Ayfer DURAL

M.S. Thesis In Mathematics

August 2009

NUMERICAL SOLUTIONS OF THE NONLOCAL BOUNDARY VALUE PROBLEMS FOR INVERSE PARABOLIC EQUATION

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

Multipoint nonlocal boundary value problems for reverse parabolic equations in a Hilbert space H with the self-adjoint positive definite operator *are* is considered. The well-posedness of this problem in Hölder spaces without a weight is established. The coercivity inequalities for solutions of multipoint nonlocal boundary value problems for reverse parabolic equations are obtained. The first order of accuracy difference scheme and the second order of accuracy difference scheme for the approximate solutions of this nonlocal boundary value problem are presented. The stability estimates, coercivity and almost coercivity inequalities for the solution of these difference schemes are established. The well-posedness of these difference schemes in Hölder spaces without a weight are proved. The Matlab implementation of these difference schemes for the multipoint nonlocal boundary value problems for reverse parabolic equations are presented. We support the theoretical results for the solution of these difference schemes by the results of numerical examples.

Keywords: Multipoint nonlocal boundary value problem, Reverse parabolic equation, Difference Schemes, Stability Estimates, Well-posedness.

TERS PARABOLİK DENKLEMLER İÇİN LOKAL OLMAYAN SINIR DEĞER PROBLEMLERİN NÜMERİK ÇÖZÜMÜ

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ÖZ

H Hilbert uzayında, pozitif tanımlı özeslenik (self-adjoind) A operatör olmak üzere çok noktalı yerel olmayan ters tip parabolik sınır değer problemleri düşünülmüştür. Bu problemlerin iyi konumlanmışlığı ağırlıksız Hölder uzaylarında doğruluğu elde edilmiştir. Çok noktalı yerel olmayan ters tip parabolik sınır değer problemlerinin çözümleri için koersatif eşitsizlikleri elde edilmiştir. Bu yerel olmayan ters tip parabolik sınır değer problemlerinin yaklaşık çözümleri için birinci dereceden ve ikinci dereceden fark şeması kurulmuştur. Bu fark şemalarının çözümü için kararlılık kestirimleri kurulmuştur. Bu fark şemalarının iyi konumlanmışlığı Hölder uzaylarında ispatlanmıştır. Bu fark şemalarının çözümleri için koersatif eşitsizlikleri, hemen hemen koersatif eşitsizlikleri sağlanmıştır. Ters tip parabolik sınır değer problemleri için fark şemasının Matlab ile çözümleri elde edilmiştir. Bu fark şemalarının çözümleri için elde edilen teorik sonuçlar, sayısal örneklerle desteklenmiştir.

Anahtar Kelimeler: Çok noktalı yerel olmayan sınır değer problemleri, Ters tip parabolik denklemler, Fark şemaları, Kararlılık kestirimleri, İyi konumlanmışlık.

DEDICATION

To My Parents

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CHAPTER 1

INTRODUCTION

It is known that most problems in fluid mechanics (dynamics, elasticity) and financial mathematics lead to partial differential equations of the parabolic type. These equations can be derived as models of physical systems and are considered as the methods for solving boundary value problems. It is known that the mixed problem for inverse parabolic equations can be solved by Fourier series method, by Fourier transform method and by Laplace transform method.

Example 1.1. *We consider the inverse parabolic problem*

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + u + (2t - 8t^2 + 8t^3 - 2t^4)\sin(x), \\ \qquad 0 < t < 1,\ 0 < t < \pi, \\ u(0,x) = u(1,x), \qquad 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1. \end{cases} \tag{1.1}$$

Solution. For the solution of problem (1.1), let us use the Fourier series method or the method of separation of variables. To do this, let $u(t,x) = v(t,x) + w(t,x)$, where $v(t,x)$ is the solution of

$$\begin{cases} \frac{\partial v}{\partial t} = -\frac{\partial^2 v}{\partial t^2} + v, 0 < t < 1,\ 0 < t < \pi, \\ v(0,x) = v(1,x), 0 \le x \le \pi, \\ v(t,0) = v(t,\pi) = 0, 0 \le t \le 1 \end{cases} \tag{1.2}$$

and $w(t,x)$ is the solution of

$$\begin{cases} \frac{\partial w}{\partial t} = -\frac{\partial^2 w}{\partial t^2} + w + (2t - 8t^2 + 8t^3 - 2t^4)\sin(x), 0 < t < 1,\ 0 < t < \pi, \\ w(0,x) = w(1,x), 0 \le x \le \pi, \\ w(t,0) = w(t,\pi) = 0, 0 \le t \le 1. \end{cases} \tag{1.3}$$

Let us first obtain the solution $v(t,x)$ of problem (1.2) . Introducing $v(t,x) = X(x)T(t) \neq 0$, taking partial derivative, and inserting into equation (1.2) , we get

$$T'(t)X(x) = -T(t)X''(x) - T(t)X(x)$$

or

$$\left(T'(t) + T(t)\right)X(x) = -T(t)X''(x).$$

Thus, we obtain

$$\frac{T'(t) + T(t)}{T(t)} = -\frac{X''(x)}{X(x)} = \lambda. \tag{1.4}$$

Using the boundary conditions in (1.2), we get

$$X(0) = X(\pi) = 0.$$

Hence, we need to solve the boundary value problem

$$X(0) = X(\pi) = 0, X''(x) - \lambda X(x) = 0, 0 < x < 1. \tag{1.5}$$

Note that the axiulary equation of (1.5) is $m^2 + \lambda = 0$. If $\lambda \leq 0$, we obtain the trivial solution. Therefore, we consider the case $\lambda > 0$. The nontrival solution of (1.5) in this case are

$$x_k(x) = \sin kx\ , \qquad k = 1, 2, \cdots .$$

Next, we need to solve

$$T'(t) - (1 + k^2)T(t) = 0.$$

Clearly, we have

$$T_k(t) = c_k e^{(1+k^2)t},\ k = 1, 2, \cdots .$$

Thus,

$$v(t,x) = \sum_{k=1}^{\infty} c_k e^{(1+k^2)t} \sin kx.$$

Using the nonlocal condition $v(1,0) = v(0,x)$, we get

$$c_k e^{(1+k^2)t} = c_k \qquad k = 1, 2, \cdots$$

and hence

$$c_k = 0\ , \qquad k = 1, 2, \cdots .$$

Therefore,

$$v(t,x) = 0.$$

Next, we obtain the solution of (1.3). We seek a solution of the form

$$w(t,x)=\sum_{k=1}^{\infty}A_k\sin kx$$

Then, it follows from (1.3) that

$$w_t+w_{xx}+w=\sum_{k=1}^{\infty}\left(A_k'(t)-k^2A_k(t)+A_k(t)\right)\sin kx=\left(2t-8t^2+8t^3-2t^4\right)\sin x$$

If $k\neq 1$, then

$$A_k'(t)+\left(-1-k^2\right)A_k(t)=0$$

or

$$A_k(t)=A_k(0)e^{(1+k^2)t}.$$

Using condition

$$A_k(1)=A_k(0),$$

we get

$$A_k(0)=0.$$

If $k=1$, then

$$A_1'(t)-2A_1(t)=2t-8t^2+8t^3-2t^4.$$

From that it follows

$$A_1(t)=A_1(0)e^{2t}+\int_0^t e^{2(t-s)}(2s-8s^2+8s^3-2s^4)ds. \tag{1.6}$$

Using the condition $w(0,x)=w(1,x)$, we obtain $A_1(0)=A_1(1)$.

From (1.6) it follows that

$$A_1(1)=A_1(0)e^2+\int_0^1 e^{2(1-s)}(2s-8s^2+8s^3-2s^4)ds$$

or

$$A_1(0)=\frac{1}{1-e^2}\int_0^1 e^{2(1-s)}(2s-8s^2+8s^3-2s^4)ds.$$

Hence,

$$A_1(t) = \frac{e^{2t}}{1-e^2}\int_0^1 e^{2(1-s)}(2s-8s^2+8s^3-2s^4)ds + \int_0^t e^{2(t-s)}(2s-8s^2+8s^3-2s^4)ds$$
$$= t^2 - 2t^3 + t^4.$$

Thus, the solution of (1.3) is

$$w(t,x) = \sum_{k=1}^{\infty} A_k \sin kx = (t^2 - 2t^3 + t^4)\sin x.$$

Finally, using (1.2), we get

$$u(t,x) = v(t,x) + w(t,x) = 0 + (t^2 - 2t^3 + t^4)\sin x.$$

Thus, $u(t,x) = (t^2 - 2t^3 + t^4)\sin(x)$ is the solution of nonlocal boundary value problem (1.1).

Similarly, we obtain the solution of the following multi-dimensional non-local boundary value problem of reverse parabolic type

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \sum_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, 0 < t < T, \\ u(0,x) = u(T,x) + \varphi(x), \quad x \in \overline{\Omega}, \\ u(t,x) = 0, \qquad x \in S, \end{cases}$$

where α_r are constants, $f(t,x)$ $(t \in [0,T], x \in \Omega)$, $\varphi(x)(x \in \overline{\Omega})$ are given smooth functions, and Ω is the unit open cube in the n-dimensional Euclidean space R^n $(0 < x_k < 1, 1 \leq k \leq n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$.

However, the method of separation of variables can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving such type of problem is the difference method.

Example 1.2. *We consider the inverse parabolic problem*

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + u + (1 + 4x^2 t - 3t)\, e^{-x^2}, & 0 < t < 1, \ -\infty < x < \infty \\ u(0,x) = u(1,x) - e^{-x^2}, & -\infty < x < \infty \end{cases} \tag{1.7}$$

Solution. For the solution of the problem in Example1.2,we use the Fourier transform method.

We take the Fourier transform of both sides of the equation.

So, our problem becomes

$$u_t = -u_{xx} + u + \left(1 + 4x^2t - 3t\right) e^{-x^2}$$

$$F\{u_t\} = -F\{u_{xx}\} + F\{u\} + F\left\{\left(1 + 4x^2t - 3t\right) e^{-x^2}\right\}$$

or

$$(F\{u(t,x)\})_t = -(is)^2 F\{u(x,t)\} + F\{u(x,t)\} + F\left\{\left(1 + 4x^2t - 3t\right) e^{-x^2}\right\}.$$

Let

$$F\{u(t,x)\} = v(t,s).$$

Then our problem becomes

$$v_t(t,s) + \left(-s^2 - 1\right) v(s,t) = F\left\{\left(1 + 4x^2t - 3t\right) e^{-x^2}\right\}.$$

By using the nonlocal boundary conditions, we get $v(t,s)$ as

$$v(t,s) = ce^{\left(s^2+1\right)(t-1)} + \frac{1}{s^2+1} F\left\{\left(1 + 4x^2t - 3t\right) e^{-x^2}\right\}.$$

Since

$$F\left\{\left(1 + 4x^2t - 3t\right) e^{-x^2}\right\} = F\left\{e^{-x^2}\right\} + F\left\{te^{-x^2}\right\} - F\left\{t\left(2 - 4x^2\right) e^{-x^2}\right\}$$

$$= F\left\{e^{-x^2}\right\} + F\left\{te^{-x^2}\right\} - F\left\{t\left(e^{-x^2}\right)''\right\}$$

$$= \left(1 + \frac{1}{s^2} - \frac{s^2}{s^2}\right) F\left\{e^{-x^2}\right\}.$$

So our problem becomes

$$v_t(t,s) + (-s^2 - 1)v(t,s) = (1 + t - s^2t) F\left\{e^{-x^2}\right\}.$$

The solution is

$$v(t,s) = e^{(s^2+1)(t-1)} v(0,s) + \int_0^t e^{(s^2+1)(t-y)} \left(1 - y - s^2y\right) F\left\{e^{-x^2}\right\} dy$$

or

$$v(t,s) = tF\left\{e^{-x^2}\right\}.$$

Now using the nonlocal boundary conditions we get $u(t,s)$ as

$$v(t,s) = tF\left\{e^{-x^2}\right\}$$

or

$$u(t,x) = F^{-1}\left\{tF\left\{e^{-x^2}\right\}\right\}.$$

Finally taking the inverse of Fourier transformation we obtain the solution for the problem as

$$u(t,x) = te^{-x^2}.$$

Note that using the same manner one obtains the solution of the following nonlocal boundary value problem for the multidimensional inverse parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} + \sum\limits_{|r|=2m} \alpha_r \frac{\partial^{|\tau|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = f(t,x), \\ 0 < t < T, x, r \in R^n, |r| = r_1 + \ldots + r_n, \\ u(0,x) = u(T,x) + \varphi(x), \quad x \in R^n, \end{cases}$$

Here, α_r are constants. $(x \in R^n)$ $f(t,x)(t \in [0,T], x \in R^n)$ are given smooth functions. However, the Fourier Transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving such type of problem is the difference method.

Example 1.3. *We consider the inverse parabolic problem*

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + u + 1 - t - e^{-x} + xe^{-x} + 2te^{-x}, \quad 0 < t < 1, \quad 0 < x < \infty \\ u(0,x) = u(1,x) - 1 + (1+x)e^{-x}, \quad 0 < x < \infty, \\ u(t,0) = 0, \quad u_x(t,0) = 0, \quad 0 \leq t \leq 1. \end{cases}$$

Solution. Here, we will use the Laplace transform method to solve the problem

$$u_t = -u_{xx} + u + 1 - t - e^{-x} + xe^{-x} + 2te^{-x},$$

we can write

$$L\{u_t\} = -L\{u_{xx}\} + L\{u\} + L\{1 - t - e^{-x} + xe^{-x} + 2te^{-x}\}$$

or

$$(L\{u(t,x)\})_t = -s^2 L\{u(t,x)\} + su(t,0) + u_x(t,0) + L\{u(t,x)\} + \frac{1-t}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2t}{s+1}.$$

Let

$$L\{u(t,x)\} = v(t,s).$$

Then our problem becomes

$$v_t(t,s) + s^2 v(t,s) - v(t,s) = \frac{1-t}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2t}{s+1}$$

or

$$v_t(t,s) + (s^2-1)v(t,s) = \frac{1-t}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2t}{s+1}.$$

Now the complementary solution is

$$v_t(t,s) + (s^2-1)v(t,s) = 0$$

or

$$v_c(t,s) = e^{-(s^2+1)t}c.$$

The particular solution is

$$\begin{aligned} v_p(t,s) &= \frac{1}{s^2-1}\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2}\left(-\frac{1}{s} + \frac{2}{s+1}\right)\right) + \frac{1}{s^2-1}\left(-\frac{1}{s} + \frac{2}{s+1}\right)t \\ &= \left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)t. \end{aligned}$$

So,

$$v(t,s) = e^{-(s^2+1)t}c + \left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)t.$$

Then,

$$u_t = -u_{xx} + u + 1 - t - e^{-x} + xe^{-x} + 2te^{-x}.$$

By taking the Laplace transform of both sides of the last differential equation, we obtain

$$L\{u_t\} = -L\{u_{xx}\} + L\{u\} + L\{1 - t - e^{-x} + xe^{-x} + 2te^{-x}\}$$

or

$$(L\{u(t,x)\})_t = -s^2 L\{u(t,x)\} + su(t,0) + u_x(t,0) + L\{u(t,x)\} + \frac{1-t}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2t}{s+1}.$$

Let

$$L\{u(t,x)\} = v(t,s).$$

Then, our problem becomes

$$v_t(t,s) + s^2 v(t,s) - v(t,s) = \frac{1-t}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2t}{s+1}$$

or

$$v_t\,(t,s)+(s^2-1)v\,(t,s)=\frac{1-t}{s}-\frac{1}{s+1}+\frac{1}{(s+1)^2}+\frac{2t}{s+1}.$$

So,

$$v(t,s)=e^{-(s^2+1)t}c+\frac{1}{s^2-1}(\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}-\frac{1}{(s+1)^2}(-\frac{1}{s}+\frac{2}{s+1}))+\frac{1}{s^2-1}(-\frac{1}{s}+\frac{2}{s+1})t$$

by using the nonlocal boundary condition, we get

$$v(s,t)=\frac{1}{s(s+1)^2}t$$

or

$$v(t,s)=\left(\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\right)t.$$

Hence taking the inverse Laplace transform

$$\begin{aligned}u(t,x)=L^{-1}\left\{v(t,s)\right\}&=\left(L^{-1}\left\{\frac{1}{s}\right\}-L^{-1}\left\{\frac{1}{s+1}\right\}-L^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right)t\\&=\left(1-e^{-x}-xe^{-x}\right)t=t-(1+x)e^{-x}t.\end{aligned}$$

So,

$$u(t,x)=\left(1-(1+x)e^{-x}\right)t$$

is the solution of the given nonlocal boundary value problem.

Finally taking the inverse of Laplace we obtain

$$\begin{cases}\frac{\partial u}{\partial t}+\sum\limits_{|r|=2m}\alpha_r\frac{\partial^{|r|}u}{\partial x_1^{r_1}...\partial x_n^{r_n}}=f(t,x),\\ 0<t<T, x\in\Omega^+, |r|=r_1+...+r_n,\\ u(0,x)=u(T,x)+\Psi(x),\quad x\in\overline{\Omega}^+,\end{cases}$$

Here, α_r are constants. $(x\in\Omega^+)$ $f(t,x)(t\in[0,T],x\in\Omega^+),\Psi(x)$ are given smooth functions. Here $\Omega^+$is the unit open cube in the n-dimensional Euclidean space R^n $(0<x_1<\infty, 1\leq k\leq n)$ with boundary S,$\overline{\Omega}^+=\Omega^+\cup S$.

However, the Laplace Transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving such type of problem is the difference method.

In the present work the nonlocal boundary value problem

$$\begin{cases}\frac{du(t)}{dt}-Au(t)=f(t)\qquad(0\leq t\leq 1),\\ u(1)=\sum\limits_{k=1}^{p}\alpha_k u(\theta_k)+\varphi,\\ 0\leq\theta_1<\theta_2<\cdots<\theta_p<1\end{cases}$$

for differential equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. The well-posedness of this problem in Holder spaces is established. The coercivity inequalities for the solutions of the boundary value problems for reverse parabolic equations are obtained. The first and second order of accuracy difference schemes for the approximate solutions of this nonlocal boundary value problem are presented. The well-posedness of these difference schemes in Holder spaces is established. In applications, the coercivity inequalities for the solutions of difference schemes for the approximate solutions of the nonlocal boundary value problems for reverse parabolic equation are obtained.The theoretical statements for the solution of this difference schemes are supported by the results of numerical experiments.

We briefly describe the contents of the various chapters.

First chapter is the introduction.

Second chapter presents all the elementary Hilbert space theory that is needed for this work.

Third chapter considers the multipoint nonlocal boundary value problems. The well-posedness of these problems in the space of smooth functions is established. Moreover, new coercivity estimates for the solution of parabolic differential equations are obtained.

Fourth chapter is devoted to the Rothe difference scheme for the approximate solution of the abstract reverse parabolic equation in a Hilbert space with the nonlocal boundary condition. The stability, coercivity and almost coercivity estimates for the solution of the difference scheme are established. Furthermore, using the abstract result, new coercivity inequalities for the solution of multipoint nonlocal boundary value difference equations of reverse parabolic type are obtained.

Fifth chapter is about the well-posedness of the second order of accuracy difference scheme for approximately solving the multipoint nonlocal boundary value differential equation.

Sixth chapter is about Numerical Analysis. The matlab implementation of first and second order of accuracy difference schemes for two multipoint nonlocal boundary value problem for reverse parabolic equations are presented. The theoretical results for the solution of these differential solutions are supported by the results of numerical examples.

Seven chapter is the conclusiouns.

CHAPTER 2

BASIC ELEMENTS OF HILBERT SPACES

In this section, we provide the necessary definitions and facts about Hilbert spaces. For more information, we refer to [Krein, S.G., 1966].

2.1 Inner Product and Hilbert Space

Definition 2.1. An inner product space is a couple $(H, < \cdot >)$, where H is a complex vector space and $< \cdot >: H \times H \to \mathbb{C}$ is a complex-valued function with the following properties:

1. $< x, x > \geq 0$ and $< x, x >= 0 \Longleftrightarrow x = 0$,
2. $< x, y >= \overline{\langle y, x \rangle}$ for all $x, y \in H$,
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in H$ and $\alpha \in \mathbb{C}$,
4. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in H$,

where 0 is the zero vector of H, and the bar denotes the complex conjugate.

$\langle x, y \rangle$ is called the inner product of the vector x and y. A Hilbert space is a complete inner product space, where the norm on H is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. Thus, inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

2.2 Bounded Linear Operators Between Hilbert Spaces

Definition 2.2. Let H_1 and H_2 be two Hilbert spaces. A *linear operator* $A : H_1 \to H_2$ is a function with the following property: for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in H_1$,

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay.$$

Let $D(A) = \{x \in H_1, \exists Ax \in H_2\}$ and $R(A) = \{y = Ax, \forall x \in D(A)\}$ denote respectively the *domain* of A and the *range* of A.

A linear operator $A : H \rightarrow H$ is called *bounded*, if there is a real number $M > 0$ such that

$$\|Ax\|_H \leq M \ \|x\|_H \text{ for all } x \in H.$$

For bounded linear operator $A : H \rightarrow H$,

$$\|A\| = \inf\{M : \|Ax\|_H \leq M \|x\|_H \, \forall x \in H\}$$

is called the *norm* of A.

Theorem 2.1. *The norm of the bounded linear operator A satisfies the following*

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| .$$

□

2.3 The Adjoint of an Operator

Definition 2.3. If $A : H_1 \rightarrow H_2$ is a linear operator, where H_1 and H_2 are Hilbert spaces, then the Hilbert *adjoint operator* A^* of A is the operator

$$A^* : H_2 \rightarrow H_1,$$

such that for all $x \in H_1$ and $y \in H_2$

$$\langle Ax, y \rangle = \langle x, A^* y \rangle .$$

Theorem 2.2. *The Hilbert adjoint operator A^* of bounded A is unique and bounded linear operator with the norm*

$$\|A^*\| = \|A\| .$$

□

Definition 2.4. A bounded linear operator $A : H \longrightarrow H$ on a Hilbert space H is called *self-adjoint* if $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in H$.

Definition 2.5. A self-adjoint operator A is called *positive* denoted as $A \geq 0$ if $(Ax, x) \geq 0$ for all $x \in H$.

Definition 2.6. Let $A : D(A) \longrightarrow H$ be a linear operator with $\overline{D(A)} = H$. A is called a *symmetric* if $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in D(A)$.

If A is symmetric and $D(A) = D(A^*)$, then A is a self-adjoint operator.

2.4 Spectrum

Definition 2.7. Let H be a Hilbert space and $A : H \longrightarrow H$ be a linear operator with $D(A) \subset H$. We associate the operator $A_\lambda = A - \lambda I$, where $\lambda \in \mathbb{C}$ and I is the identity operator on $D(A)$.

If A_λ has an inverse, we denote it by $R_\lambda(A)$ and we call it the *resolvent* operator of A, or simply, resolvent of A.

$$R_\lambda(A) = (A - \lambda I)^{-1}.$$

Definition 2.8. Let A be a linear operator with the $D(A)\, subset H$ and H is a Hilbert space. $\lambda \in \mathbb{C}$ is called a *regular* value of A if

(R1) $R_\lambda(A)$ exists.

(R2) $R_\lambda(A)$ is bounded.

(R3) $R_\lambda(A)$ is defined on a set which is dense in H.

The *resolvent* set $\rho(A)$ of A is the set of all regular values of A. Its complement $\sigma(A) = C - \rho(A)$ is called spectrum of A , and a $\lambda \in \sigma(A)$ is called spectral value of A. Furthermore, the spectrum $\rho(A)$ is partitioned into three disjoint sets as follows.

The *point spectrum* or *discrete spectrum* $\sigma_p(A)$ is the set such that $R_\lambda(A)$ does not exist. $\lambda \in \sigma(A)$ is called an *eigenvalue* of A.

The *continuous spectrum* $\sigma_c(A)$ is the set such that $R_\lambda(A)$ exists and satisfies (R3) but not (R2), that is $R_\lambda(T)$ unbounded.

The *residual spectrum* $\sigma_r(A)$ is the set such that $R_\lambda(A)$ exists (and may be bounded or not) but does not satisfy (R3), that is the domain of $R_\lambda(A)$ is not dense in H.

If $A_\lambda x = (A - \lambda I)\, x = 0$ for some $x \neq 0$, then $\lambda \in \sigma_p(A)$, by definition, that is, λ is an eigenvalue of A.

The vector x is called an *eigenvector* of A corresponding to eigenvalue λ. The subspace of $D(A)$ consisting of 0 and all eigenvectors of A corresponding to an eigenvalue λ of A is called the *eigenspace* of A corresponding to that eigenvalue λ.

$$\sigma(A) = \sigma_c(A) \cup \sigma_p(A) \cup \sigma_r(A),$$

$$\sigma(A) \cup \rho(A) = C.$$

Definition 2.9. Let H be a Hilbert space over the field of real numbers and for any $x \in H$, let $\|x\|$ denote the norm of x. Let J be any interval of the real line R. A function $x : J \to H$ is called an *abstract function.*

A function $x(t)$ is said to be *continuous* at the point $t_0 \in J$, if

$$\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0.$$

If $x : J \to H$ is continuous at each point of J, then we say that x is continuous on J and we write $x \in C\left[J, H\right]$.

Definition 2.10. Let H, H_1, H_2 be Hilbert spaces. A bilinear operator $P : H \times H_1 \to H_2$ with norm less than or equal to 1 i.e.

$$\|P\left(x, y\right)\| \leq \|x\| \| y\| ,$$

is called a *product operator*. We write $P\left(x, y\right) = xy$. Let $x : [a, b] \to H$ and $y : [a, b] \to H_1$ be two bounded functions such that the product $x(t)y(t) \in H_2$, for each $t \in [a, b]$ is linear in both x, y, and

$$\|x(t)y(t)\| \leq \|x(t)\| \, \|y(t)\|$$

(for example, $x(t) = A(t)$ is an operator with domain $D\left[A\left(t\right)\right] \supset H_1$, or one of the function x, y is a scalar function).

Let us denote the partition $(a = t_0 < t_1 < t_2 < ... < t_n = b)$ together with the points τ_i $(t_i < \tau_1 < t_{i+1}, i = 0, 1, 2, ..., n-1)$ by π and set $|\pi|$=$\max_i |t_{i+1} - t_i|$. The *Stieltjes sum* is

$$S_\pi = \sum_{i=1}^{n-1} x\left(\tau_i\right)\left[y\left(t_{i+1}\right) - y\left(t_i\right)\right].$$

If the $\lim S_\pi$ exist as $|\pi| \to 0$ and defines an element I in H_2 independent of π, then we call I the *Stieltjes integral* of the function $x\left(t\right)$ with respect to the function $y\left(t\right)$, and is denoted by

$$\int_a^b x\left(t\right) dy\left(t\right).$$

Theorem 2.3. *If $x \in C\left(\left[a, b\right], H\right)$ and $y : [a, b] \to H_1$ is of bounded variation on $[a, b]$, then the Stieltjes integral*

$$\int_a^b x\left(t\right) dy\left(t\right) \quad \textit{exists.}$$

□

Consider the function $y:[a,b]\to H_1$ and the partition

$$\pi : a = t_0 < t_1 < t_2 < ... < t_n = b.$$

Form the sum

$$V=\sum_{i=1}^{n-1}\|y(t_{i+1})-y(t_i)\|.$$

The supremum of the set of all possible sums V is called the (strong) *total variation* of the function $y(t)$ on the interval $[a,b]$ and is denoted by $V_a^b(y)$. If $V_a^b(y)<\infty$, then $y(t)$ is called an *abstract function of bounded variation* on $[a,b]$.

Example 2.4. *If $x\in C([a,b],H)$ and $y:[a,b]\to H_1$ is of bounded variation on $[a,b]$, then*

$$\left\|\int_a^b x(t)\,dy(t)\right\| \le \int_a^b \|x(t)\|\,dV_a^t[y(t)] \le \max_{t\in[a,b]}\|x(t)\|\,V_a^b[y(t)].$$

2.5 Projection Operator. Spectral Family

Let H be a Hilbert space and C be a closed subspace of H. Then, H can be written as the direct sum of C and its orthogonal complement $C^{\perp}$. More precisely,

$$H = C\oplus C^{\perp} \tag{2.1}$$

$$x = y+z \quad , \quad \textit{where}\ \ y\in C, z\in C^{\perp}.$$

Since the sum is direct, y is uniquely determined for each given $x\in H$. Thus, the spectral representation of unit matrix defines a linear operator $P:H\to H$ by $x\mapsto y=Px$. P is called an *orthogonal projection* or projection on H.

Theorem 2.4. *A bounded linear operator $P:H\longrightarrow H$ on a Hilbert space H is projection if and only if P is self-adjoint and idempotent i.e. $P^2=P$.* □

Recall the spectral family from dimensional case. Let A be an $n\times n$ matrix with n distinct eigenvalues, say $\lambda_1<\cdots<\lambda_n$. Then, A has an orthogonal set of n vectors $x_1,\cdots,x_n$, where x_j corresponds to λ_j and for convenience we write these vectors as column vectors. This basis for H has a unique representation:

$$x=\sum_{j=1}^{n}\gamma_j x_j\ , \qquad \gamma_j=(x,x_j)=x^T\overline{x_j},$$

x_j is an eigenvector of A, so that we have $Ax_j = \lambda_j x_j$.

$$Ax = \sum_{j=1}^{n} \lambda_i \gamma_j x_j.$$

We can define an operator

$$P_j : H \longrightarrow H,$$

$$x \longrightarrow \gamma_j x_j \ .$$

Obviously, P_j is the projection (orthogonal projection) of H onto the eigenspace of A corresponding to λ_j .

Theorem 2.5. *(*Spectral Theorem*) Let $A : H \longrightarrow H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then there is a family of orthogonal projections $\{E(\lambda)\}, \lambda \in \mathbb{R}$ such that*

$$\lambda_1 \leq \lambda_2 \Rightarrow E(\lambda_1) E(\lambda_1) = E(\lambda_2) E(\lambda_1) = E(\lambda_1);$$

$$E(\lambda + \varepsilon) \rightarrow E(\lambda) \quad (strongly) \ \ as \ \varepsilon \rightarrow 0^+;$$

$$E(\lambda) \rightarrow 0 \quad (strongly) \ \ as \ \lambda \rightarrow -\infty,$$

and

$$E(\lambda) \rightarrow I \quad (strongly) \ \ as \ \lambda \rightarrow +\infty;$$

1. *A has the spectral representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda,$$

where E_λ is the spectral family associate with A; the integral is to be understood in the sense of uniform operator convergence, and for all $x, y \in H$.

$$\langle Ax, y \rangle = \int_{m-0}^{M} \lambda dw(\lambda) \qquad\qquad w(\lambda) = \langle E_\lambda x, y \rangle$$

where the integral is an ordinary Riemann-Stieltjes integral.

2. *If P is a polynomial in λ with real coefficients,*

$$P(\lambda) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + ... + \alpha_0$$

then the operator $P(A)$ defined by

$$P(A) = \alpha_n A^n + \alpha_{n-1} A^{n-1} + ... + \alpha_0 I$$

has the spectral representation

$$P(A) = \int_{m-0}^{M} P(\lambda)\, dE_\lambda$$

for all $x, y \in H$.

□

Theorem 2.6. *Let* $A : D(A) \longrightarrow H$ *be a self-adjoint linear operator, where* H *is a complex Hilbert space and* $D(A)$ *is dense in* H. *Then* A *has the spectral representation*

$$A = \int_{m}^{\infty} \lambda dE_\lambda \qquad and \qquad I = \int_{m}^{\infty} dE_\lambda.$$

For continuously bounded function F *on* $[m, \infty]$,

$$F(A) = \int_{m}^{M} F(\lambda)\, dE_\lambda.$$

□

From Theorem 2.6, properties of E_λ, and Stieltjes integral it follows that

$$\|F(A)x\| \leq \int_{m}^{\infty} |f(\lambda)|\, d\,\|E_\lambda x\| \leq \int_{m}^{\infty} |f(\lambda)|\, dE_\lambda \,\|x\|$$

$$\leq \sup_{m \leq \lambda < \infty} |f(\lambda)| \int_{m}^{\infty} dE_\lambda \,\|x\|$$

Thus,

$$\|F(A)x\| \leq \sup_{m \leq \lambda < \infty} |f(\lambda)|\, \|x\|$$

or

$$\|F(A)\| \leq \sup_{m \leq \lambda < \infty} |f(\lambda)|\,.$$

Example 2.5. *Let* A *be the operator defined on the Example 2.3. Prove that*

$$\|\exp(-At)\| \leq e^{-t},$$

Solution. Using the spectral representation of the self-adjoint positive defined operators we can write

$$\exp(-At)\varphi = \int_1^{\infty} \exp(-\mu t) dE_\mu \varphi,$$

where (E_μ) is the spectral family associated with A. Hence, for all $t \geq 0$ we have

$$\| \exp(-At)\|_{H\to H} \leq \sup_{1\leq\mu<\infty} |\exp(-\mu t)| = \exp(-t).$$

CHAPTER 3

MULTIPOINT NONLOCAL BOUNDARY VALUE PROBLEMS
REVERSE PARABOLIC EQUATIONS

In this chapter, we consider multipoint nonlocal boundary value problems for reverse parabolic equations. We establish the well-posedness of these problems in the space of smooth functions. In applications, we obtain new coercivity estimates for the solution of parabolic differential equations.

3.1 The Differential Problem

The role played by coercivity inequalities (well-posedness) in the study of boundary value problems for partial differential equations is well known (see, e.g., [Ladyzhenskaya O.A., Solonnikov V.A., 1967]-[Vishik M.L., Myshkis A.D., Oleinik O.A., 1959]). Well-posedness of nonlocal boundary value problems for partial differential equations parabolic and elliptic types has been studied extensively by many researchers (see, e.g., [Ashyralyev A., 2006]-[Shakhmurov V.B., 2004], and the references given therein).

In the present chapter, we study the well-posedness of the nonlocal boundary value problem

$$\begin{cases} \frac{du(t)}{dt} - Au(t) = f(t) \qquad (0 \le t \le 1), \\ u(1) = \sum_{k=1}^{p} \alpha_k u(\theta_k) + \varphi, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1 \end{cases} \tag{3.1}$$

for the differential equation in a Hilbert space H with self -adjoint positive definite operator A.

A function $u(t)$ is said to be a *solution* of the problem (3.1) if the following conditions are satisfied:

1. $u(t)$ is continuously differentiable on the segment $[0, 1]$.

2. The element $u(t)$ belongs to $D(A)$ for all $t \in [0, 1]$ and the function $Au(t)$ is continuous on the segment $[0, 1]$.

3. $u(t)$ satisfies the equation and the nonlocal boundary conditions (3.1).

A solution of problem (3.1) defined in this manner is from now referred as a solution of problem (3.1) in the space $C(H) = C([0,1], H)$ of all continuous functions $\varphi(t)$ defined on $[0,1]$ with values in H equipped with the norm

$$\| \varphi \|_{C(H)} = \max_{0 \le t \le 1} ||\varphi(t)||_H.$$

Problem (3.1) is said to be *well-posed* in $C(H)$, if the solutions of (3.1) satisfy the following *coercivity inequality*

$$\|u'\|_{C(H)} + \|Au(t)\|_{C(H)} \le M_C \left(\|f\|_{C(H)} + \|A\varphi\|_H \right)$$

where $1 \le M_C < \infty$, which is independent of $f(t) \in C(H)$, $\varphi \in D(A)$.

It is well known that problem (3.1) is ill-posed in $C(H)$.

For $\alpha \in [0,1]$, we let $C_1^\alpha(H)$ and $C^\alpha(H)$ denote the Banach spaces obtained by the completion of the set of all smooth $H-$valued functions $\varphi(t)$ on $[0,1]$ with the norms

$$\begin{aligned}
\|\varphi\|_{C_1^\alpha(H)} &= \|\varphi\|_{C(H)} + \sup_{0 \le t < t+\tau \le 1} \frac{(1-t)^\alpha \|\varphi(t+\tau) - \varphi(t)\|_H}{\tau^\alpha}, \\
\|\varphi\|_{C^\alpha(H)} &= \|\varphi\|_{C(H)} + \sup_{0 \le t < t+\tau \le 1} \frac{\|\varphi(t+\tau) - \varphi(t)\|_H}{\tau^\alpha}.
\end{aligned}$$

We say problem (3.1) is well-posed in $\mathcal{F}(H)$, if for each $f(t) \in \mathcal{F}(H)$ problem (3.1) is uniquely solvable and the following coercivity inequality hold:

$$\|u'\|_{\mathcal{F}(H)} + \|Au\|_{\mathcal{F}(H)} \le M \left(\|f\|_{\mathcal{F}(H)} + \|A\varphi\|_{H'} \right),$$

where $H' \subset H$, $M(\alpha)$ does not depend on $f(t)$ and φ.

We are interested in studying the well-posedness of problem (3.1) under the assumption

$$\sum_{k=1}^{p} |\alpha_k| \le 1. \tag{3.2}$$

In present chapter, the well-posedness of the multipoint nonlocal boundary value problem (3.1) in space $C_1^\alpha(H)$ and $C^\alpha(H)(0 < \alpha < 1)$ is established. In applications, this abstract result permits us to obtain coercivity estimates in various Hölder norms for the solutions of nonlocal boundary value problems for parabolic equations.

3.2 Theorem on well-posedness

Now, let us give some lemmas we need in the sequel. Throughout, let H be a Hilbert space, A be a positive definite self-adjoint operator with $A \geq \delta I$, where $\delta > 0$.

Lemma 3.7. *([Ashyralyev A., Sobolevskii P.E.,2004]) For every $0 < t < t + \tau \leq 1$ and $0 \leq \beta \leq 1$, we have*

$$\left\| e^{-tA} \right\|_{H \to H} \leq 1, \tag{3.3}$$

$$\left\| tAe^{-tA} \right\|_{H \to H} \leq 1, \tag{3.4}$$

$$\left\| e^{-tA} - e^{-(t+\tau)A} \right\|_{H \to H} \leq M \frac{\tau^\beta}{(t+\tau)^\beta}, \tag{3.5}$$

$$\left\| A(e^{-tA} - e^{-(t+\tau)A}) \right\|_{H \to H} \leq M \frac{\tau^\beta}{t(t+\tau)^\beta}, \tag{3.6}$$

for some $M > 0$. □

Lemma 3.8. *Suppose that assumption (3.2) holds. Then, the operator*

$$I - \sum_{k=1}^{p} \alpha_k e^{-(1-\theta_k)A}$$

has an inverse

$$T = \left(I - \sum_{k=1}^{p} \alpha_k e^{-(1-\theta_k)A} \right)^{-1}$$

and the following estimate is satisfied:

$$||T||_{H \to H} \leq \frac{1}{1 - e^{-(1-\theta_p)\delta}} \leq C(\delta, \theta_p). \tag{3.7}$$

Proof. The proof follows from the triangle inequality, assumption (3.2), and estimate

$$\left\| \left(I - \sum_{k=1}^{p} \alpha_k e^{-(1-\theta_k)A} \right)^{-1} \right\|_{H \to H} \leq \sup_{\delta \leq \lambda < \infty} \frac{1}{|1 - \sum_{k=1}^{p} \alpha_k e^{-(1-\theta_k)\lambda}|}.$$

□

Let us now obtain the formula for solution of problem (3.1). It is clear that for smooth data, the inverse Cauchy problem

$$\frac{du}{dt} - Au(t) = f(t), 0 < t < 1, u(1) = \xi$$

has a unique solution

$$u(t) = e^{-(1-t)A}u(1) - \int_t^1 e^{-(s-t)A} f(s) ds. \tag{3.8}$$

Using (3.8) and the nonlocal boundary condition

$$\xi = u(1) = \sum_{k=1}^{p} \alpha_k u(\theta_k) + \varphi,$$

it can be written as follows

$$u(1) = \sum_{k=1}^{p} \alpha_k \left\{ e^{-(1-\theta_k)A} u(1) - \int_{\theta_k}^{1} e^{-(s-\theta_k)A} f(s) ds \right\} + \varphi.$$

By Lemma 3.8, we obtain

$$u(1) = T\left(-\sum_{k=1}^{p} \alpha_k \int_{\theta_k}^{1} e^{-(s-\theta_k)A} f(s) ds + \varphi \right). \tag{3.9}$$

Thus, the nonlocal boundary value problem (3.1) is uniquely solvable and for the solution, formulas (3.8) and (3.9) are valid.

Theorem 3.9. *Assume that $\varphi \in D(A)$, $f(t) \in C_1^{\alpha}(H)$ and (3.2). Then, problem (3.1) is well-posed in $C_1^{\alpha}(H)$ and the following coercivity estimate*

$$\|u'\|_{C_1^{\alpha}(H)} + \|Au\|_{C_1^{\alpha}(H)} \leq C(\delta, \theta_p) \left(\frac{1}{\alpha(1-\alpha)} \|f\|_{C_1^{\alpha}(H)} + \|A\varphi\|_H \right)$$

is valid, where $C(\delta, \theta_p)$ is independent of φ and $f(t)$, $t \in [0,1]$.

Proof. Using formula (3.8), we get for $t \in (0,1)$

$$\begin{aligned} Au(t) &= e^{-(1-t)A} Au(1) - \int_t^1 Ae^{-(s-t)A} f(s) ds \\ &= e^{-(1-t)A} Au(1) - \int_t^1 Ae^{-(s-t)A} (f(s) - f(t)) ds + (e^{-(1-t)A} - I) f(t). \end{aligned} \tag{3.10}$$

From estimates (3.3), (3.4), and definition of $C_1^{\alpha}(H)-$norm it follows that

$$\begin{aligned} \|Au(t)\|_H &\leq \|Au(1)\|_H + \frac{\|f\|_{C_1^{\alpha}(H)}}{(1-t)^{\alpha}} \int_t^1 \frac{ds}{(s-t)^{1-\alpha}} + 2\|f\|_{C_1^{\alpha}(H)} \\ &\leq \|Au(1)\|_H + \frac{4}{\alpha} \|f\|_{C_1^{\alpha}(H)}. \end{aligned} \tag{3.11}$$

Next, let us estimate $\|Au(1)\|_H$.

Using formula (3.9), we obtain

$$\begin{aligned} Au(1) &= T\left(-\sum_{k=1}^{p} \alpha_k \int_{\theta_k}^{1} Ae^{-(s-\theta_k)A} f(s) ds + A\varphi \right) \qquad (3.12) \\ &= T\Big(\sum_{k=1}^{p} \alpha_k \Big\{ -\int_{\theta_k}^1 Ae^{-(s-\theta_k)A}(f(s) - f(\theta_k)) ds - (I - e^{-(1-\theta_k)A}) f(\theta_k) \Big\} + A\varphi\Big) \end{aligned}$$

It follows from assumption (4.2), estimates (3.3), (3.4), (3.7), the definition of $C_1^\alpha(H)-$norm and formula (3.12) that

$$\begin{aligned}\|Au(1)\|_H &\leq C(\delta,\theta_p)\left[\sum_{k=1}^{p}|\alpha_k|\left\{\frac{\|f\|_{C_1^\alpha(H)}}{(1-\theta_k)^\alpha}\int_{\theta_k}^{1}\frac{ds}{(s-\theta_k)^{1-\alpha}}+2\|f\|_{C_1^\alpha(H)}\right\}+\|A\varphi\|_H\right]\\ &\leq C(\delta,\theta_p)\left(\frac{1}{\alpha}\|f\|_{C_1^\alpha(H)}+\|A\varphi\|_H\right). \qquad (3.13)\end{aligned}$$

Thus, combining estimates (3.11), (3.13), we get

$$\|Au\|_{C(H)}\leq C(\delta,\theta_p)\left(\frac{1}{\alpha}\|f\|_{C_1^\alpha(H)}+\|A\varphi\|_H\right). \qquad (3.14)$$

Now, we estimate

$$\sup_{0\leq t<t+\tau\leq 1}\frac{(1-t)^\alpha\|Au(t+\tau)-Au(t)\|_H}{\tau^\alpha}.$$

If $1-t\leq 2\tau$, then estimate (3.14) yields

$$\|Au(t+\tau)-Au(t)\|_H\leq C(\delta,\theta_p)\frac{\tau^\alpha}{(1-t)^\alpha}\left(\frac{1}{\alpha}\|f\|_{C_1^\alpha(H)}+\|A\varphi\|_H\right). \qquad (3.15)$$

Now, let $1-t>2\tau$. From identity (3.10) it follows that

$$\begin{aligned}Au(t)-Au(t+\tau) &= (e^{-(1-t)A}-e^{-(1-t-\tau)A})Au(1)\\ &- \int_t^{t+2\tau}Ae^{-(s-t)A}(f(s)-f(t))ds\\ &+ \int_{t+\tau}^{t+2\tau}Ae^{-(s-t-\tau)A}(f(s)-f(t+\tau))ds\\ &- \int_{t+2\tau}^{1}A(e^{-(s-t)A}-e^{-(s-t-\tau)A})(f(s)-f(t)ds\\ &+ (e^{-(1-t-\tau)A}-e^{-(1-t)A})(f(t+\tau)-f(t))\\ &+ (e^{-(1-t)A}-e^{-(1-t-\tau)A})f(t+\tau)\\ &= I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t). \qquad (3.16)\end{aligned}$$

Let us first estimate $I_1(t)$. Using estimates (3.5), (3.13), we obtain

$$\begin{aligned}\|I_1(t)\|_H &\leq \|e^{-(1-t)A}-e^{-(1-t-\tau)A}\|_{H\to H}\|Au(1)\|_H\\ &\leq \frac{M\tau^\alpha}{(1-t)^\alpha}C(\delta,\theta_p)\left(\frac{1}{\alpha}\|f\|_{C_1^\alpha(H)}+\|A\varphi\|_H\right). \qquad (3.17)\end{aligned}$$

Now, from estimate (3.4) and the definition of $C_1^\alpha(H)-$norm it follows that

$$\begin{aligned}\|I_2(t)\|_H &\leq \int_t^{t+2\tau}\|Ae^{-(s-t)A}\|_{H\to H}\|f(s)-f(t)\|_H ds\\ &\leq \frac{2^\alpha\tau^\alpha}{(1-t)^\alpha\,\alpha}\|f\|_{C_1^\alpha(H)}. \qquad (3.18)\end{aligned}$$

Using estimate (3.4), the definition of $C_1^\alpha(H)-$norm, and the fact $1-t\leq 2\tau$, we get

$$\begin{aligned}\|I_3(t)\|_H &\leq \int_{t+\tau}^{t+2\tau}\|Ae^{-(s-t-\tau)A}\|_{H\to H}\|f(s)-f(t+\tau)\|_H ds\\ &\leq \frac{2^\alpha\tau^\alpha}{(1-t)^\alpha\ \alpha}\ \|f\|_{C_1^\alpha(H)}. \qquad (3.19)\end{aligned}$$

It follows from estimate (3.6) for $\beta=1$, the definition of $C_1^\alpha(H)-$norm, and the fact $s\geq t+2\tau$ that

$$\begin{aligned}\|I_4(t)\|_H &\leq \int_{t+2\tau}^{1}\|A(e^{-(s-t)A}-e^{-(s-t-\tau)A})\|_{H\to H}\|f(s)-f(t)\|_H ds\\ &\leq M\frac{2^{-1+\alpha}\ \tau^\alpha}{(1-t)^\alpha\ (1-\alpha)}\|f\|_{C_1^\alpha(H)}. \qquad (3.20)\end{aligned}$$

By estimate (3.3) and the definition of $C_1^\alpha(H)-$norm, we obtain

$$\begin{aligned}\|I_5(t)\|_H &\leq \|e^{-(1-t-\tau)A}-e^{-(1-t)A}\|_{H\to H}\|f(t+\tau)-f(t)\|_H\\ &\leq \frac{2\tau^\alpha}{(1-t)^\alpha}\|f\|_{C_1^\alpha(H)}. \qquad (3.21)\end{aligned}$$

Finally, from estimate (3.5) and the definition of $C_1^\alpha(H)-$norm it follows that

$$\begin{aligned}\|I_6(t)\|_H &\leq \|e^{-(1-t)A}-e^{-(1-t-\tau)A}\|_{H\to H}\|f(t+\tau)\|_H\\ &\leq \frac{M\tau^\alpha}{(1-t)^\alpha}\|f\|_{C_1^\alpha(H)}. \qquad (3.22)\end{aligned}$$

Hence, combining estimates (3.17)-(3.22), for $1-t>2\tau$ we get

$$\frac{(1-t)^\alpha\|Au(t+\tau)-Au(t)\|_H}{\tau^\alpha}\leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)}+\|A\varphi\|_H\right).$$

Thus,

$$\sup_{0\leq t<t+\tau\leq 1}\frac{(1-t)^\alpha\|Au(t+\tau)-Au(t)\|_H}{\tau^\alpha}\leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)}+\|A\varphi\|_H\right). \qquad (3.23)$$

From estimates (3.14), (3.23) it follows that

$$\|Au\|_{C_1^\alpha(H)}\leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)}+\|A\varphi\|_H\right). \qquad (3.24)$$

Using differential equation (3.1), estimate (3.24) and the triangle inequality, we obtain

$$\|u'\|_{C_1^\alpha(H)}\leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\|f\|_{C_1^\alpha(H)}+\|A\varphi\|_H\right). \qquad (3.25)$$

This finishes the proof of Theorem 3.9. □

Note that the spaces $C_1^\alpha(H)$ in which coercive solvability has been established, depend on the parameter $\alpha \in [0,1]$. However, the constants in the coercive inequalities depend on $\frac{1}{\alpha(1-\alpha)}$. Hence, we can not choose the parameter α freely from $[0,1]$, where problem (3.1) is well posed. As we note above problem (3.1) is ill posed in $C(H)$ ($C_1^\alpha(H)$ for $\alpha=0$). We cannot establish well-posedness of problem (3.1) in $C_1^\alpha(H)$ for $\alpha=1$.

Let $H_\alpha = H_{\alpha,\infty}(H,A)$ be the *fractional space*, consisting all $v \in H$ for which the following norm

$$\|v\|_{H_\alpha} = \|v\|_H + \sup_{\lambda>0} \|\lambda^{1-\alpha} A e^{-\lambda A} v\|_H$$

is finite.

Theorem 3.10. *Assume that $f(t) \in C^\alpha(H)$, $f(1) - \sum_{k=1}^p \alpha_k f(\theta_k) + A\varphi \in H_\alpha$ and (3.2). Then problem (3.1) is well-posed in $C^\alpha(H)$ and the following coercivity estimate*

$$\begin{aligned} & \|u'\|_{C^\alpha(H)} + \|Au\|_{C^\alpha(H)} + \|u'\|_{C(H_\alpha)} \\ \le \; & M\left(\frac{1}{\alpha}\|f(1) - \sum_{k=1}^p \alpha_k f(\theta_k) + A\varphi\|_{H_\alpha} + \frac{C(\delta,\theta_p)}{\alpha(1-\alpha)}\|f\|_{C^\alpha(H)}\right) \end{aligned}$$

holds, where M does not depend on φ and $f(t)$, $t \in [0,1]$.

Proof. From formula (3.10) it follows that for $t \in (0,1)$

$$\begin{aligned} Au(t) &= -f(t) + e^{-(1-t)A}(Au(1) + f(1)) \\ &- \int_t^1 Ae^{-(s-t)A}(f(s) - f(t))ds + e^{-(1-t)A}(f(t) - f(1)) \\ &= J_1(t) + J_2(t) + J_3(t) + J_4(t). \end{aligned} \tag{3.26}$$

Let us establish the estimate for $\|Au\|_{C^\alpha(H)}$.

Clearly, we have

$$\|J_1\|_{C^\alpha(H)} = \|f\|_{C^\alpha(H)}. \tag{3.27}$$

Using the definition of $H_\alpha-$norm and the equality

$$\int_{1-t-\tau}^{1-t} -Ae^{-sA}ds = e^{-(1-t)A} - e^{-(1-t-\tau)A},$$

we get

$$\|J_2\|_{C^\alpha(H)} \le \frac{1}{\alpha}\|Au(1) + f(1)\|_{H_\alpha}. \tag{3.28}$$

It follows from estimate (3.4) and the definition of $C^\alpha(H)-$norm that

$$\|J_3(t)\|_H \leq \frac{(1-t)^\alpha}{\alpha}\|f\|_{C^\alpha(H)}, \quad \text{for any } t \in [0,1]. \tag{3.29}$$

Hence, using estimate (3.29), we obtain

$$\|J_3\|_{C(H)} \leq \frac{1}{\alpha}\|f\|_{C^\alpha(H)}. \tag{3.30}$$

Let us now estimate

$$\sup_{0\leq t<t+\tau\leq 1} \frac{\|J_3(t+\tau)-J_3(t)\|_H}{\tau^\alpha}.$$

Let $1-t \leq 2\tau$. Then, from the triangle inequality, estimate (3.29) it follows that

$$\begin{aligned}\frac{\|J_3(t+\tau)-J_3(t)\|_H}{\tau^\alpha} &\leq \frac{\|J_3(t+\tau)\|_H + \|J_3(t)\|_H}{\tau^\alpha} \\ &\leq \frac{(1-t-\tau)^\alpha + (1-t)^\alpha}{\tau^\alpha \alpha}\|f\|_{C^\alpha(H)} \\ &\leq \frac{(2^\alpha+1)}{\alpha}\|f\|_{C^\alpha(H)}. \end{aligned} \tag{3.31}$$

Now, let $1-t > 2\tau$. Then, we can write as

$$J_3(t) - J_3(t+\tau) = J_{31}(t) + J_{32}(t) + J_{33}(t) + J_{34}(t),$$

where $J_{31}(t) = I_2(t),\ J_{32}(t) = I_3(t),\ J_{33}(t) = I_4(t)$ (see page 5), and

$$J_{34}(t) = (e^{-(1-t-\tau)A} - e^{-\tau A})(f(t+\tau) - f(t)).$$

Thus, we have

$$\|J_{31}(t)\|_H \leq \frac{2^\alpha\tau^\alpha}{\alpha}\|f\|_{C^\alpha(H)}, \tag{3.32}$$

$$\|J_{32}(t)\|_H \leq \frac{2^\alpha\tau^\alpha}{\alpha}\|f\|_{C^\alpha(H)}, \tag{3.33}$$

$$\|J_{33}(t)\|_H \leq M\frac{2^{-1+\alpha}\ \tau^\alpha}{(1-\alpha)}\|f\|_{C^\alpha(H)}. \tag{3.34}$$

Finally, from estimate (3.3) and the definition of $C^\alpha(H)-$norm it follows that

$$\|J_{34}(t)\|_H \leq 2\tau^\alpha\|f\|_{C^\alpha(H)}. \tag{3.35}$$

Therefore, estimates (3.32)-(3.35) result that for $1-t > 2\tau$,

$$\frac{\|J_3(t+\tau)-J_3(t)\|_H}{\tau^\alpha} \leq \frac{M}{\alpha(1-\alpha)}\|f\|_{C^\alpha(H)}. \tag{3.36}$$

Using estimates (3.31), (3.36), we get

$$\sup_{0\le t<t+\tau\le 1}\frac{\|J_3(t+\tau)-J_3(t)\|_H}{\tau^\alpha}\le\frac{M}{\alpha(1-\alpha)}\|f\|_{C^\alpha(H)}. \tag{3.37}$$

So, it follows from estimates (3.30), (3.37) that

$$\|J_3\|_{C^\alpha(H)}\le\frac{M}{\alpha(1-\alpha)}\|f\|_{C^\alpha(H)}. \tag{3.38}$$

By estimate (3.3) and the definition of $C^\alpha-$norm, we obtain

$$\|J_4(t)\|_H\le(1-t)^\alpha\|f\|_{C^\alpha(H)}\le\|f\|_{C^\alpha(H)},\text{ for all } t\in[0,1]. \tag{3.39}$$

Hence, estimate (3.39) results that

$$\|J_4\|_{C(H)}\le\|f\|_{C^\alpha(H)}. \tag{3.40}$$

Using estimates (3.3), (3.5), we get for all $0\le t<t+\tau\le 1$,

$$\begin{aligned}\|J_4(t+\tau)-J_4(t)\|_H &\le \|e^{-(1-t-\tau)A}-e^{-(1-t)A}\|_{H\to H}\|f(t)-f(1)\|_H\\ &+ \|e^{-(1-t-\tau)A}\|_{H\to H}\|f(t+\tau)-f(t)\|_H\\ &\le M\frac{\tau^\alpha}{(1-t)^\alpha}(1-t)^\alpha\|f\|_{C^\alpha(H)}+\tau^\alpha\|f\|_{C^\alpha(H)}\\ &\le (M+1)\tau^\alpha\|f\|_{C^\alpha(H)}\end{aligned} \tag{3.41}$$

Thus, it follows from estimate (3.41) that

$$\sup_{0\le t<t+\tau\le 1}\frac{\|J_4(t+\tau)-J_4(t)\|_H}{\tau^\alpha}\le M_1\|f\|_{C^\alpha(H)}. \tag{3.42}$$

Combining estimates (3.40), (3.42), we obtain

$$\|J_4\|_{C^\alpha(H)}\le M_1\|f\|_{C^\alpha(H)}. \tag{3.43}$$

From estimates (3.27), (3.28), (3.38) and (3.43) it follows that

$$\|Au\|_{C^\alpha(H)}\le M\left(\frac{1}{\alpha}\|Au(1)+f(1)\|_{H_\alpha}+\frac{1}{\alpha(1-\alpha)}\|f\|_{C^\alpha(H)}\right). \tag{3.44}$$

Using the triangle inequality, estimate (3.44), and differential equation (3.1), we get

$$\|u'\|_{C^\alpha(H)}\le M\left(\frac{1}{\alpha}\|Au(1)+f(1)\|_{H_\alpha}+\frac{1}{\alpha(1-\alpha)}\|f\|_{C^\alpha(H)}\right). \tag{3.45}$$

Now, let us establish the estimate for $\|u'\|_{C(H_\alpha)}$.

Formula (3.8) and differential equation (3.1) result that for $t \in (0,1)$

$$\begin{aligned} u'(t) &= e^{-(1-t)A}(Au(1)+f(1)) + e^{-(1-t)A}(f(t)-f(1)) \\ &- \int_t^1 Ae^{-(s-t)A}(f(s)-f(t))ds \\ &= G_1(t)+G_2(t)+G_3(t). \end{aligned}$$

Using estimate (3.3) and the definition of $H_\alpha-$norm, we obtain

$$\|G_1(t)\|_{H_\alpha} \le \|Au(1)+f(1)\|_{H_\alpha}. \tag{3.46}$$

Next, from estimate (3.4) and the definition of $C^\alpha-$norm it follows that

$$\|G_2(t)\|_{H_\alpha} \le \|f\|_{C^\alpha(H)}. \tag{3.47}$$

Finally, by estimate (3.3) and the definition of $C^\alpha-$norm, we get

$$\|G_3(t)\|_{H_\alpha} \le \frac{1}{1-\alpha}\|f\|_{C^\alpha(H)}. \tag{3.48}$$

Hence, estimates (3.46)-(3.48) result that

$$\|u'\|_{C^\alpha(H)} \le M\left(\|Au(1)+f(1)\|_{H_\alpha} + \frac{1}{1-\alpha}\|f\|_{C(H_\alpha)}\right). \tag{3.49}$$

Therefore, combining estimates (3.44), (3.45) and (3.49), we obtain

$$\begin{aligned} &\|u'\|_{C^\alpha(H)} + \|Au\|_{C^\alpha(H)} + \|u'\|_{C(H_\alpha)} \\ &\le M\left(\frac{1}{\alpha}\|Au(1)+f(1)\|_{H_\alpha} + \frac{1}{\alpha(1-\alpha)}\|f\|_{C^\alpha(H)}\right) \end{aligned} \tag{3.50}$$

Let us now estimate $\|Au(1)+f(1)\|_{H_\alpha}$. From formula (3.9) it follows that

$$\begin{aligned} Au(1)+f(1) &= T\{-\sum_{k=1}^{p}\alpha_k \int_{\theta_k}^{1} Ae^{-(s-\theta_k)A}(f(s)-f(\theta_k))ds \\ &+ \sum_{k=1}^{p}\alpha_k e^{-(1-\theta_k)A}(f(\theta_k)-f(1)) \\ &+ f(1) - \sum_{k=1}^{p}\alpha_k f(\theta_k) + A\varphi\} \\ &= P_1+P_2+P_3. \end{aligned}$$

Using estimates (3.4), (3.7), the assumption (3.2), and the definition of $C^\alpha(H)-$norm, we get

$$\|P_1\|_{H_\alpha} \le C(\delta,\theta_p)\frac{1}{1-\alpha}\|f\|_{C^\alpha(H)}, \tag{3.51}$$

$$\|P_2\|_{H_\alpha} \leq C(\delta,\theta_p)\|f\|_{C^\alpha(H)}. \tag{3.52}$$

$$\|P_3\|_{H_\alpha} = \|f(1) - \sum_{k=1}^{p} \alpha_k f(\theta_k) + A\varphi\|_{H_\alpha}. \tag{3.53}$$

Estimates (3.50), (3.51)-(3.53) concludes the proof of Theorem 3.10. □

3.3 Application

In this section, we consider the applications of Theorem 3.9 and Theorem 3.10.

First, the nonlocal boundary value problem of parabolic type

$$\begin{cases} u_t + (a(x)u_x)_x - \delta u = f(t,x), \ 0 < t < 1, \ 0 < x < 1, \\ u(1,x) = \sum\limits_{m=1}^{p} \alpha_m u(\theta_m, x) + \varphi(x), 0 \leq x \leq 1, \\ 0 \leq \theta_1 < \theta_2 < \cdots < \theta_p < 1, \\ u(t,0) = u(t,1), \ u_x(t,0) = u_x(t,1), \ 0 \leq t \leq 1 \end{cases} \tag{3.54}$$

under assumption (3.2) is considered. The problem (3.54) has a unique smooth solution $u(t,x)$ for (3.2), $\delta > 0$ and the smooth functions $a(x) \geq a > 0 \quad (x \in (0,1))$, $\varphi(x)$ $(x \in [0,1])$ and $f(t,x) \quad (t, x \in [0,1])$. This allows us to reduce the nonlocal boundary values problem (3.54) to the nonlocal boundary value problem (3.1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by (3.54).Let us give a number of corollaries of the abstract Theorem 3.9.

Theorem 3.11. *For solutions of the problem (3.54), we have the following stability inequalities*

$$\| u_t \|_{C_1^\alpha(L_2([0,1]))} + \| u \|_{C_1^\alpha(W_2^2([0,1]))}$$
$$\leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)} \| f \|_{C_1^\alpha(L_2([0,1]))} + \|\varphi\|_{W_2^2([0,1])}\right),$$

where $C(\delta,\theta_p)$ does not depend on $\varphi(x)$, $f(t,x)$. □

Theorem 3.12. *Let*

$$(a(x)\varphi_x(x))_x - \delta\varphi(x) = f(1,x) - \sum_{k=1}^{p} \alpha_k f(\theta_k, x).$$

Then, for solutions of the problem (3.54), we have the following stability inequalities

$$\| u_t \|_{C^\alpha(L_2([0,1]))} + \| u \|_{C^\alpha(W_2^2([0,1]))} \leq \frac{C(\delta,\theta_p)}{\alpha(1-\alpha)} \| f \|_{C^\alpha(L_2([0,1]))},$$

where $C(\delta,\theta_p)$ is independent of $\varphi(x)$, $f(t,x)$.

The proofs of Theorem 3.11 and Theorem 3.12 are based on the abstract Theorem 3.9, Theorem 3.10, and the symmetry properties of the space operator generated by problem (3.54).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n = \{x = (x_1, \cdots, , x_n) : 0 < x_i < 1, i = 1, \cdots, n\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, the boundary value problem for the multi-dimensional parabolic equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \sum\limits_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} - \delta u = f(t,x), \\ x = (x_1, \ldots, x_n) \in \Omega, \ 0 < t < 1, \\ u(1,x) = \sum\limits_{i=1}^{p} \alpha_i u(\theta_i, x) + \varphi(x), x \in \overline{\Omega}, \\ 0 < \theta_1 < \theta_2 < \cdots < \theta_p \leq 1, \\ u(t,x) = 0, \ x \in S, \ 0 \leq t \leq 1 \end{cases} \tag{3.55}$$

under assumption (3.2) is considered. Here $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,1), \ x \in \Omega)$ are given smooth functions and $a_r(x) \geq a > 0$,$\delta > 0$.

We consider the Hilbert space $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \left(\int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \right)^{\frac{1}{2}}.$$

The problem (3.55) has a unique smooth solution $u(t,x)$ for (3.2) and the smooth functions $\varphi(x)$, $a_r(x)$ and $f(t,x)$. This allows us to reduce the problem (3.55) to the nonlocal boundary value problem (3.1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by (3.55).

Theorem 3.13. *For the solutions of the problem (3.55), the following stability inequalities*

$$\| u_t \|_{C_1^{\alpha}(L_2(\overline{\Omega}))} + \| u \|_{C_1^{\alpha}(W_2^2(\overline{\Omega}))} \leq C(\delta, \theta_p) \left(\frac{1}{\alpha(1-\alpha)} \| f \|_{C_1^{\alpha}(L_2(\overline{\Omega}))} + \|\varphi\|_{W_2^2(\overline{\Omega})} \right),$$

hold, where $C(\delta, \theta_p)$ does not depend on $\varphi(x)$, $f(t,x)$. □

Theorem 3.14. *Let*

$$\sum_{r=1}^{n} (a_r(x) \varphi_{x_r}(x))_{x_r} - \delta\varphi = f(1,x) - \sum_{k=1}^{p} \alpha_k f(\theta_k, x).$$

Then, for the solutions of the problem (3.55), the following stability inequalities

$$\| u_t \|_{C^\alpha(L_2(\overline{\Omega}))} + \| u \|_{C^\alpha(W_2^2(\overline{\Omega}))} \leq \frac{C(\delta,\theta_p)}{\alpha(1-\alpha)} \| f \|_{C^\alpha(L_2(\overline{\Omega}))},$$

hold, where $C(\delta,\theta_p)$ *is independent of* $\varphi(x)$, $f(t,x)$.

The proofs of Theorem 3.13 and Theorem 3.14 are based on the abstract Theorem 3.9, Theorem 3.10, and the symmetry properties of the operator A^x defined by formula (3.55) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3.15. *([Sobolevskii P.E., 1975]) For the solutions of the elliptic differential problem*

$$\begin{cases} A^x u(x) = \omega(x), x \in \Omega, \\ u(x) = 0, x \in S, \end{cases}$$

the following coercivity inequality holds :

$$\sum_{r=1}^{n} ||u_{x_r x_r}||_{L_2(\overline{\Omega})} \leq M||\omega||_{L_2(\overline{\Omega})}.$$

□

CHAPTER 4

WELL-POSEDNESS OF THE ROTHE DIFFERENCE FOR REVERSE PARABOLIC EQUATIONS

We consider the Rothe difference scheme for the approximate solution of the abstract parabolic equation in a Hilbert space with the nonlocal boundary condition. Theorems on stability estimates, coercivity and almost coercivity estimates for the solution of this difference scheme are established. In applications, new coercivity inequalities for the solution of multi-point nonlocal boundary value difference equations of parabolic type are obtained.

In the previous chapter, we considered the abstract nonlocal boundary value problem

$$\begin{cases} \frac{du(t)}{dt} - Au(t) = f(t) \qquad (0 \leq t \leq 1), \\ u(1) = \sum_{k=1}^{p} \alpha_k u(\theta_k) + \varphi, \\ 0 \leq \theta_1 < \theta_2 < \cdots < \theta_p < 1 \end{cases} \tag{4.1}$$

in a Hilbert space H with self-adjoint positive definite operator A, under the assumption

$$\sum_{k=1}^{p} |\alpha_k| \leq 1. \tag{4.2}$$

The well-posedness of multi-point nonlocal boundary value problem (4.1) in spaces $C_1^{\alpha}(H)$ and $C^{\alpha}(H)$ was established. Moreover, as applications, these abstract results enabled us to obtain new coercivity estimates in various Hölder norms for the solutions of nonlocal boundary value problems for parabolic equations.

In the present chapter, our focus is the well-posedness of the first order of accuracy Rothe difference scheme

$$\begin{cases} \tau^{-1}(u_k - u_{k-1}) - Au_{k-1} = \varphi_k, \varphi_k = f(t_k), \\ t_k = k\tau, \ 1 \leq k \leq N, \ N\tau = 1, \\ u_N = \sum_{m=1}^{p} \alpha_m u_{\ell_m} + \varphi, \\ \ell_m = \left[\frac{\theta_m}{\tau}\right], 1 \leq m \leq p \end{cases} \tag{4.3}$$

for approximately solving problem (4.1).

Let $[0,1]_\tau = \{t_k = k\tau, k = 1, \cdots, N, \ N\tau = 1\}$ be the *uniform grid space* with step size $\tau > 0$, where N is a fixed positive integer.

Throughout the chapter, $\mathcal{F}([0,1]_\tau, H)$ denotes the linear space of grid functions $\varphi^\tau = \{\varphi_k\}_1^N$ with values in the Hilbert space H.

Let $\mathcal{C}_\tau(H) = \mathcal{C}([0,1]_\tau, H)$ be the Banach space of bounded grid functions with the norm

$$\|\varphi^\tau\|_{\mathcal{C}_\tau(H)} = \max_{1\le k\le N} \|\varphi_k\|_H.$$

For $\alpha \in [0,1]$, let $\mathcal{C}^\alpha(H) = \mathcal{C}^\alpha([0,1]_\tau, H)$ and $\mathcal{C}_1^\alpha(H) = \mathcal{C}_1^\alpha([0,1]_\tau, H)$ be respectively the Hölder space and the weighted Hölder space with the following norms

$$\begin{aligned}
\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)} &= \|\varphi^\tau\|_{\mathcal{C}_\tau(H)} + \max_{1\le k<k+r\le N} \frac{\|\varphi_{k+r} - \varphi_k\|_H}{(r\tau)^\alpha}, \\
\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} &= \|\varphi^\tau\|_{\mathcal{C}_\tau(H)} + \max_{1\le k<k+r\le N} \frac{((N-k)\tau)^\alpha \|\varphi_{k+r} - \varphi_k\|_H}{(r\tau)^\alpha}.
\end{aligned}$$

We say that difference problem (4.3) is *stable* in $\mathcal{F}([0,1]_\tau, H)$, if we have the following *stability estimate*

$$\|\{u_{k-1}\}_1^N\|_{\mathcal{F}([0,1]_\tau,H)} \le M \left(\|\varphi^\tau\|_{\mathcal{F}([0,1]_\tau,H)} + \|\varphi\|_H\right),$$

where M is independent of φ^τ, φ and τ.

Difference problem (4.3) is said to be *well-posed* in $\mathcal{F}([0,1]_\tau, H)$, if for every $\varphi^\tau \in \mathcal{F}([0,1]_\tau, H)$ problem (4.3) is uniquely solvable and we have the following *coercivity estimate*:

$$\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_1^N\|_{\mathcal{F}([0,1]_\tau,H)} + \|\left\{Au_{k-1}\right\}_1^N\|_{\mathcal{F}([0,1]_\tau,H)} \le M \left(\|\varphi^\tau\|_{\mathcal{F}([0,1]_\tau,H)} + \|A\varphi\|_{H'}\right)$$

where $H' \subset H$, M does not depend on φ^τ, φ and τ.

Throughout the chapter, M shall indicate positive constants which can be different from time to time and we are not interested to precise. We shall write $M(\alpha, \beta, \cdots)$ to stress the fact that the constant depends only on $\alpha, \beta, \cdots$.

In this chapter, we prove the well-posedness of the multi-point nonlocal boundary value problems (4.3) in the spaces $\mathcal{C}_1^\alpha(H)$ and $\mathcal{C}^\alpha(H)$. Furthermore, we apply these abstract results to obtain new coercivity estimates in various Hölder norms for the solutions of nonlocal boundary value problems for parabolic equations.

4.1 The First Order of Accuracy Difference Scheme

Let us start with some lemmas we need below.

Lemma 4.16. *([Ashyralyev A., Sobolevskii P.E., 1994]) The following estimates hold:*

$$\|t^k A^k e^{-tA}\|_{H\to H} \le M,\ t>0,\ k\ge 0, \tag{4.4}$$

$$\|R^k\|_{H\to H} \le \frac{1}{(1+\delta\tau)^k},\ k\ge 1, \tag{4.5}$$

$$\|\tau A R^k\|_{H\to H} \le \frac{1}{k},\ k\ge 1, \tag{4.6}$$

$$\|A^\beta (R^{k+r}-R^k)\|_{H\to H} \le M\frac{(r\tau)^\gamma}{(k\tau)^{\beta+\gamma}},\ 1\le k<k+r\le N,\ \beta\in\{0,1\}, 0\le\gamma\le 1, \tag{4.7}$$

for some $M,\delta>0$, which are independent of τ, where τ is a positive small number and $R=(I+\tau A)^{-1}$ is the resolvent of A. □

Lemma 4.17. *Assume that (4.2) holds. Then, the operator*

$$I-\sum_{k=1}^{p}\alpha_k R^{N-\left[\frac{\theta_k}{\tau}\right]} \tag{4.8}$$

has an inverse

$$T_\tau = \left(I-\sum_{k=1}^{p}\alpha_k R^{N-\left[\frac{\theta_k}{\tau}\right]}\right)^{-1}$$

and the following estimate is satisfied:

$$\| T_\tau \|_{H\to H} \le C(\delta,\theta_p). \tag{4.9}$$

Proof. The proof of estimate (4.9) is based on the triangle inequality, assumption (4.2), and the estimate

$$\left\|\left(I-\sum_{k=1}^{p}\alpha_k R^{N-\left[\frac{\theta_k}{\tau}\right]}\right)^{-1}\right\|_{H\to H} \le \sup_{\delta\le\mu}\frac{1}{\left|1-\sum_{k=1}^{p}\alpha_k(1+\tau\mu)^{-N+\left[\frac{\theta_p}{\tau}\right]}\right|}.$$

□

Let us now obtain the formula for the solution of problem (4.3). It is clear that the first order of accuracy difference scheme

$$\begin{cases} \tau^{-1}(u_k-u_{k-1})-Au_{k-1}=\varphi_k,\ \varphi_k=f(t_k), \\ t_k=k\tau,\ 1\le k\le N,\ N\tau=1, \\ u_N=\sum\limits_{m=1}^{p}\alpha_m u_{\ell_m}+\varphi, \\ \ell_m=\left[\frac{\theta_m}{\tau}\right], 1\le m\le p \end{cases} \tag{4.10}$$

has a solution and the following formula holds:

$$u_k = R^{N-k}u_N - \sum_{j=k+1}^{N} R^{j-k}\varphi_j\tau,\ 0 \le k \le N-1. \tag{4.11}$$

Applying formula (4.11) and the nonlocal boundary condition

$$\xi = u_N = \sum_{m=1}^{p} \alpha_m u_{\ell_m} + \varphi,$$

we can write

$$\xi = \sum_{k=1}^{p} \alpha_k \left(R^{N-\ell_k}\xi - \sum_{j=\ell_k+1}^{N} R^{j-\ell_k}\varphi_j\tau \right) + \varphi.$$

Using Lemma 4.17, we get

$$u_N = T_\tau \left(-\sum_{k=1}^{p} \sum_{j=\ell_k+1}^{N} \alpha_k R^{j-\ell_k}\varphi_j\tau + \varphi \right). \tag{4.12}$$

Hence, difference equation (4.10) is uniquely solvable and for the solution, formulas (4.11) and (4.12) are valid.

Theorem 4.18. *Suppose that (4.2) holds and $\varphi \in H$. Then, for the solution of difference scheme (4.10) the following stability estimate*

$$\max_{0\le k\le N} \|u_k\|_H \le C(\delta,\theta_p)\left(\|\varphi\|_H + \|\varphi^\tau\|_{\mathcal{C}_\tau(H)}\right). \tag{4.13}$$

holds, where $C(\delta,\theta_p)$ is independent of τ, φ, and φ^τ.

Proof. From estimate (4.5), formula (4.11), and $N\tau = 1$ it follows that

$$\max_{0\le k\le N-1} \|u_k\|_H \le \|u_N\|_H + \max_{1\le j\le N} \|\varphi_j\|_H.$$

Using assumption (4.2), estimates (4.5), (4.9), formula (4.12), and $N\tau = 1$, we obtain

$$\|u_N\|_H \le C_1(\delta,\theta_p)\left(\|\varphi\|_H + \|\varphi^\tau\|_{\mathcal{C}_\tau(H)}\right).$$

From these estimates it follows (4.13).

This concludes the proof of Theorem 4.18. □

It is well-known that problem (4.1) in the space $C([0,1],H)$ is not well-posed for the general positive definite self-adjoint operator A and Hilbert space H. Hence, the well-posedness of difference problem (4.10) in $\mathcal{C}([0,1]_\tau,H)$ norm does not take place uniformly with respect to $\tau > 0$.

Theorem 4.19. *Let (4.2) hold and $\varphi \in D(A)$. Then, for the solution of difference problem (4.10), the almost coercivity inequality*

$$\| \left\{\tau^{-1}(u_k - u_{k-1})\right\}_1^N \|_{\mathcal{C}_\tau(H)} + \|\{Au_{k-1}\}_1^N\|_{\mathcal{C}_\tau(H)}$$
$$\leq C(\delta, \theta_p)(\min\{\ln \frac{1}{\tau}\ ,\ 1 + \ln \|A\|_{H\to H}\} \cdot \|\varphi^\tau\|_{\mathcal{C}_\tau(H)} + \|A\varphi\|_H) \qquad (4.14)$$

is valid, where $C(\delta, \theta_p)$ does not depend on τ, φ, and φ^τ.

Proof. Using formula (4.11), estimate (4.5), we get for $1 \leq k \leq N$

$$\|Au_{k-1}\|_H \ \leq \ \|Au_N\|_H + \|\varphi^\tau\|_{\mathcal{C}_\tau(H)} \sum_{j=k}^{N} \|\tau A R^{j-k+1}\|_{H\to H}. \qquad (4.15)$$

It follows from Theorem 1.2 [Ashyralyev A., Sobolevskii P.E., 1994] on page 87 that

$$\sum_{j=k}^{N} \|\tau A R^{j-k+1}\|_{H\to H} \ = \ \sum_{m=1}^{N-k+1} \tau||AR^m||_{H\to H} \leq \sum_{m=1}^{N} \tau||AR^m||_{H\to H}$$
$$\leq \ M \min\left\{\ln \frac{1}{\tau}, 1 + |\ln \parallel A \parallel_{H\to H}|\right\}. \qquad (4.16)$$

By formula (4.12), estimate (4.9), and assumption (4.2), we obtain

$$\|Au_N\|_H \leq C(\delta, \theta_p)(\|\varphi^\tau\|_{\mathcal{C}_\tau(H)} \min\{\ln \frac{1}{\tau}, 1 + |\ln \parallel A \parallel_{H\to H}|\} + \|A\varphi\|_H). \qquad (4.17)$$

Thus, from estimates (4.15)-(4.17) it follows that

$$\| \{Au_{k-1}\}_1^N \|_{\mathcal{C}_\tau(H)} \leq C(\delta, \theta_p) \left(\|\varphi^\tau\|_{\mathcal{C}_\tau(H)} \min\left\{\ln \frac{1}{\tau}, 1 + |\ln \parallel A \parallel_{H\to H}|\right\} + \|A\varphi\|_H \right). \qquad (4.18)$$

Using difference equation (4.10), the triangle inequality, and estimate (4.18), we get estimate (4.14).

This finishes the proof of Theorem 4.19. □

Theorem 4.20. *Suppose that (4.2) holds and $\varphi \in D(A)$. Then, the solution of difference scheme (4.10) satisfy the following stability estimate*

$$\| \left\{\tau^{-1}(u_k - u_{k-1})\right\}_1^N \|_{\mathcal{C}_1^\alpha(H)} + \| \{Au_{k-1}\}_1^N \|_{\mathcal{C}_1^\alpha(H)} \leq C(\delta, \theta_p) \left(\frac{1}{\alpha(1-\alpha)} \|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H \right), \qquad (4.19)$$

where $C(\delta, \theta_p)$ is independent of τ, φ, and φ^τ.

Proof. It follows from formula (4.11) and identity

$$\tau AR = I - R \qquad (4.20)$$

that for $1 \le k \le N$

$$Au_{k-1} = R^{N-k+1}Au_N - \sum_{j=k}^{N} \tau AR^{j-k+1}(\varphi_j - \varphi_{k-1}) + (R^{N-k+1} - I)\varphi_{k-1}. \qquad (4.21)$$

Using estimates (4.5), (4.6), and the definition of $\mathcal{C}_1^\alpha(H)-$norm, we get for $1 \le k \le N$

$$\begin{aligned} \|Au_{k-1}\|_H &\le \|Au_N\|_H + \frac{\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{((N-k+1)\tau)^\alpha} \sum_{j=k}^{N} \frac{\tau}{((j-k+1)\tau)^{1-\alpha}} + 2\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} \\ &\le \|Au_N\|_H + \frac{4}{\alpha}\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}. \qquad (4.22) \end{aligned}$$

Now, we estimate $\|Au_N\|_H$.

From formula (4.12) and $\tau AR = I - R$ it follows that

$$Au_N = T_\tau\{-\sum_{k=1}^{p} \alpha_k(\sum_{j=\ell_k+1}^{N} \tau AR^{j-\ell_k}(\varphi_j - \varphi_{\ell_k}) + (I - R^{N-\ell_k})\varphi_{\ell_k}) + A\varphi\}.$$

Hence, by estimates (4.5), (4.6), (4.9), the definition of $\mathcal{C}_1^\alpha(H)-$norm, and assumption (4.2), we obtain

$$\|Au_N\|_H \le C(\delta, \theta_p) \left(\frac{4}{\alpha}\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H \right). \qquad (4.23)$$

Thus, from estimates (4.22), (4.23) it follows that

$$\|\{Au_{k-1}\}_1^N\|_{\mathcal{C}_\tau(H)} \le C(\delta, \theta_p) \left(\frac{1}{\alpha}\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H \right). \qquad (4.24)$$

Let us now estimate

$$\max_{1 \le k < k+r \le N} \frac{((N-k+1)\tau)^\alpha \|Au_{k-1+r} - Au_{k-1}\|_H}{(r\tau)^\alpha}.$$

First, let $N - k + 1 \le 2r$. By estimate (4.24) and the triangle inequality, we obtain

$$\frac{((N-k+1)\tau)^\alpha \|Au_{k-1+r} - Au_{k-1}\|_H}{(r\tau)^\alpha} \le C(\delta, \theta_p) \left(\frac{1}{\alpha}\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H \right). \qquad (4.25)$$

Next, let $N - k + 1 > 2r$. From formula (4.11) it follows that

$$\begin{aligned} Au_{k-1} - Au_{k-1+r} &= (R^{N-k+1} - R^{N-k+1-r})Au_N - \sum_{j=k}^{k+2r-2} \tau AR^{j-k+1}(\varphi_j - \varphi_{k-1}) \\ &+ \sum_{j=k+r}^{k+2r-2} \tau AR^{j-(k-1+r)}(\varphi_j - \varphi_{k-1+r}) \\ &- \sum_{j=k+2r-1}^{N} \tau A(R^{j-k+1} - R^{j-(k-1+r)})(\varphi_j - \varphi_{k-1}) \\ &+ (I - R^{r-1})(\varphi_{k-1+r} - \varphi_{k-1}) + (R^{N-k+1} - R^{N-(k-1+r)})\varphi_{k-1} \\ &= I_1(k) + I_2(k) + I_3(k) + I_4(k) + I_5(k) + I_6(k). \qquad (4.26) \end{aligned}$$

We first estimate $I_1(k)$. Using estimates (4.7) for $\beta = 0$, (4.23), and the fact $N-k+1 > 2r$, we get

$$\|I_1(k)\|_H \leq C(\delta, \theta_p)\frac{(r\tau)^\alpha}{((N-k+1)\tau)^\alpha}\left(\frac{1}{\alpha}\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H\right). \tag{4.27}$$

Next, it follows from estimate (4.6) and the definition of $\mathcal{C}_1^\alpha(H)-$norm that

$$\begin{aligned}\|I_2(k)\|_H &\leq \frac{\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{((N-k+1)\tau)^\alpha}\sum_{j=k}^{k+2r-2}\frac{\tau}{((j-k+1)\tau)^{1-\alpha}} \\ &\leq \frac{2^\alpha}{\alpha}\frac{(r\tau)^\alpha}{((N-k+1)\tau)^\alpha}\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} \end{aligned}\tag{4.28}$$

By using estimate (4.6), the definition of $\mathcal{C}_1^\alpha(H)-$norm, and the fact $N-k+1 > 2r$, we obtain

$$\begin{aligned}\|I_3(k)\|_H &\leq \frac{2^\alpha\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{((N-k+1)\tau)^\alpha}\sum_{j=k+r}^{k+2r-2}\frac{\tau}{((j-(k-1+r))\tau)^{1-\alpha}} \\ &\leq \frac{2^\alpha\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{((N-k+1)\tau)^\alpha}\frac{(r\tau)^\alpha}{\alpha}. \end{aligned}\tag{4.29}$$

It follows from estimate (4.7) for $\beta = 1$, the definition of $\mathcal{C}_1^\alpha(H)-$norm, and the fact $j-k+1 \geq 2r$ that

$$\begin{aligned}\|I_4(k)\|_H &\leq M\frac{2^\alpha\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{((N-k+1)\tau)^\alpha}r\tau\sum_{j=k+2r-1}^{N}\frac{\tau}{((j-(k-1+r))\tau)^{2-\alpha}} \\ &\leq M\frac{2^\alpha\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{((N-k+1)\tau)^\alpha}\frac{(r\tau)^\alpha}{(1-\alpha)} \end{aligned}\tag{4.30}$$

Using estimate (4.5) and the definition of $\mathcal{C}_1^\alpha(H)-$norm, we obtain

$$\|I_5(k)\|_H \leq \frac{2\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{((N-k+1)\tau)^\alpha}(r\tau)^\alpha. \tag{4.31}$$

Finally, from estimate (4.7) for $\beta = 0$ and the fact $N-k+1 > 2r$ it follows that

$$\|I_6(k)\|_H \leq 2^\alpha\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}. \tag{4.32}$$

Thus, combining estimates (4.27)-(4.32), we get for $N-k+1 > 2r$

$$\frac{((N-k+1)\tau)^\alpha\|Au_{k-1+r} - Au_{k-1}\|_H}{(r\tau)^\alpha} \leq C(\delta, \theta_p)\left(\frac{\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{\alpha(1-\alpha)} + \|A\varphi\|_H\right). \tag{4.33}$$

From estimates (4.25) and (4.33) it follows that

$$\max_{1\leq k<k+r\leq N}\frac{((N-k+1)\tau)^\alpha\|Au_{k-1+r} - Au_{k-1}\|_H}{(r\tau)^\alpha} \leq C(\delta, \theta_p)\left(\frac{\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)}}{\alpha(1-\alpha)} + \|A\varphi\|_H\right) \tag{4.34}$$

Combining estimates (4.24), (4.34), we obtain that

$$\|\{Au_{k-1}\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H\right). \tag{4.35}$$

Hence, estimate (4.19) follows from difference equation (4.10), estimate (4.35) and the triangle inequality.

This concludes the proof of Theorem 4.20. □

Let $H_\alpha = H_{\alpha,\infty}(H,A)$ be the *fractional space*, consisting all $v \in H$ for which the following norm

$$\|v\|_{H_\alpha} = \|v\|_H + \sup_{\lambda>0}\|\lambda^{1-\alpha}Ae^{-\lambda A}v\|_H$$

is finite.

Theorem 4.21. *Assume that* $\varphi_N - \sum_{k=1}^p \alpha_k\varphi_{\ell_k} + A\varphi \in H_\alpha$ *and (4.2). Then, problem (4.10) is well-posed in* $\mathcal{C}^\alpha(H)$ *and the following coercivity estimate*

$$\begin{aligned}&\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\|_{\mathcal{C}^\alpha(H)} + \|\left\{Au_{k-1}\right\}_1^N\|_{\mathcal{C}^\alpha(H)} + \|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\|_{\mathcal{C}_\tau(H_\alpha)}\\ &\leq M\left(\frac{1}{\alpha}\|\varphi_N - \sum_{k=1}^p\alpha_k\varphi_{\ell_k} + A\varphi\|_{H_\alpha} + \frac{C(\delta,\theta_p)}{\alpha(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}\right)\end{aligned}$$

holds, where M *does not depend on* φ, φ^τ, *and* τ.

Proof. Let us establish the estimate for $\|\{Au_{k-1}\}_1^N\|_{\mathcal{C}^\alpha(H)}$. Similar arguments introduced in the proof of estimate (4.24) result that

$$\|\{Au_{k-1}\}_1^N\|_{\mathcal{C}_\tau(H)} \leq C(\delta,\theta_p)\left(\frac{1}{\alpha}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)} + \|A\varphi\|_H\right). \tag{4.36}$$

Next, we estimate

$$\max_{1\leq k<k+r\leq N}\frac{\|Au_{k-1+r} - Au_{k-1}\|_H}{(r\tau)^\alpha}.$$

Using formula (4.11), we obtain for $1 \leq k \leq N$ that

$$\begin{aligned}Au_{k-1} &= -\varphi_{k-1} + R^{N-k+1}(Au_N + \varphi_N)\\ &- \sum_{j=k}^N \tau AR^{j-k+1}(\varphi_j - \varphi_{k-1}) + R^{N-k+1}(\varphi_{k-1} - \varphi_N)\\ &= J_1(k) + J_2(k) + J_3(k) + J_4(k).\end{aligned} \tag{4.37}$$

It is clear that

$$\|J_1\|_{\mathcal{C}^\alpha(H)} = \|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.38}$$

Let us now estimate $\|J_2\|_{\mathcal{C}^\alpha(H)}$. To alleviate the notation, let $v = (Au_N + \varphi_N)$.

From the definition of $H_\alpha-$norm, the equality

$$\sum_{j=k}^{k+r-1} -\tau AR^{N-j} = R^{N-(k-1+r)} - R^{N-k+1},$$

and the formula connecting the resolvent of the generator of a semigroup with the semigroup it follows that

$$\begin{aligned} \|J_2(k+r) - J_2(k)\|_H &\leq \|v\|_{H_\alpha} \int_0^\infty \sum_{j=k}^{k+r-1} \frac{1}{(N-j-1)!} t^{N-j-1} e^{-t} \frac{e^{\frac{-\tau t\delta}{2}} \tau dt}{(\frac{\tau t}{2})^{1-\alpha}} \\ &\leq 2^{1-\alpha} \|v\|_{H_\alpha} (r\tau)^\alpha \frac{2}{\alpha}. \end{aligned} \tag{4.39}$$

Thus, using estimate (4.39), we get

$$\|J_2\|_{\mathcal{C}^\alpha(H)} \leq \frac{4}{\alpha} \|v\|_{H_\alpha}. \tag{4.40}$$

It follows from estimate (4.6), the definition of $C^\alpha(H)-$norm, and $N\tau = 1$ that

$$\begin{aligned} \|J_3(k)\|_H &\leq \|\varphi^\tau\|_{\mathcal{C}^\alpha(H)} \sum_{j=k}^{N} \frac{\tau}{((j-k+1)\tau)^{1-\alpha}} \\ &\leq \frac{((N-k+1)\tau)^\alpha}{\alpha} \|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}, \end{aligned} \tag{4.41}$$

for all k.

Hence, using estimate (4.41), we obtain

$$\|J_3\|_{\mathcal{C}_\tau(H)} \leq \frac{1}{\alpha} \|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.42}$$

Next, we estimate

$$\max_{1\leq k<k+r\leq N} \frac{\|J_3(k+r) - J_3(k)\|_H}{(r\tau)^\alpha}.$$

First, let us consider the case $N - k + 1 \leq 2r$. Using the triangle inequality, estimate (4.41), we get

$$\frac{\|J_3(k+r) - J_3(k)\|_H}{(r\tau)^\alpha} \leq \frac{(2^\alpha + 1)}{\alpha} \|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.43}$$

Next, we consider the case $N - k + 1 > 2r$. We can write as

$$J_3(k) - J_3(k+r) = J_{31}(k) + J_{32}(k) + J_{33}(k) + J_{34}(k),$$

where $J_{31}(k) = I_2(t),\ J_{32}(k) = I_3(k),\ J_{33}(k) = I_4(k)$ (see equation (4.26)), and

$$J_{34}(k) = (R^{r-1} - R^{N-(k-1+r)})(\varphi_{k-1+r} - \varphi_{k-1}).$$

So, we have

$$\|J_{31}(k)\|_H \le \frac{2^\alpha (r\tau)^\alpha}{\alpha}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}, \tag{4.44}$$

$$\|J_{32}(k)\|_H \le \frac{2^\alpha (r\tau)^\alpha}{\alpha}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}, \tag{4.45}$$

$$\|J_{33}(k)\|_H \le M\frac{2^{-1+\alpha}\ (r\tau)^\alpha}{(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.46}$$

Finally, using estimate (4.5) and the definition of $\mathcal{C}^\alpha(H)-$norm, we get

$$\|J_{34}(k)\|_H \le 2(r\tau)^\alpha\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.47}$$

Hence, it follows from estimates (4.44)-(4.47) that for $N-k+1 > 2r$,

$$\frac{\|J_3(t+\tau) - J_3(t)\|_H}{(r\tau)^\alpha} \le \frac{M}{\alpha(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.48}$$

Combining estimates (4.43), (4.48), we get

$$\max_{1\le k<k+r\le N}\frac{\|J_3(k+r) - J_3(k)\|_H}{(r\tau)^\alpha} \le \frac{M}{\alpha(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.49}$$

Thus, estimates (4.42), (4.49) result that

$$\|J_3\|_{\mathcal{C}^\alpha(H)} \le \frac{M}{\alpha(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.50}$$

Using estimate (4.5) and the definition of $\mathcal{C}^\alpha(H)-$norm, we obtain

$$\|J_4(k)\|_H \le ((N-k+1)\tau)^\alpha\|\varphi^\tau\|_{C^\alpha(H)} \le \|\varphi^\tau\|_{\mathcal{C}^\alpha(H)},\ \text{for all } k. \tag{4.51}$$

Hence, estimate (4.51) gives

$$\|J_4\|_{\mathcal{C}_\tau(H)} \le \|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.52}$$

By using estimates (4.5), (4.7) for $\beta = 0$, we get for all $1 \le k < k+r \le N$

$$\begin{aligned}\|J_4(k+r) - J_4(k)\|_H &\le \|R^{N-(k-1+r)} - R^{N-k+1}\|_{H\to H}\|\varphi_{k-1+r} - \varphi_N\|_H \\ &+ \|R^{N-k+1}\|_{H\to H}\|\varphi_{k-1+r} - \varphi_{k-1}\|_H \\ &\le (M+1)(r\tau)^\alpha\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}.\end{aligned} \tag{4.53}$$

So, from estimate (4.53) it follows that

$$\max_{1\le k<k+r\le N}\frac{\|J_4(k+r)-J_4(k)\|_H}{(r\tau)^\alpha}\le M_1\|\varphi^\tau\|_{C^\alpha(H)}. \tag{4.54}$$

Thus, by combining estimates (4.52), (4.54), we obtain

$$\|J_4\|_{\mathcal{C}^\alpha(H)}\le M_1\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}. \tag{4.55}$$

From estimates (4.36), (4.38), (4.40), (4.50), and (4.55) it results that

$$\|\{Au_{k-1}\}_1^N\|_{\mathcal{C}^\alpha(H)}\le M\left(\frac{1}{\alpha}\|Au_N+\varphi_N\|_{H_\alpha}+\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}\right). \tag{4.56}$$

Hence, using the triangle inequality, estimate (4.56), and difference equation (4.10), we get

$$\begin{aligned}&\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\|_{\mathcal{C}^\alpha(H)}\\&\le M\left(\frac{1}{\alpha}\|Au_N+\varphi_N\|_{H_\alpha}+\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}\right).\end{aligned} \tag{4.57}$$

Let us now establish the estimate for $\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\|_{\mathcal{C}(H_\alpha)}$.

It results from formula (4.11) and difference equation (4.10) that for all k,

$$\begin{aligned}\frac{u_k-u_{k-1}}{\tau}&=R^{N-(k-1)}(Au_N+\varphi_N)+R^{N-(k-1)}(\varphi_k-\varphi_N)\\&-\sum_{j=k}^{N}\tau AR^{j-(k-1)}(\varphi_j-\varphi_k)\\&=G_1(k)+G_2(k)+G_3(k).\end{aligned}$$

Using estimate (4.5) and the definition of $H_\alpha-$norm, we obtain

$$\|G_1(k)\|_{H_\alpha}\le\|Au_N+\varphi_N\|_{H_\alpha}. \tag{4.58}$$

Now, using the definition of $H_\alpha-$norm and the formula connecting the resolvent of the generator of a semigroup with the semigroup, we get

$$\begin{aligned}\|G_2(k)\|_{H_\alpha}&=\sup_{\lambda>0}\|\lambda^{1-\alpha}Ae^{-\lambda A}R^{N-k+1}(\varphi_k-\varphi_N)\|_H\\&=\sup_{\lambda>0}\|\lambda^{1-\alpha}Ae^{-\lambda A}\int_0^\infty\frac{t^{N-k}e^{-t}}{(N-k)!}e^{-\tau tA}(\varphi_k-\varphi_N)dt\|_H\\&\le\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}.\end{aligned} \tag{4.59}$$

Next, let us estimate $\|G_3(k)\|_{H_\alpha}$. Let $\lambda > 0$. From estimates (4.4), (4.5), (4.7) for $\beta = 1$, and identity (4.20) it follows that

$$\|\tau A e^{-\lambda A} A R^{j-k+1}\|_{H\to H} \leq \min\left\{\frac{\tau}{((j-k)\tau)^2}, \frac{\tau}{\lambda^2}\right\} \leq M\frac{\tau}{((j-k)\tau+\lambda)^2}. \tag{4.60}$$

Using estimate (4.60) and the definition of $\mathcal{C}^\alpha(H)-$norm, we get

$$\begin{aligned} \|G_3(k)\|_{H_\alpha} &\leq M\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)} \sup_{\lambda>0} \lambda^{1-\alpha} \sum_{j=k}^{N} \frac{\tau}{((j-k)\tau+\lambda)^{2-\alpha}} \\ &\leq M\frac{\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}}{1-\alpha}. \end{aligned} \tag{4.61}$$

Hence, combining estimates (4.58)-(4.61), we obtain

$$\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\|_{\mathcal{C}(H_\alpha)} \leq M\left(\|Au_N+\varphi_N\|_{H_\alpha} + \frac{\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}}{1-\alpha}\right). \tag{4.62}$$

Thus, estimates (4.56), (4.57), and (4.62) result that

$$\begin{aligned} &\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\|_{\mathcal{C}^\alpha(H)} + \|\{Au_k\}_1^N\|_{\mathcal{C}^\alpha(H)} + \|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\|_{\mathcal{C}(H_\alpha)} \\ &\leq M\left(\frac{1}{\alpha}\|Au_N+\varphi_N\|_{H_\alpha} + \frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}\right). \end{aligned} \tag{4.63}$$

Now, we estimate $\|Au_N+\varphi_N\|_{H_\alpha}$. Using formula (4.12), we get

$$\begin{aligned} Au_N+\varphi_N &= T_\tau\{-\sum_{k=1}^{p}\alpha_k \sum_{j=\ell_k+1}^{N} \tau A R^{j-\ell_k}(\varphi_j-\varphi_{\ell_k}) \\ &+ \sum_{k=1}^{p}\alpha_k R^{N-\ell_k}(\varphi_{\ell_k}-\varphi_N) \\ &+ \varphi_N - \sum_{k=1}^{p}\alpha_k\varphi_{\ell_k} + A\varphi\} \\ &= P_1+P_2+P_3. \end{aligned}$$

It follows from estimates (4.6), (4.9), (4.60), assumption (4.2), and the definition of $\mathcal{C}^\alpha(H)$-norm that

$$\|P_1\|_{H_\alpha} \leq \frac{C(\delta,\theta_p)}{(1-\alpha)}\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}, \tag{4.64}$$

$$\|P_2\|_{H_\alpha} \leq C(\delta,\theta_p)\|\varphi^\tau\|_{\mathcal{C}^\alpha(H)}, \tag{4.65}$$

$$\|P_3\|_{H_\alpha} = C(\delta,\theta_p)\|\varphi_N - \sum_{k=1}^{p}\alpha_k\varphi_{\ell_k} + A\varphi\|_{H_\alpha}. \tag{4.66}$$

Therefore, estimates (4.63), (4.64)-(4.66) finishes the proof of Theorem 4.21. □

4.2 Application

We, in this section, consider applications of Theorem 4.20 and Theorem 4.21.

First, let us consider the nonlocal boundary value problem for one dimensional parabolic equation

$$\begin{cases} u_t + (a(x)u_x)_x - \delta u = f(t,x), \ 0 < t < 1, \ 0 < x < 1, \\ u(1,x) = \sum\limits_{m=1}^{p} \alpha_m u(\theta_m, x) + \varphi(x), 0 \le x \le 1, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1, \\ u(t,0) = u(t,1), \ u_x(t,0) = u_x(t,1), \ 0 \le t \le 1 \end{cases} \tag{4.67}$$

under assumption (4.2), where $\delta > 0$, $a(x) \ge a > 0 \ \ (x \in (0,1))$, $\varphi(x)$ $(x \in [0,1])$ and $f(t,x)$ $(t, x \in [0,1])$ are smooth functions.

The discretization of problem (4.67) is carried out in two steps. In the first step, we define the grid space

$$[0,1]_h = \{x = x_n : x_n = nh, \ 0 \le n \le M, \ Mh = 1\}.$$

Let us introduce the Hilbert space $L_{2h} = L_2([0,1]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ defined on $[0,1]_h$, equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x \in [0,1]_h} |\varphi(x)|^2 h \right)^{1/2}.$$

To the differential operator A generated by problem (4.67), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,n} + \delta \varphi_n\}_1^{M-1} \tag{4.68}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} \frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), \ 0 < t < 1, \ x \in [0,1]_h, \\ u^h(1,x) = \sum\limits_{m=1}^{p} \alpha_m u^h(\theta_m, x) + \varphi(x), x \in [0,1]_h, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1. \end{cases} \tag{4.69}$$

In the second step, we replace (4.69) with the difference scheme (4.10)

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} - A_h^x u_{k-1}^h(x) = f_k^h(x), \\ f_k^h(x) = f^h(t_k, x),\ t_k = k\tau,\ 1 \leq k \leq N,\ x \in [0,1]_h, \\ u_N^h(x) = \sum\limits_{m=1}^{p} \alpha_m u_{\ell_m}^h(x) + \varphi(x), x \in [0,1]_h, \\ 0 \leq \theta_1 < \theta_2 < \cdots < \theta_p < 1. \end{cases} \tag{4.70}$$

Theorem 4.22. *Let τ and h be sufficiently small numbers. Then, the solutions of difference scheme (4.70) satisfy the following coercivity stability estimate:*

$$\begin{aligned} &\| \left\{\tau^{-1}(u_k^h - u_{k-1}^h)\right\}_1^N \|_{C_1^\alpha([0,1]_\tau, L_{2h})} + \| \left\{u_{k-1}^h\right\}_1^N \|_{C_1^\alpha([0,1]_\tau, W_{2h}^2)} \\ &\leq C(\delta, \theta_p) \left(\frac{1}{\alpha(1-\alpha)} \| \left\{f_k^h\right\}_1^N \|_{C_1^\alpha([0,1]_\tau, W_{2h}^2)} + \|\varphi^h\|_{W_{2h}^2} \right) \end{aligned}$$

hold, where $C(\delta, \theta_p)$ is independent of τ, $f_k^h(x)$, and $\varphi^h(x)$, $1 \leq k \leq N-1$. □

Theorem 4.23. *Let*

$$A_h^x \varphi^h(x) = f_1^h(x) - \sum_{k=1}^{p} \alpha_m f_{\ell_m}^h(x).$$

Then, for solutions of the problem (4.70), we have the following stability inequalities

$$\begin{aligned} &\| \left\{\tau^{-1}(u_k^h - u_{k-1}^h)\right\}_1^N \|_{C^\alpha([0,1]_\tau, L_{2h})} + \| \left\{u_{k-1}^h\right\}_1^N \|_{C^\alpha([0,1]_\tau, W_{2h}^2)} \\ &\leq \frac{C(\delta, \theta_p)}{\alpha(1-\alpha)} \|\{f_k^h\}_1^N\|_{C^\alpha(H)} \end{aligned}$$

holds, where M does not depend on φ, φ^τ, and τ. □

The proof of Theorem 4.22, Theorem 4.23 is based on the abstract Theorem 4.20, Theorem 4.21 and the symmetry properties of the difference operator A_h^x defined by formula (4.68).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n = \{x = (x_1, \cdots, , x_n) : 0 < x_i < 1, i = 1, \cdots, n\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, the boundary value problem for the multi-dimensional parabolic equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \sum\limits_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = f(t,x), \\ x = (x_1, \ldots, x_n) \in \Omega,\ 0 < t < 1, \\ u(1,x) = \sum\limits_{i=1}^{p} \alpha_i u(\theta_i, x) + \varphi(x), x \in \overline{\Omega}, \\ 0 < \theta_1 < \theta_2 < \cdots < \theta_p \leq 1, \\ u(t,x) = 0,\ x \in S,\ 0 \leq t \leq 1 \end{cases} \tag{4.71}$$

under assumption (4.2) is considered. Here $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions and $a_r(x) \geq a > 0$.

The discretization of problem (4.71) is carried out in two steps.

In the first step, define the grid space $\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \cdots, h_n m_n); m = (m_1, \cdots, m_n),\ 0 \leq m_r \leq N_r,\ h_r N_r = 1, r = 1, \cdots, n\},\ \Omega_h = \widetilde{\Omega}_h \cap \Omega,\ S_h = \widetilde{\Omega}_h \cap S$.

Let L_{2h} denote the Hilbert space

$$L_{2h} = L_2(\widetilde{\Omega}_h) = \left\{ \varphi^h(x) : \left(\sum_{x \in \widetilde{\Omega}_h} |\varphi^h(x)|^2 h_1 \cdots h_n \right)^{1/2} < \infty \right\}.$$

The differential operator A in (4.71) is replaced with

$$A_h^x u^h(x) = -\sum_{r=1}^{n} \left(a_r(x) u^h_{\bar{x}_r} \right)_{x_r, j_r}, \tag{4.72}$$

where the difference operator A_h^x is defined on those grid functions $u^h(x) = 0$, for all $x \in S_h$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}.

Using (4.71), we get

$$\begin{cases} \frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in \widetilde{\Omega}_h, \\ u^h(1,x) = \sum\limits_{m=1}^{p} \alpha_m u^h(\theta_m, x) + \varphi^h(x), x \in \widetilde{\Omega}_h, \\ 0 \leq \theta_1 < \theta_2 < \cdots < \theta_p < 1. \end{cases} \tag{4.73}$$

From (4.73) it follows that

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} - A_h^x u_{k-1}^h(x) = \varphi_k^h(x), \\ \varphi_k^h(x) = f^h(t_k, x), t_k = k\tau,\ 1 \leq k \leq N,\ x \in \widetilde{\Omega}_h, \\ u_N^h(x) = \sum\limits_{m=1}^{p} \alpha_m u_{\ell_m}^h(x) + \varphi^h(x), x \in \widetilde{\Omega}_h, \\ 0 \leq \theta_1 < \theta_2 < \cdots < \theta_p < 1. \end{cases} \tag{4.74}$$

Theorem 4.24. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then, the solutions of difference scheme (4.74) satisfy the following coercivity stability estimate:*

$$\begin{aligned} &\| \left\{ \tau^{-1}(u_k^h - u_{k-1}^h) \right\}_1^N \|_{C_1^\alpha([0,1]_\tau, L_{2h})} + \| \left\{ u_{k-1}^h \right\}_1^N \|_{C_1^\alpha([0,1]_\tau, W_{2h}^2)} \\ &\leq C(\delta, \theta_p) \left(\frac{1}{\alpha(1-\alpha)} \| \left\{ f_k^h \right\}_1^N \|_{C_1^\alpha([0,1]_\tau, W_{2h}^2)} + \|\varphi^h\|_{W_{2h}^2} \right) \end{aligned}$$

hold, where $C(\delta, \theta_p)$ is independent of τ, $f_k^h(x)$, and $\varphi^h(x)$, $1 \leq k \leq N-1$. □

Theorem 4.25. *Let* $A_h^x\varphi^h(x) = \varphi_1^h(x) - \sum_{k=1}^{p}\alpha_m\varphi_{\ell_m}^h(x)$. *Then, for solutions of the problem (4.74), we have the following stability inequalities*

$$\| \left\{\tau^{-1}(u_k^h - u_{k-1}^h)\right\}_1^N \|_{\mathcal{C}^\alpha([0,1]_\tau, L_{2h})} + \| \left\{u_{k-1}\right\}_1^N \|_{\mathcal{C}^\alpha([0,1]_\tau, W_{2h}^2)} \leq \frac{C(\delta,\theta_p)}{\alpha(1-\alpha)} \|\{f_k^h\}_1^N\|_{\mathcal{C}^\alpha(H)}$$

holds, where M *does not depend on* φ^h, f_k^h, h, *and* τ. □

The proof of Theorem 4.24, Theorem 4.25 is based on the abstract Theorem 4.20, Theorem 4.21, and the symmetry properties of the difference operator A_h^x defined by formula (4.72), and the following theorem:

Theorem 4.26. *([Sobolevskii P.E. S, 1975]) For the solutions of the elliptic differential problem*

$$\begin{cases} A_h^x u^h(x) = \omega^h(x), \ x \in \widetilde{\Omega}_h, \\ u^h(x) = 0, \ x \in S_h, \end{cases}$$

the following coercivity inequality holds :

$$\sum_{r=1}^{n} \left\|(u_k^h)_{\overline{x}_r\overline{x}_r,j_r}\right\|_{L_{2h}} \leq M||\omega^h||_{L_{2h}}.$$

□

CHAPTER 5

WELL-POSEDNESS OF THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME DIFFERENCE FOR REVERSE PARABOLIC EQUATIONS

In this chapter, we establish the well-posedness of the second order of accuracy difference scheme

$$\begin{cases} \tau^{-1}(u_k - u_{k-1}) - A(I + \frac{\tau A}{2})u_{k-1} = (I + \frac{\tau A}{2})\varphi_{k-1}, \\ \varphi_{k-1} = f(t_{k-\frac{1}{2}}),\ t_k = k\tau,\ 1 \le k \le N,\ N\tau = 1, \\ u_N = \sum_{k=1}^{p} \alpha_k \left\{(I + (\theta_k - \ell_m\tau)A)u_{\ell_m} + \varphi_{\ell_m}(\theta_k - \ell_m\tau)\right\} + \varphi, \\ \ell_m = \left[\frac{\theta_m}{\tau}\right], 1 \le m \le p \end{cases} \tag{5.1}$$

for approximately solving problem (5.1).

5.1 The Second Order Difference Scheme

We state the following lemmas we need in the sequel.

Lemma 5.27. *([Ashyralyev, A., 1987]) The following estimates hold.*

$$\|D^m - e^{-m\tau A}\|_{H\to H} \le \frac{M\tau^2}{(m\tau)^2},\ m \ge 1, \tag{5.2}$$

$$\|(\tau A)^\alpha D\|_{H\to H} \le 1,\ \alpha \in \{0, 1, 2\} \tag{5.3}$$

$$\|(\tau A)^\alpha D(I + \frac{\tau A}{2})\|_{H\to H} \le 1,\ \alpha \in \{0, 1\} \tag{5.4}$$

$$\|(I + \tau A)D(I + \frac{\tau A}{2})\|_{H\to H} \le 2, \tag{5.5}$$

$$\|(I + \frac{\tau A}{2})D(I + \frac{\tau A}{2})\|_{H\to H} \le 1, \tag{5.6}$$

$$\|(\tau A)^\beta D^m\|_{H\to H} \le \frac{1}{m^\beta},\ m \ge 1,\ 0 \le \beta \le 1, \tag{5.7}$$

$$\|A^\beta(D^{m+r} - D^m)\|_{H\to H} \le M\frac{(r\tau)^\gamma}{(m\tau)^{\beta+\gamma}},\ 1 \le m < m + r \le N,\ 0 \le \beta, \gamma \le 1, \tag{5.8}$$

for some $M, \delta > 0$ independent of τ, where τ is a positive small number and $D = (I + \tau A + \frac{(\tau A)^2}{2})^{-1}$. □

Clearly, we have

$$\tau A D(I + \frac{\tau A}{2}) = I - D. \tag{5.9}$$

Lemma 5.28. *Let assumption (4.2) hold. Then, the operator*

$$I-\sum_{k=1}^{p}\alpha_k(I+d_kA)D^{N-\left[\frac{\theta_k}{\tau}\right]} \tag{5.10}$$

has an inverse

$$T'_\tau=\left(I-\sum_{k=1}^{p}\alpha_k(I+d_kA)D^{N-\left[\frac{\theta_k}{\tau}\right]}\right)^{-1}$$

and the following estimate is satisfied:

$$\| T'_\tau \|_{H\to H}\leq C(\delta,\theta_p). \tag{5.11}$$

Proof. The proof of estimate (5.11) follows from the estimates (3.7), (5.7) for $\beta=1$, and the triangle inequality. □

Now, we obtain the formula for the solution of problem (5.12). Clearly, the second order of accuracy difference scheme

$$\begin{cases} \tau^{-1}(u_k-u_{k-1})-A(I+\frac{\tau A}{2})u_{k-1}=(I+\frac{\tau A}{2})\varphi_{k-1}, \\ \varphi_{k-1}=f(t_{k-\frac{1}{2}}),\ t_k=k\tau,\ 1\leq k\leq N,\ N\tau=1, \\ u_N=\sum_{m=1}^{p}\alpha_m\left\{(I+d_kA)u_{\ell_m}+d_m\varphi_{\ell_m}\right\}+\varphi, \\ d_m=\theta_m-\left[\frac{\theta_m}{\tau}\right]\tau,\ \ell_m=\left[\frac{\theta_m}{\tau}\right],\ 1\leq m\leq p \end{cases} \tag{5.12}$$

has a solution and the following formula holds:

$$u_k=D^{N-k}u_N-\sum_{j=k+1}^{N}D^{j-k}(I+\frac{\tau A}{2})\varphi_{j-1}\tau,\ 1\leq k\leq N-1. \tag{5.13}$$

Using formula (5.13) and the nonlocal boundary condition

$$\xi=u_N=\sum_{m=1}^{p}\alpha_m\left\{(I+d_mA)u_{\ell_m}+d_m\varphi_{\ell_m}\right\}+\varphi,$$

we obtain

$$\begin{aligned} \xi &= \sum_{m=1}^{p}\alpha_m(I+d_mA)\left(D^{N-\ell_m}\xi-\sum_{j=\ell_m+1}^{N}D^{j-\ell_m}(I+\frac{\tau A}{2})\varphi_{j-1}\tau\right) \\ &+ \sum_{m=1}^{p}\alpha_m d_m\varphi_{\ell_m}+\varphi. \end{aligned}$$

It follow from Lemma 5.28 that

$$\begin{aligned} u_N &= T'_\tau\{-\sum_{k=1}^{p}\sum_{j=\ell_k+1}^{N}\alpha_k(I+d_kA)D^{j-\ell_k}(I+\frac{\tau A}{2})\varphi_{j-1}\tau \\ &+ \sum_{k=1}^{p}\alpha_k d_k\varphi_{\ell_k}+\varphi\}. \end{aligned} \tag{5.14}$$

Thus, difference equation (5.12) is uniquely solvable and solutions satisfy formulas (5.13) and (5.14).

Theorem 5.29. *Let assumption (4.2) hold and $\varphi \in D(A)$. Then, the solution of the difference scheme (5.12) satisfy the following stability estimate*

$$\| \{u_k\}_1^N \|_{C_\tau(H)} \leq C(\delta,\theta_p)\left(\|\varphi\|_H + \| \{\varphi_k\}_1^N \|_H\right), \tag{5.15}$$

where $C(\delta,\theta_p)$ does not depend on τ, φ, and φ^τ.

Proof. Using estimates (5.3), (5.4) for $\alpha = 0$, formula (5.13), and $N\tau = 1$, we get for $1 \leq k \leq N-1$ that

$$\|u_k\|_H \leq \|u_N\|_H + \max_{1\leq k\leq N} \|\varphi_j\|_H.$$

From assumption (4.2), estimates (5.3) for $\alpha = 0$, (5.6), (5.11), formula (5.14), and $N\tau = 1$ it follows estimate (5.15). □

Theorem 5.30. *The solution of difference problem (5.12) satisfies the following almost coercivity inequality*

$$\begin{aligned}
&\| \left\{\tau^{-1}(u_k - u_{k-1})\right\}_2^N \|_{C_\tau(H)} + \|\{Au_k\}_1^N\|_{C_\tau(H)} \\
&\leq C(\delta,\theta_p)\left(\min\{\ln(\frac{1}{\tau}), 1+\ln\|A\|_{H\to H}\}\cdot \| \{\varphi_k\}_1^N \|_{C_\tau(H)} + \|A\varphi\|_H\right),
\end{aligned} \tag{5.16}$$

where $C(\delta,\theta_p)$ is independent of on τ, φ, and φ^τ.

Proof. From formula (5.13), estimate (5.3) for $\alpha = 0$ it follows that for $1 \leq k \leq N-1$

$$\begin{aligned}
\|Au_k\|_H &\leq \|D^{N-k}\|_{H\to H}\|Au_N\|_H + \sum_{j=k+1}^{N} \|\tau AD^{j-k}(I+\frac{\tau A}{2})\|_{H\to H}\|\varphi_{j-1}\|_H \\
&\leq \|Au_N\|_H + \| \{\varphi_k\}_1^N \|_{C_\tau(H)} \sum_{j=k+1}^{N} \|\tau AD^{j-k}(I+\frac{\tau A}{2})\|_{H\to H}.
\end{aligned}$$

Following the arguments of Theorem 1.2 [Ashyralyev A., Sobolevskii P.E., 1994] on page 87, using the estimates (5.4) for $\alpha = 1$, (5.7) for $\beta = 1$, we get

$$\begin{aligned}
&\sum_{j=k+1}^{N} \|\tau AD^{j-k}(I+\frac{\tau A}{2})\|_{H\to H} = ||AD(I+\frac{\tau A}{2})||_{H\to H} \\
&+ \sum_{m=2}^{N-k-1} \tau||AD^m(I+\frac{\tau A}{2})||_{H\to H} \leq \min\{\ln(\frac{1}{\tau}), 1+\ln\|A\|_{H\to H}\}.
\end{aligned} \tag{5.17}$$

By assumption (4.2), estimates (5.3) for $\alpha = 2$, (5.4) for $\alpha = 0$, (5.7) for $\beta = 1, m = 1$, and $N\tau = 1$, we obtain

$$\|Au_N\|_H \leq C(\delta, \theta_p)(\|\{\varphi_k\}_1^N\|_{\mathcal{C}_\tau(H)} \min\{\ln \frac{1}{\tau}, 1 + |\ln \| A \|_{H \to H}|\} + \|A\varphi\|_H). \quad (5.18)$$

Hence, combining estimates (5.17), (5.18), we get

$$\| \{Au_k\}_1^N \|_{\mathcal{C}_\tau(H)} C(\delta, \theta_p)(\|\{\varphi_k\}_1^N\|_{\mathcal{C}_\tau(H)} \min\{\ln \frac{1}{\tau}, 1 + |\ln \| A \|_{H \to H}|\} + \|A\varphi\|_H). \quad (5.19)$$

Thus the estimate (5.16) follows from difference equation (5.12), the triangle inequality, and estimate (5.19).

This is end of the proof of Theorem 5.29. □

Theorem 5.31. *Assume that (4.2) holds and $\varphi \in D(A)$. Then, the solution of the difference scheme (5.12) satisfy the following stability estimate*

$$\begin{aligned} &\| \{\tau^{-1}(u_k - u_{k-1})\}_2^N \|_{\mathcal{C}_1^\alpha(H)} + \| \{Au_k\}_1^N \|_{\mathcal{C}_1^\alpha(H)} \\ &\leq C(\delta, \theta_p) \left(\frac{1}{\alpha(1-\alpha)} \| \{\varphi_k\}_1^N \|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H \right) \end{aligned} \quad (5.20)$$

hold, where $C(\delta, \theta_p)$ does not depend on τ, φ, and φ^τ.

Proof. Using formula (5.13) and identity (5.9), we get for $1 \leq k \leq N-1$ that

$$\begin{aligned} Au_k &= D^{N-k} Au_N - \sum_{j=k+1}^{N} \tau A D^{j-1-k} D(I + \frac{\tau A}{2})(\varphi_{j-1} - \varphi_k) \\ &+ (D^{N-k} - I)\varphi_k. \end{aligned} \quad (5.21)$$

Hence, it follows from estimates (5.3), (5.4) for $\alpha = 0$, (5.7) for $\beta = 1$, and the definition of $\mathcal{C}_1^\alpha(H)-$norm that

$$\begin{aligned} \|Au_k\|_H &\leq \|Au_N\|_H + \frac{\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha} \sum_{j=k+1}^{N} \frac{\tau}{((j-1-k)\tau)^{1-\alpha}} + 2\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \\ &\leq \|Au_N\|_H + \frac{\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha} \int_0^{(N-k)\tau} \frac{dx}{x^{1-\alpha}} + 2\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \\ &\leq \|Au_N\|_H + \frac{4}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}. \end{aligned} \quad (5.22)$$

Next, let us estimate $\|Au_N\|_H$.

Using formula (5.14) and identity (5.9), we obtain

$$\begin{aligned} Au_N &= T'_\tau\{-\sum_{k=1}^{p}\alpha_k(I+d_kA)\sum_{j=\ell_k+1}^{N}\tau AD^{j-\ell_k}(I+\frac{\tau A}{2})\varphi_{j-1}+\sum_{k=1}^{p}\alpha_k d_k A\varphi_{\ell_k}+\varphi\} \\ &= T'_\tau\{-\sum_{k=1}^{p}\alpha_k(I+d_kA)\sum_{j=\ell_k+1}^{N}\tau AD^{j-\ell_k}(I+\frac{\tau A}{2})(\varphi_{j-1}-\varphi_{\ell_k}) \\ &- \sum_{k=1}^{p}\alpha_k\varphi_{\ell_k}+\sum_{k=1}^{p}\alpha_k(I+d_kA)D^{N-\ell_k}\varphi_{\ell_k}+A\varphi\}. \end{aligned} \tag{5.23}$$

Thus, from estimates (5.5), (5.7) for $\beta=1$, (5.11), and the definition of $\mathcal{C}_1^\alpha(H)-$norm it follows that

$$\begin{aligned} \|Au_N\|_H &\leq C(\delta,\theta_p)\{\sum_{k=1}^{p}|\alpha_k|\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \\ &\times \{\frac{1}{((N-\ell_k)\tau)^\alpha}\sum_{j=\ell_k+1}^{N}\frac{2\tau}{((j-1-\ell_k)\tau)^{1-\alpha}}+2\}+\|A\varphi\|_H\} \\ &\leq C(\delta,\theta_p)\{\sum_{k=1}^{p}|\alpha_k|\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \\ &\times \{\frac{1}{((N-\ell_k)\tau)^{1-\alpha}}\int_0^{(N-\ell_k)\tau}\frac{dx}{x^{1-\alpha}}+1\}+\|A\varphi\|_H\}. \end{aligned}$$

Hence, using assumption (4.2), we obtain

$$\|Au_N\|_H \leq C(\delta,\theta_p)\left(\frac{1}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}+\|A\varphi\|_H\right). \tag{5.24}$$

It follows from estimates (5.22), (5.24) that

$$\|\{Au_k\}_1^N\|_{\mathcal{C}_\tau(H)} \leq C(\delta,\theta_p)\left(\frac{1}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}+\|A\varphi\|_H\right). \tag{5.25}$$

We now estimate

$$\max_{1\leq k<k+r\leq N}\frac{((N-k)\tau)^\alpha\|Au_{k+r}-Au_k\|_H}{(r\tau)^\alpha}.$$

Let $N-k\leq 2r$. Using estimate (5.25) and the triangle inequality, we get

$$\frac{((N-k)\tau)^\alpha\|Au_{k+r}-Au_k\|_H}{(r\tau)^\alpha} \leq C(\delta,\theta_p)\left(\frac{1}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}+\|A\varphi\|_H\right). \tag{5.26}$$

Let us now consider $N-k>2r$. It follows from formula (5.13) that

$$\begin{aligned} Au_k - Au_{k+r} &= (D^{N-k} - D^{N-k-r})Au_N \\ &- \sum_{j=k+1}^{k+2r} \tau A D^{j-1-k} D(\tau A)(I + \frac{\tau A}{2})(\varphi_{j-1} - \varphi_k) \\ &- \sum_{j=k+2r+1}^{N} \tau A (D^{j-1-k} - D^{j-1-(k+r)}) D(I + \frac{\tau A}{2})(\varphi_{j-1} - \varphi_k) \\ &+ \sum_{j=k+r+1}^{k+2r} \tau A D^{j-1-(k+r)} D(I + \frac{\tau A}{2})(\varphi_{j-1} - \varphi_{k+r}) \\ &+ (I - D^{r-1})(\varphi_{k+r} - \varphi_k) + (D^{N-k} - D^{N-(k+r)})\varphi_k \\ &= P_1(k) + P_2(k) + P_3(k) + P_4(k) + P_5(k) + P_6(k). \end{aligned} \tag{5.27}$$

Let us start with estimate $P_1(k)$. By using estimates (5.8) for $\beta = 0$, (5.24), and the fact $N-k>2r$, we obtain

$$\begin{aligned} \|P_1(k)\|_H &\leq \|D^{-(N-k)} - D^{-(N-(k+r))}\|_{H\to H} \|Au_N\|_H \\ &\leq M \frac{(r\tau)^\alpha}{((N-(k+r))\tau)^\alpha} C(\delta, \theta_p) \left(\frac{1}{\alpha} \|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H \right) \\ &\leq C(\delta, \theta_p) \frac{(r\tau)^\alpha}{((N-k)\tau)^\alpha} \left(\frac{1}{\alpha} \|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H \right). \end{aligned} \tag{5.28}$$

From estimates (5.4) for $\alpha = 0$, (5.7) for $\beta = 1$, and the definition of $\mathcal{C}_1^\alpha(H)-$norm it follows that

$$\begin{aligned} \|P_2(k)\|_H &\leq \sum_{j=k+1}^{k+2r} \|\tau A D^{j-1-k}\|_{H\to H} \|D(I + \frac{\tau A}{2})\|_{H\to H} \|\varphi_{j-1} - \varphi_k\|_H \\ &\leq \frac{\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha} \sum_{j=k+1}^{k+2r} \frac{\tau}{((j-1-k)\tau)^{1-\alpha}} \\ &\leq \frac{\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha} \int_0^{2r\tau} \frac{dx}{x^{1-\alpha}} \\ &= \frac{2^\alpha}{\alpha} \frac{(r\tau)^\alpha}{((N-k)\tau)^\alpha} \|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \end{aligned} \tag{5.29}$$

Using estimates (5.4) for $\alpha = 0$, (5.8) for $\beta = 1$, the definition of $\mathcal{C}_1^\alpha(H)-$norm, and the fact $j-k \geq 2r+1$, we get

$$\begin{aligned}
\|P_3(k)\|_H &\leq \sum_{j=k+2r+1}^{N} \|\tau A(D^{j-1-k} - D^{j-1-(k+r)})\|_{H\to H}\|D(I+\frac{\tau A}{2})\|_{H\to H}\|\varphi_{j-1}-\varphi_k\|_H \\
&\leq \sum_{j=k+2r+1}^{N} \tau M \frac{r\tau}{((j-1-(k+r))\tau)^2}\frac{((j-1-k)\tau)^\alpha}{((N-k)\tau)^\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \\
&\leq \sum_{j=k+2r+1}^{N} M \frac{r\tau^2}{(j-1-(k+r))^{2-\alpha}}\left(\frac{j-1-k}{j-1-(k+r)}\right)^\alpha \frac{1}{((N-k)\tau)^\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \\
&\leq M\frac{2^\alpha\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha} r\tau \sum_{j=k+2r+1}^{N} \frac{\tau}{((j-1-(k+r))\tau)^{2-\alpha}} \\
&\leq M\frac{2^\alpha\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha} r\tau \int_{r\tau}^{1}\frac{dx}{x^{2-\alpha}} \\
&\leq M\frac{2^\alpha\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha}\frac{(r\tau)^\alpha}{1-\alpha}
\end{aligned} \tag{5.30}$$

Estimates (5.4) for $\alpha = 0$, (5.7) for $\beta = 1$, and the definition of $\mathcal{C}_1^\alpha(H)-$norm, and the fact $N-k > 2r$ result that

$$\begin{aligned}
\|P_4(k)\|_H &\leq \sum_{j=k+r+1}^{k+2r} \|\tau A D^{j-1-(k+r)}\|_{H\to H}\|D(I+\frac{\tau A}{2})\|_{H\to H}\|\varphi_{j-1}-\varphi_{k+r}\|_H \\
&\leq \sum_{j=k+r+1}^{k+2r} \frac{1}{(j-1-(k+r))}\frac{((j-1-(k+r))\tau)^\alpha\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-(k+r))\tau)^\alpha} \\
&= \sum_{j=k+r+1}^{k+2r} \frac{\tau}{((j-1-(k+r))\tau)^{1-\alpha}}\frac{\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha}\frac{((N-k)\tau)^\alpha}{((N-(k+r))\tau)^\alpha} \\
&\leq \frac{2^\alpha\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{((N-k)\tau)^\alpha}\int_0^{r\tau}\frac{dx}{x^{1-\alpha}} = \frac{2^\alpha}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}\frac{(r\tau)^\alpha}{((N-k)\tau)^\alpha}.
\end{aligned} \tag{5.31}$$

It follows from estimate (5.3) for $\alpha = 0$ and the definition of $\mathcal{C}_1^\alpha(H)-$norm that

$$\|P_5(k)\|_H \leq \|I-D^{r-1}\|_{H\to H}\|\varphi_{k+r}-\varphi_k\|_H \leq \frac{2((N-k)\tau)^\alpha\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}}{(r\tau)^\alpha}. \tag{5.32}$$

Finally, using estimate (5.8) for $\beta = 0$ and the fact $N-k > 2r$ it follows that

$$\|P_6(k)\|_H \leq \|D^{N-k} - D^{N-(k+r)}\|_{H\to H}\|\varphi_k\|_H \leq 2^\alpha\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)}. \tag{5.33}$$

Combining estimates (5.28)-(5.33), for $N-k > 2r$ we get

$$\frac{((N-k)\tau)^\alpha\|Au_{k+r}-Au_k\|_H}{(r\tau)^\alpha} \leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H\right). \tag{5.34}$$

From estimates (5.26) and (5.34) it follows that

$$\begin{aligned}
&\max_{1\leq k<k+r\leq N}\frac{((N-k)\tau)^\alpha\|Au_{k+r}-Au_k\|_H}{(r\tau)^\alpha} \\
&\leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H\right).
\end{aligned} \tag{5.35}$$

Thus, combining estimates (5.25), (5.35), we obtain

$$\|\{Au_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} \leq C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}_1^\alpha(H)} + \|A\varphi\|_H\right). \tag{5.36}$$

Therefore, we obtain estimate (5.20) by using difference equation (5.12), estimate (5.36) and the triangle inequality. □

Let $H'_\alpha = H'_{\alpha,\infty}(H,A)$ denote the *fractional space*, consisting all $v \in H$ for which the following norm

$$\|v\|_{H'_\alpha} = \|v\|_H + \|A^\alpha v\|_H$$

is finite.

Recall that

$$A^\alpha v = A^{\alpha-1}Av = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty s^{\alpha-1}(s+A)^{-1}Avds. \tag{5.37}$$

Theorem 5.32. *Suppose $\varphi^\tau = \{\varphi_k\}_1^N \in \mathcal{C}^\alpha(H)$, $\varphi_N - \sum_{k=1}^p \alpha_k\varphi_{\ell_k} + A\varphi \in H'_\alpha$ and (3.2). Then, problem (5.12) is well-posed in $\mathcal{C}^\alpha(H)$ and the following coercivity estimate*

$$\begin{aligned}
&\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_2^N\|_{\mathcal{C}^\alpha(H)} + \|\left\{Au_k\right\}_1^N\|_{\mathcal{C}_\tau^\alpha(H)} + \|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_2^N\|_{\mathcal{C}_\tau(H'_\alpha)}\\
&\leq M\left(\|\varphi_N - \sum_{k=1}^p \alpha_k\varphi_{\ell_k} + A\varphi\|_{H'_\alpha} + \frac{C(\delta,\theta_p)}{\alpha^2(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}\right)
\end{aligned}$$

holds, where M does not depend on φ, φ^τ, and τ.

Proof. First, we establish the estimate for $\|\{Au_k\}_1^N\|_{\mathcal{C}^\alpha(H)}$. By similar arguments given in the proof of estimate (5.25), we get

$$\|\{Au_k\}_1^N\|_{\mathcal{C}_\tau(H)} \leq C(\delta,\theta_p)\left(\frac{1}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)} + \|A\varphi\|_H\right). \tag{5.38}$$

Let us now estimate

$$\max_{1\leq k<k+r\leq N}\frac{\|Au_{k+r}-Au_k\|_H}{(r\tau)^\alpha}.$$

From formula (5.13) it follows that for all $1 \leq k \leq N-1$

$$\begin{aligned}
Au_k &= -\varphi_k + D^{N-k}(Au_N + \varphi_N)\\
&- \sum_{j=k+1}^N \tau AD^{j-1-k}D(I+\frac{\tau A}{2})(\varphi_{j-1}-\varphi_k)\\
&+ D^{N-k}(\varphi_k - \varphi_N)\\
&= Q_1(k) + Q_2(k) + Q_3(k) + Q_4(k).
\end{aligned} \tag{5.39}$$

Clearly, we have

$$\|Q_1\|_{\mathcal{C}^\alpha(H)} = \|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}. \tag{5.40}$$

Next, we estimate $\|Q_2\|_{\mathcal{C}^\alpha(H)}$. Let $v = (Au_N + \varphi_N)$.

Note that

$$\begin{aligned} D^{N-k} - D^{N-(k+r)} &= \sum_{j=k+1}^{k+r} -\tau A D^{N-j} D(I + \frac{\tau A}{2}) \\ &= \sum_{m=1}^{r} -\tau D^{N-k-m} A^{1-\alpha} D(I + \frac{\tau A}{2}) A^\alpha \end{aligned} \tag{5.41}$$

Thus, from the spectral theorem for A, the definition of H'_α and estimate (5.4) for $\alpha = 0$ it follows that

$$\begin{aligned} \|Q_2(k) - Q_2(k+r)\|_H &\leq \sum_{m=1}^{r} \|\tau D^{N-k-m} A^{1-\alpha} D(I + \frac{\tau A}{2})\|_{H\to H} \|A^\alpha v\|_H \\ &\leq \|v\|_{H'_\alpha} \sum_{m=1}^{r} \tau \sup_{\delta \leq \mu < \infty} \frac{\mu^{1-\alpha}(1 + \frac{\mu\tau}{2})}{(1 + \mu\tau + \frac{(\mu\tau)^2}{2})^{N-k-m+1}} \\ &\leq M\|v\|_{H'_\alpha} \sum_{m=1}^{r} \frac{\tau}{((N-k-m)\tau)^{1-\alpha}} \\ &\leq M\|v\|_{H'_\alpha} \int_\tau^{r\tau} \frac{dx}{((N-k)\tau - x)^{1-\alpha}}. \end{aligned} \tag{5.42}$$

Considering the case $N - k \leq 2r$ and $N - k < 2r$ separately, estimate (5.42) results that

$$\|Q_2(k) - Q_2(k+r)\|_H \leq \frac{M}{\alpha} \|v\|_{H'_\alpha} (r\tau)^\alpha. \tag{5.43}$$

Hence, it follows from (5.43) that

$$\|Q_2\|_{\mathcal{C}^\alpha(H)} \leq \frac{4}{\alpha} \|v\|_{H'_\alpha}. \tag{5.44}$$

Now, let us estimate $\|Q_3\|_{\mathcal{C}^\alpha(H)}$. Using estimate (5.7) for $\beta = 1$, and the definition of $C^\alpha(H)-$norm that

$$\begin{aligned} \|Q_3(k)\|_H &\leq \sum_{j=k+1}^{N} \|\tau A D^{j-1-k}(\tau A)\|_{H\to H} \|\varphi_{j-1} - \varphi_k\|_H \\ &\leq \|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)} \sum_{j=k+1}^{N} \frac{\tau}{((j-1-k)\tau)^{1-\alpha}} \\ &\leq \|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)} \int_\tau^{(N-k)\tau} \frac{\tau}{((j-1-k)\tau)^{1-\alpha}} \\ &\leq \frac{((N-k)\tau)^\alpha}{\alpha} \|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}, \quad \text{for all } k \end{aligned} \tag{5.45}$$

Thus, it follows from estimate (5.41) that

$$\|Q_3\|_{\mathcal{C}_\tau(H)} \leq \frac{1}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}. \tag{5.46}$$

Let us now estimate

$$\max_{1\leq k<k+r\leq N} \frac{\|Q_3(k+r)-Q_3(k)\|_H}{(r\tau)^\alpha}.$$

Let $N-k \leq 2r$. Then, by the the triangle inequality, estimate (5.41), we obtain

$$\begin{aligned}
\frac{\|Q_3(k+r)-Q_3(k)\|_H}{(r\tau)^\alpha} &\leq \frac{\|Q_3(k+r)\|_H+\|Q_3(k)\|_H}{(r\tau)^\alpha} \\
&\leq \frac{((N-k-r)\tau)^\alpha+((N-k)\tau)^\alpha}{\alpha(r\tau)^\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)} \\
&\leq \frac{(2^\alpha+1)}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}.
\end{aligned} \tag{5.47}$$

Next, let us consider the case $N-k>2r$. Easily, we have

$$Q_3(k)-Q_3(k+r)=Q_{31}(k)+Q_{32}(k)+Q_{33}(k)+Q_{34}(k),$$

where $Q_{31}(k)=P_2(t),\ Q_{32}(k)=P_3(k),\ Q_{33}(k)=P_4(k)$ (see equation (5.27)), and

$$Q_{34}(k)=-(D^r(\tau A)-D^{N-(k+r)})(\tau A)(\varphi_{k+r}-\varphi_k)).$$

Hence, we get

$$\|Q_{31}(k)\|_H \leq \frac{2^\alpha(r\tau)^\alpha}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}, \tag{5.48}$$

$$\|Q_{32}(k)\|_H \leq \frac{2^\alpha(r\tau)^\alpha}{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}, \tag{5.49}$$

$$\|Q_{33}(k)\|_H \leq M\frac{2^{-1+\alpha}\ (r\tau)^\alpha}{(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}. \tag{5.50}$$

By estimate (5.3) for $\alpha=0$ and the definition of $\mathcal{C}^\alpha(H)-$norm, we get

$$\|Q_{34}(k)\|_H \leq 2(r\tau)^\alpha\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}. \tag{5.51}$$

From estimates (5.48)-(5.51) it follows that for $N-k>2r$

$$\frac{\|Q_3(t+\tau)-Q_3(t)\|_H}{(r\tau)^\alpha} \leq \frac{M}{\alpha(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}. \tag{5.52}$$

Estimate (5.3) for $\alpha=0$ and the definition of $\mathcal{C}^\alpha-$norm result that for all k,

$$\|Q_4(k)\|_H \leq ((N-k)\tau)^\alpha\|\{\varphi_k\}_1^N\|_{C^\alpha(H)} \leq \|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)} \tag{5.53}$$

So, estimate (5.53) gives us

$$\|Q_4\|_{\mathcal{C}_\tau(H)} \leq \|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}. \tag{5.54}$$

It follows from estimates (5.3) for $\alpha = 0$, (5.8) for $\beta = 0$, and the definition of $\mathcal{C}^{\alpha}-$norm that for all $1 \leq k < k+r \leq N$,

$$\begin{aligned}
\|Q_4(k+r) - Q_4(k)\|_H &\leq \|Q^{N-(k+r)}(\tau A) - Q^{N-k}(\tau A)\|_{H\to H}\|\varphi_{k+r} - \varphi_N\|_H \\
&+ \|Q^{N-k}(\tau A)\|_{H\to H}\|\varphi_{k+r} - \varphi_k\|_H \\
&\leq \frac{M(r\tau)^{\alpha}}{((N-k)\tau)^{\alpha}}((N-k)\tau)^{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^{\alpha}(H)} \\
&+ (r\tau)^{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^{\alpha}(H)} \\
&\leq (M+1)(r\tau)^{\alpha}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^{\alpha}(H)}. \qquad (5.55)
\end{aligned}$$

Thus, using estimate (5.55), we obtain

$$\max_{1\leq k<k+r\leq N} \frac{\|Q_4(k+r) - Q_4(k)\|_H}{(r\tau)^{\alpha}} \leq M_1\|\{\varphi_k\}_1^N\|_{C^{\alpha}(H)}. \qquad (5.56)$$

From estimates (5.54), (5.56) it results that

$$\|Q_4\|_{\mathcal{C}^{\alpha}(H)} \leq M_1\|\{\varphi_k\}_1^N\|_{\mathcal{C}^{\alpha}(H)}. \qquad (5.57)$$

Combining estimates (5.40), (5.44), (5.52), and (5.57) it results that

$$\|\{Au_k\}_1^N\|_{\mathcal{C}^{\alpha}(H)} \leq M\left(\frac{1}{\alpha}\|Au_N + \varphi_N\|_{H'_{\alpha}} + \frac{1}{\alpha(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^{\alpha}(H)}\right). \qquad (5.58)$$

From the triangle inequality, estimate (5.58), and difference equation (5.12), we get

$$\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_2^N\|_{\mathcal{C}^{\alpha}(H)} \leq M\{\frac{1}{\alpha}\|Au_N + \varphi_N\|_{H'_{\alpha}} + \frac{1}{\alpha(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^{\alpha}(H)}\}. \quad (5.59)$$

Let us now establish the estimate for $\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_2^N\|_{\mathcal{C}(H'_{\alpha})}$.

Using difference equation (5.12) and formula (5.13), we obtain that for all k,

$$\begin{aligned}
\frac{u_k - u_{k-1}}{\tau} &= (I + \frac{\tau A}{2})D^{N-(k-1)}(Au_N + \varphi_N) \\
&+ (I + \frac{\tau A}{2})D^{N-(k-1)}(\varphi_{k-1} - \varphi_N) \\
&- (I + \frac{\tau A}{2})\sum_{j=k}^{N}\tau AD^{j-(k-1)}(I + \frac{\tau A}{2})(\varphi_{j-1} - \varphi_{k-1}) \\
&= S_1(k) + S_2(k) + S_3(k).
\end{aligned}$$

It follows from estimates (5.3), (5.4) for $\alpha = 0$, and the definition of $H'_{\alpha}-$norm that

$$\|S_1(k)\|_{H'_{\alpha}} \leq \|Au_N + \varphi_N\|_{H'_{\alpha}}. \qquad (5.60)$$

By estimates (5.4) for $\alpha = 0$, (5.7) for $\beta = \alpha$, and the definition of $H'_\alpha-$norm, we get

$$\|S_2(k)\|_{H'_\alpha} \leq M\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}. \tag{5.61}$$

Now, from estimates (5.6), (5.7) for $\beta = 1$, and the definition of $\mathcal{C}^\alpha(H)-$norm it gives

$$\begin{aligned}\|S_3(k)\|_H &\leq \sum_{j=k+1}^{N} \tau\|AD^{j-k}\|_{H\to H}\|(I + \frac{\tau A}{2})D(I + \frac{\tau A}{2})\|_{H\to H}\|\varphi_{j-1} - \varphi_{k-1}\|_H \\ &\leq \|\{\varphi_m\}_1^N\|_{\mathcal{C}^\alpha(H)} \sum_{j=k+1}^{N} \frac{\tau}{((j-k)\tau)^{1-\alpha}} \leq \frac{\|\{\varphi_m\}_1^N\|_{\mathcal{C}^\alpha(H)}}{\alpha} \end{aligned}\tag{5.62}$$

Next, using the spectral theorem for A, formula (5.37), and the definition of $\mathcal{C}^\alpha(H)$-norm, we get

$$\begin{aligned}\|A^\alpha S_3(k)\|_H &\leq \sum_{j=k+1}^{N} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\ &\times \int_0^\infty s^{\alpha-1} \sup_{\delta\leq\mu<\infty} \frac{\tau\mu(1+\frac{\tau\mu}{2})^2\|\{\varphi_m\}_1^N\|_{\mathcal{C}^\alpha(H)}((j-k)\tau)^\alpha}{(\mu+s)(1+\tau\mu+\frac{(\tau\mu)^2}{2})^{j-k+1}} ds \\ &\leq \frac{\|\{\varphi_m\}_1^N\|_{\mathcal{C}^\alpha(H)}}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{ds}{s^{1-\alpha}(\delta+s)} \sum_{j=k+1}^{N} \frac{\tau}{((j-k)\tau)^{1-\alpha}} \\ &\leq \frac{\|\{\varphi_m\}_1^N\|_{\mathcal{C}^\alpha(H)}}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{ds}{s^{1-\alpha}(\delta+s)} \int_0^1 \frac{dx}{x^{1-\alpha}} \\ &\leq \frac{C(\delta)}{\alpha^2(1-\alpha)}\|\{\varphi_m\}_1^N\|_{\mathcal{C}^\alpha(H)} \end{aligned}\tag{5.63}$$

Estimates (5.62), (5.63) result that

$$\|S_3(k)\|_{H'_\alpha} \leq \frac{C(\delta)}{\alpha^2(1-\alpha)}\|\{\varphi_m\}_1^N\|_{\mathcal{C}^\alpha(H)} \tag{5.64}$$

Combining estimates (5.60)-(5.64), we obtain

$$\| \left\{\tau^{-1}(u_k - u_{k-1})\right\}_2^N \|_{\mathcal{C}(H'_\alpha)} \leq M\{\|Au_N + \varphi_N\|_{H'_\alpha} + \frac{\|\{\varphi_m\}_1^N\|_{\mathcal{C}^\alpha(H)}}{\alpha^2(1-\alpha)}\} \tag{5.65}$$

Now, let us establish estimate $\|Au_N + \varphi_N\|_{H'_\alpha}$. From formula (5.14) it follows that

$$\begin{aligned}Au_N + \varphi_N &= T'_\tau\{-\sum_{k=1}^{p} \alpha_k(I + d_kA) \sum_{j=\ell_k+1}^{N} \tau AD^{j-\ell_k}(I + \frac{\tau A}{2})(\varphi_{j-1} - \varphi_{\ell_k}) \\ &+ \sum_{k=1}^{p} \alpha_k(I + d_kA)D^{N-\ell_k}(\varphi_{\ell_k} - \varphi_N) \\ &+ \varphi_N - \sum_{k=1}^{p} \alpha_k\varphi_{\ell_k} + A\varphi\} \\ &= U_1 + U_2 + U_3.\end{aligned}$$

Estimates (5.5), (5.8) for $\beta = 1$, (5.11), formula (5.37), assumption (3.2), and the definition of $\mathcal{C}^\alpha(H)-$norm give us

$$\|U_1\|_{H'_\alpha} \leq \frac{C(\delta,\theta_p)}{\alpha^2(1-\alpha)}\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}, \tag{5.66}$$

$$\|U_2\|_{H'_\alpha} \leq C(\delta,\theta_p)\|\{\varphi_k\}_1^N\|_{\mathcal{C}^\alpha(H)}, \tag{5.67}$$

$$\|U_3\|_{H'_\alpha} = C(\delta,\theta_p)\|\varphi_N - \sum_{k=1}^{p}\alpha_k\varphi_{\ell_k} + A\varphi\|_{H'_\alpha}. \tag{5.68}$$

Thus, estimates (5.65), (5.66)-(5.68) concludes the proof of Theorem 5.32. □

CHAPTER 6

NUMERICAL RESULTS

We consider the reverse type parabolic problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial x^2} = f(t,x), \quad 0 < x < 1, \ 0 < t < 1, \\ u(0,x) = u(1,x) + \rho(x), \quad 0 \leq x \leq 1, \\ u(t,0) = u(t,1) = 0, \quad 0 \leq t \leq 1, \\ \rho(x) = 0. \end{cases} \tag{6.1}$$

The exact solution of this problem is $u(t,x) = t^2(1-t)^2 \sin \pi x$.

For approximate solutions of the nonlocal value problem,we will use the first order of accuracy and a second order of accuracy difference schemes. We have the first order or second order difference equations with respect to n with matrix coefficients. To solve this difference equations we have applied a procedure of modified Gauss elimination method for difference equations with respect to n with matrix coefficients. The results of numerical experiments permit us to show that the second order of accuracy difference schemes are more accurate comparing with the first order of accuracy difference scheme.

6.1 First order of accuracy of difference scheme

For approximate solution of nonlocal boundary-value problem (6.1), consider the set $[0,1]_\tau \times [0,\pi]_h$of a family of grid points depending on the small parameters $\tau$and h

$$\begin{aligned} [0,1]_\tau \times [0,1]_h &= \{(t,x_n) : t_k = k\tau, 1 \leq k \leq N-1, \\ N\tau &= 1, x_n = nh, 1 \leq n \leq M-1, Mh = 1\}. \end{aligned}$$

Applying Rother difference scheme and formula

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = 0(h^2), \tag{6.2}$$

we get the first order of accuracy in t for the approximate solutions of the nonlocal boundary value problem (6.1)

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} = f(t_k, x_n), 1 \le k \le N, 1 \le n \le M-1, \\ f(t_k, x_n) = t^2(1-t)^2 \sin \pi x, \\ u_0^k = u_M^k = 0, 0 \le k \le N, \\ u_n^0 = u_n^N = \rho(x_n), 0 \le n \le M \\ \rho(x_n) = 0. \end{cases}$$

or

$$\begin{cases} \left(\frac{1}{h^2}\right) u_{n+1}^{k-1} + \left(-\frac{1}{\tau} - \frac{2}{h^2}\right) u_n^{k-1} + \left(\frac{1}{\tau}\right) u_n^k + \left(\frac{1}{h^2}\right) u_{n-1}^{k-1} = f(t_k, x_n), \\ 1 \le k \le N-1, 1 \le n \le M-1, \\ u_0^k = u_M^k = 0, 0 \le k \le N, \\ u_n^0 = u_n^N = \rho(x_n), 0 \le n \le M. \end{cases}$$

So,

$$\begin{cases} Au_{n+1} + Bu_n + Cu_{n-1} = R\varphi_n, 1 \le n \le M-1, \\ u_0 = \overrightarrow{0}, u_M = \overrightarrow{0}. \end{cases}$$

Denote

$$a = \left(\frac{1}{h^2}\right), \; b = \left(-\frac{1}{\tau} - \frac{2}{h^2}\right), \; c = \left(\frac{1}{\tau}\right).$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & ... & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & a & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & -1 \\ b & c & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & ... & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & ... & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & b & c & 0 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & -1 \end{bmatrix}_{(N+1)\times(N+1)}$$

and C=-A.

$$R = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} , u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1\times(1)} , s = n-1, n, n+1.$$

$$\begin{aligned}
\varphi_n &= \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times(1)}, \\
\varphi_n^0 &= \rho(x_n), 1 \le n \le M-1, \\
\varphi_n^k &= f(t_k, x_n), 1 \le k \le N, 1 \le n \le M-1.
\end{aligned}$$

Using Gauss elimination procedure, we can obtain the approximate solution of the problem.

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}, n = M-1, \cdots, 2, 1,$$

$\alpha_n (n = 1, \cdots, M-1)$ are $(N+1) \times (N+1)$ square matrix and $\beta_n (n = 1, \cdots, M-1)$ are $(N+1) \times 1$ column matrices.

For the solution of difference equations we need to find $\alpha_1$andβ_1.

We can find them from $u_0 = \overrightarrow{0} = \alpha_1 u_1 + \beta_1$,

Thus, we have

$u_0 = \overrightarrow{0} = \alpha_1 u_1 + \beta_1$,we can obtain

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{(N+1)\times(1)}$$

Formulas for $\alpha_{n+1}, \beta_{n+1}$:

$$\begin{aligned}
\alpha_{n+1} &= -(B + C\alpha_n)^{-1}A, \\
\beta_{n+1} &= (B + C\alpha_n)^{-1}(R\varphi_n - C\beta_n), n = 1, 2, 3, ..., M-1.
\end{aligned}$$

So,

$$u_M = \overrightarrow{0},$$

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, n = M-1, \cdots, 2, 1.$$

Algorithm

1. Step Input time increment $\tau = \frac{1}{N}$ and space increment h=$\frac{1}{M}$.
2. Step Use the first order of accuracy difference scheme and write in matrix form

$$Au_{n+1} + Bu_n + Cu_{n-1} = R\varphi_n, 1 \leq n \leq M-1.$$

3. Step Determine the entries of the matrices A, B, C and R.
4. Step Find α_1, β_1.
5. Step Compute $\alpha_{n+1}, \beta_{n+1}$.
6. Step Compute u_n 's $(n = M-1, ..., 2, 1)$, $\left(u_M = \widetilde{0}\right)$ using the following formula

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}.$$

6.2 Matlap Implementation of the First Order of Accuracy Difference Scheme

```
function [table,es,p]=EulerRotherMethod(N,M)
% Computers numerical solution of the equation by
% The Euler Rother Method.
if nargin < 1; N= 40 ; M= 40 ; end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%1
tau=1/N;
h=1/M;
aaa=1; %u(0)=aaa.u(1)+alx(x)
```

```
v = 1/(h^2);
for i=2:N+1;
A(i,i)=v;
end ;
alfa =(1/tau)-2/(h^2);
for i=2:N+1;
B(i,i)=alfa;
end;
beta = - 1/ tau ;
for i=1:N;
B(i+1,i)= beta ;
end;
B(1,1)=1; B(1,N+1)=-aaa ;
C=A;
for i=1:N+1;
D(i,i)=1;
end ;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 2
alpha{1} = zeros(N+1,N+1) ; betha{1} = zeros(N+1,1) ;
'fii(:,j) :j-th column matrix' ;
for j=1:M;
x=j*h;
for k=1:N+1;
t=(k-1)*tau ; 'to avoid zero indices';
fii( k, j:j ) = f(t,x); 'right side function ';
end;
fii(1,j:j)=rox(x); 'given sub function ';
```

```
end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 3
'alpha(:,:,j) : j. alpha and betha(:,j) :j-th betha' ;
for j=1:M-1;
Q=inv(B+C*alpha{j});
alpha{j+1}= - Q*A ;
betha{j+1}= Q*(D*( fii(:,j))-(C* betha{j}) );
end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 4
'COMPUTE U(n)';
U( N+1,M:M ) = 0; % U(M)=0 ;
for z = M-1:-1:1 ;
U(:,z) = alpha{z+1}* U(:,z+1) + betha{z+1};
end;
for z = 1:M ;
p(:,z+1)=U(:,z);'U(0)=0';
end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 5
'EXACT SOLUTION OF THIS PDE' ;
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j) = exact(t,x);
end;
end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%6
```

```
'ERROR ANALYSIS' ;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxes,maxapp,maxerror,relativeerror]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 7
table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 8
%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%%%
q=min(min(table)); w=max(max(table));
figure;
surf(es); title('EXACT SOLUTION'); view(-60,16);
set(gca,'ZLim',[q w]);
rotate3d;
figure;
surf(p); title('EULER-ROTHER'); rotate3d ;view(-60,16);
set(gca,'ZLim',[q w]);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%9
%%%%%%%%%%%%%%%%%%%%SUB FUNCTIONS%%%%%
function rx=rox(x)
E=exp(1);
rx=0;
function estx=exact(t,x)
E=exp(1);
estx=t^2*(1-t)^2*sin(pi*x);
function ftx=f(t,x)
```

```
E=exp(1);

ftx=sin(pi*x)*(2*t-6*t^2+4*t^3-t^2*pi^2+2*t^3*pi^2-t^4*pi^2)
```

6.3 Second order of accuracy of difference scheme

First, we consider again the nonlocal boundary value problem (6.1). Applying the second order difference scheme, formulas (6.2) and

$$\frac{u(x_{n+2})-4u(x_{n+1})+6u(x_n)-4u(x_{n-1})+u(x_{n-4})}{h^4}-u^{(4)}(x_n)=o(h^2),$$

$$\frac{2u(0)-5u(h)+4u(2h)-u(3h)}{\tau^2}-u''(0)=o(h^2),$$
$$\frac{2u(1)-5u(1-h)+4u(1-2h)-u(1-3h)}{\tau^2}-u''(1)=o(h^2).$$

we get the second order of accuracy in t for the approximate solutions of the nonlocal boundary value problem (6.1)

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau}+\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}-\frac{\tau}{2h^4}\left[u_{n+2}^{k-1}-4u_{n+1}^{k-1}+6u_n^{k-1}-4u_{n-1}^{k-1}+u_{n-2}^{k-1}\right]=\varphi_n^k, \\ \varphi_n^k=f\left(t_k-\frac{\tau}{2},x_n\right)-\frac{\tau}{2h^2}\left(f\left(t_k-\frac{\tau}{2},x_{n+1}\right)-2f\left(t_k-\frac{\tau}{2},x_n\right)+f\left(t_k-\frac{\tau}{2},x_{n-1}\right)\right), \\ f(t,x)=\sin\pi x\left[2t-6t^2+4t^3-\pi^2\left(t^2-(1-t)^2\right)\right];1\le k\le N, 2\le n\le M-2, \\ u_0^k=u_M^k=0, 0\le k\le N, \\ u_1^k=\frac{4}{5}u_2^k-\frac{1}{5}u_3^k, 0\le k\le N, \\ u_{M-1}^k=\frac{4}{5}u_{M-2}^k-\frac{1}{5}u_{M-3}^k, 0\le k\le N, \\ u_n^0-u_n^N=\rho(x_n), 0\le k\le M, \\ \rho(x)=0. \end{cases}$$

We will write in the matrix form. We will resort the system

$$\begin{cases} \left(-\frac{\tau}{2h^4}\right)u_{n+2}^{k-1} + \left(\frac{1}{h^2} + 4\frac{\tau}{2h^4}\right)u_{n+1}^{k-1} + \left(-\frac{1}{\tau}\right)u_n^{k-1} + \left(-\frac{1}{\tau} - \frac{2}{h^2} - \frac{3\tau}{h^4}\right)u_n^{k-1} \\ + \left(\frac{1}{h^2} + 4\frac{\tau}{2h^4}\right)u_{n-1}^{k-1} + \left(-\frac{\tau}{2h^4}\right)u_{n-2}^{k} = \varphi_n^k, 1 \le k \le N, 2 \le n \le M-2, \\ u_0^k = u_M^k = 0, 0 \le k \le N, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k, 0 \le k \le N, \\ u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k, 0 \le k \le N, \\ u_n^0 - u_n^N = \rho(x_n), 0 \le k \le M, \\ \varphi_n^0 = \rho(x_n), 1 \le n \le M, \\ \rho(x) = 0. \end{cases}$$

We have that

$$\begin{cases} Au_{n+2} + Bu_{n+1} + Cu_n + Du_{n-1} + Eu_{n-2} = R\varphi_n, 2 \le n \le M-2, \\ u_0 = 0, u_m = 0, \\ u_1 = \frac{4}{5}u_2 - \frac{1}{5}u_3, \\ u_{M-1} = \frac{4}{5}u_{M-2} - \frac{1}{5}u_{M-3}, \end{cases}$$

where

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}.$$

$$A = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ x & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & x & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & x & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & x & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & x & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ y & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & y & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & y & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & y & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & y & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & -1 \\ p & z & 0 & \ldots & 0 & 0 \\ 0 & p & z & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & . & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & z & 0 \\ 0 & 0 & 0 & \ldots & p & z \end{bmatrix}_{(N+1)\times(N+1)}$$

where

$$\begin{aligned} x &= -\frac{\tau}{2h^4}, \\ y &= \frac{1}{h^2} + \frac{2\tau}{h^4}, \\ p &= \frac{1}{\tau}, \\ z &= -\frac{1}{\tau} - \frac{2}{h^2} - \frac{3\tau}{h^4}, \end{aligned}$$

and $R = B, E = A$,

$$R = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & . & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ u_s^3 \\ \vdots \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1)\times(1)}, s = n-2, n-1, n, n+1, n+2.$$

For the solution of the last matrix equation, we use the modified variant Gauss elimination method. We seek a solution of the matrix equation of the matrix equation by the following form:

$$\begin{cases} u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}u_{n+2} + \gamma_{n+1}, n = M-2, \cdots, 2, 1, 0, \\ u_M = 0, \\ u_{M-1} = \left[\left(\beta_{M-2} + 5I\right) - \left(4I - \alpha_{M-2}\right)\alpha_{M-1}\right]^{-1}\left[\left(4I - \alpha_{M-2}\right)\gamma_{M-1} - \gamma_{M-2}\right], \end{cases}$$

where

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \beta_1 = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

$$\gamma_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(N+1)\times(1)}, \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(N+1)\times(1)}$$

$$\alpha_2 = \begin{bmatrix} \frac{4}{5} & 0 & 0 & ... & 0 \\ 0 & \frac{4}{5} & 0 & ... & 0 \\ 0 & 0 & \frac{4}{5} & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & 0 & 0 & ... & \frac{4}{5} \end{bmatrix}_{(N+1)\times(N+1)} , \beta_2 = \begin{bmatrix} -\frac{1}{5} & 0 & 0 & ... & 0 \\ 0 & -\frac{1}{5} & 0 & ... & 0 \\ 0 & 0 & -\frac{1}{5} & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & 0 & 0 & 0 & -\frac{1}{5} \end{bmatrix}_{(N+1)\times(N+1)}$$

and

$$\begin{aligned} \beta_{n+1} &= -\left(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}(A), \\ \alpha_{n+1} &= -\left(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \gamma_{n+1} &= -\left(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}\left(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}\right), \\ n &= 2, \cdots, M-2. \end{aligned}$$

Algorithm

1. Step; Input time increment $\tau = \frac{1}{N}$ and space increment $h = \frac{1}{M}$.

2. Step; Substitute the second order difference approximations into the equations and write these equations in matrix from to obtain the equality

$$Au_{n+2} + Bu_{n+1} + Cu_n + Du_{n-1} + Eu_{n-2} = R\varphi_n, \quad 2 \leq n \leq M-2.$$

3. Step; Determine the entries of the matrices A, B, C, E and R.

4. Step; $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_{2,}\gamma_2$ are put.

5. Step; Compute $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$, using the following formulas ($n = 2$to $M-2$),

$$\begin{aligned} \beta_{n+1} &= -\left(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}(A), \\ \alpha_{n+1} &= -\left(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}(B + D\beta_n + E\alpha_{n-1}\beta_n), \\ \gamma_{n+1} &= -\left(C + D\alpha_n + E\beta_{n-1} + E\alpha_{n-1}\alpha_n\right)^{-1}\left(R\varphi_n - D\gamma_n - E\alpha_{n-1}\gamma_n - E\gamma_{n-1}\right), \end{aligned}$$

$n = 2, \cdots, M-2.$

6. Step; Compute

$$\begin{aligned}u_M &= 0,\\ u_{M-1} &= \left[\left(\beta_{M-2}+5I\right)-\left(4I-\alpha_{M-2}\right)\alpha_{M-1}\right]^{-1}\left[\left(4I-\alpha_{M-2}\right)\gamma_{M-1}-\gamma_{M-2}\right],\\ u_{M-2} &= \left[\left(4I-\alpha_{M-2}\right)\right]^{-1}\left[\left(\beta_{M-2}+5I\right)u_{M-1}+\gamma_{M-2}\right].\end{aligned}$$

7. Step; compute u_n 's $(n = M-3, \cdots, 2, 1)$, $\left(u_M = \widetilde{0}\right)$ using the following formula

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}u_{n+2} + \gamma_{n+1}.$$

6.4 Matlap Implementation on Second Order of Accuracy Difference Scheme

```
function [table,es,p]=secondorder(N,M)
% Computers numerical solution of the equation
% Ut+Uxx=ftx(t,x);
if nargin < 1; N= 40 ; M= 40 ; end;
close;close;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%1
tau=1/N; h=1/M;
aaa=1; %u(0)=aaa.u(1)+alx(x)
x= -tau/(2*(h^4));     A=zeros(N+1,N+1);
for i=2:N+1;     A(i,i-1)=x;     end ;
E=A ;
y = 1/(h^2) +2*tau/(h^4) ;B=zeros(N+1,N+1);
for i=2:N+1;     B(i,i-1)=y;     end ;
D=B ;
z = -1/tau - 2 /(h^2) - 3*tau/(h^4) ; C=zeros(N+1,N+1);
for i=2:N+1;     C(i,i-1)= z ;     end;
s = 1/tau ;
```

```
for i=1:N;          C(i+1,i+1)= s ;      end;
C(1,1)=1;       C(1,N+1)=-1 ;
R=eye(N+1,N+1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%2
alpha{1} = zeros(N+1,N+1) ;
betha{1} = zeros(N+1,N+1) ;
gamma{1}= zeros(N+1,1) ;
alpha{2} = (4/5)*eye(N+1) ;
betha{2} = (-1/5)*eye(N+1);
gamma{2} = zeros(N+1,1);
'fi(j) = fi(k,j) hesaplanıyor ' ;
for j=1:M ;
x=j*h;
for k=2:N+1 ;
t =(k-1)*tau - tau/2;
fii( k, j:j ) = rsf(t,x,tau)-tau/2/h^2*(rsf(t,x+h,tau)-2*rsf(t,x,tau)+rsf(t,x-h,tau));
end;
fii(1,j:j)=rox(x);
end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%3
'alpha(N+1,N+1,j) ve betha(N+1,j) ler hesaplanacak' ;
for n = 2:M-2 ;
K=C+D*alpha{n}+E*betha{n-1}+E*alpha{n-1}*alpha{n} ;
betha{n+1} = - inv(K)*(A) ;
alpha{n+1} = - inv(K)*(B +D*betha{n}+E*alpha{n-1}*betha{n});
gamma{n+1} = inv(K)*( R*fii(:,n:n) - D*gamma{n} ...
-E*alpha{n-1}*gamma{n} - E*gamma{n-1} );
```

```
end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%4
'EXACT SOLUTION OF THIS PDE' ;
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(j-1)*h;
es(k,j) =exact(t,x);
end;
end ;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% 6
'INITIAL VALUES OF U IS OBTAINED HERE' ;
I=eye(N+1); U(1:N+1,M:M ) = 0 ;
U(:, M-1) = inv( betha{M-2} + 5*I - (4*I-alpha{M-2} )*alpha{M-1} )*...
( 4*gamma{M-1} - alpha{M-2}*gamma{M-1} - gamma{M-2});
%%%%%COMPUTE U(n)%%%%%%%
for z = M-2:-1:1 ;%%%%M-3;
U(:,z )=alpha{z+1}*U(:,z+1)+betha{z+1}*U(:,z+2)+gamma{z+1};
end;
for z = 1 : M ; p(:,z+1)=U(:,z); end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%5
%table=[es;p];table(1:2:end,:)=es; table(2:2:end,:)=p;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%6
%%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%%%%
%q=min(min(table)); w=max(max(table));
figure;surf(es); title('EXACT SOLUTION'); view(-60,16);
%set(gca,'ZLim',[q w]);
```

```
rotate3d;
figure;surf(p); title('Second order of approximate solution');
rotate3d ;view(-60,16); %set(gca,'ZLim',[q w]);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%7
'ERROR ANALYSIS' ;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap=[maxes,maxapp,maxerror,relativeerror]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%8
%%%%%%% SUB FUNCTIONS- SECOND ORDER %%%%%%%%%%%%
function rx=rox(x)
E=exp(1);
rx=0;
function estx=exact(t,x)
E=exp(1);
estx=t^2*(1-t)^2*sin(pi*x);
function rsftx=rsf(t,x,tau)
E=exp(1);
rsftx=sin(pi*x)*(2*t-6*t^2+4*t^3-t^2*pi^2+2*t^3*pi^2-t^4*pi^2);
```

The Errors

Now, we will give the results of the numerical analysis. In order to get the solution, we use MATLAB programs. The numerical solutions are recorded for different values of N=M and u_n^k represents the numerical solutions of these difference schemes at (t_k, x_n). For their comparison, the errors computed by

$$E_M^N = \max_{-N \le k \le N, 1 \le n \le M-1} \left| u(t_k, x_n) - u_n^k \right|$$

Table gives the error analysis between the exact solution and solutions derived by difference schemes. Table is constructed for N=M=20, 40 and 60 respectively.

Table 1. Error analysis for the exact solution $u(t, x)$.

Difference schemes	N=M=20	N=M=40	N=M=60
The first order accuracy difference scheme	0.0037	0.0017	0.0011
The second order accuracy difference scheme	0.0009	0.0002	0.0001

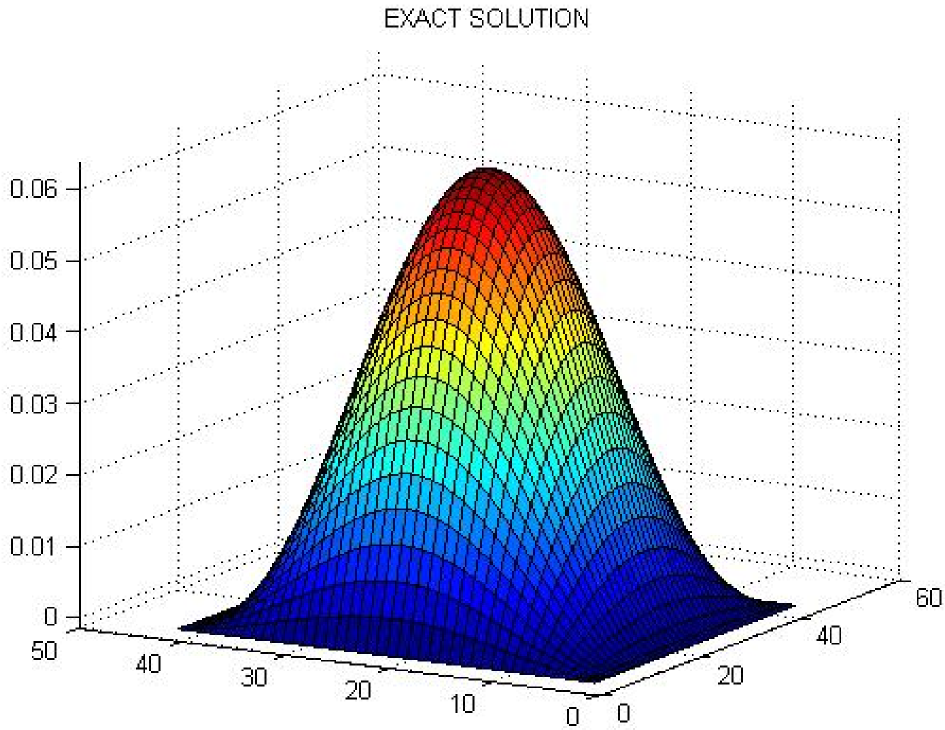

Figure 1: Exact Solution

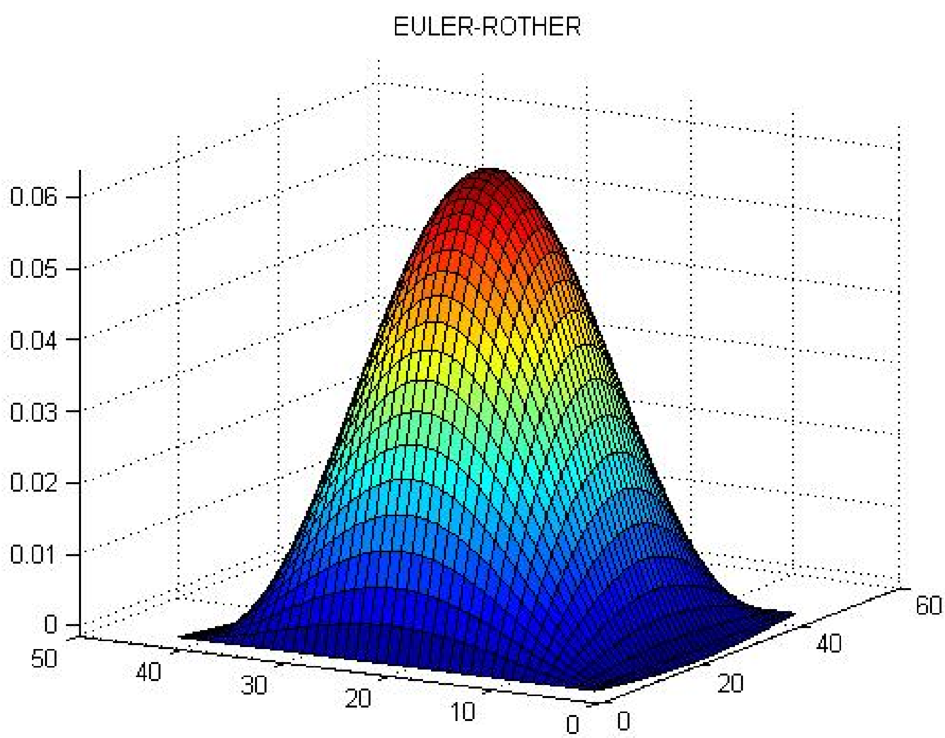

Figure 2: First order of accuracy diference scheme

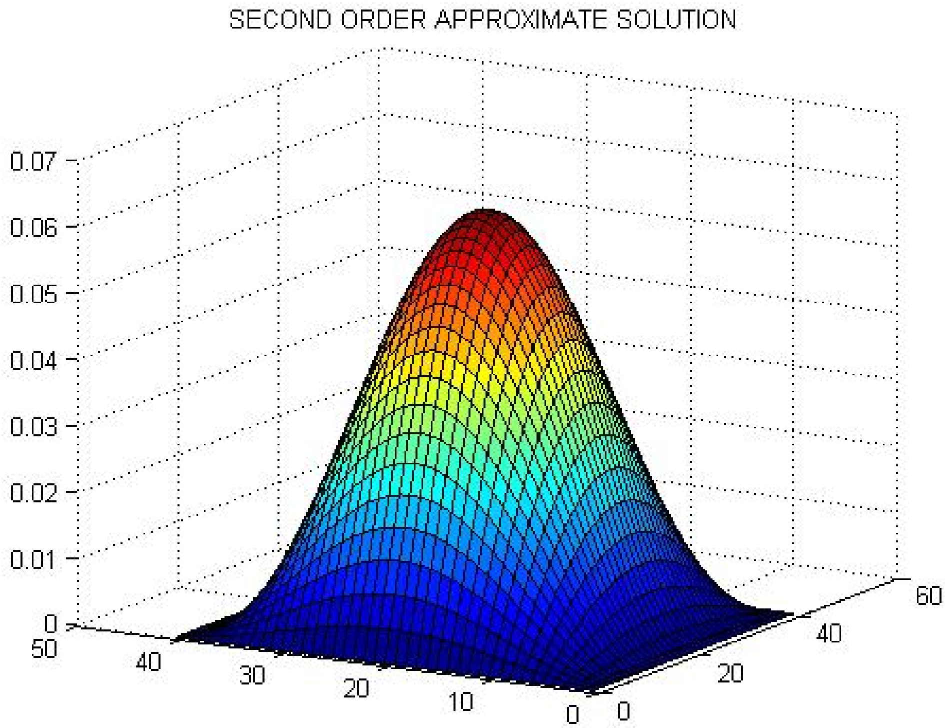

Figure 3: Second order of accuracy diference scheme

CHAPTER 7

CONCLUSIONS

This work is devoted to study the well-posedness of multi-point nonlocal boundary value parabolic differential problems of reverse type. The following original results are obtained:

- The abstract theorem on the coercive stability estimate for the solution of multi-point nonlocal boundary value parabolic differential equation of reverse type in the weighted Hölder space is proved.
- The abstract theorem on the coercive stability estimate for the solution of multi-point nonlocal boundary value parabolic differential equation of reverse type in the Hölder space is proved.
- The first and second order of accuracy difference schemes for the approximate solutions of multi-point nonlocal boundary value parabolic differential equation of reverse type are presented.
- Theorems on the stability estimates, almost coercive stability estimates, and coercive stability estimates for the solution of difference schemes for multi-point nonlocal boundary value parabolic differential equation of reverse type are proved.
- The MATLAB implementation of these difference schemes is presented.
- The theoretical results for the solution of these difference schemes are supported by the results of numerical examples.

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