NONHOMOGENEOUS WAVE PROPAGATION

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

In this study, we examine the wave equation, which is a second-order linear differential equation of the hyperbolic type, specifically in Functionally Graded Materials (FGMs) using Adomian Decomposition Method (ADM) that has been used to solve linear and nonlinear functional equations. The Adomian decomposition method is explained extensively and its applications on different types of application problems are given in examples. The main purpose of this study to apply ADM for the homogeneous wave equation with variable coefficients in the nonhomogeneous material known as FGMs. Since boundary conditions are nonhomogeneous, problem is separated into two problems and solved by superposition method: Homogeneous (eigenvalue problem) and nonhomogeneous wave problem with homogenous BCs. It is used the generalized Fourier series expansion to solve the nonhomogeneous problem using eigenfunction expansion method. At the end of the problem, solution method and the numerical coding are checked for homogeneous type materials by the close form solution. Then, for different types of nonhomogeneity parameter, results are represented graphically.

Keywords: Adomian polynomials, Adomian Decomposition Method, Functionally Graded Materials, Eigenfunction Expansion Method, Wave Equation.

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ÖZ

Bu çalışmada fonksiyonel derecelendirilmiş malzemelerde hiperbolik ikinci derece lineer diferansiyel dalga denklemini lineer ve lineer olmayan fonksiyonel denklemleri çözmek için kullanılan Adomian dekompozisyon metodu kullanarak inceliyoruz. Adomian dekompozisyon metodu geniş olarak açıklanır ve uygulamaları farklı uygulama problemleri üzerinden örneklerle verilir. Bu çalışmanın asıl amacı fonksiyonel derecelendirilmiş malzemeler olarak bilinen homojen olmayan bir malzemede değişken katsayılı homojen dalga denklemine Adomian dekompozisyon metodunu uygulamaktır. Sınır şartları homojen olmadığı için, problem iki probleme ayrılır ve süperpozisyon metodu ile çözülür: Homojen sınır şartları ile homojen(özdeğer problemi) ve homojen olmayan dalga denklemi. Homojen olmayan problemi çözmek için genelleştirilmiş Fourier seri açılımı kullanılır. Problemin sonunda, homojen malzemeler için çözüm metodu ve sayısal kodlama kapalı form çözümü ile kontrol edilir. Sonra, farklı homojen olmayan parametreler için sonuçlar grafikle gösterilir.

Anahtar Kelimeler: Adomian polinomları, Adomian Dekompozisyon Metodu, Fonksiyonel Derecelendirilmiş Malzememeler, Özdeğer Problemleri, Dalga Denklemi.

DEDICATION

To my parents

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I express sincere appreciation to Asist. Prof. Ali SAHİN for his guidance, understanding, motivation and patience.

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INTRODUCTION

In 1972, the general idea of structural gradients first was advanced for composites and polymeric materials. Bever and Duwez examined various types of gradients composites and their properties. They proposed that composites materials may have had gradients, which took place in local properties of the composite, in their structural of characteristics. Therefore, in engineering applications, it was very interested in gradient composites (Bever and Duwez, 1972). Various models were suggested for gradients in composition, in filament concentration, and in polymerization along with possible applications for the resulting graded structures. However, there was no actual investigation about how to design, fabricate, and evaluate graded structures until the 1980s.

In 1985, the use of continuous texture control was proposed in order to increase the adhesion strength and minimize the thermal stresses in the ceramic coatings and joints being developed for the reusable rocket engine. The developers realized that this continuous control of a property could be extended to the more general concept that could be applied to impart new properties and functions to any material by gradually changing its texture or composition. At this time, the concept of the material ingredient was introduced for designing such materials.

In 1986, these types of materials were termed functionally gradient materials, which soon became abbreviated to the now familiar, FGM. In 1995, as a consequence of a discussion at the third international symposium on FGMs held in Lausanne in 1994, it was decided to change the full name to functionally graded materials.

Functionally Graded Material (FGM) is a new type of material concept, which is developed and widely used to reduce the thermal stresses in space craft and the next generation fission reactors applications. Because of its greatest advantage by having graded functions of toughness and other material properties though metal/ceramic composition of the body, FGM became a key technology in material science.

In Functionally Graded Materials (FGMs) both the composition and the structure gradually change over the volume, resulting in corresponding changes in the properties of the material. It is a conceptual unit for constructing the FGM that includes various aspect of its chemical composition physical state and geometrical configuration. Material ingredients, which probably express the overall concept best, can resemble biological units such as cells and tissues. For example; bamboo, shell, tooth and bone all have graded structures consisting of biological material ingredients (Hirai, 1996), (Mortensen and Suresh, 1995), (Mortensen and Suresh, 1998). Also these graded structures are investigated by many researchers and given some studies. Nogata and Takahashi have studied about the ingenious construction of bamboo which is a self-optimizing graded structure constructed with a cell based sensing system for external mechanical stimuli and to help in the understanding of the principles and design in biological materials which are functionally graded composites. Also, it was seen that a bamboo can generate electrical signals (Nagata and Takahashi, 1995). Woo and Meguid (2001) provide an analytic solution for shallow shells made of FGMs under transverse mechanical loads and a temperature field. The basic equations are obtained for shallow shells made of FGM using the von Karman theory and solved using the Fourier series (Woo and Meguid, 2001). Zhao and Liew (2009) studied the nonlinear functionally graded ceramic-metal shell panels under mechanical and thermal loading. They examined the characteristic of the displacement and the axial stress in panels under mechanical and thermal loading, also material properties for the nonlinear response of shell panels.

Many studies are made about stress, deformation, stability and vibration problems of FGM beams, plates and shells. The problem of thermal stress distributions in a FGM beam are solved by Sankar and Tzang using a simple Bernoulli-Euler beam theory developed (Sankar and Tzeng, 2002). Jin studied about the problem of transient heat transfer in a FGM strip. The asymptotic solution of the problem was obtained in closed form by subdividing the strip into a number of homogeneous layers (Jin, 2002). Transient heat transfer problems are considered by Ootao and Tanigawa. They analyzed the transient one-dimensional temperature distribution using the method of Laplace transformation (Ootao and Tanigawa, 2004). Chen and Zou solved same problem using Galerkin boundary element method (Chen et al., 2002). Chen and Tong analyzed the sensitivity of the steady-state and transient heat conduction of FGMs using the finite element method (Chen and Tong, 2004).

The other studies are about the solution of the problem of wave propagation in FGM. One of them is acoustic wave propagation in FGM plates which were studied by Lefebvre et al. and the problem of wave was solved by using Legendre polynomials (Lefebvre et al., 2001). Also, another numerical integral technique based on confluent hypergeometric functions (CHFs) is proposed by Liu et al. for the problem of wave propagation in FGM structure. Using this technique, a part of integrand in the integral is estimated by piecewise polynomials (Liu et al., 2001). Analysis of two-dimensional stress wave propagation problems in FGMs which have two distinct models, which are considered as layered metal-ceramic and randomly distributed ceramic particles, was presented by Berezovski et al. They compared the models and showed the differences between characteristics of wave fields in the distinct models applying composite wave-propagation algorithm (Berezovski et al., 2003). In a cylinder made of FGM, problems of transient waves are analyzed presenting a method known as a hybrid numerical method. This method is applied to examine FGM cylinders, and its efficiency is proved (Han et al., 2000).

The most familiar FGM is compositionally graded from a refractory ceramic to a metal. It can incorporate incompatible functions such as the heat, wear, and oxidation resistance of ceramic with the high toughness, high strength, machinability, and bonding capability of metals without severe internal thermal stress.

Pores are also important material ingredients of FGMs. A gradual increase in the pore distribution from the interior to the surface can impart many properties such as mechanical shock resistance, thermal insulation, catalytic efficiency, and the relaxation of thermal stress.

There are many applications of FGMs concept in various research and industrial fields. In the engineering applications to cutting tools, machine parts, and engine components, incompatible functions such as heat, wear, and corrosion resistance plus toughness, and machinability are incorporated into a single part. For example, throwaway chips for cutting tools made of graded tungsten carbide/cobalt and titanium carbonitride have been developed and commercialized that incorporate the desirable properties of high machining speed, high feed rates, and long life. Various combinations of these ordinarily incompatible functions can be applied to create new materials for aerospace, chemical plants, and nuclear energy reactors.

The FGM concept is also applicable to functional materials. The application of FGMs to biomaterials is growing in importance. Over 2500 surgical operations to incorporate graded hip prostheses have been successfully performed in Japan over the past twelve years (Miyamoto, 1999).

As a consequence; USA and mostly Japan in which development and research of FGMs as functional graded materials are carried on (Koizumi, 1997) have given interest to studies of FGMs. Many researchers and developments projects which made since 1980’s proved the majority of FGM compared to ordinary materials.

CHAPTER 2

WAVE EQUATION IN MATERIALS

2.1. WAVE EQUATIONS

In this part of the study, it will be examined the wave equation which is a second-order linear partial differential equation of the hyperbolic type. The wave equation describes the shape and the movement of waves which are given a set of boundary conditions such as the initial shape of the wave, or the evolution of a force affecting the wave. The solution of the wave equation can be obtained exactly by D'Alembert's method, or using a Fourier transform method, or via separation of variables. In the following sections, it will be derived one-dimensional and two-dimensional wave equations and, using separation of variables method, the solution of some physical applications will be shown.

2.2. DERIVATION OF THE ONE-DIMENSIONAL WAVE EQUATION

Let $u = u(x,t)$ be the solution of the wave equation. Then, the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

represents displacement along the $x-$axis with respect to time. To obtain the partial differential equation (2.1), we consider a small piece of the vibrating string. Let T_1 and T_2 be the tensions at the points P and Q of the string. The horizontal components of the tension must be constant because there is no motion in horizontal direction. Thus, we determine by using the notation shown in Fig.1.

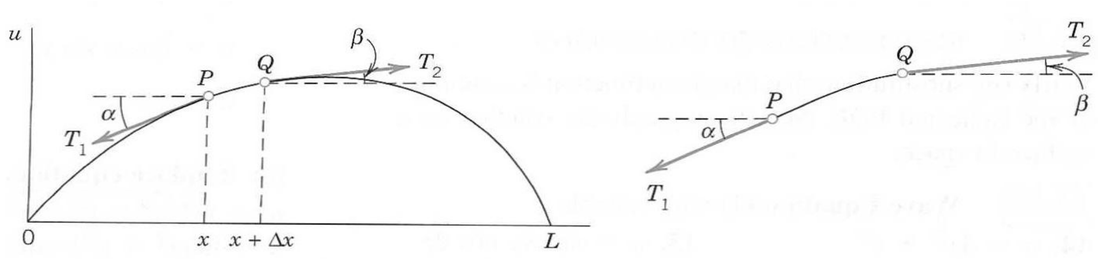

Figure 1: Deflected at fixed time t

$$\sum_x F = 0 \quad \Rightarrow \quad T_2 \cos\beta - T_1 \cos\alpha = 0 \quad \Rightarrow \quad T_2 \cos\beta = T_1 \cos\alpha = T = \text{constant.}$$

In vertical direction we have three forces, one of them is gravity force and the others are vertical components of the tension. By Newton's second law, it can be written that

$$\sum_u F = ma$$

where m is the mass and a is the acceleration. Let w denote the linear mass (kg/m), Δx the change in position (m) and g the gravity force $\left(m/s^2\right)$. We get the following

$$T_2 \sin\beta - T_1 \sin\alpha - wg\left(\Delta x\right) = ma \;\; \Rightarrow \; T_2 \sin\beta - T_1 \sin\alpha - wg(\Delta x) = w(\Delta x)\frac{\partial^2 u}{\partial t^2},$$

$$\Rightarrow \; T(\tan\beta - \tan\alpha) - wg(\Delta x) = w(\Delta x)\frac{\partial^2 u}{\partial t^2} \qquad (2.2)$$

since

$$\tan\beta = \frac{T_2 \sin\beta}{T_2 \cos\beta} = \frac{T_2 \sin\beta}{T} \;\Rightarrow\; T_2 \sin\beta = T \tan\beta,$$

$$\tan\alpha = \frac{T_1\sin\alpha}{T_1\cos\alpha} = \frac{T_1\sin\alpha}{T} \;\Rightarrow\; T_1\sin\alpha = T\tan\alpha\ .$$

Here $\tan\alpha$ and $\tan\beta$ are the slopes of the curve of the string x and $x+\Delta x$, that is

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x \quad \text{and} \quad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}.$$

Dividing Equation (2.2) by Δx, we obtain

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] - wg = w\frac{\partial^2 u}{\partial t^2}\ . \tag{2.3}$$

If we let Δx approach zero,

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] = \frac{\partial^2 u}{\partial x^2} \tag{2.4}$$

we get

$$T\frac{\partial^2 u}{\partial x^2} - wg = w\frac{\partial^2 u}{\partial t^2} \tag{2.5}$$

and dividing both sides by w

$$\frac{T}{w}\frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}\ . \tag{2.6}$$

Since $\frac{T}{w} \gg g$, $g \approx 0$, then

$$\frac{T}{w}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\,. \tag{2.7}$$

Therefore, we can rewrite Equation (2.7) as

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{2.8}$$

where $\frac{T}{w} = c^2$. The notation c^2 for the physical constant $\frac{T}{w}$ has been chosen to indicate that this constant is positive.

2.3. DERIVATION OF TWO-DIMENSIONAL WAVE EQUATION

Let us consider the motion of a stretched membrane to derive two-dimensional wave equation. Our derivation will be similar as the case of the vibrating string. We consider the forces acting on a small part of the membrane as shown in Fig. 2. The tension T is the force per unit length. We first calculate the horizontal components of the forces. These components are obtained as

$$-T\Delta x\cos\theta, \quad T\Delta x\cos\eta, \quad -T\Delta y\cos\alpha, \quad T\Delta y\cos\beta.$$

Here the angles are very small, so their cosines are close to 1. Hence the horizontal components at opposite sides are approximately equal. Therefore, we can neglect the motion in horizontal direction since the values of them are very small. The vertical components of the forces are determined which are parallel to the $yu-$plane are determined as

$$T\Delta y\sin\beta \qquad \text{and} \qquad -T\Delta y\sin\alpha.$$

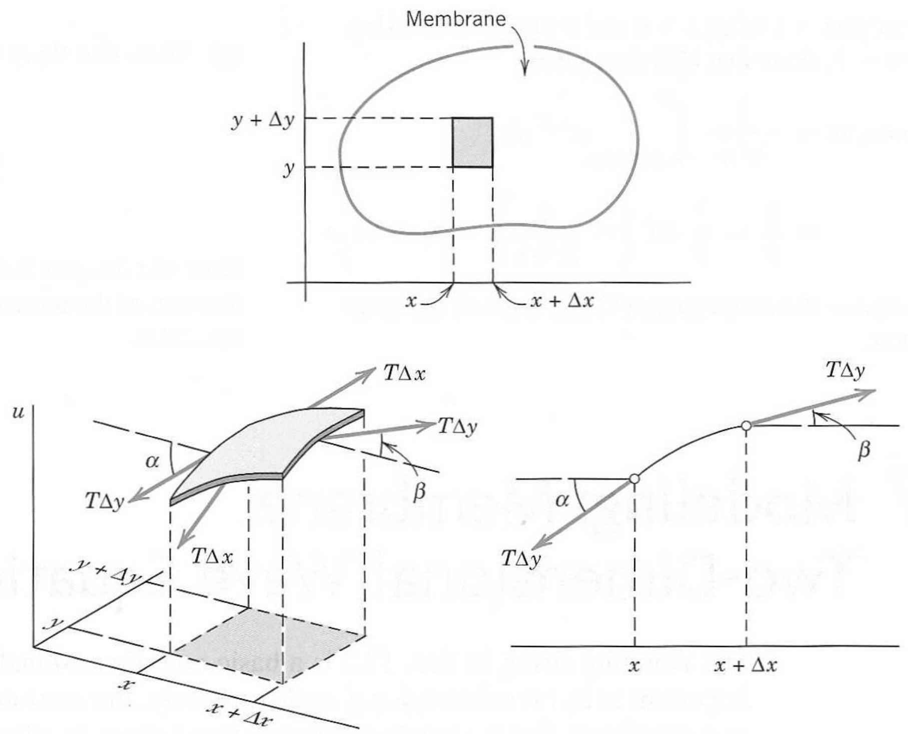

Figure 2: Vibrating membrane

We can replace the sine of the vertical components by their tangents because of the angles which are so small. Hence the result becomes like

$$T\Delta y(\sin\beta-\sin\alpha)\approx T\Delta y(\tan\beta-\tan\alpha),$$

$$=T\Delta y\left[\left(\frac{\partial u}{\partial x}\right)_{(x+\Delta x,y_1)}-\left(\frac{\partial u}{\partial x}\right)_{(x,y_2)}\right] \tag{2.9}$$

where subscripts x denote partial derivatives and values of y_1 and y_2 are between y and $y+\Delta y$. Similarly, we can calculate the other vertical components which are parallel to the $xu-$plane. The result is shown as

$$T\Delta x\left[\left(\frac{\partial u}{\partial y}\right)_{(x_1,\Delta y+y)}-\left(\frac{\partial u}{\partial y}\right)_{(x_2,y)}\right] \tag{2.10}$$

where x_1 and x_2 are between x and $x+\Delta x$.

By Newton's second law the sum of the forces are equal to the mass $w\Delta A$ $(\Delta A=\Delta x\Delta y)$ times the acceleration $\partial^2 u/\partial t^2$:

$$w(\Delta x\Delta y)\frac{\partial^2 u}{\partial t^2}=T\Delta y\left[\left(\frac{\partial u}{\partial x}\right)_{(x+\Delta x,y_1)}-\left(\frac{\partial u}{\partial x}\right)_{(x,y_2)}\right]+T\Delta x\left[\left(\frac{\partial u}{\partial y}\right)_{(x_1,\Delta y+y)}-\left(\frac{\partial u}{\partial y}\right)_{(x_2,y)}\right]$$

dividing by $w\Delta x\Delta y$

$$\frac{\partial^2 u}{\partial t^2}=\frac{T}{w}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{(x+\Delta x,y_1)}-\left(\frac{\partial u}{\partial x}\right)_{(x,y_2)}}{\Delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{(x_1,\Delta y+y)}-\left(\frac{\partial u}{\partial y}\right)_{(x_2,y)}}{\Delta y}\right]. \tag{2.11}$$

If we let Δx and Δy approach zero, we obtain the two-dimensional wave equation as

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2.12}$$

where $c^2 = T/w$.

2.4. SOLUTIONS OF WAVE EQUATIONS

Example 1: Consider the homogeneous wave equation with Dirichlet BCs

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L\,, \quad t > 0\,,$$

$$\text{BCs: } u(0,t) = 0\,, \qquad u(L,t) = 0\,,$$

$$\text{ICs: } u_t(x,0) = g(x), \quad u(x,0) = f(x)\,.$$

This problem can be solved by using separation of variables method since partial differential equation (PDE) and BCs are homogeneous. Starting with the definition of separation of variable method

$$u(x,t) = X(x)T(t)$$

and substituting it into PDE such as

$$T''(t)X(x) = c^2 X''(x)T(t),$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = \lambda$$

where λ is a constant ratio. The BVP on spatial coordinate can be solved in terms of the value of λ .

***CASE 1*:** For $\lambda > 0$, then say $\lambda = \mu^2$ and the solution of the second order ordinary differential equation

$$X''(x) - \mu^2 X(x) = 0\,, \qquad X(0) = X(L) = 0$$

can be solved as follows:

$$m^2 - \mu^2 = 0 \Rightarrow m = \pm\mu$$

$$X(x) = Ae^{\mu x} + Be^{-\mu x} \quad \Rightarrow \quad X(0) = A + B = 0 \Rightarrow A = -B,$$

$$\Rightarrow \quad X(L) = Ae^{\mu L} + Be^{-\mu L} = 0,$$

$$\Rightarrow \quad -Be^{\mu L} + Be^{-\mu L} = 0,$$

$$\Rightarrow \quad B\left(e^{-\mu L} - e^{\mu L}\right) = 0 \quad \text{here} \quad B = 0 \;\vee\; \left(e^{-\mu L} - e^{\mu L}\right) = 0.$$

But $\left(e^{-\mu L} - e^{\mu L}\right)$ is never zero because of exponential. So $B = 0$ and $A = 0$. We can say it is a trivial solution.

***CASE 2*:** For $\lambda = 0$**,**

$$X''(x) = 0 \quad \Rightarrow \quad X(x) = Cx + D, \qquad X(0) = X(L) = 0,$$

$$X(0) = D = 0, \; X(L) = CL = 0 \quad \Rightarrow L \neq 0$$

So, we obtain trivial solution.

***CASE 3*:** For $\lambda < 0$, let us say $\lambda = -\mu^2$

$$X''(x) + \mu^2 X(x) = 0, \qquad X(0) = X(L) = 0.$$

The solution can be obtained as follows:

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

after using the BCs, we can obtain eigenvalues and eigenfunctions, respectively, as

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \sin\left(\frac{n\pi}{L}x\right), \qquad n = 1, 2, 3, \ldots$$

Then the solution for $T''(t) + \mu^2 c^2 T(t) = 0$ can be written as

$$T_n(t) = C_n \cos\left(\frac{n\pi}{L}ct\right) + D_n \sin\left(\frac{n\pi}{L}ct\right).$$

The solution for each n is given by

$$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right)\left\{C_n \cos\left(\frac{n\pi}{L}ct\right) + D_n \sin\left(\frac{n\pi}{L}ct\right)\right\}$$

using superposition, the general solution can be obtained as

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left\{c_n \cos\left(\frac{n\pi}{L}ct\right) + d_n \sin\left(\frac{n\pi}{L}ct\right)\right\}$$

where $c_n = B_n C_n$, $d_n = B_n D_n$.

We use the initial conditions for determining the coefficients c_n and d_n

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow \quad c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx,$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} d_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow \quad \frac{cn\pi}{L} d_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx,$$

$$\Rightarrow \quad d_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx \ .$$

The solution is

$$u(x,t) = u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left\{\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right)\cos\left(\frac{n\pi}{L}ct\right)\right.$$
$$\left. + \left(\frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx\right)\sin\left(\frac{n\pi}{L}ct\right)\right\}$$

Example 2: Consider the homogeneous wave equation with Neumann type BCs

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L, \quad t > 0,$$

BCs: $u_x(0,t) = 0, \qquad u_x(L,t) = 0,$

ICs: $u_t(x,0) = g(x), \quad u(x,0) = f(x).$

Using separation of variables method

$$u(x,t) = X(x)T(t),$$

and substituting it into PDE like

$$T''(t)X(x) = c^2 X''(x)T(t),$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = \lambda$$

where λ is a constant ratio.

CASE 1: For $\lambda > 0$, then there is a trivial solution.

CASE 2: For $\lambda = 0$,

$$X''(x) = 0 \Rightarrow X_0(x) = Ax + B, \qquad X'(0) = X'(L) = 0,$$

$X_0(x) = B$,arbitrary constant, that can be chosen as 1.

The solution is $X_0(x) = 1$.

CASE 3: For $\lambda < 0$, let us say $\lambda = -\mu^2$ and the solution will be obtained as follows:

$$X''(x) + \mu^2 X(x) = 0, \qquad X_x(0) = X_x(L) = 0,$$

$$X(x) = a\cos(\mu x) + b\sin(\mu x)$$

after using the boundary conditions we can obtain eigenvalues and eigenfunctions, respectively, as

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \cos\left(\frac{n\pi}{L}x\right), \qquad n = 1, 2, 3, \ldots$$

Then the solution for $T''(t) + \mu^2 c^2 T(t) = 0$ can be written as

$$T_n(t) = C_n \cos\left(\frac{n\pi}{L}ct\right) + D_n \sin\left(\frac{n\pi}{L}ct\right).$$

The solution for each n is

$$u_n(x,t) = 1(C_0 t + D_0) + a_n \cos\left(\frac{n\pi}{L}x\right)\left\{C_n \cos\left(\frac{n\pi}{L}ct\right) + D_n \sin\left(\frac{n\pi}{L}ct\right)\right\}.$$

Using superposition method, the solution is

$$u(x,t) = (C_0 t + D_0) + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left\{c_n \cos\left(\frac{n\pi}{L}ct\right) + d_n \sin\left(\frac{n\pi}{L}ct\right)\right\}$$

where $c_n = a_n C_n$ and $d_n = a_n D_n$.

To determine the coefficients C_0, D_0, c_n and d_n, let us use the initial conditions

$$u(x,0) = f(x) = D_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow \; D_0 = \frac{2}{L}\int_0^L f(x)\,dx,$$

$$\Rightarrow \; c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx,$$

$$u_t(x,0) = g(x) = C_0 + \sum_{n=1}^{\infty} \frac{cn\pi}{L} d_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow \; C_0 = \frac{2}{L}\int_0^L g(x)\,dx,$$

$$\Rightarrow\; d_n = \left(\frac{L}{cn\pi}\right)\frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi}{L}x\right)dx .$$

The general solution of the Neumann problem is given as

$$u(x,t) = \frac{2}{L}\left(t\int_0^L g(x)\,dx + \int_0^L f(x)\,dx\right)$$

$$+\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{L}x\right)\left\{\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right)\cos\left(\frac{n\pi}{L}ct\right)\right.$$
$$\left. +\left(\frac{2}{cn\pi}\int_0^L g(x)\cos\left(\frac{n\pi}{L}x\right)dx\right)\sin\left(\frac{n\pi}{L}ct\right)\right\}$$

Example 3: Consider a wave equation on a plane

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right) \qquad 0 < x < a, \quad 0 < y < b, \quad t > 0 .$$

BCs: $u(0,y,t) = 0, \quad u(a,y,t) = 0 \qquad$ for $\quad 0 \le y \le b, \quad t \ge 0,$

$u(x,0,t) = 0, \quad u(x,b,t) = 0 \qquad$ for $\quad 0 \le x \le a, \quad t \ge 0 .$

ICs: $u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y) .$

Defining the separation of variables method

$$u(x,y,t) = X(x)Y(y)T(t)$$

and substituting it into PDE like

$$XYT'' = c^2\left(X''YT + XY''T\right),$$

it will be obtained as

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y} = \alpha .$$

The both sides must be equal to a constant. We consider negative separation constants only

$$\frac{X''}{X} = -\mu^2, \qquad \frac{Y''}{Y} = -\lambda^2 \quad \text{where} \quad \alpha = -\mu^2 - \lambda^2 .$$

We solve two Sturm-Liouville problems and obtain their eigenvalues and eigenfunctions, respectively, as follows:

$$X'' + \mu^2 X = 0, \qquad X(0) = X(a) = 0$$

with eigenvalues and eigenfunctions, respectively, as

$$\mu^2 = \left(\frac{n\pi}{a}\right)^2, \qquad X_n(x) = \sin\left(\frac{n\pi}{a}x\right), \qquad n = 1, 2, 3, \ldots$$

and

$$Y'' + \lambda^2 Y = 0, \qquad Y(0) = Y(b) = 0$$

with eigenvalues and eigenfunctions, respectively, as

$$\lambda^2 = \left(\frac{m\pi}{b}\right)^2, \qquad Y_m(y) = \sin\left(\frac{m\pi}{b}y\right), \quad m = 1, 2, 3, \ldots .$$

The solution of each Sturm-Liouville problem can be expressed as

$$X(x) = A_n \sin\left(\frac{n\pi}{a}x\right) \quad \text{and} \quad Y(y) = B_m \sin\left(\frac{m\pi}{b}y\right).$$

Finally, the solution can be shown as

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_n B_m \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)T(t)$$

or

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)T(t)$$

where $A_{mn} = B_m A_n$.

The solution for $T(t)$ is obtained from the equation

$$T'' + c^2\left(\lambda^2 + \mu^2\right)T = 0$$

as

$$T(t) = C_{mn}\cos\left(\sqrt{c^2\left(\lambda^2+\mu^2\right)}t\right) + D_{mn}\sin\left(\sqrt{c^2\left(\lambda^2+\mu^2\right)}t\right).$$

Assume $k_{mn}^2 = c^2\left(\lambda^2+\mu^2\right)$ then

$$T(t) = C_{mn}\cos(k_{mn}t) + D_{mn}\sin(k_{mn}t).$$

The general solution can be written as

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{a_{mn}\cos(k_{mn})t + b_{mn}\sin(k_{mn})t\right\}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

where $A_{mn}C_{mn} = a_{mn}$ and $A_{mn}D_{mn} = b_{mn}$.

To find the coefficients, we use the initial conditions

$$u(x,y,0) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right),$$

$$u_t(x,y,0) = g(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k_{mn}b_{mn}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

and we obtain the values of coefficients as

$$a_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)dxdy,$$

$$k_{mn}b_{mn} = \frac{4}{ab}\int_0^b\int_0^a g(x,y)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)dxdy,$$

$$b_{mn} = \frac{4}{k_{mn}ab}\int_0^b\int_0^a g(x,y)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)dxdy$$

where

$$k_{mn} = c\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}.$$

CHAPTER 3

ADOMIAN DECOMPOSITION METHOD

3.1. DEFINITION OF DECOMPOSITION

In the 1980s, a new powerful method was introduced by George Adomian (March 21, 1922-1996), who was the Armenian-American mathematician and also an aerospace engineer, for solving linear and nonlinear functional equations such as algebraic, ordinary and partial differential, integral etc. This method has been known as the Adomian Decomposition Method (ADM) which has been proved to be effective and reliable for handling linear and nonlinear equations.

The advantage of this method is that it solves the problems directly without the need of linearization, perturbation or any other transformation. And also, it reduces the massive computation works required by most other methods.

Convergence of the Adomian method is discussed by many researchers. For instance, Kalla proposed that the Adomian method can be applied to nonlinear Volterra integral equation due to convergence of the method. It can be understood that convergence analysis can approximate the maximum absolute error of the Adomian series solution (El-Kalla, 2008). Abbaoui and Cherruault proved the convergence of ADM for the solution of differential equation and the findings of roots of nonlinear functional equations. According to Adomian's method, he developed a technique which easily finds the exact solution by a series given and proved the convergence of the method by using the fixed-point theorem. But this proof is not enough to apply to real problems, especially partial differential equations. So, Cherruault presents a hypothes to the Adomian's method. For this, some examples are given and solved such as the nonlinear problems, the semi-linear problems and the linear problems by using the Adomian's method. At the end of the problems, the hypotheses verified that the Adomian method is rapidly convergent if the boundary conditions are well chosen (Abbaoui and Cherruault, 1994a,b). Cherruault and Adomian present a new proof about convergence of Adomian's method. They reach some results about the speed of Adomian method and thus, we can solve the nonlinear functional equations (Cherrualt and Adomian, 1993).

Many modifications related to the ADM have been proposed by many researchers and applied to the problems. One method developed by Wazwaz was called the modified Adomian decomposition method. The aim of this method is to use two iterations only without the need of Adomian polynomials in order to provide the exact solution (Wazwaz, 1999a). The other method is known as Improved Adomian Decomposition Method (IADM) by Abassy. Using IADM changes the formulation of Adomian polynomials and provides improvement over the standard Adomian method. Also, Abassy compares the ADM and the IADM by using some problems according to the convergence. And as a result, he expressed that the results of IADM are more accurate than the results of ADM and more convergent in several steps. It can be said that the IADM solves the drawbacks in the standard Adomian decomposition method. This method is applied for the analytic treatment of nonlinear initial value problems (Abassy, 2010). Luo proposed an efficient modification to the ADM, a two step Adomian decomposition method (TSADM) which facilitated the calculations. Namely, this method reduces the volume of computational work and obtains sometimes the exact solution after only one iteration. Moreover, the TSADM may not need the Adomian polynomials to obtain the exact solution. The method gives efficient solutions for systems of nonhomogeneous differential and integral equations, hyperbolic partial differential equations and singular initial value problems. Also, it shows us that this method has more advantage than the standard Adomian method (Luo, 2005). Gejji and Jafari presented the new iterative method to solve linear or nonlinear partial differential equations of integer and fractional order. The results of the examples given are compared by other iterative methods such as the Adomian method, homotopy perturbation method and variational iteration method. It can be understood that this method is more close the exact solution than the others (Gejji and Jafari, 2006). As regards the ADM, it is used to solve a wide range of physical problems in various engineering fields such as vibration and wave equation (Allan and Al-Khaled, 2006), heat and mass transfer (Chiu and Chen, 2002), fluid flow (Allan and Syam, 2005).

In recent years, some applications of the perturbation techniques have been studied by scientists and engineers. And these techniques are compared by the ADM. The one is homotopy perturbation method (HPM). Jian-Lin Li presents the comparison between the ADM and the HPM methods (Li, 2009). The other comparison is presented between the ADM and the Runga-Kutta methods for approximate solutions of predator prey model equations (Edwards et al., 1997). Also, Wazwaz compared the ADM and the Taylor series method by using some particular examples, and showed that the decomposition produced reliable results with less iteration, whereas the Taylor series method suffered from computational difficulties (Wazwaz, 1998b).

Thus, we see that the Adomian decomposition method has been used for solving linear and nonlinear functional equations up to now. In this study, we will see formula of ADM and examples of the Adomian's method how to use in the application.

To introduce this form of method, we start with $Fu = g$, where F and g are a nonlinear ordinary differential operator and a given function, respectively. We represent form of F with linear and nonlinear terms. We write the linear term as $Lu + Ru$ where L is easily invertible and R is the remainder of the linear operator. The nonlinear term is represented by Nu. Thus, the equation is written as

$$Lu + Ru + Nu = g\,,$$

$$Lu = g - Ru - Nu\,, \tag{3.1}$$

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{3.2}$$

We define L^{-1} as the n-fold definite integration operator from 0 to t:

$$L^{-1} = \int_0^t \cdots \int_0^t (.)\, dt^n \tag{3.3}$$

Solving u from (3.2) and we can have

$$u = h - L^{-1}Ru - L^{-1}Nu \tag{3.4}$$

where the function h is determined by integrating the source term g and using the given initial conditions. The unknown function u is expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.5}$$

and the nonlinear term Nu can be decomposed by infinite series of polynomials as

$$f(u) = Nu = \sum_{n=0}^{\infty} A_n \tag{3.6}$$

where $A_n\ (n = 0,1,\dots)$ are the Adomian polynomials of $u_0, u_1, \dots, u_j$ given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} \lambda^k u_k \right)\right]_{\lambda=0} \quad , \quad n = 0,1,2,3,\dots. \tag{3.7}$$

and formulated following

$$A_0 = f(u_0), \tag{3.8}$$

$$A_1 = u_1 f'(u_0), \tag{3.9}$$

$$A_2 = u_2 f'(u_0) + \frac{1}{2!}u_1^2 f''(u_0), \tag{3.10}$$

$$A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!}u_1^3 f'''(u_0), \tag{3.11}$$

$$A_4 = u_4 f'(u_0) + \left[\frac{1}{2!}u_2^2 + u_1 u_3\right] f''(u_0) + \frac{1}{2!}u_1^2 u_2 f'''(u_0) + \frac{1}{4!}u_1^4 f^{(4)}(u_0), \tag{3.12}$$

$$A_5 = u_5 f'(u_0) + \left[u_2 u_3 + u_1 u_4\right] f''(u_0) + \left[\frac{1}{2!}u_1 u_2^2 + \frac{1}{2!}u_1^2 u_3\right] f'''(u_0)$$

$$+ \frac{1}{3!}u_1^3 u_2 f^{(4)}(u_0) + \frac{1}{5!}u_1^5 f^{(5)}(u_0), \tag{3.13}$$

$$A_6 = u_6 f'(u_0) + \left[\frac{1}{2!}u_3^2 + u_2u_4 + u_1u_5\right] f''(u_0) + \left[\frac{1}{3!}u_2^3 + u_1u_2u_3 + \frac{1}{2!}u_1^2u_4\right] f'''(u_0)$$

$$+\left[\frac{1}{2!}u_1^2 \frac{1}{2!}u_2^2 \frac{1}{3!}u_1^3u_3\right] f^{(4)}(u_0) + \frac{1}{4!}u_1^4u_2 f^{(5)}(u_0) + \frac{1}{6!}u_1^6 f^{(6)}(u_0), \qquad (3.14)$$

From (3.4), we define

$u_0 = h$,

$u_1 = -L^{-1}Ru_0 - L^{-1}Nu_0$,

$u_2 = -L^{-1}Ru_1 - L^{-1}Nu_1$,

⋮

$$u_{k+1} = -L^{-1}Ru_k - L^{-1}Nu_k \ , \qquad k \geq 0 \qquad (3.15)$$

and we can calculate $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$, if the series is convergent.

3.2. EXAMPLES OF ADOMIAN POLYNOMIALS

Example 1: Calculate A_0, A_1, A_2, A_3 and A_4 for $Nu = u^5$.

$A_0 = u_0^5$,

$A_1 = 5u_1u_0^4$,

$A_2 = 5u_0^4u_2 + 10u_0^3 + u_1^2$,

$A_3 = 5u_0^4u_3 + 20u_0^3u_1u_2 + 10u_0^2u_1^3$,

$A_4 = 5u_0^4u_4 + 5u_1^4u_0 + 10u_0^3u_2^2 + 20u_0^3u_1u_3 + 30u_0^2u_1^2u_2$.

Example 2: Determine A_0, A_1, A_2, A_3 and A_4 for $Nu = u^{\alpha}$ where α is a decimal.

$$A_0 = u_0^{\alpha},$$

$$A_1 = \alpha u_0^{\alpha-1} u_1,$$

$$A_2 = \alpha u_0^{\alpha-1} u_2 + \frac{1}{2}\alpha(\alpha-1) u_0^{\alpha-2} u_1^2,$$

$$A_3 = \alpha u_0^{\alpha-1} u_3 + \alpha(\alpha-1) u_0^{\alpha-2} u_1 u_2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2) u_0^{\alpha-3} u_1^3,$$

$$A_4 = \alpha u_0^{\alpha-1} u_4 + \alpha(\alpha-1) u_0^{\alpha-2}\left(\frac{1}{2}u_2^2 + u_1 u_3\right) + \frac{1}{2}\alpha(\alpha-1)(\alpha-2) u_0^{\alpha-3} u_1^2 u_2$$

$$+ \frac{1}{24}\alpha(\alpha-1)(\alpha-2)(\alpha-3) u_0^{\alpha-4} u_1^4.$$

Example 3: Obtain A_0, A_1, A_2 and A_3 for $N(u) = u^3 + u^2$.

$$A_0 = u_0^3 + u_0^2,$$

$$A_1 = 3u_0^2 u_1 + 2u_0 u_1,$$

$$A_2 = 3u_0 u_1^2 + 3u_0^2 u_2 + u_1^2 + 2u_0 u_2,$$

$$A_3 = u_1^3 + 6u_0 u_1 u_2 + 3u_0^2 u_3 + 2u_0 u_3 + 2u_1 u_2.$$

Example 4: Solve A_0, A_1, A_2 and A_3 for $N(u) = uu_x$.

$$A_0 = u_0 u_{0_x},$$

$$A_1 = u_1 u_{0_x} + u_0 u_{1_x},$$

$$A_2 = u_2 u_{0_x} + u_1 u_{1_x} + u_0 u_{2_x},$$

$$A_3 = u_3 u_{0_x} + u_2 u_{1_x} + u_1 u_{2_x} + u_0 u_{3_x} .$$

3.3. APPLICATIONS OF ADOMIAN DECOMPOSITION METHOD

Example 5: Consider the linear equation

$$u_t = x^2 t u_{xx} , \tag{3.16}$$

$$\text{BCs: } u(0,t) = 0 , \qquad u(1,t) = e^t ,$$

$$\text{IC: } u(x,0) = x^2 .$$

The equation (3.16) is written in an operator form as

$$L_t u = x^2 t u_{xx} \tag{3.17}$$

where $L_t = \dfrac{\partial}{\partial t}$ and the inverse operator is L_t^{-1} where $L_t^{-1} = \int_0^t (.) dt$. L_t^{-1} is applied to the both sides of (3.17) and by using the initial condition. It can be written that

$$u(x,t) = u(x,0) + L_t^{-1}\left(x^2 t u_{xx}\right) \tag{3.18}$$

then

$$u_0 = x^2 ,$$

$$u_1 = x^2 \int_0^t \left(t \frac{\partial^2 u_0}{\partial x^2} \right) dt = x^2 t^2 ,$$

$$u_2 = x^2 \int_0^t \left(t \frac{\partial^2 u_1}{\partial x^2} \right) dt = \frac{1}{2} x^2 t^4 ,$$

$$u_3 = x^2 \int_0^t \left(t \frac{\partial^2 u_2}{\partial x^2} \right) dt = \frac{1}{6} x^2 t^6 ,$$

$$u_4 = x^2 \int_0^t \left(t \frac{\partial^2 u_3}{\partial x^2} \right) dt = \frac{1}{24} x^2 t^8 .$$

As a result, the solution $u(x,t)$ is obtained in a series form as

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=x^2+x^2t^2+\frac{1}{2}x^2t^4+\frac{1}{6}x^2t^6+\frac{1}{24}x^2t^8+\ldots$$

and the series converge to

$$u(x,t)=x^2e^t\,.$$

Example 6 : Consider the nonlinear differential equation

$$u_t=\frac{1}{x}u_{xx}+\frac{1}{x}u^2 \quad \text{on} \quad (0,l)\times(0,1), \tag{3.19}$$

BCs: $u(0,t)=0$, $u_x(l,t)=\dfrac{1}{1-t}$,

IC: $u(x,0)=x$.

The equation (3.19) is determined in an operator form as

$$L_tu=\frac{1}{x}u_{xx}+\frac{1}{x}u^2 \tag{3.20}$$

where $L_t=\dfrac{\partial}{\partial t}$ and inverse operator of L_t is L_t^{-1} where $L_t^{-1}=\int_0^t(.)dt$. Inverse operator is applied to the both sides of (3.20) and by using the initial condition. It can be written as

$$u=u(x,0)+L_t^{-1}\left(\frac{1}{x}u_{xx}+\frac{1}{x}u^2\right)$$

and the nonlinear term $N(u)$ which is defined as u^2 is expressed in the form of Adomian's polynomials as follows:

$$u_0=x,$$

$$A_0=x^2 \qquad \Rightarrow \qquad u_1=\frac{1}{x}\int_0^t\frac{\partial^2u_0}{\partial x^2}dt+\frac{1}{x}\int_0^tA_0dt=xt,$$

$$A_1 = 2x^2 t \quad \Rightarrow \quad u_2 = \frac{1}{x}\int_0^t \frac{\partial^2 u_1}{\partial x^2} dt + \frac{1}{x}\int_0^t A_1 dt = xt^2,$$

$$A_2 = 3x^2 t^2 \quad \Rightarrow \quad u_3 = \frac{1}{x}\int_0^t \frac{\partial^2 u_2}{\partial x^2} dt + \frac{1}{x}\int_0^t A_2 dt = xt^3,$$

$$A_3 = 4x^2 t^3 \quad \Rightarrow \quad u_4 = \frac{1}{x}\int_0^t \frac{\partial^2 u_3}{\partial x^2} dt + \frac{1}{x}\int_0^t A_3 dt = xt^4$$

$$\vdots$$

then, the solution is given by

$$u(x,t) = x\left(1 + t + t^2 + ...\right) = x\sum_{n=0}^{\infty} t^n, \qquad t \in (0,1),$$

and we have

$$u(x,t) = x\left(\frac{1}{1-t}\right).$$

Example 7: Consider the ordinary differential equation

$$u'(x) - u(x) = x\cos x - x\sin x + \sin x, \quad u(0) = 0.$$

L^{-1} is applied to both sides of the ordinary differential equation. Then, it is found that

$$u(x) = x\sin x + x\cos x - \sin x + L^{-1}\left(u(x)\right)$$

Then, we have iterations of $u(x)$ as follows:

$$u_0 = x\sin x + x\cos x - \sin x,$$

$$u_1 = \int_0^x u_0 dt = -x\cos x + \sin x + x\sin x + 2\cos x - 2,$$

$$u_2 = \int_0^x u_1(t) dt = -x\sin x - 2\cos x - x\cos x + 3\sin x - 2x + 2.$$

$$\vdots$$

As a result, by eliminating the like terms it is obtained the exact solution which is given by

$$u(x) = x\sin x.$$

Example 8: Consider the homogeneous hyperbolic equation

$$u_{tt} - u_{xx} + \alpha u = 0$$

with the initial and boundary conditions

$$u(0,x) = \phi(x) = 0, \qquad u_t(0,x) = \varphi(x) = \sin(2\pi x),$$

$$u(t,1) - u(t,0) = 0, \qquad \int_0^1 u(t,x)\,dx = 0$$

where

$$u(t,1) = u(t,0) = 0 \qquad \text{and} \qquad \alpha = 1 - 4\pi^2.$$

Shortly, we can rewrite the equation as

$$u_{tt} - u_{xx} + \alpha u = 0,$$

$$u(0,x) = 0, \quad u_t(0,x) = \sin(2\pi x),$$

$$u(t,1) - u(t,0) = 0, \quad u_x(t,1) - u_x(t,0) = 0$$

where

$$u_x(t,1) = u_x(t,0) = 2\pi\sin t.$$

By using the ADM

$$u_0 = t\sin(2\pi x),$$

$$u_1 = -\frac{t^3}{3!}\sin(2\pi x),$$

$$u_2 = \frac{t^5}{5!}\sin(2\pi x),$$

$$u_n = (-1)^n \frac{t^{2n+1}}{(2n+1)!} \sin(2\pi x)$$

then $u(x,t)$ is written as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sin(2\pi x)\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - .....\right],$$

$$= \sin(2\pi x)\sin t .$$

Example 9: Solve the nonhomogeneous Fredholm integral equation

$$u(x) = \cos^{-1} x - x + \int_0^1 x u(t)\, dt . \qquad (3.21)$$

ADM can also be applied to integral equations. Without loss of generality, we consider the Fredholm integral equation. Then, substituting the series decomposition

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

into both sides of equation (3.21), we obtain

$$u_0 = \cos^{-1} x - x,$$

$$u_1 = x\int_0^1 u_0(t)\, dt = \frac{x}{2},$$

$$u_2 = x\int_0^1 u_1(t)\, dt = \frac{x}{4},$$

$$u_3 = x\int_0^1 u_2(t)\, dt = \frac{x}{8}$$

$$\vdots$$

Then, we obtain the solution in a series form which is given by

$$u(x) = \cos^{-1} x - x + \frac{x}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...\right),$$

and so, the solution in a closed form is obtained by evaluating the geometric series

$$u(x) = \cos^{-1} x .$$

Example 10: Consider the nonlinear Klein-Gordon

$$u_{tt} - u_{xx} + u^2 = 6xt\left(x^2 - t^2\right) + x^6 t^6$$

with the initial conditions

$$u(x,0) = 0 , \quad u_t(x,0) = 0 .$$

In this problem, we will solve by using the Modified Decomposition Method. Without loss of generality, we use operator of ADM and get

$$Lu = 6xt\left(x^2 - t^2\right) + x^6 t^6 + u_{xx} - u^2 \qquad (3.22)$$

where L is a second order differential operator with respect to t and inverse operator L^{-1} is given by $L^{-1} = \int_0^t \int_0^t (.)\, dt dt$ applying inverse operator L^{-1} to the both sides of (3.22), it is obtained

$$u(x,t) = f(x) + L^{-1}\left(u_{xx} - u^2\right)$$

where the function $f(x)$ is defined as

$$f(x) = x^3 t^3 - \frac{3}{10} x t^5 + \frac{1}{56} x^6 t^8 .$$

The modified decomposition method suggests that the function $f(x)$ be decomposed into two parts, that is to say $f_0(x)$ and $f_1(x)$ such that

$$f(x) = f_0(x) + f_1(x)$$

then we define the solution $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

where the components $u_0, u_1, u_{2,} \dots$.

The zeroth component $u_0(x)$ is defined only by $f_0(x)$. The remaining part $f_1(x)$ of $f(x)$ is added to the component $u_1(x)$. Then

$$u_0(x,t) = x^3 t^3,$$

$$u_1(x,t) = -\frac{3}{10} x t^5 + \frac{1}{56} x^6 t^8 + L^{-1}\left(u_{0_{xx}} - u_0^2\right),$$

$$= -\frac{3}{10} x t^5 + \frac{1}{56} x^6 t^8 + L^{-1}\left(6 x t^3 - x^6 t^6\right),$$

$$= 0,$$

and as a result, $u_k = 0$, $k \geq 2$.

Finally, the solution is stated in a closed form as

$$u(x,t) = x^3 t^3.$$

Example 11: Consider the nonlinear partial differential equation

$$u_x + u_y + u_z = u^2$$

with the nonhomogeneous initial condition

$$u(x, y, 0) = x + y.$$

We solve the problem by using the ADM and write the equation in the operator form as

$$L_x + L_y + L_z = u^2$$

$$L_z = -L_x - L_y + u^2 \tag{3.23}$$

where

$$L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}, \quad L_z = \frac{\partial}{\partial z}$$

then L_z^{-1} is applied to the both sides of (3.23)

where

$$L_z^{-1} = \int_0^z (.)\, dz,$$

$$u(x, y, z) = u(x, y, 0) - L_z^{-1}(L_x) - L_z^{-1}\left(L_y\right) + L_z^{-1}\left(u^2\right),$$

$$u(x,y,z) = x + y - L_z^{-1}(L_x) - L_z^{-1}(L_y) + L_z^{-1}(u^2),$$

$$\sum_{n=0}^{\infty} u(x,y,z) = x + y - L_z^{-1}(L_x) - L_z^{-1}(L_y) + L_z^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{3.24}$$

where

$$u^2(x,y,z) = \sum_{n=0}^{\infty} A_n .$$

Using the initial condition along with (3.24) it can be obtained

$$u_{n+1}(x,y,z) = u_0(x,y,z) + \int_0^z \left(-L_x u_n - L_y u_n + A_n\right) dz \quad \text{for} \quad n > 0 .$$

Thus

$u_0 = x + y$,

$A_0 = x^2 + 2xy + y^2$,

$u_1 = \int_0^z \left(-L_x u_0 - L_y u_0 + A_0\right) dz = -2z + zx^2 + 2xyz + zy^2$,

$A_1 = -4zx - 4zy + 2zx^3 + 6zyx^2 + 6zxy^2 + 2zy^3$,

$u_2 = \int_0^z \left(-L_x u_1 - L_y u_1 + A_1\right) dz = -4z^3 x - 4z^2 y + z^2 x^3 + 3z^2 x^2 y + 3z^2 xy^2 + z^2 y^3$,

$A_2 = -12x^2 z^2 - 24z^2 xy - 12z^2 y^2 + 3z^2 x^4 + 12z^2 x^3 y + 18z^2 x^2 y^2 + 12z^2 xy^3 + 3z^2 y^4 + 4z^4$,

$$u_3 = \int_0^z \left(-L_x u_2 - L_y u_2 + A_2\right) dz = 4z^3 - 6z^3 x^2 - 12z^3 xy - 6z^3 y^2 + z^3 x^4 + 4z^3 x^3 y$$
$$+6z^3 x^2 y^2 + 4z^3 xy^3 + z^3 y^4,$$

$\vdots$

at the end, we have the approximate solution as

$$u(x,y,z) = u_0 + u_1 + u_2 + \ldots = \frac{x + y - 2z}{\left[1 - z(x + y - 2z)\right]} .$$

CHAPTER 4

NONHOMOGENEOUS WAVE PROPAGATION

4.1. INTRODUCTION

In this study, the homogeneous wave equation with variable coefficients in the nonhomogeneous material, which is named as Functionally Graded Materials (FGMs), is assumed. Our aim is to solve this equation by using Adomian Decomposition Method (ADM). At the end, the same problem is also solved by eigenfunction expansion method to compare the results. As a conclusion, it is seen that both method agreed with same results and shown graphically.

4.2. SOLUTION OF WAVE EQUATION IN FGMs

Let us consider the wave equation

$$c^2\rho(x)\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(E(x)\frac{\partial u}{\partial x}\right),\qquad \rho(x)=\rho_0 e^{-\alpha x},\qquad E(x)=E_0 e^{\alpha x} \tag{4.1}$$

with the boundary conditions

$$u(a,t)=0,\qquad u(b,t)=\kappa t^2,\qquad t\geq 0 \tag{4.2}$$

where κ is constant and the initial conditions

$$u(x,0)=0,\qquad u_t(x,0)=0,\qquad a\leq x\leq b. \tag{4.3}$$

In Equation (4.1), $\rho(x)$ represents material density and $E(x)$ represents the material mechanical properties given in x direction. The parameter α is a constant which is named as an inhomogeneity parameter of the FGM. The parameters ρ_0 and E_0 are constants and c^2 is the propagation speed of the wave that is also assumed to be constant.

To solve the problem, let us write $u(x,t)$ in (4.1) as a summation of two functions

$$u(x,t) = v(x,t) + w(x,t) \tag{4.4}$$

where $v(x,t)$ and $w(x,t)$ are the solutions of the nonhomogeneous and homogeneous wave equations, respectively. These two terms will be calculated one by one. First, we will solve homogeneous wave equation problem with the nonhomogeneous boundary conditions. The solution of the problem, $w(x,t)$, satisfies the equation which is given by

$$\frac{d}{dx}\left(e^{\alpha x}\frac{dw}{dx}\right) = 0 \tag{4.5}$$

along with the boundary conditions

$$w(a,t) = 0, \qquad w(b,t) = \kappa t^2, \qquad t \ge 0 . \tag{4.6}$$

The solution of (4.5) can be obtained as follows:

$$\alpha e^{-\alpha x}\frac{dw}{dx} + \frac{d^2 w}{dx^2}e^{\alpha x} = 0 \qquad \Rightarrow \qquad \frac{d^2 w}{dx^2} + \alpha\frac{dw}{dx} = 0 \qquad \text{say} \qquad \frac{dw}{dx} = \eta , \tag{4.7}$$

$$\frac{d\eta}{dx} + \alpha\eta = 0 \qquad \Rightarrow \qquad \eta = Ce^{-\alpha x} \qquad \Rightarrow \qquad \frac{dw}{dx} = Ce^{-\alpha x} . \tag{4.8}$$

Finally, the general solution of (4.5) can be given as

$$w(x,t) = K - \frac{C}{\alpha}e^{-\alpha x} \tag{4.9}$$

and after using the boundary conditions the particular solution can be obtained as

$$w(x,t) = \kappa t^2 X(x) \tag{4.10}$$

where the function $X(x)$ is defined as

$$X(x)=\frac{e^{-\alpha a}-e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}\,. \tag{4.11}$$

Now, the solution (4.10) along with (4.4) can be substituted into (4.1) to obtain the new differential equation to be solved

$$c^2\rho_0 e^{-\alpha x}\left[v_{tt}+w_{tt}\right]=\frac{\partial}{\partial x}\left[E_0 e^{\alpha x}\left(v_x+w_x\right)\right],$$

$$c^2\rho_0 e^{-\alpha x}\left[v_{tt}+w_{tt}\right]=E_0\alpha e^{\alpha x}\left(v_x+w_x\right)+E_0 e^{\alpha x}\left(v_{xx}+w_{xx}\right),$$

$$v_{tt}+2\kappa X(x)=\sigma^2 e^{2\alpha x}\left\{\left[\alpha\left(v_x+\alpha\kappa t^2\frac{e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}\right)\right]\right\}+v_{xx}-\alpha^2\kappa t^2\frac{e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}$$

where $\sigma^2=E_0\big/c^2\rho_0$. If we continue to simplify the expression above it will be obtained as

$$v_{tt}=-2\kappa X(x)+\sigma^2 e^{2\alpha x}\alpha v_x+\sigma^2\alpha^2\kappa t^2\frac{e^{2\alpha x}e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}+\sigma^2 e^{2\alpha x}v_{xx}-\sigma^2\alpha^2\kappa t^2\frac{e^{2\alpha x}e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}},$$

$$v_{tt}=-2\kappa\left(\frac{e^{-\alpha a}-e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}\right)+\sigma^2 e^{2\alpha x}\alpha v_x+\sigma^2 e^{2\alpha x}v_{xx},$$

$$v_{tt}=-2\kappa\left(\frac{e^{-\alpha a}-e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}\right)+\sigma^2 e^{2\alpha x}\left(v_{xx}+\alpha v_x\right) \tag{4.12}$$

with homogeneous boundary and initial conditions, respectively,

$$v(a,t)=0,\qquad v(b,t)=0\;,\qquad t\geq 0, \tag{4.13}$$

$$v(x,0)=0,\qquad v_t(x,0)=0,\qquad a\leq x\leq b\,. \tag{4.14}$$

Now, the Adomian decomposition method will be used to solve (4.12). We express the equation with some operators such as;

$$L_t v=L_x v-G(x) \tag{4.15}$$

where

$$L_t=\frac{\partial^2}{\partial t^2}\,,\qquad L_t^{-1}=\int_0^t\int_0^t(.)\,dt\,dt\,, \tag{4.16}$$

$$L_x = \sigma^2 e^{\alpha x} \frac{\partial}{\partial x}\left(e^{\alpha x} \frac{\partial}{\partial x} \right), \qquad G(x) = 2\kappa \frac{e^{-\alpha a} - e^{-\alpha x}}{e^{-\alpha a} - e^{-\alpha b}}. \tag{4.17}$$

If we apply the ADM directly to (4.12), we can calculate only one term that $v_1(x,t) = 0$. So, we will express the function $G(x)$ as generalized Fourier series to obtain other terms of the iteration. For this, we will solve Sturm-Liouville boundary value problem such as

$$\frac{d}{dx}\left(e^{\alpha x} \frac{d\Omega(x)}{dx} \right) + \frac{\mu^2}{e^{\alpha x}} \Omega(x) = 0 \tag{4.18}$$

with homogeneous boundary conditions $\Omega(a) = \Omega(b) = 0$. Let us start to solve the boundary value problem by differentiating (4.18) as

$$e^{\alpha x} \frac{d^2\Omega(x)}{dx^2} + \alpha e^{\alpha x} \frac{d\Omega(x)}{dx} + \mu^2 \frac{\Omega(x)}{e^{\alpha x}} = 0,$$

$$e^{2\alpha x} \frac{d^2\Omega(x)}{dx^2} + \alpha e^{2\alpha x} \frac{d\Omega(x)}{dx} + \mu^2 \Omega(x) = 0 \tag{4.19}$$

which is the second-order ordinary differential equation with variable coefficients. Then, we write z instead of $e^{-\alpha x}$ and determine the derivatives of z as follows:

$$e^{-\alpha x} = z \quad \Rightarrow \quad \frac{d\Omega}{dx} = \frac{d\Omega}{dz}\frac{dz}{dx} = \Omega'(z)(-\alpha)e^{-\alpha x} = -\Omega'(z)\alpha z,$$

$$\frac{d^2\Omega}{dz^2} = \frac{d}{dx}\left[-\Omega'(z)\alpha z \right] = \frac{d}{dz}\left[-\Omega'(z)\alpha z \right]\frac{dz}{dx},$$

$$= \left[-\Omega''(z)\alpha z - \Omega'(z)\alpha \right](-\alpha z),$$

$$= \alpha^2 z^2 \Omega''(z) + \alpha^2 z \Omega'(z)$$

by substituting all terms into (4.19), we obtain ODE with constant coefficients

$$z^{-2}\left[\alpha^2 z^2 \Omega''(z) + \alpha^2 z \Omega'(z) \right] + \alpha z^{-2}\left[-\alpha z \Omega'(z) \right] + \mu^2 \Omega(z) = 0,$$

$$\alpha^2 \Omega''(z) + \alpha^2 z^{-1}\Omega'(z) - \alpha^2 z^{-1}\Omega'(z) + \mu^2 \Omega(z) = 0,$$

$$\alpha^2 \Omega''(z) + \mu^2 \Omega(z) = 0. \tag{4.20}$$

After solving characteristic equation, two complex roots are obtained as

$$\alpha^2 m^2 + \mu^2 = 0 \quad \Rightarrow \quad m^2 = -\frac{\mu^2}{\alpha^2} \quad \Rightarrow \quad m = \pm\frac{\mu}{\alpha} i$$

and the solution of the (4.20) with respect to z is given by

$$\Omega(z) = a_1 \cos\left(\frac{\mu}{\alpha} z\right) + a_2 \sin\left(\frac{\mu}{\alpha} z\right).$$

Finally, the corresponding general solution is

$$\Omega(x) = a_1 \cos\left(\frac{\mu}{\alpha} e^{-\alpha x}\right) + a_2 \sin\left(\frac{\mu}{\alpha} e^{-\alpha x}\right) . \tag{4.21}$$

By using boundary conditions, one can find the unknown coefficients a_1 and a_2 as follows:

$$\Omega(a) = a_1 \cos\left(\frac{\mu}{\alpha} e^{-\alpha a}\right) + a_2 \sin\left(\frac{\mu}{\alpha} e^{-\alpha a}\right) = 0,$$

$$\Omega(b) = a_1 \cos\left(\frac{\mu}{\alpha} e^{-\alpha b}\right) + a_2 \sin\left(\frac{\mu}{\alpha} e^{-\alpha b}\right) = 0,$$

$$a_1 = -a_2 \tan\left(\frac{\mu}{\alpha} e^{-\alpha a}\right),$$

$$a_2 \sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right)\right) = 0.$$

if $a_2 = 0$ then $a_1 = 0$ so it is trivial solution. If $a_2 \neq 0$ then

$$\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right)\right) = 0,$$

$$\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right)\right) = \sin(n\pi), \quad n \in \mathbb{Z},$$

$$\frac{\mu}{\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right) = n\pi ,$$

$$\mu = \frac{n\pi\alpha}{e^{-\alpha b} - e^{-\alpha a}}$$

thus, we have eigenvalues

$$\mu^2 = \frac{n^2\pi^2\alpha^2}{\left(e^{-\alpha b} - e^{-\alpha a}\right)^2} . \tag{4.22}$$

To obtain eigenfunctions we follow the process

$$\Omega(x) = -a_2 \tan\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)\cos\left(\frac{\mu}{\alpha}e^{-\alpha x}\right) + a_2 \sin\left(\frac{\mu}{\alpha}e^{-\alpha x}\right),$$

$$\Omega(x) = -a_2 \frac{\sin\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}\cos\left(\frac{\mu}{\alpha}e^{-\alpha x}\right) + a_2 \sin\left(\frac{\mu}{\alpha}e^{-\alpha x}\right),$$

$$\Omega(x) = \frac{-a_2 \sin\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)\cos\left(\frac{\mu}{\alpha}e^{-\alpha x}\right) + a_2 \sin\left(\frac{\mu}{\alpha}e^{-\alpha x}\right)\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)},$$

$$\Omega(x) = a_2 \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x} - e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha}\left(e^{-\alpha a}\right)\right)} \tag{4.23}$$

and the eigenfunctions are

$$\Omega_n(x) = \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x} - e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}, \qquad n = 1, 2, 3, \dots \tag{4.24}$$

Finally, the function $G(x)$ given in (4.17) is written according to $\Omega_n(x)$ that is a generalized Fourier series representation of $G(x)$ as;

$$G(x) = \sum_{n=1}^{\infty} a_n \Omega_n(x) = \sum_{n=1}^{\infty} a_n \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x} - e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} \tag{4.25}$$

where a_n is called as coefficients of generalized Fourier series of $G(x)$. To calculate coefficients, it can be used orthogonality properties and the result is

$$a_n = \frac{\int_a^b G(x)\Omega_n(x)e^{-\alpha x}dx}{\int_a^b \Omega_n^2(x)e^{-\alpha x}dx}. \tag{4.26}$$

Now, let us evaluate integrals in numerator and denominator in (4.26), respectively:

$$\int_a^b G(x)\Omega_n(x)e^{-\alpha x}dx = \int_a^b 2\kappa \frac{e^{-\alpha a}-e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}} \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} e^{-\alpha x}dx\,,$$

$$= \frac{2\kappa}{\alpha\left(e^{-\alpha a}-e^{-\alpha b}\right)\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} \int_a^b \left(e^{-\alpha a}-e^{-\alpha x}\right)\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)e^{-\alpha x}dx$$

by using the integration by parts, the numerator will be resulted as

$$\int_a^b G(x)\Omega_n(x)e^{-\alpha x}dx = \frac{2\kappa}{\mu} \frac{\cos\left(\frac{\mu}{\alpha}\left(e^{-\alpha b}-e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} + \frac{2\kappa\alpha}{\mu^2} \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha b}-e^{-\alpha a}\right)\right)}{\left(e^{-\alpha a}-e^{-\alpha b}\right)\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}\,.$$

Similarly, let us evaluate the denominator as follows:

$$\int_a^b \Omega_n^2(x)e^{-\alpha x}dx = \int_a^b \frac{\sin^2\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)}{\cos^2\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} e^{-\alpha x}dx$$

$$= \int_a^b \frac{1-\cos\left(\frac{2\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)}{2\cos^2\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} e^{-\alpha x}dx$$

using change of variable $\omega = e^{-\alpha x} - e^{-\alpha a}$ then

$$\int_a^b \Omega_n^2(x)e^{-\alpha x}dx = \int_0^{e^{-\alpha b}-e^{-\alpha a}} -\frac{1-\cos\left(\frac{2\mu}{\alpha}(\omega)\right)}{2\alpha\cos^2\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} d\omega\,,$$

$$= -\left.\frac{\omega - \frac{\alpha}{2\mu}\sin\left(\frac{2\mu}{\alpha}\omega\right)}{2\alpha\cos^2\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}\right|_0^{e^{-\alpha b}-e^{-\alpha a}}\,,$$

$$= -\left(\frac{\left(e^{-\alpha b} - e^{-\alpha a}\right) - \frac{\alpha}{2\mu}\sin\left(\frac{2\mu}{\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right)\right)}{2\alpha\cos^2\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} \right)$$

Finally, the coefficients a_n can be obtained as

$$a_n = \frac{2\kappa\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)\left[\frac{\cos\left(\frac{\mu}{\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right)\right)}{\mu} - \frac{\alpha\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right)\right)}{\mu^2\left(e^{-\alpha a} - e^{-\alpha b}\right)}\right]}{-\frac{1}{2\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right) + \frac{1}{4\mu}\sin\left(\frac{2\mu}{\alpha}\left(e^{-\alpha b} - e^{-\alpha a}\right)\right)}. \quad (4.27)$$

Finally, it's simple form is

$$a_n = \frac{4\kappa(-1)^{n+1}\cos\left(\frac{n\pi e^{\alpha b}}{e^{\alpha b} - e^{\alpha a}}\right)}{n\pi}. \quad (4.28)$$

Now, by expressing the function $v(x,t)$ as an infinite series and defining an inverse operator of L_t respectively,

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \qquad L_t^{-1} = \int_0^t \int_0^t (.)\,dt\,dt \quad (4.29)$$

we can apply ADM to (4.12) like

$$L_t^{-1}(L_t v) = L_t^{-1}(L_x v) - L_t^{-1}(G(x)) \quad (4.30)$$

and

$$v(x,t) = v(x,0) + v_t(x,0)t - L_t^{-1}(G(x)) + L_t^{-1}(L_x v). \quad (4.31)$$

We obtain the zeroth component as

$$v_0 = v(x,0) + v_t(x,0)t - L_t^{-1}(G(x)) = -\frac{t^2}{2}\sum_{n=1}^{\infty} a_n \Omega_n(x) \quad (4.32)$$

and the other components can be found by using the following recursive formula like

$$v_{j+1}(x,t) = L_t^{-1}\left(L_x v_j\right), \qquad j = 0,1,2,\ldots. \quad (4.33)$$

The other term of $v(x,t)$ can be found recursively as follows:

$$v_1(x,t) = L_t^{-1}(L_x v_0) = \int_0^t \int_0^t \sigma^2 e^{\alpha x} \frac{\partial}{\partial x}\left(e^{\alpha x} \frac{\partial}{\partial x}\left[-\frac{t^2}{2!} \sum_{n=1}^{\infty} a_n \Omega_n(x) \right]\right) dt dt,$$

$$= -\frac{t^4}{4!}\sigma^2 \sum_{n=1}^{\infty} a_n e^{2\alpha x}\left[\alpha \Omega_n'(x) + \Omega_n''(x)\right],$$

$$= \frac{t^4}{4!}\sigma^2 \sum_{n=1}^{\infty} a_n \mu^2 \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x} - e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha} e^{-\alpha a}\right)}, \tag{4.34}$$

$$v_2(x,t) = L_t^{-1}(L_x v_1) = \int_0^t \int_0^t \sigma^2 e^{\alpha x} \frac{\partial}{\partial x}\left(e^{\alpha x} \frac{\partial}{\partial x}\left[\frac{t^4}{4!}\sigma^2 \mu^2 \sum_{n=1}^{\infty} a_n \Omega_n(x) \right]\right) dt dt,$$

$$= \frac{t^6}{6!}\sigma^4 \sum_{n=1}^{\infty} a_n \mu^2 e^{2\alpha x}\left[\alpha \Omega_n'(x) + \Omega_n''(x)\right],$$

$$= -\frac{t^6}{6!}\sigma^4 \sum_{n=1}^{\infty} a_n \mu^4 \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x} - e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha} e^{-\alpha a}\right)}, \tag{4.35}$$

$$v_3(x,t) = \frac{t^8}{8!}\sigma^6 \sum_{n=1}^{\infty} a_n \mu^6 \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x} - e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha} e^{-\alpha a}\right)}, \tag{4.36}$$

and so on. Finally, the j th component is written as,

$$v_j(x,t) = L_t^{-1}\left(L_x v_{j-1}\right) = (-1)^{j+1} \frac{t^{2j+2}}{(2j+2)!}(\sigma\mu)^{2j} \sum_{n=1}^{\infty} a_n \frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x} - e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha} e^{-\alpha a}\right)}. \tag{4.37}$$

Let us try to simplify (4.37) as follows:

$$-\frac{t^2}{2!} + \frac{t^4}{4!}\mu^2\sigma^2 - \frac{t^6}{6!}\mu^4\sigma^4 + \ldots = \frac{(\mu\sigma)^2}{(\mu\sigma)^2}\left[-\frac{t^2}{2!} + \frac{t^4}{4!}\mu^2\sigma^2 - \frac{t^6}{6!}\mu^4\sigma^4 + \ldots \right],$$

$$= \frac{1}{(\mu\sigma)^2}\left[-\frac{(\mu\sigma t)^2}{2!}+\frac{(\mu\sigma t)^4}{4!}-\frac{(\mu\sigma t)^6}{6!}+\dots\right],$$

$$= \frac{1}{(\mu\sigma)^2}\left[1-1-\frac{(\mu\sigma t)^2}{2!}+\frac{(\mu\sigma t)^4}{4!}-\frac{(\mu\sigma t)^6}{6!}+\dots\right],$$

$$= \frac{\cos(\mu\sigma t)-1}{(\mu\sigma)^2}.$$

Using the simplification above and substituting the value of a_n into (4.37), it can be written that

$$u(x,t)=\sum_{n=1}^{\infty}-\frac{(-1)^n 4\kappa}{n\pi}\left(\frac{\cos(\mu\sigma t)-1}{(\mu\sigma)^2}\right)\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)+\kappa t^2 X(x). \tag{4.38}$$

4.3 SOLUTION OF WAVE EQUATION WITH EIGENFUNCTION EXPANSION METHOD

In this part of the study, we will solve the problem defined in (4.1)-(4.3) by using eigenfunction expansion method. Let us consider again

$$c^2\rho(x)\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(E(x)\frac{\partial u}{\partial x}\right), \qquad \rho(x)=\rho_0 e^{-\alpha x}, \qquad E(x)=E_0 e^{\alpha x} \tag{4.39}$$

with boundary and initial conditions

$$u(a,t)=0, \qquad u(b,t)=\kappa t^2, \qquad a\le x\le b,$$

$$u(x,0)=0, \qquad u_t(x,0)=0, \qquad t\ge 0.$$

To solve the problem of (4.39), we propose a solution such as

$$u(x,t)=V(x,t)+W(x,t) \tag{4.40}$$

which is written as a summation of two functions where the function $W(x,t)$ satisfies the homogeneous equation given by

$$\frac{d}{dx}\left(e^{\alpha x}\frac{dW}{dx}\right)=0 \tag{4.41}$$

along with the nonhomogeneous boundary conditions

$$W(a,t)=0, \qquad W(b,t)=\kappa t^2, \qquad t\ge 0$$

The solution of (4.41) is obtained as

$$W(x,t)=\kappa t^2 X(x) \tag{4.42}$$

where the function $X(x)$ is defined as

$$X(x)=\frac{e^{-\alpha a}-e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}.$$

Substituting (4.42) along with (4.40) into (4.39), we obtain the new nonhomogeneous differential equation with the homogeneous initial and boundary conditions

$$\frac{\partial^2 V}{\partial t^2}=-2\kappa\left(\frac{e^{-\alpha a}-e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}\right)+\sigma^2 e^{2\alpha x}\left(\frac{\partial^2 V}{\partial x^2}+\alpha\frac{\partial V}{\partial x}\right) \tag{4.43}$$

$$V(a,t)=0, \quad V(b,t)=0, \quad t\ge 0,$$

$$V(x,0)=0, \quad V_t(x,0)=0, \quad a\le x\le b.$$

The solution of the homogeneous differential equation

$$\frac{d}{dx}\left(e^{\alpha x}\frac{d\Omega(x)}{dx}\right)+\frac{\mu^2}{e^{\alpha x}}\Omega(x)=0 \tag{4.44}$$

along with the homogeneous boundary conditions

$$\Omega(a)=\Omega(b)=0.$$

gives eigenvalues and eigenfunctions of (4.44), respectively, as

$$\mu^2=\frac{n^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}, \qquad \Omega_n(x)=\frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}, \qquad n=1,2,3,.. \tag{4.45}$$

Let us define

$$V(x,t)=\sum_{n=1}^{\infty}a_n(t)\Omega_n(x)=\sum_{n=1}^{\infty}a_n(t)\frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)} \tag{4.46}$$

and substitute (4.46) into (4.43) and then simplify as follows:

$$\sum_{n=1}^{\infty}\frac{d^2a_n(t)}{dt^2}\Omega_n(x)=\sigma^2e^{2\alpha x}\left(\sum_{n=1}^{\infty}a_n(t)\frac{d^2\Omega_n(x)}{dx^2}+\alpha\sum_{n=1}^{\infty}a_n(t)\frac{d\Omega_n(x)}{dx}\right)-G(x), \tag{4.47}$$

$$\sum_{n=1}^{\infty}\frac{d^2a_n(t)}{dt^2}\frac{\sin(n\beta)}{\cos(n\theta)}$$

$$=\sigma^2e^{2\alpha x}\left(\sum_{n=1}^{\infty}a_n(t)\frac{d^2}{dx^2}\left(\frac{\sin(n\beta)}{\cos(n\theta)}\right)+\alpha\sum_{n=1}^{\infty}a_n(t)\frac{d}{dx}\left(\frac{\sin(n\beta)}{\cos(n\theta)}\right)\right)-G(x),$$

$$\sum_{n=1}^{\infty}\frac{d^2a_n(t)}{dt^2}\frac{\sin(n\beta)}{\cos(n\theta)}-\frac{\sigma^2e^{\alpha x}\alpha^2\pi}{\left(e^{-\alpha b}-e^{-\alpha a}\right)}\sum_{n=1}^{\infty}na_n(t)\frac{\cos(n\beta)}{\cos(n\theta)}$$

$$+\frac{\sigma^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}\sum_{n=1}^{\infty}n^2a_n(t)\frac{\sin(n\beta)}{\cos(n\theta)}+\frac{\sigma^2e^{\alpha x}\alpha^2\pi}{e^{-\alpha b}-e^{-\alpha a}}\sum_{n=1}^{\infty}na_n(t)\frac{\cos(n\beta)}{\cos(n\theta)}=-G(x),$$

$$\sum_{n=1}^{\infty}\frac{d^2a_n(t)}{dt^2}\frac{\sin(n\beta)}{\cos(n\theta)}+\frac{\sigma^2\pi^2\alpha^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}\sum_{n=1}^{\infty}n^2a_n(t)\frac{\sin(n\beta)}{\cos(n\theta)}=-G(x).$$

Finally, it is obtained a differential equation in terms of an unknown function $a_n(t)$ such that

$$\sum_{n=1}^{\infty}\left(\frac{d^2}{dt^2}a_n(t)+\frac{\sigma^2\pi^2\alpha^2n^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}a_n(t)\right)\frac{\sin(n\beta)}{\cos(n\theta)}=-G(x) \tag{4.48}$$

where

$$\beta=\frac{\pi\left(e^{-\alpha x}-e^{-\alpha a}\right)}{e^{-\alpha b}-e^{-\alpha a}}, \tag{4.49}$$

$$\theta=\frac{\pi e^{-\alpha a}}{e^{-\alpha b}-e^{-\alpha a}}, \tag{4.50}$$

$$G(x)=2\kappa\frac{e^{-\alpha a}-e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}}=\frac{2\kappa\beta}{\pi}. \tag{4.51}$$

Using orthogonality properties of eigenfunction, $\sin(n\beta)$, over the interval $[0,\pi]$, the summation can be reduced to a single differential equation in terms of $a_n(t)$ as follows:

$$\sum_{n=1}^{\infty}\left(\frac{d^2a_n(t)}{dt^2}+\frac{\sigma^2\pi^2\alpha^2n^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}a_n(t)\right)\int_0^{\pi}\sin(m\beta)\frac{\sin(n\beta)}{\cos(n\theta)}d\beta=-\int_0^{\pi}\sin(m\beta)\frac{2\kappa\beta}{\pi}d\beta,$$

$$\left(\frac{d^2a_n(t)}{dt^2}+\frac{\sigma^2\pi^2\alpha^2n^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}a_n(t)\right)\frac{\pi}{2\cos(n\theta)}=\frac{2\kappa}{n}(-1)^n,$$

$$\frac{d^2a_n(t)}{dt^2}+\frac{\sigma^2\pi^2\alpha^2n^2}{\left(e^{-\alpha b}-e^{-\alpha a}\right)^2}a_n(t)=\frac{4\kappa\cos(n\theta)}{n\pi}(-1)^n. \quad (4.52)$$

The solution of the constant coefficient differential equation given in (4.52) is obtained like

$$a_n(t)=C_1\sin\left(\frac{\sigma\pi\alpha n}{e^{-\alpha b}-e^{-\alpha a}}t\right)+C_2\cos\left(\frac{\sigma\pi\alpha n}{e^{-\alpha b}-e^{-\alpha a}}t\right)+\frac{4\kappa\cos(n\theta)}{n\pi\mu^2\sigma^2}(-1)^n. \quad (4.53)$$

After using initial conditions, the coefficients C_1 and C_2 can be determined as

$$C_1=0 \quad \text{and} \quad C_2=-\frac{4\kappa\cos(n\theta)}{n\pi\mu^2\sigma^2}(-1)^n$$

and the final result of the function $a_n(t)$ can be obtained as

$$a_n(t)=-\frac{4\kappa\cos(n\theta)}{n\pi\mu^2\sigma^2}(-1)^n\left(\cos\left(\frac{\sigma\pi\alpha n}{e^{-\alpha b}-e^{-\alpha a}}t\right)-1\right). \quad (4.54)$$

Finally, when the value of $a_n(t)$ is substituted into (4.46) then the solution of the partial differential equation given in (4.43) can be obtained like

$$V(x,t)=\sum_{n=1}^{\infty}\left[-\frac{4\kappa\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}{n\pi\mu^2\sigma^2}(-1)^n\left(\cos\left(\frac{\sigma\pi\alpha n}{e^{-\alpha b}-e^{-\alpha a}}t\right)-1\right)\right]\frac{\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)}{\cos\left(\frac{\mu}{\alpha}e^{-\alpha a}\right)}.$$

$$V(x,t)=\sum_{n=1}^{\infty}\left[\frac{(-1)^n4\kappa}{n\pi\mu^2\sigma^2}\left(1-\cos\left(\frac{\sigma\pi\alpha n}{e^{-\alpha b}-e^{-\alpha a}}t\right)\right)\right]\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right). \quad (4.55)$$

The general solution of the problem given in beginning of the study is obtained by substituting the value of $V(x,t)$ into (4.40), along with $W(x,t)$ given in (4.42). The solution can be given as follows:

$$u(x,t)=\sum_{n=1}^{\infty}-\frac{(-1)^n 4\kappa}{n\pi}\left(\frac{(\cos(\sigma\mu t)-1)}{(\mu\sigma)^2}\right)\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)+\kappa t^2\frac{e^{-\alpha a}-e^{-\alpha x}}{e^{-\alpha a}-e^{-\alpha b}},$$

$$u(x,t)=\sum_{n=1}^{\infty}-\frac{(-1)^n 4\kappa}{n\pi}\left(\frac{(\cos(\sigma\mu t)-1)}{(\mu\sigma)^2}\right)\sin\left(\frac{\mu}{\alpha}\left(e^{-\alpha x}-e^{-\alpha a}\right)\right)+\kappa t^2 X(x). \quad (4.56)$$

CONCLUSIONS

In this study, the general formula of Adomian polynomials and Adomian decomposition method are examined in all aspects. The analysis of the method is explained and different application examples are used to explain the ADM.

The main purpose of this subject is to prove that ADM can be applied to second order wave equation with variable coefficient in nonhomogeneous material known as functionally graded materials (FGMs). Also, same problem is solved by using eigenfunction expansion method. When we compare the results, both methods give same solution. And, the solution of the problem is showed by tables and graphics on different nonhomogeneity parameters. Then, at the end of the problem solution method and numerical results are controlled for homogeneous materials by solution of close form. As a result, it is clear that the method is applicable and efficient for our problem.

In the future, this problem can be examined for different type homogeneous or nonhomogeneous equations. For instance, the exponential form can be changed and put another form instead of our main problem or homogeneous type of problem can become nonhomogeneous type.

Table 1: Comparison of approximate solution with close-form solution for $\alpha = 10^{-5}$

x	$\varepsilon = u_C(x,t) - u(x,t)$		
	$t=1$	$t=5$	$t=10$
0	0	0	0
0.1	0.99×10^{-6}	0.11×10^{-4}	0.44×10^{-4}
0.2	0.19×10^{-5}	0.22×10^{-4}	0.79×10^{-4}
0.3	0.27×10^{-5}	0.27×10^{-4}	0.10×10^{-3}
0.4	0.33×10^{-5}	0.32×10^{-4}	0.11×10^{-3}
0.5	0.37×10^{-5}	0.33×10^{-4}	0.12×10^{-3}
0.6	0.38×10^{-5}	0.32×10^{-4}	0.11×10^{-3}
0.7	0.35×10^{-5}	0.28×10^{-4}	0.10×10^{-3}
0.8	0.28×10^{-5}	0.22×10^{-4}	0.79×10^{-4}
0.9	0.17×10^{-5}	0.12×10^{-4}	0.44×10^{-4}
1.0	0	0	0

Table 2: Values of $u(x,t)$ for different time values for $\alpha = 0.5$

x	$u(x,t)$		
	$t=1$	$t=5$	$t=10$
0	0	0	0
0.1	0.0831	3.0891	12.3565
0.2	0.1620	6.0276	24.1105
0.3	0.2416	8.8228	35.2935
0.4	0.3313	11.4816	45.9442
0.5	0.4295	14.0108	56.0883
0.6	0.5350	16.4168	65.7498
0.7	0.6458	18.7118	74.9505
0.8	0.7611	20.9048	83.7124
0.9	0.8794	22.9994	92.0556
1.0	1.0000	25.0000	100.0000

Table 3: Values of $u(x,t)$ for different time values for $\alpha = 1.0$

x	$u(x,t)$		
	$t=1$	$t=5$	$t=10$
0	0	0	0
0.1	0.1400	3.7630	15.0525
0.2	0.2667	7.1681	28.6726
0.3	0.3813	10.2492	40.9966
0.4	0.4850	13.0368	52.1478
0.5	0.5797	15.5595	62.2378
0.6	0.6707	17.8418	71.3677
0.7	0.7592	19.9073	79.6286
0.8	0.8436	21.7757	87.1046
0.9	0.9241	23.4672	93.8727
1.0	1.0000	25.0000	100.0000

Table 4: Values of $u(x,t)$ for different time values for $\alpha = 1.5$

x	$u(x,t)$		
	$t=1$	$t=5$	$t=10$
0	0	0	0
0.1	0.1790	4.4767	17.9069
0.2	0.3331	8.3298	33.3197
0.3	0.4658	11.6463	46.5919
0.4	0.5800	14.5008	58.0231
0.5	0.6782	16.9578	67.8675
0.6	0.7629	19.0755	76.3449
0.7	0.8357	20.9014	83.6446
0.8	0.8982	22.4750	89.9297
0.9	0.9526	23.8314	95.3412
1.0	1.0000	25.0000	100.0000

Table 5: Values of $u(x,t)$ for different time values for $\alpha = 2.0$

x	$u(x,t)$		
	$t=1$	$t=5$	$t=10$
0	0	0	0
0.1	0.2058	5.2336	20.9356
0.2	0.3743	9.5185	38.0859
0.3	0.5122	13.0266	52.1355
0.4	0.6251	15.8999	63.6440
0.5	0.7180	18.2557	73.0699
0.6	0.7961	20.1870	80.7897
0.7	0.8616	21.7697	87.1118
0.8	0.9165	23.0668	92.2890
0.9	0.9622	24.1295	96.5285
1.0	1.0000	25.0000	100.0000

Table 6: Values of $u(x,t)$ for different time values for $\alpha = 2.5$

x	$u(x,t)$		
	$t=1$	$t=5$	$t=10$
0	0	0	0
0.1	0.2239	6.0197	24.0790
0.2	0.4015	10.7079	42.8368
0.3	0.5462	14.3590	57.4519
0.4	0.6629	17.2033	68.8382
0.5	0.7562	19.4206	77.7083
0.6	0.8303	21.1489	84.6177
0.7	0.8889	22.4957	89.9997
0.8	0.9352	23.5453	94.1918
0.9	0.9715	24.3630	97.4570
1.0	1.0000	25.0000	100.0000

Table 7: Values of $u(x,t)$ for different time values for $\alpha = 3.0$

x	$u(x,t)$		
	$t=1$	$t=5$	$t=10$
0	0	0	0
0.1	0.2546	6.8177	27.2711
0.2	0.4511	11.8685	47.4741
0.3	0.6019	15.6102	62.4411
0.4	0.7165	18.3821	73.5311
0.5	0.8030	20.4357	81.7484
0.6	0.8680	21.9576	87.8368
0.7	0.9166	23.0857	92.3477
0.8	0.9529	23.9216	95.6897
0.9	0.9799	24.5410	98.1657
1.0	1.0000	25.0000	100.0000

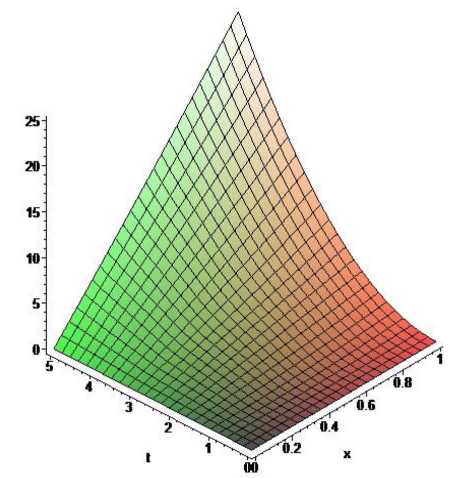

Figure 1: Function of $u(x,t)$ for exponential form when $\alpha = 0$

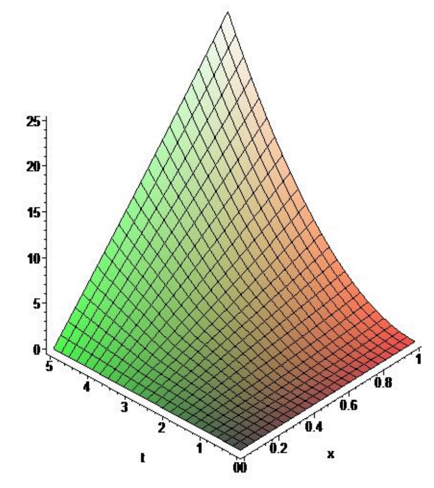

Figure 2: Function of $u(x,t)$ for exponential form when $\alpha = 10^{-5}$

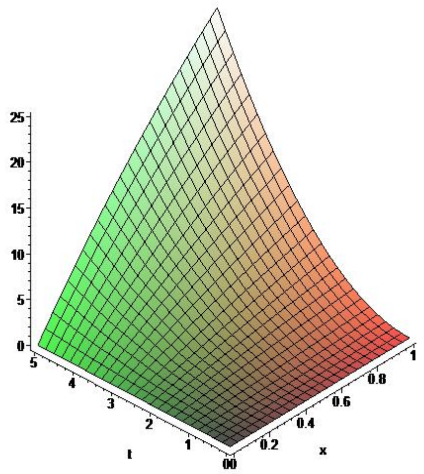

Figure 3: Function of $u(x,t)$ for exponential form when $\alpha = 0.5$

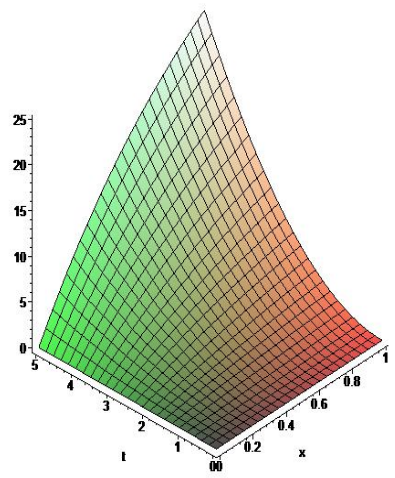

Figure 4: Function of $u(x,t)$ for exponential form when $\alpha = 1.0$

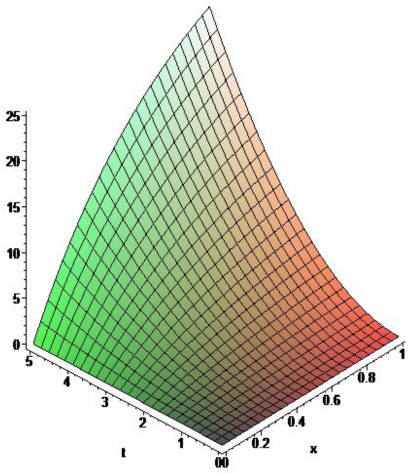

Figure 5: Function of $u(x,t)$ for exponential form when $\alpha = 1.5$

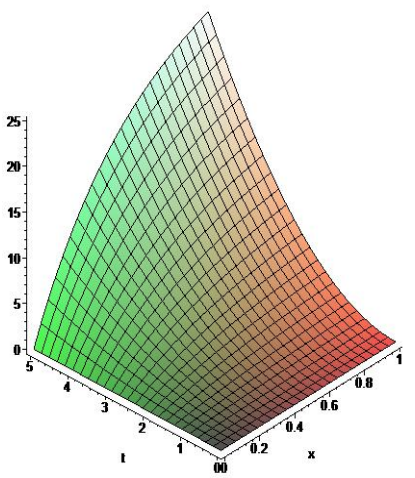

Figure 6: Function of $u(x,t)$ for exponential form when $\alpha = 2.0$

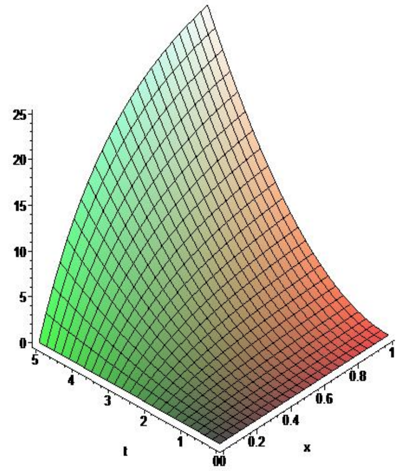

Figure 7: Function of $u(x,t)$ for exponential form when $\alpha = 2.5$

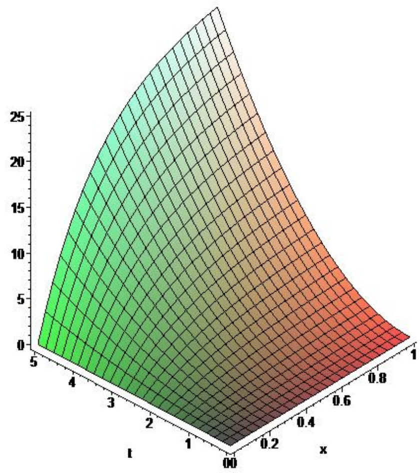

Figure 8: Function of $u(x,t)$ for exponential form when $\alpha = 3.0$

APPENDIX A

SOLUTION OF WAVE EQUATION FOR $\alpha = 0$

In this part of the study, it is examined a wave equation for homogeneous material, $\alpha = 0$, by using eigenfunction expansion method

$$\frac{\partial^2 u}{\partial t^2} = \sigma^2 \frac{\partial^2 u}{\partial x^2}, \qquad \sigma^2 = E_0 / c^2 \rho_0 , \tag{A.1}$$

Subject to the following boundary and initial conditions

$$u(a,t) = 0, \qquad u(b,t) = \kappa t^2, \qquad a \le x \le b,$$

$$u(x,0) = 0 \qquad , \qquad u_t(x,0) = 0, \qquad t \ge 0.$$

For the solution of the problem given in (A.1), it is proposed a solution

$$u(x,t) = V(x,t) + W(x,t) \tag{A.2}$$

to obtain homogeneous type BCs. The function $W(x,t)$ that satisfies the homogeneous equation given by

$$\frac{d^2 W}{dx^2} = 0 \tag{A.3}$$

along with the nonhomogeneous boundary conditions

$$W(a,t) = 0, \qquad W(b,t) = \kappa t^2, \qquad t \ge 0$$

has a solution

$$W(x,t) = \kappa t^2 \frac{x-a}{b-a}. \tag{A.4}$$

Substituting (A.4) along with (A.2) into (A.1), it is obtained the new nonhomogeneous differential equation

$$\frac{\partial^2 V}{\partial t^2} = -2\kappa\left(\frac{x-a}{b-a}\right) + \sigma^2\left(\frac{\partial^2 V}{\partial x^2}\right), \tag{A.5}$$

Subject to homogeneous boundary and initial conditions, respectively,

$$V(a,t)=0\,,\qquad V(b,t)=0\,,\qquad t\geq 0\,,$$

$$V(x,0)=0\,,\qquad V_t(x,0)=0\,,\qquad a\leq x\leq b\,.$$

From this point on, the method of eigenfunction expansion can be applied for the solution of (A.5). First, solving the equation

$$\frac{d^2\Omega(x)}{dx^2} + \mu^2\Omega(x) = 0 \tag{A.6}$$

subject to the homogeneous boundary conditions

$$\Omega(a) = \Omega(b) = 0$$

gives eigenvalues and eigenfunctions of (A.6), respectively, as

$$\mu^2 = \frac{n^2\pi^2}{(b-a)^2}\,,\ \Omega_n(x) = \frac{\sin(\mu x - \mu a)}{\cos(\mu a)}\,,\qquad n = 1,2,3,\ldots \tag{A.7}$$

Let us apply the method of eigenfunction expansion by defining $a_n(t)$ as an unknown function such as

$$V(x,t) = \sum_{n=1}^{\infty} a_n(t)\Omega_n(x) = \sum_{n=1}^{\infty} a_n(t)\frac{\sin(\mu x - \mu a)}{\cos(\mu a)} \tag{A.8}$$

and substituting (A.8) into (A.5) gives that

$$\sum_{n=1}^{\infty}\frac{d^2 a_n(t)}{dt^2}\Omega_n(x) = -2\kappa\frac{x-a}{b-a} - \sigma^2\mu^2\sum_{n=1}^{\infty} a_n(t)\Omega_n(x)\,. \tag{A.9}$$

Then, it is obtained a differential equation in terms of an unknown function $a_n(t)$ such that

$$\sum_{n=1}^{\infty}\left[a_n''(t)+\sigma^2\mu^2 a_n(t)\right]\frac{\sin(\mu x-\mu a)}{\cos(\mu a)}=-2\kappa\frac{x-a}{b-a},$$

$$\sum_{n=1}^{\infty}\left[a_n''(t)+\sigma^2\mu^2 a_n(t)\right]\frac{\sin(n\theta)}{\cos(\mu a)}=-2\kappa\frac{x-a}{b-a}$$

where

$$n\theta=\frac{n\pi(x-a)}{b-a}.$$

Using orthogonality properties of eigenfunctions, $\sin(n\theta)$, over the interval $[0,\pi]$, the summation can be reduced to a single differential equation in terms of $a_n(t)$ as follows:

$$\sum_{n=1}^{\infty}\left[a_n''(t)+\sigma^2\mu^2 a_n(t)\right]\int_0^{\pi}\sin(m\theta)\frac{\sin(n\theta)}{\cos(\mu a)}d\theta=-\int_0^{\pi}\sin(m\theta)\frac{2\kappa\theta}{\pi}d\theta,$$

$$\left[a_n''(t)+\sigma^2\mu^2 a_n(t)\right]\frac{\pi}{2\cos(\mu a)}=\frac{2\kappa(-1)^n}{n}. \qquad \text{(A.10)}$$

The solution of the constant coefficient differential equation given in (A.10) is obtained like

$$a_n(t)=C_1\sin(\sigma\mu t)+C_2\cos(\sigma\mu t)+(-1)^n\frac{4\kappa\cos(\mu a)}{n\pi\mu^2\sigma^2}. \qquad \text{(A.11)}$$

After using initial conditions, the coefficients C_1 and C_2 can be determined as

$$C_1=0 \qquad \text{and} \qquad C_2=-\frac{4\kappa\cos(\mu a)}{n\pi\mu^2\sigma^2}(-1)^n$$

and, finally, the function $a_n(t)$ can be obtained as

$$a_n(t)=\frac{4\kappa\cos(\mu a)}{n\pi\mu^2\sigma^2}(-1)^n\left(1-\cos(\sigma\mu t)\right). \qquad \text{(A.12)}$$

When the value of $a_n(t)$ is substituted into (A.8) then the solution of the partial differential equation given in (A.5) can be obtained like

$$V(x,t)=\sum_{n=1}^{\infty}\left[\frac{4\kappa(-1)^n}{n\pi\mu^2\sigma^2}\left(1-\cos(\sigma\mu t)\right)\right]\sin(\mu x-\mu a). \tag{A.13}$$

The general solution of the problem is obtained by substituting the value of $V(x,t)$ into (A.2), along with $W(x,t)$ given in (A.4). The solution can be given as follows:

$$u(x,t)=\sum_{n=1}^{\infty}\frac{(-1)^n 4\kappa}{n\pi(\mu\sigma)^2}\left(1-\cos(\sigma\mu t)\right)\sin(\mu x-\mu a)+\kappa t^2\frac{x-a}{b-a}. \tag{A.14}$$

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