POSITIVE OPERATOR'S METHOD FOR DIFFERENCE SCHEMES

by

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MAY 2009

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A thesis submitted to

the Graduate Institute of Sciences and Engineering

of

Fatih University

in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

May 2009
Istanbul,Turkey

APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ACKNOWLEDGMENT

I am glad to take this opportunity to thank firstly my supervisor Prof. Dr. Allaberen ASHYRALYEV for his genuine help and very special encouragement throughout this research.

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M. S. Thesis - Mathematics
May 2009

Supervisor: Prof. Dr. Allaberen ASHYRALYEV

ABSTRACT

In the present work a fourth order differential operator A^x defined by the formula

$$A^x u = \frac{d^4 u}{dx^4} + \delta u,$$

with domain

$$D(A^x) = \left\{ u \in C^4[0,1] : u(0) = u(1) = 0,\ u''(0) = u''(1) = 0 \right\}$$

and a fourth order difference operator A_h^x defined by the formula

$$\begin{cases} A_h^x u^h = \left\{ \frac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^2} + \delta u_k \right\}_2^{N-2}, \\ u_0 = u_N = 0, \quad -u_2 + 2u_1 - u_0 = u_{N-2} - 2u_{N-1} + u_N = 0 \end{cases}$$

are studied. Here $\delta > 0$.The positivity of differential operator A^x in $C[0,1]$ and of the difference operator A_h^x in C_h are established.

In applications the stability, the almost coercive stability and the coercive stability estimates for the solution of difference schemes in difference analogues of Holder spaces are obtained.

Keywords: Positivity, stability estimate, almost coercive stability estimate, coercive stability estimate.

FARK ŞEMALARI İÇİN POSİTİF OPERATÖRÜ METODU

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Yüksek Lisans Tezi - Matematik
Mayıs 2009

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ÖZET

Bu çalışmada,

$$A^x u = \frac{d^4u}{dx^4} + \delta u$$

formülüyle verilen ve tanım kümesi

$$D(A^x) = \left\{ u \in C^4[0,1] : u(0) = u(1) = 0,\ u''(0) = u''(1) = 0 \right\}$$

olan dördüncü derece A^x differensiyal operatörü ile

$$\begin{cases} A_h^x u^h = \left\{ \frac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^2} + \delta u_k \right\}_2^{N-2}, \\ \\ u_0 = u_N = 0, \ \ -u_2 + 2u_1 - u_0 = u_{N-2} - 2u_{N-1} + u_N = 0 \end{cases}$$

verilen dördüncü derece A_h^x fark operatörü araştırılmıştır. A^x differensiyal operatörünün $C[0,1]$ uzayındaki pozitifliği ve A_h^x fark operatörünün C_h uzayındaki pozitifliği incelenmiştir.

Uygulamalarda, fark şemalarının Holder uzayındaki fark analoguyla çözümü icin kararlılık kestirimleri, hemen hemen koersif kestirimleri ve koersif kestirimleri elde edilmiştir.

Anahtar Kelimeler: Pozitiflik, kararlılık kestirimi, hemen hemen koersif kestirimi, koersif kestirimi.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

It is a well -known that various local and non-local boundary value problems for PDE can be reduced to the abstract boundary value problem for ordinary differential equation in a Banach space E with an unbounded differential operator A. The study of various properties of partial differential equation is based on a positivity property of this differential operator in a Banach space. The positivity of wider class of differential operators have been studied by Yosida K. (Japan), Kato T. (Japan, USA), Agmon S. (Israel), Friedman A. (USA), Solomyak M.Z. (USSR, Israel), Sobolevskii P.E. (USSR, Israel), Stewart H.B. (USA) at all.

Important progress has been made in the study of positive operators from the viewpoint of the stability analysis of higher order accuracy difference schemes for partial differential equations. It is a well-known that the most useful methods for stability analysis of difference schemes are difference analogue of maximum principle and energy method. The application of theory of positive difference operators permits us to investigate the stability and coercive stability properties of difference schemes in various norms for partial differential equations specially when we can not be able to use of maximum principle and energy method.

It is known that fourth order differential equations can be solved by the Fourier series method, by the Fourier transform method and by the Laplace transform method.

Now let us consider some examples.

First let us consider the following simple fourth order boundary value problem

$$\begin{cases} \frac{\partial^4 u(x)}{\partial x^4} + u(x) = (\pi^4 + 1)\sin \pi x, \ 0 < x < 1, \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0. \end{cases} \tag{1.1}$$

For the solution of problem (1.1), we use the Fourier series method. Let

$$u(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos k\pi x + B_k \sin k\pi x.$$

Substituting in the equation we get

$$\frac{A_0}{2}+\sum_{k=1}^{\infty}\left(k^4\pi^4+1\right)\left(A_k\sin k\pi x+B_k\sin k\pi x\right)=\left(\pi^4+1\right)\sin\pi x$$

and

$$A_0=0,\ B_k=0 \text{ for } k=1,2,\dots\ ,$$

for $k\neq 1$

$$\left(k^4\pi^4+1\right)A_k\sin k\pi x=0.$$

Therefore

$$A_k=0,$$

for $k=1$

$$\left(\pi^4+1\right)A_1\sin\pi x=\left(\pi^4+1\right)\sin\pi x.$$

Thus

$$A_1=1,$$

and

$$u(x)=\sin\pi x$$

is the solution of the fourth order boundary value problem (1.1)

Note that using the same manner one obtains the solution of the following $2m$ order boundary value problem

$$\begin{cases}(-1)^m\frac{d^{2m}u}{dx^{2m}}(x)+\sigma u(x)=f(x),\ 0<x<1,\\ u^{(2k)}(0)=u_0^{(k)},\ u^{(2k)}(1)=u_1^{(k)},\\ K=0,1,2,...,m\end{cases}$$

where σ and $f(x)$ are given smooth functions. However, the method of Fourier series can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving boundary value problems with dependent coefficients is difference method.

Second, we consider the fourth order differential equation

$$\begin{cases}\frac{\partial^4u(x)}{\partial x^4}+u(x)=2e^{-x},\quad -0<x<\infty,\\ u(0)=1,\ u^{'}(0)=-1,\ u^{''}(0)=1,\ u^{'''}(0)=-1.\end{cases}\tag{1.2}$$

Here, we will use the Laplace transform method to solve the problem (1.2). Let

$$u(s) = L\{u(x)\}.$$

So our problem becomes

$$\left(s^4+1\right)u(s)+1-s+s^2-s^3 = \frac{2}{1+s},$$

and

$$u(s) = \frac{1}{1+s}.$$

Finally taking the inverse of Laplace we obtain

$$u(x) = e^{-x}$$

is the solution of the given fourth order differential equation (1.2).

Note that using the same manner one obtains the solution of the following $2m$ order differential equation

$$\begin{cases} (-1)^m \frac{d^{2m}u}{dx^{2m}}(x) + \sigma u(x) = f(x), \ 0 < x < \infty, \\ \\ u^{(K)}(0) = u_0^{(K)}, \ 0 \le K \le 2m-1 \end{cases}$$

where σ and $f(x)$ are given smooth functions. However, the Laplace transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving differential equations with dependent coefficients is difference method.

Third, we consider the following fourth order differential equation

$$\frac{d^4u(x)}{dx^4} + u(x) = \left(13-48x^2+16x^4\right)e^{-x^2}, \ x \in \mathbb{R}^1. \tag{1.3}$$

We will use the Fourier transform method to solve the problem (1.3). Let

$$u(s) = F\{u(x)\}.$$

We take the Fourier transform of both sides of the equation (1.3) and, we get

$$F\left\{\frac{d^4u(x)}{dx^4}\right\} + F\{u(x)\} = F\left\{\left(13-48x^2+16x^4\right)e^{-x^2}\right\}.$$

From that it follows

$$\left((is)^4+1\right)u(s)=F\left\{\left(e^{-x^2}\right)''''\right\}+F\left\{e^{-x^2}\right\}.$$

Then our problem becomes

$$u(s)=F\left\{e^{-x^2}\right\}.$$

Finally taking the inverse of Fourier transformation we obtain the solution for the problem (1.3) as

$$u(x)=e^{-x^2}.$$

Note that using the same manner one obtains the solution of the following $2m$ order differential equation

$$(-1)^m\frac{d^{2m}u}{dx^{2m}}(x)+\sigma u(x)=f(x),\ x\in\mathbb{R}^n$$

where σ, $f(x)$ are given smooth functions. However, the Fourier transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving fouth order differential equations with dependent coefficients is difference method.

Now let us give the definition of positive operators.

Definition 1.1 *The operator A is said to be strongly positive if its spectrum $\sigma(A)$ lies in the interior of the sector of angle ϕ, $0<2\phi<\pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1(\phi)=\{\rho e^{i\phi}:0\le\rho\le\infty\}$ and $S_2(\phi)=\{\rho e^{-i\phi}:0\le\rho\le\infty\}$, and outside of the resolvent $(\lambda-A)^{-1}$ is subject to the bound*

$$\left\|(\lambda-A)^{-1}\right\|_{E\to E}\le\frac{M(\phi)}{1+|\lambda|}.$$

The infimum of all such angles ϕ is called the spectral angle of the strongly positive operator A and is denoted by $\phi(A)=\phi(A,E)$. Since the spectrum $\sigma(A)$ is a closed set, it lies inside the sector formed by the rays $S_1(\phi(A))$ and $S_2(\phi(A))$, and some neighborhood of the apex of this sector does not intersect $\sigma(A)$. We shall consider contours $\Gamma=\Gamma(\phi,r)$ composed by the rays $S_1(\phi)$, $S_2(\phi)$ and an arc of circle of radius r centered at the origin; ϕ and r will be chosen so that $\sigma(A)<|\sigma|<\pi/2$ and the arc of circle of radius r lies in the resolvent set $\rho(A)$ of the operator A.

In the present work a fourth order differential operator A^x defined by the formula

$$A^xu=\frac{d^4u}{dx^4}+\delta u,$$

with domain

$$D\left(A^{x}\right)=\left\{u\in C^{2}\left[0,1\right]:u\left(0\right)=u\left(1\right)=0,\ u^{''}\left(0\right)=u^{''}\left(1\right)=0\right\}$$

and a fourth order difference operator A_h^x defined by the formula

$$\begin{aligned} A_h^x u^h &= \left\{\frac{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}}{h^2}+\delta u_k\right\}_2^{N-2}, \\ u^h &= \left\{u_k\right\}_0^N \end{aligned}$$

are studied. Here $\delta>0$.The positivity of differential operator A^x in $C\left[0,1\right]$ and of the difference operator A_h^x in C_h are established.

Let us briefly describe the contents of the various sections. It consists four chapters. First chapter is the introduction. Second chapter consists of three sections. A brief survey of all investigations in this area can be found in the first section. In the second section the Green's function is constructed. Third section is devoted to the study of the positivity of the operator A with constant coefficients generated by the nonlocal boundary value problem in C Banach space. The third chapter consists of three sections. A brief survey of all investigations in this area can be found in the first section. In the second section the Green's function is constructed. Third section is devoted to the study of the positivity of the operator A_h with constant coefficients generated by the nonlocal boundary value problem in C_h Banach space. The last chapter is the conclusions.

CHAPTER 2

POSITIVITY OF THE FOURTH ORDER DIFFERENTIAL OPERATORS

2.1 Introduction

Let us consider a differential operator A^x defined by the formula

$$A^x u = \frac{d^4 u}{dx^4} + \delta u,$$

with domain

$$D(A^x) = \left\{ u \in C^4[0,1] : u(0) = u(1) = 0,\ u''(0) = u''(1) = 0 \right\}.$$

Here $\delta > 0$.

We introduce $C[0,1]$, the space of all continous functions $\varphi(x)$ defined on $[0,1]$ with the following norm

$$\|\varphi\|_{C[0,1]} = \max_{0 \le x \le 1} |\varphi(x)|.$$

We will investigate the resolvent of the operator A^x, that is, in solving the equation

$$A^x u + \lambda u = f$$

or

$$\frac{d^4 u(x)}{dx^4} + \delta u(x) + \lambda u(x) = f(x),$$

$$u(0) = u(1) = 0,\ u''(0) = u''(1) = 0.$$

2.2 Green's function of the operator A^x

Lemma 2.1 *For all λ the equation*

$$A^x u + \lambda u = f \tag{2.1}$$

is uniquely solvable and the following formula holds

$$u(x) = (A^x + \lambda)^{-1} f(x) = \int_0^1 J(x, s, l; \lambda + \delta) f(l)\, dl, \tag{2.2}$$

where

$$J(x, s, l; \lambda + \delta) = \frac{1}{8k^2} \int_0^1 \{T_1 (e^{-k(1+i)(1-x)} - e^{-k(1+i)(1+x)})$$

$$\times (e^{-k(1+i)(1-s)} - e^{-k(1+i)(1+s)})$$
$$- \left(e^{-k(1+i)|x-s|} - e^{-k(1+i)(x+s)}\right)$$
$$\times T_2 (e^{-k(1-i)(1-s)} - e^{-k(1-i)(1+s)})$$
$$\times (e^{-k(1-i)(1-l)} - e^{-k(1-i)(1+l)})$$
$$- \left(e^{-k(1+i)(s-l)} - e^{-k(1+i)(s+l)}\right)\}\, ds$$

for $0 \le l \le s \le 1$, and

$$J(x, s, l; \lambda + \delta) = \frac{1}{8k^2} \int_0^1 \{T_1 (e^{-k(1+i)(1-x)} - e^{-k(1+i)(1+x)})$$

$$\times (e^{-k(1+i)(1-s)} - e^{-k(1+i)(1+s)})$$
$$- \left(e^{-k(1+i)|x-s|} - e^{-k(1+i)(x+s)}\right)$$
$$\times T_2 (e^{-k(1-i)(1-s)} - e^{-k(1-i)(1+s)})$$
$$\times (e^{-k(1-i)(1-l)} - e^{-k(1-i)(1+l)})$$
$$- \left(e^{-k(1+i)(l-s)} - e^{-k(1+i)(s+l)}\right)\}\, ds$$

for $0 \le s \le l \le 1$. Here

$$T_1 = \left(1 - e^{-2k(1+i)}\right)^{-1},\ T_2 = \left(1 - e^{-2k(1-i)}\right)^{-1},\ k = \frac{(\lambda + \delta)^{\frac{1}{4}}}{\sqrt{2}}.$$

Proof. We see that the problem (2.1) can be obviously written as two boundary value problems for the second order linear differential equations

$$-\frac{d^2 u}{dx^2} + \mu u = z(x),\ 0 < x < 1,$$

$$u(0) = u(1) = 0,\ u''(0) = u''(1) = 0,$$

and

$$-\frac{d^2 z}{dx^2} + \mu z = f(x),\ 0 < x < 1.$$

Here $\mu = i\sqrt{\lambda + \delta}$.

We have the following formula

$$T_1 \left\{ \left(e^{-k(1+i)(x)} - e^{-k(1+i)(2-x)}\right) \varphi_1 \right.$$
$$+ \left(e^{-k(1+i)(1-x)} - e^{-k(1+i)(1+x)}\right) \psi_1$$
$$- \frac{1}{2k(1+i)} \left(e^{-k(1+i)(1-x)} - e^{-k(1+i)(1+x)}\right)$$
$$\left. \times \int_0^1 (e^{-k(1+i)(1-s)} - e^{-k(1+i)(1+s)}) z(s)\, ds \right\}$$
$$+ \frac{1}{2k(1+i)} \int_0^1 \left(e^{-k(1+i)|x-s|} - e^{-k(1+i)(x+s)}\right) z(s)\, ds, \tag{2.3}$$

$$T_1 = \left(1 - e^{-2k(1+i)}\right)^{-1}$$

for the solution of the boundary value problem

$$-\frac{d^2 u}{dx^2} - \mu u = z(x),\ 0 < x < 1,\ u(0) = \varphi_1,\ u(1) = \psi_1,$$

for second order linear differential equations. Applying formula (2.3) and boundary conditions

$$u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0,$$

we get

$$\varphi_1 = 0,\ \psi_1 = 0$$

and

$$z(0) = 0,\ z(1) = 0. \tag{2.4}$$

We have the following formula

$$T_2\left\{\left(e^{-k(1-i)(x)}-e^{-k(1-i)(2-x)}\right)\varphi_2\right.$$
$$+\left(e^{-k(1-i)(1-x)}-e^{-k(1-i)(1+x)}\right)\psi_2$$
$$-\frac{1}{2k(1-i)}\left(e^{-k(1-i)(1-x)}-e^{-k(1-i)(1+x)}\right)$$
$$\left.\times\int_0^1(e^{-k(1-i)(1-l)}-e^{-k(1-i)(1+l)})f\left(l\right)dl\right\}$$
$$+\frac{1}{2k(1-i)}\int_0^1\left(e^{-k(1-i)|x-l|}-e^{-k(1-i)(x+l)}\right)f\left(l\right)dl, \tag{2.5}$$

$$T_2=\left(1-e^{-2k(1-i)}\right)^{-1}$$

for the solution of the boundary value problem

$$-\frac{d^2z}{dx^2}+\mu z=f\left(x\right),\ 0<x<1,\ z\left(0\right)=\varphi_2,\ z\left(1\right)=\psi_2,$$

for second order linear differential equations. Applying formula (2.5) and boundary conditions

$$z\left(0\right)=0, z(1)=0,$$

we get

$$\varphi_2=0,\ \psi_2=0. \tag{2.6}$$

Finally, applying formulas (2.3), (2.4), (2.5), (2.6), we obtain

$$\left(A^x+\lambda\right)^{-1}f\left(x\right)=\frac{1}{8k^2}\int_0^1\int_0^1\{T_1(e^{-k(1+i)(1-x)}-e^{-k(1+i)(1+x)})$$

$$\times(e^{-k(1+i)(1-s)}-e^{-k(1+i)(1+s)})$$
$$-\left(e^{-k(1+i)|x-s|}-e^{-k(1+i)(x+s)}\right)$$
$$\times T_2(e^{-k(1-i)(1-s)}-e^{-k(1-i)(1+s)})$$
$$\times(e^{-k(1-i)(1-l)}-e^{-k(1-i)(1+l)})$$
$$-\left(e^{-k(1+i)|s-l|}-e^{-k(1+i)(s+l)}\right)\}f(l)dsdl.$$

Lemma 2.1 is proved. ■

The function $J(x,s,l;\lambda+\delta)$ is called the Green's function of the resolvent equation (2.1).

Thus, we obtain the formula for the resolvent $(\lambda I + A^x)^{-1}$ in the case $\lambda \geq 0$. In the same way we can obtain a formula for the resolvent $(\lambda I + A^x)^{-1}$ in the case of complex λ. But we need to obtain that k^2, $T_{\ 1}$, $T_{\ 2}$ are not equal to zero.

2.3 Positivity of A^x in $C[0,1]$

Theorem 2.1 *For all* λ, $\lambda \in R_\varphi = \left\{\lambda : |\arg\lambda| \leq \varphi,\ \varphi < \frac{\pi}{2}\right\}$ *the resolvent* $(\lambda I + A^x)^{-1}$ *defined by the formula* (2.2) *is subject to the bound*

$$\left\|(\lambda I + A^x)^{-1}\right\|_{C[0,1]\to C[0,1]} \leq M(\varphi,\delta)(1+|\lambda|)^{-1},$$

where $M(\varphi,\delta)$ *does not depend on* λ.

The proof of this theorem is based on the following lemmas.

Lemma 2.2 *Let*

$$\psi = \arg(k) = \arg(\frac{\sqrt[4]{\lambda+\delta}}{\sqrt{2}}).$$

Then

$$\psi \leq \frac{\varphi}{4}.$$

Proof. It is easy to see that

$$\begin{aligned}\lambda+\delta &= \delta+\rho\cos\varphi+i\rho\sin\varphi,\\ \arg(\lambda+\delta) &= \arctan(\frac{\rho\sin\varphi}{\delta+\rho\cos\varphi})\\ &\leq \arctan(\frac{\rho\sin\varphi}{\rho\cos\varphi})\\ &= \arctan(\tan\varphi)\\ &= \varphi.\end{aligned}$$

Therefore

$$\begin{aligned}\arg(k) &= \frac{\arg(\lambda+\delta)}{4}\\ &\leq \frac{\varphi}{4}.\end{aligned}$$

Lemma 2.2 is proved. ■

Lemma 2.3 *The following estimate holds*

$$|\lambda+\delta| \geq \sqrt{\cos\varphi}(\delta+\rho).$$

Proof. Using the notations of lemma 2.2, we get

$$\begin{aligned} |\lambda+\delta| &= \left\{(\delta+\rho\cos\varphi)^2+(\rho\sin\varphi)^2\right\}^{\frac{1}{2}} \\ &= \left\{\delta^2+2\rho\delta\cos\varphi+\rho^2\cos^2\varphi+\rho^2\sin^2\varphi\right\}^{\frac{1}{2}} \\ &= \left\{\delta^2+2\rho\delta\cos\varphi+\rho^2\right\}^{\frac{1}{2}} \\ &\geq \sqrt{\cos\varphi}\left\{\delta^2+2\rho\delta+\rho^2\right\}^{\frac{1}{2}} \\ &= \sqrt{\cos\varphi}(\delta+\rho). \end{aligned}$$

Lemma 2.3 is proved. ■

Lemma 2.4 *The following estimates are satisfied*

$$\left|e^{-k(1+i)(1-x)}-e^{-k(1+i)(1+x)}\right| \leq 2e^{-|k|(\cos\psi-\sin\psi)(1-x)},$$

$$\left|e^{-k(1+i)(1-s)}-e^{-k(1+i)(1+s)}\right| \leq 2e^{-|k|(\cos\psi-\sin\psi)(1-s)},$$

$$\left|e^{-k(1-i)(1-s)}-e^{-k(1-i)(1+s)}\right| \leq 2e^{-|k|(\cos\psi+\sin\psi)(1-s)},$$

$$\left|e^{-k(1-i)(1-l)}-e^{-k(1-i)(1+l)}\right| \leq 2e^{-|k|(\cos\psi+\sin\psi)(1-l)}.$$

Proof. Using semigroup property of exponential function, we get

$$\left|e^{-k(1+i)(1-x)}-e^{-k(1+i)(1+x)}\right| = \left|e^{-k(1+i)(1-x)}\right|\left|1-e^{-2k(1+i)x}\right|$$

$$\begin{aligned} &\leq e^{-|k|(\cos\psi-\sin\psi)(1-x)}\left(1+\left|e^{-2k(1+i)x}\right|\right) \\ &= e^{-|k|(\cos\psi-\sin\psi)(1-x)} \\ &\quad \times\left(1+e^{-2|k|(\cos\psi-\sin\psi)x}\right) \\ &\leq 2e^{-|k|(\cos\psi-\sin\psi)(1-x)}. \end{aligned}$$

Using semigroup property, we get

$$\left|e^{-k(1+i)(1-s)}-e^{-k(1+i)(1+s)}\right|=\left|e^{-k(1+i)(1-s)}\right|\left|1-e^{-2k(1+i)s}\right|$$

$$\begin{aligned}&\leq\ e^{-|k|(\cos\psi-\sin\psi)(1-s)}\left(1+\left|e^{-2k(1+i)s}\right|\right)\\&\leq\ 2e^{-|k|(\cos\psi-\sin\psi)(1-s)},\end{aligned}$$

$$\left|e^{-k(1-i)(1-s)}-e^{-k(1-i)(1+s)}\right|=\left|e^{-k(1-i)(1-s)}\right|\left|1-e^{-2k(1-i)s}\right|$$

$$\begin{aligned}&\leq\ e^{-|k|(\cos\psi+\sin\psi)(1-s)}\left(1+\left|e^{-2k(1-i)s}\right|\right)\\&\leq\ 2e^{-|k|(\cos\psi+\sin\psi)(1-s)},\end{aligned}$$

and

$$\left|e^{-k(1-i)(1-l)}-e^{-k(1-i)(1+l)}\right|=\left|e^{-k(1-i)(1-l)}\right|\left|1-e^{-2k(1-i)l}\right|$$

$$\begin{aligned}&\leq\ e^{-|k|(\cos\psi+\sin\psi)(1-l)}\left(1+\left|e^{-2k(1-i)l}\right|\right)\\&\leq\ 2e^{-|k|(\cos\psi+\sin\psi)(1-l)}.\end{aligned}$$

Lemma 2.4 is proved. ■

Lemma 2.5 *The following estimates hold*

$$\left|e^{-k(1+i)|x-s|}-e^{-k(1+i)(x+s)}\right|\leq 2e^{-|k|(\cos\psi-\sin\psi)(x-s)}$$

for $0\leq s\leq x$,

$$\left|e^{-k(1+i)|x-s|}-e^{-k(1+i)(x+s)}\right|\leq 2e^{-|k|(\cos\psi-\sin\psi)(s-x)}$$

for $0\leq x\leq s$, *and*

$$\left|e^{-k(1-i)|s-l|}-e^{-k(1-i)(s+l)}\right|\leq 2e^{-|k|(\cos\psi+\sin\psi)(s-l)}$$

for $0\leq l\leq s$,

$$\left|e^{-k(1-i)|s-l|}-e^{-k(1-i)(s+l)}\right|\leq 2e^{-|k|(\cos\psi+\sin\psi)(l-s)}$$

for $0\leq s\leq l$.

Proof. for $0 \le s \le x$

$$\left|e^{-k(1+i)|x-s|} - e^{-k(1+i)(x+s)}\right| = \left|e^{-k(1+i)(x-s)} - e^{-k(1+i)(x+s)}\right|.$$

Using semigroup property, we get

$$\left|e^{-k(1+i)(x-s)} - e^{-k(1+i)(x+s)}\right| = \left|e^{-k(1+i)(x-s)}\right| \left|1 - e^{-2k(1+i)s}\right|$$

$$\begin{aligned} &\le e^{-|k|(\cos\psi - \sin\psi)(x-s)} \left(1 + e^{-2k(1+i)s}\right) \\ &\le 2e^{-|k|(\cos\psi - \sin\psi)(x-s)}, \end{aligned}$$

for $0 \le x \le s$

$$\left|e^{-k(1+i)|x-s|} - e^{-k(1+i)(x+s)}\right| = \left|e^{-k(1+i)(s-x)} - e^{-k(1+i)(x+s)}\right|$$

$$\begin{aligned} &= \left|e^{-k(1+i)(s-x)}\right| \left|1 - e^{-2k(1+i)x}\right| \\ &\le e^{-|k|(\cos\psi - \sin\psi)(s-x)} \left(1 + e^{-2k(1+i)x}\right) \\ &\le 2e^{-|k|(\cos\psi - \sin\psi)(s-x)}, \end{aligned}$$

and for $0 \le l \le s$

$$\left|e^{-k(1-i)|s-l|} - e^{-k(1-i)(s+l)}\right| = \left|e^{-k(1-i)(s-l)} - e^{-k(1-i)(s+l)}\right|$$

$$\begin{aligned} &= \left|e^{-k(1-i)(s-l)}\right| \left|1 - e^{-2k(1-i)l}\right| \\ &\le e^{-|k|(\cos\psi + \sin\psi)(s-l)} \left(1 + e^{-2k(1-i)l}\right) \\ &\le 2e^{-|k|(\cos\psi + \sin\psi)(s-l)}, \end{aligned}$$

and using the notations of lemma 2.5 for $0 \le s \le l$, we get

$$\left|e^{-k(1-i)|s-l|} - e^{-k(1-i)(s+l)}\right| = \left|e^{-k(1-i)(l-s)} - e^{-k(1-i)(s+l)}\right|$$

$$\begin{aligned} &= \left|e^{-k(1-i)(l-s)}\right| \left|1 - e^{-2k(1-i)s}\right| \\ &\le e^{-|k|(\cos\psi + \sin\psi)(l-s)} \left(1 + e^{-2k(1-i)s}\right) \\ &\le 2e^{-|k|(\cos\psi + \sin\psi)(l-s)}. \end{aligned}$$

Lemma 2.5 is proved. ■

Lemma 2.6 *The following estiamtes for T_1 and T_2 are satisfied.*

$$|T_1| \leq \frac{1}{1-e^{-2|k|(\cos\psi-\sin\psi)}},$$

$$|T_2| \leq \frac{1}{1-e^{-2|k|(\cos\psi+\sin\psi)}}.$$

Proof. Using the formula of T_1, we get

$$\begin{aligned} |T_1| &= \frac{1}{|1-e^{-2k(1+i)}|} \\ &\leq \frac{1}{1-|e^{-2k(1+i)}|} \\ &= \frac{1}{1-e^{-2|k|(\cos\psi-\sin\psi)}}, \end{aligned}$$

and using the formula of T_2, we get

$$\begin{aligned} |T_2| &= \frac{1}{|1-e^{-2k(1-i)}|} \\ &\leq \frac{1}{1-|e^{-2k(1-i)}|} \\ &= \frac{1}{1-e^{-2|k|(\cos\psi+\sin\psi)}}. \end{aligned}$$

Lemma 2.6 is proved. ■

Lemma 2.7 *Let*

$$\frac{4}{1-e^{-2|k|(\cos\psi+\sin\psi)}}e^{-|k|(\cos\psi-\sin\psi)(1-s)}\left(1-e^{-|k|(\cos\psi+\sin\psi)}\right)$$
$$+2\left[1-e^{-|k|(\cos\psi+\sin\psi)s}+\left(1-e^{-|k|(\cos\psi+\sin\psi)(1-s)}\right)\right]=K_1,$$

and

$$\frac{4}{1-e^{-2|k|(\cos\psi-\sin\psi)}}e^{-|k|(\cos\psi-\sin\psi)(1-x)}\left(1-e^{-|k|(\cos\psi-\sin\psi)}\right)$$
$$+2\left(2-e^{-|k|(\cos\psi-\sin\psi)x}-e^{-|k|(\cos\psi-\sin\psi)}\right)=K_2.$$

The following estimates hold.

$$K_1,\ K_2 \leq 8$$

Proof. Using the following estimates,

$$e^{-|k|(\cos\psi-\sin\psi)(1-s)} \leq 1,$$

$$\frac{\left(1-e^{-|k|(\cos\psi+\sin\psi)}\right)}{1-e^{-2|k|(\cos\psi+\sin\psi)}} \le 1,$$

$$\left[1-e^{-|k|(\cos\psi+\sin\psi)s}-\left(1-e^{-|k|(\cos\psi+\sin\psi)(1-s)}\right)\right] \le 2,$$

we get

$$K_1 \le 8,$$

and using

$$e^{-|k|(\cos\psi-\sin\psi)(1-x)} \le 1,$$

$$\frac{1-e^{-|k|(\cos\psi-\sin\psi)}}{1-e^{-2|k|(\cos\psi-\sin\psi)}} \le 1,$$

$$2-e^{-|k|(\cos\psi-\sin\psi)x}-e^{-|k|(\cos\psi-\sin\psi)} \le 2.$$

Then, we get

$$K_2 \le 8.$$

Lemma 2.7 is proved. ■

Lemma 2.8 *The following estimate is satisfied.*

$$\frac{1}{\cos\frac{\varphi}{2}} \le \frac{1}{\sqrt{\cos\varphi}}.$$

Proof. Using trigonometric identity of cosine function, we get

$$\begin{aligned}\frac{1}{\cos\varphi} &= \frac{1}{\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}} \\ &\ge \frac{1}{\cos^2\frac{\varphi}{2}},\end{aligned}$$

$$\frac{1}{\cos\frac{\varphi}{2}} \le \frac{1}{\sqrt{\cos\varphi}}.$$

Lemma 2.8 is proved. ■

Using the triangle inequality, we get

$$\left|(A+\lambda I)^{-1} f(x)\right| \le \frac{1}{8|k|^2}\int_0^1\int_0^1 \left\{|T_1|\left|e^{-k(1+i)(1-x)}-e^{-k(1+i)(1+x)}\right|\right.$$

$$\times\left|e^{-k(1+i)(1-s)}-e^{-k(1+i)(1+s)}\right|$$
$$+\left|e^{-k(1+i)|x-s|}-e^{-k(1+i)(x+s)}\right|\}$$
$$\times\left|T_1\right|\left|e^{-k(1-i)(1-s)}-e^{-k(1-i)(1+s)}\right|$$
$$\times\left|e^{-k(1-i)(1-l)}-e^{-k(1-i)(1+l)}\right|$$
$$+\left(e^{-k(1+i)|s-l|}-e^{-k(1+i)(s+l)}\right)\}\,dsdl\max_{0\le l\le 1}|f(l)|\,.$$

Using the estimates on Lemmas 2.4, 2.5, 2.6, we get

$$\left|(A+\lambda I)^{-1}f(x)\right|\le\frac{1}{8|k|^2}\int_0^1\left\{\frac{4}{1-e^{-2|k|(\cos\psi-\sin\psi)}}e^{-|k|(\cos\psi-\sin\psi)(1-x)}\right.$$

$$\times e^{-|k|(\cos\psi-\sin\psi)(1-s)}$$
$$+\left|e^{-k(1+i)|x-s|}-e^{-k(1+i)(x+s)}\right|\}$$
$$\times\frac{1}{|k|(\cos\psi+\sin\psi)}\left\{\frac{4}{1-e^{-2|k|(\cos\psi+\sin\psi)}}\right.$$
$$\times e^{-|k|(\cos\psi-\sin\psi)(1-s)}$$
$$\times\left(1-e^{-|k|(\cos\psi+\sin\psi)}\right)+2\left[1-e^{-|k|(\cos\psi+\sin\psi)s}\right.$$
$$\left.\left.+\left(1-e^{-|k|(\cos\psi+\sin\psi)(1-s)}\right)\right]\right\}ds\max_{0\le l\le 1}|f(l)|\,,$$

using the estimate on lemma 2.7 we get,

$$\left|(A+\lambda I)^{-1}f(x)\right|\le\frac{1}{8|k|^2}\int_0^1\left\{\frac{4}{1-e^{-2|k|(\cos\psi-\sin\psi)}}e^{-|k|(\cos\psi-\sin\psi)(1-x)}\right.$$

$$\times e^{-|k|(\cos\psi-\sin\psi)(1-s)}$$
$$+\left|e^{-k(1+i)|x-s|}-e^{-k(1+i)(x+s)}\right|\}$$
$$\times\frac{8}{|k|(\cos\psi+\sin\psi)}ds\max_{0\le l\le 1}|f(l)|$$
$$\le\ \frac{1}{8|k|^2}\frac{1}{|k|(\cos\psi-\sin\psi)}\left\{\frac{4}{1-e^{-2|k|(\cos\psi-\sin\psi)}}\right.$$
$$\times e^{-|k|(\cos\psi-\sin\psi)(1-x)}\left(1-e^{-|k|(\cos\psi-\sin\psi)}\right)$$
$$\left.+2\left(2-e^{-|k|(\cos\psi-\sin\psi)x}-e^{-|k|(\cos\psi-\sin\psi)}\right)\right\}$$
$$\times\frac{8}{|k|(\cos\psi+\sin\psi)}\|f\|_{C[0,1]}\,.$$

using the estimates on lemmas 2.2, 2.3, 2.7, 2.8 we get,

$$\left|(A+\lambda I)^{-1} f(x)\right| \leq \frac{8}{|k|^4(\cos\psi+\sin\psi)(\cos\psi-\sin\psi)}\|f\|_{C[0,1]}$$

$$\begin{aligned}
&\leq \frac{8}{|k|^4\cos 2\psi}\|f\|_{C[0,1]} \\
&= \frac{32}{|\lambda+\delta|\cos 2\psi}\|f\|_{C[0,1]} \\
&\leq \frac{32}{(|\lambda|+\delta)\cos\frac{\varphi}{2}\sqrt{\cos\varphi}}\|f\|_{C[0,1]} \\
&\leq \frac{32}{(|\lambda|+\delta)\cos\varphi}\|f\|_{C[0,1]} \\
&\leq \frac{M(\varphi,\delta)}{1+|\lambda|}\|f\|_{C[0,1]}\,.
\end{aligned}$$

It follows that,

$$\left\|(A+\lambda I)^{-1} f\right\|_{C[0,1]} \leq \frac{M(\varphi,\delta)}{1+|\lambda|}\|f\|_{C[0,1]}\,,$$

and

$$\left\|(A+\lambda I)^{-1} f\right\|_{C[0,1]\to C[0,1]} \leq \frac{M(\varphi,\delta)}{1+|\lambda|}.$$

Here $M(\varphi,\delta)$ does not depend on λ.

Theorem 2.1 is proved.

Now, we consider the nonlocal boundary-value problem for two dimensional elliptic equation

$$\begin{cases}
-\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}+\delta u=f(t,x)\,,\ 0<t<T,\ 0<x<1, \\
u(0,x)=\varphi(x)\,,\ u(T,x)=\psi(x)\,,\ 0\leq x\leq 1, \\
u(t,0)=u(t,1)=0,\ u_{xx}(t,0)=u_{xx}(t,1)=0,\ 0\leq t\leq T,
\end{cases} \tag{2.7}$$

where $\varphi(x)\,,\ \psi(x)$ and $f(t,x)$ are given sufficiently smooth functions and $\delta>0$ is a sufficiently large number.

Theorem 2.2 *For the solution of the boundary value problem (2.7) the following coercive inequalities are valid:*

$$\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{C_{0T}^{\alpha,\alpha}([0,T],C[0,1])}+\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{C_{0T}^{\alpha,\alpha}([0,T],C[0,1])}$$

$$\leq M\left[\left\|\varphi^{(4)}\right\|_{C[0,1]}+\left\|\psi^{(4)}\right\|_{C[0,1]}+\frac{1}{\alpha\left(1-\alpha\right)}\left\|f\right\|_{C_{0T}^{\alpha,\alpha}([0,T],C[0,1])}\right]$$

where M is independent of $\alpha, f(t,x), \varphi(x), \psi(x)$.

The proof of Theorem 2.2 is based on the positivity of the operator A^x generated by problem (2.7) and on the following theorem about coercivity inequalities in $C_{0T}^{\alpha,\alpha}\left([0,T],E\right)$ for the solution of the abstract boundary-value problem for the differential equation.

$$-v''\left(t\right)+Av\left(t\right)=f\left(t\right)\quad\left(0\leq t\leq T\right),v\left(0\right)=v_0,v\left(T\right)=v_T \tag{2.8}$$

in an arbitrary Banach space E with the linear positive operator A.

Theorem 2.3 *Suppose $v_0, v_T\in D\left(A\right)$, $f\left(t\right)\in C_{0,T}^{\alpha,\alpha}\left(E\right)\left(0<\alpha<1\right)$. Then the boundary value problem (2.8) is well posed in the Holder space $C_{0,T}^{\alpha,\alpha}\left(E\right)$, if A is the positive operator in the Banach space E. For the solution $v\left(t\right)$ in $C_{0,T}^{\alpha,\alpha}\left(E\right)$ of the boundary-value problem the coercive inequality*

$$\left\|v''\right\|_{C_{0,T}^{\alpha,\alpha}(E)}+\left\|Av\right\|_{C_{0,T}^{\alpha,\alpha}(E)}\leq\frac{M}{\alpha\left(1-\alpha\right)}\left\|f\right\|_{C_{0,T}^{\alpha,\alpha}(E)}+M\left[\left\|Av_0\right\|_E+\left\|Av_T\right\|_E\right]$$

holds, where M does not depend on α, v_0, v_T and $f\left(t\right)$.Here, we denote $C_{0,T}^{\alpha,\alpha}\left([0,T],E\right)$

the Banach space obtained by completion of the set of smooth $E-$valued functions $\varphi(t)$ on $[0,T]$ in the norm

$$\left\|\varphi\right\|_{C_{0T}^{\alpha,\alpha}([0,T],E)}=\max_{0\leq t\leq T}\left\|\varphi\left(t\right)\right\|_E$$

$$+\sup_{0\leq t\leq t+\tau\leq T}\frac{\left(T-t\right)^{\alpha}\left(t+\tau\right)^{\alpha}\left\|\varphi\left(t+\tau\right)-\varphi\left(t\right)\right\|}{\tau^{\alpha}}.$$

CHAPTER 3

POSITIVITY OF THE FOURTH ORDER DIFFERENCE OPERATORS

3.1 Introduction

Let us define the grid space $[0,1]_h = \{x_k = kh,\ 0 \leq k \leq N,\ Nh = 1\}$, N is a fixed positive integer. The number h is called the step of the grid space. A function $\varphi^h = \{\varphi_k\}_0^N$ defined on $[0,1]_h$ will be called a grid function. To the operator A^x we assign the difference operator A_h^x of a first order of approximation defined by the formula

$$\begin{aligned} A_h^x u^h &= \left\{ \frac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^2} + \delta u_k \right\}_2^{N-2}, \\ u^h &= \{u_k\}_0^N, \end{aligned} \tag{3.1}$$

which acts on grid functions defined on $[0,1]_h$ with

$$u_0 = u_N = 0,\ \ -u_2 + 2u_1 - u_0 = u_{N-2} - 2u_{N-1} + u_N = 0.$$

We denote by $C_h = C[0,1]_h$ and equipped with the norm

$$\left\| v^h \right\|_{C_h} = \max_{2 \leq k \leq N-2} |v_k|.$$

We will investigate the resolvent operator A_h^x in solving the equation

$$A_h^x u^h + \lambda u^h = f^h \tag{3.2}$$

or

$$\frac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^2} + \delta u_k + \lambda u_k = f_k,$$

$$f_k = f(x_k),\ 2 \leq k \leq N-2,$$

$$u_0 = u_N = 0,\ \ -u_2 + 2u_1 - u_0 = u_{N-2} - 2u_{N-1} + u_N = 0.$$

The positivity of the difference operator A_h^x in C_h is established.

3.2 Green's function of the operator A_h^x

Lemma 3.1 *For all* λ *the equaiton* (3.2) *is uniquely solvable, and the following formula holds.*

$$u^h = (A_h^x + \lambda)^{-1} f^h = \left\{ \sum_{l=1}^{N-1} J(k,j,l;\lambda+\delta) f_l h \right\}_0^N, \tag{3.3}$$

where

$$J(k,j,l;\lambda+\delta) = \frac{\left(1-R_1^{2N}\right)(1+\mu_1 h)(1+\mu_2 h)}{\mu_1\mu_2 R_1 R_2 (2+\mu_1 h)(2+\mu_2 h)}$$

$$\times \{ [([T_{h1}(1-R_1)^2$$
$$\times \sum_{j=1}^{N-1} \left(R_1^{N-j-1} - R_1^{N+j-1} \right) R_2^{N-j-1} - 1 \Big]$$
$$\times \left[\sum_{j=1}^{N-1} h \left(T_{h1} \left(R_1^{N-k} - R_1^{N+k} \right) \right. \right.$$
$$\times \left. \left(R_1^{N-j} - R_1^{N+j} \right) - \left(R_1^{|k-j|} - R_1^{k+j} \right) \right) R_2^{j-1} \Big]$$

$$- \left[T_{h1}(1-R_1)^2 \sum_{j=1}^{N-1} \left(R_1^{N-j-1} - R_1^{N+j-1} \right) R_2^{j-1} - R_2^{N-2} \right]$$
$$\times \left[\sum_{j=1}^{N-1} h \left(T_{h1} \left(R_1^{N-k} - R_1^{N+k} \right) \left(R_1^{N-j} - R_1^{N+j} \right) \right. \right.$$
$$- \left. \left(R_1^{|k-j|} - R_1^{k+j} \right) \right) R_2^{N-j-1} \Big])$$

$$\times \left(T_{h1}(1-R_1)^2 \sum_{j=1}^{N-1} \left(R_1^{j-1} - R_1^{2N-j-1} \right) [T_{h2} \right.$$
$$\times \left(R_2^{N-j} - R_2^{N+j} \right) \left(R_2^{N-l} - R_2^{N+l} \right) - \left(R_2^{|j-l|} - R_2^{j+l} \right) \Big]$$
$$+ \left(1-R_2^2\right) \left(R_2^{l-1} - R_2^{2N-l-1} \right))$$

$$- \left(\left[T_{h1} \left(1 - R_1\right)^2 \sum_{j=1}^{N-1} \left(R_1^{j-1} - R_1^{2N-j-1} \right) R_2^{N-j-1} - R_2^{N-2} \right] \right.$$
$$\times \left[\sum_{j=1}^{N-1} h(T_{h1} \left(R_1^{N-k} - R_1^{N+k} \right) \left(R_1^{N-j} - R_1^{N+j} \right) \right.$$
$$\left. - \left(R_1^{|k-j|} - R_1^{k+j} \right)) R_2^{j-1} \right]$$

$$+ \left[T_{h1} \left(1 - R_1\right)^2 \sum_{j=1}^{N-1} \left(R_1^{j-1} - R_1^{2N-j-1} \right) R_2^{j-1} - 1 \right]$$
$$\times \sum_{j=1}^{N-1} h \left(T_{h1} \left(R_1^{N-k} - R_1^{N+k} \right) \left(R_1^{N-j} - R_1^{N+j} \right) \right.$$
$$\left. \left. - \left(R_1^{|k-j|} - R_1^{k+j} \right) \right) R_2^{N-j-1} \right] \right)$$

$$\times \left(T_{h1} \left(1 - R_1\right)^2 \sum_{j=1}^{N-1} \left(R_1^{N-j-1} - R_1^{N+j-1} \right) [T_{h2} \right.$$
$$\times \left(R_2^{N-j} - R_2^{N+j} \right) \left(R_2^{N-l} - R_2^{N+l} \right) - \left(R_2^{|j-l|} - R_2^{j+l} \right) \Big]$$
$$+ \left(1 - R_2^2\right) \left(R_2^{N-l-1} - R_2^{N+l-1} \right)) \Big\}$$

$$\times \Big\{ \left(I - R_2^{2N-4} - R_1^{2N} + R_1^{2N} R_2^{2N-4} \right) + h \left(2\mu_1^2 R_1 \right.$$
$$\times \left(R_1^N + R_2^{N-2} \right) \frac{R_1^{N-1} - R_2^{N-1}}{\left(\mu_2 - \mu_1\right) R_2} - \left(1 + R_1^N R_2^{N-2} \right)$$
$$\times \left. \frac{1 - R_1^{N-1} R_2^{N-1}}{\left(\mu_1 + \mu_2 + h\mu_1\mu_2\right) R_2} \right] - \left(1 - R_1\right) \mu_1^3 R_1$$
$$\times \left[\left(\frac{1 - R_1^{N-1} R_2^{N-1}}{\left(\mu_1 + \mu_2 + h\mu_1\mu_2\right) R_2} \right)^2 \right.$$
$$\left. \left. - \left(\frac{R_1^{N-1} - R_2^{N-1}}{\left(\mu_2 - \mu_1\right) R_2} \right)^2 \right] \right\}^{-1}$$

$$+ \sum_{j=1}^{N-1} h^3 \left[T_{h1} \left(R_1^{N-k} - R_1^{N+k} \right) \left(R_1^{N-j} - R_1^{N+j} \right) \right.$$
$$\left. - \left(R_1^{|k-j|} - R_1^{k+j} \right) \right] \left[T_{h2} \left(R_2^{N-j} - R_2^{N+j} \right) \right.$$
$$\left. \times \left(R_2^{N-l} - R_2^{N+l} \right) - \left(R_2^{|j-l|} - R_2^{j+l} \right) \right] \Big]$$

for $1 \leq l \leq N-1$ *and* $0 \leq k \leq N$. *Here*

$$\begin{aligned} T_{h1} &= \left(1-R_1^{2N}\right)^{-1},\ R_1 = (1+\mu_1)^{-1}, \\ \mu_1 &= \frac{1}{2}\left(ih\sqrt{\lambda+\delta}+\sqrt{i\sqrt{\lambda+\delta}\left(4+ih^2\sqrt{\lambda+\delta}\right)}\right), \end{aligned}$$

and

$$\begin{aligned} T_{h2} &= \left(1-R_2^{2N}\right)^{-1},\ R_2 = (1+\mu_2)^{-1}, \\ \mu_1 &= \frac{1}{2}\left(-ih\sqrt{\lambda+\delta}+\sqrt{-i\sqrt{\lambda+\delta}\left(-4+ih^2\sqrt{\lambda+\delta}\right)}\right). \end{aligned}$$

Proof. We see that the problem (3.2) can be obviously written as the equivalent boundary value problem for the second order linear difference equations

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}+\mu u_k = z_k,\ 1 \leq k \leq N-1, \\ u_0 = u_N = 0,\ \ -u_2+2u_1-u_0 = u_{N-2}-2u_{N-1}+u_N = 0, \\ -\frac{z_{k+1}-2z_k+z_{k-1}}{h^2}-\mu z_k = f_k,\ 1 \leq k \leq N-1. \end{cases}$$

Here $\mu = \sqrt{\lambda+\delta}.$

We have the following formula

$$\begin{aligned} u_k &= T_{h1}\left\{\left(R_1^k-R_1^{2N-k}\right)\varphi_1+\left(R_1^{N-k}-R_1^{N+k}\right)\psi_1 \right. \\ &\left. -\frac{R_1^{N-k}-R_1^{N+k}}{1-R_1^2}\sum_{j=1}^{N-1}h^2\left(R_1^{N-j}-R_1^{N+j}\right)z_j\right\} \\ &+\frac{1}{1-R_1^2}\sum_{j=1}^{N-1}h^2\left(R_1^{|k-j|}-R_1^{k+j}\right)z_j \end{aligned} \tag{3.4}$$

for the solution of the boundary value problem

$$-\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}+\mu u_k = z_k,$$

$$z_k = z\left(x_k\right),\ 1 \leq k \leq N-1,$$

$$u_0 = \varphi_1,\ u_N = \psi_1$$

for second order of linear difference equations.

We have the following formula

$$z_k = T_{h2}\left\{\left(R_2^k - R_2^{2N-k}\right)\varphi_2 + \left(R_2^{N-k} - R_2^{N+k}\right)\psi_2 - \frac{R_2^{N-k} - R_2^{N+k}}{1-R_2^2}\sum_{l=1}^{N-1} h^2\left(R_2^{N-l} - R_2^{N+l}\right) f_l\right\} + \frac{1}{1-R_2^2}\sum_{l=1}^{N-1} h^2\left(R_2^{|k-l|} - R_2^{k+l}\right) f_l \tag{3.5}$$

for the solution of the boundary value problem

$$-\frac{z_{k+1} - 2z_k + z_{k-1}}{h^2} - \mu z_k = f_k,$$

$$f_k = f\left(x_k\right),\ 1 \le k \le N-1,$$

$$z_0 = \varphi_2,\ z_N = \psi_2$$

for second order linear difference equations.

Applying (3.4), (3.5) and the boundary conditions $u_0 = 0$, $u_N = 0$ we get,

$$\begin{aligned} u_k = & -\frac{T_{h2}}{1-R_1^2}\sum_{j=1}^{N-1} h^2\left[T_{h1}\left(R_1^{N-k} - R_1^{N+k}\right)\left(R_1^{N-j} - R_1^{N+j}\right)\right. \\ & \left.-\left(R_1^{|k-j|} - R_1^{k+j}\right)\right]\left[\left(R_2^j - R_2^{2N-j}\right)\varphi_2 + \left(R_2^{N-j} - R_2^{N+j}\right)\psi_2\right] \\ & + \frac{1}{\left(1-R_1^2\right)\left(1-R_2^2\right)}\sum_{j=1}^{N-1}\sum_{l=1}^{N-1} h^4\left[T_{h1}\left(R_1^{N-k} - R_1^{N+k}\right)\right. \\ & \left.\times\left(R_1^{N-j} - R_1^{N+j}\right) - \left(R_1^{|k-j|} - R_1^{k+j}\right)\right]\left[T_{h2}\left(R_2^{N-j} - R_2^{N+j}\right)\right. \\ & \left.\times\left(R_2^{N-l} - R_2^{N+l}\right) - \left(R_2^{|j-l|} - R_2^{j+l}\right)\right] f_l, \end{aligned} \tag{3.6}$$

applying the boundary conditions $-u_2 + u_1 - u_0 = u_{N-2} - 2u_{N-1} + u_N = 0$ we get,

$$\begin{aligned} & -\frac{T_{h2}}{1-R_1^2}\sum_{j=1}^{N-1} h^2\left[T_{h1}\left(-R_1^{N-2} + R_1^{N+2} + 2R_1^{N-1} - 2R_1^{N+1}\right)\right. \\ & \left.\times\left(R_1^{N-j} - R_1^{N+j}\right) - \left(-R_1^{|j-2|} + R_1^{j+2} + 2R_1^{j-1} - 2R_1^{j+1}\right)\right] \\ & \times\left[\left(R_2^j - R_2^{2N-j}\right)\varphi_2 + \left(R_2^{N-j} - R_2^{N+j}\right)\psi_2\right] \end{aligned}$$

$$+\frac{1}{\left(1-R_1^2\right)\left(1-R_2^2\right)}\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h^4\left[T_{h1}\left(-R_1^{N-2}+R_1^{N+2}\right.\right.$$
$$\left.+2R_1^{N-1}-2R_1^{N+1}\right)\left(R_1^{N-j}-R_1^{N+j}\right)-\left(-R_1^{|j-2|}+R_1^{j+2}\right.$$
$$\left.\left.+2R_1^{N-1}-2R_1^{N+1}\right)\right]$$

$$\times\left[T_{h2}\left(R_2^{N-j}-R_2^{N+j}\right)\left(R_2^{N-l}-R_2^{N+l}\right)-\left(R_2^{|j-l|}-R_2^{j+l}\right)f_l\right]=0,$$

and

$$-\frac{T_{h2}}{1-R_1^2}\sum_{j=1}^{N-1}h^2\left[T_{h1}\left(R_1^2-R_1^{2N-2}-2R_1+2R_1^{2N-1}\right)\right.$$
$$\left.\times\left(R_1^{N-j}-R_1^{N+j}\right)-\left(R_1^{|N-j-2|}-R_1^{N+j-2}-2R_1^{N-j-1}+2R_1^{N+j-1}\right)\right]$$
$$\times\left[\left(R_2^{j}-R_2^{2N-j}\right)\varphi_2+\left(R_2^{N-j}-R_2^{N+j}\right)\psi_2\right]$$

$$+\frac{1}{\left(1-R_1^2\right)\left(1-R_2^2\right)}\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h^4\left[T_{h1}\left(R_1^2-R_1^{2N-2}\right.\right.$$
$$\left.-2R_1+2R_1^{2N-1}\right)\left(R_1^{N-j}-R_1^{N+j}\right)-\left(R_1^{|N-j-2|}-R_1^{N+j-2}\right.$$
$$\left.\left.+-2R_1^{N-j-1}+2R_1^{N+j-1}\right)\right]$$

$$\times\left[T_{h2}\left(R_2^{N-j}-R_2^{N+j}\right)\left(R_2^{N-l}-R_2^{N+l}\right)-\left(R_2^{|j-l|}-R_2^{j+l}\right)f_l\right]=0.$$

Solving last system of equations we obtain,

$$\varphi_2 = \frac{R_1^{-2}R_2^{-2}}{1-R_2^2}\left\{\left[T_{h1}\left(1-R_1\right)^2\sum_{j=1}^{N-1}\left(R_1^{j}-R_1^{2N-j}\right)\right.\right. \tag{3.7}$$
$$\left.\times\left(R_2^{N-j}-R_2^{N+j}\right)-\left(R_2^{N-1}-R_2^{N+1}\right)\right]$$
$$\times\left[-T_{h1}\left(1-R_1\right)^2\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h^2\left(R_1^{N-j}-R_1^{N+j}\right)\right.$$

$$\times \left[T_{h2}\left(R_2^{N-j}-R_2^{N+j}\right)\left(R_2^{N-l}-R_2^{N+l}\right)-\left(R_2^{|j-l|}-R_2^{j+l}\right)\right]f_l$$
$$\left.-T_{h2}\left(1-R_2^2\right)\sum_{l=1}^{N-1}h^2\left(R_2^{N-l-1}-R_2^{N+l-1}\right)f_l\right]$$
$$+\left[T_{h1}\left(1-R_1\right)^2\sum_{j=1}^{N-1}\left(R_1^{N-j}-R_1^{N+j}\right)\left(R_2^{N-j}-R_2^{N+j}\right)\right.$$

$$\left.-\left(R_2-R_2^{2N-1}\right)\right]\left[T_{h1}\left(1-R_1\right)^2\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h^2\right.$$
$$\left(R_1^{j}-R_1^{2N-j}\right)\left[T_{h2}\left(R_2^{N-j}-R_2^{N+j}\right)\left(R_2^{N-l}-R_2^{N+l}\right)\right.$$
$$\left.-\left(R_2^{|j-l|}-R_2^{j+l}\right)\right]f_l+T_{h2}\left(1-R_2^2\right)$$
$$\left.\left.\times\sum_{l=1}^{N-1}h^2\left(R_2^{l-1}-R_2^{2N-l-1}\right)f_l\right]\right\}$$

$$\times\left\{1-R_2^{2N-4}+2T_{h1}\left(1-R_1\right)^2\left[\left(R_1^N+R_2^{N-2}\right)\right.\right.$$
$$\left.\times\frac{R_1^{N-1}-R_2^{N-1}}{R_1-R_2}-\left(1+R_1^NR_2^{N-2}\right)\frac{1-R_1^{N-1}R_2^{N-1}}{1-R_1R_2}\right]$$
$$+T_{h1}\left(1-R_1\right)^4\left[\left(\frac{1-R_1^{N-1}R_2^{N-1}}{1-R_1R_2}\right)^2\right.$$
$$\left.\left.-\left(\frac{R_1^{N-1}-R_2^{N-1}}{R_1-R_2}\right)^2\right]\right\}^{-1},$$

and

$$\psi_2 = \frac{R_1^{-2}R_2^{-2}}{1-R_2^2}\left\{\left[T_{h1}\left(1-R_1\right)^2\sum_{j=1}^{N-1}\left(R_1^{N-j}-R_1^{N+j}\right)\right.\right. \tag{3.8}$$
$$\left.\times\left(R_2^{j}-R_2^{2N-j}\right)-\left(R_2^{N-1}-R_2^{N+1}\right)\right]$$
$$\times\left[T_{h1}\left(1-R_1\right)^2\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h^2\left(R_1^{j}-R_1^{2N-j}\right)\right.$$

$$\times \left[T_{h2}\left(R_2^{N-j}-R_2^{N+j}\right)\left(R_2^{N-l}-R_2^{N+l}\right)-\left(R_2^{|j-l|}-R_2^{j+l}\right)\right] f_l$$
$$+T_{h2}\left(1-R_2^2\right)\sum_{l=1}^{N-1} h^2\left(R_2^{l-1}-R_2^{2N-l-1}\right) f_l\Bigg]$$
$$-\left[T_{h1}\left(1-R_1\right)^2\sum_{j=1}^{N-1}\left(R_1^{j}-R_1^{2N-j}\right)\left(R_2^{j}-R_2^{2N-j}\right)\right.$$

$$-\left(R_2-R_2^{2N-1}\right)\Big]\left[-T_{h1}\left(1-R_1\right)^2\sum_{j=1}^{N-1}\sum_{l=1}^{N-1} h^2\right.$$
$$\times\left(R_1^{N-j}-R_1^{N+j}\right)\left[T_{h2}\left(R_2^{N-j}-R_2^{N+j}\right)\left(R_2^{N-l}-R_2^{N+l}\right)\right.$$
$$-\left(R_2^{|j-l|}-R_2^{j+l}\right)\Big] f_l-T_{h2}\left(1-R_2^2\right)$$
$$\times\sum_{l=1}^{N-1} h^2\left(R_2^{N-l-1}-R_2^{N+l-1}\right)\Bigg]\Bigg\}$$

$$\times\Big\{1-R_2^{2N-4}+2T_{h1}\left(1-R_1\right)^2\Big[\left(R_1^N+R_2^{N-2}\right)$$
$$\times\frac{R_1^{N-1}-R_2^{N-1}}{R_1-R_2}-\left(1+R_1^N R_2^{N-2}\right)\frac{1-R_1^{N-1}R_2^{N-1}}{1-R_1R_2}\Bigg]$$
$$+T_{h1}\left(1-R_1\right)^4\left[\left(\frac{1-R_1^{N-1}R_2^{N-1}}{1-R_1R_2}\right)^2\right.$$
$$\left.-\left(\frac{R_1^{N-1}-R_2^{N-1}}{R_1-R_2}\right)^2\right]\Bigg\}^{-1}.$$

Let

$$\begin{aligned} R \;=\; & \Big\{1-R_2^{2N-4}+2T_{h1}\left(1-R_1\right)^2\Big[\left(R_1^N+R_2^{N-2}\right) \\ & \times\frac{R_1^{N-1}-R_2^{N-1}}{R_1-R_2}-\left(1+R_1^N R_2^{N-2}\right)\frac{1-R_1^{N-1}R_2^{N-1}}{1-R_1R_2}\Bigg] \\ & +T_{h1}\left(1-R_1\right)^4\left[\left(\frac{1-R_1^{N-1}R_2^{N-1}}{1-R_1R_2}\right)^2\right. \\ & \left.-\left(\frac{R_1^{N-1}-R_2^{N-1}}{R_1-R_2}\right)^2\right]\Bigg\}^{-1}, \end{aligned}$$

$$\Delta = I - R_2^{2N-4} - R_1^{2N} + R_1^{2N} R_2^{2N-4},$$

and

$$
\begin{aligned}
\beta &= 2\mu_1^2 R_1 \left[\left(R_1^N + R_2^{N-2}\right) \frac{R_1^{N-1} - R_2^{N-1}}{\left(\mu_2 - \mu_1\right) R_2} - \left(1 + R_1^N R_2^{N-2}\right)\right. \\
&\quad \left. \times \frac{1 - R_1^{N-1} R_2^{N-1}}{\left(\mu_1 + \mu_2 + h\mu_1\mu_2\right) R_2}\right] - \left(1 - R_1\right) \mu_1^3 R_1 \\
&\quad \times \left[\left(\frac{1 - R_1^{N-1} R_2^{N-1}}{\left(\mu_1 + \mu_2 + h\mu_1\mu_2\right) R_2}\right)^2 - \left(\frac{R_1^{N-1} - R_2^{N-1}}{\left(\mu_2 - \mu_1\right) R_2}\right)^2\right].
\end{aligned}
$$

Then

$$
R = T_{h1}^{-1} \Delta^{-1} \left(1 + h\beta\Delta^{-1}\right)^{-1}.
$$

$$
\begin{aligned}
R &= T_{h1}^{-1} \left\{\left(1 - R_2^{2N-4}\right)\left(1 - R_1^{2N}\right) + 2\left(1 - \frac{1}{1 + \mu_1 h}\right)\right. \\
&\quad \times \left[\left(R_1^N + R_2^{N-2}\right)\left(R_1^{N-1} - R_2^{N-1}\right) \frac{1 - \frac{1}{1+\mu_1 h}}{\frac{1}{1+\mu_1 h} - \frac{1}{1+\mu_2 h}}\right. \\
&\quad \left. - \left(1 + R_1^N R_2^{N-2}\right)\left(1 - R_1^{N-1} R_2^{N-1}\right) \frac{1 - \frac{1}{1+\mu_1 h}}{1 - \frac{1}{1+\mu_1 h} \frac{1}{1+\mu_2 h}}\right] \\
&\quad + \left(1 - R_1\right)\left(1 - \frac{1}{1 + \mu_1 h}\right)\left[\left(1 - R_1^{N-1} R_2^{N-1}\right)^2\right. \\
&\quad \times \left(\frac{1 - \frac{1}{1+\mu_1 h}}{1 - \frac{1}{1+\mu_1 h} \frac{1}{1+\mu_2 h}}\right)^2 - \left(R_1^{N-1} - R_2^{N-1}\right)^2 \\
&\quad \left. \left. \times \left(\frac{1 - \frac{1}{1+\mu_1 h}}{\frac{1}{1+\mu_1 h} - \frac{1}{1+\mu_2 h}}\right)^2\right]\right\}^{-1} \\
&\quad \left. \left. \times \left(\frac{1 - \frac{1}{1+\mu_1 h}}{\frac{1}{1+\mu_1 h} - \frac{1}{1+\mu_2 h}}\right)^2\right]\right\}^{-1} \\
&= T_{h1}^{-1} (\Delta + h\beta)^{-1},
\end{aligned}
$$

and

$$
\Delta^{-1} - (\Delta + h\beta)^{-1} = h\beta\Delta^{-1}(\Delta + h\beta)^{-1},
$$

$$
(\Delta + h\beta)^{-1} = \Delta^{-1}\left(1 + h\beta\Delta^{-1}\right)^{-1}.
$$

Finally, applying the formulas (3.6), (3.7), (3.8) and formula for R, we obtain

$$
\begin{aligned}
u_k &= \frac{\left(1-R_1^{2N}\right)(1+\mu_1 h)(1+\mu_2 h)}{\mu_1\mu_2 R_1 R_2 (2+\mu_1 h)(2+\mu_2 h)} \\
&\quad \times \left\{ \left[\left(\left[T_{h1} (1-R_1)^2 \right. \right. \right. \right. \\
&\quad \times \sum_{j=1}^{N-1} \left(R_1^{N-j-1} - R_1^{N+j-1} \right) R_2^{N-j-1} - 1 \Bigg] \\
&\quad \times \left[\sum_{j=1}^{N-1} h \left(T_{h1} \left(R_1^{N-k} - R_1^{N+k} \right) \right. \right. \\
&\quad \times \left(R_1^{N-j} - R_1^{N+j} \right) - \left(R_1^{|k-j|} - R_1^{k+j} \right) \Big) R_2^{j-1} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&- \left[T_{h1} (1-R_1)^2 \sum_{j=1}^{N-1} \left(R_1^{N-j-1} - R_1^{N+j-1} \right) R_2^{j-1} - R_2^{N-2} \right] \\
&\times \left[\sum_{j=1}^{N-1} h \left(T_{h1} \left(R_1^{N-k} - R_1^{N+k} \right) \left(R_1^{N-j} - R_1^{N+j} \right) \right. \right. \\
&- \left(R_1^{|k-j|} - R_1^{k+j} \right) \Big) R_2^{N-j-1} \Bigg] \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(T_{h1} (1-R_1)^2 \sum_{j=1}^{N-1} \sum_{l=1}^{N-1} h \left(R_1^{j-1} - R_1^{2N-j-1} \right) \left[T_{h2} \right. \right. \\
&\times \left(R_2^{N-j} - R_2^{N+j} \right) \left(R_2^{N-l} - R_2^{N+l} \right) - \left(R_2^{|j-l|} - R_2^{j+l} \right) \Big] f_l \\
&+ \left(1 - R_2^2 \right) \sum_{l=1}^{N-1} h \left(R_2^{l-1} - R_2^{2N-l-1} \right) f_l \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\left[T_{h1} (1-R_1)^2 \sum_{j=1}^{N-1} \left(R_1^{j-1} - R_1^{2N-j-1} \right) R_2^{N-j-1} - R_2^{N-2} \right] \right. \\
&\times \left[\sum_{j=1}^{N-1} h \left(T_{h1} \left(R_1^{N-k} - R_1^{N+k} \right) \left(R_1^{N-j} - R_1^{N+j} \right) \right. \right. \\
&- \left(R_1^{|k-j|} - R_1^{k+j} \right) \Big) R_2^{j-1} \Bigg]
\end{aligned}
$$

$$+\left[T_{h1}\left(1-R_{1}\right)^{2}\sum_{j=1}^{N-1}\left(R_{1}^{j-1}-R_{1}^{2N-j-1}\right)R_{2}^{j-1}-1\right]$$

$$\times\sum_{j=1}^{N-1}h\left(T_{h1}\left(R_{1}^{N-k}-R_{1}^{N+k}\right)\left(R_{1}^{N-j}-R_{1}^{N+j}\right)\right.$$

$$\left.\left.\left.-\left(R_{1}^{|k-j|}-R_{1}^{k+j}\right)\right)R_{2}^{N-j-1}\right]\right)$$

$$\times\left(T_{h1}\left(1-R_{1}\right)^{2}\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h\left(R_{1}^{N-j-1}-R_{1}^{N+j-1}\right)\left[T_{h2}\right.\right.$$

$$\left.\times\left(R_{2}^{N-j}-R_{2}^{N+j}\right)\left(R_{2}^{N-l}-R_{2}^{N+l}\right)-\left(R_{2}^{|j-l|}-R_{2}^{j+l}\right)\right]f_{l}$$

$$\left.\left.+\left(1-R_{2}^{2}\right)\sum_{l=1}^{N-1}h\left(R_{2}^{N-l-1}-R_{2}^{N+l-1}\right)f_{l}\right)\right\}$$

$$\times\Delta^{-1}\left(1+h\beta\Delta^{-1}\right)^{-1}$$

$$+\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h^{2}\left[T_{h1}\left(R_{1}^{N-k}-R_{1}^{N+k}\right)\left(R_{1}^{N-j}-R_{1}^{N+j}\right)\right.$$

$$\left.-\left(R_{1}^{|k-j|}-R_{1}^{k+j}\right)\right]\left[T_{h2}\left(R_{2}^{N-j}-R_{2}^{N+j}\right)\right.$$

$$\left.\left.\times\left(R_{2}^{N-l}-R_{2}^{N+l}\right)-\left(R_{2}^{|j-l|}-R_{2}^{j+l}\right)\right]f_{l}\right].$$

Lemma 3.1 is proved. ■

The grid function $J(k,j,l;\lambda+\delta)$ is called the Green's function of the resolvent equation (3.2).

Thus, we obtain the formula for the resolvent $(\lambda I+A_h^x)^{-1}$ in the case $\lambda\geq 0$. In the same way we can obtain a formula for the resolvent $(\lambda I+A_h^x)^{-1}$ in the case of complex λ. But we need to obtain that $(2+\mu_1 h),(2+\mu_2 h),\Delta,$ and $(1+h\beta\Delta^{-1})$ are not equal to zero.

3.3 Positivity of A_h^x in C_h

Theorem 3.1 *For all* λ, $\lambda\in R_{\varphi}=\left\{\lambda:|\arg\lambda|\leq\varphi,\quad|\lambda|=\rho,\quad\varphi\leq\frac{\pi}{2}\right\}$ *the resolvent* $(\lambda I+A_h^x)^{-1}$ *defined by the formula* (3.3) *is subject to the bound*

$$\left\|(\lambda I+A_h^x)^{-1}\right\|_{C_h\to C_h}\leq M\left(\varphi,\delta\right)\left(1+|\lambda|\right)^{-1},$$

where $M(\varphi,\delta)$ *does not depend on* h.

The proof of this theorem is based on the following lemmas

Lemma 3.2 *The following estimates hold*

$$|\mu_1| \geq \sqrt[4]{|\lambda+\delta|},$$

and

$$|\mu_2| \geq \sqrt[4]{|\lambda+\delta|}.$$

Proof. Using the formula for μ_1, we have that

$$\left|\frac{\mu_1}{\sqrt{i}\sqrt[4]{\lambda+\delta}}\right| = \left|\frac{h}{2}\sqrt{i}\sqrt[4]{\lambda+\delta} + \sqrt{1+i\frac{h^2}{4}\sqrt{\lambda+\delta}}\right|.$$

We denote that $\lambda+\delta = re^{i\varphi}$, here $r = |\lambda+\delta|$. Then

$$i\sqrt{\lambda+\delta} = \sqrt{r}e^{i\psi_1},$$

here $\psi_1 = \frac{\varphi}{2}+\frac{\pi}{2}$.

Now since h is sufficiently small, we have

$$\begin{aligned}\left|\frac{\mu_1}{\sqrt{i}\sqrt[4]{\lambda+\delta}}\right| &= \left|\frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_1}{2}} + \sqrt{1+\frac{h^2}{4}\sqrt{r}e^{i\psi_1}}\right| \\ &\geq \left|\frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_1}{2}} + 1 - \frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_1}{2}}\right| \\ &\geq 1.\end{aligned}$$

Therefore

$$\begin{aligned}|\mu_1| &\geq \left|\sqrt{i}\sqrt[4]{\lambda+\delta}\right| \\ &= \left|\sqrt[4]{\lambda+\delta}\right|.\end{aligned}$$

Using the notations above, we get

$$\begin{aligned}\left|\frac{\mu_2}{\sqrt{-i\sqrt[4]{\lambda+\delta}}}\right| &= \left|\frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_2}{2}} + \sqrt{1+\frac{h^2}{4}\sqrt{r}e^{i\psi_2}}\right| \\ &\geq \left|\frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_2}{2}} + 1 - \frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_2}{2}}\right| \\ &\geq 1.\end{aligned}$$

$$-i\sqrt{\lambda+\delta} = \sqrt{r}e^{i\psi_2},$$

since h is sufficiently small, we get

$$\begin{aligned}\left|\frac{\mu_2}{\sqrt{-i\sqrt[4]{\lambda+\delta}}}\right| &= \left|\frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_2}{2}} + \sqrt{1+\frac{h^2}{4}\sqrt{r}e^{i\psi_2}}\right| \\ &\geq \left|\frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_2}{2}} + 1 - \frac{h}{2}\sqrt[4]{r}e^{i\frac{\psi_2}{2}}\right| \\ &\geq 1.\end{aligned}$$

Therefore

$$\begin{aligned}|\mu_2| &\geq \left|\sqrt{-i\sqrt[4]{\lambda+\delta}}\right| \\ &= \left|\sqrt[4]{\lambda+\delta}\right|.\end{aligned}$$

Lemma 3.2 is proved. ■

Lemma 3.3 *The following intervals for μ_1 and μ_2 are satisfied.*

$$\frac{\pi}{4} \leq \arg\mu_1 \leq \frac{3\pi}{8},$$

and

$$-\frac{\pi}{4} \leq \arg\mu_2 \leq \frac{\pi}{8}.$$

Proof. We denote $\mu_1 = |\mu_1|\, e^{i\arg\mu_1}$. Since h is sufficiently small,

$$\begin{aligned}\arg\mu_1 &= \arg(\sqrt{i\sqrt[4]{\lambda+\delta}}) \\ &\leq \frac{\varphi}{4} + \frac{\pi}{4},\end{aligned}$$

then

$$\frac{\pi}{4} \leq \arg\mu_1 \leq \frac{3\pi}{8},$$

and, we denote $\mu_2 = |\mu_2| e^{i \arg \mu_2}$. Since h is sufficiently small,

$$\begin{aligned} \arg \mu_2 &= \arg(\sqrt{-i}\sqrt[4]{\lambda+\delta}) \\ &\leq \frac{\varphi}{4} - \frac{\pi}{4}, \end{aligned}$$

then

$$-\frac{\pi}{4} \leq \arg \mu_2 \leq \frac{\pi}{8}.$$

Lemma 3.3 is proved. ■

Lemma 3.4 *The following estimates hold.*

$$\begin{aligned} |R_1| &\leq \frac{1}{1+\sqrt[4]{|\lambda+\delta|}h\cos\varphi_1} \\ &\leq 1, \end{aligned}$$

where $\varphi_1 = \arg \mu_1$, *and*

$$\begin{aligned} |R_2| &\leq \frac{1}{1+\sqrt[4]{|\lambda+\delta|}h\cos\varphi_2} \\ &\leq 1, \end{aligned}$$

where $\varphi_2 = \arg \mu_2$.

Proof. We have the following estimate for $|1+\mu_1 h|$, $|1+\mu_2 h|$

$$\begin{aligned} |1+\mu_1 h| &= \left\{(1+h|\mu_1|\cos\varphi_1)^2 + (h|\mu_1|\sin\varphi_1)^2\right\}^{\frac{1}{2}} \\ &= \left\{1+2h|\mu_1|\cos\varphi_1 + h^2|\mu_1|^2\right\}^{\frac{1}{2}} \\ &\geq 1+h|\mu_1|\cos\varphi_1, \end{aligned}$$

$$\begin{aligned} |1+\mu_2 h| &= \left\{(1+h|\mu_2|\cos\varphi_2)^2 + (h|\mu_2|\sin\varphi_2)^2\right\}^{\frac{1}{2}} \\ &= \left\{1+2h|\mu_2|\cos\varphi_2 + h^2|\mu_2|^2\right\}^{\frac{1}{2}} \\ &\geq 1+h|\mu_2|\cos\varphi_2. \end{aligned}$$

Using the formula of R_1, R_2 and the notations of lemma 3.3, we obtain

$$\begin{aligned} |R_1| &= \frac{1}{|1+\mu_1 h|} \\ &\leq \frac{1}{1+h\,|\mu_1|\cos\varphi_1} \\ &\leq 1, \end{aligned}$$

and

$$\begin{aligned} |R_2| &= \frac{1}{|1+\mu_2 h|} \\ &\leq \frac{1}{1+h\,|\mu_2|\cos\varphi_2} \\ &\leq 1. \end{aligned}$$

Lemma 3.4 is proved. ■

Lemma 3.5 *The following estimate holds.*

$$\frac{\left|1-R_1^{2N}\right| |1+\mu_1 h|\,|1+\mu_2 h|}{|2+\mu_1 h|\,|2+\mu_2 h|} \leq 2.$$

Proof.

$$\begin{aligned} \left|1-R_1^{2N}\right| &\leq 1+\left|R_1^{2N}\right| \\ &\leq 2. \end{aligned}$$

Let $\varphi_1 = \arg\mu_1$. Using the notations of Lemma 3.3, we obtain

$$\begin{aligned} \left|\frac{1+\mu_1 h}{2+\mu_1 h}\right| &= \frac{\sqrt{1+|\mu_1|\,h\,(\cos\varphi_1+i\sin\varphi_1)}}{\sqrt{2+|\mu_1|\,h\,(\cos\varphi_1+i\sin\varphi_1)}} \\ &= \sqrt{\frac{(1+|\mu_1|\,h\cos\varphi_1)^2+(|\mu_1|\,h\sin\varphi_1)^2}{(2+|\mu_1|\,h\cos\varphi_1)^2+(|\mu_1|\,h\sin\varphi_1)^2}} \\ &= \sqrt{\frac{1+2\,|\mu_1|\,h\cos\varphi_1+|\mu_1|^2\,h^2}{4+2\,|\mu_1|\,h\cos\varphi_1+|\mu_1|^2\,h^2}} \\ &\leq 1. \end{aligned}$$

Let $\varphi_2 = \arg\mu_2$. Using the notations of Lemma 3.3, we obtain

$$\begin{aligned}\left|\frac{1+\mu_2 h}{2+\mu_2 h}\right| &= \frac{\sqrt{1+|\mu_2|\,h\,(\cos\varphi_2 + i\sin\varphi_2)}}{\sqrt{2+|\mu_2|\,h\,(\cos\varphi_2 + i\sin\varphi_2)}} \\ &= \sqrt{\frac{(1+|\mu_2|\,h\cos\varphi_2)^2 + (|\mu_2|\,h\sin\varphi_2)^2}{(2+|\mu_2|\,h\cos\varphi_2)^2 + (|\mu_2|\,h\sin\varphi_2)^2}} \\ &= \sqrt{\frac{1+2\,|\mu_2|\,h\cos\varphi_2 + |\mu_2|^2 h^2}{4+2\,|\mu_2|\,h\cos\varphi_2 + |\mu_2|^2 h^2}} \\ &\le 1.\end{aligned}$$

Lemma 3.5 is proved. ■

Lemma 3.6 *The following estimates hold.*

$$\left|R_1^{N-j-1} - R_1^{N+j-1}\right| \le 2\left|R_1^{N-j-1}\right|,$$

$$\left|R_1^{N-k} - R_1^{N+k}\right| \le 2,$$

$$\left|R_1^{j-1} - R_1^{2N-j-1}\right| \le 2\left|R_1^{j-1}\right|,$$

$$\left|R_2^{N-j} - R_2^{N+j}\right| \le 2\left|R_2^{N-j}\right|,$$

$$\left|R_2^{N-l} - R_2^{N+l}\right| \le 2\left|R_2^{N-l}\right|,$$

$$\left|R_2^{l-1} - R_2^{2N-l-1}\right| \le 2\left|R_2^{l-1}\right|.$$

Proof. Since $1 \le j \le N-1$, we have

$$\begin{aligned}\left|R_1^{N-j-1} - R_1^{N+j-1}\right| &= \left|R_1^{N-j-1}\right|\left|1 - R_1^{2j}\right| \\ &\le \left|R_1^{N-j-1}\right|\left(1 + \left|R_1^{2j}\right|\right) \\ &\le 2\left|R_1^{N-j-1}\right|,\end{aligned}$$

since $0 \le k \le N$, we have

$$\begin{aligned}\left|R_1^{N-k} - R_1^{N+k}\right| &= \left|R_1^{N-k}\right|\left|1 - R_1^{2k}\right| \\ &\le \left|R_1^{N-k}\right|\left(1 + \left|R_1^{2k}\right|\right) \\ &\le 2,\end{aligned}$$

$$\begin{aligned}
\left|R_1^{j-1} - R_1^{2N-j-1}\right| &= \left|R_1^{j-1}\right| \left|1 - R_1^{2(N-j)}\right| \\
&\leq \left|R_1^{j-1}\right| \left(1 + \left|R_1^{2(N-j)}\right|\right) \\
&\leq 2\left|R_1^{j-1}\right|,
\end{aligned}$$

$$\begin{aligned}
\left|R_2^{N-j} - R_2^{N+j}\right| &\leq \left|R_2^{N-j}\right| \left|1 - R_2^{2j}\right| \\
&\leq \left|R_2^{N-j}\right| \left(1 + \left|R_2^{2j}\right|\right) \\
&\leq 2\left|R_2^{N-j}\right|,
\end{aligned}$$

since $1 \leq l \leq N-1$, we have

$$\begin{aligned}
\left|R_2^{N-l} - R_2^{N+l}\right| &\leq \left|R_2^{N-l}\right| \left|1 - R_2^{2l}\right| \\
&\leq \left|R_2^{N-l}\right| \left(1 + \left|R_2^{2l}\right|\right) \\
&\leq 2\left|R_2^{N-l}\right|,
\end{aligned}$$

$$\begin{aligned}
\left|R_2^{l-1} - R_2^{2N-l-1}\right| &\leq \left|R_2^{l-1}\right| \left|1 - R_2^{2(N-l)}\right| \\
&\leq \left|R_2^{l-1}\right| \left(1 + \left|R_2^{2(N-l)}\right|\right) \\
&\leq 2\left|R_2^{l-1}\right|.
\end{aligned}$$

Lemma 3.6 is proved. ■

Lemma 3.7 *The following estimates hold.*

$$\left|R_1^{|k-j|} - R_1^{k+j}\right| \leq 2\left|R_1^{k-j}\right|$$

for $1 \leq j \leq k$,

$$\left|R_1^{|k-j|} - R_1^{k+j}\right| \leq 2\left|R_1^{j-k}\right|$$

for $k+1 \leq j \leq N-1$,

$$\left|R_1^{|j-l|} - R_1^{j+l}\right| \leq 2\left|R_1^{j-l}\right|$$

for $1 \leq l \leq j$ *and*

$$\left|R_1^{|j-l|} - R_1^{j+l}\right| \leq 2\left|R_1^{l-j}\right|$$

for $j{+}1 \leq l \leq N-1$.

Proof. for $1 \leq j \leq k$:

$$\begin{aligned}\left|R_1^{|k-j|} - R_1^{k+j}\right| &= \left|R_1^{k-j} - R_1^{k+j}\right| \\ &= \left|R_1^{k-j}\right|\left(1 - R_1^{2j}\right) \\ &\leq 2\left|R_1^{k-j}\right|,\end{aligned}$$

for $k+1 \leq j \leq N-1$:

$$\begin{aligned}\left|R_1^{|k-j|} - R_1^{k+j}\right| &= \left|R_1^{j-k} - R_1^{k+j}\right| \\ &= \left|R_1^{j-k}\right|\left(1 - R_1^{2k}\right) \\ &\leq 2\left|R_1^{j-k}\right|,\end{aligned}$$

for $1 \leq l \leq j$:

$$\begin{aligned}\left|R_1^{|j-l|} - R_1^{j+l}\right| &= \left|R_1^{j-l} - R_1^{l+j}\right| \\ &= \left|R_1^{j-l}\right|\left(1 - R_1^{2l}\right) \\ &\leq 2\left|R_1^{j-l}\right|,\end{aligned}$$

for $j{+}1 \leq l \leq N-1$

$$\begin{aligned}\left|R_1^{|j-l|} - R_1^{j+l}\right| &= \left|R_1^{l-j} - R_1^{l+j}\right| \\ &= \left|R_1^{l-j}\right|\left(1 - R_1^{2j}\right) \\ &\leq 2\left|R_1^{l-j}\right|.\end{aligned}$$

Lemma 3.7 is proved. ■

Lemma 3.8 *The following estimates hold.*

$$|T_{h1}| \leq \frac{1}{1 - |R_1|^{2N}},$$

and

$$|T_{h2}| \leq \frac{1}{1 - |R_2|^{2N}}.$$

Proof. Using the formula of T_{h1}, we get

$$\begin{aligned} |T_{h1}| &= \frac{1}{|1-R_1^{2N}|} \\ &\leq \frac{1}{1-|R_1|^{2N}}. \end{aligned}$$

Using the formula of T_{h2}, we get

$$\begin{aligned} |T_{h2}| &= \frac{1}{|1-R_2^{2N}|} \\ &\leq \frac{1}{1-|R_2|^{2N}}. \end{aligned}$$

Lemma 3.8 is proved. ■

Lemma 3.9 *The following estimate holds.*

$$\left|\Delta^{-1}\right| \left|\left(1+h\beta\Delta^{-1}\right)^{-1}\right| \leq C(\varphi,\delta).$$

Proof. Using the formula for Δ^{-1}, we get

$$\begin{aligned} \left|\Delta^{-1}\right| &= \left|1-R_2^{2N-4}-R_1^{2N}+R_1^{2N}R_2^{2N-4}\right|^{-1} \\ &\leq C_1. \end{aligned}$$

Using the formula for β, we get

$$\begin{aligned} |\beta| &= \left|2\mu_1^2R_1\left[\left(R_1^N+R_2^{N-2}\right)\frac{R_1^{N-1}-R_2^{N-1}}{(\mu_2-\mu_1)R_2}-\left(1+R_1^NR_2^{N-2}\right)\right.\right. \\ &\quad \times \left.\frac{1-R_1^{N-1}R_2^{N-1}}{(\mu_1+\mu_2+h\mu_1\mu_2)R_2}\right]-(1-R_1)\mu_1^3R_1 \\ &\quad \times \left.\left[\left(\frac{1-R_1^{N-1}R_2^{N-1}}{(\mu_1+\mu_2+h\mu_1\mu_2)R_2}\right)^2-\left(\frac{R_1^{N-1}-R_2^{N-1}}{(\mu_2-\mu_1)R_2}\right)^2\right]\right| \\ &\leq C_2. \end{aligned}$$

Therefore for sufficiently small h, we obtain

$$\left|\Delta^{-1}\right| \left|\left(1+h\beta\Delta^{-1}\right)^{-1}\right| \leq \frac{|\Delta^{-1}|}{1-h|\beta||\Delta^{-1}|}$$

$$
\begin{aligned}
&\leq \frac{C_1}{1-hC_1C_2} \\
&= C(\varphi,\delta).
\end{aligned}
$$

Lemma 3.9 is proved. ■

Lemma 3.10 *The following estimates hold.*

$$\frac{h}{1-|R_1|} \leq \frac{1}{|\mu_1|\cos\varphi_1},$$

$$\frac{|1-R_1|}{1-|R_1|} \leq \frac{1}{\cos\varphi_1},$$

where $\varphi_1 = \arg\mu_1,$ *and*

$$\frac{h}{1-|R_2|} \leq \frac{1}{|\mu_2|\cos\varphi_2},$$

where $\varphi_2 = \arg\mu_2.$

Proof. Using the notations on Lemma 3.4, and since h is sufficiently small, we obtain

$$
\begin{aligned}
\frac{h}{1-|R_1|} &\leq \frac{h\left(1+|\mu_1|\,h\cos\varphi_1\right)}{|\mu_1|\,h\cos\varphi_1} \\
&\leq \frac{1}{|\mu_1|\cos\varphi_1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{|1-R_1|}{1-|R_1|} &\leq \frac{\left|1-\frac{1}{1+\mu_1 h}\right|}{1-\frac{1}{1+|\mu_1|h\cos\varphi_1}} \\
&= \frac{1+|\mu_1|\,h\cos\varphi_1}{|1+\mu_1 h|\cos\varphi_1} \\
&\leq \frac{1+|\mu_1|\,h\cos\varphi_1}{1+|\mu_1|\,h\cos\varphi_1}\frac{1}{\cos\varphi_1} \\
&= \frac{1}{\cos\varphi_1},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{h}{1-|R_2|} &\leq \frac{h\left(1+|\mu_2|\,h\cos\varphi_2\right)}{|\mu_2|\,h\cos\varphi_2} \\
&\leq \frac{1}{|\mu_2|\cos\varphi_2}.
\end{aligned}
$$

Lemma 3.10 is proved. ■

Lemma 3.11 *Let*

$$\begin{aligned} L_1 &= \frac{4h\,|R_2|^{N+1-j}}{1-|R_2|}\frac{1-|R_2|^{N-1}}{1-|R_2|^{2N}} \\ &+2h\frac{|R_2|-|R_2|^{N-j}+1-|R_2|^j}{1-|R_2|}, \end{aligned}$$

$$\begin{aligned} L_2 &= \frac{4h\,|R_1|}{1-|R_1|}\frac{1-|R_1|^{N-1}}{1-|R_1|^{2N}} \\ &+2h\frac{|R_1|-|R_1|^{N-k}+1-|R_1|^k}{1-|R_1|}, \end{aligned}$$

$$\begin{aligned} L_3 &= \frac{64\,|1-R_1|}{1-|R_1|}\frac{1-|R_1|^{N-1}}{1-|R_1|^{2N}} \\ &\times\frac{1}{|\mu_2|\cos\varphi_2}+\frac{8}{|\mu_2|\cos\varphi_2}, \end{aligned}$$

Then, the following estimates hold.

$$L_1 \le \frac{8}{|\mu_2|\cos\varphi_2},$$

$$L_2 \le \frac{8}{|\mu_1|\cos\varphi_1},$$

$$L_3 \le \frac{72}{|\mu_2|\cos\varphi_1\cos\varphi_2},$$

Proof. Using the estimate of Lemma 3.10 and the following estimates

$$|R_2|^{N+1-j} \le 1,$$

$$\frac{1-|R_2|^{N-1}}{1-|R_2|^{2N}} \le 1,$$

$$\left(|R_2|-|R_2|^{N-j}+1-|R_2|^j\right) \le 2,$$

we get

$$L_1 \le \frac{8}{|\mu_2| \cos\varphi_2}.$$

Using the estimate of Lemma 3.10 and the following estimates

$$|R_1| \le 1,$$

$$\frac{1-|R_1|^{N-1}}{1-|R_1|^{2N}} \le 1,$$

$$\left(|R_1| - |R_1|^{N-k} + 1 - |R_1|^k\right) \le 2,$$

we obtain

$$L_2 \le \frac{8}{|\mu_1| \cos\varphi_1}.$$

Using the estimate of Lemma 3.10 and the following estimate

$$\frac{1-|R_1|^{N-1}}{1-|R_1|^{2N}} \le 1,$$

we have that

$$\begin{aligned} L_3 &\le \frac{64}{|\mu_2| \cos\varphi_1 \cos\varphi_2} + \frac{8}{|\mu_2| \cos\varphi_2} \\ &\le \frac{72}{|\mu_2| \cos\varphi_1 \cos\varphi_2}. \end{aligned}$$

Lemma 3.11 is proved. ■

Lemma 3.12 *The following estimates hold.*

$$\left(\frac{4\,|1-R_1|}{1-|R_1|}\frac{1-|R_1|^{N-1}}{1-|R_1|^{2N}} + 1\right) \le \frac{5}{\cos\varphi_1},$$

$$\frac{4h}{1-|R_2|}\frac{1-|R_2|^{N-1}}{1-|R_2|^{2N}} \le \frac{4}{|\mu_2| \cos\varphi_2}.$$

Proof. Using the estimate of Lemma 3.10 and the following estimate

$$\frac{1-|R_1|^{N-1}}{1-|R_1|^{2N}} \le 1,$$

we get

$$\left(\frac{4\,|1-R_1|}{1-|R_1|}\frac{1-|R_1|^{N-1}}{1-|R_1|^{2N}} + 1\right) \le \frac{4}{\cos\varphi_1} + 1 \le \frac{5}{\cos\varphi_1}.$$

Using the estimate of Lemma 3.10 and the following estimate

$$\frac{1-|R_2|^{N-1}}{1-|R_2|^{2N}} \le 1,$$

we get

$$\frac{4h}{1-|R_2|}\frac{1-|R_2|^{N-1}}{1-|R_2|^{2N}} \le \frac{4}{|\mu_2|\cos\varphi_2}.$$

Lemma 3.12 is proved. ■

Using the triangle inequality, we get

$$\left|(\lambda I + A_h^x)^{-1} f_h\right| \le \left|\frac{\left(1-R_1^{2N}\right)(1+\mu_1 h)(1+\mu_2 h)}{\mu_1\mu_2 R_1 R_2 (2+\mu_1 h)(2+\mu_2 h)}\right|$$

$$\times \{ [([|T_{h1}| \, |1-R_1|^2$$
$$\times \sum_{j=1}^{N-1} \left|R_1^{N-j-1} - R_1^{N+j-1}\right| \left|R_2^{N-j-1}\right| + 1 \Big]$$
$$\times \Bigg[\sum_{j=1}^{N-1} h \left(|T_{h1}| \left|R_1^{N-k} - R_1^{N+k}\right|\right.$$
$$\left.\times \left|R_1^{N-j} - R_1^{N+j}\right| + \left|R_1^{|k-j|} - R_1^{k+j}\right|\right) \left|R_2^{j-1}\right| \Bigg]$$

$$+ \left[|T_{h1}| \, |1-R_1|^2 \sum_{j=1}^{N-1} \left|R_1^{N-j-1} - R_1^{N+j-1}\right| \left|R_2^{j-1}\right| + \left|R_2^{N-2}\right| \right]$$
$$\times \left[\sum_{j=1}^{N-1} h \Bigg(|T_{h1}| \left|R_1^{N-k} - R_1^{N+k}\right| \left|R_1^{N-j} - R_1^{N+j}\right| \right.$$
$$\left. + \left|R_1^{|k-j|} - R_1^{k+j}\right| \Bigg) \left|R_2^{N-j-1}\right| \right] \Big)$$

$$\times \Bigg(|T_{h1}| \, |1-R_1|^2 \sum_{j=1}^{N-1} \sum_{l=1}^{N-1} h \left|R_1^{j-1} - R_1^{2N-j-1}\right| [|T_{h2}|$$
$$\times \left|R_2^{N-j} - R_2^{N+j}\right| \left|R_2^{N-l} - R_2^{N+l}\right| + \left|R_2^{|j-l|} - R_2^{j+l}\right|]$$
$$+ \left|1-R_2^2\right| \sum_{l=1}^{N-1} h \left|R_2^{l-1} - R_2^{2N-l-1}\right| \Bigg)$$

$$+\left(\left[|T_{h1}|\,|1-R_1|^2\sum_{j=1}^{N-1}\left|R_1^{j-1}-R_1^{2N-j-1}\right|\left|R_2^{N-j-1}\right|+\left|R_2^{N-2}\right|\right]\right.$$
$$\times\left[\sum_{j=1}^{N-1}h\left(|T_{h1}|\left|R_1^{N-k}-R_1^{N+k}\right|\left|R_1^{N-j}-R_1^{N+j}\right|\right.\right.$$
$$\left.\left.+\left|R_1^{|k-j|}-R_1^{k+j}\right|\right)\left|R_2^{j-1}\right|\right]$$

$$+\left[|T_{h1}|\,|1-R_1|^2\sum_{j=1}^{N-1}\left|R_1^{j-1}-R_1^{2N-j-1}\right|\left|R_2^{j-1}\right|+1\right]$$
$$\times\sum_{j=1}^{N-1}h\left(|T_{h1}|\left|R_1^{N-k}-R_1^{N+k}\right|\left|R_1^{N-j}-R_1^{N+j}\right|\right.$$
$$\left.\left.+\left|R_1^{|k-j|}-R_1^{k+j}\right|\right)\left|R_2^{N-j-1}\right|\right]\right)$$

$$\times\left(|T_{h1}|\left|1-R_2^2\right|\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h\left|R_1^{N-j-1}-R_1^{N+j-1}\right|\left[|T_{h2}|\right.\right.$$
$$\left.\times\left|R_2^{N-j}-R_2^{N+j}\right|\left|R_2^{N-l}-R_2^{N+l}\right|+\left|R_2^{|j-l|}-R_2^{j+l}\right|\right]$$
$$\left.\left.+\left|1-R_1^2\right|\sum_{l=1}^{N-1}h\left|R_2^{N-l-1}-R_2^{N+l-1}\right|\right)\right\}$$
$$\times\left|\Delta^{-1}\right|\left|\left(1+h\beta\Delta^{-1}\right)^{-1}\right|$$

$$+\sum_{j=1}^{N-1}\sum_{l=1}^{N-1}h^2\left[|T_{h1}|\left|R_1^{N-k}-R_1^{N+k}\right|\left|R_1^{N-j}-R_1^{N+j}\right|\right.$$
$$\left.+\left|R_1^{|k-j|}-R_1^{k+j}\right|\right]\left[|T_{h2}|\left|R_2^{N-j}-R_2^{N+j}\right|\right.$$
$$\left.\times\left|R_2^{N-l}-R_2^{N+l}\right|+\left|R_2^{|j-l|}-R_2^{j+l}\right|\right]\right]\max_{1\le l\le N-1}|f_l|\,.$$

Using the estimates on Lemmas 3.4, 3.5, 3.7 and 3.8, we get

$$\left|\left(\lambda I+A_h^x\right)^{-1}f_h\right|\le\frac{2}{|\mu_1|\,|\mu_2|\,|R_1|\,|R_2|}$$

$$\left[\left\{2\left(\left[\frac{4\left|1-R_1\right|}{1-\left|R_1\right|}\frac{1-\left|R_1\right|^{N-1}}{1-\left|R_1\right|^{2N}}\right.\right.\right.\right.$$

$$\left.+1\right]\times\left[\frac{4h\left|R_1\right|}{1-\left|R_1\right|}\frac{1-\left|R_1\right|^{N-1}}{1-\left|R_1\right|^{2N}}\right.$$

$$\left.\left.+2h\frac{\left|R_1\right|-\left|R_1\right|^{N-k}+1-\left|R_1\right|^{k}}{1-\left|R_1\right|}\right]\right)$$

$$\times\left(\frac{4\left|1-R_1\right|}{1-\left|R_1\right|^{2N}}\sum_{j=1}^{N-1}\left|R_1\right|^{j-1}+\left|R_1\right|^{N-j-1}\right.$$

$$\times\left[\frac{4h\left|R_2\right|^{N+1-j}}{1-\left|R_2\right|}\frac{1-\left|R_2\right|^{N-1}}{1-\left|R_2\right|^{2N}}\right.$$

$$\left.+2h\frac{\left|R_2\right|-\left|R_2\right|^{N-j}+1-\left|R_2\right|^{j}}{1-\left|R_2\right|}\right]$$

$$\left.\left.+\frac{8h}{1-\left|R_2\right|}\frac{1-\left|R_2\right|^{N-1}}{1-\left|R_2\right|^{2N}}\right)\right\}$$

$$\times\left|\Delta^{-1}\right|\left|\left(1+h\beta\Delta^{-1}\right)^{-1}\right|$$

$$+\sum_{j=1}^{N-1}\left[\frac{4h\left|R_1\right|^{N-j}}{1-\left|R_1\right|^{2N}}+h\left|R_1^{|k-j|}-R_1^{k+j}\right|\right]$$

$$\times\left[\frac{4h\left|R_2\right|^{N+1-j}}{1-\left|R_2\right|}\frac{1-\left|R_2\right|^{N-1}}{1-\left|R_2\right|^{2N}}\right.$$

$$\left.\left.+2h\frac{\left|R_2\right|-\left|R_2\right|^{N-j}+1-\left|R_2\right|^{j}}{1-\left|R_2\right|}\right]\right]\max_{1\leq l\leq N-1}\left|f_l\right|.$$

Using the estimates on Lemmas 3.11, 3.12 and, we obtain

$$\left|(\lambda I+A_h^x)^{-1}f_h\right|\leq\frac{2}{\left|\mu_1\right|\left|\mu_2\right|\left|R_1\right|\left|R_2\right|}\left\{\frac{80}{\left|\mu_1\right|\cos^2\varphi_1}\right.$$

$$\left(\frac{64\left|1-R_1\right|}{1-\left|R_1\right|}\frac{1-\left|R_1\right|^{N-1}}{1-\left|R_1\right|^{2N}}\right.$$
$$\left.\times\frac{1}{\left|\mu_2\right|\cos\varphi_2}+\frac{8}{\left|\mu_2\right|\cos\varphi_2}\right)$$
$$\times\left|\Delta^{-1}\right|\left|\left(1+h\beta\Delta^{-1}\right)^{-1}\right|$$

$$+\frac{8}{\left|\mu_2\right|\cos\varphi_2}\left(\frac{4h\left|R_1\right|}{1-\left|R_1\right|}\frac{1-\left|R_1\right|^{N-1}}{1-\left|R_1\right|^{2N}}\right.$$
$$\left.\left.+2h\frac{\left|R_1\right|-\left|R_1\right|^{N-k}+1-\left|R_1\right|^{k}}{1-\left|R_1\right|}\right)\right\}\left\|f\right\|_{C_h}.$$

Using the estimates of Lemmas 3.9 and 3.11, we obtain

$$\left|\left(\lambda I+A_h^x\right)^{-1}f_h\right|\leq\frac{M_1}{\left|\mu_1\right|^2\left|\mu_2\right|^2\cos\varphi_1\cos\varphi_2\left|R_1\right|\left|R_2\right|}\left\|f\right\|_{C_h}.$$

Since

$$\mu_1^2\mu_1^2=\frac{\lambda+\delta}{R_1R_2},$$

and

$$\begin{aligned}\cos\varphi_1\cos\varphi_2&\geq\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2\\&=\cos(\varphi_1-\varphi_2)\\&=\cos\frac{\varphi}{2},\end{aligned}$$

we have

$$\left|\left(\lambda I+A_h^x\right)^{-1}f_h\right|\leq\frac{M_1}{(\lambda+\delta)\cos\varphi}\left\|f\right\|_{C_h}.$$

Using estimates of Lemmas 2.3 and 3.2 we get

$$\begin{aligned}\left|\left(\lambda I+A_h^x\right)^{-1}f_h\right|&\leq\frac{M_1}{(\delta+|\lambda|)\sqrt{\cos\varphi}\cos\frac{\varphi}{2}}\left\|f\right\|_{C_h}\\&\leq\frac{M_1}{(\delta+|\lambda|)\cos\varphi}\left\|f\right\|_{C_h}.\end{aligned}$$

It follows that,

$$\left\|\left(\lambda I+A_h^x\right)^{-1}f_h\right\|_{C_h}\leq\frac{M\left(\varphi,\delta\right)}{1+|\lambda|}\left\|f\right\|_{C_h},$$

and

$$\left\|(\lambda I + A_h^x)^{-1}\right\|_{C_h \to C_h} \leq \frac{M\,(\varphi,\delta)}{1+|\lambda|}.$$

Here $M\,(\varphi,\delta)$ does not depend on h.

Theorem 3.1 is proved.

Now, we consider the difference scheme of the second order of accuracy

$$\begin{cases} \frac{1}{\tau^2}\left(u_{k+1}^n - 2u_k^n + u_{k-1}^n\right) \\ +\frac{1}{h^4}\left(u_k^{n+2} - 4u_k^{n+1} + 6u_k^n - 4u_k^{n-1} + u_k^{n-2}\right) + \delta u_k^n = \varphi_k^n, \\ \varphi_k^n = f\,(t_k, x_n)\,,\ t_k = k\tau,\ x_n = nh, \\ 1 \leq k \leq N-1,\ 2 \leq n \leq M-2,\ N\tau = T,\ Mh = 1, \\ u_0^n = \varphi^n,\ u_N^n = \psi^n,\ \varphi^n = \varphi\,(x_n)\,, \\ \psi^n = \psi\,(x_n)\,,\ x_n = nh,\ 0 \leq n \leq M, \\ u_k^0 = u_k^M = 0,\ \ -u_k^2 + 2u_k^1 - u_k^0 = u_k^{M-2} - 2u_k^{M-1} + u_k^M = 0, \\ 0 \leq k \leq N, \end{cases} \tag{3.9}$$

for the approximate solution of the boundary value problem (3.9).

Theorem 3.2 *The solution of the difference scheme (3.9) satisfy the following stability estimate*

$$\begin{aligned} \left\|u^{\tau,h}\right\|_{C_\tau^{\alpha,\alpha}(C_h)} &\leq M\left[\left\|\varphi^h\right\|_{C_h} + \left\|\psi^h\right\|_{C_h} + \left\|f^{\tau,h}\right\|_{C_\tau^{\alpha,\alpha}(C_h)}\right], \\ 0 &\leq \alpha \leq 1, \end{aligned}$$

where M does not depend on $f^{\tau,h}$, φ^h, ψ^h, α, h and τ.

The proof of Theorem 3.2 is based on the positivity of the operator A_h^x in C_h and generated by problem (3.9) and on the following theorem about stability inequalities in

$C_{\tau}^{\alpha,\alpha}(E)$ for the solution of the difference scheme

$$\begin{cases} \frac{1}{\tau^2}\left(u_{k+1}-2u_k+u_{k-1}\right)+Au_k=\varphi_k, \\ \\ \varphi_k=f\left(t_k\right),\ t_k=k\tau,\ 1\leq k\leq N-1,\ N\tau=T, \\ \\ u_0=\varphi,\ u_N=\psi \end{cases} \tag{3.10}$$

Theorem 3.3 *The solution of the difference scheme (3.10) satisfy the following stability estimate*

$$\begin{aligned} \|u^{\tau}\|_{C_{\tau}^{\alpha,\alpha}(E)} &\leq M\left[\|\varphi\|_E+\|\psi\|_E+\|f^{\tau}\|_{C_{\tau}^{\alpha,\alpha}(E)}\right], \\ 0 &\leq \alpha\leq 1, \end{aligned}$$

where M does not depend on f^{τ}, φ, ψ, α and τ.Here, we denote $C_{\tau}^{\alpha,\alpha}(E)\,(0\leq\alpha\leq 1)$ the Banach space of the mesh $E-$valued functions $\varphi^{\tau}=\{\varphi_k\}_{k=1}^{N-1}$ with the norm

$$\|\varphi^{\tau}\|_{C_{\tau}^{\alpha,\alpha}(E)}=\|\varphi^{\tau}\|_{C_{\tau}(E)}+$$

$$+\sup_{1\leq k\leq k+r\leq N-1}\frac{(N\tau-k\tau)^{\alpha}(k+r)^{\alpha}\left\|\varphi_{k+r}-\varphi_k\right\|_E}{\tau^{\alpha}},$$

$$\|\varphi^{\tau}\|_{C_{\tau}(E)}=\max_{1\leq k\leq N-1}\|\varphi_k\|_E\,.$$

Theorem 3.4 *The solution of the difference scheme (3.9) satisfy the following almost coercitive stability estimates:*

$$\begin{aligned} &\left\|\left\{\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\}_{k=1}^{N-1}\right\|_{C_{\tau}(C_h)} \\ &+\left\|\left\{\left\{h^{-4}\left(u_k^{n+2}-4u_k^{n+1}+6u_k^n-4u_k^{n-1}+u_k^{n-2}\right)\right\}_{n=2}^{M-2}\right\}_{k=1}^{N-1}\right\|_{C_{\tau}(C_h)} \\ &\leq M\left[\left\|\left\{h^{-4}\left(\varphi^{n+2}-4\varphi^{n+1}+6\varphi^n-4\varphi^{n-1}+\varphi^{n-2}\right)\right\}_{n=2}^{M-2}\right\|_{C_h}\right. \\ &+\left\|\left\{h^{-4}\left(\psi^{n+2}-4\psi^{n+1}+6\psi^n-4\psi^{n-1}+\psi^{n-2}\right)\right\}_{n=2}^{M-2}\right\|_{C_h} \\ &+\ln\frac{1}{\tau+h}\left\|f^{\tau,h}\right\|_{C_{\tau}(C_h)} \end{aligned}$$

where M does not depend on $f^{\tau,h}$, φ^h, ψ^h, h and τ.

The proof of Theorem 3.4 is based on the positivity of the operator A_h^x in C_h and on the following theorem on almost coercivity inequality for the solution of problem (3.10) in $C_\tau(E)$.

Theorem 3.5 *The solution of the difference scheme (3.10) satisfy the following almost coercivity inequality*

$$\left\| \left\{ \frac{1}{\tau^2}\left(u_{k+1}-2u_k+u_{k-1}\right) \right\}_{k=1}^{N-1} \right\|_{C_\tau(E)} + \left\| \left\{ Au_k \right\}_{k=1}^{N-1} \right\|_{C_\tau(E)}$$

$$\leq M\left[\|A\varphi\|_E + \|A\psi\|_E + \min\left\{ \ln\frac{1}{\tau}, 1+\left|\ln\|A\|_{E\to E}\right| \right\} \|f^\tau\|_{C_\tau(E)} \right],$$

where M does not depend on f^τ, φ, ψ and τ.

Theorem 3.6 *The solution of the difference scheme (3.9) satisfy the coercivity estimates*

$$\left\| \left\{ \tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right) \right\}_{k=1}^{N-1} \right\|_{C_\tau^{\alpha,\alpha}(C_h)}$$

$$+\left\| \left\{ \left\{ h^{-4}\left(u_k^{n+2}-4u_k^{n+1}+6u_k^n-4u_k^{n-1}+u_k^{n-2}\right) \right\}_{n=2}^{M-2} \right\}_{k=1}^{N-1} \right\|_{C_\tau^{\alpha,\alpha}(C_h)}$$

$$\leq M\left[\left\| \left\{ h^{-4}\left(\varphi^{n+2}-4\varphi^{n+1}+6\varphi^n-4\varphi^{n-1}+\varphi^{n-2}\right) \right\}_{n=2}^{M-2} \right\|_{C_h} \right.$$

$$+\left\| \left\{ h^{-4}\left(\psi^{n+2}-4\psi^{n+1}+6\psi^n-4\psi^{n-1}+\psi^{n-2}\right) \right\}_{n=2}^{M-2} \right\|_{C_h}$$

$$+\frac{1}{\alpha(1-\alpha)}\left\|f^{\tau,h}\right\|_{C_\tau^{\alpha,\alpha}(C_h)}, \quad 0<\alpha<1,$$

where M does not depend on f^τ, φ^h, ψ^h, h, α and τ.

The proof of Theorem 3.6 is based on the positivity of the operator A_h^x in C_h and on the following theorem on coercivity inequality for the solution of problem (3.10) in $C_\tau^{\alpha,\alpha}(E)$

Theorem 3.7 *The solution of the difference scheme (3.10) satisfy the following coercivity inequality*

$$\left\| \left\{ \frac{1}{\tau^2}\left(u_{k+1}-2u_k+u_{k-1}\right) \right\}_{k=1}^{N-1} \right\|_{C_\tau^{\alpha,\alpha}(E)} + \left\| \left\{ Au_k \right\}_{k=1}^{N-1} \right\|_{C_\tau^{\alpha,\alpha}(E)}$$

$$\leq M\left[\|A\varphi\|_E + \|A\psi\|_E + \frac{1}{\alpha(1-\alpha)}\|f^\tau\|_{C_\tau^{\alpha,\alpha}(E)} \right],$$

where M does not depend on f^τ, φ, ψ, α and τ.

Note that in a similar manner we can construct the difference schemes of a high order of accuracy generated by Taylor's decomposition of the function on three points with respect to one variable for approximate solutions of the boundary value problem (3.9). Abstract theorems given above permit us to obtain the stability, the almost coercive stability and the coercive stability estimates for the solutions of these difference schemes.

CHAPTER 4

CONCLUSION

This work is devoted to the study of the positivity of differential operator A^x in $C[0,1]$ and of the difference operator A_h^x in C_h. The following original results are obtained:

- Green's function of the differential operator A^x defined by the formula

$$A^x u = \frac{d^4 u}{dx^4} + \delta u,$$

where $\delta > 0$, with the domain

$$D(A^x) = \left\{ u \in C^4[0,1] : u(0) = u(1) = 0,\ u''(0) = u''(1) = 0 \right\}$$

is constructed.

- The positivity of the fourth order differential operator A^x in $C[0,1]$ is proved.

- The coercive stability estimates for the solution of two-dimensional elliptic differential problem defined by the formula

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + \delta u = f(t,x),\ 0 < t < T,\ 0 < x < 1, \\ u(0,x) = \varphi(x),\ u(T,x) = \psi(x),\ 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0,\ u_{xx}(t,0) = u_{xx}(t,1) = 0,\ 0 \le t \le T, \end{cases}$$

where $\varphi(x)$, $\psi(x)$ and $f(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number, in Holder spaces are obtained.

- Green's function of the difference opeartor A_h^x defined by the formula

$$\begin{cases} A_h^x u^h = \left\{ \frac{u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}}{h^2} + \delta u_k \right\}_2^{N-2}, \\ u_0 = u_N = 0,\ \ - u_2 + 2u_1 - u_0 = u_{N-2} - 2u_{N-1} + u_N = 0 \end{cases}$$

where $\delta > 0$, is constructed.

- The positivity of the fourth order difference operator A_h^x in C_h is proved.

- The stability, the almost coercive stability and the coercive stability estimates for the solution of difference scheme

$$\begin{cases} \frac{1}{\tau^2}\left(u_{k+1}^n - 2u_k^n + u_{k-1}^n\right) \\ +\frac{1}{h^4}\left(u_k^{n+2} - 4u_k^{n+1} + 6u_k^n - 4u_k^{n-1} + u_k^{n-2}\right) + \delta u_k^n = \varphi_k^n, \\ \varphi_k^n = f\left(t_k, x_n\right),\ t_k = k\tau,\ x_n = nh, \\ 1 \le k \le N-1,\ 2 \le n \le M-2,\ N\tau = T,\ Mh = 1, \\ u_0^n = \varphi^n,\ u_N^n = \psi^n,\ \varphi^n = \varphi\left(x_n\right), \\ \psi^n = \psi\left(x_n\right),\ x_n = nh,\ 0 \le n \le M, \\ u_k^0 = u_k^M = 0,\ \ -u_k^2 + 2u_k^1 - u_k^0 = u_k^{M-2} - 2u_k^{M-1} + u_k^M = 0, \\ 0 \le k \le N \end{cases}$$

in difference analogues of Holder spaces are obtained.

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