for i=2:N; A(i,i+1)=a; end;

A(N+1,N+1)=0; A;

C=A;

for i=2:N ; B(i,i-1)= b ; end;

for i=2:N ; B(i,i)= c ; end;

for i=2:N ; B(i,i+1)= d ; end;

B(1,1)=1;B(1,(N/2)+1)=al;B(1,N+1)=-al;

B(N+1,1)=-1; B(N+1,2)=1; B(N+1,(N/2)+1)=al;

B(N+1,N/2)=-al; B(N+1,N) =al ; B(N+1,N+1)=-al;

B;

for i=1:N+1; D(i,i)=1; end ;

D;

'fii(j) finding ' ;

for j=1:M+1;

x=((j-1)*h);

fii(1,j:j) =-(7/32)*sin(pi*x) ;

fii(N+1,j:j) =tau*(-(9/16)*sin(pi*x));

for k=2:N;

fii(k,j:j) =6*((k-1)*tau)*(sin(pi*x))+((pi)^2)*(((k-1)*tau)^3)*(sin(pi*x));

end;
```

```
end;
fii;

alpha(N+1,N+1,1:1)= 0 ;
betha(N+1,1:1) = 0 ;
for j=1:M-1;
alpha( :, :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j))*(-A) ;
betha( :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j ) )*(D*fii(:, j:j )...
- C * betha(:, j:j ) );
end;
U( N+1,1, M:M ) = 0;
for z = M-1:-1:1 ;
U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 ) + betha(:,z+1:z+1);
end;
for z = 1:M ;
p(:,z+1:z+1)=U(:,:,z:z);
end;
'EXACT SOLUTION OF THIS PROBLEM' ;
for j=1:M+1 ;
for k=1:N+1;
x=(j-1)*h;
es( k, j:j )=(((k-1)*tau)^3)* sin((pi)*x);
end;
end;
es;
% 'ERROR ANALYSIS' ;
maxes=max(max(es)) ;
```

```
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxes,maxapp,maxerror,relativeerror]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;
axis tight;
title 'EXACT SOLUTION';
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p) ; rotate3d ;
title 'DIFFERENCE SCHEMES SOLUTION';
axis tight;
```

4.2 The Second Order of Accuracy in Time Difference Scheme

We consider again the nonlocal boundary-value problem (4.1). Applying the formulas

$$\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-u"(t_k)=O(\tau^2),$$
$$\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2}-u"(x_n)=O(h^2)$$

and using the second order of accuracy in t implicit difference scheme for(3.5)wave equation, we obtain the difference scheme second order of accuracy in t and in x for approximate solutions of the nonlocal boundary value problem (4.1)

$$\begin{cases} \frac{U_n^{k+1}-2U_n^k+U_{n-1}^{k-1}}{\tau^2}-\frac{U_{n+1}^k-2U_n^k+U_{n-1}^k}{2h^2}-\frac{U_{n+1}^{k+1}-2U_n^{k+1}+U_{n-1}^{k+1}}{4h^2} \\ -\frac{U_{n+1}^{k-1}-2U_n^{k-1}+U_{n-1}^{k-1}}{4h^2}=\varphi_n^k, \\ \varphi_n^k=6(k\tau)\sin(\pi x_n)+\pi^2(k\tau)^3\sin(\pi x_n), \\ x_n=nh,\ t_k=k\tau,\ 1\le k\le N-1,\ 1\le n\le M-1, \\ U_n^0=\frac{1}{4}U^N-\frac{1}{4}U^{(N/2)+1}-\frac{7}{32}\sin\pi x, 1\le n\le M-1, \\ \left(\frac{I+\tau^2A_h^x}{\tau}\right)\left[\left(\frac{I+\tau^2A_h^x}{\tau}\right)\left(u_1^h(x_n)-u_0^h(x_n)\right)\right. \\ \left.+\frac{\tau}{2}(f^h(0,x_n)-A_h^x u_0^h(x_n)\right]= \\ \frac{1}{4}(2\tau)^{-1}\left(u_{N-2}^h(x_n)-4u_{N-1}^h(x_n)+3u_N^h(x_n)\right) \\ -\frac{1}{4}(2\tau)^{-1}\left(u_{(N/2)+1}^h(x_n)-u_{(N/2)-1}^h(x_n)\right)- \\ \frac{9}{16}\sin(\pi x_n),\ 1\le n\le M-1, \\ x_n=nh,\ t_k=k\tau,\ 1\le k\le N-1,\ 1\le n\le M-1, \\ u_0^k=u_M^k=0, 0\le k\le N, \\ u_1^k=\frac{4}{5}u_2^k-\frac{1}{5}u_3^k,\ u_{M-1}^k=\frac{4}{5}u_{M-2}^k-\frac{1}{5}u_{M-3}^k,\ 0\le k\le N. \end{cases} \tag{4.4}$$

where

We have again the $(N+1)\times(M+1)$ system of linear equations. We will write it in the matrix form. We will resort the system

$$\begin{cases} \left(\frac{-1}{4h^2}\right)U_{n+1}^{k+1} + \left(-\frac{1}{2h^2}\right)U_{n+1}^{k} + \left[-\frac{1}{4h^2}\right]U_{n+1}^{k-1} \\ +\left(\frac{1}{\tau^2}+\frac{1}{2h^2}\right)U_n^{k+1} + \left[-\frac{2}{\tau^2}+\frac{1}{h^2}\right]U_n^k + \left(\frac{1}{\tau^2}+\frac{1}{2h^2}\right)U_n^{k-1} \\ +\left[-\frac{1}{4h^2}\right]U_{n-1}^{k+1} + \left(-\frac{1}{2h^2}\right)U_{n-1}^{k} + \left[-\frac{1}{4h^2}\right]U_{n-1}^{k-1} = \varphi_n^k, \\ 1\le k\le N-1,\ \ 1\le n\le M-1 \\ \varphi_n^k = \begin{cases} 6t\sin(\pi x_n)+\pi^2t^3\sin(\pi x_n), \\ 1\le k\le N-1, \\ 0,\ k=N, \end{cases}, \\ x_n = nh,\ t_k = k\tau,\ 1\le k\le N-1,\ 1\le n\le M-1, \\ U_0^k = U_M^k = 0, 0\le k\le N, \\ U_1^k = \frac{4}{5}U_2^k - \frac{1}{5}U_3^k,\ U_{M-1}^k = \frac{4}{5}U_{M-2}^k - \frac{1}{5}U_{M-3}^k,\ 0\le k\le N. \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdots \\ \cdots \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}.$$

$$\varphi_n^k = \begin{cases} 6t\sin(\pi x_n)+\pi^2t^3\sin(\pi x_n), \\ 1\le k\le N-1, \\ 0,\ k=N, \end{cases}$$

$$\varphi_n^0 = -\frac{7}{32}\sin(\pi x_n), \varphi_n^N = -\frac{9}{16}\sin(\pi x_n),$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ e & e & 0 & 0 & . & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & . & & 0 & 0 & 0 \\ z & w & z & 0 & . & & 0 & 0 & 0 \\ . & z & w & z & . & & . & . & . \\ 0 & 0 & z & w & . & & 0 & 0 & 0 \\ . & . & . & . & . & & . & . & . \\ . & . & . & . & . & z & w & z & . \\ 0 & 0 & 0 & 0 & . & & z & w & z \\ f & g & 0 & . & . & & . & . & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & ... & 1/4 & 0 & ... & 0 & 0 & 0 & -1/4 \\ x & y & x & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & x & y & x & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & x & y & x & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & ... & 0 & 0 & ... & x & y & x & 0 \\ 0 & 0 & 0 & 0 & & ... & 0 & 0 & ... & & x & y & x \\ m & n & 0 & 0 & & ... & -k & k & ... & & -k & p & -r \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$D = B,\ E = A,\ R = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix},$$

We have again the matrix equation (4.4) with new data:

$$x = \frac{1}{\tau^2} + \frac{1}{2h^2}, y = -\frac{2}{\tau^2} + \frac{1}{h^2}, z = -\frac{1}{4h^2},$$
$$w = -\frac{1}{2h^2}, e = \frac{\tau^3}{16h^4}, f = -\frac{\tau^3}{4h^4},$$

$$g = -\frac{\tau}{2h^2} - \frac{\tau^3}{4h^4}, \ m = -\frac{1}{\tau} + \frac{6\tau^3}{16h^4}, n = \frac{1}{\tau} + \frac{\tau}{h^2} + \frac{6\tau^3}{16h^4},$$
$$k = \frac{1}{8\tau}, \ p = \frac{1}{2\tau}, r = \frac{3}{8\tau},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ \ldots \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)} , \quad s = \ n-1, \ n, \ n+1.$$

For the solution of the last matrix equation, we will use the modified variant Gauss elimination method. We seek a solution of the matrix equation by the following form

$$\left\{ \ U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1} U_{n+2} + \gamma_{n+1}, \quad n = M-2, ..., 2, 1, 0, \right. \tag{4.5}$$

where α_j , β_j, $(j = 1 : M-1)$ are $(N+1) \times (N+1)$ square matrices and γ_j-s are $(N+1) \times 1$ column matrices. And $\alpha_1, \ \beta_1, \ \gamma_1, \ \alpha_2, \ \beta_2, \ \gamma_2$:

$$\begin{cases} \beta_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(A), \\ \alpha_{n+1} = -(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \gamma_{n+1} = +(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n)^{-1} \\ \times(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}), \end{cases} \tag{4.6}$$

where $n = 2 : M-2$. Here

$$\alpha_1 = \begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix}, \ \beta_1 = \begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix},$$

$$\gamma_1 = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \end{bmatrix},$$

$$\alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & ... & 0 \\ 0 & \frac{4}{5} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \frac{4}{5} \end{bmatrix}, \ \beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & ... & 0 \\ 0 & -\frac{1}{5} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & -\frac{1}{5} \end{bmatrix},$$

$$\begin{cases} U_M = \tilde{0}, \\ U_{M-1} = [(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\alpha_{M-1}]^{-1}[(4I - \alpha_{M-2})\gamma_{M-1} - \gamma_{M-2}], \\ U_{M-2} = [(4I - \alpha_{M-2})]^{-1}[(\beta_{M-2} + 5I)U_{M-1} + \gamma_{M-2}], \\ U_0 = \tilde{0}. \end{cases}$$

Matlab Implementation of the Second Order of Accuracy Difference Scheme

```
function secondorderA(N,M)

close; close;

if nargin<1; N=10; M=10; end

al=1/4;

at=3/4;

tau=1/N; h=1/M ;

x= (1/(tau^2))+(1/(2*(h^2)));

y= (-2/(tau^2))+(1/(h^2)) ;

z= -(1/(4*(h^2)));

w= (-1/(2*(h^2)));
```

```
for i=1:N ;
A(i,i)=0;end;
A(N+1,1) = tau^3/(16*(h^4)) ;
A(N+1,2)=tau^3/(16*(h^4));
A(N+1,N+1) =0; A(1,1)=0;A;
for i=2:N ;
B(i,i-1)=z ;
B(i,i) =w ;
B(i,i+1)=z ; end;
B(1,1) =0 ;
B(N+1,1)=-((tau^3)/(4*(h^4)));
B(N+1,2)=(-tau/(2*(h^2))+(-tau^3)/(4*(h^4))); B;
for i=2:N ; C(i,i-1)= x ; end;
for i=2:N; C(i,i)= y ; end;
for i=2:N ; C(i,i+1)= x ; end;
C(1,1)=1 ;C(1,N+1)=-1/4 ; C(1,(N/2)+1)=1/4;
C(N+1,1)=(-1/tau)+(6*tau^3)/(16*(h^4));
C(N+1,2)=(1/tau)+(tau/(h^2))+(6*tau^3)/(16*(h^4)) ;
C(N+1,(N/2))=-(al/(2*tau));
C(N+1,(N/2)+2)=(al/(2*tau));
C(N+1,N+1)=-at/(2*tau);C(N+1,N)=1/(2*tau) ;
C(N+1,N-1)=-al/(2*tau);
C;
for i=2:N ;
D(i,i-1)=z ;
D(i,i) =w ;
D(i,i+1)=z ; end;
```

```
D(1,1) =0 ;

D(N+1,1)=-(tau^3)/(4*(h^4));

D(N+1,2)=((-2*tau*(h^2))-(tau^3))/(4*(h^4));D;

for i=1:N ;

E(i,i)=0;end;

E(N+1,1) = tau^3/(16*(h^4)) ;

E(N+1,2)=tau^3/(16*(h^4));

E(N+1,N+1) =0; E(1,1)=0;

E;

for i=1:N+1; R(i,i)=1 ; end;

R;

alpha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;

betha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;

gamma(N+1,1:1)= 0 ;

alpha(1:N+1,1:N+1,2:2) = (4/5)*eye(N+1) ;

betha(1:N+1,1:N+1,2:2) = (-1/5)*eye(N+1);

gamma(N+1,2:2)= 0 ;

'fii(j) finding ' ;

for j=1:M+1;

x=((j-1)*h);

fii(1,j:j) =-(7/32)*sin(pi*x) ;

fii(N+1,j:j) =-(9/16)*sin(pi*x);

for k=2:N;

fii(k,j:j) =6*((k-1)*tau)*(sin(pi*x))+((pi)^2)...

*(((k-1)*tau)^3)*(sin(pi*x));

end;

end;
```

```
fii;
for n = 2:M-2 ;
bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)...
+ E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;
betha(:,:,n+1:n+1 ) = -inv( bebek )*(A) ;
alpha(:,:,n+1:n+1) = -inv(bebek )*(B +D*betha(:,:,n:n)...
+ E * alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*...
(R*fii(:,n:n) - D * gamma(:,n:n)-E * alpha(:,:,n-1:n-1)...
* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
U(1:N+1,1:N+1)=nan;
U( 1:N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2)...
+ 5*eye(N+1))- (4*eye(N+1)-alpha(:,:,M-2:M-2))...
*alpha(:,:,M-1:M-1))*((4*eye(N+1)-alpha(:,:,M-2:M-2))...
*gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) =inv(4*eye(N+1)-alpha(:,:,M-2:M-2))*...
((betha(:,:,M-2:M-2)+5*eye(N+1))*U(:,M-1:M-1)...
+gamma(:,M-2:M-2));
for z = M-3:-1:1;
U(:,z:z )=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)...
+betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamma(:,z+1:z+1);end;
for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;
'EXACT SOLUTION OF THIS PROBLEM' ;
```

```
for j=1:M+1 ;
for k=1:N+1;
x=(j-1)*h;
es( k, j:j )=(((k-1)*tau)^3)* sin((pi)*x);
end;
end;
es;
% 'ERROR ANALYSIS' ;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxes,maxapp,maxerror,relativeerror]
%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;
axis tight;
title 'EXACT SOLUTION';
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
```

```
surf(p) ; rotate3d ;

title 'DIFFERENCE SCHEMES SOLUTION'; axis tight;
```

4.3 The Second Order of Accuracy in Time Difference Scheme Generated by $\mathbf{A}^2$

Applying the formulas

$$\frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2}-v"(x_n)=O(h^2),$$
$$\frac{u(x_{n+2})-4u(x_{n+1})+6u(x_n)-4u(x_{n-1})+u(x_{n-2})}{h^4}-u^{(iv)}(x_n)=O(h^2),$$

and

$$\frac{2u(0)-5u(h)+4u(2h)-u(3h)}{h^2}-u''(0)=O(h^2),$$
$$\frac{2u(1)-5u(1-h)+4u(1-2h)-u(1-3h)}{h^2}-u''(1)=O(h^2)$$

and using the second order of accuracy in t implicit difference scheme (3.6) for wave equation, we obtain the difference scheme second order 0f accuracy in t and in x for approximate solutions of the nonlocal boundary value problem (4.1)

$$\begin{cases}
\frac{U_n^{k+1}-2U_n^k+U_n^{k-1}}{\tau^2}-\frac{U_{n+1}^k-2U_n^k+U_{n-1}^k}{h^2} \\
+\frac{\tau^2}{4}\left(\frac{U_{n+2}^{k+1}-4U_{n+1}^{k+1}+6U_n^{k+1}-4U_{n-1}^{k+1}+U_{n-2}^{k+1}}{h^4}-2\frac{U_{n+1}^{k+1}-2U_n^{k+1}+U_{n-1}^{k+1}}{h^2}\right) \\
=\varphi_n^k=6(k\tau)\sin(\pi x_n)+\pi^2(k\tau)^3\sin(\pi x_n), \\
1\leq k\leq N-1, 2\leq n\leq M-2, \\
\\
U_n^0=\frac{1}{4}U^N-\frac{1}{4}U^{(N/2)+1}-\frac{7}{32}\sin\pi x, 1\leq n\leq M-1, \\
\\
\left(\frac{I+\tau^2A_h^x}{\tau}\right)\left[\left(\frac{I+\tau^2A_h^x}{\tau}\right)\left(U_1^h(x_n)-U_0^h(x_n)\right)\right. \\
\left.+\frac{\tau}{2}(f^h(0,x_n)-A_h^xU_0^h(x_n)\right]= \\
\frac{1}{4}(2\tau)^{-1}\left(U_{N-2}^h(x_n)-4U_{N-1}^h(x_n)+3U_N^h(x_n)\right) \\
-\frac{1}{4}(2\tau)^{-1}\left(U_{(N/2)+1}^h(x_n)-U_{(N/2)-1}^h(x_n)\right)- \\
\frac{9}{16}\sin(\pi x_n),\ 1\leq n\leq M-1, \\
\\
U_0^k=U_M^k=0, 0\leq k\leq N, \\
U_1^k=\frac{4}{5}U_2^k-\frac{1}{5}U_3^k,\ U_{M-1}^k=\frac{4}{5}U_{M-2}^k-\frac{1}{5}U_{M-3}^k,\ 0\leq k\leq N.
\end{cases} \tag{4.7}$$

We have the $(N+1)\times(M+1)$ system of linear equations. We will write it in the matrix form. We will resort the system

$$
\begin{cases}
\frac{\tau^2}{4h^4}U_{n+2}^{k+1} + \left[-\frac{1}{h^2}\right]U_{n+1}^{k} + \left[-\frac{\tau^2}{h^4} - \frac{\tau^2}{2h^2}\right]U_{n+1}^{k+1} + \left[\frac{1}{\tau^2}\right]U_n^{k-1} + \left[-\frac{2}{\tau^2} + \frac{2}{h^2}\right]U_n^k \\
+\left[\frac{1}{\tau^2} + \frac{3\tau^2}{2h^4} + \frac{\tau^2}{h^2}\right]U_n^{k+1} + \left[-\frac{\tau^2}{h^4} - \frac{\tau^2}{2h^2}\right]U_{n-1}^{k+1} + \left[-\frac{1}{h^2}\right]U_{n-1}^{k} + \frac{\tau^2}{4h^4}U_{n-2}^{k+1} = \varphi_n^k, \\
= \varphi_n^k = 6(k\tau)\sin(\pi x_n) + \pi^2(k\tau)^3\sin(\pi x_n), \\
1 \le k \le N-1, 2 \le n \le M-2, \\
\\
U_n^0 = \frac{1}{4}U^N - \frac{1}{4}U^{(N/2)+1} - \frac{7}{32}\sin \pi x, 1 \le n \le M-1, \\
\\
\left(\frac{I+\tau^2 A_h^x}{\tau}\right)\left[\left(\frac{I+\tau^2 A_h^x}{\tau}\right)\left(U_1^h(x_n) - U_0^h(x_n)\right) + \frac{\tau}{2}(f^h(0,x_n) - A_h^x U_0^h(x_n)\right] = \\
\frac{1}{4}(2\tau)^{-1}\left(U_{N-2}^h(x_n) - 4U_{N-1}^h(x_n) + 3U_N^h(x_n)\right) - \frac{1}{4}(2\tau)^{-1}\left(U_{(N/2)+1}^h(x_n) - U_{(N/2)-1}^h(x_n)\right) - \\
\frac{9}{16}\sin(\pi x_n),\ 1 \le n \le M-1, \\
\\
U_0^k = U_M^k = 0, 0 \le k \le N, \\
U_1^k = \frac{4}{5}U_2^k - \frac{1}{5}U_3^k,\ U_{M-1}^k = \frac{4}{5}U_{M-2}^k - \frac{1}{5}U_{M-3}^k,\ 0 \le k \le N.
\end{cases}
$$

We have that

$$
\begin{cases}
A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \\
\ 2 \le n \le M-2, \\
U_0 = \sigma,\ \ U_M = \sigma,\ \ U_3^k = 4U_2^k - 5U_1^k,\quad U_{M-3}^k = 4U_{M-2}^k - 5U_{M-1}^k, \\
0 \le k \le N.
\end{cases}
$$

We denote

$$
x = \frac{\tau^2}{4h^4},\ y = -\frac{\tau^2}{h^4},\ z = \frac{1}{\tau^2} + \frac{3\tau^2}{2h^4},\ w = -\frac{1}{h^2},\ v = \frac{1}{\tau^2},
$$
$$
t = -\frac{2}{\tau^2} + \frac{2}{h^2},
$$

$$
e = \frac{\tau^3}{16h^4}, w = \frac{-1}{h^2}, a = -\frac{\tau^3}{4h^4}, b = -\frac{\tau}{2h^2} - \frac{\tau^3}{4h^4},
$$

$$
m = -\frac{1}{\tau} + \frac{6\tau^3}{16h^4}, n = \frac{1}{\tau} + \frac{\tau}{h^2} + \frac{6\tau^3}{16h^4},\ k = \frac{1}{8\tau},
$$
$$
p = \frac{1}{2\tau}, r = \frac{3}{8\tau},
$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ ... \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\varphi_n^k = \begin{cases} 6t\sin(\pi x_n) + \pi^2 t^3 \sin(\pi x_n), \\ \ 1 \le k \le N-1, \\ f(t_{k+1}, x_n), \ 1 \le k \le N-1, \\ 0, \ k = N, \end{cases}$$
$$\varphi_n^0 = -\frac{7}{32}\sin(\pi x_n), \varphi_n^N = -\frac{9}{16}\sin(\pi x_n),$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & x & 0 & 0 & . & 0 & 0 & 0 \\ . & . & x & . & . & . & . & . \\ 0 & 0 & 0 & x & . & 0 & 0 & 0 \\ 0 & . & 0 & 0 & 0 & x & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & 0 & x \\ e & e & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & w & y & 0 & . & 0 & 0 & 0 & 0 \\ . & . & w & .y & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & z & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & w & y & 0 & 0 \\ . & . & . & . & . & . & w & y & . \\ 0 & 0 & 0 & 0 & . & 0 & 0 & w & y \\ a & b & 0 & . & . & . & . & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & ... & ... & ... & 1/4 & 0 & ... & 0 & 0 & 0 & -1/4 \\ v & t & z & . & ... & & 0 & ... & . & . & . & . \\ 0 & v & t & z & ... & & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & v & t & ... & & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & & ... & ... & 0 & 0 & 0 & 0 \\ & & & & & & & ... & v & t & z & 0 \\ 0 & 0 & ... & 0 & ... & & 0 & ... & 0 & v & t & z \\ m & n & . & . & & -k & k & ... & 0 & -k & p & -r \end{bmatrix}, \text{ and } D = B, \ E = A,$$

$$R=\begin{bmatrix}1 & 0 & ... & 0\\ 0 & 1 & ... & 0\\ ... & ... & ... & ...\\ 0 & 0 & ... & 1\end{bmatrix}, U_s=\begin{bmatrix}U_s^0\\ U_s^1\\ ....\\ ....\\ U_s^{N-1}\\ U_s^N\end{bmatrix}, wheres = n\pm 2, n\pm 1, n.$$

For the solution of the last matrix equation, we will use the modified variant Gauss elimination method. We seek a solution of the matrix equation by the following form

$$\left\{\ U_n=\alpha_{n+1}U_{n+1}+\beta_{n+1}U_{n+2}+\gamma_{n+1},\quad n=M-2,...,2,1,0,\right.$$

where α_j , β_j, $(j = 1 : M-1)$ are $(N+1)\times(N+1)$ square matrices and γ_j-s are $(N+1)\times 1$ column matrices. And $\alpha_1,\ \beta_1,\ \gamma_1,\ \alpha_2,\ \beta_2,\ \gamma_2$:

$$\alpha_1=\begin{bmatrix}0 & 0 & 0 & ... & 0\\ 0 & 0 & 0 & ... & 0\\ 0 & 0 & 0 & ... & 0\\ . & . & . & . & .\\ 0 & 0 & 0 & ... & 0\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_1=\begin{bmatrix}0 & 0 & 0 & ... & 0\\ 0 & 0 & 0 & ... & 0\\ 0 & 0 & 0 & ... & 0\\ . & . & . & . & .\\ 0 & 0 & 0 & ... & 0\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\gamma_1=\begin{bmatrix}0\\ 0\\ 0\\ ...\\ 0\end{bmatrix}_{(N+1)\times(1)},\ \gamma_2=\begin{bmatrix}0\\ 0\\ 0\\ ...\\ 0\end{bmatrix}_{(N+1)\times(1)},$$

$$\alpha_2=\begin{bmatrix}\frac{4}{5} & 0 & 0 & ... & 0\\ 0 & \frac{4}{5} & 0 & ... & 0\\ 0 & 0 & \frac{4}{5} & ... & 0\\ . & . & . & . & .\\ 0 & 0 & 0 & ... & \frac{4}{5}\end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & 0 & \dots & 0 \\ 0 & -\frac{1}{5} & 0 & \dots & 0 \\ 0 & 0 & -\frac{1}{5} & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & -\frac{1}{5} \end{bmatrix}_{(N+1)\times(N+1)} .$$

Using the equality $U_s = \alpha_{s+1}U_{s+1} + \beta_{s+1}U_{s+2} + \gamma_{s+1}$, (for $s = n,\ n-1,\ n-2$) and the equality

$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n$, we can write

$$\begin{aligned}
&[A + C\beta_{n+1} + D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1}]U_{n+2} \\
&+[B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n + E\alpha_{n-1}\alpha_{n+1} + E\alpha_{n-1}\beta_n \\
&+E\beta_{n-1}\alpha_{n+1}]U_{n+1} \\
&+C\gamma_{n+1} + D\alpha_n\gamma_{n+1} + D\gamma_n + E\alpha_{n-1}\alpha_n\gamma_{n+1} + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\gamma_{n+1} \\
&+E\gamma_{n-1} = R\varphi_n.
\end{aligned}$$

The last equation is satisfied if it is to be selected:

$$\begin{cases}
A + C\beta_{n+1} + D\alpha_n\beta_{n+1} + E\alpha_{n-1}\alpha_n\beta_{n+1} + E\beta_{n-1}\beta_{n+1} = 0, \\
B + C\alpha_{n+1} + D\alpha_n\alpha_{n+1} + D\beta_n \\
+E\alpha_{n-1}\alpha_{n+1} + E\alpha_{n-1}\beta_n + E\beta_{n-1}\alpha_{n+1} = 0, \\
C\gamma_{n+1} + D\alpha_n\gamma_{n+1} + D\gamma_n + E\alpha_{n-1}\alpha_n\gamma_{n+1} + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\gamma_{n+1} \\
+E\gamma_{n-1} = R\varphi_n, \\
2 \le n \le M-2.
\end{cases}$$

Then we obtain the formulas for $\alpha_{n+1},\ \beta_{n+1},\ \gamma_{n+1}$:

$$\begin{aligned}
\alpha_{n+1} &= \left[C + D\alpha_n + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\right]^{-1} \\
&\quad \times \left[-B - D\beta_n - E\alpha_{n-1}\beta_n\right], \\
\beta_{n+1} &= -\left[C + D\alpha_n + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\right]^{-1} A, \\
\gamma_{n+1} &= \left[C + D\alpha_n + E\alpha_{n-1}\alpha_n + E\beta_{n-1}\right]^{-1} \\
&\quad \times \left[R\varphi_n - D\gamma_n + E\alpha_{n-1}\gamma_n + E\gamma_{n-1}\right].
\end{aligned}$$

For solution of the last difference equation we need to find U_M, U_{M-1}, U_{M-2} :

Applying the formulas

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1},\ n = M-3, ..., 2, 1, 0,$$

we obtain U_{n-s} $(n = M-3, ..., 2, 1, 0)$.

Algorithm

1. **Step** Input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$
2. **Step** Use the second order of accuracy difference scheme and write in matrix form;

$$A\,U_{n+2} + B\,U_{n+1} + C\,U_n + DU_{n-1} + EU_{n-2} = R\varphi_n, \quad 2 \leq n \leq M-2,$$

3. **Step** Determine the entries of the matrices $A,\ B,\ C,\ D,\ E$ and R.
4. **Step** Find $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$.
5. **Step** Compute $\alpha_{n+1},\ \beta_{n+1},\ \gamma_{n+1}$.
6. **Step** Find $U_M,\ U_{M-1},\ U_{M-2}$.
7. **Step;** Compute U_n-s (n=M-3,...,2,1), using the following formula:

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}U_{n+2} + \gamma_{n+1}.$$

Matlab Implementation of the Second Order of Accuracy Difference Scheme Generated by A²

```
function secondorderA(N,M)

close; close;

if nargin<1; N=10; M=10; end

al=1/4;

at=3/4;

tau=1/N; h=1/M ;

x= (tau^2)/(4*(h^4));

y= -(tau^2)/(h^4) ;

z= (1/tau^2)+(3*(tau^2))/(2*(h^4));
```

```
t= (-2/tau^2)+(2/h^2);

v= 1/tau^2;

w= -1/h^2;

for i=2:N; A(i,i+1) = x ; end ;

A(N+1,1) = tau^3/(16*(h^4)) ;

A(N+1,2)=tau^3/(16*(h^4)); A(N+1,N+1) =0; A(1,1)=0;A;

E=A;

for i=2:N ; B(i,i) =w ; end;

for i=2:N ; B(i,i+1)= y ; end;

B(1,1) =0 ; B(N+1,1)=-((tau^3)/(4*(h^4)));

B(N+1,2)=(-tau/(2*(h^2))+(-tau^3)/(4*(h^4))); B;

D=B;

for i=2:N ; C(i,i-1)=v ; end;

for i=2:N ; C(i,i) =t ; end;

for i=2:N ; C(i,i+1)= z ; end;

C(1,1)=1 ;C(1,N+1)=-1/4 ; C(1,(N/2)+1)=1/4;

C(N+1,1)=(-1/tau)+(6*tau^3)/(16*(h^4));

C(N+1,2)=(1/tau)+(tau/(h^2))+(6*tau^3)/(16*(h^4)) ;

C(N+1,(N/2))=-(al/(2*tau));

C(N+1,(N/2)+2)=(al/(2*tau));

C(N+1,N+1)=-at/(2*tau);

C(N+1,N)=1/(2*tau) ;C(N+1,N-1)=-al/(2*tau);

C;

for i=2:N ; D(i,i) =w ; end;

for i=2:N ; D(i,i+1)= y ; end;

D(1,1) =0 ; D(N+1,1)=-((tau^3)/(4*(h^4)));

D(N+1,2)=(-tau/(2*(h^2))+(-tau^3)/(4*(h^4))); D;
```

```
for i=2:N; E(i,i+1) = x ; end ;

E(N+1,1) = tau^3/(16*(h^4)) ;

E(N+1,2)=tau^3/(16*(h^4)); E(N+1,N+1) =0;

E(1,1)=0; E;

for i=1:N+1; R(i,i)=1 ; end;

R;

alpha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;

betha(1:N+1,1:N+1,1:1) = 0*eye(N+1) ;

gamma(N+1,1:1)= 0 ;

alpha(1:N+1,1:N+1,2:2) = (4/5)*eye(N+1) ;

betha(1:N+1,1:N+1,2:2) = (-1/5)*eye(N+1);

gamma(N+1,2:2)= 0 ;

'fii(j) finding ' ;

for j=1:M+1;

x=((j-1)*h);

fii(1,j:j) =-(7/32)*sin(pi*x) ;

fii(N+1,j:j) =-(9/16)*sin(pi*x);

for k=2:N;

fii(k,j:j) =6*((k-1)*tau)*(sin(pi*x))+((pi)^2)*(((k-1)*tau)^3)*(sin(pi*x));

end;

end;

fii;

for n = 2:M-2 ;

bebek = C + D*alpha(:,: , n:n ) + E*betha(:,:,n-1 : n-1)...

+ E*alpha(:,:,n-1:n-1)*alpha(:,:, n:n) ;

betha(:,:,n+1:n+1 ) = -inv( bebek )*(A) ;

alpha(:,:,n+1:n+1) = -inv(bebek )*(B +D*betha(:,:,n:n)...
```

```
+ E * alpha(:,:,n-1:n-1)* betha(:,:,n) ) ;
gamma(:,n+1:n+1) = inv( bebek )*...
(R*fii(:,n:n) - D * gamma(:,n:n)-E * alpha(:,:,n-1:n-1)...
* gamma(:,n:n) - E*gamma(:, n-1 : n-1) ) ;
end;
U(1:N+1,1:N+1)=nan;
U( 1:N+1, M:M ) = 0 ;
U( :, M-1:M-1 ) = inv( (betha(:,:,M-2:M-2) ...
+ 5*eye(N+1))- (4*eye(N+1)-alpha(:,:,M-2:M-2))*alpha(:,:,M-1:M-1))
*((4*eye(N+1)-alpha(:,:,M-2:M-2))*...
gamma(: , M-1:M-1)- gamma(: , M-2:M-2) );
U(: , M-2:M-2 ) =inv(4*eye(N+1)-alpha(:,:,M-2:M-2))*((betha(:,:,M-2:M-2)+5*eye(N+1))*...
U(:,M-1:M-1)+gamma(:,M-2:M-2));
for z = M-3:-1:1;
U(:,z:z )=alpha(:,:,z+1:z+1)*U(:,z+1:z+1)...
+betha(:,:,z+1:z+1)*U(:,z+2:z+2)+gamma(:,z+1:z+1);end;
for z = 1 : M ;
p(:,z+1:z+1)=U(:,z:z);
end;
'EXACT SOLUTION OF THIS PROBLEM' ;
for j=1:M+1 ;
for k=1:N+1;
x=(j-1)*h;
es( k, j:j )=(((k-1)*tau)^3)* sin((pi)*x);
end;
end;
es;
```

```
% 'ERROR ANALYSIS' ;

maxes=max(max(es)) ;

maxapp=max(max(p)) ;

maxerror=max(max(abs(es-p)));

relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );

cevap = [maxes,maxapp,maxerror,relativeerror]

%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure ;

m(1,1)=min(min(p))-0.01;

m(2,2)=nan;

surf(m);

hold;

surf(es) ; rotate3d ;

axis tight;

title 'EXACT SOLUTION';

figure ;

m(1,1)=min(min(p))-0.01;

m(2,2)=nan;

surf(m);

hold;

surf(p) ; rotate3d ;

title 'DIFFERENCE SCHEMES SOLUTION';

axis tight;
```

4.4 Numerical Analysis

We consider the nonlocal boundary-value problem for wave equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = 6t\sin\pi x + \pi^2 t^3 \sin\pi x,\ 0 < t < 1,\ 0 < x < 1, \\ u(0,x) = \frac{1}{4}u(1,x) - \frac{1}{4}u(\frac{1}{2},x) + \psi(x), \\ \psi(x) = -\frac{7}{32}\sin\pi x, 0 \le x \le 1, \\ u_t(0,x) = \frac{1}{4}u_t(1,x) - \frac{1}{4}u_t(\frac{1}{2},x) + \varphi(x), \\ \varphi(x) = -\frac{9}{16}\sin\pi x,\ 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0,\ \ -1 \le t \le 1. \end{cases} \tag{4.8}$$

The exact solution is u(t,x)=t^3 sin πx.

For approximate solutions of the nonlocal boundary-value problem (4.1), we will use the first and the second order of accuracy difference schemes with $\tau = \frac{1}{14}, h = \frac{\pi}{44}, \alpha = 0.1$. We have the second order or fourth order difference equations to respect in n with matrix coefficients. To solve this difference equations we have applied a procedure of modification Gauss elimination method. The exact and numerical solutions are given in the following table 4.9. TABLE

Now, we will give the results of the numerical analysis. For their comparison, the errors computed by

$$E_M^N = \max_{1 \le k \le N-1} \left(\sum_{n=1}^{M-1} \left| u(t_k, x_n) - U_n^k \right|^2 h \right)^{\frac{1}{2}}$$

of the numerical solutions, where $u(t_k, x_n)$ represents the exact solution and U_n^k represents the numerical solution at (t_k, x_n).

The result are shown in the following table. TABLE: Comparison of the errors of different difference schemes for N=M=20.

Difference schemes	E$_M^N$
The first order of accuracy difference scheme (4.2)	0.1052
The second order of accuracy difference scheme (4.4)	0.0621
The second order of accuracy difference scheme (4.7)	0.0609

(4.9)

Table 1.

Thus, the second order of accuracy difference schemes were more accurate compare with the first order of accuracy difference scheme.

The first figure is the exact solution,the second figure is the solution of the first order of accuracy difference scheme, the third figure is the solution of second order of accuracy difference scheme .The first line is the exact solution,the second line is the solution of the first order of accuracy difference scheme, the third line is the solution of second order of accuracy difference scheme and the fourth line is the solution of second order of accuracy difference scheme generated by A^2.

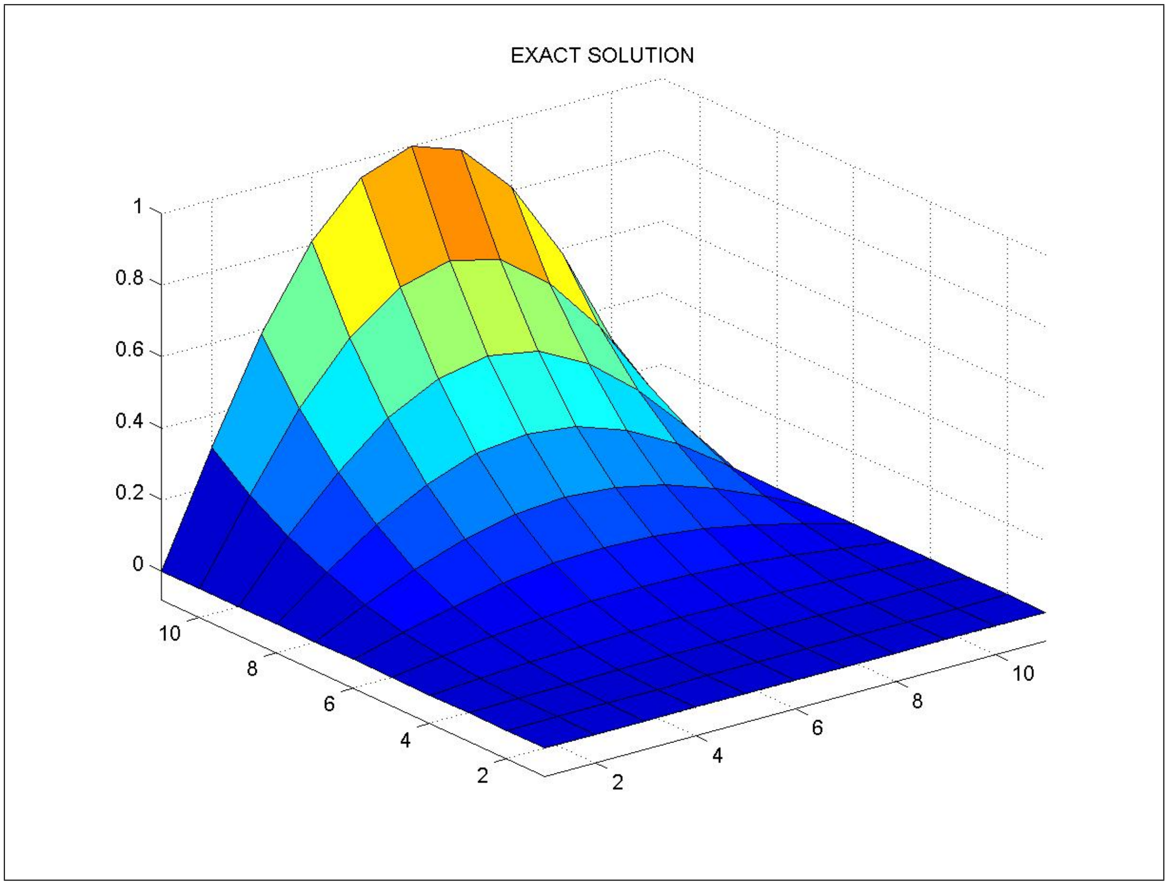
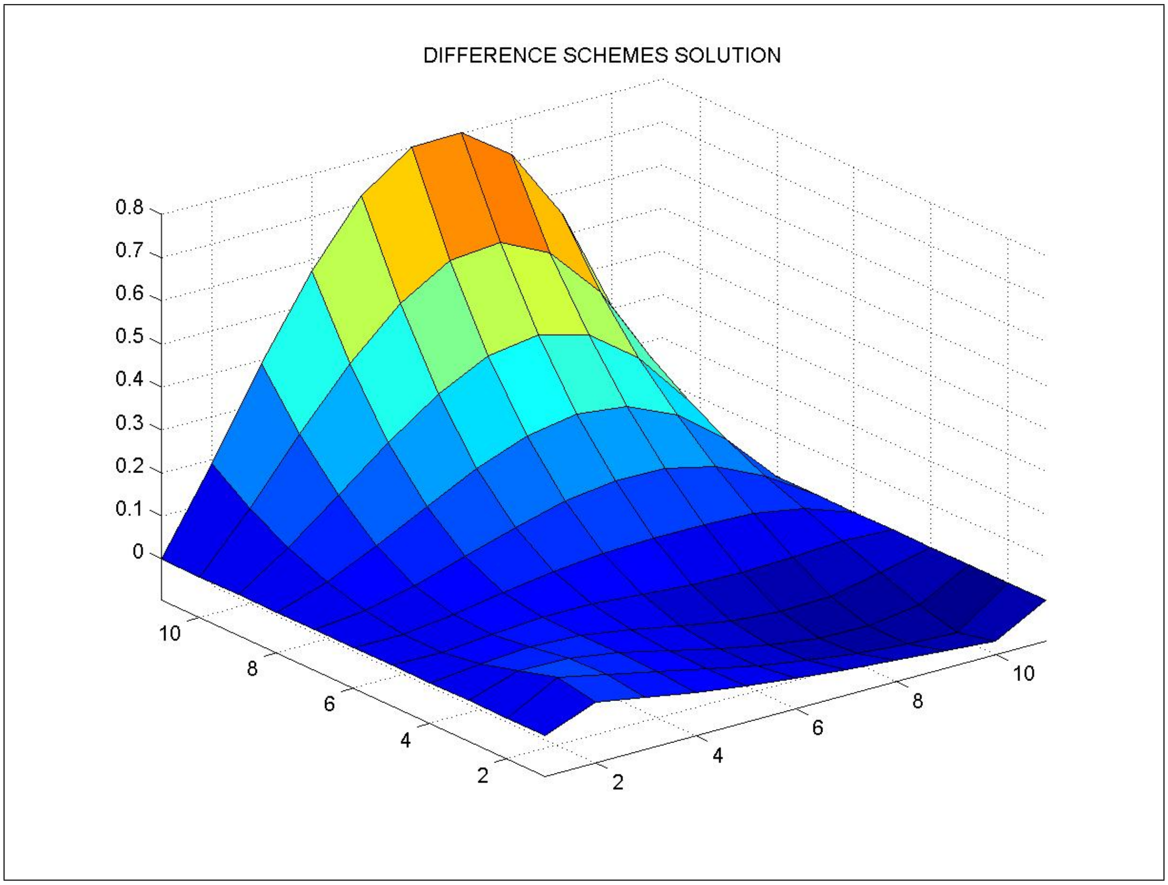
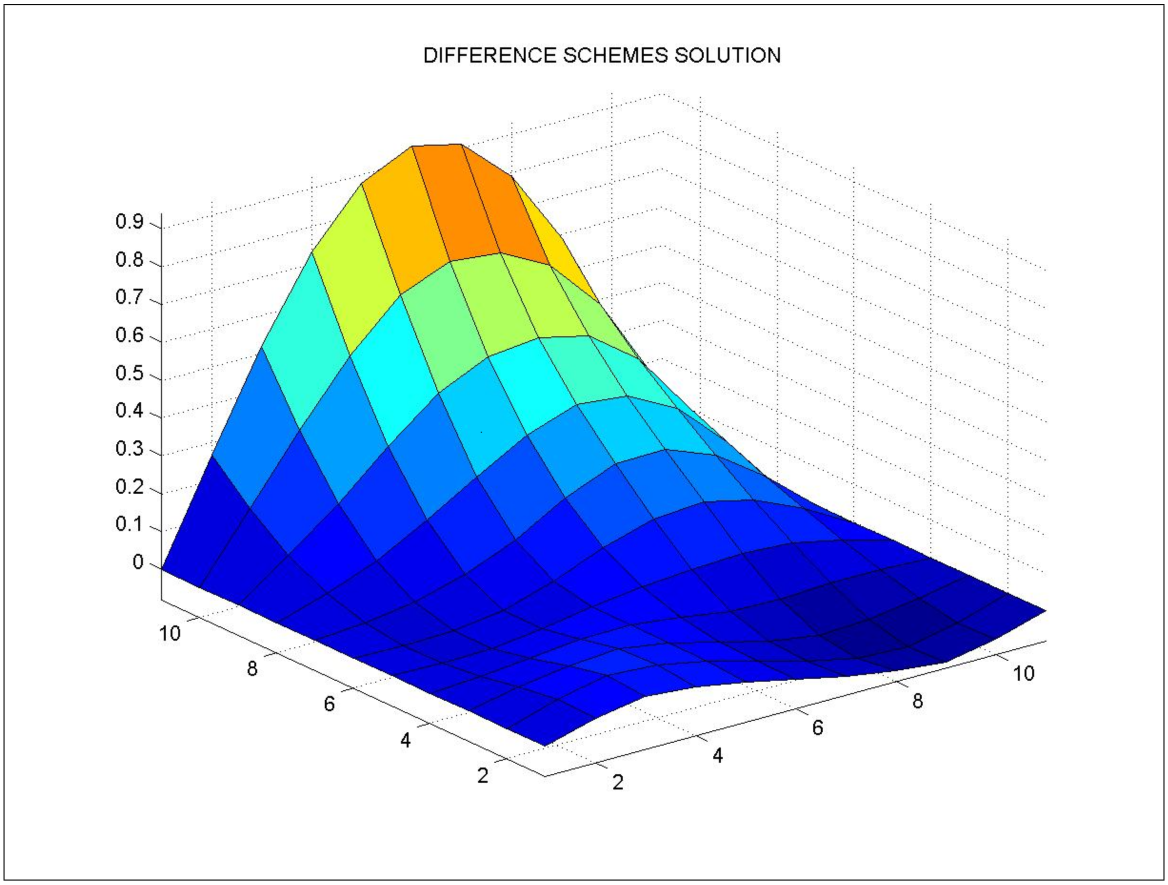


Figure1.


DIFFERENCE SCHEMES SOLUTION
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
10
8
6
4
2
2
4
6
8
10

(4.10)

Figure2.


DIFFERENCE SCHEMES SOLUTION
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
10
8
6
4
2
2
4
6
8
10

(4.11)

Figure3.

CHAPTER 5

CONCLUSIONS

This work is devoted to the study of the stability of the nonlocal boundary value problem for hyperbolic equations.We obtain the following original results:

- The stability estimates for solution of the multipoint nonlocal boundary value problem for abstract hyperbolic equations

in a Hilbert space H with self -adjoint positive definite operator A are obtained.

- The stability estimates for the solutions of the two types of nonlocal boundary value problems for hyperbolic equations are obtained.

- The first and second order of accuracy difference schemes generated by the integer power of A approximately solving this abstract nonlocal boundary value problem

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_{k+1}=f_k, f_k=f(t_k), \\ t_k=k\tau,\ 1\leq k\leq N-1, N\tau=1, \\ u_0=\sum\limits_{r=1}^{n}\alpha_r u_{\left[\frac{\lambda_r}{\tau}\right]}+\varphi, \\ (I+\tau^2A)\tau^{-1}(u_1-u_0)=\sum\limits_{r=1}^{n}\beta_r\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]}\right)\frac{1}{\tau}+\psi, \end{cases} \tag{5.1}$$

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k, \\ f_k=f(t_k), \quad t_k=k\tau,\ 1\leq k\leq N-1, N\tau=1, \\ u_0=\sum\limits_{\frac{\lambda_r}{\tau}\in Z}\alpha_r u_{\frac{\lambda_r}{\tau}}+\sum\limits_{\frac{\lambda_r}{\tau}\notin Z}\alpha_r\left(u_{\left[\frac{\lambda_r}{\tau}\right]}+\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]}\right)\left(\frac{\lambda_r}{\tau}-\left[\frac{\lambda_r}{\tau}\right]\right)\right)+\varphi, \\ (I+\frac{\tau^2A}{2})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0) \\ =\sum\limits_{r\neq n,\frac{\lambda_r}{\tau}\notin Z}\alpha_r\left(\frac{1}{2\tau}\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]-1}\right)+\left(f_{\left[\frac{\lambda_r}{\tau}\right]}-Au_{\left[\frac{\lambda_r}{\tau}\right]}\right)\left(\frac{\lambda_r}{\tau}-\left[\frac{\lambda_r}{\tau}\right]\right)\right) \\ +\alpha_n\left(\frac{1}{2\tau}\left(3u_{\left[\frac{\lambda_n}{\tau}\right]}-4u_{\left[\frac{\lambda_n}{\tau}\right]-1}+u_{\left[\frac{\lambda_n}{\tau}\right]-2}\right)+\left(f_{\left[\frac{\lambda_n}{\tau}\right]}-Au_{\left[\frac{\lambda_n}{\tau}\right]}\right)\left(\frac{\lambda_n}{\tau}-\left[\frac{\lambda_n}{\tau}\right]\right)\right) \\ +\sum\limits_{r\neq n,\frac{\lambda_r}{\tau}\in Z}\alpha_r\frac{1}{2\tau}\left(u_{\frac{\lambda_r}{\tau}+1}-u_{\frac{\lambda_r}{\tau}-1}\right)++\alpha_n\frac{1}{2\tau}\left(u_{\frac{\lambda_n}{\tau}}-4u_{\frac{\lambda_n}{\tau}-1}+3u_{\frac{\lambda_n}{\tau}-2}\right)+\psi, \\ f_0=f(0), f_N=f(1), \end{cases} \tag{5.2}$$

$$
\begin{cases}
\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k, \\
f_k=f(t_k), \quad t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\
u_0=\sum\limits_{\frac{\lambda_r}{\tau}\in Z}\alpha_r u_{\frac{\lambda_r}{\tau}}+\sum\limits_{\frac{\lambda_r}{\tau}\notin Z}\alpha_r\left(u_{\left[\frac{\lambda_r}{\tau}\right]}+\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]}\right)\left(\frac{\lambda_r}{\tau}-\left[\frac{\lambda_r}{\tau}\right]\right)\right)+\varphi, \\
(I+\frac{\tau^2 A}{4})[(I+\frac{\tau^2 A}{4})\tau^{-1}(u_1-u_0)-\frac{\tau}{2}(f_0-Au_0)] \\
=\sum\limits_{r\neq n,\frac{\lambda_r}{\tau}\notin Z}\alpha_r\left(\frac{1}{2\tau}\left(u_{\left[\frac{\lambda_r}{\tau}\right]+1}-u_{\left[\frac{\lambda_r}{\tau}\right]-1}\right)+\left(f_{\left[\frac{\lambda_r}{\tau}\right]}-Au_{\left[\frac{\lambda_r}{\tau}\right]}\right)\left(\frac{\lambda_r}{\tau}-\left[\frac{\lambda_r}{\tau}\right]\right)\right) \\
+\alpha_n\left(\frac{1}{2\tau}\left(3u_{\left[\frac{\lambda_n}{\tau}\right]}-4u_{\left[\frac{\lambda_n}{\tau}\right]-1}+u_{\left[\frac{\lambda_n}{\tau}\right]-2}\right)+\left(f_{\left[\frac{\lambda_n}{\tau}\right]}-Au_{\left[\frac{\lambda_n}{\tau}\right]}\right)\left(\frac{\lambda_n}{\tau}-\left[\frac{\lambda_n}{\tau}\right]\right)\right) \\
+\sum\limits_{r\neq n,\frac{\lambda_r}{\tau}\in Z}\alpha_r\frac{1}{2\tau}\left(u_{\frac{\lambda_r}{\tau}+1}-u_{\frac{\lambda_r}{\tau}-1}\right)++\alpha_n\frac{1}{2\tau}\left(3u_{\frac{\lambda_n}{\tau}}-4u_{\frac{\lambda_n}{\tau}-1}+u_{\frac{\lambda_n}{\tau}-2}\right)+\psi, \\
f_0=f(0), f_N=f(1)
\end{cases}
\tag{5.3}
$$

are described.

-The stability estimates for the solutions of these difference schemes are obtained.

- The numerical analysis is given. We show that the theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

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