Fatma S.Ö. TETİKOĞLU

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DIFFERENCE SCHEMES FOR NEUMANN BIDSATZE SAMARSKII PROBLEMS

by

Fatma Songül ÖZESENLİ TETİKOĞLU

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Fatma Songül Özesenli Tetikoğlu

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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Prof. Dr. Mustafa BAYRAM
Head of Department

This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Prof. Dr. Allaberen ASHYRALYEV
Supervisor

Examining Committee Members

Prof. Dr. Allaberen ASHYRALYEV	______________________
Prof. Dr. Fatma TİRYAKİ	______________________
Assoc. Prof. Dr. Yaşar SÖZEN	______________________

It is approved that this thesis has been written in compliance with the formatting rules laid down by the Graduate Institute of Sciences and Engineering.

____

Assist. Prof. Dr. Nurullah ARSLAN
Director

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M. S. Thesis - Mathematics
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ABSTRACT

Neumann Bitsadze Samarskii nonlocal boundary value problem for the elliptic differential equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. The well-posedness of this problem in Hölder spaces without a weight is established. The coercivity inequalities for solutions of the nonlocal boundary value problem for the elliptic equation are obtained. The first order of accuracy difference scheme for the approximate solutions of this nonlocal boundary value problem is presented. The stability estimates, coercivity and almost coercivity inequalites for the solution of this difference scheme are established. The well-posedness of this difference scheme in Hölder spaces without a weight is proved. The Matlab implementation of this difference scheme for the elliptic equation is presented. The theoretical statements for the solution of this difference scheme is supported by the results of numerical examples.

Keywords: Neumann Bitsadze Samarskii Problem, Elliptic Equation, Difference Schemes, Stability Estimates.

NEUMANN BIDSATZE SAMARSKII TİPTEKİ PROBLEMLER İÇİN FARK ŞEMALARI

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ÖZ

Hilbert uzayında özeşlenik pozitif tanımlı A operatörlü diferansiyel denklemlerinin yerel olmayan Neumann Bitsadze Samarskii sınır değer problemi ele alınmıştır. Bu sınır değer probleminin iyi konumlanmışlığı ağırlıksız Hölder uzaylarında doğruluğu ortaya konulmuştur. Neumann Bitsadze Samarskii eliptik denkleminin çözümü için koersatif eşitsizlikleri elde edilmiştir. Bu yerel olmayan Neumann Bitsadze Samarskii sınır değer probleminin yaklaşık çözümü için birinci dereceden fark şeması kurulmuştur. Bu fark şemasının çözümü için kararlılık kestirimleri kurulmuştur. Bu fark şemasının iyi konumlanmışlığı Hölder uzaylarında kanıtlanmıştır. Fark şemasının çözümü için koersatif eşitsizlikleri, hemen hemen koersatif eşitsizlikleri sağlanmıştır. Eliptik denklemler için fark şemasının Matlab ile çözümleri elde edilmiştir. Bu fark şemasının çözümü için bulunan teorik sonuçlar, sayısal örneklerle desteklenmiştir.

Anahtar Kelimeler: Neumann Bitsadze Samarskii Problemi, Eliptik Denklem, Yerel Olmayan Sınır değer Problemi, Fark Şemaları, Kararlılık.

DEDICATION

To HIM

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CHAPTER 1

INTRODUCTION

Many problems in fluid mechanics, dynamics, elasticity and other areas of engineering, physics and biological systems lead to partial differential equations of elliptic type. Methods of solutions Neumann Bitsadze Samarskii nonlocal boundary value problems for elliptic differential equations have been studied extensively by many researches (see [Berikelashvili.G. K., 2003], [Soldatov A. P., 2006], [Gordeziani D.G., 1970], [Ilyın V., Moiseev E., 1990], [Makarov V. L., Lazurchak I. I., and Bandyrskii B. I., 2003], [Kilbas A. A., and Repin O. A., 2003], [Bitsadze A.V., Samarskii A.A., 1969], Ashyralyev A. and Altay N., 2006], [Kapanadze D.V., 1987], [Ashyralyev A., 2008], Berikelashvili G.K., 2003], [Gurbanov I.A. and Dosiev A.A., 1984], [Balakishiyev B.B. and Dosiev A.A., 1991], [Dosiev A.A. and Ashirov B.S., 1991], Ashyralyev A. and Öztürk E., 2008], [Öztürk E., 2008] and the references therein). The main aim of this work is to investigate the stability of difference schemes of approximate solutions of Neumann Bitsadze Samarskii nonlocal boundary value problems for partial differential equations of elliptic type.

It is known that the Neumann Bitsadze Samarskii nonlocal boundary value problem for elliptic equations can be solved analytically by Fourier series, Fourier transform and Laplace transform methods. Now, let us illustrate these three different analytical methods by examples.

Example 1.1. Consider the following simple nonlocal boundary value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = \exp(-\pi t)\cos(\pi x),\ 0 < t < 1,\ 0 < x < 1, \\ u_t\,(0, x) = -\pi\cos(\pi x), 0 \le x \le 1 \\ u_t\,(1, x) = u_t(\frac{1}{2}, x) + \pi\cos(\pi x)\left(\exp(-\frac{\pi}{2}) - \exp(-\pi)\right),\ 0 \le x \le 1, \\ u_x\,(t, 0) = u_x\,(t, 1) = 0,\ 0 \le t \le 1. \end{cases} \tag{1.1}$$

For the solution of the problem (1.1), we use the Fourier series method. In order to solve the problem we need to separate $u(t, x)$ into two parts

$$u(t, x) = v(t, x) + w(t, x), \tag{1.2}$$

where $v(t, x)$ is the solution of the problem

$$\begin{cases} -\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + v = 0,\ 0 < t < 1,\ \ 0 < x < 1, \\ v_t\,(0, x) = -\pi\cos(\pi x), 0 \le x \le 1 \\ v_t\,(1, x) = v_t(\frac{1}{2}, x) + \pi\cos(\pi x)\left(\exp(-\frac{\pi}{2}) - \exp(-\pi)\right), 0 \le x \le 1, \\ v_x\,(t, 0) = v_x\,(t, 1) = 0,\ 0 \le t \le 1. \end{cases} \tag{1.3}$$

and

$$\begin{cases} -\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + w = \exp(-\pi t)\cos(\pi x),\ \ 0 < t < 1,\ 0 < x < 1, \\ w_t\,(0, x) = 0,\ \ w_t\,(1, x) = w_t(\frac{1}{2}, x),\ \ 0 < x < 1, \\ w_x\,(t, 0) = w_x\,(t, \pi) = 0,\ \ 0 \le t \le 1. \end{cases} \tag{1.4}$$

Now, let us obtain the solution of (1.3), by the method of separation of variables. To do this a solution of the form

$$v(t, x) = T(t)X(x) \neq 0$$

is suggested. Taking the partial derivatives and substituting the result in (1.3), we obtain

$$\frac{-T''(t) + T\,(t)}{T(t)} - \frac{X''(x)}{X(x)} = 0,$$

or

$$\frac{-T''(t)+T(t)}{T(t)}=\frac{X''(x)}{X(x)}=\lambda. \tag{1.5}$$

The boundary conditions presented in (1.3) require $X'(0)=X'(1)=0$. Hence, from (1.5) we have the ordinary differential equation

$$X''(x)=\lambda X(x),\ X'(0)=X'(1)=0. \tag{1.6}$$

If $\lambda\geq 0$, then the boundary value problem (1.6) has only trivial solution $X(x)=0$. For $\lambda<0$, the nontrivial solutions of the boundary value problem (1.6) are

$$X_k(x)=\cos(k\pi x),\quad \text{where } k=1,2,3,\cdots,\quad \lambda=-k^2\pi^2.$$

So, the nontrivial solutions of the boundary value problem (1.6) are

$$X_k(x)=\cos k\pi x,\ \lambda=-k^2\pi^2\ \text{ where } k=1,2,3,\cdots.$$

The other ordinary differential equation presented in (1.5) is

$$-T''(t)+T(t)=\lambda T(t),$$

with $\lambda_k=-k^2,\ k=1,2,\cdots$. The solution of this ordinary differential equation is

$$T_k(t)=D_k e^{\sqrt{1+k^2\pi^2}t}+E_k e^{-\sqrt{1+k^2\pi^2}t},\quad \text{where } k=1,2,3,\cdots.$$

Thus,

$$v(t,x)=\sum_{k=1}^{\infty}v_k(t,x)=\sum_{k=1}^{\infty}\left(A_k e^{\sqrt{1+k^2\pi^2}t}+B_k e^{-\sqrt{1+k^2\pi^2}t}\right)\cos k\pi x.$$

Using the nonlocal boundary conditions

$$v_t(0,x)=-\pi\cos\pi x,\ v_t(1,x)=v_t(\frac{1}{2},x)+\pi\cos\pi x\left(\exp(-\frac{\pi}{2})-\exp(-\pi)\right),$$

we obtain

$$A_1=\frac{\pi}{\sqrt{2}}\left(\frac{-e^{-\sqrt{2}}+e^{-\frac{\sqrt{2}}{2}}+e^{-\pi}-e^{-\frac{\pi}{2}}}{e^{\frac{\sqrt{2}}{2}}-e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-\frac{\sqrt{2}}{2}}}\right),$$

$$B_1=\frac{\pi}{\sqrt{2}}\left(\frac{-e^{\sqrt{2}}+e^{\frac{\sqrt{2}}{2}}+e^{-\pi}-e^{-\frac{\pi}{2}}}{e^{\frac{\sqrt{2}}{2}}-e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-\frac{\sqrt{2}}{2}}}\right),$$

$$A_k=B_k=0,\ k=2,3,\cdots.$$

Hence, the solution of (1.3) is

$$v(t,x)=\frac{\pi}{\sqrt{2}}\left(\left(\frac{-e^{-\sqrt{2}}+e^{-\frac{\sqrt{2}}{2}}+e^{-\pi}-e^{-\frac{\pi}{2}}}{e^{\frac{\sqrt{2}}{2}}-e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-\frac{\sqrt{2}}{2}}}\right)e^{\sqrt{1+\pi^2}t}\right.$$

$$+\left(\frac{-e^{\sqrt{2}}+e^{\frac{\sqrt{2}}{2}}+e^{-\pi}-e^{-\frac{\pi}{2}}}{e^{\frac{\sqrt{2}}{2}}-e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-\frac{\sqrt{2}}{2}}}\right)e^{-\sqrt{1+\pi^2}t}\Bigg)\sin x.$$

Second, we obtain the solution of (1.4). We seek a solution of the form

$$w\left(t,x\right)=\sum_{k=1}^{\infty}C_k\left(t\right)\sin kx.$$

Then

$$-w_{tt}-w_{xx}+w=\sum_{k=1}^{\infty}\left[-C_k''\left(t\right)+\left(1+k^2\pi^2\right)C_k\left(t\right)\right]\cos k\pi x$$
$$=\exp(-\pi t)\cos\pi x.$$

If $k\neq 1$, then

$$-C_k''\left(t\right)+\left(1+k^2\pi^2\right)C_k\left(t\right)=0.$$

So, we obtain

$$C_k\left(t\right)=c_1e^{\sqrt{1+k^2\pi^2}t}+c_2e^{-\sqrt{1+k^2\pi^2}t}.$$

Using the nonlocal boundary conditions

$$C_k(0)=0,\ C_k(1)=C_k(\frac{1}{2}),$$

we get $c_1=c_2=0$ and $C_k(t)=0$.

If $k=1$, then

$$-C_1''\left(t\right)+(1+\pi^2)C_1\left(t\right)=\exp(-\pi t).$$

So, we obtain

$$C_1\left(t\right)=c_1e^{\sqrt{1+\pi^2}t}+c_2e^{-\sqrt{1+\pi^2}t}+\exp(-\pi t).$$

Using the nonlocal boundary conditions

$$C_1(0)=0,\ C_1(1)=C_1(\frac{1}{2}),$$

we get

$$c_1=-\frac{\pi}{\sqrt{2}}\left(\frac{-e^{-\sqrt{2}}+e^{-\frac{\sqrt{2}}{2}}+e^{-\pi}-e^{-\frac{\pi}{2}}}{e^{\frac{\sqrt{2}}{2}}-e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-\frac{\sqrt{2}}{2}}}\right),$$

$$c_2=-\frac{\pi}{\sqrt{2}}\left(\frac{-e^{\sqrt{2}}+e^{\frac{\sqrt{2}}{2}}+e^{-\pi}-e^{-\frac{\pi}{2}}}{e^{\frac{\sqrt{2}}{2}}-e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-\frac{\sqrt{2}}{2}}}\right)$$

and from particular solution, we get

$$C_1\left(t\right)=\exp(-\pi t).$$

Thus the solution of (1.4) is

$$w\left(t,x\right)=\left(-\frac{\pi}{\sqrt{2}}\left(\frac{-e^{-\sqrt{2}}+e^{-\frac{\sqrt{2}}{2}}+e^{-\pi}-e^{-\frac{\pi}{2}}}{e^{\frac{\sqrt{2}}{2}}-e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-\frac{\sqrt{2}}{2}}}\right)e^{\sqrt{1+\pi^2}t}\right.$$
$$\left.-\frac{\pi}{\sqrt{2}}\left(\frac{-e^{\sqrt{2}}+e^{\frac{\sqrt{2}}{2}}+e^{-\pi}-e^{-\frac{\pi}{2}}}{e^{\frac{\sqrt{2}}{2}}-e^{\sqrt{2}}+e^{-\sqrt{2}}-e^{-\frac{\sqrt{2}}{2}}}\right)e^{-\sqrt{1+\pi^2}t}\right)\cos\pi x+\exp(-\pi t)\cos\pi x.$$

Finally, using (1.2), we obtain

$$u\left(t,x\right)=v\left(t,x\right)+w\left(t,x\right)=\exp(-\pi t)\cos\pi x.$$

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2}-\sum\limits_{r=1}^{n}\alpha_r\frac{\partial^2 u(t,x)}{\partial x_r^2}+\delta u=f(t,x), \\ x=(x_1,\dots,x_n)\in\overline{\Omega}, 0<t<T, \\ \frac{\partial u}{\partial n}\Big|_S=0, \\ u(0,x)=\varphi(x),\ \ x\in\overline{\Omega}, \\ u_t(T,x)=\beta u_t(\lambda,x)+\psi(x), x\in\overline{\Omega}, |\beta|\le 1, 0\le\lambda<T, \end{cases}$$

where β is constant and $f(t,x)$ $(t\in[0,T],\ x\in\overline{\Omega})$, $\varphi(x),\psi(x)$ $(x\in\overline{\Omega})$ are given smooth functions. Here Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<1, 1\le k\le n)$ with boundary S, $\overline{\Omega}=\Omega\cup S$.

However, the method of separation of variables can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Example 1.2. Now, we will consider the application of Laplace transformation method to the problem

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u=-\exp(-t-x),\quad 0<t<1,\ 0<x<\infty, \\ u_t\left(0,x\right)=\exp(-x),\ u_t(1,x)=u_t(\tfrac{1}{2},x)-\exp(-1-x)+\exp(-\tfrac{1}{2}-x),\ 0<x<\infty, \\ u\left(t,0\right)=\exp(-t), u_x(t,0)=-\exp(-t),\ 0\le t\le 1. \end{cases} \tag{1.7}$$

We denote

$$\mathbf{L}\left\{u(t,x)\right\}=v(t,s).$$

Then using the properties of the Laplace transform, we obtain

$$-v_{tt}\left(t,s\right)+\left(1-s^{2}\right)v\left(t,s\right)=-\exp(-t)\frac{s^{2}}{s+1}.$$

Solving it we can write

$$v(t,s)=c_{1}e^{\sqrt{1-s^{2}}t}+c_{2}e^{-\sqrt{1-s^{2}}t}-\exp(-t)\frac{s^{2}}{s+1}.$$

Now, using the nonlocal boundary conditions (1.7) which are transformed to

$$v_{t}\left(0,s\right)=-\frac{1}{s+1},\ v_{t}\left(1,s\right)=v_{t}(\frac{1}{2},s)+(\exp(-1)+\exp(-\frac{1}{2}))\frac{1}{(s+1)},$$

we obtain

$$v(t,s)=\exp(-t)\frac{1}{s+1}.$$

Finally, taking the inverse Laplace transform of this equation, we obtain

$$u\left(t,x\right)=\mathbf{L}^{-1}\left\{v\left(t,s\right)\right\}=\mathbf{L}^{-1}\left\{\exp(-t)\frac{1}{s+1}\right\}$$

$$=\exp(-t)\mathbf{L}^{-1}\left\{\frac{1}{(s+1)}\right\}=e^{-t-x}.$$

Hence, the solution of (1.7) is

$$u\left(t,x\right)=e^{-t-x}.$$

Note that using the same procedure one obtains the solution of the following nonlocal boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} + \delta u = f(t,x), \\ x = (x_1, \ldots, x_n) \in \overline{\Omega}^+, 0 < t < T, \\ u_t(0,x) = \varphi(x), \\ u_t(T,x) = \beta u_t(\lambda, x) + \psi(x), x \in \overline{\Omega}^+, |\beta| \leq 1, 0 < \lambda < T, \\ u(t,x) \mid_{x_k=0} = 0, \frac{\partial u(t,x)}{\partial x_k} \mid_{x_k=0} = 0, \\ 1 \leq k \leq n, \ x \in \overline{\Omega}^+, 0 \leq t \leq T, \end{cases}$$

where β is constant , $f(t,x)$ $(t \in [0,T], \ x \in \overline{\Omega}^+)$, $\varphi(x), \psi(x)$ $(x \in \overline{\Omega}^+)$ are given smooth functions. Here Ω^+ is the open set in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \leq k \leq n)$ with boundary S^+, $\overline{\Omega}^+ = \Omega^+ \cup S^+$.

However, Laplace transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method.

Example 1.3. The last example is a nonlocal boundary value problem solved by using Fourier transform method. Consider the problem

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 2\exp(-t-x^2) - 4x^2\exp(-t-x^2), \ 0 < t < 1, \ \ -\infty < x < \infty, \\ u_t\left(0,x\right) = -e^{-x^2}, -\infty < x < \infty. \\ u_t(1,x) = u_t(\frac{1}{2},x) + \exp(-1-x^2) + \exp(-\frac{1}{2}-x^2), \ \ -\infty < x < \infty. \end{cases} \tag{1.8}$$

We denote

$$v\left(t,s\right) = \mathbf{F}\left\{u\left(t,x\right)\right\}.$$

Then, taking the Fourier transform of both sides of the differential equation in (1.8), we obtain

$$v_{tt}\left(t,s\right) - \left(s^2+1\right) v\left(t,s\right) = \mathbf{F}\left\{2\exp(-t-x^2) - 4x^2\exp(-t-x^2)\right\}$$

$$= \mathbf{F}\{\exp(-x^2)''\}\exp(-t).$$

Solving it we can write

$$v(t,s) = c_1 e^{\sqrt{s^2+1}t} + c_2 e^{-\sqrt{s^2+1}t} + (t^2+1)\mathbf{F}\left\{e^{-x^2}\right\}\exp(-t).$$

Now, using the nonlocal boundary conditions in (1.8) which are transformed to

$$v_t(0,s) = -\mathbf{F}\left\{e^{-x^2}\right\},\ v_t(1,s) = v_t(\frac{1}{2},s) - (e^{-1}+e^{-\frac{1}{2}})\mathbf{F}\left\{e^{-x^2}\right\},$$

we get

$$v(t,s) = \exp(-t)\mathbf{F}\left\{e^{-x^2}\right\}.$$

Finally, taking the inverse of Fourier transformation we obtain the solution for the problem (1.8) as

$$u(t,x) = \exp(-t-x^2).$$

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the 2m-th order multidimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \sum\limits_{|r|=2m} a_r \frac{\partial^{|r|}u}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} + \delta u = f(t,x), \\ 0<t<T, x, r, \delta \in \mathbb{R}^n,\ |r| = r_1+\cdots+r_n, \\ u_t(0,x) = \varphi(x), x\epsilon\mathbb{R}^n, \\ u_t(T,x) = \beta u_t(\lambda,x)+\psi(x),\ x\epsilon\mathbb{R}^n\ ,\ |\beta|\le 1, 0\le\lambda<1, \end{cases}$$

where β is constant, $f(t,x)$ $(t\in[0,T],\ x\in R^n)$, $\varphi(x),\psi(x)$ $(x\in R^n)$ are given smooth functions. However, the Fourier transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in t and in the space variables is difference method, which is basically realized by digital computers and known to be numerical method. However the stability of difference schemes used in numerical methods need to be proved or justified theoretically.

It is known that various Neumann Bitsadze Samarskii problems for elliptic equations can be reduced to the Neumann Bitsadze Samarskii problem for differential equation in a Hilbert space H with self-adjoint positive definite operator A. In the present work, the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u(t)}{dt^2} + Au(t) = f(t),\ 0 < t < 1, \\ \\ u_t(0) = \varphi, u_t(1) = \beta u_t(\lambda) + \psi, \\ \\ 0 \leq \lambda < 1,\ |\beta| \leq 1 \end{cases}$$

for the differential equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. The first and second orders of accuracy difference schemes are constructed. Applying the operator approach, the stability of difference schemes for the approximate solution of differential equations are obtained. The theoretical statements for the solution of this difference schemes are supported by the results of numerical examples.

Let us briefly describe the contents of the various sections of the thesis. It consists of eight chapters.

First chapter is the introduction.

Second chapter presents two sections. In the first section, the well-posedness of this problem in Hölder space without a weight is examined. In the second section, this abstract results permit us to obtain the coercivity stability inequality for the solutions of elliptic differential equations.

Third chapter consists of two sections. In the first section, the stable first order of accuracy difference scheme approximately solving the nonlocal boundary value Neumann Bitsadze Samarskii problem for elliptic equations in a Hilbert space H with self-adjoint positive definite operator A is presented. The stability estimate for the solution of the difference scheme of the nonlocal boundary Neumann Bitsadze Samarskii problem for elliptic equations is obtained. The coercivity and almost coercivity inequalities for the solution of the difference scheme of the nonlocal boundary Neumann Bitsadze Samarskii problem for elliptic equations are presented. Finally, the second order of accuracy difference scheme approximately solving the nonlocal boundary value Neumann Bitsadze Samarskii problem for elliptic equations is presented.Theorems on stability, almost coercitive stability and coercitive stability are formulated.In the second section, this abstract results permit us to obtain the stability estimates, almost coercive stability estimates and coercive stability estimates for the solution of the difference scheme for elliptic equations.

Fourth chapter devoted to the numerical results. The first and second orders of accuracy difference schemes are studied. A matlab program is given to conclude that the second order of accuracy is more accurate. Figures and tables are included.

Fifth chapter contains conclusions.

Sixth chapter is the programming for given applications.

CHAPTER 2

NEUMANN BITSADZE SAMARSKII NONLOCAL BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL EQUATIONS

2.1 THE DIFFERENTIAL PROBLEM

We consider the Bitsadze Samarskii nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u(t)}{dt^2} + Au(t) = f(t),\ 0 < t < 1, \\ u'(0) = \varphi,\ u'(1) = \beta u'(\lambda) + \psi, \\ |\beta| \leq 1,\ 0 \leq \lambda < 1, \end{cases} \tag{2.1}$$

for the differential equation in a Hilbert space H with the self-adjoint positive definite operator A.

Let us give there lemmas from [Sobolevskii P.E., 1969 and Öztürk E.,2008] that will be needed below. Here and in the future we will put $B = A^{\frac{1}{2}}$.

Lemma 2.1. The following estimates hold:

$$\left\| B^{\alpha} e^{-tB} \right\|_{H \longrightarrow H} \leq t^{-\alpha},\ 0 \leq \alpha \leq 1, \tag{2.2}$$

$$\left\| (I - e^{-2B})^{-1} \right\|_{H \longrightarrow H} \leq M, \tag{2.3}$$

where M is a positive constant.

Lemma 2.2. For any $0 \leq t < t + \tau \leq 1$ and $0 \leq \alpha \leq 1$ one has the inequality

$$\left\| e^{-tB} - e^{-(t+\tau)B} \right\|_{H \to H} \leq M \frac{\tau^{\alpha}}{(\tau + t)^{\alpha}}, \tag{2.4}$$

where M does not depend on α, t and τ.

Lemma 2.3. Let

$$D = \beta(I - e^{-2B})^{-1}(e^{-(1-\lambda)B} - e^{-(1+\lambda)B}).$$

Then the operator $B^2(I - D)$ has an inverse

$$K = B^{-2}\,(I - D)^{-1}\,,$$

and the following estimate is satisfied

$$\|K\|_{H\to H} \leq M, \tag{2.5}$$

where M is a positive constant.

Now, we will obtain the formula for the solution of the problem (2.1). It is clear that the boundary value problem for elliptic equation

$$-\frac{d^2u(t)}{dt^2} + Au(t) = f(t),\ \ 0 < t < 1, u(0) = u_0, u(1) = u_1 \tag{2.6}$$

has a unique solution [Sobolevskii P. E., 1969]

$$\begin{aligned} u(t) = T\left\{(e^{-tB} - e^{-(2-t)B})u(0) + (e^{-(1-t)B} - e^{-(1+t)B})u(1)\right. \\ \left. -(e^{-(1-t)B} - e^{-(1+t)B})(2B)^{-1}\int_0^1 (e^{-(1-s)B} - e^{-(1+s)B})f(s)ds\right\} \\ +(2B)^{-1}\int_0^1 (e^{-|t-s|B} - e^{-(t+s)B})f(s)ds. \end{aligned} \tag{2.7}$$

Here

$$T = (I - e^{-2B})^{-1}.$$

Using formula (2.7), we obtain

$$\begin{aligned} u'(t) = T\left\{-B(e^{-tB} + e^{-(2-t)B})u(0) + B(e^{-(1-t)B} + e^{-(1+t)B})u(1)\right. \\ \left. -\frac{1}{2}(e^{-(1-t)B} + e^{-(1+t)B})\int_0^1 (e^{-(1-s)B} - e^{-(1+s)B})f(s)ds\right\} \\ +\frac{1}{2}\left[-\int_0^t e^{-(t-s)B}f(s)ds\right. \end{aligned} \tag{2.8}$$

$$+\int_t^1 e^{-(s-t)B} f(s)ds + \int_0^1 e^{-(t+s)B} f(s)ds \Bigg] .$$

Using (2.8) and the nonlocal boundary conditions, we get

$$-TB(I+e^{-2B})u(0) + 2BTe^{-B}u(1)$$

$$= T\int_0^1 (e^{-(2-s)B} - e^{-sB}) f(s)ds + \varphi,$$

$$\left[-2TBe^{-B} + \beta\left(e^{-\lambda B} + e^{-(2-\lambda)B}\right)\right] u(0)$$

$$-\left[TB(I+e^{-2B}) - TB\beta(e^{-(1-\lambda)B} - e^{-(1+\lambda)B})\right] u(1)$$

$$= T\int_0^1 (e^{-(1-s)B} - e^{-(1+s)B}) f(s)ds$$

$$-T\frac{\beta}{2}(e^{-(1-\lambda)B} + e^{-(1+\lambda)B})\int_0^1 (e^{-(1-s)B} - e^{-(1+s)B}) f(s)ds$$

$$+\frac{\beta}{2}\Bigg(-\int_0^\lambda e^{-(\lambda-s)B} f(s)ds$$

$$+\int_\lambda^1 e^{-(s-\lambda)B} f(s)ds + \int_0^1 e^{-(s+\lambda)B} f(s)ds\Bigg) + \psi.$$

From Lemma 2.3 it follows that

$$B^2\left(I - \beta T(e^{-(1-\lambda)B} - e^{-(1+\lambda)B})\right)$$

has an inverse. Therefore

$$u(0) = -\left(I - \beta T(e^{-(1-\lambda)B} - e^{-(1+\lambda)B})\right)^{-1} \tag{2.9}$$

$$\times\left\{T^2\left(I + e^{-2B} - \beta\left(e^{-(1-\lambda)B} - e^{-(1+\lambda)B}\right)\right)\right.$$

$$\times\int_0^1 B^{-1}(e^{-(2-s)B} - e^{-sB}) f(s)ds$$

$$+T\left(I+e^{-2B}-\beta\left(e^{-(1-\lambda)B}-e^{-(1+\lambda)B}\right)\right)B^{-1}\varphi$$

$$-2T^2\int_0^1 B^{-1}(e^{-(2-s)B}-e^{-(2+s)B})f(s)ds$$

$$+\beta T^2\left(e^{-(1-\lambda)B}-e^{-(1+\lambda)B}\right)\int_0^1 B^{-1}(e^{-(2-s)B}-e^{-(2+s)B})f(s)ds$$

$$-\beta e^{-B}T\left(-\int_0^\lambda B^{-1}e^{-(\lambda-s)B}f(s)ds+\int_\lambda^1 B^{-1}e^{-(s-\lambda)B}f(s)ds\right.$$

$$\left.+\int_0^1 B^{-1}e^{-(s+\lambda)B}f(s)ds\right)-2e^{-B}TB^{-1}\psi\Bigg\}$$

and

$$u(1)=-\left(I-\beta T(e^{-(1-\lambda)B}-e^{-(1+\lambda)B})\right)^{-1} \tag{2.10}$$

$$\Bigg\{-T^2(I+e^{-2B})\int_0^1 B^{-1}(e^{-(1-s)B}-e^{-(1+s)B})f(s)ds$$

$$+T\frac{\beta}{2}(I+e^{-2B})\left(-\int_0^\lambda B^{-1}e^{-(\lambda-s)B}f(s)ds\right.$$

$$\left.+\int_\lambda^1 B^{-1}e^{-(s-\lambda)B}f(s)ds+\int_0^1 B^{-1}e^{-(s+\lambda)B}f(s)ds\right)$$

$$+2T^2\int_0^1 B^{-1}(e^{-(3-s)B}-e^{-(1+s)B})f(s)ds$$

$$-\beta T^2\left(e^{-\lambda B}-e^{-(2-\lambda)B}\right)\int_0^1 B^{-1}(e^{-(2-s)B}-e^{-sB})f(s)ds$$

$$-\left[-2TB^{-1}e^{-B}+\beta TB^{-1}\left(e^{-\lambda B}-e^{-(2-\lambda)B}\right)\right]\varphi$$

$$-TB^{-1}(I+e^{-2B})\psi\Big\}.$$

Eventually, if the function $f(t)$ continuously differentiable on [0,1], φ, $\psi \in D(A^{\frac{1}{2}})$ and formulas (2.7), (2.9) and (2.10) give a solution of problem (2.1).

A function $u(t)$ is called a solution of problem (2.1) if the following conditions are satisfied:

i. $u(t)$ is twice continuously differentiable in the segment $[0,1]$.

ii. The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$, and the function $Au(t)$ is continuous on $[0,1]$.

iii. $u(t)$ satisfies the equation and nonlocal boundary condition (2.1).

Let us denote by $C^{\alpha}([0,1],H)$, $0<\alpha<1$, the Banach spaces obtained by completion of the set of all smooth H values functions $\varphi(t)$ on $[0,1]$ in the norms

$$\|\varphi\|_{C^{\alpha}([0,1],H)} = \|\varphi\|_{C([0,1],H)} + \sup_{0\leq t<t+\tau\leq 1} \frac{\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}},$$

where $C([0,1],\ H)$ stands for the Banach space of all continous functions $\varphi(t)$ defined on $[0,1]$ with the norm

$$\|\varphi\|_{C([0,1],H)} = \max_{0\leq t\leq 1} \|\varphi(t)\|_H .$$

A solution of problem (2.1) defined in this manner will from now on be referred to as a solution of problem (2.1) in the space $C([0,1],H)$.

We say that the problem (2.1) is well-posed in $C([0,1],H)$, if there exists the unique solution $u(t)$ in $C([0,1],H)$ of problem (2.1) for any $f(t)\in C([0,1],H)$ and the following coercivity inequality is satisfied:

$$\|u''\|_{C([0,1],H)} + \|Au\|_{C([0,1],H)} \leq M_c\left[\|f\|_{C([0,1],H)} + \left\|A^{\frac{1}{2}}\varphi\right\|_H + \left\|A^{\frac{1}{2}}\psi\right\|_H\right],$$

where M_c does not depend on $f(t)$ and φ, ψ. Unfortunately, the problem (2.1) is ill-posed in the space $C([0,1],H)$.

We say that the problem (2.1) is well-posed in $C^{\alpha}([0,1],H)$, if there exists a unique solution $u(t)$ in $C^{\alpha}([0,1],H)$ of problem (2.1) for any $f(t)\in C^{\alpha}([0,1],H)$ and the following coercivity inequality is satisfied:

$$\|u''\|_{C^{\alpha}([0,1],H)} + \|Au\|_{C^{\alpha}([0,1],H)} \leq M_c(\alpha)\left[\|f\|_{C^{\alpha}([0,1],H)} + \left\|A^{\frac{1}{2}}\varphi\right\|_{H'} + \left\|A^{\frac{1}{2}}\psi\right\|_{H'}\right],$$

where $H'\subset H$ and $M_c(\alpha)$ does not depend on $f(t)$ and φ, ψ.

With the help of the self-adjoint positive definite operator B in a Hilbert space H, the Banach space $E_{\alpha}=E_{\alpha}(B,H)(0<\alpha<1)$ consists of those $v\in H$ for which the norm

$$||v||_{E_{\alpha}} = \sup_{z>0} z^{1-\alpha}||Be^{(-zB)}v||_H + ||v||_H$$

is finite.

Theorem 2.1. Suppose $A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi \in E_{\alpha(H,B)}$. Then the boundary value problem (2.1) is well posed in a Hölder space $C^{\alpha}\left([0,1],H\right)$ and the following coercivity inequality holds:

$$\|u''\|_{C^{\alpha}([0,1],H)} + \|Au\|_{C^{\alpha}([0,1],H)}$$
$$\leq M\left[\frac{1}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}([0,1],H)} + \left\|A^{\frac{1}{2}}\varphi\right\|_{E_{\alpha(H,B)}} + \left\|A^{\frac{1}{2}}\psi\right\|_{E_{\alpha(H,B)}}\right],$$

where M does not depend on $f(t)$ and α, φ, ψ.

Proof. The boundary value problem (2.6) is well posed in a Hölder space $C^{\alpha}\left([0,1],H\right)$ and the following coercivity inequality holds [Ashyralyev A., Sobolevskii P.E., 2004]

$$\|u''\|_{C^{\alpha}([0,1],H)} + \|Au\|_{C^{\alpha}([0,1],H)} \tag{2.11}$$
$$\leq \frac{M}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}([0,1],H)} + M\left[\|Au(0)-f(0)\|_{E_{\alpha(H,B)}} + \|Au(1)-f(1)\|_{E_{\alpha(H,B)}}\right].$$

Then, the proof of this theorem is based on the estimate (2.11) and on the estimates

$$\|Au(0)-f(0)\|_{E_{\alpha(H,B)}} \tag{2.12}$$
$$\leq \frac{M}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}([0,1],H)} + M\left[\left\|A^{\frac{1}{2}}\varphi\right\|_{E_{\alpha(H,B)}} + \left\|A^{\frac{1}{2}}\psi\right\|_{E_{\alpha(H,B)}}\right]$$

and

$$\|Au(1)-f(1)\|_{E_{\alpha(H,B)}} \tag{2.13}$$
$$\leq \frac{N}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}([0,1],H)} + N\left[\left\|A^{\frac{1}{2}}\varphi\right\|_{E_{\alpha(H,B)}} + \left\|A^{\frac{1}{2}}\psi\right\|_{E_{\alpha(H,B)}}\right].$$

Therefore we will prove (2.12) and (2.13).

First, we estimate $\|Au(0)-f(0)\|_{E_{\alpha(H,B)}}$.

Applying the formula (2.9), we can write

$$Au(0)-f(0) = \left(I - \beta T(e^{-(1-\lambda)B} - e^{-(1+\lambda)B})\right)^{-1}$$
$$\times\left\{T^2\left(I + e^{-2B} - \beta\left(e^{-(1-\lambda)B} - e^{-(1+\lambda)B}\right)\right)\right.$$
$$\times\left[\int_0^1 Be^{-(2-s)B}\left(f(s)-f(1)\right)ds + \int_0^1 Be^{-(2-s)B}f(1)ds\right.$$
$$\left.-\int_0^1 Be^{-sB}\left(f(s)-f(0)\right)ds - \int_0^1 Be^{-sB}f(0)ds\right]$$

$$+T\left(I+e^{-2B}-\beta\left(e^{-(1-\lambda)B}-e^{-(1+\lambda)B}\right)\right)B\varphi$$

$$-2T^2\left[\int_0^1 Be^{-(2-s)B}(f(s)-f(1))ds+\int_0^1 Be^{-(2-s)B}f(1)\right.$$

$$\left.-\int_0^1 Be^{-(2+s)B}(f(s)-f(0)ds-\int_0^1 Be^{-(2+s)B}f(0)ds\right]$$

$$+\beta T^2\left(e^{-(1-\lambda)B}-e^{-(1+\lambda)B}\right)$$

$$\times\left[\int_0^1 Be^{-(2-s)B}(f(s)-f(1))ds+\int_0^1 Be^{-(2-s)B}f(1)ds\right.$$

$$\left.-\int_0^1 Be^{-(2+s)B}(f(s)-f(0))ds-\int_0^1 Be^{-(2+s)B}f(0)ds\right]$$

$$-\beta e^{-B}T\left(-\int_0^\lambda Be^{-(\lambda-s)B}(f(s)-f(\lambda))ds-\int_0^\lambda Be^{-(\lambda-s)B}f(\lambda)ds\right.$$

$$+\int_\lambda^1 Be^{-(s-\lambda)B}(f(s)-f(\lambda))+\int_\lambda^1 Be^{-(s-\lambda)B}f(\lambda)ds$$

$$\left.+\int_0^1 Be^{-(s+\lambda)B}(f(s)-f(0))ds+\int_0^1 Be^{-(s+\lambda)B}f(0)ds\right)$$

$$-2e^{-B}TB^{-1}\psi\big\}.$$

Using the identities

$$\int_0^\lambda Be^{-(\lambda-s)B}ds=I-e^{-\lambda B}, \tag{2.14}$$

$$\int_\lambda^1 Be^{-(s-\lambda)B}ds=I-e^{-(1-\lambda)B},\ 0<\lambda<1, \tag{2.15}$$

we get

$$Au(0)-f(0)=\left(I-\beta T(e^{-(1-\lambda)B}-e^{-(1+\lambda)B})\right)^{-1}T^2$$

$$\times\big\{\left[-e^{-B}+e^{-2B}+e^{-3B}-e^{-4B}\right.$$

$$\left.-\beta(-e^{-(1+\lambda)B}+e^{-(2-\lambda)B}+e^{-(3+\lambda)B}-e^{-(4-\lambda)B}\right]f(0)$$

$$-\left[-e^{-B}+e^{-2B}+e^{-3B}-e^{-4B}\right]f(1)$$

$$-\beta\left[-e^{-(1+\lambda)B}+e^{-(2-\lambda)B}+e^{-(3+\lambda)B}-e^{-(4-\lambda)B}\right]f(\lambda)$$

$$-\left[B\varphi-e^{-4B}B\varphi-\beta\left(e^{-(1-\lambda)B}+e^{-(1+\lambda)B}-e^{-(3-\lambda)B}-e^{-(3+\lambda)B}\right)B\varphi\right.$$

$$\left.-2(e^{-B}-e^{-3B})B\psi\right]$$

$$+\int_0^1 B(e^{-sB}-e^{-(2+s)B})(f(s)-f(0))ds$$

$$-\beta\left(e^{-(1-\lambda)B}+e^{-(1+\lambda)B}\right)\int_0^1 B(e^{-sB}-e^{-(2+s)B})(f(s)-f(0))ds$$

$$+\beta\int_0^\lambda B(e^{-(\lambda+s+1)B}-e^{-(\lambda+s+3)B})(f(s)-f(0))ds$$

$$-\int_0^1 B\left(e^{-(4-s)B}-e^{-(2-s)B}\right)(f(s)-f(1))ds$$

$$-\int_0^\lambda B(e^{-(\lambda-s+1)B}-e^{-(\lambda-s+3)B})(f(s)-f(\lambda))ds$$

$$\left.+\beta\int_\lambda^1 B(e^{-(s-\lambda+1)B}-e^{-(s-\lambda+3)B})(f(s)-f(\lambda))ds\right\}$$

$$=J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8.$$

Here

$$J_1=P^{-1}T^2\left\{\left[-e^{-B}+e^{-2B}+e^{-3B}-e^{-4B}\right.\right.$$

$$\left.-\beta(-e^{-(1+\lambda)B}+e^{-(2-\lambda)B}+e^{-(3+\lambda)B}-e^{-(4-\lambda)B}\right]f(0)$$

$$-\left[-e^{-B}+e^{-2B}+e^{-3B}-e^{-4B}\right]f(1)$$

$$\left.-\beta\left[-e^{-(1+\lambda)B}+e^{-(2-\lambda)B}+e^{-(3+\lambda)B}-e^{-(4-\lambda)B}\right]f(\lambda)\right\}$$

$$J_2=-P^{-1}T^2\left[B\varphi-e^{-4B}B\varphi\right.$$

$$-\beta\left(e^{-(1-\lambda)B}+e^{-(1+\lambda)B}-e^{-(3-\lambda)B}-e^{-(3+\lambda)B}\right)B\varphi$$

$$\left.-2(e^{-B}-e^{-3B})B\psi\right]$$

$$J_3=P^{-1}T^2\int_0^1 B(e^{-sB}-e^{-(2+s)B})(f(s)-f(0))ds$$

$$J_4=-P^{-1}T^2\beta\left(e^{-(1-\lambda)B}+e^{-(1+\lambda)B}\right)$$

$$\times \int_0^1 B(e^{-sB} - e^{-(2+s)B})(f(s) - f(0))ds$$

$$J_5 = P^{-1}T^2\beta \int_0^\lambda B(e^{-(\lambda+s+1)B} - e^{-(\lambda+s+3)B})(f(s) - f(0))ds$$

$$J_6 = -P^{-1}T^2 \int_0^1 B\left(e^{-(4-s)B} - e^{-(2-s)B}\right)(f(s) - f(1))ds$$

$$J_7 = -P^{-1}T^2 \int_0^\lambda B(e^{-(\lambda-s+1)B} - e^{-(\lambda-s+3)B})(f(s) - f(\lambda))ds$$

$$J_8 = P^{-1}T^2\beta \int_\lambda^1 B(e^{-(s-\lambda+1)B} - e^{-(s-\lambda+3)B})(f(s) - f(\lambda))ds$$

Here

$$P = \left(I - \beta T(e^{-(1-\lambda)B} - e^{-(1+\lambda)B})\right).$$

Let us estimate J_k for $k = 1, 2, \cdots, 8$ separately. We start with J_1.

Using the estimates (2.2) and (2.5), we show that

$$\|J_1\|_H \leq \left\|P^{-1}\right\|_{H\to H} \left\|T^2\right\|_{H\to H} \{[|| - e^{-B} + e^{-2B}||_{H\to H}$$

$$+||e^{-3B} - e^{-4B}||_{H\to H} + |\beta|. \left(||e^{-(1+\lambda)B}||_{H\to H} + ||e^{-(2-\lambda)B}||_{H\to H}\right.$$

$$\left.+||e^{-(3+\lambda)B}||_{H\to H} + ||e^{-(4-\lambda)B}||_{H\to H}\right)] \, ||f(0)||_H$$

$$+ \left[|| - e^{-B} + e^{-2B}||_{H\to H} + ||e^{-3B} + e^{-4B}||_{H\to H}\right] ||f(1)||_H$$

$$+|\beta| \left[||e^{-(1+\lambda)B}||_{H\to H} + ||e^{-(2-\lambda)B}||_{H\to H}\right.$$

$$\left.+||e^{-(3+\lambda)B}||_{H\to H} + ||e^{-(4-\lambda)B}||_{H\to H}\right] ||f(\lambda)||_H$$

$$\leq M_1 \max_{0\leq t\leq 1} ||f(t)||_H \leq M_1||f||_{C([0,1],H)} \leq M||f||_{C^\alpha([0,1],H)}.$$

Thus we proved that

$$\|J_1\|_H \leq M||f||_{C^\alpha([0,1],H)}.$$

Next, let us estimate $z^{1-\alpha} \|Be^{-z}J_1\|_H$.

$$z^{1-\alpha} \left\|Be^{-z}J_1\right\|_H \leq z^{1-\alpha} \left\|P^{-1}\right\|_{H\to H} \left\|T^2\right\|_{H\to H}$$

$$\times \{ [||B(-e^{-(z+1)B} + e^{-(2+z)B})||_{H\to H}$$

$$+||Be^{-(3+z)B}||_{H\to H}+||Be^{-(4+z)B}||_{H\to H}$$

$$+|\beta|\big(\big(||Be^{-(1+\lambda+z)B}||_{H\to H}+||Be^{-(2-\lambda+z)B}||_{H\to H}$$

$$+||Be^{-(3+\lambda+z)B}||_{H\to H}+||Be^{-(4-\lambda+z)B}||_{H\to H}\big)\big]\,||f(0)||_H$$

$$+\left[||Be^{-(z+1)B}||_{H\to H}+||Be^{-(2+z)B}||_{H\to H}\right.$$

$$\left.+||Be^{-(3+z)B}||_{H\to H}+||Be^{-(4+z)B}||_{H\to H}\right]||f(1)||_H$$

$$+|\beta|\left[||Be^{-(1+\lambda+z)B}||_{H\to H}+||Be^{-(2-\lambda+z)B}||_{H\to H}\right.$$

$$\left.+||Be^{-(3+\lambda+z)B}||_{H\to H}+||Be^{-(4-\lambda+z)B}||_{H\to H}\right]||f(\lambda)||_H$$

$$\leq Mz^{1-\alpha}\left[\frac{1}{(1+z)}+\frac{1}{(3+z)}+\frac{1}{(1+\lambda+z)}+\frac{1}{(2-\lambda+z)}\right.$$

$$+\frac{1}{(3+\lambda+z)}+\frac{1}{(4-\lambda+z)}+\frac{1}{(1+z)}+\frac{1}{(3+z)}$$

$$\left.+\frac{1}{(1+\lambda+z)}+\frac{1}{(2-\lambda+z)}+\frac{1}{(3+\lambda+z)}+\frac{1}{(4-\lambda+z)}\right]\max_{0\leq t\leq 1}||f(t)||_H$$

$$\leq M\left[4\cdot 1^{-\alpha}+4\cdot 2^{-\alpha}+2(2-\lambda)^{-\alpha}+2(4-\lambda)^{-\alpha}\right]||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M||f||_{C^{\alpha}([0,1],H)}.$$

Thus we proved that

$$\|J_2\|_{E_\alpha}\leq M|f||_{C^{\alpha}([0,1],H)}.$$

Now let us estimate J_2. Using the estimates (2.2), (2.5) and the definition of the norm of space $C^{\alpha}\left([0,1],H\right)$, we get

$$\|J_2\|_H\leq ||P^{-1}||_{H\to H}||T^2||_{H\to H}\left[||B\varphi||_H+||e^{-4B}||_{H\to H}||B\varphi||_H\right.$$

$$+|\beta|\left(||e^{-(1-\lambda)B}||_{H\to H}+||e^{-(1+\lambda)B}||_{H\to H}\right.$$

$$\left.+||e^{-(3-\lambda)B}||_{H\to H}+||e^{-(3+\lambda)B}||_{H\to H}\right)||B\varphi||_H$$

$$\left.+2||e^{-B}-e^{-3B}||_{H\to H}||B\psi||_H\right\}\leq M\left(||B\varphi||_H+||B\psi||_H\right).$$

Let us now estimate $z^{1-\alpha}\left\|Be^{-z}J_2\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-z}J_2\right\|_H\leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\left[||Be^{-zB}B\varphi||_H+||Be^{-(4+zB)}||_{H\to H}||B\varphi||_H\right.$$

$$+|\beta|\left(||Be^{-(1-\lambda+z)B}||_{H\to H}+||Be^{-(1+\lambda+z)B}||_{H\to H}\right.$$

$$+||Be^{-(3-\lambda+z)B}||_{H\to H}+||Be^{-(3+\lambda+z)B}||_{H\to H})\big)\,||B\varphi||_H$$

$$+2||B(e^{-(1+z)B}-e^{-(3+z)B})||_{H\to H}||B\psi||_H\}$$

$$\leq M\sup z^{1-\alpha}\left(||Be^{-zB}B\varphi||_H+||B\psi||_H\right)$$

$$M\left(||B\varphi||_{E_\alpha}+||B\psi||_{E_\alpha}\right).$$

So, we conclude that

$$\|J_2\|_{E_\alpha}\leq M(||B\varphi||_{E_\alpha}+||B\psi||_{E_\alpha}).$$

Now, let us estimate J_3. Using the estimates (2.2), (2.5) and the definition of the norm of space $C^\alpha\left([0,1],H\right)$, we obtain

$$\|J_3\|_H\leq\left\|P^{-1}\right\|_{H\to H}\left\|T^2\right\|_{H\to H}$$

$$\times\int\limits_0^1\left(||Be^{-sB}||_{H\to H}+||Be^{-(2+s)B}||\right)||f(s)-f(0)||ds$$

$$\leq M_3\int\limits_0^1\left(\frac{s^\alpha}{s}+\frac{s^\alpha}{2+s}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\leq M_3\int\limits_0^1\left(\frac{1}{s^{1-\alpha}}+\frac{1}{(2+s)^{1-\alpha}}\right)ds||f||_{C^\alpha([0,1],H)}\leq\frac{M_4}{\alpha}||f||_{C^\alpha([0,1],H)}.$$

Thus we proved that

$$\|J_3\|_H\leq\frac{M_4}{\alpha}||f||_{C^\alpha([0,1],H)}.$$

Now let us estimate $z^{1-\alpha}\left\|Be^{-z}J_3\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-z}J_3\right\|_H\leq z^{1-\alpha}\left\|P^{-1}\right\|_{H\to H}||T^2||_{H\to H}$$

$$\times\int\limits_0^1\left(||B^2e^{-(z+s)B}||_{H\to H}+||B^2e^{-(2+s+z)B}||_{H\to H}\right)||f(s)-f(0)||_Hds$$

$$\leq z^{1-\alpha}M_5\int\limits_0^1\left(\frac{s^\alpha}{(s+z)^2}+\frac{s^\alpha}{(2+s+z)^2}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\leq z^{1-\alpha}M_5\int\limits_0^1\left(\frac{1}{(s+z)^{2-\alpha}}+\frac{1}{(2+s+z)^{2-\alpha}}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\leq\frac{M_6}{1-\alpha}||f||_{C^\alpha([0,1],H)}.$$

Hence, we have that

$$\|J_3\|_{E_\alpha}\leq\frac{M_4}{\alpha}||f||_{C^\alpha([0,1],H)}+\frac{M_6}{1-\alpha}||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_7}{\alpha(1-\alpha)}||f||_{C^\alpha([0,1],H)} \leq M||f||_{C^\alpha([0,1],H)}.$$

Next, we estimate J_4. Using the estimates (2.2), (2.5) and the definition of the norm of space $C^\alpha\left([0,1],H\right)$, we obtain

$$\|J_4\|_H \leq ||P^{-1}||_{H\to H}||T^2||_{H\to H}|\beta|\left(||e^{-(1-\lambda)B}+e^{-(1+\lambda)B}||_{H\to H}\right)$$

$$\times\int_0^1\left(||Be^{-sB}||_{H\to H}-||Be^{-(2+s)B}||_{H\to H}\right)||f(s)-f(0)||_H ds$$

$$\leq M_8\int_0^1\left(\frac{s^\alpha}{s}-\frac{s^\alpha}{2+s}\right)ds||f||_{C^\alpha([0,1],H)} \leq \frac{M_9}{\alpha}||f||_{C^\alpha([0,1],H)}.$$

Therefore,

$$\|J_4\|_H \leq \frac{M_9}{\alpha}||f||_{C^\alpha([0,1],H)}.$$

Using the same manner, we can estimate $z^{1-\alpha}\left\|Be^{-zB}J_4\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}J_4\right\|_H \leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times|\beta|\left(||e^{-(1-\lambda)B}+e^{-(1+\lambda)B}||_{H\to H}\right)$$

$$\times\int_0^1\left(||B^2e^{-(s+z)B}||_{H\to H}+||B^2e^{-(2+s+z)B}||_{H\to H}\right)||f(s)-f(0)||_H ds$$

$$\leq z^{1-\alpha}M_{10}\int_0^1\left(\frac{s^\alpha}{(s+z)^2}-\frac{s^\alpha}{(2+s+z)^2}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{11}}{1-\alpha}||f||_{C^\alpha([0,1],H)}.$$

Thus,

$$\|J_4\|_{E_\alpha} \leq \frac{M_9}{\alpha}||f||_{C^\alpha([0,1],H)}+\frac{M_{11}}{1-\alpha}||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{12}}{\alpha(1-\alpha)}||f||_{C^\alpha([0,1],H)}.$$

Hence, we have

$$\|J_4\|_{E_\alpha} \leq M||f||_{C^\alpha([0,1],H)}.$$

Now let us estimate J_5. Using estimates (2.2), (2.5) and the definition of the norm of space $C^{\alpha}\left([0,1],H\right)$, we obtain

$$\left\|J_5\right\|_H \leq ||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times|\beta|\int_0^{\lambda}(||Be^{-(\lambda+s+1)B}||_{H\to H}+||Be^{-(\lambda+s+3)B}||_{H\to H})||f(s)-f(0)||_H ds$$

$$\leq M_{13}\int_0^{\lambda}\left(\frac{s^{\alpha}}{(\lambda+s+1)}+\frac{s^{\alpha}}{(\lambda+s+3)}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M_{13}\int_0^{\lambda}\left(\frac{1}{(\lambda+s+1)^{1-\alpha}}+\frac{1}{(\lambda+s+3)^{1-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{14}}{\alpha}||f||_{C^{\alpha}([0,1],H)}$$

So,

$$\left\|J_5\right\|_H \leq \frac{M_{14}}{\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Now, let us estimate $z^{1-\alpha}\left\|Be^{-zB}J_5\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}J_5\right\|_H \leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}|\beta|$$

$$\times\int_0^{\lambda}(||B^2e^{-(\lambda+s+z+1)B}||_{H\to H}+||B^2e^{-(\lambda+s+z+3)B}||_{H\to H}||f(s)-f(0)||_H ds$$

$$\leq z^{1-\alpha}M_{15}\int_0^{\lambda}\left(\frac{s^{\alpha}}{(\lambda+s+z+1)^2}+\frac{s^{\alpha}}{(\lambda+s+z+3)^2}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$z^{1-\alpha}M_{15}\int_0^{\lambda}\left(\frac{1}{(\lambda+s+z+1)^{2-\alpha}}+\frac{1}{(\lambda+s+z+3)^{2-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{16}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Thus,

$$\left\|J_5\right\|_{E_{\alpha}} \leq \frac{M_{14}}{\alpha}||f||_{C^{\alpha}([0,1],H)}+\frac{M_{16}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{17}}{\alpha(1-\alpha)}||f||_{C^{\alpha}([0,1],H)}.$$

Hence,

$$\left\|J_5\right\|_H \leq M\left\|f\right\|_{C^{\alpha}([0,1],H)}.$$

Let us estimate J_6. Using estimates (2.2), (2.5) and the definition of the norm of space $C^{\alpha}([0,1],H)$, we get

$$\|J_6\|_H \leq ||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\int_0^1 (||Be^{-(4-s)B}||_{H\to H}+||Be^{-(2-s)B}||_{H\to H})||f(s)-f(1)||_H ds$$

$$\leq M_{18}\int_0^1 \left(\frac{(1-s)^{\alpha}}{(4-s)}+\frac{(1-s)^{\alpha}}{(2-s)}\right)||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M_{18}\int_0^1 \left(\frac{1}{(4-s)^{1-\alpha}}+\frac{1}{(2-s)^{1-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{19}}{\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Now, let us estimate $z^{1-\alpha}\left\|Be^{-zB}J_6\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}J_6\right\|_H \leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\int_0^1 (||B^2e^{-(4-s+z)B}||_{H\to H}+||B^2e^{-(2-s+z)B}||_{H\to H})||f(s)-f(1)||_H ds$$

$$\leq z^{1-\alpha}M_{20}\int_0^1 \left(\frac{(1-s)^{\alpha}}{(4-s+z)^2}+\frac{(1-s)^{\alpha}}{(2-s+z)^2}\right)||f||_{C^{\alpha}([0,1],H)}$$

$$\leq z^{1-\alpha}M_{20}\int_0^1 \left(\frac{1}{(4-s+z)^{2-\alpha}}+\frac{1}{(2-s+z)^{2-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{21}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

So, we have

$$\|J_6\|_{E_\alpha} \leq \frac{M_{19}}{\alpha}||f||_{C^{\alpha}([0,1],H)}+\frac{M_{21}}{\alpha-1}||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{22}}{\alpha(1-\alpha)}||f||_{C^{\alpha}([0,1],H)}.$$

Therefore, we conclude that

$$\|J_6\|_{E_\alpha} \leq M||f||_{C^{\alpha}([0,1],H)}.$$

Now, we estimate J_7. Using estimates (2.2), (2.5) and the definition of the norm of space

$C^{\alpha}\left([0,1],H\right)$, we obtain

$$\|J_7\|_H \leq ||P^{-1}||_{H\rightarrow H}||T^2||_{H\rightarrow H}$$

$$\times\int\limits_0^{\lambda}(||Be^{-(\lambda-s+1)B}||_{H\rightarrow H}+||Be^{-(\lambda-s+3)B}||_{H\rightarrow H})||f(s)-f(\lambda)||_H ds$$

$$\leq M_{23}\int\limits_0^{\lambda}\left(\frac{(\lambda-s)^{\alpha}}{(\lambda-s+1)}+\frac{(\lambda-s)^{\alpha}}{(\lambda-s+3)}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$M_{23}\int\limits_0^{\lambda}\left(\frac{1}{(\lambda-s+1)^{1-\alpha}}+\frac{1}{(\lambda-s+3)^{1-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{24}}{\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Next, let us estimate $z^{1-\alpha}\left\|Be^{-zB}J_7\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}J_7\right\|_H \leq z^{1-\alpha}||P^{-1}||_{H\rightarrow H}||T^2||_{H\rightarrow H}$$

$$\times\int\limits_0^{\lambda}(||B^2e^{-(\lambda-s+1+z)B}||_{H\rightarrow H}+||B^2e^{-(\lambda-s+z+3)B}||_{H\rightarrow H})||f(s)-f(\lambda)||_H ds$$

$$\leq z^{1-\alpha}M_{25}\int\limits_0^{\lambda}\left(\frac{(\lambda-s)^{\alpha}}{(\lambda-s+1)^2}+\frac{(\lambda-s)^{\alpha}}{(\lambda-s+3)^2}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{26}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Hence, we get

$$\|J_7\|_{E_\alpha} \leq \frac{M_{24}}{\alpha}||f||_{C^{\alpha}([0,1],H)}+\frac{M_{26}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)} \leq \frac{M_{27}}{\alpha(1-\alpha)}||f||_{C^{\alpha}([0,1],H)}.$$

Therefore,

$$\|J_7\|_{E_\alpha} \leq M||f||_{C^{\alpha}([0,1],H)}.$$

Now, let us estimate J_8. Estimates (2.2), (2.5) and the definition of the norm of space

$C^{\alpha}\left([0,1],H\right)$ give

$$\|J_8\|_H \leq ||P^{-1}||_{H\rightarrow H}||T^2||_{H\rightarrow H}$$

$$\times|\beta|\int\limits_{\lambda}^{1}(||Be^{-(s-\lambda+1)B}||_{H\rightarrow H}+||Be^{-(s-\lambda+3)B}||_{H\rightarrow H})||f(s)-f(\lambda)||_H ds$$

$$\leq M_{28}\int\limits_{\lambda}^{1}\left(\frac{(s-\lambda)^{\alpha}}{(s-\lambda+1)}+\frac{(s-\lambda)^{\alpha}}{(s-\lambda+3)}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{29}}{\alpha}||f||_{C^{\alpha}([0,1],H)}$$

Now, let us estimate $z^{1-\alpha}\left\|Be^{-z}J_8\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-z}J_8\right\|_H \leq z^{1-\alpha}||P||_{H\to H}||T^2||_{H\to H}|\beta|$$

$$\times\int_{\lambda}^{1}(||B^2e^{-(s-\lambda+1+z)B}||_{H\to H}+||B^2e^{-(s-\lambda+3+z)B}||)||f(s)-f(\lambda)||_H ds$$

$$\leq z^{1-\alpha}M_{30}\int_{\lambda}^{1}\left(\frac{(s-\lambda)^{\alpha}}{(s-\lambda+1+z)^2}+\frac{(s-\lambda)^{\alpha}}{(s-\lambda+3+z)^2}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{31}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}$$

So,

$$\left\|J_7\right\|_{E_\alpha} \leq \frac{M_{29}}{\alpha}||f||_{C^{\alpha}([0,1],H)}+\frac{M_{31}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{32}}{\alpha(1-\alpha)}||f||_{C^{\alpha}([0,1],H)}.$$

Therefore,

$$\left\|J_8\right\|_{E_\alpha} \leq M||f||_{C^{\alpha}([0,1],H)}.$$

Combining the estimates for J_k, for $k=1,\cdots,8$ in $E_\alpha(B,H)$ and (2.11) , we get (2.12).

Second, we will estimate $\left\|Au(1)-f(1)\right\|_{E_{\alpha(H,B)}}$.

Applying the formula (2.10), we can write

$$Au(1)-f(1) = -\left(I-\beta T(e^{-(1-\lambda)B}-e^{-(1+\lambda)B})\right)^{-1}$$

$$\times\left\{-T^2(I+e^{-2B})\left[\int_0^1 Be^{-(1-s)B}(f(s)-f(1))ds+\int_0^1 Be^{-(1-s)B}f(1)ds\right.\right.$$

$$\left.-\int_0^1 Be^{-(1+s)B}(f(s)-f(0))ds-\int_0^1 Be^{-(1+s)B}f(0)ds\right]$$

$$+T^2\frac{\beta}{2}(I+e^{-2B})\left(e^{-(1-\lambda)B}+e^{-(1+\lambda)B}\right)$$

$$\times\left[\int_0^1 Be^{-(1-s)B}(f(s)-f(1))ds+\int_0^1 Be^{-(1-s)B}f(1)ds\right.$$

$$\left.-\int_0^1 Be^{-(1+s)B}(f(s)-f(0))ds-\int_0^1 Be^{-(1+s)B}f(s)ds\right]$$

$$+2T^2\left[\int_0^1 Be^{-(3-s)B}(f(s)-f(1))ds+\int_0^1 Be^{-(3-s)B}f(1)\right.$$

$$-\int_0^1 Be^{-(1+s)B}(f(s)-f(0))ds-\int_0^1 Be^{-(1+s)B}f(0)ds$$

$$-\beta T^2\left(e^{-\lambda B}-e^{-(2-\lambda)B}\right)\left[\int_0^1 Be^{-(2-s)B}(f(s)-f(1))ds+\int_0^1 Be^{-(2-s)B}f(1)ds\right.$$

$$-\int_0^1 Be^{-sB}(f(s)-f(0))ds-\int_0^1 Be^{-sB}f(0)ds$$

$$+T\frac{\beta}{2}(I+e^{-2B})\left(-\int_0^\lambda Be^{-(\lambda-s)B}(f(s)-f(\lambda))ds-\int_0^\lambda Be^{-(\lambda-s)B}f(\lambda)ds\right.$$

$$+\int_\lambda^1 Be^{-(s-\lambda)B}(f(s)-f(\lambda))ds+\int_\lambda^1 Be^{-(s-\lambda)B}f(\lambda)ds$$

$$+\int_0^1 Be^{-(s+\lambda)B}(f(s)-f(0))ds+\int_0^1 Be^{-(s+\lambda)B}f(0)ds$$

$$-\left[-2TBe^{-B}+\beta TB\left(e^{-\lambda B}-e^{-(2-\lambda)B}\right)\right]\varphi-TB(I+e^{-2B})\psi\Big\}.$$

Using identities (2.14) and (2.15), we get

$$Au(1)-f(1)=P^{-1}T^2\Big\{\Big[-e^{-B}+e^{-2B}+e^{-3B}-e^{-4B}$$

$$+\beta\left(\frac{3}{2}\left(e^{-(1-\lambda)B}-e^{-(1+\lambda)B}-e^{-(3-\lambda)B}+e^{-(3+\lambda)B}\right)\right)\Bigg)$$

$$-\frac{1}{2}\left(e^{-(2-\lambda)B}-e^{-(2+\lambda)B}-e^{-(4-\lambda)B}+e^{-(4+\lambda)B}\right)\Bigg)\Bigg]f(1)$$

$$-\Big[-e^{-B}+e^{-2B}+e^{-3B}-e^{-4B}$$

$$-\frac{\beta}{2}\Big(-e^{-(1+\lambda)B}-e^{-(2-\lambda)B}+e^{-(2+\lambda)B}-e^{-(3-\lambda)B}-e^{-(3+\lambda)B}$$

$$+e^{-(4-\lambda)B}+e^{-(5-\lambda)B}+2e^{-(1-2\lambda)B}+2e^{-(3-2\lambda)B}\Big)\Big]f(0)$$

$$-\frac{\beta}{2}\left[-e^{-(1-\lambda)B}+2e^{-(3-\lambda)B}+e^{-(4+\lambda)B}-e^{-(5-\lambda)B}\right]f(\lambda)$$

$$-e^{-\lambda B}(f(\lambda)-f(0))$$

$$+\Big[B\psi-e^{-4B}B\psi-2e^{-B}B\varphi+2e^{-3B}B\varphi$$

$$+\beta\left(e^{-\lambda B}-e^{-(2-\lambda)B}+e^{-(2+\lambda)B}+e^{-(4-\lambda)B}\right)B\varphi\Big]$$

$$+\int_0^1 B(e^{-(1-s)B} - e^{-(3-s)B})(f(s) - f(1))ds$$

$$-\frac{\beta}{2}\left(e^{-(1-\lambda)B} + e^{-(1+\lambda)B}\right)\int_0^1 B(e^{-(1-s)B} + e^{-(3-s)B})(f(s) - f(1))ds$$

$$+\beta\left(e^{-\lambda B} + e^{-(2-\lambda)B}\right)\int_0^1 Be^{-(2-s)B}(f(s) - f(1))ds$$

$$-\int_0^1 B(e^{-(3+s)B} - e^{-(1+s)B})(f(s) - f(0))ds$$

$$+\frac{\beta}{2}\left(e^{-(1-\lambda)B} + e^{-(1+\lambda)B}\right)\int_0^1 B(e^{-(3+s)B} + e^{-(1+s)B})(f(s) - f(0))ds$$

$$+\frac{\beta}{2}(I - e^{-2B})\int_0^1 B(e^{-(s+\lambda)B} - e^{-(s+\lambda+2)B})(f(s) - f(0))ds$$

$$-\beta\left(e^{-\lambda B} + e^{-(2-\lambda)B}\right)\int_0^1 Be^{-sB}(f(s) - f(0))ds$$

$$-\frac{\beta}{2}(I - e^{-2B})\int_0^\lambda B(e^{-(\lambda-s)B} + e^{-(\lambda-s+2)B})(f(s) - f(\lambda))ds$$

$$\left.-\frac{\beta}{2}(I - e^{-2B})\int_\lambda^1 B(e^{-(s-\lambda)B} + e^{-(s-\lambda+2)B})(f(s) - f(\lambda))ds\right\}$$

$$= F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 + F_{10} + F_{11}.$$

Here

$$F_1 = P^{-1}T^2\left\{\left[-e^{-B} + e^{-2B} + e^{-3B} - e^{-4B}\right.\right.$$

$$+\beta\left(\frac{3}{2}\left(e^{-(1-\lambda)B} - e^{-(1+\lambda)B} - e^{-(3-\lambda)B} + e^{-(3+\lambda)B}\right)\right)$$

$$\left.\left.-\frac{1}{2}\left(e^{-(2-\lambda)B} - e^{-(2+\lambda)B} - e^{-(4-\lambda)B} + e^{-(4+\lambda)B}\right)\right)\right] f(1)$$

$$-\left[-e^{-B} + e^{-2B} + e^{-3B} - e^{-4B}\right.$$

$$-\frac{\beta}{2}\left(-e^{-(1+\lambda)B} - e^{-(2-\lambda)B} + e^{-(2+\lambda)B} - e^{-(3-\lambda)B} - e^{-(3+\lambda)B}\right.$$

$$\left.\left.+e^{-(4-\lambda)B} + e^{-(5-\lambda)B} + 2e^{-(1-2\lambda)B} + 2e^{-(3-2\lambda)B}\right)\right] f(0)$$

$$-\frac{\beta}{2}\left[-e^{-(1-\lambda)B} + 2e^{-(3-\lambda)B} + e^{-(4+\lambda)B} - e^{-(5-\lambda)B}\right] f(\lambda)$$

$$\left.-e^{-\lambda B}(f(\lambda) - f(0))\right\}$$

$$F_2 = P^{-1}T^2 \left[B\psi - e^{-4B}B\psi - 2e^{-B}B\varphi + 2e^{-3B}B\varphi\right.$$

$$\left.+\beta\left(e^{-\lambda B} - e^{-(2-\lambda)B} + e^{-(2+\lambda)B} + e^{-(4-\lambda)B}\right)B\varphi\right]$$

$$F_3 = P^{-1}T^2 \int_0^1 B(e^{-(1-s)B} - e^{-(3-s)B})(f(s) - f(1))ds$$

$$F_4 = -P^{-1}T^2\frac{\beta}{2}\left(e^{-(1-\lambda)B} + e^{-(1+\lambda)B}\right)$$

$$\times \int_0^1 B(e^{-(1-s)B} + e^{-(3-s)B})(f(s) - f(1))ds$$

$$F_5 = P^{-1}T^2\beta\left(e^{-\lambda B} + e^{-(2-\lambda)B}\right)\int_0^1 Be^{-(2-s)B}(f(s) - f(1))ds$$

$$F_6 = -P^{-1}T^2 \int_0^1 B(e^{-(3+s)B} - e^{-(1+s)B})(f(s) - f(0))ds$$

$$F_7 = -P^{-1}T^2\frac{\beta}{2}\left(e^{-(1-\lambda)B} + e^{-(1+\lambda)B}\right)$$

$$\times \int_0^1 B(e^{-(3+s)B} + e^{-(1+s)B})(f(s) - f(0))ds$$

$$F_8 = -P^{-1}T^2\frac{\beta}{2}(I - e^{-2B})$$

$$\times \int_0^1 B(e^{-(s+\lambda)B} - e^{-(s+\lambda+2)B})(f(s) - f(0))ds$$

$$F_9 = -P^{-1}T^2\beta\left(e^{-\lambda B} + e^{-(2-\lambda)B}\right)\int_0^1 Be^{-sB}(f(s) - f(0))ds$$

$$F_{10} = -P^{-1}T^2\frac{\beta}{2}(I - e^{-2B})$$

$$\times \int_0^\lambda B(e^{-(\lambda-s)B} + e^{-(\lambda-s+2)B})(f(s) - f(\lambda))ds$$

$$F_{11} = -P^{-1}T^2\frac{\beta}{2}(I - e^{-2B})$$

$$\times \int_\lambda^1 B(e^{-(s-\lambda)B} + e^{-(s-\lambda+2)B})(f(s) - f(\lambda))ds.$$

Second, let us estimate F_k for $k = 1, 2, \cdots, 11$ separately. We start with F_1. Using the estimates (2.2) and (2.5), we obtain

$$\|F_1\|_H \le ||P^{-1}||_{H\to H}||T^2||_{H\to H}\left\{\left[|| - e^{-B} + e^{-2B}||_{H\to H} + ||e^{-3B} - e^{-4B}||_{H\to H}\right.\right.$$

$$+|\beta|\left(\frac{3}{2}\left(||e^{-(1-\lambda)B}-e^{-(1+\lambda)B}||_{H\to H}+||e^{-(3-\lambda)B}-e^{-(3+\lambda)B}||_{H\to H}\right)\right)$$

$$+\frac{1}{2}\left(||e^{-(2-\lambda)B}-e^{-(2+\lambda)B}||_{H\to H}+||e^{-(4-\lambda)B}-e^{-(4+\lambda)B}||_{H\to H}\right)\Big)\Big]\,||f(1)||_H$$

$$+\left[||-e^{-B}+e^{-2B}||_{H\to H}+||e^{-3B}-e^{-4B}||_{H\to H}\right.$$

$$+\frac{|\beta|}{2}\left(||e^{-(1+\lambda)B}||_{H\to H}+||e^{-(2-\lambda)B}||_{H\to H}+||e^{-(2+\lambda)B}||_{H\to H}\right.$$

$$+||e^{-(3-\lambda)B}||_{H\to H}+||e^{-(3+\lambda)B}||_{H\to H}+||e^{-(4-\lambda)B}||_{H\to H}+||e^{-(5-\lambda)B}||_{H\to H}\Big)$$

$$\left.+2||e^{-(1-2\lambda)B}||_{H\to H}+2||e^{-(3-2\lambda)B}||_{H\to H}\Big)\right]||f(0)||_H$$

$$+\frac{|\beta|}{2}\left[||e^{-(1-\lambda)B}||_{H\to H}+2||e^{-(3-\lambda)B}||_{H\to H}\right.$$

$$\left.+||e^{-(4+\lambda)B}||_{H\to H}+||e^{-(5-\lambda)B}||_{H\to H}\right]||f(\lambda)||_H$$

$$+||e^{-\lambda B}(f(\lambda)-f(0))||_H\Big\}$$

$$\leq M_1\max_{0\leq t\leq 1}||f(t)||\leq M||f||_{C^\alpha([0,1],H)}.$$

Thus,

$$\|F_1\|_H\leq M||f||_{C^\alpha([0,1],H)}.$$

Now, let us estimate $z^{1-\alpha}\left\|Be^{-zB}F_1\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_1\right\|_H\leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\Big\{\Big[||B(-e^{-(z+1)B}+e^{-(2+z)B})||_{H\to H}+||B(e^{-(3+z)B}-e^{-(4+z)B})||_{H\to H}$$

$$+|\beta|\frac{3}{2}\Big(||B(e^{-(1-\lambda+z)B}-e^{-(1+\lambda+z)B})||_{H\to H}$$

$$+||B(e^{-(3-\lambda+z)B}-e^{-(3+\lambda+z)B})||_{H\to H}\Big)$$

$$+\frac{1}{2}\Big(||B(e^{-(2-\lambda+z)B}-e^{-(2+\lambda+z)B})||_{H\to H}$$

$$+||B(e^{-(4-\lambda+z)B}-e^{-(4+\lambda+z)B})||_{H\to H}\Big)\Big]\,||f(1)||_H$$

$$+\Big[||B(-e^{-(1+z)B}+e^{-(2+z)B})||_{H\to H}+||B(e^{-(3+z)B}-e^{-(4+z)B})||_{H\to H}$$

$$+\frac{|\beta|}{2}\Big(||Be^{-(1+\lambda+z)B}||_{H\to H}+||Be^{-(2-\lambda+z)B}||_{H\to H}$$

$$+||Be^{-(2+\lambda+z)B}||_{H\to H}+||Be^{-(3-\lambda+z)B}||_{H\to H}$$

$$+||Be^{-(3+\lambda+z)B}||_{H\to H}+||Be^{-(4-\lambda+z)B}||_{H\to H}+||Be^{-(5-\lambda+z)B}||_{H\to H}$$

$$+2||Be^{-(1-2\lambda+z)B}||_{H\to H}+2||Be^{-(3-2\lambda+z)B}||_{H\to H}\Big)\Big]\,||f(0)||_H$$

$$+\frac{|\beta|}{2}\Big[||Be^{-(1-\lambda+z)B}||_{H\to H}+2||Be^{-(3-\lambda+z)B}||_{H\to H}$$

$$+||Be^{-(4+\lambda+z)B}||_{H\to H}+||Be^{-(5-\lambda+z)B}||_{H\to H}\Big]\,||f(\lambda)||_H$$

$$+||Be^{-(\lambda+z)B}||_{H\to H}||f(\lambda)-f(0))||_H\}$$

$$\leq M_2 z^{1-\alpha}\left[\frac{2}{z+1}+\frac{2}{z+3}+\beta\left(\frac{4}{z-\lambda+1}\right.\right.$$

$$+\frac{2}{z-\lambda+2}+\frac{6}{z-\lambda+3}+\frac{2}{z-\lambda+4}$$

$$+\frac{2}{z-\lambda+5}+\frac{1}{z+1}+\frac{1}{z+3}+\frac{1}{z+\lambda+1}$$

$$+\frac{1}{z+\lambda+2}+\frac{1}{z+\lambda+3}+\frac{1}{z+\lambda+4}$$

$$\left.\left.+\frac{2}{z-2\lambda+1}+\frac{2}{z-2\lambda+3}\right)\right]\max_{0\leq t\leq 1}||f(t)||_H+Mz^{1-\alpha}\frac{1}{z+\lambda}\lambda^{\alpha}||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M_2\left[2\cdot 1^{-\alpha}+4\cdot 3^{-\alpha}+\beta(4(1-\lambda)^{-\alpha}+2(2-\lambda)^{-\alpha}\right.$$

$$+6(3-\lambda)^{-\alpha}+2(4-\lambda)^{-\alpha}+2(5-\lambda)^{-\alpha}+(2+\lambda)^{-\alpha}$$

$$+(3+\lambda)^{-\alpha}+(4+\lambda)^{-\alpha}$$

$$\left.+2(1-2\lambda)^{-\alpha}+2(3-2\lambda)^{-\alpha}\right]||f||_{C^{\alpha}([0,1],H)}+M||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M||f||_{C^{\alpha}([0,1],H)}.$$

Hence,

$$\|F_1\|_{E_\alpha}\leq M||f||_{C^{\alpha}([0,1],H)}.$$

Next, let us estimate F_2. Using the estimates (2.2) and (2.5), we get

$$\|F_2\|_H\leq||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\left[||B\psi||_H+||e^{-4B}B\psi||_H+2||e^{-B}B\varphi||_H+2||e^{-3B}B\varphi||_H\right.$$

$$\left.+|\beta|\left(||e^{-\lambda B}-e^{-(2+\lambda)B}||_{H\to H}+||e^{-(2-\lambda)B}-e^{-(4-\lambda)B}||_{H\to H}\right)||B\varphi||_H\right.$$

$$\leq M\left[||B\varphi||_{E_\alpha}+||B\psi||_{E_\alpha}\right].$$

Thus,

$$\|F_2\|_H\leq M\left[||B\varphi||_{E_\alpha}+||B\psi||_{E_\alpha}\right].$$

Let us now estimate $z^{1-\alpha}\left\|Be^{-zB}F_2\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_2\right\|_H\leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\left[||Be^{-z}||_{H\to H}||B\psi||_H-||Be^{-(4+z)B}||_{H\to H}||B\psi||_H\right.$$

$$-2||Be^{-(z+1)B}||_{H\to H}||B\varphi||_H+2||Be^{-(3+z)B}||_{H\to H}||B\varphi||_H$$

$$+|\beta|\left(||B(e^{-(\lambda+1+z)B}-e^{-(2+\lambda+z)B})||_{H\to H}\right.$$

$$\left.-||B(e^{-(2-\lambda+z)B}-e^{-(4-\lambda+z)B})||_{H\to H}\right)||B\varphi||_H$$

$$\leq M\left[||B\varphi||_{E_\alpha}+||B\psi||_{E_\alpha}\right].$$

Therefore,

$$\|F_2\|_{E_\alpha}\leq M\left[||B\varphi||_H+||B\psi||_H\right].$$

Now, let us estimate F_3. Using estimates (2.2) and (2.5), we get

$$\|F_3\|_H\leq||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\int_0^1(||Be^{-(1-s)B}||+||Be^{-(3-s)B}||)||f(s)-f(1)||ds$$

$$\leq M_3\int_0^1\left(\frac{(1-s)^\alpha}{1-s}+\frac{(1-s)^\alpha}{3-s}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\leq M_3\int_0^1\left(\frac{1}{(1-s)^{1-\alpha}}+\frac{1}{(3-s)^{1-\alpha}}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\leq\frac{M_4}{\alpha}||f||_{C^\alpha([0,1],H)}.$$

Hence,

$$\|F_3\|_H\leq\frac{M_4}{\alpha}||f||_{C^\alpha([0,1],H)}.$$

Next, let us estimate $z^{1-\alpha}\left\|Be^{-zB}F_3\right\|_H$

$$z^{1-\alpha}\left\|Be^{-zB}F_3\right\|_H\leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\int_0^1(||B^2e^{-(1-s+z)B}||+||B^2e^{-(3-s+z)B}||)||f(s)-f(1)||ds$$

$$\leq M_5z^{1-\alpha}\left(\int_0^1\frac{(1-s)^\alpha}{(1-s+z)^2}+\frac{(1-s)^\alpha}{(3-s+z)^2}\right)||f||_{C^\alpha([0,1],H)}ds$$

$$\leq M_5z^{1-\alpha}\left(\int_0^1\frac{1}{(1-s+z)^{2-\alpha}}+\frac{1}{(3-s+z)^{2-\alpha}}\right)||f||_{C^\alpha([0,1],H)}$$

$$\leq\frac{M_6}{1-\alpha}||f||_{C^\alpha([0,1],H)}.$$

Thus, we proved that

$$\|F_3\|_{E_\alpha}\leq\frac{M_4}{\alpha}||f||_{C^\alpha([0,1],H)}+\frac{M_6}{1-\alpha}||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_7}{\alpha\left(1-\alpha\right)}||f||_{C^{\alpha}([0,1],H)}.$$

Therefore,

$$\|F_3\|_{E_\alpha} \leq M||f||_{C^{\alpha}([0,1],H)}.$$

Now, let us estimate F_4. Using the estimates (2.2) and (2.5), we obtain

$$\|F_4\|_H \leq ||P||_{H\to H}||T^2||_{H\to H}\frac{|\beta|}{2}||e^{-(1-\lambda)B}+e^{-(1+\lambda)B}||$$

$$\times\int_0^1 (||B(e^{-(1-s)B}||+||Be^{-(3-s)B}||)||f(s)-f(1)||ds$$

$$\leq M_8\int_0^1 \left(\frac{(1-s)^{\alpha}}{1-s}+\frac{(1-s)^{\alpha}}{3-s}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_9}{\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Hence,

$$\|F_4\|_H \leq \frac{M_9}{\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Next, we estimate $z^{1-\alpha}\left\|Be^{-zB}F_4\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_4\right\|_H \leq z^{1-\alpha}||P||_{H\to H}||T^2||_{H\to H}\frac{|\beta|}{2}||e^{-(1-\lambda)B}+e^{-(1+\lambda)B}||$$

$$\times\int_0^1 (||B^2(e^{-(1-s+z)B}||+||B^2e^{-(3-s+z)B}||)||f(s)-f(1)||ds$$

$$\leq M_{10}\int_0^1 \left(\frac{(1-s)^{\alpha}}{(1-s+z)^2}-\frac{(1-s)^{\alpha}}{(3-s+z)^2}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M_{10}\int_0^1 \left(\frac{1}{(1-s+z)^{2-\alpha}}-\frac{1}{(3-s+z)^{2-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{11}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Thus,

$$\|F_4\|_{E_\alpha} \leq \frac{M_9}{\alpha}||f||_{C^{\alpha}([0,1],H)}+\frac{M_{11}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{12}}{\alpha\left(1-\alpha\right)}||f||_{C^{\alpha}([0,1],H)}.$$

Therefore,

$$\|F_4\|_{E_\alpha} \leq M||f||_{C^{\alpha}([0,1],H)}.$$

Now, let us estimate F_5. Using estimates (2.2) and (2.5), we get

$$\|F_5\|_H \leq ||P^{-1}||_{H\to H}||T^2||_{H\to H}|\beta|||e^{-\lambda B}+e^{-(2-\lambda)B}||_{H\to H}$$

$$\times\int\limits_0^1 ||Be^{-(2-s)B}||_{H\to H}||f(s)-f(1)||ds$$

$$\leq M_{13}\int\limits_0^1 \frac{(1-s)^\alpha}{(2-s)}||f||_{C^\alpha([0,1],H)}ds \leq M_{13}\int\limits_0^1 \frac{ds}{(2-s)^{1-\alpha}}||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{14}}{\alpha}||f||_{C^\alpha([0,1],H)}.$$

Next, let us estimate $z^{1-\alpha}\left\|Be^{-zB}F_5\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_5\right\|_H. \leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}|\beta|||B(e^{-\lambda B}+e^{-(2-\lambda)B})||_{H\to H}$$

$$\times\int\limits_0^1 ||B^2e^{-(2-s+z)B}||_{H\to H}||f(s)-f(1)||_{H\to H}ds$$

$$\leq M_{15}\int\limits_0^1 \frac{(1-s)^\alpha}{(2-s+z)^2}||f||_{C^\alpha([0,1],H)}ds$$

$$\leq M_{15}\int\limits_0^1 \frac{ds}{(2-s+z)^{2-\alpha}}||f||_{C^\alpha([0,1],H)} \leq \frac{M_{16}}{1-\alpha}||f||_{C^\alpha([0,1],H)}.$$

So,

$$\|F_5\|_{E_\alpha} \leq \frac{M_{14}}{\alpha}||f||_{C^\alpha([0,1],H)}+\frac{M_{16}}{1-\alpha}||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{17}}{\alpha\,(1-\alpha)}||f||_{C^\alpha([0,1],H)}.$$

Therefore,

$$\|F_5\|_{E_\alpha} \leq M||f||_{C^\alpha([0,1],H)}.$$

Let us now estimate F_6. Using estimates (2.2) and (2.5), we obtain

$$\|F_6\|_H \leq ||P||_{H\to H}||T^2||_{H\to H}$$

$$\times\int\limits_0^1 (||Be^{-(3+s)B}||_{H\to H}-||Be^{-(1+s)B}||_{H\to H})||f(s)-f(0)||_H ds$$

$$\leq M_{18}\int\limits_0^1 \left(\frac{s^\alpha}{(3+s)}+\frac{s^\alpha}{(1+s)}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\leq M_{18}\int\limits_0^1 \left(\frac{1}{(3+s)^{1-\alpha}}+\frac{1}{(1+s)^{1-\alpha}}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{19}}{\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Now, let us estimate $z^{1-\alpha}\left\|Be^{-zB}F_6\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_6\right\|_H \leq z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}$$

$$\times\int_0^1(||B^2e^{-(3+s+z)B}||_{H\to H}+||B^2e^{-(1+s+z)B}||_{H\to H})||f(s)-f(0)||_H ds$$

$$\leq z^{1-\alpha}M_{20}\int_0^1\left(\frac{s^{\alpha}}{(3+s+z)^2}+\frac{s^{\alpha}}{(3+s+z)^2}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq z^{1-\alpha}M_{20}\int_0^1\left(\frac{1}{(3+s+z)^{2-\alpha}}+\frac{1}{(1+s+z)^{2-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{21}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Thus

$$\|F_6\|_{E_\alpha} \leq \frac{M_{19}}{\alpha}||f||_{C^{\alpha}([0,1],H)}+\frac{M_{21}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{22}}{\alpha\left(1-\alpha\right)}||f||_{C^{\alpha}([0,1],H)}.$$

Therefore,

$$\|F_6\|_{E_\alpha} \leq M||f||_{C^{\alpha}([0,1],H)}.$$

Let us estimate F_7.

$$\|F_7\|_{E_\alpha} = \|F_7\|_H+\sup_{z>0}z^{1-\alpha}\left\|Be^{-zB}F_7\right\|_H.$$

Using estimates (2.2) and (2.5), we obtain

$$\|F_7\|_H \leq ||P^{-1}||_{H\to H}||T^2||_{H\to H}\frac{|\beta|}{2}||e^{-(1-\lambda)B}+e^{-(1+\lambda)B}||_{H\to H}$$

$$\times\int_0^1(||Be^{-(3+s)B}||_{H\to H}+||Be^{-(1+s)B}||_{H\to H})||f(s)-f(0)||_H ds$$

$$\leq M_{23}\int_0^1\left(\frac{s^{\alpha}}{(3+s)}+\frac{s^{\alpha}}{(1+s)}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M_{23}\int_0^1\left(\frac{1}{(3+s)^{1-\alpha}}+\frac{1}{(1+s)^{1-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{24}}{\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Next, let us estimate $z^{1-\alpha}\left\|Be^{-zB}F_7\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_7\right\|_H \le z^{1-\alpha}||P^{-1}||_{H\to H}||T^2||_{H\to H}\frac{|\beta|}{2}||e^{-(1-\lambda)B}+e^{-(1+\lambda)B}||_{H\to H}$$

$$\times\int_0^1(||B^2e^{-(3+s+z)B}||_{H\to H}+||B^2e^{-(1+s+z)B}||_{H\to H})||f(s)-f(0)||_H ds$$

$$\le M_{25}\int_0^1\left(\frac{s^\alpha}{(3+s+z)^2}+\frac{s^\alpha}{(1+s+z)^2}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\le M_{25}\int_0^1\left(\frac{1}{(3+s+z)^{2-\alpha}}+\frac{1}{(1+s+z)^{2-\alpha}}\right)||f||_{C^\alpha([0,1],H)}$$

$$\le \frac{M_{26}}{1-\alpha}||f||_{C^\alpha([0,1],H)}.$$

Hence,

$$\|F_7\|_{E_\alpha}\le\frac{M_{24}}{\alpha}||f||_{C^\alpha([0,1],H)}+\frac{M_{26}}{1-\alpha}||f||_{C^\alpha([0,1],H)}$$

$$\le\frac{M_{27}}{\alpha(1-\alpha)}||f||_{C^\alpha([0,1],H)}.$$

Thus,

$$\|F_7\|_{E_\alpha}\le N||f||_{C^\alpha([0,1],H)}.$$

Now, let us estimate F_8. Using estimates (2.2) and (2.5), we obtain

$$\|F_8\|_H\le||P^{-1}||_{H\to H}||T^2||_{H\to H}\frac{|\beta|}{2}||I-e^{-2B}||_{H\to H}$$

$$\times\int_0^1(||Be^{-(s+\lambda)B}||_{H\to H}+||Be^{-(s+\lambda+2)B}||_{H\to H})||f(s)-f(0)||_H ds$$

$$\le M_{28}\int_0^1\left(\frac{s^\alpha}{(s+\lambda)}+\frac{s^\alpha}{(s+\lambda+2)}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\le M_{28}\int_0^1\left(\frac{1}{(s+\lambda)^{1-\alpha}}+\frac{1}{(s+\lambda+2)^{1-\alpha}}\right)ds||f||_{C^\alpha([0,1],H)}$$

$$\le\frac{M_{29}}{\alpha}||f||_{C^\alpha([0,1],H)}$$

Now, we estimate $z^{1-\alpha}\left\|Be^{-zB}F_8\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_8\right\|_H\le z^{1-\alpha}||P||_{H\to H}||T^2||_{H\to H}\frac{|\beta|}{2}||I-e^{-2B}||_{H\to H}$$

$$\times\int_0^1(||B^2e^{-(s+\lambda+z)B}||_{H\to H}+||B^2e^{-(s+\lambda+z+2)B}||_{H\to H})||f(s)-f(0)||_H ds$$

$$\leq z^{1-\alpha} M_{30} \int_0^1 \left(\frac{s^\alpha}{(\lambda+s+z)^2} + \frac{s^\alpha}{(\lambda+s+z+2)^2} \right) ds ||f||_{C^\alpha([0,1],H)}$$

$$\leq z^{1-\alpha} M_{30} \int_0^1 \left(\frac{1}{(\lambda+s+z)^{2-\alpha}} + \frac{1}{(\lambda+s+z+2)^{2-\alpha}} \right) ds ||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{31}}{1-\alpha} ||f||_{C^\alpha([0,1],H)}.$$

So,

$$\|F_8\|_{E_\alpha} \leq \frac{M_{29}}{\alpha} ||f||_{C^\alpha([0,1],H)} + \frac{M_{31}}{1-\alpha} ||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{32}}{\alpha\,(1-\alpha)} ||f||_{C^\alpha([0,1],H)}.$$

Therefore,

$$\|F_8\|_{E_\alpha} \leq M ||f||_{C^\alpha([0,1],H)}.$$

Now, let us estimate F_9. Using estimates (2.2) and (2.5), we get

$$\|F_9\|_H \leq ||P||_{H\to H} ||T^2||_{H\to H} |\beta| ||e^{-\lambda B} + e^{-(2-\lambda)B}||_{H\to H}$$

$$\times \int_0^1 ||Be^{-sB}||_{H\to H} ||f(s)-f(0)||_H ds$$

$$\leq M_{33} \int_0^1 \frac{s^\alpha}{s} ds ||f||_{C^\alpha([0,1],H)} \leq M_{33} \int_0^1 \frac{ds}{s^{1-\alpha}} ||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{34}}{\alpha} ||f||_{C^\alpha([0,1],H)}.$$

Now let us estimate $z^{1-\alpha} \left\| Be^{-zB} F_9 \right\|_H$.

$$z^{1-\alpha} \left\| Be^{-zB} F_9 \right\|_H \leq z^{1-\alpha} ||P^{-1}||_{H\to H} ||T^2||_{H\to H} |\beta| ||e^{-\lambda B} + e^{-(2-\lambda)B}||_{H\to H}$$

$$\times \int_0^1 ||B^2 e^{-(s+z)B}||_{H\to H} ||f(s)-f(0)||_H ds$$

$$\leq z^{1-\alpha} M_{35} \int_0^1 \frac{s^\alpha}{(s+z)^2} ds ||f||_{C^\alpha([0,1],H)}$$

$$\leq z^{1-\alpha} M_{35} \int_0^1 \frac{ds}{(s+z)^{2-\alpha}} ||f||_{C^\alpha([0,1],H)} \leq \frac{M_{36}}{1-\alpha} ||f||_{C^\alpha([0,1],H)}.$$

Hence,

$$\|F_9\|_{E_\alpha} \leq \frac{M_{34}}{\alpha} ||f||_{C^\alpha([0,1],H)} + \frac{M_{36}}{1-\alpha} ||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{37}}{\alpha\left(1-\alpha\right)}||f||_{C^{\alpha}([0,1],H)}.$$

Therefore,

$$\|F_9\|_{E_\alpha} \leq M||f||_{C^{\alpha}([0,1],H)}.$$

We now estimate F_{10}. Using estimates (2.2) and (2.5), we obtain

$$\|F_{10}\|_H \leq ||P||_{H\to H}||T^2||_{H\to H}\frac{|\beta|}{2}||I-e^{-2B}||_{H\to H}$$

$$\times\int\limits_0^\lambda (||B(e^{-(\lambda-s)B}||_{H\to H}+||Be^{-(\lambda-s+2)B}||_{H\to H})||f(s)-f(\lambda)||_H ds$$

$$\leq M_{38}\int\limits_0^\lambda\left(\frac{(\lambda-s)^\alpha}{(\lambda-s)}+\frac{(\lambda-s)^\alpha}{(\lambda-s+2)}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M_{38}\int\limits_0^\lambda\left(\frac{1}{(\lambda-s)^{1-\alpha}}+\frac{1}{(\lambda-s+2)^{1-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{39}}{\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

Now, let us estimate $z^{1-\alpha}\left\|Be^{-zB}F_{10}\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_{10}\right\|_H \leq z^{1-\alpha}||P||_{H\to H}||T^2||_{H\to H}\frac{|\beta|}{2}||I-e^{-2B}||_{H\to H}$$

$$\times\int\limits_0^\lambda (||B^2(e^{-(\lambda-s+z)B}||_{H\to H}+||B^2e^{-(\lambda-s+z+2)B}||_{H\to H})||f(s)-f(\lambda)||_H ds$$

$$\leq M_{40}\int\limits_0^\lambda\left(\frac{(\lambda-s)^\alpha}{(\lambda-s+z)^2}+\frac{(\lambda-s)^\alpha}{(\lambda-s+z+2)^2}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq M_{40}\int\limits_0^\lambda\left(\frac{1}{(\lambda-s+z)^{2-\alpha}}+\frac{1}{(\lambda-s+z+2)^{2-\alpha}}\right)ds||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{41}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}.$$

So,

$$\|F_{10}\|_{E_\alpha} \leq \frac{M_{39}}{\alpha}||f||_{C^{\alpha}([0,1],H)}+\frac{M_{41}}{1-\alpha}||f||_{C^{\alpha}([0,1],H)}$$

$$\leq \frac{M_{42}}{\alpha\left(1-\alpha\right)}||f||_{C^{\alpha}([0,1],H)}.$$

Thus,

$$\|F_{10}\|_{E_\alpha} \leq M||f||_{C^{\alpha}([0,1],H)}$$

Finally, let us estimate $\|F_{11}\|_{E_\alpha}$.

$$\|F_{11}\|_{E_\alpha} = \|F_{11}\|_H + \sup_{z>0} z^{1-\alpha}\left\|Be^{-zB}F_{11}\right\|_H$$

$$\|F_{11}\|_H \leq ||P||_{H\to H}||T||_{H\to H}\frac{|\beta|}{2}||I-e^{-2B}||_{H\to H}$$

$$\times\int_\lambda^1 (||B(e^{-(s-\lambda)B}||_{H\to H} + ||Be^{-(s-\lambda+2)B}||_{H\to H}||f(s)-f(\lambda)||_H ds$$

$$\leq M_{43}\int_\lambda^1 \left(\frac{(s-\lambda)^\alpha}{(s-\lambda)^2} + \frac{(s-\lambda)^\alpha}{(s-\lambda+2)^2}\right) ds||f||_{C^\alpha([0,1],H)}$$

$$\leq M_{43}\int_\lambda^1 \left(\frac{1}{(s-\lambda)^{2-\alpha}} + \frac{1}{(s-\lambda+2)^{2-\alpha}}\right) ds||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{45}}{1-\alpha}||f||_{C^\alpha([0,1],H)}.$$

Now, let us estimate $z^{1-\alpha}\left\|Be^{-zB}F_{11}\right\|_H$.

$$z^{1-\alpha}\left\|Be^{-zB}F_{11}\right\|_H . \leq z^{1-\alpha}||P^{-1}||_{H\to H}||T||_{H\to H}\frac{|\beta|}{2}||I-e^{-2B}||_{H\to H}$$

$$\times\int_\lambda^1 (||B^2(e^{-(s-\lambda+z)B}||_{H\to H} + ||B^2e^{-(s-\lambda+z+2)B}||_{H\to H}||f(s)-f(\lambda)||_H ds$$

$$\leq M_{46}\int_\lambda^1 \left(\frac{(s-\lambda)^\alpha}{(s-\lambda+z)^2} + \frac{(s-\lambda)^\alpha}{(s-\lambda+z+2)^2}\right) ds||f||_{C^\alpha([0,1],H)}$$

$$\leq M_{46}\int_\lambda^1 \left(\frac{1}{(s-\lambda+z)^{2-\alpha}} + \frac{1}{(s-\lambda+z+2)^{2-\alpha}}\right) ds||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{47}}{1-\alpha}||f||_{C^\alpha([0,1],H)}.$$

Hence, we proved that

$$\|F_{11}\|_{E_\alpha} \leq \frac{M_{45}}{\alpha}||f||_{C^\alpha([0,1],H)} + \frac{M_{47}}{1-\alpha}||f||_{C^\alpha([0,1],H)}$$

$$\leq \frac{M_{48}}{\alpha\,(1-\alpha)}||f||_{C^\alpha([0,1],H)}.$$

Therefore,

$$\|F_{11}\|_{E_\alpha} \leq M||f||_{C^\alpha([0,1],H)}.$$

Combining the estimates F_i for $i = 1,\cdots,11$ in $E_\alpha(B,H)$ and (2.11), we get (2.13). So, Theorem 2.1 is proved.

2.2 APPLICATIONS

We consider the mixed boundary value problem for the elliptic equation

$$\begin{cases} -u_{tt} - (a(x)u_x)_x + \delta u = f(t,x), 0 < t < 1, 0 < x < 1, \\ \\ u(t,0) = u(t,1),\ \ u_x(t,1) = u_x(t,0), 0 \leq t \leq 1, \\ \\ u_t(0,x) = 0, u_t(1,x) = \beta u_t(\lambda,x), 0 \leq x \leq 1, \\ \\ 0 \leq \lambda < 1,\ \ |\beta| \leq 1, \end{cases} \tag{2.16}$$

where $a(x)$ and $f(t,x)$ are given sufficiently smooth functions $a(x) \geq a > 0$, $\delta = const > 0$. The problem has a unique solution $u(t,x)$. This allows us to reduce the mixed problem (2.16) to the nonlocal boundary value problem (2.1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A defined by (2.16) .

Theorem 2.2. The solutions of the nonlocal boundary value problem (2.16) satisfy the coercivity inequality

$$\|u_{tt}\|_{C^\alpha([0,1],L_2[0,1])} + \|u\|_{C^\alpha([0,1],W_2^2[0,1])}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \|f\|_{C^\alpha([0,1],L_2[0,1])} .$$

Here M does not depend on $f(t,x)$.

The proof of Theorem 2.2 is based on the abstract Theorem 2.1 and the symmetry properties of the space operator generated by the problem.

Second, let Ω be the unit open cube in $\mathbb{R}^n (x = (x_1, \cdots, x_n) : 0 < x_k < 1, 1 \leq k \leq n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, the Bitsadze-Samarskii type mixed boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -u_{tt}-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u=f(t,x), 0<t<1,\ x=(x_1,\ldots,x_n)\in\Omega, \\ u_t(0,x)=0,\ u_t(1,x)=\beta u_t(\lambda,x),\ x\in\overline{\Omega}, \\ |\beta|\leq 1,\ 0\leq\lambda<1, \frac{\partial u}{\partial \vec{n}}(t,x)\mid_{x\epsilon S}=0. \end{cases} \tag{2.17}$$

is considered. The problem has unique smooth solution $u(t,x)$ for the smooth $\alpha_r(x)\geq a>0$ $(x\in\Omega)$ and $f(t,x)$ $(t\in(0,1), x\in\overline{\Omega})$ functions. This allows us to reduce the mixed problem (2.17) to the nonlocal boundary problem (2.1) in a Hilbert space $H=L_2(\overline{\Omega})$ of the all integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{\int\ldots\int\limits_{x\in\overline{\Omega}} |f(x)|^2\, dx_1\ldots dx_n\right\}^{\frac{1}{2}}$$

with a self -adjoint positive definite operator A defined by (2.17) .

Theorem 2.3. The solutions of the nonlocal boundary value problem (2.17) satisfy the coercivity inequality

$$\|u_{tt}\|_{C^\alpha([0,1],L_2(\overline{\Omega}))}+\|u\|_{C^\alpha([0,1],W_2^2(\overline{\Omega}))}$$

$$\leq\frac{M}{\alpha(1-\alpha)}\|f\|_{C^\alpha([0,1],L_2(\overline{\Omega}))}.$$

Here M does not depend on $f(t,x)$.

The proof of Theorem 2.3 is based on the abstract Theorem 2.1 and the symmetry properties of the space operator generated by the problem (2.17) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 2.4. For the solution of the elliptic differential problem

$$-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u=w(x), x\in\Omega,$$

$$\frac{\partial u}{\partial \vec{n}}(x)=0,\ x\in S$$

the following coercivity inequalities [Sobolevskii, P. E.,1975]

$$\sum_{r=1}^{n} \left\| u_{x_r x_r} \right\|_{L_2\left(\overline{\Omega}\right)} \leq M \left\| w \right\|_{L_2\left(\overline{\Omega}\right)}$$

are valid.

CHAPTER 3

FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

3.1 FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

Let us associate the nonlocal boundary- value problem (2.1) with the corresponding first order of accuracy difference scheme

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1}-2u_k+u_{k-1}]+Au_k=f_k,\ f_k=f(t_k),\ t_k=k\tau,\ 1\leq k\leq N-1, \\ \frac{u_1-u_0}{\tau}=\varphi, \\ \frac{u_N-u_{N-1}}{\tau}=\beta\frac{u_{\left[\frac{\lambda}{\tau}\right]+1}-u_{\left[\frac{\lambda}{\tau}\right]}}{\tau}+\psi,\ N\tau=1. \end{cases} \tag{3.1}$$

It is known that study of discretization over time only of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A is replaced by the difference operators A_h that act in the Hilbert spaces H_h and are uniformly self-adjoint positive definite in h for $0\leq h\leq h_0$.

It is known (see [Krein,1966]) that for a self-adjoint positive definite operator A, it follows that $B=\frac{1}{2}(\tau A+\sqrt{4A+\tau^2A^2})$ is self-adjoint positive definite and $R=(I+\tau B)^{-1}$ which is defined on the whole space H is a bounded operator. Here, I is the identity operator. Now, let us give some lemmas that will be needed below.

Lemma 3.1. The estimates hold (see [Ashyralyev A., and Sobolevskii P. E., 2004])

$$\left\|e^{-k\tau A^{\frac{1}{2}}}-R^k\right\|_{H\longrightarrow H}\leq\frac{M\tau}{k\tau},\ k\geq 1, \tag{3.2}$$

$$\left\|R^k\right\|_{H\to H}\leq M(1+\delta\tau)^{-k},\ k\tau\left\|BR^k\right\|_{H\to H}\leq M,\ k\geq 1,\ \delta>0, \tag{3.3}$$

$$\|\ B^{\beta}(R^{k+r}-R^k)\ \|_{H\to H}\leq M\frac{(r\tau)^{\alpha}}{(k\tau)^{\alpha+\beta}}, 1\leq k<k+r\leq N,\ 0\leq\alpha,\ \beta\leq 1, \tag{3.4}$$

$$||(I-R^{2N})^{-1}||_{H\to H}\leq M. \tag{3.5}$$

Lemma 3.2. Suppose A is the self-adjoint positive definite operator in Hilbert space H. Then the following estimate holds (see [Sobolevskii P.E., 1977]):

$$\sum_{j=1}^{N-1} \left\| (I-R)R^{j-1} \right\|_{H\to H} \leq M \min\left(\ln(\frac{1}{\tau}), 1+\left|\ln \|B\|_{H\to H}\right| \right), \tag{3.6}$$

where M does not depend on τ.

Lemma 3.3. The operator

$$I-R^{2N-2}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1}+R^{N+\left[\frac{\lambda}{\tau}\right]-1}\right)$$

has an inverse

$$P_\tau = \left[I-R^{2N-2}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1}+R^{N+\left[\frac{\lambda}{\tau}\right]-1}\right)\right]^{-1}$$

and the following estimate is satisfied (see[Öztürk E.,2008]) :

$$\|P_\tau\|_{H\longrightarrow H} \leq M, \tag{3.7}$$

where M is independent of τ.

Lemma 3.4. For any f_k, $1\leq k\leq N-1$ the solution of the problem (3.1) exists and the following formula holds

$$u_k = (I-R^{2N})^{-1}\left\{\left(R^k-R^{2N-k}\right)\varphi+\left(R^{N-k}-R^{N+k}\right)u_N \right. \tag{3.8}$$

$$\left. -\left(R^{N-k}-R^{N+k}\right)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-1-i}-R^{N-1+i}\right)f_i\tau\right\}$$

$$+(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|-1}-R^{k+i-1}\right)f_i\tau \text{ for } k=1,\cdots,N-1,$$

$$u_0 = P_\tau(I+\tau B)(2I+\tau B)^{-1}B^{-1}$$

$$\times\left\{\left[(I+R)\left(R^{N-2}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)f_i\tau\right.\right.\right.$$

$$\left.\left.+\left[I-R^{2N-1}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1}+R^{N+\left[\frac{\lambda}{\tau}\right]}\right)\right]\sum_{i=1}^{N-1}R^{i-1}f_i\tau\right)\right]$$

$$+\left(R^{N-1}+R^N\right)\beta\left[-\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}R^{\left[\frac{\lambda}{\tau}\right]-i}f_i\tau\right.$$

$$\left.+\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}R^{i-\left[\frac{\lambda}{\tau}\right]-1}f_i\tau+\sum_{i=1}^{N-1}R^{\left[\frac{\lambda}{\tau}\right]+i}f_i\tau\right]$$

$$-(I+R)^2R^{N-2}\beta\tau f_{\left[\frac{\lambda}{\tau}\right]}\Big\}$$

$$-P_\tau(I-R)^{-1}\left[I+R^{2N-1}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1}+R^{N+\left[\frac{\lambda}{\tau}\right]}\right)\right]\tau\varphi$$

$$+P_\tau(I-R)^{-1}(R^{N-1}+R^N)\tau\psi$$

and

$$u_N=-P_\tau(I+\tau B)(2I+\tau B)^{-1}B^{-1}$$

$$\times\left\{\left[-(I+R)+\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1}+R^{N+\left[\frac{\lambda}{\tau}\right]-1}\right)\right]\right.$$

$$\times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)f_i\tau-\beta(I+R^{2N-1})$$

$$\times\left[-\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}R^{\left[\frac{\lambda}{\tau}\right]-i}f_i\tau\right.$$

$$\left.+\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}R^{i-\left[\frac{\lambda}{\tau}\right]-1}f_i\tau+\sum_{i=1}^{N-1}R^{\left[\frac{\lambda}{\tau}\right]+i}f_i\tau\right]$$

$$-\beta(I+R^{2N-1})\tau(I+R)R^{-1}f_{\left[\frac{\lambda}{\tau}\right]}$$

$$\left.-\left[R^{N-1}-R^N-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]}+R^{2N-\left[\frac{\lambda}{\tau}\right]-1}\right)\right]\sum_{i=1}^{N-1}(R^{i-1}+R^i)f_i\tau\right\}$$

$$-P_\tau(I-R)^{-1}\left[R^{N+1}+R^N-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]}+R^{2N-\left[\frac{\lambda}{\tau}\right]-1}\right)\right]\varphi\tau$$

$$+P_\tau(I-R)^{-1}(I+R^{2N-1})\tau\psi\ .$$

Proof. It is clear that the boundary value problem for the second order difference equation

$$\begin{cases}-\frac{1}{\tau^2}[u_{k+1}-2u_k+u_{k-1}]+Au_k=f_k, 1\le k\le N-1, N\tau=1,\\ \\ u_0,\ u_N \text{ are given}\end{cases}\tag{3.9}$$

has a solution and the following formula holds (see[Sobolevskii P.E., 1977]):

$$u_k=(I-R^{2N})^{-1}\left\{\left(R^k-R^{2N-k}\right)u_0+\left(R^{N-k}-R^{N+k}\right)u_N\right.\tag{3.10}$$

$$\left.-\left(R^{N-k}-R^{N+k}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)f_i\tau\right\}$$

$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|}-R^{k+i}\right)f_i\tau.$$

Applying formula (3.10) and nonlocal boundary conditions

$$u_1 - u_0 = \tau\varphi,$$

and

$$u_N - u_{N-1} = \beta\left(u_{\left[\frac{\lambda}{\tau}\right]+1} - u_{\left[\frac{\lambda}{\tau}\right]}\right) + \tau\psi,$$

we obtain

$$(I - R^{2N})^{-1}(R - I)(I + R^{2N-1})u_0 - (I - R^{2N})^{-1}(R - I)\left(R^{N-1} + R^N\right)u_N \tag{3.11}$$
$$= (I - R^{2N})^{-1}(I + \tau B)(2I + \tau B)^{-1}B^{-1}$$
$$\times\left[\left(R^{N-1} - R^{N+1}\right)\sum_{i=1}^{N-1}\left(R^{N-i} - R^{N+i}\right)f_i\tau\right.$$
$$\left.-(I - R^{2N})\sum_{i=1}^{N-1}\left(R^{i-1} - R^{i+1}\right)f_i\tau\right] + \tau\varphi$$

and

$$-(I - R^{2N})^{-1}(I - R)\left[R^{N-1} + R^N - \beta\left(R^{\left[\frac{\lambda}{\tau}\right]} + R^{2N-\left[\frac{\lambda}{\tau}\right]-1}\right)\right]u_0 \tag{3.12}$$
$$+(I - R^{2N})^{-1}(I - R)\left[I + R^{2N-1} - \beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1} + R^{N+\left[\frac{\lambda}{\tau}\right]}\right)\right]u_N$$
$$= (I - R^{2N})^{-1}(I + \tau B)(2I + \tau B)^{-1}B^{-1}$$
$$\times\left[R^{-1} - R - \beta(I - R)\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1} + R^{N+\left[\frac{\lambda}{\tau}\right]}\right)\right]\sum_{i=1}^{N-1}\left(R^{N-i} - R^{N+i}\right)f_i\tau$$
$$+\beta(I + \tau B)(2I + \tau B)^{-1}B^{-1}\left[(R - I)\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1} R^{\left[\frac{\lambda}{\tau}\right]-i}f_i\tau\right.$$
$$\left.+(R^{-1} - I)\sum_{i=\left[\frac{\lambda}{\tau}\right]+1}^{N-1} R^{i-\left[\frac{\lambda}{\tau}\right]}f_i\tau - (R - I)\sum_{i=1}^{N-1} R^{\left[\frac{\lambda}{\tau}\right]+i}f_i\tau\right]$$
$$+\beta(I + \tau B)(2I + \tau B)^{-1}B^{-1}(R - I)f_{\left[\frac{\lambda}{\tau}\right]}\tau + \tau\psi.$$

From (3.12) and (3.11) it follows

$$u_0 = P_\tau(I + \tau B)(2I + \tau B)^{-1}B^{-1}\left\{(I + R)\left[R^{N-2}\sum_{i=1}^{N-1}\left(R^{N-i} - R^{N+i}\right)f_i\tau\right.\right. \tag{3.13}$$
$$\left.+\left[I - R^{2N-1} - \beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1} + R^{N+\left[\frac{\lambda}{\tau}\right]}\right)\right]\sum_{i=1}^{N-1} R^{i-1}f_i\tau\right]$$
$$+\left(R^{N-1} + R^N\right)\beta\left[-\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1} R^{\left[\frac{\lambda}{\tau}\right]-i}f_i\tau\right.$$

$$+\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1} R^{i-\left[\frac{\lambda}{\tau}\right]-1} f_i\tau + \sum_{i=1}^{N-1} R^{\left[\frac{\lambda}{\tau}\right]+i} f_i\tau\Bigg]$$

$$+(I+R)^2R^{N-2}\beta\tau f_{\left[\frac{\lambda}{\tau}\right]}\Big\}$$

$$-P_\tau(I-R)^{-1}\left[I+R^{2N-1}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1}+R^{N+\left[\frac{\lambda}{\tau}\right]}\right)\right]\tau\varphi$$

$$+P_\tau(I-R)^{-1}(R^{N-1}+R^N)\tau\psi$$

and

$$u_N = -P_\tau(I+\tau B)(2I+\tau B)^{-1}B^{-1} \tag{3.14}$$

$$\times\left\{\left[-(I+R)+\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]-1}-R^{N+\left[\frac{\lambda}{\tau}\right]-1}\right)\right]\right.$$

$$\times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)f_i\tau-\beta(I+R^{2N-1})\Bigg[-\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1} R^{\left[\frac{\lambda}{\tau}\right]-i} f_i\tau$$

$$+\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1} R^{i-\left[\frac{\lambda}{\tau}\right]-1} f_i\tau + \sum_{i=1}^{N-1} R^{\left[\frac{\lambda}{\tau}\right]+i} f_i\tau\Bigg]$$

$$-\beta(I+R^{2N-1})\tau(I+R)R^{-1}f_{\left[\frac{\lambda}{\tau}\right]}$$

$$\left.-\left[R^{N-1}-R^N-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]}+R^{2N-\left[\frac{\lambda}{\tau}\right]-1}\right)\right]\sum_{i=1}^{N-1}(R^{i-1}+R^i)f_i\tau\right\}$$

$$-P_\tau(I-R)^{-1}\left[R^{N+1}+R^N-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]}+R^{2N-\left[\frac{\lambda}{\tau}\right]-1}\right)\right]\varphi\tau$$

$$+P_\tau(I-R)^{-1}(I+R^{2N-1})\tau\psi.$$

Lemma 3.4 is proved.

Let $\mathcal{F}([0,1]_\tau, H)$ be the linear space of mesh functions $f^\tau=\{f_k\}_1^{N-1}$ with values in the Hilbert space H. We denote $C([0,1]_\tau, H)$ and $C^\alpha([0,1]_\tau, H)$, $0<\alpha<1$ Banach spaces with the norms

$$\|f^\tau\|_{C([0,1]_\tau,H)} = \max_{1\le k\le N-1}\|f_k\|_H\,,$$

$$\|f^\tau\|_{C^\alpha([0,1]_\tau,H)} = \|f^\tau\|_{C([0,1]_\tau,H)}$$

$$+\sup_{1\le k\le k+r\le N-1}\frac{\|f_{k+r}-f_k\|_H}{(r\tau)^\alpha}$$

The nonlocal boundary value problem (3.1) is said to be stable in $\mathcal{F}([0,1]_\tau, H)$ if we have the inequality

$$\|u^\tau\|_{\mathcal{F}([0,1]_\tau,H)} \le M\left[\|f^\tau\|_{\mathcal{F}([0,1]_\tau,H)}+\|\varphi\|_H+\|\psi\|_H\right],$$

where M does not depend on f^τ, φ, ψ and τ.

Theorem 3.1. The solution of the difference scheme (3.1) satisfy the stability estimate

$$\|u^\tau\|_{C([0,1]_\tau,H)} \leq M\left[\|f^\tau\|_{C([0,1]_\tau,H)} + \|\varphi\|_H + \|\psi\|_H\right], \tag{3.15}$$

where M does not depend on f^τ, φ, ψ and τ.

Proof. By [Sobolevskii, P. E.]

$$\|u^\tau\|_{C([0,1]_\tau,H)} \leq M\left[\|f^\tau\|_{C([0,1]_\tau,H)} + \|u_0\|_H + \|Ru_N\|_H\right] \tag{3.16}$$

is proved for the solution of difference scheme (3.1). Then, the proof of (3.15) is based on (3.16) and on the estimates

$$\|u_0\|_H \leq M\left[\|f^\tau\|_{C([0,1]_\tau,H)} + \|\varphi\|_H + \|\psi\|_H\right]$$

and

$$\|Ru_N\|_H \leq M\left[\|f^\tau\|_{C([0,1]_\tau,H)} + \|\varphi\|_H + \|\psi\|_H\right]. \tag{3.17}$$

Using formula (3.8) and estimates (3.3), (3.4), (3.5) and (3.7), we get

$$\begin{aligned}
\|u_0\|_H \leq{}& \|P_\tau\|_{H\to H}\,||(I+\tau B)(2I+\tau B)^{-1}||_{H\to H}||B^{-1}||_{H\to H} \qquad (3.18)\\
&\times\Bigg\{||I+R||_{H\to H}\Bigg[||R^{N-2}||_{H\to H}\sum_{i=1}^{N-1}\left(||R^{N-i}||_{H\to H}+||R^{N+i}||_{H\to H}\right)||f_i||_H\tau\\
&+\left[||I-R^{2N-1}||_{H\to H}+|\beta|\left(||R^{N-\left[\frac{\lambda}{\tau}\right]-1}||_{H\to H}+||R^{N+\left[\frac{\lambda}{\tau}\right]}||_{H\to H}\right)\right]\\
&\times\sum_{i=1}^{N-1}||R^{i-1}||_{H\to H}||f_i||_H\tau\Bigg)\Bigg]+\left(||R^{N-1}||_{H\to H}+||R^{N}||_{H\to H}\right)|\beta|\\
&\times\Bigg[\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}||R^{\left[\frac{\lambda}{\tau}\right]-i}||_{H\to H}||f_i||_H\tau+\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}||R^{i-\left[\frac{\lambda}{\tau}\right]-1}||_{H\to H}||f_i||_H\tau\\
&+\sum_{i=1}^{N-1}||R^{\left[\frac{\lambda}{\tau}\right]+i}||_{H\to H}||f_i||_H\tau\Bigg]\\
&+||(I+R)^2||_{H\to H}||R^{N-2}||_{H\to H}|\beta|\tau||f_i||_H\\
&+||P_\tau||_{H\to H}\left[||I+R^{2N-1}||_{H\to H}\right.\\
&\left.+|\beta|\left(||R^{N-\left[\frac{\lambda}{\tau}\right]-2}||_{H\to H}+||R^{N+\left[\frac{\lambda}{\tau}\right]-1}||_{H\to H}\right)\right]||\varphi||_{H\to H}
\end{aligned}$$

$$+||P_\tau||_{H\to H}(||R^{N-1}||_{H\to H}+||R^N||_{H\to H})||\psi||_{H\to H}$$

$$\leq M\left[\max_{1\leq k\leq N-1}\|f_k\|_H+\|\varphi\|_H+\|\psi\|_H\right]$$

and

$$\|Ru_N\|_H\leq\|P_\tau\|_{H\to H}||(2I+\tau B)^{-1}(I+\tau B)||_{H\to H}||B^{-1}||_{H\to H}$$
$$\times\left\{||R+R^2||_{H\to H}+|\beta|\left(||R^{N-[\frac{\lambda}{\tau}]}||_{H\to H}+||R^{N+[\frac{\lambda}{\tau}]}||_{H\to H}\right)\right.$$
$$\times\sum_{i=1}^{N-1}\left(||R^{N-i}||_{H\to H}+||R^{N+i}||_{H\to H}\right)||f_i||_H\tau$$
$$+|\beta|||R+R^{2N}||_{H\to H}\left[\sum_{i=1}^{[\frac{\lambda}{\tau}]-1}||R^{[\frac{\lambda}{\tau}]-i}||_{H\to H}||f_i||_H\tau\right.$$
$$+\sum_{i=[\frac{\lambda}{\tau}]}^{N-1}||R^{i-[\frac{\lambda}{\tau}]-1}||_{H\to H}||f_i||_H\tau$$
$$\left.+\sum_{i=1}^{N-1}||R^{[\frac{\lambda}{\tau}]+i}||_{H\to H}||f_i||_H\tau\right]$$
$$+|\beta|||I+R^{2N-1}||_{H\to H}\tau||I+R||_{H\to H}||f_{[\frac{\lambda}{\tau}]}||_H$$
$$+\left[||R^N+R^{N+1}||_{H\to H}+|\beta|\left(||R^{[\frac{\lambda}{\tau}]+1}||_{H\to H}+||R^{2N-[\frac{\lambda}{\tau}]}||_{H\to H}\right)\right]$$
$$\left.\times\sum_{i=1}^{N-1}(||R^{i-1}||_{H\to H}+||R^i||_{H\to H})||f_i||_H\tau\right\}$$
$$+||P_\tau||_{H\to H}\left[||R^{N+2}||_{H\to H}+||R^{N+1}||_{H\to H}\right.$$
$$\left.+|\beta|\left(||R^{[\frac{\lambda}{\tau}]+1}||_{H\to H}+||R^{2N-[\frac{\lambda}{\tau}]}||_{H\to H}\right)\right]||\varphi||_{H\to H}$$
$$+||P_\tau||_{H\to H}||R+R^{2N}||_{H\to H}||\psi||_{H\to H}$$
$$\leq M\left[\max_{1\leq k\leq N-1}\|f_k\|_H+\|\varphi\|_H+\|\psi\|_H\right].$$

Hence, Theorem 4.1 is proved.

Theorem 3.2. The solution of the difference problem (3.1) in $C([0,1]_\tau, H)$ obey the almost coercive inequality

$$\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\|_{C([0,1]_\tau,H)}+\|\{Au_k\}_1^{N-1}\|_{C([0,1]_\tau,H)} \tag{3.19}$$
$$\leq M\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{H\to H}|\right\}\|f^\tau\|_{C([0,1]_\tau,H)}+\|B\varphi\|_H+\|B\psi\|_H\right].$$

Here M does not depend on τ, ψ, φ and $f_k, 1\leq k\leq N-1$.

Proof. By [Ashyralyev A, Sobolevskii P. E, 2004],

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau, H)} + \| \{Au_k\}_1^{N-1} \|_{C([0,1]_\tau, H)} \tag{3.20}$$
$$\leq M \left[\min\left\{ \ln\frac{1}{\tau}, 1 + |\ln \| B \|_{H\to H}| \right\} \| f^\tau \|_{C([0,1]_\tau, H)} + \|Au_0\|_H + \|ARu_N\|_H \right]$$

is proved for the solutions of the boundary value problem

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1} - 2u_k + u_{k-1}] + Au_k = f_k, 1 \leq k \leq N-1, \\ \\ u_0,\ u_N \text{ are given},\ N\tau = 1. \end{cases} \tag{3.21}$$

Using estimates (3.3), (3.5) and the formulas (3.13) and (3.14), we obtain

$$\| Au_0 \|_H \tag{3.22}$$
$$\leq M_1 \left(\min\left\{ \ln\frac{1}{\tau}, 1 + |\ln \| B \|_{H\to H}| \right\} \| f^\tau \|_{C([0,1]_\tau, H)} + \|B\varphi\|_H + ||B\psi||_H \right)$$

and

$$\| ARu_N \|_H \tag{3.23}$$
$$\leq M_1 \left(\min\left\{ \ln\frac{1}{\tau}, 1 + |\ln \| B \|_{H\to H}| \right\} \| f^\tau \|_{C([0,1]_\tau, H)} + \|B\varphi\|_H + ||B\psi||_H \right)$$

for solutions of the boundary value problem (3.1).

First, let us prove(3.22)

$$Au_0 = J_1^0 + J_2^0,$$

where

$$J_1^0 = -P_\tau \left\{ \left[I + R^{2N-1} - \beta \left(R^{N-[\frac{\lambda}{\tau}]-1} + R^{N+[\frac{\lambda}{\tau}]} \right) \right] B\varphi \right.$$
$$\left. - \left(R^{N-1} - R^N \right) B\psi \right\},$$

$$J_2^0 = P_\tau(I + \tau B)(2I + \tau B)^{-1} \left\{ (I + R) \left[R^{N-1} \sum_{i=1}^{N-1} B \left(R^{N-i} - R^{N+i} \right) f_i\tau \right. \right.$$
$$\left. + \left[R - R^{2N} - \beta \left(R^{N-[\frac{\lambda}{\tau}]} + R^{N+[\frac{\lambda}{\tau}]+1} \right) \right] \sum_{i=1}^{N-1} BR^{i-1} f_i\tau \right]$$
$$+ \left(R^N + R^{N+1} \right) \beta \left[- \sum_{i=1}^{[\frac{\lambda}{\tau}]-1} BR^{[\frac{\lambda}{\tau}]-i} f_i\tau \right.$$

$$+\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1} BR^{i-\left[\frac{\lambda}{\tau}\right]-1}f_i\tau+\sum_{i=1}^{N-1} R^{\left[\frac{\lambda}{\tau}\right]+i}f_i\tau\Bigg]$$

$$+R^{N-1}(I+R)^2\beta B\tau f_{\left[\frac{\lambda}{\tau}\right]}\Bigg\}.$$

To this end it suffices to show that

$$\left\|J_1^0\right\|_H \leq M_1[\|B\varphi\|_H+\|B\psi\|_H] \tag{3.24}$$

and

$$\left\|J_2^0\right\|_H \leq M_1\min\left\{\ln\frac{1}{\tau},1+|\ln\parallel B\parallel_{H\to H}|\right\}\parallel f^\tau\parallel_{C([0,1]_\tau,H)}. \tag{3.25}$$

Estimate (3.24) follows from the estimates (3.4), and (3.7). Using estimates, we get

$$\left\|J_1^0\right\|_H \leq ||P_\tau||_{H\to H}||\left\{\left[||I+R^{2N-1}||\right.\right.$$

$$\left.+|\beta|\left(||R^{N-\left[\frac{\lambda}{\tau}\right]-1}||_{H\to H}+||R^{N+\left[\frac{\lambda}{\tau}\right]}||_{H\to H}\right)\right]||B\varphi||_{H\to H}$$

$$\left.+\left(||R^{N-1}||_{H\to H}+||R^N||_{H\to H}\right)||B\psi||_{H\to H}\right\}$$

$$\leq M_1[\|B\varphi\|_H+\|B\psi\|_H].$$

Now, from estimates (3.4) and (3.7), we obtain

$$\left\|J_2^0\right\|_H \leq \|P_\tau\|_{H\to H}||(I+\tau B)(2I+\tau B)^{-1}||_{H\to H}$$

$$\times\left\{||I+R||_{H\to H}\left[||R^{N-1}||_{H\to H}\sum_{i=1}^{N-1}(||BR^{N-i}||_{H\to H}+||BR^{N+i}||_{H\to H})||f_i||_H\tau\right.\right.$$

$$+\left[||I||_{H\to H}+||R^{2N-1}||_{H\to H}+|\beta|\left(||R^{N-\left[\frac{\lambda}{\tau}\right]-1}||_{H\to H}+||R^{N+\left[\frac{\lambda}{\tau}\right]}||_{H\to H}\right)\right]$$

$$\left.\left.\times\sum_{i=1}^{N-1}||BR^i||_{H\to H}||f_i||_H\tau\right)\right]$$

$$+\left(||R^{N-1}||_{H\to H}+||R^N||_{H\to H}\right)|\beta|$$

$$\times\left[+\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]}||BR^{\left[\frac{\lambda}{\tau}\right]-i+1}||_{H\to H}||f_i||_H\tau\right.$$

$$+\sum_{i=\left[\frac{\lambda}{\tau}\right]+1}^{N-1}||BR^{i-\left[\frac{\lambda}{\tau}\right]}||_{H\to H}||f_i||_H\tau$$

$$+\sum_{i=1}^{N-1}||BR^{\left[\frac{\lambda}{\tau}\right]+i+1}||_{H\to H}||f_i||_H\tau\Bigg]$$

$$+||BR^{N-1}||_{H\to H}||I+R^2||_{H\to H}|\beta|\tau||f_{\left[\frac{\lambda}{\tau}\right]}||_H$$

$$\leq M\sum_{j=1}^{N-1}\tau||BR^{j-1}||_{H\to H}||f^\tau||_{C([0,1]_\tau,H)}.$$

From the last estimate and estimate (3.6), it follows estimate (3.25).

Second, let us prove (3.23).

$$ARu_N=J_1^N+J_2^N,$$

where

$$J_1^N=-P_\tau\left\{\left[R^{N+2}+R^{N+1}-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]+1}+R^{2N-\left[\frac{\lambda}{\tau}\right]}\right)\right]B\varphi\right.$$

$$\left.+\left(R+R^{2N}\right)B\psi\right\}$$

$$J_2^N=-P_\tau(I+\tau B)(2I+\tau B)^{-1}$$

$$\times\left\{\left[-(R^2+R)+\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}-R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right]\sum_{i=1}^{N-1}B\left(R^{N-i}-R^{N+i}\right)f_i\tau\right.$$

$$+\beta\left(R+R^{2N}\right)\left[\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}BR^{\left[\frac{\lambda}{\tau}\right]-i+1}f_i\tau-\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}BR^{i-\left[\frac{\lambda}{\tau}\right]}f_i\tau\right.$$

$$\left.-\sum_{i=1}^{N-1}BR^{\left[\frac{\lambda}{\tau}\right]+i+1}f_i\tau\right]+\beta(I+R^{2N-1})BR\tau(I+R)f_{\left[\frac{\lambda}{\tau}\right]}$$

$$-\left[R^{N+1}+R^{N+2}-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]+2}+R^{2N-\left[\frac{\lambda}{\tau}\right]+1}\right)\right]$$

$$\left.\times\sum_{i=1}^{N-1}B\left(R^{i-1}+R^i\right)f_i\tau\right\}.$$

To this end it suffices to show that

$$\left\|J_1^N\right\|_H\leq M_1[\|B\varphi\|_H+\|B\psi\|_H] \tag{3.26}$$

and

$$\left\|J_2^N\right\|_H\leq M_1\min\left\{\ln\frac{1}{\tau},1+|\ln\parallel B\parallel_{H\to H}|\right\}\parallel f^\tau\parallel_{C([0,1]_\tau,H)}. \tag{3.27}$$

Estimate (3.26) follows from the estimates (3.4), and (3.7). Using the estimates we get,

$$\left\|J_1^N\right\|_H \leq ||P_\tau||_{H\to H}\left\{\left[||R^{N+2}||_{H\to H} + ||R^{N+1}||_{H\to H}\right.\right.$$

$$+|\beta|\left(||R^{\left[\frac{\lambda}{\tau}\right]+1}||_{H\to H} + ||R^{2N-\left[\frac{\lambda}{\tau}\right]}||_{H\to H}\right)\Big] \, ||B\varphi||_{H\to H}$$

$$\left. +||R+R^{2N}||_{H\to H}||B\psi||_{H\to H}\right\}$$

$$\leq M_1[\|B\varphi\|_H + \|B\psi\|_H].$$

Now, from estimates (3.4) and (3.7), we obtain

$$\left\|J_2^N\right\|_H \leq \|P_\tau\|_{H\to H}\,||(I+\tau B)(2I+\tau B)^{-1}||_{H\to H}$$

$$\times\left\{\left[||R^2+R||_{H\to H} + |\beta|\left(||R^{N-\left[\frac{\lambda}{\tau}\right]+1}||_{H\to H} + ||R^{N+\left[\frac{\lambda}{\tau}\right]+1}||_{H\to H}\right)\right]\right.$$

$$\times\sum_{i=1}^{N-1}(||BR^{N-i}||_{H\to H} + ||BR^{N+i}||_{H\to H})||f_i||_H\tau$$

$$+|\beta|||R+R^{2N}||_{H\to H}\left[\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]}||BR^{\left[\frac{\lambda}{\tau}\right]-i+1}||_{H\to H}||f_i||_H\tau\right.$$

$$+\sum_{i=\left[\frac{\lambda}{\tau}\right]+1}^{N-1}||BR^{i-\left[\frac{\lambda}{\tau}\right]}||_{H\to H}||f_i||_H\tau$$

$$\left.+\sum_{i=1}^{N-1}||BR^{\left[\frac{\lambda}{\tau}\right]+i+1}||_{H\to H}||f_i||_H\tau\right]$$

$$+|\beta|||I+R^{2N}||_{H\to H}||RB||_{H\to H}\tau||f_{\left[\frac{\lambda}{\tau}\right]}||_H$$

$$+\left[||R^{N+1}||_{H\to H} + ||R^{N+2}||_{H\to H} + |\beta|\left(||R^{\left[\frac{\lambda}{\tau}\right]+2}||_{H\to H} + ||R^{2N-\left[\frac{\lambda}{\tau}\right]+1}||_{H\to H}\right)\right]$$

$$\left.\times\sum_{i=1}^{N-1}(||BR^{i-1}||_{H\to H} + ||BR^{i}||_{H\to H})||f_i||_H\tau\right\}$$

$$\leq M\sum_{j=1}^{N-1}\tau||BR^j||_{H\to H}||f^\tau||_{C([0,1]_\tau,H)}.$$

From the last estimate and estimate (3.6), it follows estimate (3.27).

Hence, from estimates (3.22) and (3.23), Theorem 4.2 is proved.

Theorem 3.3. The difference problem (3.1) is well posed in Hölder spaces $C^\alpha([0,1]_\tau, H)$ and the following coercivity inequality holds:

$$\|\ \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\ \|_{C^\alpha([0,1]_\tau,H)} + \|\ \{Au_k\}_1^{N-1}\ \|_{C^\alpha([0,1]_\tau,H)} \qquad (3.28)$$

$$\leq M\frac{1}{\alpha(1-\alpha)}\left\|f^{\tau}\right\|_{C^{\alpha}([0,1]_{\tau},H)}+M[||B\varphi||_{E_{\alpha}(B,H)}+||B\psi||_{E_{\alpha}(B,H)}].$$

Proof. By [Ashyralyev A, Sobolevskii P.E, 2004],

$$\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\|_{C^{\alpha}([0,1]_{\tau},H)} \tag{3.29}$$

$$\leq M\left[||ARu_0-f_1||_{E_{\alpha}(B,H)}+||ARu_N-f_{N-1}||_{E_{\alpha}(B,H)}\right.$$
$$\left.+\frac{1}{\alpha(1-\alpha)}\| f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}\right]$$

is proved for the solutions of the nonlocal boundary value problem (3.21). The proof of this theorem is based on the estimate (3.29) and

$$\| ARu_0-f_1\|_{E_{\alpha}(B,H)} \tag{3.30}$$

$$\leq M_1\left[\frac{1}{\alpha(1-\alpha)}\| f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}+M||B\varphi||_{E_{\alpha}(B,H)}+||B\psi||_{E_{\alpha}(B,H)}\right]$$

and

$$\| ARu_N-f_{N-1}\|_{E_{\alpha}(B,H)} \tag{3.31}$$

$$\leq M_1\left[\frac{1}{\alpha(1-\alpha)}\| f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}+M||B\varphi||_{E_{\alpha}(B,H)}+||B\psi||_{E_{\alpha}(B,H)}\right].$$

Thus, we will prove (3.30) and (3.31) for the solutions of the nonlocal boundary value problem (3.1).

First, applying the formula (3.13), we can write

$$Au_0-f_1=P_{\tau}(I+\tau B)(2I+\tau B)^{-1}\left\{(I+R)\left[R^N\left(\sum_{i=1}^{N-1}\tau BR^{N-i}(f_i-f_{N-1})\right.\right.\right.$$

$$\left.+\sum_{i=1}^{N-1}\tau BR^{N-i}f_{N-1}-\sum_{i=1}^{N-1}\tau BR^{N+i}(f_i-f_1)-\sum_{i=1}^{N-1}\tau BR^{N+i}f_1\right)$$

$$+\left[R^2-R^{2N+1}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}+R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right)\right]$$

$$\left.\times\left(\sum_{i=1}^{N-1}\tau BR^{i-1}(f_i-f_1)+\sum_{i=1}^{N-1}\tau BR^{i-1}f_1\right)\right]$$

$$-\left(R^N+R^{N+1}\right)\beta\left[R\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}(f_i-f_{\left[\frac{\lambda}{\tau}\right]})+R\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}f_{\left[\frac{\lambda}{\tau}\right]}\right.$$

$$-\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau BR^{i-\left[\frac{\lambda}{\tau}\right]}(f_i-f_{\left[\frac{\lambda}{\tau}\right]})-\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau BR^{i-\left[\frac{\lambda}{\tau}\right]}f_{\left[\frac{\lambda}{\tau}\right]}$$

$$-R\sum_{i=1}^{N-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]+i}(f_i-f_1)-R\sum_{i=1}^{N-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}f_1\Bigg]$$

$$+(I+R)^2R^{N-1}\beta Bf_{\left[\frac{\lambda}{\tau}\right]}\Big\}-f_1$$

$$-P_\tau\left[R+R^{2N}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]}+R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right)\right]B\varphi$$

$$+P_\tau\left(R^N+R^{N+1}\right)B\psi.$$

Using the simple formulas

$$\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}=I-R^{\left[\frac{\lambda}{\tau}\right]-1}, \tag{3.32}$$

$$\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau BR^{i-\left[\frac{\lambda}{\tau}\right]}=R^{-1}\left(I-R^{N-\left[\frac{\lambda}{\tau}\right]}\right), \tag{3.33}$$

we obtain that

$$Au_0-f_1=P_\tau\left(R^N+R^{N+1}\right)B\psi$$

$$-P_\tau\left[R+R^{2N}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]}+R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right)\right]B\varphi$$

$$+P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^N+R^{N+1})\sum_{i=1}^{N-1}\tau BR^{N-i}(f_i-f_{N-1})$$

$$-P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^N+R^{N+1})\sum_{i=1}^{N-1}\tau BR^{N+i}(f_i-f_1)$$

$$+P_\tau(I+\tau B)(2I+\tau B)^{-1}(I+R)$$

$$\times\left[R^2+R^{2N+1}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}+R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right)\right]\sum_{i=1}^{N-1}\tau BR^{i-1}(f_i-f_1)$$

$$-P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^{N+1}+R^{N+2})\beta\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}(f_i-f_{\left[\frac{\lambda}{\tau}\right]})$$

$$+P_\tau(I+\tau B)(2I+\tau B)^{-1}\beta(R^{N+1}+R^{N+2})\sum_{i=1}^{N-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]+i}(f_i-f_1)$$

$$+P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^N+R^{N+1})\beta\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau BR^{i-\left[\frac{\lambda}{\tau}\right]}(f_i-f_{\left[\frac{\lambda}{\tau}\right]})$$

$$+P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^N+R^{N+1})(I-R^{N-1})f_{N-1}$$

$$+P_\tau(I+\tau B)(2I+\tau B)^{-1}(I+R)R\beta$$

$$\times\left[R^N-R^{N-2}+R^{\left[\frac{\lambda}{\tau}\right]+N-1}-R^{2N-\left[\frac{\lambda}{\tau}\right]-2}+(R^{N-1}+R^N)BR\right]f_{\left[\frac{\lambda}{\tau}\right]}$$

$$+P_\tau(I+\tau B)(2I+\tau B)^{-1}\Big[R^{2N-1}(I+R^N)(I+R)$$

$$-\beta\Big[(I+R)(R^{2N+\left[\frac{\lambda}{\tau}\right]}-R^{N+\left[\frac{\lambda}{\tau}\right]+1})$$

$$+(R^N-R)(R^{N-\left[\frac{\lambda}{\tau}\right]-1}+R^{N+\left[\frac{\lambda}{\tau}\right]+2})\Big]\Big]f_1$$

$$=\sum_{z=1}^{9}J_k^z,$$

where

$$J_k^1=P_\tau\left(R^N+R^{N+1}\right)B\psi$$

$$-P_\tau\left[R+R^{2N}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]}+R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right)\right]B\varphi,$$

$$J_k^2=P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^N+R^{N+1})\sum_{i=1}^{N-1}\tau BR^{N-i}(f_i-f_{N-1}),$$

$$J_k^3=-P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^N+R^{N+1})\sum_{i=1}^{N-1}\tau BR^{N+i}(f_i-f_1),$$

$$J_k^4=P_\tau(I+\tau B)(2I+\tau B)^{-1}(I+R)$$

$$\times\left[R^2+R^{2N+1}-\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}+R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right)\right]\sum_{i=1}^{N-1}\tau BR^{i-1}(f_i-f_1),$$

$$J_k^5=P_\tau(I+\tau B)(2I+\tau B)^{-1}\beta(R^{N+1}+R^{N+2})\sum_{i=1}^{N-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]+i}(f_i-f_1),$$

$$J_k^6=-P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^{N+1}+R^{N+2})\beta\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}(f_i-f_{\left[\frac{\lambda}{\tau}\right]})$$

$$+P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^N+R^{N+1})\beta\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau BR^{i-\left[\frac{\lambda}{\tau}\right]}(f_i-f_{\left[\frac{\lambda}{\tau}\right]}),$$

$$J_k^7=P_\tau(I+\tau B)(2I+\tau B)^{-1}(R^N+R^{N+1})(I-R^{N-1})f_{N-1},$$

$$J_k^8=P_\tau(I+\tau B)(2I+\tau B)^{-1}\Big[R^{2N-1}(I+R^N)(I+R)$$

$$-\beta\Big[(I+R)(R^{2N+\left[\frac{\lambda}{\tau}\right]}-R^{N+\left[\frac{\lambda}{\tau}\right]+1})$$

$$+(R^N-R)(R^{N-\left[\frac{\lambda}{\tau}\right]-1}+R^{N+\left[\frac{\lambda}{\tau}\right]+2})\Big]\Big]f_1,$$

$$J_k^9 = P_\tau(I+\tau B)(2I+\tau B)^{-1}(I+R)R\beta$$
$$\times \left[R^N - R^{N-2} + R^{\left[\frac{\lambda}{\tau}\right]+N-1} - R^{2N-\left[\frac{\lambda}{\tau}\right]-2} + (R^{N-1}+R^N)BR\right] f_{\left[\frac{\lambda}{\tau}\right]}.$$

Now let us estimate $J_m^\tau = \{J_k^m\}_{k=1}^{N-1}$ for $m = 1, 2, 3, \cdots, 9$ seperately.We start with J_k^1. Using estimates (3.3), (3.7), we get

$$\left\|J_k^1\right\|_H = \|P_\tau\|_{H\to H}\left(\left\|R^N\right\|_{H\to H} + \left\|R^{N+1}\right\|_{H\to H}\right)\|B\psi\|_H$$
$$+\|P_\tau\|_{H\to H}\left[\left\|R+R^{2N}\right\|_{H\to H}\right.$$
$$\left.+|\beta|\left(\left\|R^{N-\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H} + \left\|R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right\|_{H\to H}\right)\right]\|B\varphi\|_H$$
$$\leq M[\|B\varphi\|_{E_\alpha(B,H)} + \|B\psi\|_{E_\alpha(B,H)}].$$

Thus, we proved

$$\left\|J_k^1\right\|_H \leq M[\|B\varphi\|_H + \|B\psi\|_H].$$

Now let us estimate $z^\alpha||(z+B)^{-1}BJ_k^1||_H$ for any $z > 0$.

$$z^\alpha||(z+B)^{-1}BJ_k^1||_H \leq z^\alpha\|P_\tau\|_{H\to H}$$
$$\left(\left\|BR^N\right\|_{H\to H} + \left\|BR^{N+1}\right\|_{H\to H}\right)\left\|B(z+B)^{-1}B\psi\right\|_H$$
$$+\|P_\tau\|_{H\to H}\left[\left\|BR\right|| + ||BR^{2N}\right\|_{H\to H}$$
$$\left.+|\beta|\left(\left\|BR^{N-\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H} + \left\|BR^{N+\left[\frac{\lambda}{\tau}\right]+2}\right\|_{H\to H}\right)\right]\left\|B(z+B)^{-1}B\varphi\right\|_H$$
$$\leq M_2[\|B\varphi\|_H + \|B\psi\|_H].$$

Thus, we proved that

$$\left\|J_k^1\right\|_{E_\alpha(B,H)} \leq M_1[\|B\varphi\|_{E_\alpha(B,H)} + \|B\psi\|_{E_\alpha(B,H)}].$$

Now, let us estimate J_k^2. Using estimates (3.3), (3.7), we obtain that

$$\left\|J_k^2\right\|_H \leq \|P_\tau\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$
$$\times(\left\|R^N\right\|_{H\to H} + \left\|R^{N+1}\right\|_{H\to H})\sum_{i=1}^{N-1}\tau\left\|BR^{N-i}\right\|_{H\to H}\|f_i - f_{N-1}\|_{H\to H}$$

$$\leq M_3\sum_{i=1}^{N-1}\tau\frac{((N-1)\tau - i\tau)^\alpha}{((N-1)-i)\tau}\|f^\tau\|_{C^\alpha([0,1]_\tau,H)}.$$

The sum enclosed in the right- hand side square brackets is the lower Darboux integral sum for the integral

$$\int_0^1 \frac{ds}{(1-s)^{1-\alpha}}.$$

It follows that

$$\left\|J_k^2\right\|_H \leq M_4 \frac{1}{\alpha} \left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)} \leq M_5 \frac{1}{\alpha(1-\alpha)} \left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

Now let us estimate $z^\alpha||(z+B)^{-1}BJ_k^2||_H$ for any $z>0$.

$$z^\alpha||(z+B)^{-1}BJ_k^2||_H \leq z^\alpha \left\|P_\tau\right\|_{H\to H} \left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$(\left\|BR^N\right\|_{H\to H} + \left\|BR^{N+1}\right\|_{H\to H}) \sum_{i=1}^{N-1} \tau \left\|B(z+B)^{-1}R^{N-i}\right\|_{H\to H} \left\|f_i - f_{N-1}\right\|_H$$

$$\leq M_6 \frac{z^\alpha}{z+1} \sum_{i=1}^{N-1} \tau \frac{((N-1)\tau - i\tau)^\alpha}{((N-1)-i)\tau} \left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)} \leq M_7 \frac{1}{\alpha(1-\alpha)} \left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

So, we proved that

$$\left\|J_k^2\right\|_{E_\alpha(B,H)} \leq M_1 \frac{1}{\alpha(1-\alpha)} \left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

Now let us estimate J_k^3. Using estimates (3.3), (3.7), and the definition of the norm of space $C^\alpha([0,1]_\tau, H)$, we get

$$\left\|J_k^3\right\|_H \leq \left\|P_\tau\right\|_{H\to H} \left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H} (\left\|R^N\right\|_{H\to H} + \left\|R^{N+1}\right\|_{H\to H})$$

$$\times \sum_{i=1}^{N-1} \tau \left(||BR^{N+i}||_{H\to H}\right) ||f_i - f_1||_H$$

$$\leq M_5 \sum_{i=1}^{N-1} \tau \left(\frac{i^\alpha}{(N+i)\tau}\right) \left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}$$

The sum enclosed in the right- hand side square brackets is the lower Darboux integral sum for the integral

$$\int_0^1 \frac{ds}{(1+s)} = \ln 2.$$

Thus,

$$\left\|J_k^3\right\|_H \leq M_6 \ln 2 \left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)} \leq M \left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

Now let us estimate $z^\alpha||(z+B)^{-1}BJ_k^3||_H$ for any $z>0$.

$$z^\alpha||(z+B)^{-1}BJ_k^3||_H \leq z^\alpha \left\|P_\tau\right\|_{H\to H} \left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$(\left\|BR^N\right\|_{H\to H}+\left\|BR^{N+1}\right\|_{H\to H})$$

$$\times\sum_{i=1}^{N-1}\tau\left(||B(z+B)^{-1}R^{\left[\frac{\lambda}{\tau}\right]+i}||_{H\to H}+||B(z+B)^{-1}R^{N+i}||_{H\to H}\right)||f_i-f_1||_H$$

$$\leq M\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

So, we proved that

$$\left\|J_k^3\right\|_{E_\alpha(B,H)}\leq M_1\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

Then, we estimate J_k^4. Using the estimates (3.3), (3.7) and the definition of the norm space $C^\alpha([0,1]_\tau\,,H)$, we obtain that

$$\left\|J_k^4\right\|_H\leq\left\|P_\tau\right\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|(I+R)\right\|_{H\to H}$$

$$\times\left[\left\|R^2+R^{2N+1}\right\|_{H\to H}+|\beta|\left(\left\|R^{N-\left[\frac{\lambda}{\tau}\right]+1}+R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right\|_{H\to H}\right)\right]$$

$$\times\sum_{i=1}^{N-1}\tau\left\|BR^{i-1}\right\|_{H\to H}\left\|f_i-f_1\right\|_H$$

$$\leq M_7\sum_{i=1}^{N-1}\frac{\tau(i-1)^\alpha}{\tau(i-1)}\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}\leq M\frac{1}{(1-\alpha)}\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

Now let us estimate $z^\alpha||(z+B)^{-1}BJ_k^4||_H$.

$$z^\alpha||(z+B)^{-1}BJ_k^4||_H\leq z^\alpha\left\|P_\tau\right\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|(I+R)\right\|_{H\to H}$$

$$\times\left[\left\|R^2+R^{2N+1}\right\|_{H\to H}+|\beta|\left(\left\|R^{N-\left[\frac{\lambda}{\tau}\right]+1}+R^{N+\left[\frac{\lambda}{\tau}\right]+2}\right\|_{H\to H}\right)\right]$$

$$\times\sum_{i=1}^{N-1}\tau\left\|B(z+B)^{-1}R^{i-1}\right\|_{H\to H}\left\|f_i-f_1\right\|_H$$

$$\leq M_8\sum_{i=1}^{N-1}\frac{\tau(i-1)^\alpha}{\tau(i-1)}\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}\leq M\frac{1}{(1-\alpha)}\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

So, we proved that

$$\left\|J_k^4\right\|_{E_\alpha(B,H)}\leq M_1\frac{1}{(1-\alpha)}\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

In a similar manner, we can prove that

$$\left\|J_k^5\right\|_H\leq M_1\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)},$$

Let us estimate J_k^6. Estimates (3.3), (3.7) and the definition of the norm space $C^\alpha([0,1]_\tau\,,H)$ give

$$\left\|J_k^6\right\|_H\leq\left\|P_\tau\right\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|R^{N+1}+R^{N+2}\right\|_{H\to H}$$

$$\times|\beta|\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau\left\|BR^{\left[\frac{\lambda}{\tau}\right]-i}\right\|_{H\to H}\left\|f_i-f_{\left[\frac{\lambda}{\tau}\right]}\right\|_H$$

$$+\|P_\tau\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|R^N+R^{N+1}\right\|_{H\to H}$$

$$\times|\beta|\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau\left\|BR^{i-\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H}\left\|f_i-f_{\left[\frac{\lambda}{\tau}\right]}\right\|_H$$

$$\leq M_9\left[\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau\frac{\left(\left[\frac{\lambda}{\tau}\right]-i\right)^\alpha}{\left(\left[\frac{\lambda}{\tau}\right]-i\right)\tau}+\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau\frac{\left(i-\left[\frac{\lambda}{\tau}\right]\right)^\alpha}{\left(i-\left[\frac{\lambda}{\tau}\right]\right)\tau}\right]\|f^\tau\|_{C^\alpha([0,1]_\tau,H)}$$

$$\leq M\frac{1}{(1-\alpha)}\|f^\tau\|_{C^\alpha([0,1]_\tau,H)}.$$

Now let us estimate $z^\alpha||(z+B)^{-1}BJ_k^6||_H$.

$$z^\alpha||(z+B)^{-1}BJ_k^6||_H\leq z^\alpha\|P_\tau\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$\times(\left\|BR^N\right||_{H\to H}+||BR^{N+1}\right\|_{H\to H})$$

$$\times|\beta|\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau\left\|B(z+B)^{-1}R^{\left[\frac{\lambda}{\tau}\right]-i}\right\|_{H\to H}\left\|f_i-f_{\left[\frac{\lambda}{\tau}\right]}\right\|_H$$

$$+\|P_\tau\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left(\left\|BR^N\right||_{H\to H}+||BR^{N+1}\right\|_{H\to H}\right)$$

$$\times|\beta|\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau\left\|B(z+B)^{-1}R^{i-\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H}\left\|f_i-f_{\left[\frac{\lambda}{\tau}\right]}\right\|_H$$

$$\leq M_9\left[\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau\frac{\left(\left[\frac{\lambda}{\tau}\right]-i\right)^\alpha}{\left(\left[\frac{\lambda}{\tau}\right]-i\right)\tau}+\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau\frac{\left(i-\left[\frac{\lambda}{\tau}\right]\right)^\alpha}{\left(i-\left[\frac{\lambda}{\tau}\right]\right)\tau}\right]\|f^\tau\|_{C^\alpha([0,1]_\tau,H)}$$

$$\leq M\frac{1}{(1-\alpha)}\|f^\tau\|_{C^\alpha([0,1]_\tau,H)}.$$

So we proved that

$$\left\|J_k^6\right\|_{E_\alpha(B,H)}\leq M_1\frac{1}{(1-\alpha)}\|f^\tau\|_{C^\alpha([0,1]_\tau,H)}.$$

Now, let us estimate J_k^7. Using estimates (3.3), (3.7), we obtain that

$$\left\|J_k^7\right\|_H\leq\|P_\tau\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$\times\left\|R^N+R^{N+1}\right\|_{H\to H}\left\|I-R^{N-1}\right\|_{H\to H}\|f_{N-1}\|_H$$

$$\leq M_{10}\max_{0<k<N}\|f_k\|_H\leq M\|f^\tau\|_{C^\alpha([0,1]_\tau,H)}.$$

Now let us estimate $z^\alpha||(z+B)^{-1}BJ_k^7||_H$.

$$z^\alpha||(z+B)^{-1}BJ_k^7||_H\leq z^\alpha\|P_\tau\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$\times \left\|BR^{N}\right\|_{H\to H}+\left\|BR^{N+1}\right\|_{H\to H}\left\|I-R^{N-1}\right\|_{H\to H}\left\|f_{N}\right\|_{H}$$

$$\leq M_{11}\max_{0<k<N}\left\|f_{k}\right\|_{H}\leq M\left\|f^{\tau}\right\|_{C^{\alpha}([0,1]_{\tau},H)}.$$

So, we proved that

$$\left\|J_{k}^{7}\right\|_{E_{\alpha}(B,H)}\leq M_{1}\left\|f^{\tau}\right\|_{C^{\alpha}([0,1]_{\tau},H)}.$$

In a similar manner, we can prove that

$$\left\|J_{k}^{8}\right\|_{H}\leq M_{1}\left\|f^{\tau}\right\|_{C^{\alpha}([0,1]_{\tau},H)},$$

and

$$\left\|J_{k}^{9}\right\|_{H}\leq M_{1}\left\|f^{\tau}\right\|_{C^{\alpha}([0,1]_{\tau},H)},$$

Combining estimates for J_{k}^{m}, for $m=1,\cdots,9$ in $E_{\alpha}(B,H)$, (3.30) is proved.

Second, applying the formula (3.14), we can write

$$ARu_{N}-f_{N-1}=-P_{\tau}\left[R^{N+2}+R^{N+1}-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]+1}+R^{2N-\left[\frac{\lambda}{\tau}\right]}\right)\right]B\varphi$$

$$+P_{\tau}(R+R^{2N})B\psi$$

$$-P_{\tau}(I+\tau B)(2I+\tau B)^{-1}\left\{\left[R^{2}+R+\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}-R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right]\right.$$

$$\times\left[\sum_{i=1}^{N-1}\tau BR^{N-i}(f_{i}-f_{N-1})+\sum_{i=1}^{N-1}\tau BR^{N-i}f_{N-1}\right.$$

$$\left.-\sum_{i=1}^{N-1}\tau BR^{N+i}(f_{i}-f_{1})-\sum_{i=1}^{N-1}\tau BR^{N+i}f_{1}\right]$$

$$-\beta(R+R^{2N})(I-R)$$

$$\times\left[R\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}(f_{i}-f_{\left[\frac{\lambda}{\tau}\right]})+R\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}f_{\left[\frac{\lambda}{\tau}\right]}\right.$$

$$-\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau BR^{i-\left[\frac{\lambda}{\tau}\right]}(f_{i}-f_{\left[\frac{\lambda}{\tau}\right]})-\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau BR^{i-\left[\frac{\lambda}{\tau}\right]}f_{\left[\frac{\lambda}{\tau}\right]}$$

$$\left.-R\sum_{i=1}^{N-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]+i}(f_{i}-f_{1})-R\sum_{i=1}^{N-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]+i}f_{1}\right]$$

$$-\beta(I+R^{2N-1})R\tau B(I+R)f_{\left[\frac{\lambda}{\tau}\right]}$$

$$-\left[R^{N+1}+R^{N+2}-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]+2}+R^{2N-\left[\frac{\lambda}{\tau}\right]+1}\right)\right](I-R^{2})$$

$$\left.\times\left[\sum_{i=1}^{N-1}\tau BR^{i-1}(f_{i}-f_{1})+\sum_{i=1}^{N-1}\tau BR^{i-1}f_{1}\right]\right\}-f_{N-1}$$

Now, using formulas (3.32) and (3.33), we obtain

$$ARu_N - f_{N-1} = -P_\tau \left[R^{N+2} + R^{N+1} - \beta \left(R^{\left[\frac{\lambda}{\tau}\right]+1} + R^{2N-\left[\frac{\lambda}{\tau}\right]}\right)\right] B\varphi$$

$$+P_\tau (R + R^{2N}) B\psi$$

$$-P_\tau (I + \tau B)(2I + \tau B)^{-1}$$

$$\times \left[R^2 + R + \beta \left(R^{N-\left[\frac{\lambda}{\tau}\right]+1} - R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right] \sum_{i=1}^{N-1} \tau B R^{N-i} (f_i - f_{N-1})$$

$$+P_\tau (I + \tau B)(2I + \tau B)^{-1} (R + R^{2N}) \beta$$

$$\times \left[R \sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1} \tau B R^{\left[\frac{\lambda}{\tau}\right]-i} (f_i - f_{\left[\frac{\lambda}{\tau}\right]}) - \sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1} \tau B R^{i-\left[\frac{\lambda}{\tau}\right]} (f_i - f_{\left[\frac{\lambda}{\tau}\right]})\right]$$

$$+P_\tau (I + \tau B)(2I + \tau B)^{-1} (I + R)$$

$$\times \left[R^2 + R + \beta \left(R^{N-\left[\frac{\lambda}{\tau}\right]+1} - R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right] \sum_{i=1}^{N-1} \tau B R^{i-1} (f_i - f_1)$$

$$+P_\tau (I + \tau B)(2I + \tau B)^{-1}$$

$$\times \left[R^2 + R + \beta \left(R^{N-\left[\frac{\lambda}{\tau}\right]+1} - R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right] \sum_{i=1}^{N-1} \tau B R^{N+i} (f_i - f_1)$$

$$-P_\tau (2I + \tau B)^{-1} (R + R^{2N}) \beta \sum_{i=1}^{N-1} \tau B R^{\left[\frac{\lambda}{\tau}\right]+i} (f_i - f_1)$$

$$-P_\tau (I + \tau B)(2I + \tau B)^{-1}$$

$$\times \left[R^2 + R + \beta \left(R^{N-\left[\frac{\lambda}{\tau}\right]+1} - R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right] (I - R^{N-1}) f_{N-1}$$

$$+P_\tau (I + \tau B)(2I + \tau B)^{-1} (I + R)$$

$$\times \left[R^N + R^{N+1} - \beta \left(R^{\left[\frac{\lambda}{\tau}\right]+1} + R^{2N-\left[\frac{\lambda}{\tau}\right]}\right)\right] (R^{N-1} - I) f_1$$

$$+P_\tau (I + \tau B)(2I + \tau B)^{-1} (I + R^{2N-1}) \beta$$

$$\times \left[2R^2 - R - R^{\left[\frac{\lambda}{\tau}\right]+1} + R^{N-\left[\frac{\lambda}{\tau}\right]}\right] f_{\left[\frac{\lambda}{\tau}\right]}$$

$$= \sum_{z=1}^{8} F_k^z$$

where

$$F_k^1 = -P_\tau \left[R^{N+2} + R^{N+1} - \beta \left(R^{\left[\frac{\lambda}{\tau}\right]+1} + R^{2N-\left[\frac{\lambda}{\tau}\right]}\right)\right] B\varphi$$

$$+P_\tau (R + R^{2N}) B\psi$$

$$F_k^2 = -P_\tau (I + \tau B)(2I + \tau B)^{-1}$$

$$\times\left[R^2+R+\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}-R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right]\sum_{i=1}^{N-1}\tau BR^{N-i}(f_i-f_{N-1})$$

$$F_k^3=P_\tau(I+\tau B)(2I+\tau B)^{-1}(R+R^{2N})\beta$$

$$\times\left[R\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]-i}(f_i-f_{\left[\frac{\lambda}{\tau}\right]})-\sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1}\tau BR^{i-\left[\frac{\lambda}{\tau}\right]}(f_i-f_{\left[\frac{\lambda}{\tau}\right]})\right]$$

$$F_k^4=P_\tau(I+\tau B)(2I+\tau B)^{-1}(I+R)$$

$$\times\left[R^2+R+\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}-R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right]\sum_{i=1}^{N-1}\tau BR^{i-1}(f_i-f_1)$$

$$F_k^5=P_\tau(I+\tau B)(2I+\tau B)^{-1}$$

$$\times\left[R^2+R+\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}-R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right]\sum_{i=1}^{N-1}\tau BR^{N+i}(f_i-f_1)$$

$$-P_\tau(2I+\tau B)^{-1}(R+R^{2N})\beta\sum_{i=1}^{N-1}\tau BR^{\left[\frac{\lambda}{\tau}\right]+i}(f_i-f_1)$$

$$F_k^6=P_\tau(I+\tau B)(2I+\tau B)^{-1}$$

$$\times\left[R^2+R+\beta\left(R^{N-\left[\frac{\lambda}{\tau}\right]+1}-R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right)\right](I-R^{N-1})f_N$$

$$F_k^7=P_\tau(I+\tau B)(2I+\tau B)^{-1}(I+R)$$

$$\times\left[R^N+R^{N+1}-\beta\left(R^{\left[\frac{\lambda}{\tau}\right]+1}+R^{2N-\left[\frac{\lambda}{\tau}\right]}\right)\right](R^N-I)f_1$$

$$F_k^8=P_\tau(I+\tau B)(2I+\tau B)^{-1}(I+R^{2N-1})\beta$$

$$\times\left[2R^2-R-R^{\left[\frac{\lambda}{\tau}\right]+1}+R^{N-\left[\frac{\lambda}{\tau}\right]}\right]f_{\left[\frac{\lambda}{\tau}\right]}$$

Now let us estimate $F_m^\tau=\{J_k^m\}_{k=1}^{N-1}$ for $m=1,2,3,\cdot\cdot\cdot,8$ seperately. We start with F_k^1. Using estimates (3.3), (3.7), we get

$$\left\|F_k^1\right\|_H=\|P_\tau\|_{H\to H}\left[\left\|R^{N+2}+R^{N+1}\right\|_{H\to H}\right.$$

$$+|\beta|\left(\left\|R^{\left[\frac{\lambda}{\tau}\right]+1}\right\|_{H\to H}+\left\|R^{2N-\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H}\right)\Big]\|B\varphi\|_H$$

$$+\|P_\tau\|_{H\to H}\left\|R+R^{2N}\right\|_{H\to H}\|B\psi\|_H$$

$$\leq M_2[\|B\varphi\|_H+\|B\psi\|_H].$$

Thus, we proved

$$\left\|F_k^1\right\|_H\leq M_2[\|B\varphi\|_{E_\alpha(B,H)}+\|B\psi\|_{E_\alpha(B,H)}].$$

Now let us estimate $z^{\alpha}||(z+B)^{-1}BF_k^1||_H$ for any z>0.

$$z^{\alpha}||(z+B)^{-1}BF_k^1||_H \leq z^{\alpha}\left\|P_\tau\right\|_{H\to H}$$

$$\times \left[||BR^{N+2}||_{H\to H} + ||BR^{N+1}\right\|_{H\to H}$$

$$+|\beta|\left(\left\|BR^{[\frac{\lambda}{\tau}]+1}\right\|_{H\to H} + \left\|BR^{2N-[\frac{\lambda}{\tau}]}\right\|_{H\to H}\right)\Big]\left\|B(z+B)^{-1}B\varphi\right\|_H$$

$$+\left\|P_\tau\right\|_{H\to H}\left(\|RB||_{H\to H} + ||BR^{2N}\right\|_{H\to H})\left\|B(z+B)^{-1}B\psi\right\|_H$$

$$\leq M_3[\|B\varphi\|_H + \|B\psi\|_H].$$

Thus, we proved that

$$\left\|F_k^1\right\|_{E_\alpha(B,H)} \leq M_1[\|B\varphi\|_H + \|B\psi\|_H].$$

Now, let us estimate F_k^2. Using estimates (3.3), (3.7), we obtain that

$$\left\|F_k^2\right\|_H \leq \|P_\tau\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|R^2+R\right\|_{H\to H} + |\beta|\left(\left\|R^{N-[\frac{\lambda}{\tau}]+1}\right\|_{H\to H} + \left\|R^{N+[\frac{\lambda}{\tau}]+1}\right\|_{H\to H}\right)\right]$$

$$\times\sum_{i=1}^{N-1}\tau\left\|BR^{N-i}\right\|_{H\to H}\|f_i - f_{N-1}\|_H$$

$$\leq M_4\sum_{i=1}^{N-1}\tau\frac{((N-1)\tau - i\tau)^{\alpha}}{((N-1)-i)\tau}\|f^{\tau}\|_{C^{\alpha}([0,1]_\tau,H)}.$$

The sum enclosed in the right- hand side square brackets is the lower Darboux integral sum for the integral

$$\int_0^1 \frac{ds}{(1-s)^{1-\alpha}}.$$

It follows that

$$\left\|F_k^2\right\|_H \leq M_5\frac{1}{\alpha(1-\alpha)}\|f^{\tau}\|_{C^{\alpha}([0,1]_\tau,H)}.$$

Now let us estimate $z^{\alpha}||(z+B)^{-1}BF_k^2||_H$.

$$z^{\alpha}||(z+B)^{-1}BF_k^2||_H \leq z^{\alpha}\|P_\tau\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|B(R^2+R)\right\|_{H\to H} + |\beta|\left(\left\|BR^{N-[\frac{\lambda}{\tau}]+1}\right\|_{H\to H} + \left\|BR^{N+[\frac{\lambda}{\tau}]+1}\right\|_{H\to H}\right)\right]$$

$$\times \sum_{i=1}^{N-1} \tau \left\| B(z+B)^{-1} R^{N-i} \right\|_{H\to H} \|f_i - f_{N-1}\|_H$$

$$\leq M_6 \sum_{i=1}^{N-1} \tau \frac{((N-1)\tau - i\tau)^\alpha}{((N-1)-i)\tau} \|f^\tau\|_{C^\alpha([0,1]_\tau, H)} \leq M_7 \frac{1}{\alpha(1-\alpha)} \|f^\tau\|_{C^\alpha([0,1]_\tau, H)} .$$

So we proved that

$$\left\| F_k^2 \right\|_{E_\alpha(B,H)} \leq M_1 \frac{1}{\alpha(1-\alpha)} \|f^\tau\|_{C^\alpha([0,1]_\tau, H)} .$$

Now let us estimate F_k^3. Using estimates (3.3), (3.7), and the definition of the norm of space $C^\alpha([0,1]_\tau, H)$, we get

$$\left\| F_k^3 \right\|_H \leq \|P_\tau\|_{H\to H} \left\| (I+\tau B)(2I+\tau B)^{-1} \right\|_{H\to H} \left\| R + R^{2N} \right\|_{H\to H} |\beta|$$

$$\left[||R||_{H\to H} \sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1} \tau \left\| B R^{\left[\frac{\lambda}{\tau}\right]-i} \right\|_{H\to H} \left\| f_i - f_{\left[\frac{\lambda}{\tau}\right]} \right\|_H \right.$$

$$\left. + \sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1} \tau \left\| B R^{i-\left[\frac{\lambda}{\tau}\right]} \right\|_{H\to H} \left\| f_i - f_{\left[\frac{\lambda}{\tau}\right]} \right\|_H \right]$$

$$\leq M_8 \left[\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1} \tau \frac{\left(\left[\frac{\lambda}{\tau}\right] - i\right)^\alpha}{\left(\left[\frac{\lambda}{\tau}\right] - i\right)\tau} + \sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1} \tau \frac{(i - \left[\frac{\lambda}{\tau}\right])^\alpha}{(i - \left[\frac{\lambda}{\tau}\right])\tau} \right] \|f^\tau\|_{C^\alpha([0,1]_\tau, H)}$$

$$\leq M_1 \frac{1}{(1-\alpha)} \|f^\tau\|_{C^\alpha([0,1]_\tau, H)} .$$

Thus,

$$\left\| F_k^3 \right\|_H \leq M_1 \frac{1}{(1-\alpha)} \|f^\tau\|_{C^\alpha([0,1]_\tau, H)} .$$

Now let us estimate $z^\alpha ||(z+B)^{-1} B F_k^3||_H$.

$$z^\alpha ||(z+B)^{-1} B F_k^3||_H \leq z^\alpha \|P_\tau\|_{H\to H} \left\| (I+\tau B)(2I+\tau B)^{-1} \right\|_{H\to H}$$

$$\times \left\| B(R + R^{2N}) \right\|_{H\to H} |\beta|$$

$$\left[||R||_{H\to H} \sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1} \tau \left\| B(z+B)^{-1} R^{\left[\frac{\lambda}{\tau}\right]-i} \right\|_{H\to H} \left\| f_i - f_{\left[\frac{\lambda}{\tau}\right]} \right\|_H \right.$$

$$\left. + \sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1} \tau \left\| B(z+B)^{-1} R^{i-\left[\frac{\lambda}{\tau}\right]} \right\|_{H\to H} \left\| f_i - f_{\left[\frac{\lambda}{\tau}\right]} \right\|_H \right]$$

$$\leq M_9 \left[\sum_{i=1}^{\left[\frac{\lambda}{\tau}\right]-1} \tau \frac{\left(\left[\frac{\lambda}{\tau}\right]-i\right)^{\alpha}}{\left(\left[\frac{\lambda}{\tau}\right]-i\right)\tau} + \sum_{i=\left[\frac{\lambda}{\tau}\right]}^{N-1} \tau \frac{\left(i-\left[\frac{\lambda}{\tau}\right]\right)^{\alpha}}{\left(i-\left[\frac{\lambda}{\tau}\right]\right)\tau} \right] \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}$$

$$\leq M_1 \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}.$$

So we proved that

$$\left\|F_k^3\right\|_{E_{\alpha}(B,H)} \leq M_1 \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}.$$

Then, we estimate F_k^4. Using the estimates (3.3), (3.7) and the definition of the norm space $C^{\alpha}([0,1]_{\tau}, H)$, we obtain that

$$\left\|F_k^4\right\|_H \leq \|P_{\tau}\|_{H\to H} \left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H} \|(I+R)\|_{H\to H}$$

$$\times \left[\left\|R^2+R\right\| + |\beta| \left(\left\|R^{N-\left[\frac{\lambda}{\tau}\right]+1}\right\| + \left\|R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right\| \right) \right]$$

$$\times \sum_{i=1}^{N-1} \tau \left\|BR^{i-1}\right\|_{H\to H} \|f_i - f_1\|_H$$

$$\leq M_{10} \sum_{i=1}^{N-1} \frac{\tau(i-1)^{\alpha}}{\tau(i-1)} \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)} \leq M_{11} \frac{1}{(1-\alpha)} \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}.$$

Now, let us estimate $z^{\alpha}||(z+B)^{-1}BF_k^4||_H$.

$$z^{\alpha}||(z+B)^{-1}BF_k^4||_H \leq z^{\alpha} \|P_{\tau}\|_{H\to H} \left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H} \|(I+R)\|_{H\to H}$$

$$\times \left[\left\|B(R^2+R)\right\| + |\beta| \left(\left\|BR^{N-\left[\frac{\lambda}{\tau}\right]+1}\right\| + \left\|BR^{N+\left[\frac{\lambda}{\tau}\right]+1}\right\| \right) \right]$$

$$\times \sum_{i=1}^{N-1} \tau \left\|B(z+B)^{-1}R^{i-1}\right\|_{H\to H} \|f_i - f_1\|_H$$

$$\leq M_{12} \sum_{i=1}^{N-1} \frac{\tau(i-1)^{\alpha}}{\tau(i-1)} \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)} \leq M_{13} \frac{1}{(1-\alpha)} \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}.$$

So, we proved that

$$\left\|F_k^4\right\|_{E_{\alpha}(B,H)} \leq M_1 \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)}.$$

Let us estimate F_k^5. Estimates (3.3), (3.7) and the definition of the norm space $C^{\alpha}([0,1]_{\tau}, H)$ give

$$\left\|F_k^5\right\|_H \leq \|P_{\tau}\|_{H\to H} \left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$\times \left[\left\|R^2+R\right\|_{H\to H} + |\beta| \left(\left\|R^{N-\left[\frac{\lambda}{\tau}\right]+1}\right\|_{H\to H} + \left\|R^{N+\left[\frac{\lambda}{\tau}\right]+1}\right\|_{H\to H} \right) \right]$$

$$\times \sum_{i=1}^{N-1} \tau \left\|BR^{N+i}\right\|_{H\to H} \|f_i - f_1\|_H$$

$$+\left\|P_\tau\right\|_{H\to H}\left\|(2I+\tau B)^{-1}\right\|_{H\to H}\left\|R+R^{2N}\right\|_{H\to H}$$

$$\times|\beta|\sum_{i=1}^{N-1}\tau\left\|BR^{\left[\frac{\lambda}{\tau}\right]+i}\right\|_{H\to H}\left\|f_i-f_1\right\|_H$$

$$\leq M_{14}\sum_{i=1}^{N-1}\tau\left(\frac{i^\alpha}{(N+i)\tau}+\frac{i^\alpha}{(\left[\frac{\lambda}{\tau}\right]+i)\tau}\right)\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}$$

$$\leq M_{15}\frac{1}{(1-\alpha)}\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

Now let us estimate $z^\alpha||(z+B)^{-1}BF_k^5||_H$.

$$z^\alpha||(z+B)^{-1}BF_k^5||_H\leq z^\alpha\left\|P_\tau\right\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|B(R^2+R)\right\|_{H\to H}+|\beta|\left(\left\|BR^{N-\left[\frac{\lambda}{\tau}\right]+1}\right\|_{H\to H}+\left\|BR^{N+\left[\frac{\lambda}{\tau}\right]+1}\right\|_{H\to H}\right)\right]$$

$$\times\sum_{i=1}^{N-1}\tau\left\|B(z+B)^{-1}R^{N+i}\right\|_{H\to H}\left\|f_i-f_1\right\|_H$$

$$+\left\|P_\tau\right\|_{H\to H}\left\|(2I+\tau B)^{-1}\right\|_{H\to H}\left\|B(R+R^{2N})\right\|_{H\to H}$$

$$\times|\beta|\sum_{i=1}^{N-1}\tau\left\|B(z+B)^{-1}R^{\left[\frac{\lambda}{\tau}\right]+i}\right\|_{H\to H}\left\|f_i-f_1\right\|_H$$

$$\leq M_{16}\sum_{i=1}^{N-1}\tau\left(\frac{i^\alpha}{(N+i)\tau}+\frac{i^\alpha}{(\left[\frac{\lambda}{\tau}\right]+i)\tau}\right)\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}$$

$$\leq M_{17}\frac{1}{(1-\alpha)}\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

So, we proved that

$$\left\|F_k^5\right\|_{E_\alpha(B,H)}\leq M_{18}\frac{1}{(1-\alpha)}\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}\leq M_1\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

Now, let us estimate F_k^6. Using the estimates (3.3), (3.7), we obtain that

$$\left\|F_k^6\right\|_H\leq\left\|P_\tau\right\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|R^2+R\right\|_{H\to H}+|\beta|\left(\left\|R^{N-\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H}+\left\|R^{N+\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H}\right)\right]$$

$$\times\left\|I-R^{N-1}\right\|_{H\to H}\left\|f_{N-1}\right\|_H$$

$$\leq M_{19}\max_{0<k<N}\left\|f_k\right\|_H\leq M_1\left\|f^\tau\right\|_{C^\alpha([0,1]_\tau,H)}.$$

Now let us estimate $z^\alpha||(z+B)^{-1}BF_k^6||_H$.

$$z^\alpha||(z+B)^{-1}BF_k^6||_H\leq z^\alpha\left\|P_\tau\right\|_{H\to H}\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left\|(z+B)^{-1}\right\|_{H\to H}$$

$$\times\left[\left\|B(R^2-I)\right\|_{H\to H}+|\beta|\left(\left\|BR^{N-\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H}+\left\|BR^{N+\left[\frac{\lambda}{\tau}\right]}\right\|_{H\to H}\right)\right]$$

$$\times \left\| I - R^{N-1} \right\|_{H \to H} \| f_N \|_H$$

$$\leq M_{20} \max_{0<k<N} \| f_k \|_H \leq M_1 \| f^\tau \|_{C^\alpha([0,1]_\tau, H)} .$$

So we proved that

$$\left\| F_k^6 \right\|_{E_\alpha(B,H)} \leq M_1 \| f^\tau \|_{C^\alpha([0,1]_\tau, H)} .$$

In a similar manner, we can prove that

$$\left\| F_k^7 \right\|_H \leq M_1 \| f^\tau \|_{C^\alpha([0,1]_\tau, H)},$$

and

$$\left\| F_k^8 \right\|_H \leq M_1 \| f^\tau \|_{C^\alpha([0,1]_\tau, H)},$$

Combining the estimate for F_k^m, for $m = 1, \cdots, 8$ in $C^\alpha([0,1]_\tau, H)$, (3.31) is proved.

So, Theorem 3.3 is proved.

Note that the second order of accuracy difference scheme of approximate solutions of nonlocal boundary value problem (2.1) is

$$\begin{cases} -\frac{1}{\tau^2}\left[u_{k+1} - 2u_k + u_{k-1}\right] + Au_k = f_k, f_k = f(t_k), t_k = k\tau, 1 \leq k \leq N-1, \\ \frac{-3u_0 + 4u_1 - u_2}{2\tau} = \varphi, \\ \frac{u_{N-2} - 4u_{N-1} + 3u_N}{2\tau} = \beta \frac{u_{\left[\frac{\lambda}{\tau}\right]+1} - u_{\left[\frac{\lambda}{\tau}\right]-1}}{2\tau} + \psi \end{cases} \tag{3.34}$$

Theorems 3.1 - 3.3 can be formulated and established for the solution of (3.34).Really, we have that

Theorem 3.4. The solution of the difference scheme (3.34) satisfy the stability estimate

$$\| u^\tau \|_{C([0,1]_\tau, H)} \leq M \left[\| f^\tau \|_{C([0,1]_\tau, H)} + \| \varphi \|_H + \| \psi \|_H \right],$$

where M does not depend on f^τ, φ, ψ and τ.

Theorem 3.5. The solution of the difference problem (3.34) in $C([0,1]_\tau, H)$ obey the almost coercive inequality

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau, H)} + \| \{Au_k\}_1^{N-1} \|_{C([0,1]_\tau, H)}$$

$$\leq M \left[\min\left\{ \ln \frac{1}{\tau}, 1 + |\ln \| B \|_{H \to H}| \right\} \| f^\tau \|_{C([0,1]_\tau, H)} + \| B\varphi \|_H + \| B\psi \|_H \right].$$

Here M does not depend on τ, ψ, φ and $f_k, 1 \leq k \leq N-1$.

Theorem 3.6. The difference problem (3.34) is well posed in Hölder spaces $C^{\alpha}([0,1]_{\tau}, H)$ and the following coercivity inequality holds:

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C^{\alpha}([0,1]_{\tau},H)} + \| \{Au_k\}_1^{N-1} \|_{C^{\alpha}([0,1]_{\tau},H)}$$

$$\leq M \frac{1}{\alpha(1-\alpha)} \|f^{\tau}\|_{C^{\alpha}([0,1]_{\tau},H)} + M[||B\varphi||_{E_{\alpha}(B,H)} + ||B\psi||_{E_{\alpha}(B,H)}].$$

3.2 APPLICATIONS

Now, the applications of Theorem 3.1, Theorem 3.2 and Theorem 3.3 will be given. First, the nonlocal boundary value problem (2.16) for one dimensional elliptic equation is considered. The discretization of problem (2.16) is carried out in two steps. In the first step let us define the grid space

$$[0,1]_h = \{x : x_n = nh, 0 \leq n \leq M,\ Mh = 1\}.$$

We introduce the Hilbert space $L_{2h} = L_2([0,1]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_r\}_{r=1}^{M-1}$ defined on $[0,1]_h$, equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = (\sum_{x \in [0,1]_h} |\varphi^h(x)|^2 h)^{\frac{1}{2}}.$$

To the differential operator A generated by the problem (2.16) we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,n} + \delta \varphi_n\}_1^{M-1}, \tag{3.35}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_r\}_1^{M-1}$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is know that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x we arrive at the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in [0,1]_h, \\ u_t^h(0,x) = 0;\ u_t^h(1,x) = \beta u_t^h(\lambda,x),\ x \in [0,1]_h. \end{cases} \tag{3.36}$$

In the second step, (3.36) is replaced, by the difference scheme (3.1)

$$\begin{cases} -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h(x)=f_k^h(x), \\ f_k^h(x)=f^h(t_k,x),\ t_k=k\tau, 1\le k\le N-1,\ N\tau=1, x\in[0,1]_h, \\ \frac{u_1^h(x)-u_0^h(x)}{\tau}=0, x\in[0,1]_h, \\ \frac{u_N^h(x)-u_{N-1}^h(x)}{\tau}=\beta\frac{u_{\left[\frac{\lambda}{\tau}\right]+1}^h(x)-u_{\left[\frac{\lambda}{\tau}\right]}^h(x)}{\tau}, x\in[0,1]_h. \end{cases} \tag{3.37}$$

Theorem 3.7. Let τ and h be a sufficiently small numbers. Then the solutions of difference scheme (3.37) satisfy the following stability and almost coercivity estimates:

$$\|u^\tau\|_{C([0,1]_\tau,L_{2h})}\le M_1\|f^\tau\|_{C([0,1]_\tau,L_{2h})}, \tag{3.38}$$

$$\max_{1\le k\le N-1}\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\|_{L_{2H}}+\max_{1\le k\le N-1}\|\left(u_k^h\right)\|_{W_{2h}^2} \tag{3.39}$$

$$\le M_2\left[\ln\frac{1}{\tau+h}\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}\right].$$

Here, M_1 and M_2 are independent on τ, h and $f_k^h(x), 1\le k\le N-1$.

The proof of Theorem 3.7 is based on the abstract Theorems 3.1 - 3.2, on the estimate

$$\min\left\{\ln\frac{1}{\tau},1+|\ln\|\ B_h^x\|_{H\to H}|\right\}\ \le M\ln\ \frac{1}{\tau+h}, \tag{3.40}$$

as well as the symmetry properties of the difference operator A_h^x defined by the formula (3.35) in L_{2h}.

Theorem 3.8. Let τ and h be a sufficiently small numbers. Then the solutions of difference scheme (3.37) satisfy the following coercivity stability estimate:

$$\|\left\{\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\}_1^{N-1}\|_{C^\alpha([0,1]_\tau,L_{2h})}+\|\left\{u_k^h\right\}_1^{N-1}\|_{C^\alpha([0,1]_\tau,W_{2h}^2)} \tag{3.41}$$

$$\le M\frac{1}{\alpha(1-\alpha)}\|f_k^h\|_{C^\alpha([0,1]_\tau,L_{2h})}.$$

M does not depend on τ, $f_k^h(x), 1\le k\le N-1$.

The proof of Theorem 3.8 is based on the abstract Theorem 3.3 and the symmetry properties of the difference operator A_h^x defined by the formula (3.35).

Second, the Neumann-Bitsadze-Samarskii type problem for the multidimensional elliptic equation (2.17) is considered. The discretization of problem (2.17) is carried out in two steps. In the first step, let us define the grid sets

$$\overline{\Omega}_h = \{x = x_m = (h_1 m_1, \cdots, h_n m_n), m = (m_1, \cdots, m_n), 0 \leq m_r \leq N_r$$

$$h_r N_r = 1, r = 1, \cdots, n\}, \Omega_h = \overline{\Omega}_h \cap \Omega, S_h = \overline{\Omega}_h \cap S.$$

We introduce the Hilbert space $L_{2h} = L_2(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\overline{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}} = (\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n)^{1/2}.$$

To the differential operator A generated by the problem (2.17), we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{r=1}^{n} (a_r(x) u_{\bar{x}_r}^h)_{x_r, j_r} + \delta u_{x_r}^h \tag{3.42}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we arrive at the nonlocal boundary value problem for an infinite system of ordinary differential equations

$$\begin{cases} -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in \Omega_h, \\ \\ u_t^h(0,x) = 0;\ u_t^h(1,x) = \beta u_t^h(\lambda, x),\ x \in \overline{\Omega}_h. \end{cases} \tag{3.43}$$

In the second step, (3.43) is replaced by the difference scheme (3.1), we get second order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = \varphi_k^h, \\ \\ \varphi_k^h = f^h(t_k, x), t_k = k\tau, 1 \leq k \leq N-1,\ N\tau = 1, x \in \Omega_h, \\ \\ \frac{u_1^h(x) - u_0^h(x)}{\tau} = 0;\ \frac{u_{N-1}^h(x) - u_N^h(x)}{\tau} = \beta \frac{u_{\left[\frac{\lambda}{\tau}\right]+1}^h(x) - u_{\left[\frac{\lambda}{\tau}\right]}^h(x)}{\tau}, x \in \overline{\Omega}_h. \end{cases} \tag{3.44}$$

Theorem 3.9. Let τ and $|h|=\sqrt{h_1^2+...+h_n^2}$ be a sufficiently small numbers. Then, the solutions of difference scheme (3.44) satisfy the following stability and almost coercive stability estimates

$$\|u^\tau\|_{C([0,1]_\tau,L_{2h})} \leq M_1 \left[\|f^\tau\|_{C([0,1]_\tau,L_{2h})} + \|\varphi\|_{L_{2h}} + \|\psi\|_{L_{2h}}\right], \tag{3.45}$$

$$\max_{1\leq k\leq N-1} \|\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\|_{L_{2h}} + \max_{1\leq k\leq N-1} \sum_{r=1}^{n} \|\left(u_k^h\right)_{x_r\overline{x}_r,\, j_r}\|_{L_{2h}} \tag{3.46}$$

$$\leq M_2 \ln\tfrac{1}{\tau+|h|} \max_{1\leq k\leq N-1} \|f_k^h\|_{L_{2h}}.$$

Here, M_1 and M_2 are independent on τ, h and $f_k^h(x), 1\leq k\leq N-1$.

The proof of Theorem 3.9 is based on the abstract Theorems 3.1 - 3.2, on the estimate

$$\min\left\{\ln\frac{1}{\tau}, 1+|\ln\| B_h^x\|_{H\to H}|\right\} \leq M\ln\frac{1}{\tau+|h|}, \tag{3.47}$$

as well as the symmetry properties of the difference operator A_h^x defined by the formula (3.42) in L_{2h}, along with the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 3.10. For the solutions of the elliptic difference problem

$$A_h^x u^h(x) = w^h(x), x\in\Omega_h, \tag{3.48}$$

$$u^h(x)=0, x\in S_h$$

the following coercivity inequality holds (see [Sobolevskii P.E,1975]):

$$\sum_{r=1}^{n} \|(u^h)_{x_r\overline{x}_r,\, j_r}\|_{L_{2h}} \leq M\|w^h\|_{L_{2h}}.$$

Theorem 3.11. Let τ and $|h|$ be a sufficiently small numbers. Then the solutions of difference scheme (3.44) satisfy the following coercivity stability estimate:

$$\|\left\{\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\right\}_1^{N-1}\|_{C^\alpha([0,1]_\tau,L_{2h})} + \|\left\{u_k^h\right\}_1^{N-1}\|_{C^\alpha([0,1]_\tau,W_{2h}^2)}$$

$$\leq M\frac{1}{\alpha(1-\alpha)}\|\left\{f_k^h\right\}_1^{N-1}\|_{C^\alpha([0,1]_\tau,L_{2h})}.$$

M does not depend on τ, $f_k^h(x), 1\leq k\leq N-1$.

The proof of Theorem 3.11 is based on the abstract Theorem 3.3 and the symmetry properties of the difference operator A_h^x defined by the formula (3.42) and on the Theorem 3.10 on the coercivity inequality for the solution of the elliptic difference equation (3.48) in L_{2h}.

CHAPTER 4

NUMERICAL RESULTS

We consider the nonlocal boundary Bitsadze -Samarskii problem for elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u = \exp(-\pi t)\cos(\pi x), \\ \\ 0 < t < 1,\ 0 < x < 1, \\ \\ u_t(0,x) = -\pi\cos(\pi x), \\ \\ u_t(1,x) = u_t(\frac{1}{2},x) + \pi\cos(\pi x)(\exp(-\frac{\pi}{2}) - \exp(-\pi\ )), 0 \le x \le 1, \\ \\ u_x(t,0) = u_x(t,1) = 0,\ 0 \le t \le 1. \end{cases} \tag{4.1}$$

The exact solution of this problem is

$$u\,(t,x) = \exp(-\pi t)\cos(\pi x).$$

In the present chapter for the approximate solutions of the nonlocal boundary Bitsadze-Samarskii problem (4.1), we will use the first and second orders of accuracy difference schemes. We have the second order difference equations with respect to n with matrix coefficients. To solve these difference equations, we have applied a procedure of modified Gauss elimination method for difference equations with respect to n with matrix coefficients. The results of numerical experiments permit us to show that the second order of accuracy difference scheme is more accurate compared with the first order of accuracy difference scheme.

4.1 THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

For the approximate solution of the nonlocal boundary Bitsadze -Samarskii problem (4.1), we consider the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$\begin{aligned} [0,1]_\tau \times [0,\pi]_h &= \{(t_k, x_n) : t_k = k\tau, \quad 1 \le k \le N-1, \quad N\tau = 1, \\ x_n &= nh, \quad 1 \le n \le M-1, Mh = \pi\}. \end{aligned}$$

Applying the formulas

$$u'(0) = \frac{u(h) - u(0)}{h} + O(h),$$

$$u'(1) = \frac{u(1) - u(1-h)}{h} + O(h)$$

and

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O(h^2), \tag{4.2}$$

and the difference scheme (3.1), we present the following first order of accuracy difference scheme for the approximate solutions of the problem (4.1).

$$\begin{cases} -\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + u_n^k = \exp(-\pi t_k)\cos(\pi x_n), \\ 1 \le k \le N-1, 1 \le n \le M-1, \\ \frac{u_n^1 - u_n^0}{\tau} = -\pi\cos(\pi x_n), \ \ 0 \le n \le M, \\ \frac{u_n^N - u_n^{N-1}}{\tau} = \frac{u_n^{\frac{N}{2}} - u_n^{\frac{N}{2}-1}}{\tau}, \\ +\pi\cos(\pi x_n)\left(\exp(-\frac{\pi}{2}) - \exp(-\pi)\right), 0 \le n \le M, \\ u_1^k - u_0^k = u_M^k - u_{M-1}^k = 0, \ \ 0 \le k \le N, \end{cases} \tag{4.3}$$

We have $(N+1) \times (M+1)$ system of linear equations in (4.3) and we will write them in the matrix form. We can rewrite this system as the following form

$$\begin{cases} \left(\frac{1}{h^2}\right) u_{n+1}^k + \left(-\frac{2}{\tau^2} - \frac{2}{h^2} - 1\right) u_n^k + \left(\frac{1}{h^2}\right) u_{n-1}^k + \\ + \left(\frac{1}{\tau^2}\right) u_n^{k-1} + \left(\frac{1}{\tau^2}\right) u_n^{k+1} = \varphi_n^k, \\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ u_n^1 - u_n^0 = -\tau\pi\cos(\pi x_n),\ 0 \le n \le M, \\ u_n^N - u_n^{N-1} = u_n^{\frac{N}{2}} - u_n^{\frac{N}{2}-1} + \tau\pi\cos(\pi x_n)\left(\exp(-\frac{\pi}{2}) - \exp(-\pi)\right), 0 \le n \le M, \\ u_1^k - u_0^k = u_M^k - u_{M-1}^k = 0,\ 0 \le k \le N, \\ \varphi_n^k = -\exp(-\pi t_k)\cos(\pi x_n). \end{cases} \tag{4.4}$$

We denote

$$a = \frac{1}{h^2}, \quad b = -\frac{2}{\tau^2} - \frac{2}{h^2} - 1, \quad c = \frac{1}{\tau^2},$$

$$\varphi_n^k = \begin{cases} -\tau\pi\cos(\pi x_n), \quad k = 0, \\ -\exp(-\pi t_k)\cos(\pi x_n),\ 1 \le k \le N-1, \\ \tau\pi\cos(\pi x_n)\left(\exp(-\frac{\pi}{2}) - \exp(-\pi)\right),\ k = N. \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ c & b & c & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & c & b & c & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & c & 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & c & b & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & ... & 0 & c & b & c \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 1 & -1 & ... & 0 & 0 & -1 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

and

$$C = A,$$

$$D = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ \dots \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s = \ n-1, \ n, n+1.$$

Then, (4.4) can be written as

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ \\ U_1 - U_0 = \tilde{0}, U_M - U_{M-1} = \tilde{0}. \end{cases} \tag{4.5}$$

So, we have the second order difference equation with respect to n with matrix coefficients. To solve this difference equation we have applied a procedure of modified Gauss elimination method for difference equation with respect to n with matrix coefficients. Hence, we seek a solution of the matrix equation in the following form

$$U_n = \alpha_{n+1} U_{n+1} + \beta_{n+1}, \quad n = M-1, \cdot\cdot\cdot, 2, 1 \tag{4.6}$$

where $\alpha_n \ (n = 1, \cdot\cdot\cdot, M)$ are $(N+1)\times(N+1)$ square matrices and $\beta_n \ (n = 1, \cdot\cdot\cdot, M)$ are $(N+1)\times 1$ column matrices.

Here

$$\alpha_{n+1} = -\left(B + C\alpha_n\right)^{-1} A, \tag{4.7}$$

$$\beta_{n+1} = \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), \ n = 1, 2, 3, \cdot\cdot\cdot, M-1. \tag{4.8}$$

For the solution of difference equations, we need to find α_1 and β_1. We can find them from $U_0 = \alpha_1 U_1 + \beta_1$. Thus, we have

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{4.9}$$

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}_{(N+1)\times 1}.$$

For the first step, using formulas (4.7) and (4.8), we can compute α_{n+1} and β_{n+1}, $1 \leq n \leq M-1$. Thus, using formulas (4.6) and $U_M = (1-\alpha_M)^{-1}\beta_M$, we can compute U_n, $1 \leq n \leq M-1$. We can summarize the computation procedure by the following algorithm: $U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}$, $n = M-1, \cdots, 2, 1, 0$.

4.2 THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME

Applying the formulas

$$u'(0) = \frac{-u(2\tau) + 4u(\tau) - 3u(0)}{2\tau} + O(\tau^2),$$

$$u'(1) = \frac{-u(1-2\tau) + 4u(1-\tau) - 3u(1)}{2\tau} + O(\tau^2),$$

we present the following second order of accuracy difference scheme with respect t for the approximate solutions of the problem (4.1)

$$\begin{cases} -\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + u_n^k = \exp(-\pi t_k)\cos(\pi x_n), \\ 1 \leq k \leq N-1, 1 \leq n \leq M-1, \\ \frac{-3u_n^0+4u_n^1-u_n^2}{2\tau} = -\pi\cos(\pi x_n), 0 \leq n \leq M, \\ \frac{u_n^{N-2}-4u_n^{N-1}+3u_n^N}{2\tau} = \frac{u_n^{\frac{N}{2}-2}-4u_n^{\frac{N}{2}-1}+3u_n^{\frac{N}{2}}}{2\tau} \\ -\pi\cos(\pi x_n)\left(\exp\left(-\frac{\pi}{2}\right) - \exp(-\pi)\right), 0 \leq n \leq M, \\ \frac{u_1^k-u_0^k}{h} = \frac{u_M^k-u_{M-1}^k}{h} = 0, 0 \leq k \leq N. \end{cases} \tag{4.10}$$

We have $(N+1)\times(M+1)$ system of linear equations in (4.10) and we will write them in the matrix form. We can rewrite this system as the following form

$$\begin{cases} \left(\frac{1}{h^2}\right) u_{n+1}^k + \left(-\frac{2}{\tau^2} - \frac{2}{h^2} - 1\right) u_n^k + \left(\frac{1}{\tau^2}\right) u_n^{k-1} \\ + \left(\frac{1}{\tau^2}\right) u_n^{k+1} + \left(\frac{1}{h^2}\right) u_{n-1}^k = -\exp(-\pi t_k)\cos(\pi x_n), \\ 1 \le k \le N-1,\ 1 \le n \le M-1, \\ \frac{-3u_n^0 + 4u_n^1 - u_n^2}{2\tau} = -\pi\cos(\pi x_n), 0 \le n \le M, \\ \frac{u_n^{N-2} - 4u_n^{N-1} + 3u_n^N}{2\tau} = \frac{u_n^{\frac{N}{2}-2} - 4u_n^{\frac{N}{2}-1} + 3u_n^{\frac{N}{2}}}{2\tau} \\ -\pi\cos(\pi x_n)\left(\exp\left(-\frac{\pi}{2}\right) - \exp(-\pi)\right), 0 \le n \le M, \\ \frac{u_1^k - u_0^k}{h} = \frac{u_M^k - u_{M-1}^k}{h} = 0, 0 \le k \le N. \end{cases} \tag{4.11}$$

We denote

$$a = \frac{1}{h^2},\ b = \frac{1}{\tau^2},\ c = -\frac{2}{\tau^2} - \frac{2}{h^2} - 1,$$

$$\varphi_n^k = \begin{cases} -2\tau\pi\cos(\pi x_n), \quad k = 0, \\ -\exp(-\pi t_k)\cos(\pi x_n),\ 1 \le k \le N-1, \\ -2\tau\pi\cos(\pi x_n)\left(\exp\left(-\frac{\pi}{2}\right) - \exp(-\pi)\right),\ k = N. \end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & a & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} -3 & 4 & -1 & 0 & ... & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ b & c & b & 0 & ... & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & b & c & b & ... & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & ... & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & ... & c & b & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & ... & b & c & b & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & ... & 0 & b & c & b \\ 0 & 0 & 0 & 0 & ... & -1 & 4 & -3 & ... & 0 & 1 & -4 & 3 \end{bmatrix}_{(N+1)\times(N+1)},$$

and $C = A$.

$$D = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s = \begin{bmatrix} U_s^0 \\ U_s^1 \\ \ldots \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s = \ n-1, \ n, n+1.$$

Then, (4.11) can be written as

$$\begin{cases} A\,U_{n+1} + B\,U_n + C\,U_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ \\ U_1 - U_0 = \tilde{0}, U_M - U_{M-1} = \tilde{0}, \end{cases} \tag{4.12}$$

So, we have the second order difference equation with respest to n with matrix coefficients.To solve this difference equation, we use the same modified Gauss elimination method.

Now we will give the results of the numerical analysis. In order to get the solution of (4.3) and (4.10) we used MATLAB program. The errors computed by

$$E_M^N = \max_{1\le k\le N-1} \left(\sum_{n=1}^{M-1} \left| u(t_k, x_n) - u_n^k \right|^2 h \right)^{\frac{1}{2}}$$

of the numerical solutions for different values of M and N, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). The result are shown in Tables 4.1, 4.2 for $N = 20$, $M = 80$ and $N = 40$, $M = 80$ respectively.

Table 1. Comparison of the errors different difference schemes for $N = 20$, $M = 80$.

Difference schemes	E_M^N
The first order of accuracy difference scheme (4.3)	0.0366
The second order of accuracy difference scheme (4.10)	0.0189

Table 2. Comparison of the errors different difference schemes for $N = 40$, $M = 80$.

Difference schemes	E_M^N
The first order of accuracy difference scheme (4.3)	0.0159
The second order of accuracy difference scheme (4.10)	0.0047

Thus, second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

The first figure is the exact solution, the second figure is the solution of the first order of accuracy difference scheme, the third figure is the solution of second order of accuracy difference scheme.

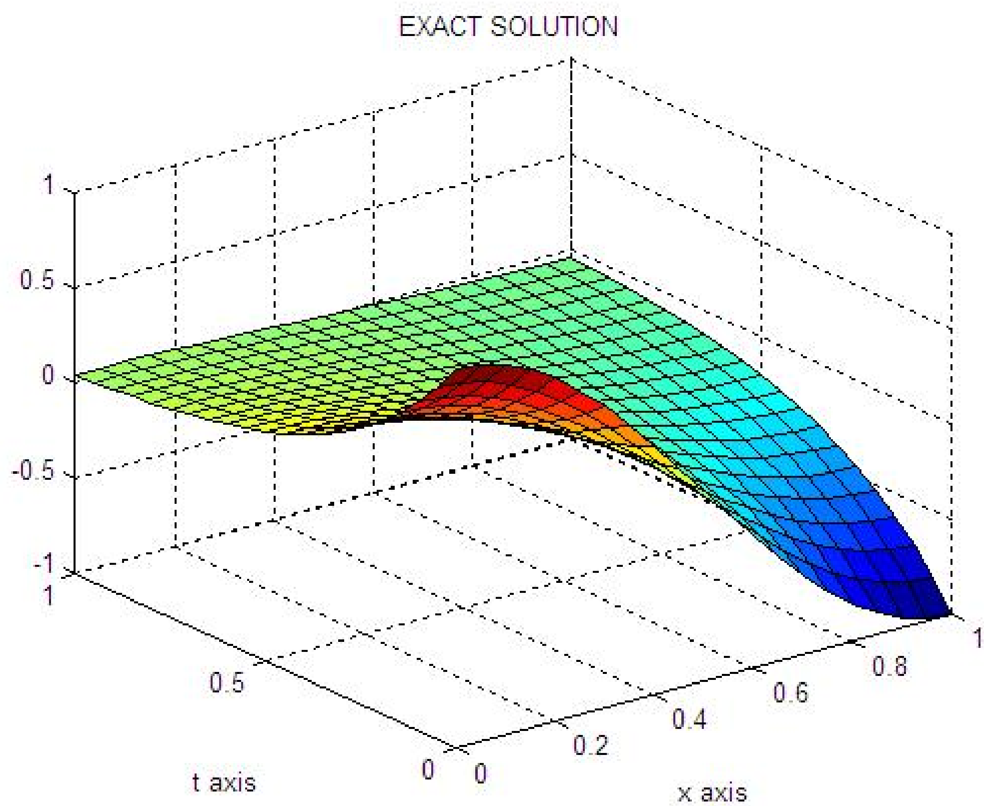

Figure 1. The exact solution

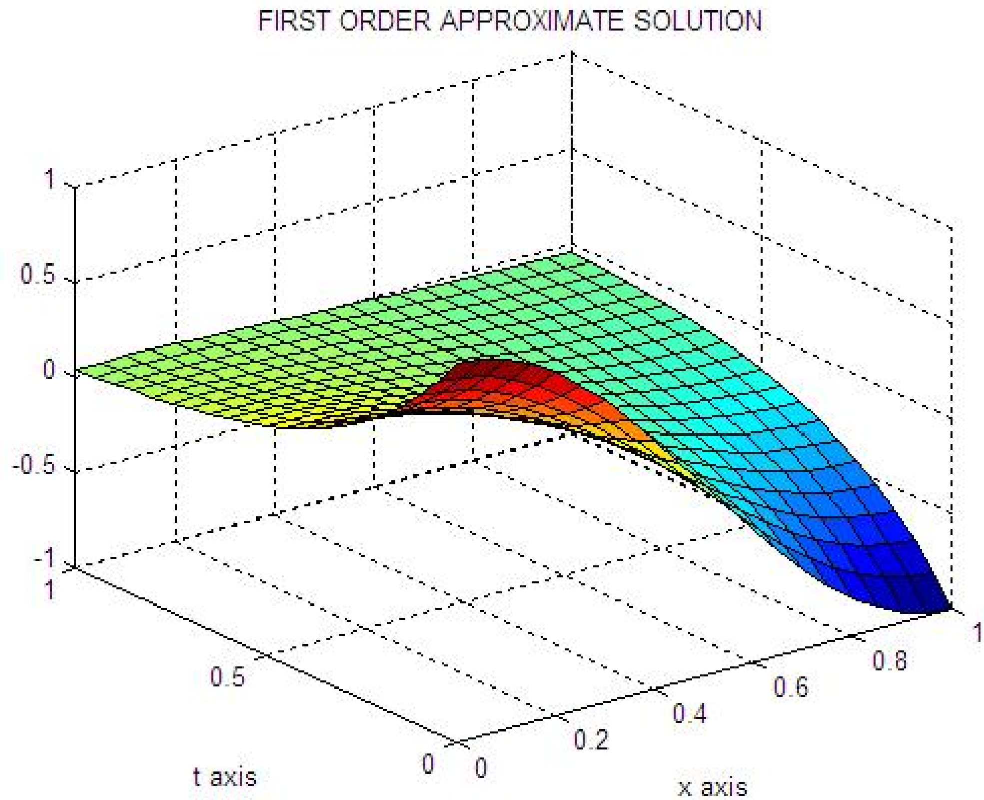

Figure 2. The first order of accuracy difference scheme

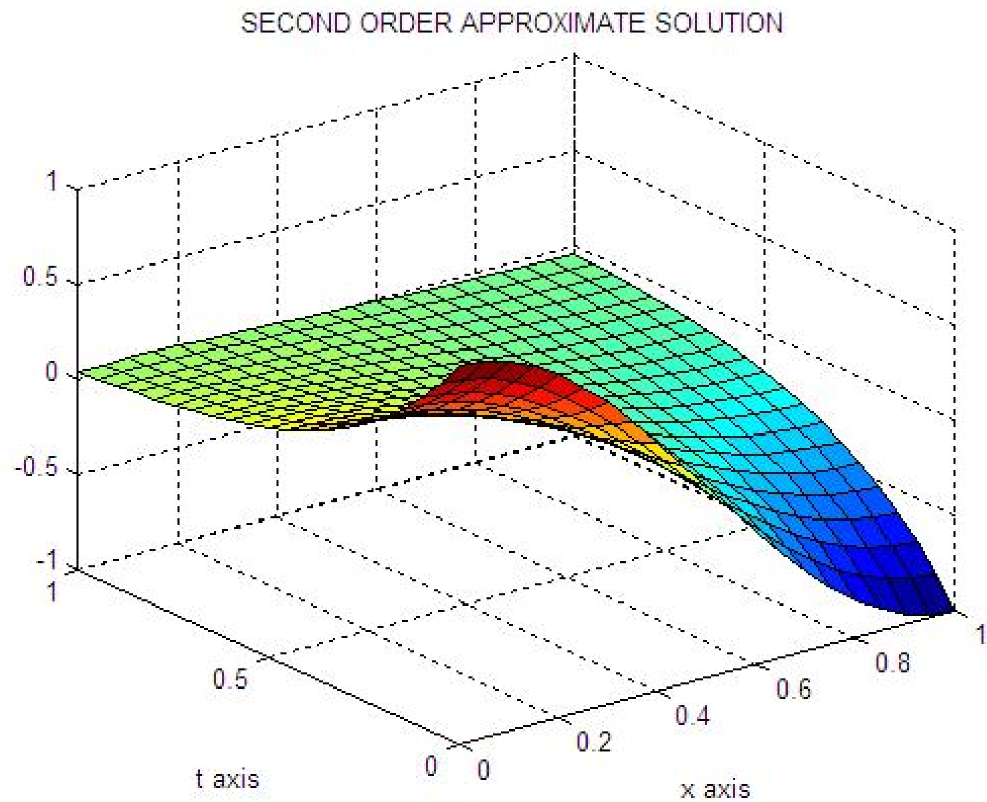

Figure 3. The second order of accuracy difference scheme

CHAPTER 5

CONCLUSIONS

This work is devoted to study the well-posedness of Neumann Bitsadze Samarskii nonlocal boundary value problem for elliptic differential and difference equations. The following original results are obtained:

- The abstract theorem on the coercive stability estimate for the solution of Neumann Bitsadze Samarskii nonlocal boundary value differential problem for the elliptic equation in the Holder space is proved.

- The theorems on coercive stability estimates for the solution of Neumann Bitsadze Samarskii nonlocal boundary for the elliptic equations are presented.

- The first and second orders of accuracy difference schemes for approximate solutions of the Neumann Bitsadze Samarskii nonlocal boundary value problems for elliptic equations are presented.

- Theorems on the stability estimates, almost coercive stability estimates and coercive stability estimates for the solution of difference scheme for elliptic equations are proved.

- The Matlab implementation of the first and second orders difference schemes are presented.

- The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

CHAPTER 6
MATLAB PROGRAMMING

In this chapter, Matlab programs are presented for the first and second orders of accuracy difference schemes for different value M and N.

8.1 MATLAB IMPLEMENTATION OF THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

```
function firstord(N,M)
    if nargin<1; close;        close; end;
    N=20; M=80;
    h=1/M; tau=1/N;
    a=1/(h^2);a
    b=(-2/(h^2))+(-2/(tau^2)-1);b
    c=1/(tau^2);c
    for k=2:N;
    A(k,k)=a; A(1,1)=0;A(N+1,N+1)=0; A;A
    end;
    for k=2:N;
    B(k,k)=b; B(k,k-1)=c; B(k,k+1)=c;
    B(N+1,N/2)=1; B(N+1,N+1)=1; B(1,1)=-1;
    B(N+1,(N/2)+1)=-1; B(N+1,N+1)=1; B(1,1)=-1;B(N+1,(N/2))=1; B(N+1,N)=-1; B(1,2)=1;B;B
    end;
    C=A;C
    for i=1:N+1; D(i,i)=1; end; D;D
    for n=1:M-1 ;
    for k=1:N+1 ;
```

```
t=(k-1)*tau;
x=(n)*h;
fii(k,n:n)=-(exp(-(pi)*t))*(cos((pi)*x));
end;
end;fii
for n=1:M-1;
x=(n)*h;
fii(1,n:n)=-(pi)*(tau)*(cos(pi*x));
fii(N+1,n:n)=(pi)*(tau)*(cos((pi)*x))*(-exp(-(pi))+exp((-(pi)/2)));
end;
fii;fii
%size(A);
%size(B);
%size(C);
I=eye(N+1,N+1);
alpha{1}=eye(N+1,N+1);
betha{1}=zeros(N+1,1);
for n=1:M-1;
alpha{n+1}=inv(B+C*alpha{n})*(-A);
betha{n+1}=inv(B+C*alpha{n})*(D*fii(:,n:n)-C*betha{n});
end;alpha{M}
%O=inv(I-alpha{M});O
U{M}=inv(I-alpha{M})*betha{M};
for Z=M-1:-1:1;
U{Z}=alpha{Z+1}*U{Z+1}+betha{Z+1};
end;U{Z}
for Z=1:M;
```

```
p(:,Z+1)=U{Z};

end;

p(:, 1)=zeros(N+1, 1);

%%%%%%%%%'EXACT SOLUTION OF THIS PDE' %%%%%%%%%%%

for n=1:M+1;

for k=1:N+1;

t=(k-1)*tau;

x=(n-1)*h;

es(k,n:n)=(exp(-(pi)*t))*(cos((pi)*x));

end;

end;

%%%%%%%%%% END EXACT SOLUTION %%%%%%%%%%%%%%%%%%%%%%

%%%%%%% ERROR ANALYSIS OF GENERAL SOL OF THE DIFF SCHEME
%%%%%%%

for i=1:N-1;

for j=1:M-1;

ftf(i, j)=p(i+1, j+1)-es(i+1, j+1);

end;

end;

fmat1=abs(ftf);

fmat2=fmat1.*fmat1*h;

fmat3=sum(fmat2, 2);

fmat4=fmat3.^(1/2);

sumerror=max(fmat4)

%%%%%%%%%%%%%%%%%%%%% END OF ERROR ANALYSIS %%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%

p;
```

```
es;
[xler,tler]=meshgrid(0:h:1, 0:tau:1);
table=[es;p]; table(1:2:end,:)=es; table(2:2:end,:)=p;
q=min(min(table));
w=max(max(table));
figure;
surf(xler,tler,es);
title('EXACT SOLUTION'); set(gca,'ZLim',[q w]);
rotate3d;
XLabel('x axis'); YLabel('t axis');
figure; surf(xler,tler,p);
title('First Order DS'); set(gca,'ZLim',[q w]);
rotate3d ;
XLabel('x axis'); YLabel('t axis');
%%%%%%%%%% END GRAPH %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

8.2 MATLAB IMPLEMENTATION OF THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME .

```
function secondord(N,M)
    if nargin<1; close;        close; end;
    N=40; M=80;
    h=1/M; tau=1/N;
    a=1/(h^2);a
    c=(-2/(h^2))+(-2/(tau^2)-1);c
    b=1/(tau^2);b
    for k=2:N;
    A(k,k)=a; A(1,1)=0;A(N+1,N+1)=0; A;A
```

```
end;
for k=2:N;
B(k,k)=c; B(k,k-1)=b; B(k,k+1)=b;
B(N+1,N/2)=4; B(N+1,N+1)=3; B(1,1)=-3;
B(N+1,(N/2)+1)=-3; B(N+1,N+1)=1;B(N+1,(N/2))=4; B(N+1,N)=-4; B(1,2)=4;
B(1,3)=-1; B(N+1,(N/2)-1)=-1;B(N+1,N-1)=3;B;B
end;
C=A;C
for i=1:N+1; D(i,i)=1; end; D;D
for n=1:M-1 ;
for k=1:N+1 ;
t=(k-1)*tau;
x=(n)*h;
fii(k,n:n)=-(exp(-(pi)*t))*(cos((pi)*x));
end;
end;fii
for n=1:M-1;
x=(n)*h;
fii(1,n:n)=-2*(pi)*(tau)*(cos(pi*x));
fii(N+1,n:n)=-2*(pi)*(tau)*(cos((pi)*x))*(-exp(-(pi))+exp((-(pi)/2)));
end;
fii;fii
%size(A);
%size(B);
%size(C);
I=eye(N+1,N+1);
alpha{1}=eye(N+1,N+1);
```

```
betha{1}=zeros(N+1,1);
for n=1:M-1;
alpha{n+1}=inv(B+C*alpha{n})*(-A);
betha{n+1}=inv(B+C*alpha{n})*(D*fii(:,n:n)-C*betha{n});
end;alpha{M}
%O=inv(I-alpha{M});O
U{M}=inv(I-alpha{M})*betha{M};
for Z=M-1:-1:1;
U{Z}=alpha{Z+1}*U{Z+1}+betha{Z+1};
end;U{Z}
for Z=1:M;
p(:,Z+1)=U{Z};
end;
p(:, 1)=zeros(N+1, 1);
%%%%%%%%%'EXACT SOLUTION OF THIS PDE' %%%%%%%%%%%
for n=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(n-1)*h;
es(k,n:n)=(exp(-(pi)*t))*(cos((pi)*x));
end;
end;
%%%%%%%%% END EXACT SOLUTION %%%%%%%%%%%%%%%%%%%%%%
%%%%%%% ERROR ANALYSIS OF GENERAL SOL OF THE DIFF SCHEME %%%%%%%
for i=1:N-1;
for j=1:M-1;
```

```
ftf(i, j)=p(i+1, j+1)-es(i+1, j+1);
end;
end;
fmat1=abs(ftf);
fmat2=fmat1.*fmat1*h;
fmat3=sum(fmat2, 2);
fmat4=fmat3.^(1/2);
sumerror=max(fmat4)
%%%%%%%%%%%%%%%%%% END OF ERROR ANALYSIS %%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%
p;
es;
[xler,tler]=meshgrid(0:h:1, 0:tau:1);
table=[es;p]; table(1:2:end,:)=es; table(2:2:end,:)=p;
q=min(min(table));
w=max(max(table));
figure;
surf(xler,tler,es);
title('EXACT SOLUTION'); set(gca,'ZLim',[q w]);
rotate3d;
XLabel('x axis'); YLabel('t axis');
figure; surf(xler,tler,p);
title('Second Order DS'); set(gca,'ZLim',[q w]);
rotate3d ;
XLabel('x axis'); YLabel('t axis');
%%%%%%%%%%% END GRAPH %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

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