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MULTIPOINT PADÉ APPROXIMATIONS AND THEIR CONVERGENCE

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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MULTIPOINT PADÉ APPROXIMATIONS AND THEIR CONVERGENCE

Cevdet Akal

M. S. Thesis - Mathematics
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Supervisor: Prof. Dr. Alexey Lukashov

ABSTRACT

Padé approximants and their generalizations play very important role in many applications. There are different ways to generalize the definition of the Padé approximants to the case of general rational interpolation. Namely, multipoint Padé approximations and Newton-Padé approximations solve the same interpolation problem but they are written in different terms.

In this study, one of the main purposes is to investigate another way of writing the solution of the rational interpolation problem. As an advantage of that way, it is supposed to get new conditions for the convergence of multipoint Padé approximations.

Keywords: rational interpolation, Newton-Padé approximation, multipoint Padé approximation

ÇOK NOKTALI PADÉ YAKLAŞTIRIMLARI VE YAKINSAMALARI

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ÖZ

Padé yaklaştırımları ve bunların genellemeleri birçok uygulamada önemli rol oynar. Genel rasyonel interpolasyon durumlarında Padé yaklaştırımlarının genellemesinde birbirinden farklı yollar vardır. Yani, çok noktalı Padé yaklaştırımları ve Newton-Padé yaklaştırımları aynı interpolasyon problemlerini çözmelerine karşın farklı formlarda yazılırlar.

Bu çalışmada, temel amaçlardan biri rasyonel interpolasyon problemlerinin çözümünün farklı formlarda yazılmasını incelemektir. Bu metodun avantajı, çok noktalı Padé yaklaştırımlarının yakınsamalarına yeni koşullar getirmesinin beklenilmesidir.

Anahtar Kelimeler: rasyonel interpolasyon, Newton-Padé yaklaştırımları, çok noktalı Padé yaklaştırımları

DEDICATION

To my family

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CHAPTER 1

INTRODUCTION

The history of continued fractions, and associated with it, the problem of Padé approximation is one of the oldest in the history of mathematics. There are very early predecessors, but the study was really started in the 18th century and came to maturity in the 19th century. The serious work started with Cauchy in his famous Cours d'analyse and Jacobi and was continued by Frobenius and Padé. A current standard work is the book by G. Baker Jr. and P. Graves-Morris ([Baker Jr. G. , Graves-Morris P., 1981b]).

The problem of rational interpolation has very long story and is not solved completely up to now even for the case of a single variable. One of widely used approaches has its origins in Padé approximants.

Recall that (Padé-Frobenius definition, ([Baker Jr. G. , Graves-Morris P., 1981a])) a unique rational function $r_{M,N}(z) = P_M(z)/Q_N(z)$, with $P_M \in \widetilde{H}_M,\ Q_N \in \widetilde{H}_N,\ \widetilde{H}_k$ being the set of polynomials of degree $\leq k$ with complex coefficients, satisfying a formal identity

$$Q_N(z)\, f(z) - P_M(z) = A_{M+N+1} z^{M+N+1} + \dots \tag{1.1}$$

where

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad (a_0 \neq 0), \tag{1.2}$$

is called *the* $[M, N]$ *Padé approximant* of the formal power series $f(z)$. Padé approximants may be considered as a particular case of rational interpolation subject to conditions $r_{M,N}^{(k)}(0) = f^{(k)}(0)\,,\ k = 0, 1, \ldots, M+N$, if the series (1.2) converges in a neighbourhood of 0 and $Q_N(0) \neq 0$.

Multipoint Padé approximants and Newton-Padé approximants are their natural generalizations for the Hermite interpolation. There are different ways to define them. For a survey one may consult Meinguet, Stahl ([Stahl H., 1996]). The first one is as follows:

Let an infinite triangular matrix of interpolation points $a_{ij} \in \bar{\mathbb{C}}$ (called interpolation scheme) be given:

$$A := \begin{pmatrix} a_{00} & & & \\ \cdots & \cdots & & \\ a_{0n} & \cdots & a_{nn} & \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{1.3}$$

Each row

$$A_n := \{a_{0n}, \ldots, a_{nn}\} \tag{1.4}$$

of the matrix A defines an interpolation set with $n+1$ interpolation points.

The rational function

$$r_{M,N} = r_{M,N}(f,\ A_{M+N}; .) = r_{M,N}(f,\ A; .) = \frac{P_{M,N}}{Q_{M,N}} \tag{1.5}$$

with $P_{M,N} \in \widetilde{H}_M,\ Q_{M,N} \in \widetilde{H}_N,$ and $Q_{M,N} \neq 0,$ is called *multipoint Padé approximant* or *linearized rational interpolant of degree* M, N to the function f at the $M+N+1$ points of the interpolation set A_{M+N} if the quotient

$$\frac{Q_{M,N} f - P_{M,N}}{\omega_{M+N}} \text{ is bounded at each point } x \in A_{M+N}. \tag{1.6}$$

Now let us define Newton-Padé approximants. Firstly, following M. A. Gallucci and W. B. Jones ([Gallucci M. A., Jones W. B., 1976]) a *formal Newton series* (FNS) is an ordered triple $[\{\alpha_n\}_0^\infty, \{\beta_n\}_0^\infty, \{f_n\}_0^\infty]$, where $\alpha_0, \alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \beta_3, \ldots$ are complex numbers (not necessarily distinct) and for each $n = 0, 1, 2, \ldots$, f_n is the polynomial

$$f_n(z) = \sum_{k=0}^{n} \alpha_k \omega_k(z), \tag{1.7}$$

where

$$\omega_0(z) = 1; \quad \omega_k(z) = \prod_{j=1}^{k} (z - \beta_j), \quad k = 1, 2, 3, \ldots, \tag{1.8}$$

and where z is a complex variable. The α_n, β_n, and f_n are called, respectively, *the nth Newton coefficient*, *interpolation point*, and *partial sum of* $[\{\alpha_n\}, \{\beta_n\}, \{f_n\}]$ and a FNS is said to *converge at* z if the sequence of partial sums $\{f_n(z)\}$ is convergent.

The Newton-Padé approximant

$$r_{M,N}(z) = \frac{P_{M,N}(z)}{Q_{M,N}(z)} \tag{1.9}$$

with $P_{M,N} \in \widetilde{H}_M$, $Q_{M,N} \in \widetilde{H}_N$ is the function of the form (1.9) such that

$$Q(z) \cdot f(z) - P(z) = d_{M+N+1} \omega_{M+N+1}(z) + \ldots. \tag{1.10}$$

Convergence theory of classical Padé approximants has a long and distinguished history dating back to Hermite's work. Essential part of the theory studies convergence of sequences of Padé approximants for special classes of functions which are defined by properties of their Maclaurin series coefficients. For the multipoint Padé approximants general convergence theory for meromorphic, Markov and more general classes of functions was developed but, to the best of our knowledge, there are no results about convergence of multipoint Padé approximants from special properties of their Newton series coefficients.

A rational function which fits given function at various points, not necessarily distinct, is called a *multipoint Padé approximant*. The associated problem of interpolation by rational functions is called the Cauchy-Jacobi problem. Multipoint Padé approximants are also called *rational interpolants*, *N-point Padé approximants*, or *Newton-Padé approximants* ([Gallucci M. A., Jones W. B., 1976]), depending on the context. Interpolation at confluent points is sometimes called *osculatory interpolation*. Multipoint Padé approximations for a sufficiently general class of functions were first studied in ([Stahl H., 1996]), while, in ([Claessens G., 1976]) and ([Baker Jr. G. , Graves-Morris P., 1981a]), such approximations were used to obtain significant results concerning best rational approximations of analytic functions.

Balk ([Balk M.B.,1960]) obtained conditions of convergence of Padé approximations which used properties of Taylor series coefficients only and applied them to study the convergence of Padé approximations of some elementary functions. The convergence of the multipoint Padé approximations for concrete functions is not studied sufficiently, it is possible to mention here recent Kandayan's paper ([Kandayan A.A., 2009]).

One of the main purposes of this study is to give an analogue of Balk's test of Padé approximants convergence for Newton-Padé approximants. To do so, we will need to use Newton form not only for numerator P_M, as in ([Baker Jr. G. , Graves-Morris P., 1981a]). Note that determinantal representations of Newton-Padé approximations given in ([Gallucci M. A., Jones W. B., 1976]) only for denominators Q_N in power form $Q_N(z) = \sum\limits_{i=0}^{N} \lambda_i z^i$. For Q_N in Newton form $Q_N(z) = \sum\limits_{k=0}^{N} b_k \omega_k(z)$, D.D. Warner ([Warner D.D., 1974]) gave representations in terms of other definition of the FNS (as the infinite triangular matrix of divided differences), and G. Claessens ([Claessens G., 1976]) established recurrence formulas using definition of FNS as above.

Here we will give determinantal representations of Newton-Padé approximants for denominators Q_N in Newton form.

CHAPTER 2

FOUNDATIONS OF THE PADE APPROXIMATIONS THEORY

2.1. Definition and Conditions for the Convergence of Padé Approximations

Suppose that we are given a power series $\sum_{i=0}^{\infty} c_i z^i$, representing a function $f(z)$, so that

$$f(z) = \sum_{i=0}^{\infty} c_i z^i. \tag{2.1}$$

This expansion is the fundamental starting point of any analysis using Padé approximants. Throughout this work, we reserve the notation c_i, $i = 0, 1, 2, \ldots$, for the given set of coefficients, and $f(z)$ is the associated function. A *Padé approximant* is a rational fraction

$$[L/M] = \frac{a_0 + a_1 z + \ldots + a_L z^L}{b_0 + b_1 z + \ldots + b_M z^M} \tag{2.2}$$

which has a Maclaurin expansion which agrees with (2.1) as far as possible.

Definition 2.1. [Baker, Graves-Morris; 1981] . If polynomials $A^{[L/M]}(z)$, $B^{[L/M]}(z)$, of degrees L, M respectively, can be found such that

$$\frac{A^{[L/M]}(z)}{B^{[L/M]}(z)} = f(z) + O\left(z^{L+M+1}\right) \tag{2.3}$$

with

$$B^{[L/M]}(0) = 1 \tag{2.4}$$

then we define $[L/M] = \frac{A^{[L/M]}(z)}{B^{[L/M]}(z)}$.

The notation emphasizes that numerator and denominator depend on both L and M. An entirely equivalent specification of the definition is to replace (2.3) by

$$A^{[L/M]}(z) - f(z)\, B^{[L/M]}(z) = O\left(z^{L+M+1}\right)$$

provided that (2.4) is retained. The notation of (2.3) and (2.4) is exclusively reserved for this purpose throughout the work, and without further explanation. So there are $L+1$ independent numerator coefficients and M independent denominator coefficients, making $L+M+1$ unknown coefficients in all. This number suggests that normally the $[L/M]$ ought to fit the power series (2.1) through the orders $1, z, z^2, \ldots, z^{[L+M]}$. In the notation of formal power series,

$$\sum_{i=0}^{\infty} c_i z^i = \frac{a_0 + a_1 z + \ldots + a_L z^L}{b_0 + b_1 z + \ldots + b_M z^M} + O\left(z^{L+M+1}\right). \tag{2.5}$$

By cross-multiplying, we find that

$$\begin{aligned} &\left(b_0 + b_1 z + \ldots + b_M z^M\right)(c_0 + c_1 z + \ldots) \\ &= a_0 + a_1 z + \ldots + a_L z^L + O\left(z^{L+M+1}\right) \end{aligned} \tag{2.6}$$

Equating the coefficients of $z^{L+1},\ z^{L+2},\ \ldots,\ z^{L+M}$, we find

$$\begin{aligned} b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + \ldots + b_0 c_{L+1} &= 0, \\ b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + \ldots + b_0 c_{L+2} &= 0, \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad &= \vdots \\ b_M c_L + b_{M-1} c_{L+1} + \ldots + b_0 c_{L+M} &= 0. \end{aligned} \tag{2.7}$$

If $j < 0$, we define $c_j = 0$ for consistency. Since $b_0 = 1$, equations (2.7) become a set of M linear equations for the M unknown denominator coefficients:

$$\begin{pmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} \\ c_{L-M+3} & c_{L-M+4} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_L & c_{L+1} & \cdots & c_{L+M-1} \end{pmatrix} \begin{pmatrix} b_M \\ b_{M-1} \\ b_{M-2} \\ \vdots \\ b_1 \end{pmatrix} = - \begin{pmatrix} c_{L+1} \\ c_{L+2} \\ c_{L+3} \\ \vdots \\ c_{L+M} \end{pmatrix}, \tag{2.8}$$

from which the b_i may be found. The numerator coefficients, $a_0,\ a_1,\ \ldots,\ a_L,$ follow immediately from (2.6) by equating the coefficients of $1,\ z\,,\ z^2,\ \ldots,\ z^L$:

$$\begin{aligned} a_0 &= c_0, \\ a_1 &= c_1 + b_1 c_{0,} \\ a_2 &= c_2 + b_1 c_1 + b_2 c_0, \qquad (2.9) \\ \vdots &= \vdots \qquad \vdots \qquad \vdots \\ a_L &= c_L + \sum_{i=1}^{\min(L,M)} b_i c_{L-i}. \qquad (2.10) \end{aligned}$$

Thus (2.8) and (2.9) normally determine the Padé numerator and denominator and are called the Padé equations; we have constructed an $[L/M]$ Padé approximant which agrees $\sum_{i=0}^{\infty} c_i z^i$ through order z^{L+M}.

If Cramer's rule is used, we may calculate $b_0 : b_1 : \ldots : b_M$ from (2.8) and hence the denominator of (2.2). Aside from a common factor, the result is

$$Q^{[L/M]}(z) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} & c_{L+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-2} & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M-1} & c_{L+M} \\ z^M & z^{M-1} & \cdots & z & 1 \end{vmatrix}. \tag{2.11}$$

We take (2.11) to define $Q^{[L/M]}(z)$ and use this convention throughout.

Again, recall that $c_j = 0$ if $j < 0$. Now consider

$$Q^{[L/M]}(z)\sum_{i=0}^{\infty} c_i z^i = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ \sum_{i=0}^{\infty} c_i z^{M+i} & \sum_{i=0}^{\infty} c_i z^{M+i-1} & \cdots & \sum_{i=0}^{\infty} c_i z^i \end{vmatrix}.$$

By subtracting z^{L+1} times the first row from the last, z^{L+2} times the second row from the last, etc., up to z^{L+M} times the penultimate row from the last, we reduce the series in the last row. They become lacunary series, with a gap of M terms missing. Using the initial terms of these series, we define

$$P^{[L/M]}(z) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ \sum_{i=0}^{L-M} c_i z^{M+i} & \sum_{i=0}^{L-M+1} c_i z^{M+i-1} & \cdots & \sum_{i=0}^{L} c_i z^i \end{vmatrix}. \tag{2.12}$$

Again, (2.12) is our notational convention.

Theorem 2.1.1. *With the definitions (2.11) and (2.12),*

$$Q^{[L/M]}(z)\sum_{i=0}^{\infty} c_i z^i - P^{[L/M]}(z) = O\left(z^{L+M+1}\right). \tag{2.13}$$

Proof We note that $\deg\left\{P^{[L/M]}\right\} \le L,\ \ \deg\left\{Q^{[L/M]}\right\} \le M$ and that remainder is

$$Q^{[L/M]}(z)\sum_{i=0}^{\infty}c_i z^i - P^{[L/M]}(z)$$

$$= \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ \sum\limits_{i=L+1}^{\infty} c_i z^{M+i} & \sum\limits_{i=L+2}^{\infty} c_i z^{M+i-1} & \cdots & \sum\limits_{i=L+M+1}^{\infty} c_i z^i \end{vmatrix} \tag{2.14}$$

$$= \sum_{i=1}^{\infty} z^{L+M+i} \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ c_{L+i} & c_{L+i+1} & \cdots & c_{L+M+i} \end{vmatrix}. \tag{2.15}$$

Equation (2.15) is occasionally a useful form for the error using Padé approximation. Equation (2.13) goes a long way towards satisfying (2.5). To this end, consider

$$C(L/M) = Q^{[L/M]}(0) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-2} \\ c_L & c_{L+1} & \cdots & c_{L+M-1} \end{vmatrix}.$$

This is called a *Hankel determinant*, because of the systematic way in which its rows are formed from the given coefficients c_i. Notice that if $Q^{[L/M]}(0) \neq 0$, then the linear equations (2.8) are nonsingular and the solution given by (2.11) is unambiguous. Furthermore, we may divide (2.13) by $Q^{[L/M]}(z)$, yielding

$$\sum_{i=0}^{\infty} c_i z^i - \frac{P^{[L/M]}(z)}{Q^{[L/M]}(z)} = O\left(z^{L+M+1}\right).$$

This result has proved the following theorem:

Theorem 2.1.2. *[Jacobi, 1846] With the definitions (2.11) and (2.12), the* $[L/M]$ *Padé approximant of* $\sum_{i=0}^{\infty} c_i z^i$ *is given by*

$$[L/M] = \frac{P^{[L/M]}(z)}{Q^{[L/M]}(z)} \tag{2.16}$$

provided $Q^{[L/M]}(0) \neq 0$.

Now we may give the classical definition, also called *the Frobenius* and *Padé Frobenius* definition.

Definition 2.2. If $P_L(z)$, $Q_M(z)$ are polynomials of orders L, M respectively, and if

$$Q_M(z) f(z) - P_L(z) = O\left(z^{L+M+1}\right), \tag{2.17}$$

then $P_L(z)/Q_M(z)$ is a Padé approximant of $f(z)$.

Note that if $Q_M(0) = 0$, then in this case

$$f(z) \neq \frac{P_L(z)}{Q_M(z)} + O\left(z^{L+M+1}\right).$$

If, with equation (2.11), $Q^{[L/M]}(0) \neq 0$, then the rescaling

$$A^{[L/M]}(z) = \frac{P^{[L/M]}(z)}{Q^{[L/M]}(0)}$$

and

$$B^{[L/M]}(z) = \frac{Q^{[L/M]}(z)}{Q^{[L/M]}(0)}$$

implies that the two definitions correspond up to an unimportant numerical factor.

We extend the notation $[L/M]$ of (2.16) as $[L/M]_f$ to emphasize approximation of $f(z)$, and as $[L/M](z)$ to emphasize the $z-$dependence. We will thus have the various forms

$$[L/M] = [L/M]_f = [L/M](z) = [L/M]_f(z)$$

available for convenience. It is common practice to display the approximants in a table, called the *Padé table.*

Table1. The Padé Table

$M \setminus L$	0	1	2	$\dots$
0	$[0/0]$	$[1/0]$	$[2/0]$	$\dots$
1	$[0/1]$	$[1/1]$	$[2/1]$	$\dots$
2	$[0/2]$	$[1/2]$	$[2/2]$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The sequence of the form $\left\{[L/M]_f\right\}$, $L = 0,\ 1,\ 2,\ \dots$ (where $M \in \mathbb{Z}_+$ is fixed), are referred to as *row sequences (*or *rows)* in the Padé table, and the sequence $\left\{[L/L]_f\right\}$, $L = 0,\ 1,\ 2,\ \dots,$ is called *the diagonal sequence* (or *the main diagonal*).

Convergence of Padé approximants is a huge area of research and includes too many interesting results to be included in this thesis.

As is known, Padé approximants are locally the best rational approximants to a given power series. These approximants are constructed directly in terms of its coefficients and enable us to realize an efficient analytic continuation of the series beyond its circle of convergence, and in a sense the poles of the approximants localize the singular points (including the poles and their multiplicities) of the extended function in the corresponding domain of convergence and on its boundary. The last property of the Padé approximants is based on the fact that all their poles are '*free*' and are determined only by the condition that the tangency to the given power series be maximal. For this reason, the Padé approximants differ substantially from rational approximants whose poles are fixed (completely or partially), and in particular from polynomial approximations, in which case all poles are fixed at the single point at infinity.

It is this property of Padé approximants, efficiently solving the problem of analytic continuation of power series, that underlines their numerous successful applications in analysis and in the study of applied problems. At present, the method of Padé approximants is one of the most promising non-linear methods of summation of a power series and localization of its singular points. Among such methods the theory of Padé approximants has thus become a completely independent branch of approximation theory, and Padé approximants themselves have found diverse applications both directly in the theory of rational approximation and in number theory, the theory of non-self-adjoint operators, the study of differential equations depending on a small parameter, and perturbation theory.

The problem of meromorphic recovery of a function f from a power series (2.1) in the so-called *maximal circle* $D_M(f)$ of $M-$meromorphy of f (in which f is meromorphic and has $\leq M$ poles) is solved by the classical theorem of Montessus de Ballore under the assumption that f has exactly M poles in $D_M(f)$ (as usual, the poles of a function are counted according to their multiplicities).

Theorem 2.1.3. *(Montessus de Ballore). Let a function f have exactly M poles in the circle $D_M(f)$ given by $|z| < R$. Then the following assertions hold.*

1. For any sufficiently large L, the Padé approximants $[L/M]_f$ of the series f have exactly M finite poles tend as $M \to \infty$ to those of the function f in $D_M(f)$, and the number of poles of $[L/M]_f$ 'attracted' by each pole of f is equal to the multiplicity of this pole.

2. The sequence $[L/M]_f$, $L = 0, 1, 2, \ldots$, converges to the function f uniformly on compact subsets of the domain D'_M obtained from D_M by removing the poles of

f.

Moreover, under the assumptions of the theorem the rate of convergence of the sequence $[L/M]_f$ *to the function* f *in* $\acute{D}_M$ *is characterized by the inequality*

$$\overline{\lim_{L\to\infty}} \left| f(z) - [L/M]_f(z) \right|^{1/L} \leq \frac{|z|}{R} < 1.$$

In the proof of his result, Montessus de Ballore heavily used the Hadamard formulae (for the radii $R = R_M(f)$ of the circles $D_M(f)$) obtained earlier directly in terms of the coefficients of the series (2.1). Namely, let

$$H_{L,M} = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L \\ \vdots & \vdots & \vdots & \vdots \\ c_L & c_{L+1} & \cdots & c_{L+M-1} \end{vmatrix} \quad \text{(we set } c_k = 0 \text{ for } k < 0\text{)}$$

The following assertion holds.

Theorem 2.1.4. *(Hadamard). For any* $M \in \mathbb{Z}_+$

$$R_M = \frac{l_M}{l_{M+1}}, \quad \textit{where } l_j = \overline{\lim_{L\to\infty}} |H_{L,j}|^{1/L}$$

$$(l_0 = 1;\ \textit{if } l_1, \ldots, l_M \neq 0 \textit{ and } l_{M+1} = 0,\ \textit{then } R_M = \infty).$$

It readily follows from Montessus de Ballore's theorem that the finite poles of the rational functions $[L/M]_f$ tend to the corresponding poles of f at the rate of a geometric progression.

In fact, the above property of the poles of the functions $[L/M]_f$ is characteristic. This follows immediately from Gonchar's complete description of the $M-$meromorphic continuation of the power series of f with the help of the Mth row of the Padé table for an arbitrary $M \in \mathbb{N}$.

In Gonchar obtained formulae ([Gonchar A.A., Rakhmanov E.A., 1987]), in terms related to the asymptotic behaviour of the finite poles of the $M-$th row of the Padé table, for the radius of the Mth circle of and for the divisor of poles of the extended function f inside this circle, and he also proved a general theorem on the convergence of the Mth row of the Padé table with respect to the (logarithmic) capacity on compact subsets of $D_M(f)$; the Montessus result follows from this theorem as a special case.

The following more general problem arises naturally: what conclusions can be made about f in the large if it is known that the finite poles of the Mth row of the Padé table tend to some points of the complex plane without any assumption about the rate of this convergence? Let us consider the first row, that is, the case $M = 1$. If $c_L \cdot c_{L+1} \neq 0$, then the only finite pole ζ_n of the rational function $[L/1]_f$ is $\zeta_L = c_L \cdot c_{L+1}$. Thus, the relation $\zeta_L \to a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is equivalent to the condition that $c_L/c_{L+1} \to a$ as $L \to \infty$, and we arrive the assumptions of the classical Fabry ratio theorem.

Theorem 2.1.5. *(Fabry). If the coefficients of the power series (2.1) satisfy the relation*

$$\lim_{L\to\infty} \frac{c_L}{c_{L+1}} = a,$$

then $z = a$ is a singular point of the sum of this series, and it belongs to the boundary of the circle of convergence $|z| < R_0$ of the series, $R_0 = |a|$.

Thus, for $M = 1$ the Fabry theorem establishes in fact a relationship between the asymptotic behaviour of the finite poles of the first row of the Padé table and the singular points of f on the boundary of the circle of holomorphy $D_0(f)$. Treatment of the analogous problem for an arbitrary $M \in \mathbb{N}$ is one of the main objectives of the present survey.

It is quite another matter when treating diagonal sequences of Padé approximants. One of the first results of general nature on convergence of these rational approximants of analytic functions is the classical Markov theorem obtained in terms of Chebyshev continued fractions for functions of the form

$$\hat{\mu}(z) := \int_S \frac{d\mu(\zeta)}{z-\zeta}, \tag{2.18}$$

where μ is a positive Borel measure with support $S = S_\mu \Subset \mathbb{R}$.

Theorem 2.1.6. *(Markov). For a function $\hat{\mu}$ of the form (2.18) with support $S_\mu \Subset \mathbb{R}$ consisting of infinitely many points, the diagonal Padé approximants $[L/L]_{\hat{\mu}}$ constructed from the coefficients of the expansion of $\hat{\mu}$ in a Laurent series at the point $z = \infty$ converge to $\hat{\mu}$ uniformly on compact subsets of the domain $\bar{\mathbb{C}} \setminus [a,b]$, where $[a,b]$ is the minimal closed interval of the real axis such that $[a,b] \supset S_\mu$.*

Thus, any Markov function (a function of the form (2.18) with $S_\mu \Subset \mathbb{R}$) can be recovered, outside the convex hull $\hat{S}_\mu = [a,b]$ of the support of the measure, from the coefficients of its Laurent expansion at the point $z = \infty$ (that is, from the moments of the measure μ).

The heart of the matter is that Markov's theorem considers the uniform convergence of the Padé main diagonal only outside the convex hull $\hat{S}_\mu$ of the support of the measure rather than in the domain $\bar{\mathbb{C}} \setminus S_\mu$ of holomorphy of the function $\hat{\mu}$, because in the most typical situation the set of limit points of those poles of the rational functions $[L/L]_{\hat{\mu}}$ coincides with $\hat{S}_\mu$. In the general case in which the support of the measure μ in (2.18) does not belong to any line, the limit points of the poles of the diagonal Padé approximants can form analytic arcs in the domain $D = \bar{\mathbb{C}} \setminus S_\mu$ and can even be dense in $\bar{\mathbb{C}}$. (More precisely, a subsequence of poles of the Padé approximants can converge to any given point of the corresponding analytic arc or to any given point of $\bar{\mathbb{C}}$, respectively.) In this situation, we face the principal question of whether or not a pole of the Padé approximants can have a limit (rather than simply a limit point) over the entire sequence $L \in \mathbb{N}$ that is distinct from any pole of f. This question is directly related to the problem of recovering the divisor of poles of a function meromorphic in $\bar{\mathbb{C}}\backslash S_\mu$ of the form

$$f = \hat{\mu} + r, \tag{2.19}$$

from the Padé diagonal, where $r \in \mathbb{C}(z)$ is a rational function holomorphic on $[a, b]$ (f is a 'rational perturbation' of the Markov function $\hat{\mu}$). The construction of the Padé approximants is essentially non-linear, and therefore the investigation of the convergence of these approximants for functions of the form (2.19) is a complicated task. A positive solution of the problem of recovering the divisor follows immediately from the existence of a subsequence (of the main diagonal) that is uniformly convergent on compact subsets of $\bar{\mathbb{C}}\backslash S_\mu$ to the meromorphic functions f with respect to the spherical metric (in which the distance is measured by the length of a shortest arc between the corresponding points on the Riemann sphere).

The Markov theorem is directly related to the results of Gonchar and Rakhmanov on the convergence of the Padé approximants for meromorphic functions f of the form

$$f = \hat{\mu} + r, \tag{2.20}$$

where $\hat{\mu}$ is a Markov function and r is a rational function holomorphic on $[a, b] = \hat{S}_\mu$ ($r \in \mathbb{C}(z) \cap H[a, b]$, so that f is a rational perturbation of $\hat{\mu}$). The construction of the Padé approximants is essentially non-linear, and therefore the investigation of convergence of such rational approximations for functions of the form (2.20) is a nontrivial problem. In Gonchar's theorem, it is claimed that the diagonal Padé approximants $[L/L]_f$ are uniformly convergent to the function f on compact subsets of the domain $\bar{\mathbb{C}}\backslash [a, b]$ with respect to the spherical metric under the assumption that $S_\mu = [a, b]$ and $\acute{\mu}(x) = d\mu/dx > 0$ almost everywhere on $[a, b]$ and the function $r \in \mathbb{C}(z)$ is holomorphic on $[a, b]$. In Rakhmanov's paper, a similar result on uniform convergence of $[L/L]_f$ with respect to the spherical metric outside $[a, b]$ is established under the assumption that $\hat{\mu}$ is an arbitrary Markov function and the function $r \in \mathbb{R}(z)$ is holomorphic on $[a, b]$.

A central place in the convergence theory is taken by the Baker-Gammel-Wills Conjecture $[BGW]$.

Conjecture 1. If $P(z)$ is a power series representing a function which is regular for $|z| \leq 1$, except for m poles within this circle and except for $z = +1$, at which point the function is assumed continuous when only points $|z| \leq 1$ are considered, then at least a subsequence of the $[L/L]$ Padé approximants converge uniformly to the function (as L tends to infinity) in the domain formed by removing the interiors of small circles with centers at these poles.

Over time, many different versions of this conjecture were proposed and studied.

Conjecture 2. If $P(z)$ is a power series which is meromorphic in $|z| \leq 1$ and continuous on the sphere in $|z| \leq 1$, then at least a subsequence of the $[M/M]$ Padé approximants is equicontinuous on the sphere in $|z| \leq 1$.

This conjecture implies that at least a subsequence of the $[M/M]$ Padé approximants converge uniformly on the sphere to $f(z)$.

A weaker version of this conjecture was proposed by Stahl.

Conjecture 3. Let the function f be algebraic and meromorphic in the unit disc D. Then there exists an infinite subsequence $L \in \mathbb{N}$ such that

$$[L/L](z) \to f(z) \text{ as } n \to \infty, \ n \in \mathbb{N} \tag{2.21}$$

holds true locally uniformly for $z \in D \backslash \{\text{poles of } f\}$.

From the point of view of workers who are trying to evaluate function values by means of Padé approximants, the sum and substance of these conjectures has been to interpret them to mean, “just disregard the approximants with defects and use the rest of them and you will be OK."

After 40 years of study by a number of workers, Lubinsky produced a counter-example to Conjecture 2. Shortly thereafter, and apparently motivated by the work of Lubinsky, Buslaev produced an algebraic counter-example to Conjectures 2 and 3.

2.2. Padé Approximations for Exponential Function

The coefficients c_i of the Maclaurin expansion of the exponential function are sufficiently simple that explicit forms of the exponential function are sufficiently simple that explicit forms of the numerator and denominator of the Padé approximants can be found. In this section, we will calculate the denominator $Q^{[L/M]}(z)$. The numerator follows by an extremely simple and elegant trick, based on the identity $\exp(-z) = \dfrac{1}{\exp(z)}$. Padé in his thesis, elaborated the properties of his approximants with special emphasis on the example of the exponential function: it is a beautiful example of how the approximants work in an ideal situation. Further properties of Padé approximants of $\exp(z)$ are to be found .

Our task is to calculate

$$Q^{[L/M]}(z) = \begin{vmatrix} \frac{1}{(L-M+1)!} & \frac{1}{(L-M+2)!} & \cdots & \frac{1}{L!} & \frac{1}{(L+1)!} \\ \frac{1}{(L-M+2)!} & \frac{1}{(L-M+3)!} & \cdots & \frac{1}{(L+1)!} & \frac{1}{(L+2)!} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{(L)!} & \frac{1}{(L+1)!} & \cdots & \frac{1}{L+M-1!} & \frac{1}{(L+M)!} \\ z^M & z^{M-1} & \cdots & z & 1 \end{vmatrix} \tag{2.22}$$

It is easier to begin with the constant term in (2.22), and so we define $C(L/M) \equiv$

$Q^{[L/M]}(0)$, which is the coefficient of the "1" in the lower right-hand corner of (2.22),

$$C(L/M) = \begin{vmatrix} \frac{1}{(L-M+1)!} & \frac{1}{(L-M+2)!} & \cdots & \frac{1}{L!} \\ \frac{1}{(L-M+2)!} & \frac{1}{(L-M+3)!} & \cdots & \frac{1}{(L+1)!} \\ \vdots & \vdots & & \vdots \\ \frac{1}{L!} & \frac{1}{(L+1)!} & \cdots & \frac{1}{(L+M-1)!} \end{vmatrix} \tag{2.23}$$

We assume that $L \geq M-1$. If this condition does not hold, the factorial functions must be suitably reinterpreted as gamma functions for the analysis to be valid. We remove the denominators from each row, by defining

$$p = \prod_{i=1}^{M} \frac{1}{(L+i-1)!},$$

and then

$$C(L/M) = p \begin{vmatrix} \frac{L!}{(L-M+1)!} & \frac{L!}{(L-M+2)!} & \cdots & L & 1 \\ \frac{(L+1)!}{(L-M+2)!} & \frac{(L+1)!}{(L-M+3)!} & \cdots & L+1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{(L+M-1)!}{L!} & \frac{(L+M-1)!}{(L+1)!} & \cdots & L+M-1 & 1 \end{vmatrix} \tag{2.24}$$

In (2.24), the determinant has M rows. Subtract the $(M-1)$th row from the Mth, then the $(M-2)$th row from the $(M-1)$th, etc. The identity

$$\frac{r!}{s!} - \frac{(r-1)!}{(s-1)!} = (r-s)\frac{(r-1)!}{s!} \tag{2.25}$$

is used repeatedly. In column 1 of (2.24), $r-s = M-1$; in column 2, $r-s = M-2$;

etc., and so one finds that

$$C\left(L/M\right)=p\left(M-1\right)!\begin{vmatrix}\frac{L!}{(L-M+1)!} & \frac{L!}{(L-M+2)!} & \cdots & L & 1\\ \frac{L!}{(L-M+2)!} & \frac{L!}{(L-M+3)!} & \cdots & 1 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ \frac{(L+M-2)!}{L!} & \frac{(L+M-2)!}{(L+1)!} & \cdots & 1 & 0\end{vmatrix}$$

$$=p\left(-1\right)^{M-1}\left(M-1\right)!\begin{vmatrix}\frac{L!}{(L-M+2)!} & \frac{L!}{(L-M+3)!} & \cdots & 1\\ \frac{(L+1)!}{(L-M+3)!} & \frac{(L+1)!}{(L-M+4)!} & \cdots & 1\\ \vdots & \vdots & & \vdots\\ \frac{(L+M-2)!}{L!} & \frac{(L+M-2)!}{(L+1)!} & \cdots & 1\end{vmatrix} \tag{2.26}$$

This is a $(M-1)\times(M-1)$ determinant with a form identical to (2.24) but with M replaced by $M-1$. Consequently, an obvious inductive argument shows that

$$C\left(L/M\right)=p\prod_{i=1}^{M}\left(-1\right)^{i-1}\left(i-1\right)!$$
$$=\left(-1\right)^{M(M-1)/2}\prod_{i=1}^{M}\frac{(i-1)!}{(L+i-1)!} \tag{2.27}$$

Thus, for the case $M=1$,

$$C\left(L/1\right)=\frac{1}{L!},$$

and for the case $M=2$,

$$C\left(L/2\right)=\begin{vmatrix}\frac{1}{(L-1)!} & \frac{1}{L!}\\ \frac{1}{L!} & \frac{1}{(L+1)!}\end{vmatrix}=\frac{-1}{L!\left(L+1\right)!}$$

The sign pattern of (2.27) distinguishes Polyá frequency series. The row operations we have performed to deduce (2.27) from (2.23) are still permissible with the form (1), except that the situation is more complicated. We consider the coefficient of

$(-z)^j$ in $Q^{[L/M]}(z)$, which is

$$(-1)^j q_j^{[L/M]} = \begin{vmatrix} \frac{1}{(L-M+1)!} & \frac{1}{(L-M+2)!} & \cdots & \frac{1}{(L-j+1)!} & \cdots & \frac{1}{(L+1)!} \\ \frac{1}{(L-M+2)!} & \frac{1}{(L-M+3)!} & \cdots & \frac{1}{(L-j+2)!} & \cdots & \frac{1}{(L+2)!} \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{1}{L!} & \frac{1}{(L+1)!} & \cdots & \frac{1}{(L+M-j)!} & \cdots & \frac{1}{(L+M)!} \end{vmatrix} \tag{2.28}$$

where the column $\begin{bmatrix} \frac{1}{(L-j+1)!} \\ \frac{1}{(L-j+2)!} \\ \vdots \\ \frac{1}{(L+M-j)!} \end{bmatrix}$ is deleted. We perform a similar analysis: define

$$\acute{p} = \prod_{i=1}^{M} \frac{1}{(L+i)!},$$

and then

$$(-1)^j q_j^{[L/M]} = \acute{p} \begin{vmatrix} \frac{(L+1)!}{(L-M+1)!} & \cdots & \frac{(L+1)!}{(L-j+1)!} & \cdots & 1 \\ \frac{(L+2)!}{(L-M+2)!} & \cdots & \frac{(L+2)!}{(L-j+2)!} & \cdots & 1 \\ \vdots & & \vdots & & \vdots \\ \frac{(L+M)!}{L!} & \cdots & \frac{(L+M)!}{(L+M-j)!} & \cdots & 1 \end{vmatrix} \tag{2.29}$$

Subtracting rows, and using the identity (2.25),

$$(-1)^j q_j^{[L/M]} = (-1)^M \acute{p} \frac{M!}{j}$$
$$\times \begin{vmatrix} \frac{(L+1)!}{(L-M+2)!} & \cdots & \frac{(L+1)!}{(L-j+2)!} & \cdots & 1 \\ \frac{(L+2)!}{(L-M+3)!} & \cdots & \frac{(L+2)!}{(L-j+3)!} & \cdots & 1 \\ \vdots & & \vdots & & \vdots \\ \frac{(L+M-1)!}{L!} & \cdots & \frac{(L+M-1)!}{(L+M-j)!} & \cdots & 1 \end{vmatrix}$$

which again is an $(M-1)\times(M-1)$ determinant with a form similar to (2.29). We make j similar reductions from (2.29) to obtain

$$
\begin{aligned}
(-1)^j \, q_j^{[L/M]} &= \pm \frac{\acute{p}}{j!} \prod_{i=1}^{j} (M-i+1)! \\
&\times \begin{vmatrix}
\frac{(L+1)!}{(L-M+j+1)!} & \cdots & \frac{(L+1)!}{L!} & 1 \\
\frac{(L+2)!}{(L-M+j+2)!} & \cdots & \frac{(L+2)!}{(L+1)!} & 1 \\
\vdots & & \vdots & \vdots \\
\frac{(L+M-j)!}{L!} & \cdots & \frac{(L+M-j)!}{(L+M-j-1)!} & 1
\end{vmatrix}
\end{aligned}
$$

Removing a common factor factor from each row,

$$
\begin{aligned}
(-1)^j \, q_j^{[L/M]} &= \pm \frac{\acute{p}}{j!} \frac{(L+M-j)!}{L!} \prod_{i=1}^{j} (M-i+1)! \\
&\times \begin{vmatrix}
\frac{L!}{(L-M+j+1)!} & \cdots & 1 \\
\frac{(L+1)!}{(L-M+j+2)!} & \cdots & 1 \\
\vdots & & \\
\frac{(L+M-j-1)!}{L!} & \cdots & 1
\end{vmatrix}
\end{aligned}
$$

The analysis now follows the familiar pattern using identity (2.25), and we deduce that

$$
\begin{aligned}
(-1)^j \, q_j^{[L/M]} &= \pm \left\{ \prod_{i=1}^{M} \frac{1}{(L+i)!} \right\} \frac{(L+M-j)!}{L!j!} \\
&\times \left\{ \prod_{i=1}^{j} (M-i+1)! \right\} \prod_{i=1}^{M-j-1} i! \\
&= \pm \frac{(L+M-j)!}{L!j!\,(M-j)!} \prod_{i=1}^{M} \frac{i!}{(L+i)!}
\end{aligned}
\tag{2.30}
$$

The sign of the right-hand side of (2.30) is easily determined to be the same as that of (2.27), because determinants (2.23) and (2.28) have the same dimension, and are expanded by the same top right-hand elements recursively. Hence

$$(-1)^j \, q_j^{[L/M]} = (-1)^{M(M-1)/2} \frac{(L+M-j)!}{L!j!\,(M-j)!} \prod_{i=1}^{M} \frac{i!}{(L+i)!} \tag{2.31}$$

Notice that (2.27) emerges as the special case with $j = 0$. Consequently, we have

$$q_j^{[L/M]} = (-1)^j \, C\,(L/M) \, \frac{(L+M-j)!}{(L+M)!} \frac{M!}{(M-j)!} \frac{1}{j!}$$

and

$$\begin{aligned} Q^{[L/M]}(z) &= C\,(L/M) \sum_{j=0}^{M} \frac{(L+M-j)!}{(L+M)!} \frac{M!}{(M-j)!} \frac{(-z)^j}{j!} \\ &= C\,(L/M) \left\{ 1 + \frac{M}{L+M} \frac{(-z)}{1!} + \frac{M\,(M-1)}{(L+M)\,(L+M-1)} \frac{(-z)^2}{2!} + \ldots \right\} \\ &= C\,(L/M) \left\{ 1 + \frac{-M}{-L-M} \frac{(-z)}{1!} + \frac{-M\,(-M+1)}{(-L-M)\,(-L-M+1)} \frac{(-z)^2}{2!} + \ldots \right\} \\ &= C\,(L/M)_1 \, F_1\,(-M; \; -L-M; \; -z)\,. \end{aligned} \tag{2.32}$$

We may deduce from (2.32) that

$$P^{[L/M]}(z) = C\,(L/M)_1 \, F_1\,(-L; \; -L-M; \; z)\,,$$

and hence the $[L/M]$ Padé approximant for $\exp(z)$ is

$$[L/M] = \frac{{}_1F_1\,(-L; \; -L-M; \; z)}{{}_1F_1\,(-M; \; -L-M; \; -z)}. \tag{2.33}$$

2.3. Balk's Results on Padé Approximations

Balk's paper contains several results on the convergence of Padé approximants. He starts with 'general propositions about convergence of double sequences'

Let a set of points in a metric space be considered as an infinite table with two entries, hence to any entry (μ, ν) ($\mu-$number of row, $\nu-$number of column $\mu, \nu = 0, 1, 2, ...$) it corresponds exactly one point $\alpha_{\mu,\nu}$ of the set; to different entries of the table it corresponds possibly equal points. This table will be denoted as: "table $\{\alpha_{\mu,\nu}\}_{\mu,\nu=0}^{\infty}$" of "table T".

The table $\{\alpha_{\mu,\nu}\}_{\mu,\nu=0}^{\infty}$ is called *convergent* if all sequences of points with different entries of the table converge and have the same limit. In other words, in the space under consideration there exists such element S, that for any sequence of different entries (μ_k, ν_k) $(k = 0, 1, 2, \ldots)$

$$\lim_{k\to\infty} \rho\left(S, \alpha_{\mu_k,\nu_k}\right) = 0.$$

If all rows of the table converge and confinal then it does not imply the convergence of the table. Even the convergence and confinality of all rows, columns and diagonals of the table. (i.e. of sequences $\{\alpha_{m,n}\}, \{\alpha_{n,m}\}, \{\alpha_{n,n+m}\}$, where m is fixed and n varies, $n = 0, 1, 2, \ldots$) are not sufficient for the convergence of the table. A counter example may be taken as the table where $\alpha_{n,2n} = 1 \quad (n = 0, 1, 2, \ldots)$ and all other elements equal to zero.

Nevertheless; to clerify the convergence of the table T, it is not necessary to consider all sequences of its elements. It is clear from the following theorem.

Theorem 2.3.1. *If all sequences $\{\alpha_{\mu_k,\nu_k}\}_{k=0}^{\infty}$ of the table T, such that*

1)$\mu_k \leq \mu_{k+1}$,

2)$\nu_k \leq \nu_{k+1}$,

3) $(\mu_{k+1} - \mu_k) + (\nu_{k+1} - \nu_k) = 1$,

converge and confinal then the table T converges.

Proof During the proof we shall use that from any infinite sequence of nonnegative integers it is possible to take an infinite nondecreasing subsequence.

Denote by S the limit of all sequences of points from the table satisfying conditions 1) -3). By contradiction, let there exist a sequence

$$\{\alpha_{\mu_k,\nu_k}\}_{k=0}^{\infty} \tag{2.34}$$

which does not converge to S.

It means that there exist a number $\varepsilon > 0$ such that the inequality

$$\rho\left(S, \alpha_{\mu_k,\nu_k}\right) > \varepsilon$$

holds for an infinite set of numbers of the sequence. Put those numbers in the ascending order. We get the infinite sequence

$$\left\{\alpha_{\mu_{k_p},\nu_{k_p}}\right\}_{p=0}^{\infty}$$

which will be denoted by

$$\left\{\alpha_{\mu'_p,\nu'_p}\right\}_{p=0}^{\infty}.$$

The inequality

$$\rho\left(S, \alpha_{\mu'_p,\nu'_p}\right) > \varepsilon \tag{2.35}$$

holds for all p.

From the sequence of nonnegative integers $\left\{\nu'_p\right\}_{p=0}^{\infty}$ we choose a nondecreasing subsequence $\left\{\nu'_{p_r}\right\}_{r=0}^{\infty}$ which will be denoted by $\{\nu''_r\}_{r=0}^{\infty}$.

Together with it, we shall consider infinite sequence infinite sequence $\{\mu''_r\}_{r=0}^{\infty}$, where $\mu_r'' = \mu_{p_r}'$. From the last sequence, as a next step we should take an infinite nondecreasing subsequence $\left\{\mu_{r_s}''\right\}$. The sequence of entries $\left\{\left(\mu_{r_s}'', \nu_{r_s}''\right)\right\}_{s=0}^{\infty}$ is denoted by $\{\mu_s''', \nu_s'''\}_{s=0}^{\infty}$. It satisfies conditions 1) -2) because of construction.

Evidently, it is always possible to include it into a sequence of entries satisfying all three conditions of the theorem. For instance:

$$\begin{aligned} &\{(\mu_0''', i_0)\}_{i_0=\nu_0'''}^{\nu_1'''}, \qquad &&\{(j_0''', \nu_1''')\}_{j_0=\mu_0'''}^{\mu_1'''}, \\ &\{(\mu_1''', i_1)\}_{i_1=\nu_1'''}^{\nu_2'''}, \qquad &&\{(j_1''', \nu_2''')\}_{j_1=\mu_1'''}^{\mu_2'''}, \ldots \end{aligned} \tag{2.36}$$

By the condition of the theorem the sequence of points with entries (2.36) converges to S, which contradicts the relation (2.35). Theorem is proved.

In some cases the set of elements of the table T which are placed in the right handside of its main diagonal more exactly the set of all elements $\alpha_{\mu,\nu}$ such that $\nu \geq \mu$ is of special interest. That set we shall call *semi-table of the table T* and denote by $\{\alpha_{\mu,\nu}\}_{\mu,\nu=0}^{\infty}\ (\nu \geq \mu)$, or shorter by "semi-table T".

If all sequences of points with different entries of the semi-table T', converge and confinal, then it is called convergent.

The investigation of the semi-table $\{\alpha_{\mu,\nu}\}_{\mu,\nu=0}^{\infty}\ (\nu \geq \mu)$ may be reduced to the investigation table $\{\beta_{\mu,\lambda}\}_{\mu,\lambda=0}^{\infty}$ if $\beta_{\mu,\lambda} = \alpha_{\mu,\lambda+\mu}\ \ (\lambda, \mu = 0, 1, 2, \ldots)$

For semi-tables the following is true.

Theorem 2.3.2. *If all sequences of the semi-table T' satisfying conditions 1) -3) of the theorem 1, converge and confinal then the semi-table T' converges.*

Proof The proof of Theorem 2 is analogous to the proof of Theorem 1.

We shall prove a stronger assertion.

Theorem 2.3.3. *Let all sequences* $\{\alpha_{\mu_k,\nu_k}\}_{k=0}^{\infty}$ *of the semi-table* $\{\alpha_{\mu,\nu}\}_{\mu,\nu=0}^{\infty}$ $(\nu \geq \mu)$

a) $\mu_k \leq \mu_{k+1}$,

b)$\nu_k \leq \nu_{k+1}$,

c) $(\mu_{k+1} - \mu_k) + (\nu_{k+1} - \nu_k) = 1$,

d) $\lim_{k\to\infty} \frac{\mu_k}{\nu_k}$ *exists*

be convergent and confinal. Then the semi-table converges.

Proof By contradiction let there exists a sequence in T in the right hand side of main diagonal a sequence $\{\alpha_{\mu_k,\nu_k}\}_{k=0}^{\infty}$, which does not converge to S, where S is the limit of all sequences satisfying conditions a) -d).

By similar considerations as in the proof of Theorem 1 we deduce that there exist a positive number ε and a sequence

$$\left\{\alpha_{\mu_r',\nu_r'}\right\}_{r=0}^{\infty} \tag{2.37}$$

such that for all its members

$$\rho\left(S, \alpha_{\mu_r',\nu_r'}\right) > \varepsilon, \quad \mu_r' \leq \mu_{r+1,}' \quad \nu_r' \leq \nu_{r+1}'. \tag{2.38}$$

since the set of numbers $\left\{\frac{\mu_r'}{\nu_r'}\right\}_{r=0}^{\infty}$ by condition of the theorem is bounded $0 \leq \frac{\mu_r'}{\nu_r'} \leq 1$, then it is possible to choose its convergent subsequence $\left\{\frac{\mu_{r_n}'}{\nu_{r_n}'}\right\}_{n=0}^{\infty}$, which is denoted by $\left\{\frac{\mu_n''}{\nu_n''}\right\}_{n=0}^{\infty}$.

The sequence of entries satisfies conditions a, b and d of theorem. It is easy to see that it is always possible to include it into a sequence of entries satisfying also the condition c).

It follows from the following remarks and the validity is almost evident.

Remark1. If (x, y) is an arbitrary point of the interval with ends $A_n\,(\mu''_n, \nu''_n)$ and $A_{n+1}\left(\mu''_{n+1}, \nu''_{n+1}\right)$, then the fraction $\frac{x}{y}$ is between μ''_n/ν''_n and μ''_{n+1}/ν''_{n+1} .

Remark2. Let $\left(\mu''_{n+1} - \mu''_n\right) + \left(\nu''_{n+1} - \nu''_n\right) = m_n$; then there exist $m_n + 1$ points $\beta_i^{(n)}\left(\gamma_i^{(n)}, \delta_i^{(n)}\right) \quad (i = 0, 1, \ldots, m_n)\,,$ such that

1) $B_0^{(n)} \equiv A_n,\;\; B_m^{(n)} \equiv A_{n+1};$

2)$\gamma_i^{(n)} \leq \gamma_{i+1}^{(n)}, \quad \delta_i^{(n)} \leq \delta_{i+1}^{(n)}, \quad \left(\gamma_{i+1}^{(n)} - \gamma_i^{(n)}\right) + \left(\delta_{i+1}^{(n)} - \delta_i^{(n)}\right) = 1;$

3) Every point $\beta_i^{(n)}$ is distant from the interval A_nA_{n+1} less than 1.

From those remarks it follows that for any point $\beta_i^{(n)}\left(\gamma_i^{(n)}, \delta_i^{(n)}\right)$ it is possible to find the pair of numbers (x, y) (point of interval A_nA_{n+1}) such that the following relations hold:

1)$\gamma_i^{(n)} = x + \theta_{i_1}^{(n)}, \quad \delta_i^{(n)} = y + \theta_{i_2}^{(n)}$

$\Big(\theta_{i_1}^{(n)}$ and $\theta_{i_2}^{(n)}$ are some numbers which are less than or equal to 1 in absolute value$\Big)$.

2)$\frac{x}{y}$ is between μ''_n/ν''_n and μ''_{n+1}/ν''_{n+1} .

Then the fraction $\left(\gamma_i^{(n)} - \theta_{i_1}^{(n)}\right) / \left(\delta_i^{(n)} - \theta_{i_2}^{(n)}\right)$ is also between μ''_n/ν''_n and μ''_{n+1}/ν''_{n+1}

.

By the condition,

$\lim_{n\to\infty} (\mu''_n/\nu''_n)$ exists, denoted by l. $\left(\text{Clearly } \lim_{n\to\infty} \gamma_i^{(n)}/\delta_i^{(n)} = l\right)$

Furthermore, it is not difficult to check that the sequence of entries

$$\left\{\left(\gamma_i^{(0)}, \delta_i^{(0)}\right)\right\}_{i=0}^{m_0}, \quad \left\{\left(\gamma_i^{(1)}, \delta_i^{(1)}\right)\right\}_{i=0}^{m_1}, \ \ldots, \left\{\left(\gamma_i^{(n)}, \delta_i^{(n)}\right)\right\}_{i=0}^{m_n}, \ldots$$

satisfies all four conditions of theorem. But then, by the condition of the theorem the sequence of points with those entries has to converge S, which contradicts relation (2.38).

Sometimes only these sequences of elements of the table T, or of the semi-table T', such that

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \nu_n = \infty$$

are of special interest.

Concerning the collection of such sequences it is possible to prove theorems which are analogues of Theorems 1-3. For that reason it is necessary only to slightly modify the proofs of Theorems 1-3.

In particular, the following proposition which we shall use in this work holds.

Theorem 2.3.4. *If all sequences of the kind* $\{\alpha_{\mu_n,\nu_n}\}_{n=0}^{\infty}, \ \lim\limits_{n\to\infty} \mu_n = \infty$, *satisfying either conditions 1-3 of the Theorem 1, or conditions a-d of Theorem 3, converge and confinal then all sequences of the kind* $\{\alpha_{\mu_n,\nu_n}\}_{n=0}^{\infty}, \ \lim\limits_{n\to\infty} \mu_n = \infty$ *in the semitable converge and confinal.*

In investigations of the convergence of double sequences sometimes it is useful the notion of equiuniform convergence of its rows or columns, diagonals, etc.

We shall say that rows of the table $\{\alpha_{\mu,\nu}\}_{\mu,\nu=0}^{\infty}$ *converge equiuniformly* to the point S if for any $\varepsilon > 0$ there exists N s.t. for all $\nu \geq N$ and for any μ the inequality

$$\rho\left(S, \alpha_{\mu,\nu}\right) < \varepsilon. \tag{2.39}$$

Theorem 2.3.5. *Any sequence* $\{\alpha_{\mu_n,\nu_n}\}_{n=0}^{\infty}$ *of points with different entries of the table* $\{\alpha_{\mu,\nu}\}_{\mu,\nu=0}^{\infty}$ *converges to the point* S *under condition* $\lim \nu_n = \infty$ *if and only if rows of the table equiuniformly converges to* S.

Proof Necessity: Let rows of the table T be not equiuniformly convergent to S. It means that there exists $\varepsilon > 0$, such that for infinite sequence of entries $(\mu_1,\nu_1), (\mu_2,\nu_2), \ldots, (\mu_n,\nu_n)$, $\lim\limits_{n\to\infty} \nu_n = \infty$ the inequality

$$\rho(S, \alpha_{\mu_n,\nu_n}) > \varepsilon. \tag{2.40}$$

holds.

But it means that the sequence $\{\alpha_{\mu_n,\nu_n}\}$ does not converge to S.

Sufficiency: Let all rows of the table T be equiuniformly convergent to S. Let $\{\alpha_{\mu_n,\nu_n}\}_{n=0}^{\infty}$ be a sequence of points on the table such that $\lim\limits_{n\to\infty} \nu_n = \infty$. Let us prove that the sequence $\{\alpha_{\mu_n,\nu_n}\}_{n=0}^{\infty}$ converges to S.

Because of equiuniformly convergence of the rows there exists N such that for all $\nu_n > N$

$$\rho(S, \alpha_{\mu_n,\nu_n}) < \varepsilon.$$

But since $\lim\limits_{n\to\infty} \nu_n = \infty$, there exists n_0 such that $\nu_n > N$ for $n > n_0$. So for any $\varepsilon > 0$ there exists n_0, such that for all $n > n_0$

$$\rho(S, \alpha_{\mu_n,\nu_n}) < \varepsilon. \tag{2.41}$$

And it means that $\lim\limits_{n\to\infty} \alpha_{\mu_n,\nu_n} = S$.

Theorem 2.3.6. *The table* T *converges to the point* S *if and only if rows of table and columns of table equiuniformly converge to* S*,simultaneously.*

Proof Necessity: Evidently follows from the former theorem.

Sufficiency: Let rows and columns of the table T equiuniformly converge to S. Then, obviously for any $\varepsilon > 0$ there exists N, such that for all $\mu > N$, $\nu > N$ the inequality

$$\rho\left(S, \alpha_{\mu,\nu}\right) < \varepsilon$$

holds.

In other words, condition (2.41) is valid for any entry (μ, ν) outside of the square $0 \leq \mu \leq N, \quad 0 \leq \nu \leq N$.

Let us prove now that any sequence $\{\alpha_{\mu_n,\nu_n}\}_{n=0}^{\infty}$ of points with different entries of the table converges to S.

Take arbitrary $\varepsilon > 0$. Then as we saw the inequality

$$\rho\left(S, \alpha_{\mu_n,\nu_n}\right) < \varepsilon. \tag{2.42}$$

holds for all points α_{μ_n,ν_n} with possible exclusion of points that are inside certain square $0 \leq \mu_n \leq N$, $0 \leq \nu_n \leq N$. But since points have different entries, then inside that square there may be only finite number of the points in this sequence $\{\alpha_{\mu_n,\nu_n}\}_{n=0}^{\infty}$.

So, inequality (2.42) holds for almost all members of the sequence and it means that it converges to S. Sufficiency is proved.

CHAPTER 3

MULTIPOINT PADE APPROXIMATIONS AND NEWTON-PADE APPROXIMATIONS

3.1. Foundations of the Multipoint Padé Approximations Theory

First, we need the basic framework of Newtonian polynomial interpolation.

Divided Differences For a function $f(z)$ is satisfying such continuity properties as are necessary, we define

$$f[z_0] = f(z_0), \tag{3.1}$$

$$f[z_0, z_1] = \frac{f(z_0) - f(z_1)}{z_0 - z_1}, \tag{3.2}$$

and other divided differences are defined recursively by

$$f[z_0, z_1, \ldots, z_{r+1}] = \frac{f[z_0, z_1, \ldots, z_{r-1}, z_r] - f[z_0, z_1, \ldots, z_{r-1}, z_{r+1}]}{z_r - z_{r+1}}, \quad r = 1, 2, \ldots. \tag{3.3}$$

Theorem 3.1.1. *(Hermite's Formula) If $f(z)$ is analytic inside and continuous on a contour Γ enclosing $z_0, z_1, \ldots, z_k$, then*

$$f[z_0, z_1, \ldots, z_r] = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\prod_{k=0}^{r} (\zeta - z_k)} d\zeta \tag{3.4}$$

Proof. The proof is by induction using (3.1), (3.2) and (3.3).

For confluent points $z_0 = z_1 = \ldots = z_r$, it is natural to define

$$f\left[z_0, z_0, \ldots, z_0\right] = \frac{f^{(r)}\left(z_0\right)}{r!}. \tag{3.5}$$

Hermite's formula easily extends to cases of partial confluence.

Corollary. $f\left[z_0, z_1, \ldots, z_r\right]$ is a totally symmetric function of all its arguments $z_0, z_1, \ldots, z_r$.

Newton's Formulas.

$$\begin{aligned} f\left(z\right) = &\sum_{i=0}^{n} f\left[z_0, z_1, \ldots, z_j\right] \prod_{k=0}^{i-1}\left(z - z_k\right) \\ &+ f\left[z_0, z_1, \ldots, z_n, z\right] \prod_{k=0}^{n}\left(z - z_k\right). \end{aligned} \tag{3.6}$$

For $n > 0$, (3.6) is an identity expressing $f\left(z\right)$ as a Newton polynomial and a remainder term. One may "deduce" the formal identity

$$f\left(z\right) = f\left[z_0\right] + \left(z - z_0\right) f\left[z_0, z_1\right] + \left(z - z_0\right)\left(z - z_1\right) f\left[z_0, z_1, z_2\right] + \ldots. \tag{3.7}$$

Whenever the remainder in (3.6) tends to zero, (3.7) becomes an identity. The proof of (3.6) by induction is straightforward. It is the interpretation of (3.6) and (3.7) that is most significant. If $z_0 = z_1 = \ldots = z_i = \ldots$, (3.6) and (3.7) become

$$f\left(z\right) = f\left(z_0\right) + \left(z - z_0\right) f'\left(z_0\right) + \frac{\left(z - z_0\right)^2}{2!} f''\left(z_0\right) + \ldots \tag{3.8}$$

$$= \sum_{i=0}^{n} \frac{\left(z - z_0\right)^i}{i!} f^{(i)}\left(z_0\right) + \frac{\left(z - z_0\right)^{n+1}}{2\pi i} \int\limits_{\Gamma} \frac{f\left(\zeta\right) d\zeta}{\left(\zeta - z_0\right)^n \left(\zeta - z\right)}. \tag{3.9}$$

Equation (3.9) holds provided Γ is a contour enclosing z, z_0 and $f(z)$ is analytic within Γ and continuous on Γ. In fact, (3.9) gives the Taylor series for $f(z)$ and its remainder. For conciseness, we make a further definition:

Definition 3.1. (Warner, D.D. ;1974) For a function $f(z)$ is satisfying such continuity properties as are necessary, then F, the **Formal Newton Series (FNS)**, is the infinite triangular matrix of divided differences defined by

$$F \equiv \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} & \dots \\ 0 & f_{11} & f_{12} & f_{13} & \dots \\ 0 & 0 & f_{22} & f_{23} & \dots \\ 0 & 0 & 0 & f_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where

$$f_{i,j} = f\left[z_i, z_{i+1}, \dots, z_j\right], \text{ for } j \geq i. \tag{3.10}$$

Then Newton's formula (3.7) becomes the formal identity

$$f(z) = f(z_0) + f_{0,1}(z - z_0) + f_{0,2}(z - z_0)(z - z_1) + \dots$$

However, the multiplication of two FNS in that form is possible only using all data $f_{i,j}$ with using the following lemma:

Lemma([Milne-Thomson L.M., 1960]) If $h(z) = f(z)\, g(z)$, then for $i \leq j$

$$h_{ij} = \sum_{k=i}^{j} f_{ik} g_{kj} = \sum_{k=i}^{j} g_{ik} f_{kj}. \tag{3.11}$$

Proof The lemma is trivially true for $n = 0$. We proceed inductively,

$$\begin{aligned}
h_{ij} &= \frac{h_{i,j-1} - h_{i+1,j}}{z_i - z_j} \\
&= \frac{1}{z_i - z_j}\left[\sum_{k=i}^{j-1} f_{ik} g_{k,j-1} - \sum_{k=i+1}^{j} f_{i+1,k} g_{kj}\right] \\
&= \frac{1}{z_i - z_j}\left[\sum_{k=i}^{j-1} f_{ik}\left(g_{k,j-1} - g_{k+1,j}\right) - \sum_{k=i+1}^{j}\left(f_{i,k-1} - f_{i+1,k}\right) g_{kj}\right] \\
&= f_{ii} g_{ij} + \frac{1}{z_i - z_j}\left[\sum_{k=i+1}^{j-1} f_{ik} g_{kj}\left(z_k - z_j\right) - \sum_{k=i+1}^{j-1} f_{ik} g_{kj}\left(z_i - z_k\right)\right] + f_{ij} g_{jj} \\
&= \sum_{k=i}^{j} f_{ik} g_{kj}. \qquad (3.12)
\end{aligned}$$

Finally, we observe that the proceeding argument is symmetric in f and g.

We now proceed to consider interpolation of a given function $f(z)$ using rational fractions which are sometimes called *interpolants*. The basic problem is to find a rational fraction

$$r^{[L/M]}(z) = u^{[L/M]}(z) / v^{[L/M]}(z) \qquad (3.13)$$

where $u^{[L/M]}(z)$ has maximum order L, $v^{[L/M]}(z)$ has maximum order M and

$$r^{[L/M]}(z_i) = f(z_i), \qquad i = 0, 1, 2, \ldots, L + M. \qquad (3.14)$$

If a solution to this basic problem exists, it is obtained by defining

$$u^{[L/M]}(z) = \sum_{j=0}^{L} u_j z^j, \qquad v^{[L/M]}(z) = \sum_{k=0}^{M} v_k z^k \qquad (3.15)$$

for specific values of L, M. Let us assume that $v_0 = 1$ is a permissible normalization for the moment. Substitution of (3.13) and (3.15) into (3.14) yields $L+M+1$ linear equations for $L+M+1$ unknown coefficients $u_0, u_1, \ldots, u_L, v_1, v_2, \ldots, v_M$. Normally, there is a unique solution leading to a rational interpolant which is uniquely defined up to a constant common factor in the numerator and denominator of (3.13). Otherwise, the equations are said to be degenerate. If the equations are degenerate but consistent, and $v^{[L/M]}(z) \neq 0$, then $u^{[L/M]}(z)$ and $v^{[L/M]}(z)$ have a common factor. Using (3.22) with $f(z) = r^{[L/M]}(z)$, it follows that the factors $(z - z_i)$, $i = 0, 1, \ldots, L + M$ are the only possible elementary common factors of $u^{[L/M]}(z)$ and $v^{[L/M]}(z)$. For each such factor $(z - z_k)$, (3.14) must be tested with $i = k$ for the proposed solution. If the linear equations are inconsistent, no rational fraction of type $[L/M]$ fits the data. As an example, we next show that no rational function of type $[1/1]$ fits the data

$$f(-1) = 1, \qquad f(0) = 1, \qquad f(1) = 3 \tag{3.16}$$

at the indicated points. The equations (3.13), (3.14) and (3.15) become

$$u_0 - u_1 = v_0 - v_1, \tag{3.17}$$

$$u_0 = v_0, \tag{3.18}$$

$$u_0 + u_1 = 3(v_0 + v_1). \tag{3.19}$$

Equations (3.17) and (3.18) imply that $u_0 = v_0$, $u_1 = v_1$, and so (3.19) implies that $u_0 = u_1 = v_0 = v_1 = 0$. Equations (3.17), (3.18) and (3.19) are degenerate. In fact, only the new value $f(1) = 1$ would render equations consistent and allow rational interpolation to be effected by a (degenerate) interpolant of type $[1/1]$.

Since Padé approximation is rational approximation with complete confluence of the interpolation points, it is interesting to note the similarity between the previous analysis and that of the existence or nonexistence of Padé approximants. Having briefly considered some of the hazards of using rational interpolation, the next theorem gives the standard solution in the nondegenerate case.

Theorem 3.1.2. *The N-point Padé approximant of type $[L/M]$ defined by interpo-*

lation at the points $z_0, z_1, \ldots, z_{L+M}$, allowing confluence, is normally given by

$$r^{[L/M]}(z) = u^{[L/M]}(z) / v^{[L/M]}(z),$$

fwhere $u^{[L/M]}(z)$ and $v^{[L/M]}(z)$ are defined by

$$u^{[L/M]}(z) = \begin{vmatrix} f_{M,L+1} & f_{M,L+2} & \cdots & f_{M,L+M} & \sum_{j=M}^{L} f_{M,j} \prod_{k=0}^{j-1}(z - z_k) \\ f_{M-1,L+1} & f_{M-1,L+2} & \cdots & f_{M-1,L+M} & \sum_{j=M-1}^{L} f_{M-1,j} \prod_{k=0}^{j-1}(z - z_k) \\ \vdots & \vdots & & \vdots & \vdots \\ f_{0,L+1} & f_{0,L+2} & \cdots & f_{0,L+M} & \sum_{j=0}^{L} f_{0,j} \prod_{k=0}^{j-1}(z - z_k) \end{vmatrix}, \tag{3.20}$$

$$v^{[L/M]}(z) = \begin{vmatrix} f_{M,L+1} & f_{M,L+2} & \cdots & f_{M,L+M} & \prod_{k=0}^{M-1}(z - z_k) \\ f_{M-1,L+1} & f_{M-1,L+2} & \cdots & f_{M-1,L+M} & \prod_{k=0}^{M-2}(z - z_k) \\ \vdots & \vdots & & \vdots & \vdots \\ f_{0,L+1} & f_{0,L+2} & \cdots & f_{0,L+M} & 1 \end{vmatrix}, \tag{3.21}$$

and the definition (3.10) has been used.

The remainder is given by

$$v^{[L/M]}(z) f(z) - u^{[L/M]}(z) = \prod_{k=0}^{L+M}(z - z_k)$$
$$\times \begin{vmatrix} f_{M,L+1} & f_{M,L+2} & \cdots & f_{M,L+M} & f[z_M, \ldots, z_{L+M}, z] \\ f_{M-1,L+1} & f_{M-1,L+2} & \cdots & f_{M-1,L+M} & f[z_{M-1}, \ldots, z_{L+M}, z] \\ \vdots & \vdots & & \vdots & \vdots \\ f_{0,L+1} & f_{0,L+2} & \cdots & f_{0,L+M} & f[z_0, \ldots, z_{L+M}, z] \end{vmatrix}. \tag{3.22}$$

If "impossible" entries in (3.20)-(3.22) occur, the following interpretation is intended:

If $j < i$, then

$$f_{i,j} = 0, \quad \sum_{k=i}^{j} (\text{term})_k = 0, \quad \text{and} \prod_{k=i}^{j} (\text{factor})_k = 1.$$

A sufficient condition for the result that $u^{[L/M]}(z_i)/v^{[L/M]}(z_i) = f(z_i)$ is that $v^{[L/M]}(z_i) \neq 0$, $i = 0, 1, \ldots, L + M$.

Proof The formulas (3.20) and (3.21) are polynomials of the appropriate orders. Using Newton's formula (3.20), it follows that

$$v^{[L/M]}(z) f(z) - u^{[L/M]}(z) = \prod_{k=0}^{L} (z - z_k) \times \begin{vmatrix} f_{M,L+1} & f_{M,L+2} & \cdots & f_{M,L+M} & f[z_M, \ldots, z_L, z] \\ f_{M-1,L+1} & f_{M-1,L+2} & \cdots & f_{M-1,L+M} & f[z_{M-1}, \ldots, z_L, z] \\ \vdots & \vdots & & \vdots & \vdots \\ f_{0,L+1} & f_{0,L+2} & \cdots & f_{0,L+M} & f[z_0, \ldots, z_L, z] \end{vmatrix}. \tag{3.23}$$

Recalling definitions (3.3) and (3.10), repeated subtraction of the jth column of (3.23) from the last for $j = 1, 2, \ldots, M$ yields (3.22). This is manifestly zero at $z_0, \ldots, z_{L+M}$. Provided $v^{[L/M]}(z_i) \neq 0$, $i = 0, 1, \ldots, L + M$, the result follows.

3.2. Newton-Padé Approximations in Gallucci-Jones Form

Definition 3.2. A *formal Newton series* (FNS) is an ordered triple $[\{\alpha_n\}_0^\infty, \{\beta_n\}_0^\infty, \{f_n\}_0^\infty]$, where $\alpha_0, \alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \beta_3, \ldots$ are complex numbers (not necessarily distinct) and for each $n = 0, 1, 2, \ldots, f_n$ is the polynomial

$$f_n(z) = \sum_{k=0}^{n} a_k \omega_k(z), \tag{3.24}$$

where

$$\omega_0(z) = 1; \quad \omega_k(z) = \prod_{j=1}^{n}(z - \beta_j), \qquad k = 1, 2, 3, \ldots,$$

and where z is a complex variable. The α_n, β_n and f_n are called, respectively, the nth Newton coefficient, interpolation point, and partial sum of $[\{\alpha_n\}, \{\beta_n\}, \{f_n\}]$ and a FNS is said to converge at z if the sequence of partial sums $\{f_n(z)\}$ is convergent. When convergent, the limit $\lim f_n(z)$ is called the value of the FNS at z. For convenience (when there is no danger of confusion) we may use the symbols f and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \omega_n(z) \tag{3.25}$$

to represent the FNS $[\{\alpha_n\}, \{\beta_n\}, \{f_n\}]$. As in many other similar situations in analysis, the symbols (3.25) are used to denote both the infinite process and the value of its limit, when it exists.

Some arithmetic operations for formal Newton series are given by the following:

Definition 3.3. Let $f(z) = \sum_{k=0}^{\infty} \alpha_k \omega_k(z)$ and $g(z) = \sum_{k=0}^{\infty} c_k \omega_k(z)$ be FNS with interpolation points $\{\beta_i\}$ and let c be a complex number. We define:

(a) $(f+g)(z) = \sum_{k=0}^{\infty} (\alpha_k + c_k)\, \omega_k(z),$

(b) $(c.f)(z) = \sum_{k=0}^{\infty} (c.\alpha_k)\, \omega_k(z),$

(c) $(z.f)(z) = \alpha_0 \beta_1 \omega_0(z) + \sum_{k=1}^{\infty} (\alpha_{k-1} + \alpha_k \beta_{k+1})\, \omega_k(z),$

(d) If $c \neq \beta_i,\ i = 1, 2, 3, \ldots,$ then $f(z) / (c - z) = \sum_{k=0}^{\infty} b_k \omega_k(z),$

where

$$b_0 = a_0 / (\beta_1 - c) ; \quad b_k = (\alpha_k - b_{k-1}) / (\beta_{k+1} - c) , \quad k = 1, 2, 3, \ldots \tag{3.26}$$

Every FNS (3.25) determines a function f defined at the points of convergence of the partial sums (3.24). Clearly, (3.25) always converges (at least) at the points $\beta_1, \beta_2, \beta_3, \ldots$. Conversely, under certain conditions a function f will determine a FNS expansion with a given sequence of interpolation points $\{\beta_i\}$.

If $\alpha_k = 0$ for $k \geq n+1$, then (3.26) is a finite (or terminating) FNS and defines a polynomial in z of degree not greater than n. Conversely, as an immediate consequence of Definition 2, every polynomial of degree n determines a unique (finite) FNS with the given sequence of interpolation points $\{\beta_i\}$. From Definition 2 it is also clear that the product (multiplication) of a FNS by a polynomial is a well-defined FNS (with the same sequence of interpolation points); the quotient (division) of a FNS by a polynomial is a well-defined FNS (with the same sequence of interpolation points) provided the (divisor) polynomial does not vanish at any of the interpolation points. The following theorem provides further useful information concerning multiplication of a FNS by polynomials.

Theorem 3.2.1. *Let* $f(z) = \sum_{k=0}^{\infty} \alpha_k \omega_k(z)$ *be a FNS with interpolation points* $\{\beta_n\}$. *If* m, ν *and* μ *are positive integers, let* $K_{\nu,\mu}^{(m)}$ *denote the sum of all products consisting of* m *factors of the* β_i*'s with*

$$\nu - \mu + 1 \leq i \leq \nu \quad (K_{\nu,\mu}^{(m)} = 0, \ \textit{if } m < 0 \textit{ or if } \mu < 1; \ K_{\nu,\mu}^{(0)} = 1 \ \textit{if } \mu \geq 1).$$

Then

(A) *For* $p = 1, 2, 3, \ldots,$

$$z^p f(z) = \sum_{k=0}^{\infty} A_{k,p} \omega_k(z) , \tag{3.27}$$

where (setting $a_i = 0$ for $i < 0$)

$$A_{k,\nu} = \sum_{j=0}^{p} \alpha_{k-j} K_{k+1,j+1}^{(p-j)}. \tag{3.28}$$

(B) *If $\nu(z) = d_0 + d_1 z + \ldots + d_m z^m, \quad m \geq 0,$ then*

$$\nu(z) f(z) = \sum_{k=0}^{\infty} b_k \omega_k(z), \tag{3.29}$$

where

$$b_k = \sum_{j=0}^{m} d_j A_{k,j}. \tag{3.30}$$

(C) *In particular, with the notation of (B),*

$$\nu(z) f(z) = \sum_{k=n+m+1}^{\infty} b_k \omega_k(z),$$

if $\alpha_i = 0$ for $i = 0, 1, \ldots, n+m$.

(D) *If $c \neq \beta_i$ for $i = 1, 2, 3, \ldots,$ and $a_k = 0$ for $k = 0, 1, \ldots, n+m,$ then*

$$f(z) / (z - c) = \sum_{k=n+m+1}^{\infty} b_k \omega_k(z),$$

where the coefficients b_k are defined by (3.26).

Proof It can be verified directly from the definition of the $K_{\nu,\mu}^{(m)}$, that

$$K_{k,j}^{(p-j)} + \beta_{k+1} K_{k+1,j+1}^{(p-1-j)} = K_{k+1,j+1}^{(p-j)}, \qquad j = 0, 1, \ldots, p; \quad p = 1, 2, 3, \ldots \tag{3.31}$$

The proof of (A) is by an induction on p. The case $p = 1$ follows immediately from

Definition 2c. Now assume that (3.27) and (3.28) is true for $1 \leq p \leq n$. Then

$$\begin{aligned} z^n f(z) = z\left(z^{n-1} f(z)\right) &= z \sum_{k=0}^{\infty} A_{k,n-1}\omega_k(z), \quad \text{(by induction hypothesis)} \\ &= A_{0,n-1}\beta_1\omega_0(z) + \sum_{k=0}^{\infty}\left(A_{k-1,n-1} + \beta_{k+1}A_{k,n-1}\right)\omega_k(z), \quad \text{(by Definition 2c)} \\ &= \sum_{k=0}^{\infty} A_{k,n}\omega_k(z), \end{aligned}$$

where the $A_{k,n}$ satisfy (3.28), since

$$A_{0,n} = A_{0,n-1}\beta_1 = \left(\alpha_0 K_{1,1}^{(n-1)}\right)\beta_1 = \alpha_0 K_{1,1}^{(n)}$$

and, for $k = 1, 2, 3, \ldots,$

$$\begin{aligned} A_{k,n} &= A_{k-1,n-1} + A_{k,n-1}\beta_{k+1} \\ &= \sum_{j=0}^{n-1}\alpha_{k-1-j}K_{k,j+1}^{(n-1-j)} + \beta_{k+1}\sum_{j=0}^{n-1}\alpha_{k-j}K_{k+1,j+1}^{(n-1-j)} \quad \text{(by induction hypothesis)} \\ &= \sum_{j=0}^{n}\alpha_{k-j}\left(K_{k+j,1}^{(n-j)} + \beta_{k+1}K_{k+1,j+1}^{(n-1-j)}\right) \\ &= \sum_{j=0}^{n}\alpha_{k-j}K_{k+1,j+1}^{(n-j)} \quad \text{(by (3.31))}. \end{aligned}$$

Part (B) is an immediate consequence of (3.27) and (3.28). Part (C) follows from (3.30) and the fact that $A_{k,j} = 0$ provided $a_i = 0$ for all $i = k,\ k-1, \ldots, k-j$. Part (D) follows immediately from (3.26) and this completes the proof.

Definition 3.4. If $u(z)$ and $v(z)$ are polynomials in z, $v(z)$ not identically zero, then (u, v) is called a rational expression. Two rational expressions (u, v) and (u^*, v^*) are said to be *equivalent*, denoted by $(u, v) \sim (u^*, v^*)$, if and only if

$$u(z)\, v^*(z) \equiv u^*(z)\, v(z); \tag{3.32}$$

they are called *equal*, denoted by $(u, v) = (u^*, v^*)$, if and only if there exists a nonzero complex number α such that

$$\alpha \cdot u(z) \equiv u^*(z), \qquad \alpha \cdot v(z) \equiv v^*(z). \tag{3.33}$$

A rational expression (u, v) is said to be *of type* $[m, n]$ if and only if the degree of u is at most n and the degree of v is at most m.

Theorem 3.2.2. *Let $f(z) = \sum a_k\omega_k(z)$ be a FNS with interpolation points $\{\gamma_i\}$ and let m and n be (fixed) nonnegative integers. Then:*

(A) *If $u(z) = c_0\omega_0(z)+c_1\omega_1(z)+\ldots+c_n\omega_n(z)$ and $v(z) = d_0+d_1z+\ldots+d_mz^m$, then a necessary and sufficient condition that the FNS $vf - u$ be the form*

$$v(z) f(z) - u(z) = b_{n+m+1}\omega_{n+m+1}(z) + b_{n+m+2}\omega_{n+m+2}(z) + \ldots \tag{3.34}$$

is that the coefficients c_j and d_j satisfy the system of equations

$$\begin{aligned} d_0A_{0,0} + d_1A_{0,1} + \ldots + d_mA_{0,m} &= c_0 \\ d_0A_{1,0} + d_1A_{1,1} + \ldots + d_mA_{1,m} &= c_1 \\ &\vdots \\ d_0A_{n,0} + d_1A_{n,1} + \ldots + d_mA_{n,m} &= c_n \end{aligned} \tag{3.35}$$

$$\begin{aligned} d_0A_{n+1,0} + d_1A_{n+1,1} + \ldots + d_mA_{n+1,m} &= 0 \\ &\vdots \\ d_0A_{n+m,0} + d_1A_{n+m,1} + \ldots + d_mA_{n+m,m} &= 0 \end{aligned} \tag{3.36}$$

where $A_{k,\nu}$ are defined by (3.28).

(**B**) There exists a unique (up to equivalence $\sim$) rational expression (u, v) of type $[m, n]$, such that the FNS $v(z) f(z) - u(z)$ has the form (3.34).

Proof (**A**) By Theorem 1, letting $c_k = 0$ for $k \geq n$, we obtain

$$v(z) f(z) - u(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{m} d_j A_{k,j} - c_k \right) \omega_k(z), \tag{3.37}$$

of which (A) is an immediate consequence. To prove (B), we note that (3.36) is a homogeneous linear system of m equations in $(m+1)$ unknowns. Hence, there exist $d_0, d_1, \ldots, d_m$, not all zero, satisfying (3.36). Having chosen such d_i, we choose the c_i to satisfy (3.35) and the resulting rational expression (u, v) is of type $[m, n]$ such that

$$v^*(z) f(z) - u^*(z) = b^*_{n+m+1} \omega_{n+m+1}(z) + b^*_{n+m+2} \omega_{n+m+2}(z) + \ldots. \tag{3.38}$$

By Theorem 1(C), $v(v^* f - u^*)$ and $v^*(vf - u)$ are both FNS whose first $n + m + 1$ coefficients are zero. Hence,

$$\begin{gathered} v^*(z) u(z) - v(z) u^*(z) \\ = v(z) [v^*(z) f(z) - u^*(z)] - v^*(z) [v(z) f(z) - u(z)] \end{gathered}$$

is also a FNS whose first $n+m+1$ coefficients vanish. But $v^* u - v u^*$ is a polynomial of degree at most $n+m$ and therefore must be identically zero. Thus $(u^*, v^*) \sim (u, v)$, which completes the proof.

Definition 3.5. Let $f(z) = \sum\limits_{k=0}^{\infty} \alpha_k \omega_k(z)$ be a FNS with interpolation points $\{\beta_i\}$. Corresponding to each ordered pair of nonnegative integers (m, n), Theorem 1 asserts the existence of a unique rational function

$$R_{m,n}(f, z) = P_{m,n}(f, z) / Q_{m,n}(f, z), \tag{3.39}$$

such that $(P_{m,n}, Q_{m,n})$ is a rational expression equivalent to a rational expression (u, v) of type $[m, n]$ satisfying (3.34). $R_{m,n}(f, z)$ is called the $[m, n]$ *Newton-Padé approximant* of $f(z)$. The doubly infinite array

$$\begin{matrix} R_{0,0}(f,z) & R_{0,1}(f,z) & R_{0,2}(f,z) & \dots \\ R_{1,0}(f,z) & R_{1,1}(f,z) & R_{1,2}(f,z) & \dots \\ R_{2,0}(f,z) & R_{2,1}(f,z) & R_{2,2}(f,z) & \dots \\ \dots & \dots & \dots & \dots \end{matrix} \tag{3.40}$$

is called the Newton-Padé table of $f(z)$.

3.3. Algorithms for Computation of Newton-Padé Approximations

Now, let us give some algorithms for calculating multipoint Padé or Newton-Padé approximations. They are based on Claessens's identities. For multipoint Padé approximation, the following theorem holds.

Theorem 3.3.1. [*Claessens, 1978*].

$$\begin{aligned} &\left[\left\{r^{[L+1/M]}(z) - r^{[L/M]}(z)\right\}^{-1} - \left\{r^{[L/M+1]}(z) - r^{[L/M]}(z)\right\}^{-1}\right](z - z_{L+M}) \\ &= \left[\left\{r^{[L/M-1]}(z) - r^{[L/M]}(z)\right\}^{-1} - \left\{r^{[L-1/M]}(z) - r^{[L/M]}(z)\right\}^{-1}\right](z - z_{L+M+1}) \end{aligned} \tag{3.41}$$

whenever the indicated quantities exist and are nondegenerate. Claessens's identity reduces to Wynn's identity in the confluent limit.

Outline proof of (3.41). We define

$$F^{[L/M]}_{1,L+M} = \begin{vmatrix} f_{M,L+1} & f_{M,L+2} & f_{M,L+M} \\ f_{M-1,L+1} & f_{M-1,L+2} & f_{M-1,L+M} \\ & & \\ f_{1,L+1} & f_{1,L+2} & f_{1,L+M} \end{vmatrix} \tag{3.42}$$

and note that $F_{1,L+M}^{[L/M]} = C(L/M)$ in the confluent limit. The subscripts of $F_{1,L+M}^{[L/M]}$ denote the indices $1, 2, \ldots, L+M$ of the interpolation points used in its construction. Using the methods in Wynn's identity, we find that

$$\begin{aligned} & r^{[L+1/M]}(z) - r^{[L/M]}(z) \\ & = \frac{(z-z_0)\ldots(z-z_{L+M+1})\,F_{0,L+M+1}^{[L+1/M+1]}F_{0,L+M}^{[L+1/M]}}{v^{[L+1/M]}(z)\,v^{[L/M]}(z)}, \end{aligned} \tag{3.43}$$

and we generalize directly to

$$\begin{aligned} & r^{[L+1/M]}(z) - r^{[L/M]}(z) \\ & = \frac{(z-z_0)\ldots(z-z_{L+M+1})\,F_{0,L+M}^{[L/M+1]}F_{0,L+M+1}^{[L+1/M+1]}}{v^{[L/M+1]}(z)\,v^{[L/M]}(z)} \end{aligned} \tag{3.44}$$

By reordering the points of (3.21), we find that

$$\begin{aligned} & v^{[L/M+1]}(z) \\ & = \begin{vmatrix} f_{M,L} & f_{M-1,L} & \cdots & f_{0,L} & \prod_{k=L+1}^{L+M+1}(z-z_k) \\ f_{M,L+1} & f_{M-1,L+1} & \cdots & f_{0,L+1} & \prod_{k=L+2}^{L+M+1}(z-z_k) \\ \vdots & \vdots & & \vdots & \vdots \\ f_{M,L+M+1} & f_{M-1,L+M+1} & \cdots & f_{0,L+M+1} & 1 \end{vmatrix} \end{aligned}$$

By applying Sylvester's identity to this, we find that

$$\begin{aligned} & v^{[L/M+1]}(z)\,F_{0,L+M}^{[L+1/M]} \\ & = v^{[L+1/M]}(z)\,F_{0,L+M}^{[L/M+1]} - (z-z_{L+M+1})\,F_{0,L+M+1}^{[L+1/M+1]}v^{[L/M]}(z). \end{aligned} \tag{3.45}$$

Hence, we deduce from (3.43), (3.44) and (3.45) that

$$\begin{aligned} & \left\{r^{[L+1/M]}(z) - r^{[L/M]}(z)\right\}^{-1} - \left\{r^{[L/M+1]}(z) - r^{[L/M]}(z)\right\}^{-1} \\ & = \frac{\left\{v^{[L/M]}(z)\right\}^2 (z-z_0)^{-1}\ldots(z-z_{L+M})^{-1}}{F_{0,L+M}^{[L+1/M]}F_{0,L+M}^{[L/M+1]}}. \end{aligned}$$

Equation (3.41) follows from a similar treatment of the right-hand side.

The generalized $\varepsilon-$algorithm is the formal identity

$$\left(z - z_{k+j+1}\right)\left[\varepsilon_{k+1}^{(j)} - \varepsilon_{k-1}^{(j+1)}\right]\left[\varepsilon_{k}^{(j+1)} - \varepsilon_{k}^{(j)}\right] = 1 \tag{3.46}$$

for indices k, j in the range $k = 0, 1, 2, \ldots$ and $j \geq -\left[k/2\right]$. The artificial initialization conditions are

$$\varepsilon_{-1}^{(j)} = 0, \qquad j = 0, 1, 2, \ldots,$$

and

$$\varepsilon_{2k}^{(-k-1)} = 0, \quad k = 0, 1, 2, \ldots. \tag{3.47}$$

The usual initialization condition, using values derived from an interpolating polynomial is

$$\varepsilon_{0}^{(j)} = r^{[j/0]}\left(z\right). \tag{3.48}$$

Elements of the $\varepsilon-$table are identified with values of rational interpolants by the formula

$$\varepsilon_{2k}^{(j)} = r^{[k+j/k]}\left(z\right), \qquad k = 0, 1, 2, \ldots, \quad j \geq -k. \tag{3.49}$$

In the sequel G. Claessens denotes the coefficients of $\omega_{0i}\left(z\right)$ in p_{mn} (resp. q_{mn}) by $\alpha_{mn}^{(i)}$ (resp. $b_{mn}^{(i)}$).

First he proves two theorems, which relate certain triples of elements in the rational Hermite interpolation table. A first theorem concerns the elements $r_{m,n} =$

P_1/Q_1, $r_{m,n-1} = P_2/Q_2$ and $r_{m+1,n-1} = P_3/Q_3$.

Theorem 3.3.2.

$$\frac{P_1}{Q_1} = \frac{\alpha_{m,n-1}^{(m)} P_3 - \alpha_{m+1,n-1}^{(m+1)} (z - z_{m+n}) P_2}{\alpha_{m,n-1}^{(m)} Q_3 - \alpha_{m+1,n-1}^{(m+1)} (z - z_{m+n}) Q_2}. \tag{3.50}$$

Proof Since $\alpha_{m+1,n-1}^{(m+1)} \neq 0$, it is clear that the denominator of the right side of (3.50) has exactly degree n.

On the other hand, since

$$\alpha_{m,n-1}^{(m)} \alpha_{m+1,n-1}^{(m+1)} - \alpha_{m+1,n-1}^{(m+1)} \alpha_{m,n-1}^{(m)} = 0$$

the numerator has at most degree m.

Now let

$$\begin{aligned} U(z) = &\left[\alpha_{m,n-1}^{(m)} Q_3(z) - \alpha_{m+1,n-1}^{(m+1)} (z - z_{m+n}) Q_2(z)\right] f(z) \\ &- \left[\alpha_{m,n-1}^{(m)} P_3(z) - \alpha_{m+1,n-1}^{(m+1)} (z - z_{m+n}) P_2(z)\right], \end{aligned}$$

or

$$\begin{aligned} U(z) = &\, \alpha_{m,n-1}^{(m)} \left[Q_3(z) f(z) - P_3(z)\right] \\ &- \alpha_{m+1,n-1}^{(m+1)} (z - z_{m+n}) \left[Q_2(z) f(z) - P_2(z)\right]. \end{aligned}$$

We will show that

$$U(z) = 0\left(\omega_{0,m+n+1}(z)\right). \tag{3.51}$$

Suppose in the set $\{z_i\}$, $i = 0, 1, \ldots, m+n$ there are l distinct points $z_{\alpha i}$, $i =$

$1, 2, \ldots, l$ with resp. multiplicity m_i. Then

$$\sum_{i=1}^{l} m_i = m + n + 1.$$

Consider also the formal ith derivative of $U(z)$,

$$\begin{aligned} U^{(i)}(z) = {} & \alpha_{m,n-1}^{(m)} \left[Q_3(z) f(z) - P_3(z) \right]^{(i)} \\ & - \alpha_{m+1,n-1}^{(m+1)} (z - z_{m+n}) \left[Q_2(z) f(z) - P_2(z) \right]^{(i)} \\ & - i\alpha_{m+1,n-1}^{(m+1)} \left[Q_2(z) f(z) - P_2(z) \right]^{(i-1)}, \end{aligned}$$

with $i \geq 1$.

Then, it is easy to conclude, using the definition of P_2/Q_2 and P_3/Q_3, that

$$U^{(j)}(z_{\alpha i}) = 0$$

for $j = 0, 1, \ldots, m_i - 1$ and $i = 1, 2, \ldots, l$, which implies (3.51).

Because of the supposed normality of the rational Hermite interpolation table and because of the unicity of the rational Hermite interpolant, the function associated with the right side of (3.50) must be equal to the rational Hermite interpolant of order $[m, n]$ (and hence the numerator has exactly degree m).

This concludes the proof.

Considering the elements $r_{m-1,n} = (P_1/Q_1)$, $r_{m,n} = (P_2/Q_2)$ and $r_{m,n-1} = (P_3/Q_3)$, we can prove in the same way the following result.

Theorem 3.3.3.

$$\frac{P_1}{Q_1} = \frac{\alpha_{m,n-1}^{(m)} P_2 - \alpha_{m,n}^{(m)} P_3}{\alpha_{m,n-1}^{(m)} Q_2 - \alpha_{m,n}^{(m)} Q_3}. \tag{3.52}$$

Note that in (3.50) because of the appearance of the factor $z - z_{m+n}$, the representation as a Newton series for the numerator and the denominator of the rational Hermite interpolant of order $[m, n]$ has been lost.

The relations (3.50) and (3.52) can be used for calculating the rational Hermite interpolants. Indeed, note that relation (3.50) enables us to go to the right in the rational Hermite interpolation table, while relation (3.52) allows us to move upwards.

A First Method: Consider the elements in the rational Hermite interpolation table lying on an ascending staircase,

$$T_k = \{r_{k,0}, r_{k-1,0}, r_{k-1,1}, \ldots, r_{0,k}\}, \tag{3.53}$$

with $k \geq 1$.

Theorem 3.3.4. *To compute the coefficients of the numerator and denominator in the sequence T_k, the following recurrence formulas exist:*

$$\begin{aligned}
\alpha^{(i)}_{k-j,j} &= \frac{\alpha^{(k-j)}_{k-j,j-1}\alpha^{(i)}_{k-j+1,j-1} - \alpha^{(k-j+1)}_{k-j+1,j-1}\left[\alpha^{(i-1)}_{k-j,j-1} - (z_k - z_i)\,\alpha^{(i)}_{k-j,j-1}\right]}{\alpha^{(k-j)}_{k-j,j-1} + (z_k - z_0)\,\alpha^{(k-j+1)}_{k-j+1,j-1}}, \\
i &= 0, 1, \ldots, k-j \\
b^{(i)}_{k-j,j} &= \frac{\alpha^{(k-j)}_{k-j,j-1}b^{(i)}_{k-j+1,j-1} - \alpha^{(k-j+1)}_{k-j+1,j-1}\left[b^{(i-1)}_{k-j,j-1} - (z_k - z_i)\,b^{(i)}_{k-j,j-1}\right]}{\alpha^{(k-j)}_{k-j,j-1} + (z_k - z_0)\,\alpha^{(k-j+1)}_{k-j+1,j-1}}, \\
i &= 0, 1, \ldots, j
\end{aligned} \tag{3.54}$$

for $j = 1, 2, \ldots, k$ and

$$\alpha^{(i)}_{k-j-1,j} = \frac{\alpha^{(k-j)}_{k-j,j-1}\alpha^{(i)}_{k-j,j} - \alpha^{(k-j)}_{k-j,j}\alpha^{(i)}_{k-j,j-1}}{\alpha^{(k-j)}_{k-j,j-1} - \alpha^{(k-j)}_{k-j,j}},$$
$$i = 0, 1, \ldots, k-j-1 \tag{3.55}$$
$$b^{(i)}_{k-j-1,j} = \frac{\alpha^{(k-j)}_{k-j,j-1}b^{(i)}_{k-j,j} - \alpha^{(k-j)}_{k-j,j}b^{(i)}_{k-j,j-1}}{\alpha^{(k-j)}_{k-j,j-1} - \alpha^{(k-j)}_{k-j,j}},$$
$$i = 0, 1, \ldots, j$$

for $j = 1, 2, \ldots, k-1$.

Proof First, we rewrite (3.50) and (3.52) in the respective forms

$$\frac{p_{k-j,j}}{q_{k-j,j}} = \frac{\alpha^{(k-j)}_{k-j,j-1}p_{k-j+1,j-1} - \alpha^{(k-j+1)}_{k-j+1,j-1}\left(z - z_k\right)p_{k-j,j-1}}{\alpha^{(k-j)}_{k-j,j-1}q_{k-j+1,j-1} - \alpha^{(k-j+1)}_{k-j+1,j-1}\left(z - z_k\right)q_{k-j,j-1}} \tag{3.56}$$

and

$$\frac{p_{k-j-1,j}}{q_{k-j-1,j}} = \frac{\alpha^{(k-j)}_{k-j,j-1}p_{k-j,j} - \alpha^{(k-j)}_{k-j,j}p_{k-j,j-1}}{\alpha^{(k-j)}_{k-j,j-1}q_{k-j,j} - \alpha^{(k-j)}_{k-j,j}q_{k-j,j-1}} \tag{3.57}$$

To determine the Newton coefficients of the numerator and denominator of $r_{k-j,j}$, we proceed as follows.

The numerator N of the right side of (3.56) can be written as

$$N = \alpha^{(k-j)}_{k-j,j-1}\sum_{i=0}^{k-j+1}\alpha^{(i)}_{k-j+1,j-1}\omega_{0i}\left(z\right) - \alpha^{(k-j+1)}_{k-j+1,j-1}\sum_{i=0}^{k-j}\alpha^{(i)}_{k-j,j-1}\omega_{0i}\left(z\right)$$
$$\times\left[\left(z_0 - z_k\right)\omega_{00}\left(z\right) + \omega_{01}\left(z\right)\right].$$

The second term on the right side becomes

$$N = \alpha_{k-j,j-1}^{(k-j)} \sum_{i=0}^{k-j+1} \alpha_{k-j+1,j-1}^{(i)} \omega_{0i}(z) - \alpha_{k-j+1,j-1}^{(k-j+1)} \times \sum_{i=0}^{k-j+1} \left[\alpha_{k-j,j-1}^{(i)} + (z_i - z_k)\, \alpha_{k-j,j-1}^{(i)} \right] \omega_{0i}(z)$$

with the convention that $\alpha_{k-j,j-1}^{(i)} = 0$, if $i < 0$ or $i > k-j$ or,

$$N = \sum_{i=0}^{k-j+1} \Big(\alpha_{k-j,j-1}^{(k-j)} \alpha_{k-j+1,j-1}^{(i)} - \alpha_{k-j+1,j-1}^{(k-j+1)} \alpha_{k-j,j-1}^{(i-1)} + (z_k - z_i)\, z_{k-j+1,j-1}^{(k-j+1)} \alpha_{k-j,j-1}^{(i)} \Big) \omega_{0i}(z).$$

Analogously, the denominator D of (3.56) can be expressed as

$$D = \sum_{i=0}^{j} \Big(\alpha_{k-j,j-1}^{(k-j)} b_{k-j+1,j-1}^{(i)} - \alpha_{k-j+1,j-1}^{(k-j+1)} b_{k-j,j-1}^{(i-1)} + (z_k - z_i)\, \alpha_{k-j+1,j-1}^{(k-j+1)} b_{k-j,j-1}^{(i)} \Big) \omega_{0i}(z),$$

with $b_{k-j,j-1}^{(i)} = 0$ if $i < 0$ or $i > j-1$.

Normalizing so that the denominator takes on the value 1 for $z = z_0$ we finally get the first set of recurrence formula (3.54). Note that the denominator in (3.54) can not vanish, since otherwise the numerator and denominator in (3.55) would have a common factor $z - z_0$, which contradicts the supposed normality of the rational Hermite interpolants. From (3.57) we immediately derive (3.55) by taking into account the normalizing condition. Again the denominator in (3.55) can not vanish for an analogous reason.

Making alternately use of (3.54) and (3.55) it is possible to construct the Newton coefficients of the elements of (3.53), since we know the first two elements as partial sums of the given Newton series.

Theorem 3.3.5. *For the coefficients of the denominators of the rational Hermite* *interpolants we have the recurrence relation*

$$\begin{aligned} b^{(i)}_{k-j+1,j-1} &= b^{(i)}_{k-j,j}\left[1+(z_k-z_0)\,B_{k-j+1,j-1}\right] \\ &+B_{k-j+1,j-1}\left[b^{(i-1)}_{k-j,j-1}-(z_k-z_i)\,b^{(i)}_{k-j,j-1}\right], \end{aligned} \tag{3.58}$$

for $i=0,1,\ldots,j-1,$ *where*

$$B_{k-j+1,j-1} = -\frac{b^{(j)}_{k-j,j}}{b^{(j)}_{k-j,j}\,(z_k-z_0)+b^{(j-1)}_{k-j,j-1}}.$$

Proof The second relation of (3.54) is our starting point. Taking into account that for $i=j$

$$b^{(i)}_{k-j,j} = -\frac{\alpha^{(k-j+1)}_{k-j+1,j-1}b^{(j-1)}_{k-j,j-1}}{\alpha^{(k-j)}_{k-j,j-1}+(z_k-z_0)\,\alpha^{(k-j+1)}_{k-j+1,j-1}},$$

we find that

$$\frac{\alpha^{(k-j+1)}_{k-j+1,j-1}}{\alpha^{(k-j)}_{k-j,j-1}} = -\frac{b^{(j)}_{k-j,j}}{b^{(j)}_{k-j,j}\,(z_k-z_0)+b^{(j-1)}_{k-j,j-1}} = B_{k-j+1,j-1}.$$

Consequently, the second relation of (3.54) becomes, after a suitable reordering,

$$\begin{aligned} b^{(i)}_{k-j+1,j-1} &= b^{(i)}_{k-j,j}\left[1+(z_k-z_0)\,B_{k-j+1,j-1}\right] \\ &+B_{k-j+1,j-1}\left[b^{(i-1)}_{k-j,j-1}-(z_k-z_i)\,b^{(i)}_{k-j,j-1}\right] \end{aligned}$$

for $i=0,1,\ldots,j-1$.

A second method: Suppose we are interested in the element of order $[m,n]$ in (3.40), then we could proceed as follows.

Calculate p_{mn} by forming the next table column by column, using the first relation of (3.54).

Table 2.

$p_{m,0}$	$p_{m,1}$	$\vdots$	$p_{m+1,n-1}$	$p_{m,n}$
$p_{m+1,0}$	$p_{m+1,1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
$p_{m+n-1,0}$	$p_{m+n-1,1}$			
$p_{m+n,0}$				

Construct the next table row by row, using relation (3.58), to get q_{mn}.

Table 3.

$q_{0,n}$	$q_{0,n+1}$	$\cdots$	$q_{0,n+m-1}$	$q_{0,n+m}$
$q_{1,n}$	$q_{1,n+1}$	$\cdots$	$q_{1,n+m-1}$	
$\cdots$	$\cdots$	$\cdots$		
$q_{m-1,n}$	$q_{m-1,n+1}$			
$q_{m,n}$				

This amounts to the following theorem.

Theorem 3.3.6. *For the computation of an arbitrary element $r_{m,n}$ of (3.40), use can be made of the following recurrence relations*

$$\begin{aligned}\alpha_{m+n-j,j}^{(i)} = & \frac{\alpha_{m+n-j,j-1}^{(m+n-j)}\alpha_{m+n-j+1,j-1}^{(i)}}{\alpha_{m+n-j,j-1}^{(m+n-j)} + (z_{m+n} - z_0)\,\alpha_{m+n-j+1,j-1}^{(m+n-j+1)}} \\ & - \frac{\alpha_{m+n-j+1,j-1}^{(m+n-j+1)}\left[\alpha_{m+n-j,j-1}^{(i-1)} - (z_{m+n} - z_i)\,\alpha_{m+n-j,j-1}^{(i)}\right]}{\alpha_{m+n-j,j-1}^{(m+n-j)} + (z_{m+n} - z_0)\,\alpha_{m+n-j+1,j-1}^{(m+n-j+1)}}\end{aligned} \tag{3.59}$$

for $i = 0, 1, \ldots, m+n-j$, and

$$\begin{aligned}b_{m+n-j+1,j-1}^{(i)} = & \, b_{m+n-j,j}^{(i)}\left[1 + (z_{m+n} - z_0)\,B_{m+n-j+1,j-1}\right] \\ & + B_{m+n-j+1,j-1}\left[b_{m+n-j,j-1}^{(i-1)} - (z_{m+n} - z_i)\,b_{m+n-j,j-1}^{(i)}\right],\end{aligned} \tag{3.60}$$

for $i = 0, 1, \ldots, j-1$, *where*

$$B_{m+n-j+1,j-1} = -\frac{b^{(j)}_{m+n-j,j}}{b^{(j)}_{m+n-j,j}\left(z_{m+n} - z_0\right) + b^{(j-1)}_{m+n-j,j-1}}.$$

Looking at the triangular structure, it is clear that this method can be of interest, if we have to know the following triangular array of rational Hermite interpolants:

$$\begin{pmatrix} r_{0,0} & r_{0,1} & \cdots & r_{0,n-1} & r_{0,n} \\ r_{1,0} & r_{1,1} & \cdots & r_{1,n-1} & \\ \cdots & \cdots & \cdots & & \\ r_{n-1,0} & r_{n-1,1} & & & \\ r_{n,0} & & & & \end{pmatrix}$$

3.4. Multipoint Padé Approximations of the Beta Function

This section is based on A. A. Kandayan's paper "Multipoint Padé Approximations of the Beta Function"

Multipoint Padé approximations for a sufficiently general class of functions were first studied in [A. A. Gonchar, G. Lopez-Lagomasino, 1978] , while, in

[A. A. Gonchar, E. A. Rakhmanov, 1987] and [A. I. Aptekarev, 2002], such approximations were used to obtain significant results concerning best rational approximations of analytic functions.

Kandayan studied multipoint approximations of the Euler integral of the first kind (or the beta function)

$$f_\alpha(z) = B(\alpha, z) = \frac{\Gamma(\alpha)\Gamma(z)}{\Gamma(\alpha+z)}.$$

Here, α is an arbitrary fixed complex parameter which is not an integer. At the points $\alpha = 0, -1, -2, \ldots$, the gamma function $\Gamma(\alpha)$ has a pole. At the points $\alpha = 1, 2, 3, \ldots$, the function $f_\alpha(z)$ becomes rational and the Padé problem for the function $f_\alpha(z)$ with the nodes (3.61) degenerates. The function $f_\alpha(z)$ is holomorphic in the domain

$$D = \mathbb{C} \backslash \{0, -1, -2, \ldots\}.$$

At the points $z = -n$, $n \in \mathbb{Z}_+$, it has simple poles. The sequence of polynomials

$$\omega_n(z) = \prod_{k=0}^{2n} (z - \beta - k) \tag{3.61}$$

defines the table of interpolation nodes, i.e., we consider the Newtonian interpolation

$$\omega_{n+1}(z) = \omega_n(z)(z - \beta - 2n - 1)(z - \beta - 2n - 2).$$

Here, β is an arbitrary fixed complex parameter. In what follows we assume without loss of generality that $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$.

Theorem 3.4.1. *The Padé problem for beta function f_α has a unique (up to normalization) solution. The degree of the denominator Q_n is necessarily equal to n.*

Proof Let us use a method that was applied earlier to Padé approximations of the exponential [E. M. Nikishin, V. N. Sorokin; 1988], but, instead of the differential operator we use the difference operator

$$(\Delta f)(z) = f(z+1) - f(z).$$

Choose an arbitrary solution of the problem. Consider the function

$$\tilde{R}_n(z) = \left(\Delta^{n+1} R_n\right)(z).$$

Since $\deg P_n \le n$, it follows that

$$\tilde{R}_n(z) = \left(\Delta^{n+1} Q_n f_\alpha\right)(z).$$

By interpolation conditions $R_n(z) = 0,\ \ z = z_0, \ldots, z_{2n+1}$ and (3.61), we have

$$\tilde{R}_n(z) = 0, \quad z = \beta + k, \quad k = 0, \ldots, n-1. \tag{3.62}$$

We use the Leibniz formula

$$(\Delta(Qf))(z) = f(z+1) \cdot (\Delta Q)(z) + Q(z) \cdot (\Delta f)(z).$$

Further,

$$(\Delta f_\alpha)(z) = -f_{\alpha+1}(z).$$

Therefore,

$$(\Delta(Q_n f_\alpha))(z) = \left(\breve{D}_\alpha Q_n\right)(z) \cdot f_{\alpha+1}(z),$$

where

$$\left(\breve{D}_\alpha Q_n\right)(z) = \frac{z}{\alpha}(\Delta Q_n)(z) - Q_n(z).$$

Thus, by induction, we obtain

$$\tilde{R}_n(z) = \tilde{Q}_n(z) \cdot f_{\alpha+n+1}(z),$$

where

$$\tilde{Q}_n = \breve{D}_{\alpha+n} \ldots \breve{D}_{\alpha+1} \breve{D}_\alpha Q_n.$$

Moreover, by $\breve{D}_\alpha$ we denote the following linear operator:

$$\breve{D}_a = \frac{z}{a}\Delta - I$$

acting in the $(n+1)$-dimensional linear space $\mathbb{C}_n[z]$ of polynomials of degree at most n. Here, the parameter α assumes the values $\alpha, \alpha+1, \ldots, \alpha+n$ and I is the identity operator. It follows from (3.62) that

$$\tilde{Q}_n(z) = \prod_{k=0}^{n-1}(z-\beta-k)$$

(up to normalization).

If we show that each of the operators $\breve{D}_{\alpha+k}$ is invertible, then the theorem will be proved. In the linear space $\mathbb{C}_n[z]$, we introduce the basis of factorial polynomials

$$v_j(z) = \frac{1}{j!}\prod_{k=0}^{j-1}(z-\beta-k), \quad j = 0, \ldots, n.$$

Let us write the matrix of the operator $\breve{D}_\alpha$ in the basis. We have

$$\breve{D}_a v_j = \frac{\beta+j-1}{a}v_{j-1} + \frac{j-a}{a}v_j,$$

where formally we set $v_{-1} \equiv 0$. Thus,

$$\breve{D}_a = \frac{1}{a}\begin{pmatrix} 0-a & \beta & 0 & \ldots & 0 & 0 \\ 0 & 1-a & \beta+1 & \ldots & 0 & 0 \\ 0 & 0 & 2-a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & n-1-a & \beta+n-1 \\ 0 & 0 & 0 & \ldots & 0 & n-a \end{pmatrix}. \tag{3.63}$$

By assumption of the problem, a is not an integer, because α is not integer as well.

Therefore, the determinant

$$\det \breve{D}_a = \frac{1}{a^{n+1}} \prod_{k=0}^{n} (k-a)$$

is nonzero.

The theorem is proved.

CHAPTER 4

NEW DETERMINANTAL REPRESENTATIONS OF NEWTON-PADE APPROXIMATIONS

4.1. Main Definitions and Facts

Lemma 4.1. Let $\{z_k\}_{k=0}^{n}$ and $\{z_k^{'}\}_{k=0}^{n}$ be explicit formulas for Newton-Padé approximations two finite sequences of (not necessarily distinct) complex numbers.

If

$$\omega_k(z) = \prod_{j=1}^{k}(z - z_j), \ k = 1, 2, \ldots, n; \quad \omega_0(z) = 1$$

and

$$P_n(z) = \left(z - z_1^{'}\right)\left(z - z_2^{'}\right)\ldots\left(z - z_n^{'}\right),$$

then

$$P_n(z) = \sum_{k=0}^{n} c_{kn}\omega_k(z),$$

where

$$c_{kn} = \sum_{i_1=1}^{k+1} \sum_{i_2=i_1}^{k+1} \cdots \sum_{i_{n-k}=i_{n-k-1}}^{k+1} \prod_{j=1}^{n-k} \left(z_{i_j} - z'_{i_j+j-1} \right),$$

$$k = 0, 1, \ldots, n-1, \qquad c_{n,n} = 1.$$

Proof. By induction, write

$$P_m(z) = P_{m-1}(z) \left(z - z'_m \right) = \sum_{k=0}^{m-1} c_{k,m-1} \omega_k(z) \left(z - z'_m \right)$$
$$= \sum_{k=0}^{m-1} c_{k,m-1} \omega_k(z) \left[(z - z_{k+1}) + \left(z_{k+1} - z'_m \right) \right].$$

Hence,

$$c_{k,m} = c_{k-1,m-1} + c_{k,m-1} \left(z_{k+1} - z'_m \right), \qquad k = 1, 2, \ldots, m-1. \tag{4.1}$$

So, for $k = 0$ we have

$$c_{0,m} = c_{0,m-1}(z_1 - z'_m) = \left(z_1 - z'_1 \right) \ldots \left(z_1 - z'_{m-1} \right) \left(z_1 - z'_m \right).$$

Now, by (4.1) and by the induction hypothesis,

$$c_{k,m} = \sum_{i_1=1}^{k} \sum_{i_2=i_1}^{k} \cdots \sum_{i_{m-k}=i_{m-k-1}}^{k} \prod_{j=1}^{m-k} \left(z_{i_j} - z'_{i_j+j-1} \right) +$$
$$+ \sum_{i_1=1}^{k+1} \sum_{i_2=i_1}^{k+1} \cdots \sum_{i_{m-k-1}=i_{m-k-2}}^{k+1} \prod_{j=1}^{m-k-1} \left(z_{i_j} - z'_{i_j+j-1} \right)$$

Combining all sums, we obtain

$$\begin{aligned} c_{k,m} &= \sum_{i_1=1}^{k}\sum_{i_2=i_1}^{k}\cdots\sum_{i_{m-k-1}=i_{m-k-2}}^{k}\sum_{i_{m-k}=i_{m-k-1}}^{k+1}\prod_{j=1}^{m-k}\left(z_{i_j}-z'_{i_j+j-1}\right) \\ &+\ldots+\sum_{i_1=1}^{k}(z_{i_1}-z'_{i_1})\prod_{j=k+2}^{m}(z_{k+1}-z'_j)+\ldots+\prod_{j=k+2}^{m}(z_{k+1}-z'_j) \\ &= \sum_{i_1=1}^{k+1}\sum_{i_2=i_1}^{k+1}\cdots\sum_{i_{m-k}=i_{m-k-1}}^{k+1}\prod_{j=1}^{m-k}\left(z_{i_j}-z'_{i_j+j-1}\right), \\ & k=1,2,\ldots,m-1; \quad m=1,2,\ldots,n. \end{aligned}$$

So lemma is proved.

Corollary 4.1.1. *Let l and m be two integers. Then, identity*

$$\omega_l(z).\omega_m(z)=\sum_{k=\max(l,m)}^{l+m}\Omega_{l,m}^{(k)}\omega_k(z),$$

where

$$\begin{aligned}\Omega_{l,m}^{(k)} = &\sum_{j_1=\max(l,m)}^{k+1}\sum_{j_2=j_1}^{k+1}\cdots\sum_{j_{l+m-i}=j_{l+m-k-1}}^{k+1}\left(z_{j_1}-z_{j_1-\max(l.m)}\right)\ldots \\ &\times\left(z_{j_{k+m-k}}-z_{j_{l+m-k}+l+m-k-1}\right),\end{aligned}$$

holds.

Proof. This is an immediate application of Lemma 1.

Lemma 4.2. If

$$f(z)=\sum_{l=0}^{\infty}a_l\omega_l(z) \tag{4.2}$$

is a finite Newton series and

$$Q_N(z)=\sum_{k=0}^{N}b_k\omega_k(z), \tag{4.3}$$

then

$$f\left(z\right)Q_N\left(z\right)=\sum_{i=0}^{\infty}e_{i,N}\omega_i\left(z\right),$$

where

$$e_{i,N}=\sum_{k=0}^{N-(N-i)_+}\beta_{i,k}b_k,\;\;\beta_{i,k}=\sum_{l=i-k}^{i}a_l\Omega_{k,l}^{(i)}. \tag{4.4}$$

Proof. By applying Corollary 2, we get

$$f\left(z\right)Q_N\left(z\right)=\sum_{l=0}^{\infty}a_l\sum_{k=0}^{N}b_k\omega_l(z)\omega_k(z)=\sum_{l=0}^{\infty}a_l\sum_{k=0}^{N}b_k\sum_{i=\max(l.k)}^{l+k}\Omega_{k,l}^{(i)}\omega_i(z).$$

Let us split the summation in three parts as follows,

$$\begin{aligned}f\left(z\right)Q_N\left(z\right)=&\sum_{l=0}^{N}a_l\sum_{k=0}^{l}b_k\sum_{i=l}^{l+k}\Omega_{k,l}^{(i)}\omega_i(z)+\sum_{l=0}^{N}a_l\sum_{k=l+1}^{N}b_k\sum_{i=k}^{l+k}\Omega_{k,l}^{(i)}\omega_i(z)+\\&+\sum_{l=N+1}^{\infty}a_l\sum_{k=0}^{N}b_k\sum_{i=l}^{l+k}\Omega_{k,l}^{(i)}\omega_i(z).\end{aligned} \tag{4.5}$$

Then by changing the order of summations in each term of (4.5), we obtain

$$\begin{aligned}\sum_{l=0}^{N}a_l\sum_{k=0}^{l}b_k\sum_{i=l}^{l+k}\Omega_{k,l}^{(i)}\omega_i(z)=&\sum_{i=0}^{N}\sum_{k=0}^{(i-1)/2}\sum_{l=i-k}^{i}a_lb_k\Omega_{k,l}^{(i)}\omega_i(z)\\+\sum_{i=0}^{N}\sum_{k=(i+1)/2}^{i}\sum_{l=k}^{i}a_lb_k\Omega_{k,l}^{(i)}\omega_i(z)&+\sum_{i=N+1}^{2N}\sum_{k=i-N}^{(i-1)/2}\sum_{l=i-k}^{N}a_lb_k\Omega_{k,l}^{(i)}\omega_i(z)\\&+\sum_{i=N+1}^{2N}\sum_{k=(i+1)/2}^{N}\sum_{l=k}^{N}a_lb_k\Omega_{k,l}^{(i)}\omega_i(z)\end{aligned} \tag{4.6}$$

$$\sum_{l=0}^{N} a_l \sum_{k=l+1}^{N} b_k \sum_{i=k}^{l+k} \Omega_{k,l}^{(i)} \omega_i(z) = \sum_{i=0}^{N} \sum_{k=(i+1)/2}^{i} \sum_{l=i-k}^{k-1} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) + \sum_{i=N+1}^{2N} \sum_{k=(i+1)/2}^{N} \sum_{l=i-k}^{k-1} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z), \tag{4.7}$$

$$\sum_{l=N+1}^{\infty} a_l \sum_{k=0}^{N} b_k \sum_{i=l}^{l+k} \Omega_{k,l}^{(i)} \omega_i(z) = \sum_{i=N+1}^{2N} \sum_{k=0}^{i-N-1} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) + \sum_{i=N+1}^{2N} \sum_{k=i-N}^{N} \sum_{l=N+1}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) + \sum_{i=2N+1}^{\infty} \sum_{k=0}^{N} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z). \tag{4.8}$$

First assume i is odd. By combining (4.7) and the second and fourth term of (4.6), we get, taking into account (4.5) and (4.8) the equality

$$\begin{aligned} f(z)\, Q_N(z) = & \sum_{i=0}^{N} \sum_{k=0}^{i} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) + \sum_{i=N+1}^{2N} \sum_{k=(i+1)/2}^{N} \sum_{l=i-k}^{N} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) \\ & + \sum_{i=N+1}^{2N} \sum_{k=0}^{i-N-1} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) + \sum_{i=N+1}^{2N} \sum_{k=i-N}^{N} \sum_{l=N+1}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) \\ & + \sum_{i=2N+1}^{\infty} \sum_{k=0}^{N} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) + \sum_{i=N+1}^{2N} \sum_{k=i-N}^{(i-1)/2} \sum_{l=i-k}^{N} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) \end{aligned} \tag{4.9}$$

Combining second, third, fourth and sixth term of (4.9), we obtain;

$$\begin{aligned} f(z)\, .Q_N(z) = & \sum_{i=0}^{N} \sum_{k=0}^{i} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) + \sum_{i=N+1}^{2N} \sum_{k=0}^{N} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) + \\ & + \sum_{i=N+1}^{\infty} \sum_{k=0}^{N} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z). \end{aligned} \tag{4.10}$$

and combining second and third terms of (4.10), we finally get

$$f(z)\, Q_N(z) = \sum_{i=0}^{\infty} \sum_{k=0}^{N-(N-i)_+} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z) \ ,$$

where $x_+ = \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases}$. Similarly if i is even we will get the same result.

Now, we may add the following to definition 3.3

$$f(z).Q_N(z) = \sum_{i=0}^{\infty} \sum_{k=0}^{N-(N-i)_+} \sum_{l=i-k}^{i} a_l b_k \Omega_{k,l}^{(i)} \omega_i(z).$$

Definition 4.1. The Henkel-Newton determinant is defined by

$$H_{M,N}^{(i)} = \begin{vmatrix} \beta_{M+1,i} & \beta_{M+1,i+1} & \dots & \beta_{M+1,N} \\ \vdots & \vdots & & \vdots \\ \beta_{M+N,i} & \beta_{M+N,i+1} & \dots & \beta_{M+N,N} \end{vmatrix}.$$

Theorem 4.1.2. *Let $f(z) = \sum_{l=0}^{\infty} a_l \omega_l(z)$ be a formal Newton series with interpolation points $\{z_i\}$ and let M and N be (fixed) nonnegative integers. Then,*

(A) If

$$P_M(z) = c_0\omega_0(z) + c_1\omega_1(z) + \ldots + c_M\omega_M(z)$$

and

$$Q_N(z) = b_0\omega_0(z) + b_1\omega_1(z) + \ldots + b_N\omega_N(z),$$

then a necessary and sufficient condition that the formal Newton series $f(z).Q_N(z) - P_M(z)$ be of the form

$$f(z)\,Q_N(z) - P_M(z) = d_{N+M+1}\omega_{N+M+1} + d_{N+M+2}\omega_{N+M+2} + \ldots \tag{4.11}$$

is that the coefficients c_j and b_j satisfy the system of equations

$$b_0\beta_{0,0} + b_1\beta_{0,1} + \ldots + b_M\beta_{0,M} = c_0 \tag{4.12}$$

$$b_0\beta_{1,0} + b_1\beta_{1,1} + \ldots + b_M\beta_{1,M} = c_1 \tag{4.13}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$b_0\beta_{N,0} + b_1\beta_{N,1} + \ldots + b_M\beta_{N,M} = c_N$$

$$b_0\beta_{N+1,0} + b_1\beta_{N+1,1} + \ldots + b_M\beta_{N+1,M} = 0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \tag{4.14}$$

$$b_0\beta_{N+M,0} + b_1\beta_{N+M,1} + \ldots + b_M\beta_{N+M,M} = 0$$

where the $\beta_{i,j}$ are defined by (4.4).

(B) There exists a unique (up to equivalence $\tilde{}$) rational expression (P, Q) of type $[m, n]$, such that the formal Newton series

$$f(z)\, Q_N(z) - P_M(z)$$

has the form (4.11).

(C) A nontrivial solution, $c_0, \ldots, c_N, \quad b_0, \ldots, b_M$, to the system of equations (4.12) is determined uniquely (up to a nonzero multiplicative constant) if and only if the Henkel-Newton determinant $H^{(1)}_{M,N} \neq 0$ where

$$H^{(1)}_{M,N} = \begin{vmatrix} \beta_{M+1,1} & \beta_{M+1,2} & \dots & \beta_{M+1,N} \\ \vdots & \vdots & & \vdots \\ \beta_{M+N,1} & \beta_{M+N,2} & \dots & \beta_{M+N,N} \end{vmatrix}$$

Proof. By previous theorem, letting $c_k = 0$ for $k \geq N$, we obtain

$$f(z)\, Q_N(z) - P_M(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{M} b_j \beta_{k,j} - c_k \right) \omega_k(z),$$

of which (A) is an immediate consequence. To prove (B), we note that (4.14) is a homogeneous linear system of m equations in $(m+1)$ unknowns. Hence, there exist $b_0, \dots, b_M$, not all zero, satisfying (4.14). Having chosen such b_i , we choose the c_i to satisfy (4.12) and the resulting rational expression (P, Q) is of type $[M, N]$ and satisfies (4.11). To prove the uniqueness of (P, Q) , we let (P^*, Q^*) denote an arbitrary rational expression of type $[M, N]$ such that

$$f(z)\, Q^*_N(z) - P^*_M(z) = d^*_{N+M+1}\omega_{N+M+1} + d^*_{N+M+2}\omega_{N+M+2} + \dots \qquad (4.15)$$

By Theorem 1(C) in Gallucci and Jones paper ([Gallucci M. A., Jones W. B., 1976]), $Q(f.Q^* - P^*)$ and $Q^*(fQ - P)$ are both formal Newton series whose first $N+M+1$ coefficients are zero. Hence,

$$\begin{aligned} &Q^*(z)P(z) - Q(z)P^*(z) \\ &= Q(z)\left[f(z)\, Q^*(z) - P^*(z)\right] - Q^*(z)\left[f(z)\, Q(z) - P(z)\right] \end{aligned}$$

ia also a formal Newton series whose first $N + M + 1$ coefficients vanish. But $Q^*P - QP^*$ is a polynomial of degree at most $N+M$ and therefore must be identically zero. Thus, $(P^*, Q^*)\tilde{}(P, Q)$ which completes the proof of (B). In proof of part (C), first let us write the equality (4.11) as

$$f(z)(b_0\omega_0(z) + \ldots + b_n\omega_n(z)) - (c_0\omega_0(z) + \ldots + c_m\omega_m) \tag{4.16}$$

$$= \sum_{i=N+M+1}^{\infty} d_i\omega_i(z) \tag{4.17}$$

Now, if we set $b_0 = 0$ then substitution $z = z_1$ gives $c_0 = 0$. Thus, equation (4.16) becomes

$$f(z)(b_1\omega_1(z) + \ldots + b_n\omega_n(z)) - (c_1\omega_1(z) + \ldots + c_m\omega_m) \tag{4.18}$$

$$= \sum_{i=N+M+1}^{\infty} d_i\omega_i(z) \tag{4.19}$$

Dividing (4.18) by $\omega_1(z)$ we get the system of equations

$$\beta_{1,1}b_1 = c_1$$

$$\beta_{2,1} + \beta_{2,2}b_2 = c_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$\beta_{M,1} + \ldots + \beta_{M,N}b_N = c_M$$

$$\beta_{M+1,1} + \ldots + \beta_{M+1,N}b_N = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\beta_{M+N,1} + \ldots + \beta_{M+N,N}b_N = 0$$

$M+N$ equations with $M+N$ unknowns as previous. So, we may write the new equation as

$$f(z)(b_1+b_2\tilde{\omega}_2(z)\ldots+b_n\tilde{\omega}_n(z))-(c_1+c_2\tilde{\omega}_2(z)\ldots+c_m\tilde{\omega}_m)$$
$$=\sum_{i=N+M+1}^{\infty} d_i\tilde{\omega}_i(z)$$

where $\tilde{\omega}_{k-1}=\prod_{i=2}^{k}(z-z_i)$. Then, the system has no nontrivial solution. So, let us set $b_0\neq 0$. Set $b_0=1$, we get the following system of equations with respect to b_k and c_k :

$$-c_0=-\beta_{0,0}$$
$$\beta_{1,1}b_1-c_1=-\beta_{1.0}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\beta_{M,1}b_1+\beta_{M,2}b_2+\ldots+\beta_{M,N}b_N-c_M=-\beta_{M.0}$$
$$\beta_{M+1,1}b_1+\beta_{M+1,2}b_2+\ldots+\beta_{M+1,N}b_N=-\beta_{M+1.0}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\beta_{M+N,1}b_1+\beta_{M+N,2}b_2+\ldots+\beta_{M+N,N}b_N=-\beta_{M+N.0}$$

So, that system is solvable if and only if the determinant

$$D_{M,N} = \begin{vmatrix} 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ \beta_{1,1} & 0 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M,1} & \beta_{M,2} & \dots & \beta_{M,N} & 0 & 0 & \dots & -1 \\ \beta_{M+1,1} & \beta_{M+1,2} & \dots & \beta_{M+1,N} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M+N,1} & \beta_{M+N,2} & \dots & \beta_{M+N,N} & 0 & 0 & \dots & 0 \end{vmatrix}$$

does not vanish.

$$D_{M,N} = \begin{vmatrix} 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ \beta_{1,1} & 0 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M,1} & \beta_{M,2} & \dots & \beta_{M,N} & 0 & 0 & \dots & -1 \\ \beta_{M+1,1} & \beta_{M+1,2} & \dots & \beta_{M+1,N} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M+N,1} & \beta_{M+N,2} & \dots & \beta_{M+N,N} & 0 & 0 & \dots & 0 \end{vmatrix}$$

is equal to

$$(-1).(-1)^{2M+N+2}. \begin{vmatrix} 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ \beta_{1,1} & 0 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M-1,1} & \beta_{M-1,2} & \dots & \beta_{M-1,N} & 0 & 0 & \dots & -1 \\ \beta_{M+1,1} & \beta_{M+1,2} & \dots & \beta_{M+1,N} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M+N,1} & \beta_{M+N,2} & \dots & \beta_{M+N,N} & 0 & 0 & \dots & 0 \end{vmatrix}$$

$$= (-1)^{2M+N+2}.(-1)^{2M+N}.\begin{vmatrix} 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ \beta_{1,1} & 0 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M-2,1} & \beta_{M-2,2} & \dots & \beta_{M-2,N} & 0 & 0 & \dots & -1 \\ \beta_{M+1,1} & \beta_{M+1,2} & \dots & \beta_{M+1,N} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M+N,1} & \beta_{M+N,2} & \dots & \beta_{M+N,N} & 0 & 0 & \dots & 0 \end{vmatrix}$$

by same manner if we continue it will be equal to

$$(-1)^{M+N}.\begin{vmatrix} \beta_{M+1,1} & \beta_{M+1,2} & \dots & \beta_{M+1,N} \\ \vdots & \vdots & & \vdots \\ \beta_{M+N,1} & \beta_{M+N,2} & \dots & \beta_{M+N,N} \end{vmatrix}$$

$$= (-1)^{M+N} H^{(1)}_{M,N}$$

By Cramer's rule, if $H_{M,N} \neq 0$ then b_k may be represented in the form

$$b_k = \frac{(-1)^{M+N}.\begin{vmatrix} \beta_{M+1,1} & \dots & \beta_{M+1,k-1} & -\beta_{M+1,0} & \beta_{M+1,k+1} & \dots & \beta_{M+1,N} \\ \beta_{M+2,1} & \dots & \beta_{M+2,k-1} & -\beta_{M+2,0} & \beta_{M+2,k+1} & \dots & \beta_{M+2,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M+N,1} & \dots & \beta_{M+N,k-1} & -\beta_{M+N,0} & \beta_{M+N,k+1} & \dots & \beta_{M+N,N} \end{vmatrix}}{(-1)^{M+N}.H^{(1)}_{M,N}}$$

$$= \frac{\begin{vmatrix} \beta_{M+1,1} & \dots & \beta_{M+1,k-1} & -\beta_{M+1,0} & \beta_{M+1,k+1} & \dots & \beta_{M+1,N} \\ \beta_{M+2,1} & \dots & \beta_{M+2,k-1} & -\beta_{M+2,0} & \beta_{M+2,k+1} & \dots & \beta_{M+2,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta_{M+N,1} & \dots & \beta_{M+N,k-1} & -\beta_{M+N,0} & \beta_{M+N,k+1} & \dots & \beta_{M+N,N} \end{vmatrix}}{H^{(1)}_{M,N}}.$$

Hence,

$$Q_N(z) = \frac{\begin{vmatrix} \omega_0(z) & \omega_1(z) & \dots & \omega_N(z) \\ \beta_{M+1,0} & \beta_{M+1,1} & \dots & \beta_{M+1,N} \\ \vdots & \vdots & & \vdots \\ \beta_{M+N,0} & \beta_{M+N,1} & \dots & \beta_{M+N,N} \end{vmatrix}}{H^{(1)}_{M,N}}.$$

From the representation for $Q_N(z)$ in (4.3) we get an analytic formula for $P_M(z)$ such that $P_M(z) = -\sum_{k=0}^{M} e_{k,N}\omega_k(z)$.

So, we have proved that the interpolation Newton-Padé problem is uniquely solvable if and only if the Henkel-Newton determinant does not vanish.

4.2. Conditions for the Convergence

Theorem 4.2.1. *Let us suppose that $\{\mu_n\}, \{\nu_n\}$ are arbitrary increasing sequences such that*

$$\begin{cases} \mu_n \le \mu_{n+1} \le \mu_n + 1, \\ \nu_n \le \nu_{n+1} \le \nu_n + 1, \\ (\mu_{n+1} - \mu_n) + (\nu_{n+1} - \nu_n) \ge 1. \end{cases} \tag{4.20}$$

If there exists a natural number N such that the series

$$\sum_{h=N}^{\infty} (-1)^{\mu_n} \frac{H^{(0)}_{\mu_n+1,\nu_n+1}}{H^{(1)}_{\mu_n,\nu_n}} \omega_{\mu_n+\nu_n+1}(z) \frac{1}{Q_{\nu_n}(z)\, Q_{\nu_{n+1}}(z)} \tag{4.21}$$

converges uniformly in a domain G which contains points $z_1, z_2, \dots,$ then the sequence $\{R_{\mu_n,\nu_n}(z)\}_{n=1}^{\infty}$ converges in G and uniformly to $f(z)$.

Proof. Consider the expression

$$P_{\mu_{n+1}}(z)\,Q_{\nu_n}(z) - P_{\mu_n}(z)\,Q_{\nu_{n+1}}(z) = \left[f(z)\,Q_{\nu_n}(z) - P_{\mu_n}(z)\right]Q_{\nu_{n+1}}(z) - \left[f(z)\,Q_{\nu_{n+1}}(z) - P_{\mu_{n+1}}(z)\right]Q_{\nu_n}(z) \tag{4.22}$$

Left hand side is a polynomial of $\omega_k(z)$ for $k \leq \mu_n + \nu_n + 1$. Second term in right hand side by definition of Newton-Padé approximants starts from the term $\omega_k(z)$ with $k \geq \mu_{n+1} + \nu_{n+1} + 1$ $(> \mu_n + \nu_n + 1)$, and first term starts with term $\omega_k(z)$ with $k \geq \mu_n + \nu_n + 1$. Then taking $z_{1,} z_{2,} \ldots,$ for z values (if a point z_j is a repetation of z_i then instead of values of expressions taking $k-$th derivatives at the point z_i), we get that in both sides of (4.22) there is only one term with $\omega_{\mu_n+\nu_n+1}(z)$.

The coefficient of $\omega_{\mu_n+\nu_n+1}(z)$ in $P_{\mu_{n+1}}(z)\,Q_{\nu_n}(z) - P_{\mu_n}(z)\,Q_{\nu_{n+1}}(z)$ is equal to the coefficient of $\omega_{\mu_n+\nu_n+1}(z)$ in the expression of $f(z)\,Q_{\nu_n}(z)$, and that last coefficient is equal to

$$e_{\mu_n+\nu_n+1,\nu_n} = \sum_{k=0}^{\nu_n} \beta_{\mu_n+\nu_n+1,k} b_k.$$

Computing this term from the right hand side of (4.22), we get

$$P_{\mu_{n+1}}(z)\,Q_{\nu_n}(z) - P_{\mu_n}(z)\,Q_{\nu_{n+1}}(z) = (-1)^{\nu_n} \frac{H^{(0)}_{\mu_n+1,\nu_n+1}}{H^{(1)}_{\mu_n,\nu_n}} \omega_{\mu_n+\nu_n+1}(z) \tag{4.23a}$$

Now comparing that expression with the expression for $Q_{\nu_n}(z)$ it is easy to see the validity of formula (4.23a). Hence,

$$R_{\mu_{n+1},\nu_{n+1}}(z) - R_{\mu_n,\nu_n}(z) = (-1)^{\nu_n} \frac{H^{(0)}_{\mu_n+1,\nu_n+1}}{H^{(1)}_{\mu_n,\nu_n}} \omega_{\mu_n+\nu_n+1}(z) \frac{1}{Q_{\nu_n}(z)\,Q_{\nu_{n+1}}(z)}$$

From here it is clear if series (4.21) converges at a point z, then at that point the sequence $\{R_{\mu_n,\nu_n}(z)\}$ converges too. Denote the limit function by $F(z)$. From the condition of theorem it follows that $Q_{\nu_n}(z) \neq 0$ in G, starting from some number N. Hence, all functions $R_{\mu_n,\nu_n}(z)$ are analytic in G for $n \geq N$. Since the sequence

$\{R_{\mu_n,\nu_n}\}$ converges uniformly on compacts in G, function $F(z)$ is also analytic in G. Moreover, for any point $z \in G$

$$\lim_{n\to\infty} \frac{d^k R_{\mu_n,\nu_n}(z)}{dz^k} = \frac{d^k F(z)}{dz^k}$$

for any natural number k. Taking $z_1, z_2, \ldots,$ for z values, we get that $F(z_j) = f(z_j)$, $j = 1, 2, \ldots$.

Using that theorem it is possible to prove the following criteria of convergence of Newton-Padé approximants.

Theorem 4.2.2. *Let D be the convergence domain for the series*

$$\sum_{n=1}^{\infty} \frac{\left|H^{(0)}_{\mu_n+1,\nu_n+1}\right|}{\left|H^{(1)}_{\mu_n,\nu_n}\right|} \left|\omega_{\mu_n+\nu_n+1}(z)\right|, \qquad \omega_n(z) = \prod_{k=0}^{n} (z - z_k).$$

Let in some domain $G \subset D$, D containing points $z_1, z_2, \ldots,$ denominators $Q_{\nu_n}(z)$, $n = 1, 2, \ldots$ of the sequence of Newton-Padé fractions $\{R_{\mu_n,\nu_n}(z)\}_{n=1}^{\infty}$ are uniformly bounded by modulus from below by a positive constant. Then, the sequence $\{R_{\mu_n,\nu_n}(z)\}_{n=1}^{\infty}$ converges in G to the function $f(z)$ generating the table and uniformly in arbitrary bounded closed domain $F \subset G$.

Proof. Since

$$\left|R_{\mu_{n+1},\nu_{n+1}}(z) - R_{\mu_n,\nu_n}(z)\right| = \left|(-1)^{\mu_n} \frac{H^{(0)}_{\mu_n+1,\nu_n+1}}{H^{(1)}_{\mu_n,\nu_n}} \omega_{\mu_n+\nu_n+1}(z) \frac{1}{Q_{\nu_n}(z) Q_{\nu_{n+1}}(z)}\right|$$
$$\leq \frac{1}{m^2} \frac{\left|H^{(0)}_{\mu_n+1,\nu_n+1}\right|}{\left|H^{(1)}_{\mu_n,\nu_n}\right|} \left|\omega_{\mu_n+\nu_n+1}(z)\right|,$$

then from the condition of theorem it follows that the series converges absolutely in G and uniformly in arbitrary compact domain F.

Hence, the result follows.

CHAPTER 5

CONCLUSION

This thesis is devoted to multipoint Padé approximants, Newton-Padé approximants and their convergence. The followings were studied.

- Padé approximations for exponential functions (review)
- Convergence of Padé tables (review)
- Multipoint Padé approximations and Newton-Padé approximations (review)
- Algorithms for computation of Newton-Padé approximations (review)
- Determinantal representations of Newton-Padé approximations and their applications (original)

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