DIFFERENCE SCHEMES FOR THE FRACTIONAL PARABOLIC INVERSE PROBLEM WITH AN UNKNOWN SOURCE FUNCTION

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

A lot of scientists and researchers are trying to enhance mathematical models of real life cases for investigating and understanding the behavior of them. Especially in many fields of engineering, some parameters of a given model is obtained by measuring the observed data at a certain point especially under unknown boundary condition. These problems can be modeled as inverse problem. Recently, some phenomena have been modeled and investigated as fractional inverse problems. In the present work, for a fractional inverse problem with an unknown time dependent source term, stability estimates are obtained by using operator theory approach. For the approximate solutions of the problem, the stable difference schemes which have first and second orders of accuracy are presented. The algorithm is tested in a one-dimensional fractional inverse problem.

Keywords: Fractional parabolic equations, inverse problem, difference schemes, stability.

BELİRSİZ BİR KAYNAK FONKSİYONLU KESİRLİ PARABOLİK TERS PROBLEMLER İÇİN FARK ŞEMALARI

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ÖZ

Bilim adamları ve araştırmacıların bir çoğu gerçek hayat sorunlarının tutumlarını anlamak ve araştırmak için bu problemlerin matematiksel modellerini geliştirmeye çalışırlar. Özellikle mühendisliğin bir çok dalında belirsiz sınır koşulları altında modelin bazı parametrelerinin değerlerinin gözlemlenen bilgiye göre elde edilmesini ele alan problemler ters problem olarak modellenmektedir. Son zamanlarda bazı problemler kesirli ters problem olarak araştırılarak modellenir. Bu araştırmada, belirsiz bir kaynak fonksiyonlu kesirli parabolik ters problemlerin sayısal çözümünde operatör metodu kullanılarak kararlılıklı kestirimleri elde edilmektedir. Bu problemin yaklaşık çözümü için birinci ve ikinci dereceden kararlılıklı fark şeması elde edilmiştir. Tek boyutlu kesirli ters problem algoritma ile test edildi.

Anahtar Kelimeler: Kesirli parabolik denklemler, ters problem, fark şemaları, kararlılık.

To my parents

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CHAPTER 1

INTRODUCTION

An inverse problem is the task that often occurs in many branches of science and mathematics where the values of some model parameters must be obtained from the observed data. The transformation from data to model parameters is a result of the interaction of a physical systems. Inverse problems arise in geophysics, medical imaging, remote sensing, ocean acoustic tomography, nondestructive testing, and astronomy. Some examples were given in temperature over-specification by Dehghan (Dehghan, 2001), in chemistry (chromatography) by Kimura and Suzuki (Kimura and Suzuki, 1993), in physics (optical tomography) by Gryazin, Klibanov and Lucas (Gryazin et al., 1999). In the inverse problems, the optimal overdetermination conditions are analyzed in some classical boundary conditions or similar conditions given at a point. A literature review is given (Cannon and Yin, 1990). (V.Isakov, 1998), some widely used numerical methods (linearization, variational regularization of the Cauchy problem, relaxation methods, layer-stripping and discrete methods) are summarized. Some inverse boundary value problems are given (Belov, 2002) and the generalized overdetermination conditions such as nonlocal, integral, and final overdetermination conditions are used (Prilepko and Kostin, 1992b), (Dehghan, 2006), (Dehghan, 2003), (Dehghan, 1999). Inverse problems arises in many fields of science and engineering such as ion transport problems, chromatography and heat determination problems with an unknown internal energy source. Different types of inverse problems have been investigated and the main results obtained in this field of research were given by many researchers such as (Hasanov, 2010), (Hasanov and Tatar, 2010), (Blasio and Lorenzi, 2007), (Orlovsky and Piskarev, 2009), (Eidelman, 1978), (Ashyralyev, 2010), (Prilepko and Kostin, 1992a).

Leibniz's note led to the appearance of the theory of derivatives and integrals of arbitrary. For three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics and useful only for mathematicians. Fractional integrals and derivatives also appear in the theory of control of dynamical systems, when the controlled system or/and the controller is described by a fractional differential equation (Podlubny, 1999). On the other hand, application of finite difference method on fractional derivatives developed the term *fractional difference derivative.* Finite Difference Method is used for solving several fractional differential equations, as (Zhang, 2011), (Singh et al., 2011), (Hua and Zhang, 2012), (Joaquin and Santos, 2011), (Ghazizadeh et al., 2010), (Ford and Morgado, 2000), (Dal, 2009).

Recently, many application areas such as fractional control of engineering systems, bioengineering applications, image and signal processing are related to fractional calculus. Methods of solutions of problems and theory of fractional calculus have been studied by many researchers (see (Samko et al., 1993), (Kilbas et al., 2006), (Ashyralyev et al., 2009), (Sobolevskii, 1975), (Ashyralyev, 2009), (Ashyralyev et al., 2011) and the references there in). Many scientists and researchers are trying to enhance mathematical models of real life cases for investigating and understanding the behavior of them. Therefore, some phenomena have been modeled and investigated as fractional inverse problems. Some references are (Murio and Meja, 2008), (Cheng et al., 2009), (Nakagawa et al., 2010), (Zhang and Xu, 2011), (K. Sakamoto, 2011), (Adams and Gelhar, 1992).

The main aim of this work is to investigate the stability of fractional inverse parabolic differential equation with the Dirichlet condition.

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a\frac{\partial^2 u(t,x)}{\partial x^2} - D_t^{1/2}u(t,x) + \sigma u\left(t,x\right) = p\left(t\right)q\left(x\right) + f\left(t,x\right), \\ 0 < x < \pi,\ 0 < t \leq T, \\ u\left(t,0\right) = u\left(t,\pi\right) = 0, 0 \leq t \leq T, \\ u\left(0,x\right) = \varphi\left(x\right),\ 0 \leq x \leq \pi, \\ u\left(t,x^*\right) = \rho\left(t\right),\ 0 \leq x^* \leq \pi, 0 \leq t \leq T, \end{cases} \tag{1.1}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $a > 0$ and $\sigma > 0$ is a sufficiently large number. Here $D_t^{1/2} = D_{0+}^{1/2}$ is the standard Riemann-Liouville's derivative of order $1/2$. Theorems on the stability of problem (1.1) are analyzed by assuming that $q(x)$ is a sufficiently smooth function, $q(0) = q(\pi) = 0$ and $q(x^*) \neq 0$.

It is known that fractional inverse parabolic differential equations can be solved by the Fourier series method, by the Fourier transform method and by the Laplace transform method. Now, let us illustrate these three different analytical methods by examples.

Example 1.1. Consider the following fractional differential equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \dfrac{\partial u(t,x)}{\partial x^2} + u(t,x) - D^{\frac{1}{2}}u(t,x) - p(t)\sin x \\ = \left(-3t - \dfrac{1}{\sqrt{\pi}}t^{\frac{-1}{2}} + \dfrac{2}{\sqrt{\pi}}t^{\frac{1}{2}}\right)\sin x, \\ 0 < t < 1,\ 0 < x < \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1, \\ u(0,x) = \sin x,\ 0 \le x \le \pi, \\ u\left(t,\frac{\pi}{2}\right) = 1 - t,\ 0 \le t \le 1. \end{cases} \tag{1.2}$$

Solution. Let us assume that $u(t,x) = g(t,x) + \sin x$ and $g(t,x)$ is the solution of the problem where

$$\begin{cases} \dfrac{\partial g(t,x)}{\partial t} - \dfrac{\partial g(t,x)}{\partial x^2} + g(t,x) - D^{\frac{1}{2}}g(t,x) - p(t)\sin x \\ = \left(\dfrac{2}{\sqrt{\pi}}t^{\frac{1}{2}} - 3t - 2\right)\sin x, 0 < t < 1,\ 0 \le x \le \pi, \\ g(t,0) = g(t,\pi) = 0,\ 0 \le t \le 1, \\ g(t,0) = 0,\ 0 \le x \le \pi, \\ g(t,\frac{\pi}{2}) = -t,\ 0 \le t \le 1. \end{cases}$$

We use Fourier series method. In order to solve the problem, we need to separate $g(t,x)$ into two parts

$$g(t,x) = v(t,x) + w(t,x), \tag{1.3}$$

where $v(t,x)$ is the solution of the homogenous problem

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} - \dfrac{\partial v(t,x)}{\partial x^2} + v(t,x) - D^{\frac{1}{2}}v(t,x) = p(t)\sin x, \\ 0 < t < 1,\ 0 < x < \pi, \\ v(t,0) = v(t,\pi) = 0,\ 0 \le t \le 1, \\ v(0,x) = \sin x,\ 0 \le x \le \pi \end{cases} \tag{1.4}$$

and $w(t,x)$ is the solution of nonhomogeneous problem

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} - \dfrac{\partial w(t,x)}{\partial x^2} + w(t,x) - D^{\frac{1}{2}} w(t,x) \\ = \left(\dfrac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - 3t - 2 + p(t) \right) \sin x, \\ 0 < t < 1, \; 0 < x < \pi, \\ w(t,0) = w(t,\pi) = 0, \; 0 \le t \le 1, \\ w(0,x) = 0, \; 0 \le x \le \pi. \end{cases} \tag{1.5}$$

Now, let us obtain the solution of problem (1.4), by the method of separation of variables. In order to do that, a solution of the form $v(t,x) = T(t)X(x) \neq 0$ is suggested. Taking the partial derivatives and substituting the results in (1.4), we obtain

$$\begin{cases} T'(t)X(x) - T(t)X''(x) - D^{\frac{1}{2}} T(t)X(x) + T(t)X(x) = 0, \\ \dfrac{T'(t) - D^{\frac{1}{2}} T(t)}{T(t)} = \dfrac{X''(x)}{X(x)} - 1 = \lambda - 1. \end{cases} \tag{1.6}$$

The boundary conditions presented in problem (1.4) require $X(0) = X(\pi) = 0$. Hence, from (1.6), we have the following Sturm-Louville problem:

$$\begin{cases} x''(x) - \lambda X(x) = 0, \\ X(0) = X(\pi) = 0. \end{cases} \tag{1.7}$$

If $\lambda \ge 0$, then the Sturm-Louville problem (1.7) has only trivial solution

$$X(x) = 0.$$

For $\lambda < 0$, the nontrivial solution of the Sturm-Louville problem (1.7) are

$$X_k(x) = \sin \sqrt[2]{-\lambda_k} x, \; \lambda_k = -k^2,$$

where $k = 1, 2, 3, \cdots$. The fractional differential equation presented in (1.6) is

$$T_k'(t) - D^{\frac{1}{2}} T_k(t) = (\lambda - 1) T_k(t)$$

with $k = 1, 2, 3, \cdots$, where $\lambda_k = -k^2$. So, we have

$$\begin{cases} T_k'(t) - D^{\frac{1}{2}} T_k(t) + (k^2 + 1) T_k(t) = 0, \\ T(0) = 0. \end{cases}$$

In order to solve, we need to take $T_k(t)$ as a summation of powers of $t^{\frac{n}{2}}$:

$$T_k = \sum_{n=0}^{\infty} a_n^k t^{\frac{n}{2}}. \tag{1.8}$$

Taking derivatives of (1.8), and substituting the results in the given system, we get

$$\sum_{n=1}^{\infty} \frac{n}{2} a_n^k(t) t^{\frac{n}{2}-1} - \sum_{n=1}^{\infty} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n+\frac{1}{2})} a_n^k(t) t^{\frac{n}{2}-\frac{1}{2}} + (k^2+1) \sum_{n=0}^{\infty} a_n^k(t) t^{\frac{n}{2}} = 0. \tag{1.9}$$

Using the initial condition, it is obtained

$$v(0,x) = 0 \Longrightarrow T_k(0) = 0 \Longrightarrow a_0^k = 0.$$

From equation (1.9), it follows that

$$\sum_{n=-1}^{\infty} \frac{n+2}{2} a_{n+2}^k t^{\frac{n}{2}} - \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n+1}{2}+1)}{\Gamma(\frac{n+1}{2}+\frac{1}{2})} a_n^k t^{\frac{n}{2}} + (k^2+1) \sum_{n=1}^{\infty} a_n^k t^{\frac{n}{2}} = 0. \tag{1.10}$$

Since the power series does not contain negative powers of t, we have $a_1^k = 0$. Equating coefficients of (1.10) , we get $a_n^k = 0$, for any $n \geqslant 2$. So,

$$v(t,x) = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} a_n^k t^{\frac{n}{2}} X_k(x).$$

Thus, the solution of problem (1.4) is $v(t,x) = 0$. In order to solve problem (1.5) , we need to take

$$w(t,x) = \sum_{n=1}^{\infty} c_n(t) \sin nx.$$

Taking partial derivatives and substituting them in (1.5), we get

$$\sin nx \left(\sum_{n=1}^{\infty} c_n'(t) + n^2 c_n(t) - D^{\frac{1}{2}} c_n(t) + 1 \right) = \left(\frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} - 3t - 2 + p(t) \right) \sin x.$$

From the initial condition

$$w(0,x) = 0,$$

we obtain

$$c_n(0) = 0, n = 1, 2, \cdots .$$

Then, for $n = 1$

$$c_1'(t) + 2c_1(t) - D^{\frac{1}{2}} c_1(t) = \left(\frac{2}{\sqrt{\pi}} - 3t - 2 + p(t) \right), c_1(0) = 0$$

and for $n \neq 1$

$$c_n'(t) + (n^2+1)c_n(t) - D^{\frac{1}{2}}c_n(t) = 0, c_n(0) = 0$$

can be written. Thus, the solution of (1.5) is $w(t,x) = c_1(t)\sin x$. Substituting $v(t,x)$ and $w(t,x)$ in the formula (1.3) , we obtain

$$g(t,x) = c_1(t)\sin x.$$

Using the overdetermined condition, we get

$$g\left(t, \frac{\pi}{2}\right) = c_1(t) = -t$$

and

$$g(t,x) = -t\sin x.$$

Putting $c_1(t) = -t$, we reach to

$$p(t) = 1 + t.$$

Substituting $g(t,x)$ in formula of $u(t,x)$

$$u(t,x) = (1-t)\sin x$$

is the solution of fractional parabolic differential equation (1.2) .

Note that using similar manner one can obtain the solution of the following problem for the multidimensional fractional parabolic equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \sum\limits_{|r|=2m} \alpha_r(x) \dfrac{\partial^{|r|}u(t,x)}{\partial x_1^{r_1}...\partial x_n^{r_n}} - D^{\frac{1}{2}}u(t,x) + \delta u(t,x) \\ \\ = f(t,x) + p(t)q(x), \\ x = (x_1, ...x_n) \in \Omega, \ 0 < t < T, \ \ |r| = r_1 + r_2 + ...r_n, \\ \\ u(0,x) = \varphi(x), \ x \in \overline{\Omega}, \\ \\ u(t,x) = 0, \ 0 < t < T, \ x \in S, \\ \\ u(t,x^*) = \omega(t), \ 0 \le t \le T, \ x^* \in \Omega \end{cases}$$

where $x \in \Omega$ and $q(x), \alpha_r(x)$ $(x \in \Omega), \varphi(x), \psi(x)$ $(x \in \overline{\Omega}), \omega(t)$ $(t \in [0,T])$ $f(t,x)(t \in (0,T),\ x \in \Omega)$ are given smooth functions, $u(t,x)$ and $p(t)$ are unknown functions $\alpha_r(x) \geqslant a > 0$. Here, Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n(0 < x_m < 1, 1 \le m \le n)$ with boundary $S, \overline{\Omega} = \Omega \cup S$. However, the method of separation of variables can be used only in the case when (1.2) has constant coefficient.

Example 1.2. Now, we will consider the application of Laplace transformation method (in x) to the following problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) - D^{\frac{1}{2}}u(t,x) - p(t)\sin x \\ = \left(-3t - \frac{1}{\sqrt{\pi}}t^{\frac{-1}{2}} + \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}\right)\sin x, \\ 0 \le t \le 1,\ 0 \le x < \infty, \\ u(t,0) = 0,\ u_x(t,0) = 1-t,\ 0 \le t \le 1, \\ u(0,x) = \sin x,\ 0 \le x < \infty, \\ u\left(t, \frac{\pi}{2}\right) = 1-t,\ 0 \le t \le 1. \end{cases} \tag{1.11}$$

Solution. We denote

$$L\left\{u(t,x)\right\} = v(t,s).$$

Then, by using the properties of the Laplace transform method,

$$L\left\{u_{xx}(t,x)\right\} = s^2 v(t,s) - sv(t,0) - v_x(t,s) \ = s^2 v(t,s) + t - 1,$$

and

$$\begin{aligned} & L\left\{u_t(t,x)\right\} - L\left\{u_{xx}(t,x)\right\} + L\left\{u(t,x)\right\} - L\left\{D^{\frac{1}{2}}u(t,x)\right\} \\ = & \left(-3t - \frac{1}{\sqrt{\pi}}t^{\frac{-1}{2}} + \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} + p(t)\right) L\left\{\sin x\right\}. \end{aligned}$$

are obtained. Then, problem (1.11) turns into the following problem

$$\begin{cases} v_t(t,s) - (s^2-1)v(t,s) - D^{\frac{1}{2}}v(t,s) - t + 1 \\ = \left(-3t - \frac{1}{\sqrt{\pi}}t^{\frac{-1}{2}} + \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} + p(t)\right)\frac{1}{s^2+1} \\ v(0,s) = \frac{1}{s^2+1}. \end{cases} \tag{1.12}$$

In order to solve problem (1.12), we take

$$v(t,s) = \sum_{k=0}^{\infty} a_k(s) t^{\frac{k}{2}} = a_0(s) + \sum_{k=1}^{\infty} a_k(s) t^{\frac{k}{2}} \tag{1.13}$$

and

$$p(t)=\sum_{k=0}^{\infty} c_k(s)t^{\frac{k}{2}}.$$

Using the initial condition of problem (1.12) and from equation (1.13) it follows that

$$a_0(s)=\frac{1}{s^2+1}$$

and

$$\sum_{k=0}^{\infty} a_k(s)\left(\tfrac{k}{2}t^{\frac{k}{2}-1}-(s^2-1)t^{\frac{k}{2}}-\frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}t^{\frac{k}{2}-\frac{1}{2}}\right)-t+1$$
$$=\left(-3t-\frac{1}{\sqrt{\pi}}t^{\frac{-1}{2}}+\frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}+\sum_{k=0}^{\infty} c_k t^{\frac{k}{2}}\right)\frac{1}{s^2+1}.$$

Equating the coefficients of $t^{\frac{k}{2}}$ for $k=-1,0,\cdots$, we obtain that

$$t^{\frac{-1}{2}}:\frac{a_1(s)}{2}-a_0(s)\frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}=\frac{-1}{\sqrt{\pi}(1+s^2)}$$
$$\Longrightarrow a_1(s)=0,$$

$$t^0: a_2(s)-(s^2-1)a_0(s)-a_1(s)\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}+1=\frac{c_0}{s^2+1}$$
$$\Longrightarrow a_2(s)=\frac{c_0-2}{s^2+1},$$

$$t^{\frac{1}{2}}: a_3(s)\tfrac{3}{2}-(s^2-1)a_1(s)-a_2(s)\frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)}=\frac{2}{\sqrt{\pi}(s^2+1)}+\frac{c_1}{s^2+1}$$
$$\Longrightarrow a_3(s)=\frac{2}{\sqrt{\pi}(s^2+1)}+\frac{2(c_0-2)}{\sqrt{\pi}(s^2+1)}+\frac{c_1}{s^2+1}.$$

$$t: 2a_4(s)-(s^2-1)a_2(s)-a_3(s)\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}-1=\frac{-3+c_2}{s^2+1}$$
$$\Longrightarrow a_4(s)=\frac{(s^2-1)(c_0-1)}{2(s^2+1)}+\frac{(c_0-2)}{2(s^2+1)}+\frac{c_1\sqrt{\pi}}{2(s^2+1)}.$$

and so on. Thus,

$$v(t,s)=\frac{1}{s^2+1}+\frac{c_0-2}{s^2+1}t+a_3(s)t^{\frac{3}{2}}+a_4(s)t^2+\cdots. \tag{1.14}$$

Taking the inverse Laplace transform of problem (1.14),

$$u(t,x)=\sin x+(c_0-2)t\sin x+L^{-1}\left\{a_3(s)\right\}t^{\frac{3}{2}}+L^{-1}\left\{a_4(s)\right\}t^2+\cdots.$$

Using overdetermination condition,

$$\begin{aligned} u\left(t, \frac{\pi}{2}\right) &= 1 + (c_0 - 2)t + L^{-1}\left\{a_3(s)t^{\frac{3}{2}} + a_4(s)t^2 + \cdots\right\}\Big]_{x=\frac{\pi}{2}} \\ &= 1 - t \Longrightarrow c_0 = 1 \end{aligned}$$

$$L^{-1}\{a_3(s)\} = L^{-1}\{a_4(s)\} = \cdots = 0$$

$$a_3(s) = a_4(s) = \cdots = 0.$$

Since $c_0 = 1$, we get

$$\Longrightarrow a_2(s) = -\frac{1}{s^2+1},$$

$$\Longrightarrow a_3(s) = \frac{c_1}{s^2+1} = 0 \Longrightarrow c_1 = 0,$$

It implies that

$$\begin{aligned} &\Longrightarrow a_4(s) = \frac{(s^2-1)(1-1) + (1-2) + 0 + c_2}{2(s^2+1)} = 0 \\ &\Longrightarrow c_2 = 1, \end{aligned}$$

and

$$c_k = 0$$

for $k \geqslant 3$. Then, we get

$$u(t, x) = (1 - t)\sin x$$

and

$$p(t) = 1 + t.$$

Note that using the same manner one can obtain the solution of the following problem for the multidimensional fractional parabolic equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \sum\limits_{|r|=2m} \alpha_r(x) \dfrac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} ... \partial x_n^{r_n}} - D^{\frac{1}{2}} u(t,x) + \delta u(t,x) \\ = f(t,x) + p(t)q(x), \\ x = (x_1, \cdots, x_n) \in \overline{\Omega^+},\ 0 < t < T,\ \ |r| = r_1 + r_2 + \cdots r_n, \\ u(0,x) = \varphi(x),\ x \in \overline{\Omega^+}, \\ u(t,x) = 0,\ u_x(t,x) = 0,\ 0 < t < T,\ x \in S, \\ u(t,x^*) = \omega(t),\ 0 \le t \le T,\ x^* \in \Omega \end{cases}$$

where $\Omega^+ = \{(x_1, x_2, .., x_n) : 0 < x_m < \infty,\ 1 \le m < n\}$ with boundary $S, \overline{\Omega}^+ = \Omega^+ \cup S$ and $q(x), \alpha_r(x)$ $(x \in \overline{\Omega}^+), \varphi(x), \psi(x)$ $(x \in \overline{\Omega}^+), \omega(t)$ $(t \in [0,T]) f(t,x)$ $(t \in (0,T),\ x \in \Omega^+)$ are given smooth functions. $u(t,x)$ and $p(t)$ are unknown functions and $\alpha_r(x) \geqslant a > 0$.

However, the Laplace transform method can be used only in the case when (1.11) has the constant coefficients.

Example 1.3. The last example is the solution of the fractional parabolic differential equation by Fourier transform method. Consider the following application of the problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) - D^{\frac{1}{2}} u(t,x) - p(t) e^{-x^2} \\ = \left(1 - t(4x^2 - 2) - \dfrac{2}{\sqrt{\pi}} t^{\frac{1}{2}}\right) e^{-x^2}, \\ 0 < t < 1,\ \ -\infty < x < \infty, \\ u(0,x) = 0,\ \ -\infty < x < \infty, \\ u\,(t,1) = te^{-1},\ 0 \le t \le 1. \end{cases} \tag{1.15}$$

Solution. Firstly, we denote $F\{u(t,x)\} = v(t,s)$. Since

$$F\{u_{xx}(t,x)\} = -s^2 v(t,x), \quad F\{u(0,x)\} = v(0,s) = 0,$$

and

$$F\left\{-(4x^2-2)e^{x^2})\right\} = s^2 F\left\{e^{-x2}\right\},$$

we can write

$$\begin{cases} v_t(t,s) + (s^2+1)v(t,s) - D^{\frac{1}{2}}v(t,s) \\ = \left(1 - \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} + p(t)\right) F\left\{e^{-x^2}\right\} + s^2 t F\left\{e^{-x^2}\right\}, \\ v(0,s) = 0. \end{cases} \tag{1.16}$$

For solving problem (1.16), let us take

$$v(t,s) = \sum_{k=0}^{\infty} a_k(s) t^{\frac{k}{2}} = a_0(s) + \sum_{k=1}^{\infty} a_k(s) t^{\frac{k}{2}}, p(t) = \sum_{k=0}^{\infty} c_k(s) t^{\frac{k}{2}}.$$

We have that

$$v(0,s) = a_0(s) = 0.$$

$$\sum_{k=1}^{\infty} a_k(s)\frac{k}{2}t^{\frac{k}{2}-1} + (s^2+1)\sum_{k=0}^{\infty} a_k(s)\frac{k}{2}t^{\frac{k}{2}} - \sum_{k=0}^{\infty} a_k(s)\frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+\frac{1}{2})}t^{\frac{k}{2}-\frac{1}{2}}$$
$$= \left(1 - \frac{\sqrt{\pi}}{2}t^{\frac{1}{2}} + \sum_{k=0}^{\infty} c_k t^{\frac{k}{2}}\right) F\left\{e^{-x^2}\right\} + s^2 t F\left\{e^{-x^2}\right\}.$$

Equating the coefficients of $t^{\frac{k}{2}}$, $k = 1, 2, 3\cdots$, we get

$$\begin{aligned} t^{\frac{-1}{2}} \quad &: \quad \frac{a_1(s)}{2} - a_0(s)\frac{\Gamma(1)}{\Gamma(\frac{1}{2})} = 0 \\ &\Longrightarrow \quad a_1(s) = 0, \end{aligned}$$

$$\begin{aligned} t^0 \quad &: \quad a_2(s) + (s^2+1)a_0(s) - a_1(s)\frac{\Gamma(\frac{3}{2})}{\Gamma(1)} = F\left\{e^{-x^2}\right\}(1+c_0) \\ &\Longrightarrow \quad a_2(s) = (1+c_0)F\left\{e^{-x^2}\right\}, \end{aligned}$$

$$\begin{aligned} t^{\frac{1}{2}} \quad &: \quad a_3(s)\frac{3}{2} + (s^2+1)a_1(s) - a_2(s)\frac{\Gamma(2)}{\Gamma(\frac{3}{2})} = \left(\frac{-2}{\sqrt[2]{\pi}} + c_1\right) F\left\{e^{-x^2}\right\} \\ &\Longrightarrow \quad a_3(s) = \frac{2}{3}\left(c_1 + c_0\frac{2}{\sqrt{\pi}}\right) F\left\{e^{-x^2}\right\}, \end{aligned}$$

$$t \quad : \quad 2a_4(s) + (s^2{+}1)a_2(s) - a_3(s)\frac{\Gamma(\frac{5}{2})}{\Gamma(2)} = \left(c_2 + s^2\right) F\left\{e^{-x^2}\right\}$$

$$\Longrightarrow \quad a_4(s) = \frac{1}{2}\left(c_2 + \frac{2}{3}c_1 - \left(s^2 + 1 - \frac{4}{3\sqrt{\pi}}\right)c_0 - 1\right) F\left\{e^{-x2}\right\},$$

and so on. Thus,

$$v(t,s) = (1+c_0)tF\left\{e^{-x^2}\right\} + \frac{2}{3}\left(c_1 + c_0\frac{2}{\sqrt{\pi}}\right)t^{\frac{3}{2}}F\left\{e^{-x^2}\right\} + \cdots.$$

Finally, taking the inverse Fourier transform of $v(t,s)$,

$$u(t,x) = F^{-1}\left\{u(t,s)\right\} = (1+c_0)te^{-x^2} + \frac{2}{3}\left(c_1 + c_0\frac{2}{\sqrt{\pi}}\right)t^{\frac{3}{2}}e^{-x^2} + \cdots,$$

is obtained. From overdetermination condition

$$u(t,1) = (1+c_0)te^{-1} + \frac{2}{3}\left(c_1 + c_0\frac{2}{\sqrt{\pi}}\right)t^{\frac{3}{2}}e^{-1} + \cdots = te^{-1},$$

it follows that

$$\Longrightarrow c_0 = 0,\ c_1 = 0,\ a_3(s) = a_4(s) = a_5(s) = \cdots = 0,$$

Since

$$\Longrightarrow a_4(s) = 0,\ k \geqslant 4$$

and

$$c_2 = 1.$$

We get

$$p(t) = c_2 t^{\frac{2}{2}} = t$$

and

$$u(t,x) = te^{-x^2}.$$

Note that using the same manner one obtains the solution of the following boundary value problem for the multidimensional parabolic equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \sum\limits_{|r|=2m} \alpha_r(x) \dfrac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + D^{\frac{1}{2}} u(t,x) + \delta u(t,x) \\ = f(t,x) + p(t) q(x), \\ x = (x_1, \cdots, x_n) \in R^n, \quad 0 < t < T, \ |r| = r_1 + r_2 + \cdots + r_n, \\ \\ u(0,x) = \varphi(x), x \in R^n, \\ \\ u(t, x^*) = \omega(t), 0 \le t \le T, \ x^* \in \Omega \subset R^n, \end{cases}$$

where $q(x)$, $\alpha_r(x)$, $\varphi(x)$ $(x \in R^n)$, $\omega(t)$ $(t \in ([0,T])$ and $f(t,x)$ $(t \in (0,T), \ x \in \Omega)$ are given smooth functions, $u(t,x)$ and $p(t)$ are unknown functions, $\alpha_r(x) \ge a > 0$ and $\delta > 0$ is a sufficiently large number.

However, the Fourier transform method can be used only in the case when it has constant coefficients. The most useful method for solving partial differential equations with dependent coefficients in x and in the space variables is the difference method.

In the present work, the well-posedness of the right-hand side identification problem for one dimensional fractional parabolic equation is considered. The difference schemes of the first and second orders of accuracy of these problems are presented. Let us briefly describe the contents of the various sections. It consists of four chapters and an appendix.

First chapter is the introduction.

Second chapter is on the stability analysis of fractional inverse parabolic problem.

Third chapter is the first and second order of difference schemes. Also numerical analysis is given.

Fourth chapter is the conclusion.

Appendix-A is Matlab program.

CHAPTER 2

STABILITY ANALYSIS

In this thesis, stability estimates for the solution of (1.1) is investigated. For the mathematical substantion, we introduce the Banach space $\overset{\circ}{C}{}^{\alpha}[0,\pi]$, $\alpha \in (0,1)$, of all continuous functions $\phi(x)$ defined on $[0,\pi]$ with $\phi(0)=\phi(\pi)=0$ satisfying a Hölder condition for which the following norm is finite

$$\|\phi\|_{\overset{\circ}{C}{}^{\alpha}[0,\pi]} = \|\phi\|_{C[0,\pi]} + \sup_{0<x<x+h<\pi} \frac{|\phi(x+h)-\phi(x)|}{h^{\alpha}},$$

where $C[0,\pi]$ is the space of all continuous function $\phi(x)$ defined on $[0,\pi]$ with the norm

$$\|\phi\|_{C[0,\pi]} = \max_{0\le x\le \pi} |\phi(x)|.$$

With the help of a positive operator A, we introduce the fractional spaces $E_{\alpha}, 0<\alpha<1$, consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_{E_{\alpha}} = \|v\|_{E} + \sup_{\lambda>0} \lambda^{1-\alpha} \|A\exp\{-\lambda A\} v\|_{E}. \tag{2.1}$$

Throughout the thesis, positive constants will be indicated by $M_i(\alpha,\beta,\cdots)$. Here variables are used to focus on the fact that the constant depends only on $\alpha,\beta,\cdots$ and the sub-index i is used to indicate a different constant.

Theorem 2.1 Let $\varphi \in \overset{\circ}{C}{}^{2\alpha+2}[0,\pi]$, $F \in C\left([0,T], \overset{\circ}{C}{}^{2\alpha}[0,\pi]\right)$ and $\rho'(t) \in C[0,T]$. Then for the solution of problem (1.1), the following coercive stability estimates

$$\|u_t\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha}[0,\pi]\right)} + \|u\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha+2}[0,\pi]\right)} \le M(x^*,q)\|\rho'\|_{C[0,T]}$$
$$+M(a,\delta,\sigma,\alpha,x^*,q,T)\left(\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,\pi]} + \|F\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha}[0,\pi]\right)} + \|\rho\|_{C[0,T]}\right),$$

$$\|p\|_{C[0,T]} \le M(x^*,q)\|\rho'\|_{C[0,T]}$$
$$+M(a,\delta,\sigma,\alpha,x^*,q,T)\left[\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,\pi]} + \|F\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha}[0,\pi]\right)} + \|\rho\|_{C[0,T]}\right]$$

hold.

Proof. Let us search for the solution of the inverse problem (1.1) in the following form (see (Borukhov and Vabishchevich, 2000))

$$u(t,x)=\eta(t)\,q(x)+w(t,x), \tag{2.2}$$

where

$$\eta(t)=\int_0^t p(s)\,ds.$$

Using the overdetermined condition, it is easy to see that

$$\eta(t)=\frac{\rho(t)-w(t,x^*)}{q(x^*)} \tag{2.3}$$

and

$$p(t)=\frac{\rho'(t)-w_t(t,x^*)}{q(x^*)}. \tag{2.4}$$

Using the identity (2.4) and the triangle inequality, it follows that

$$\begin{aligned} |p(t)| &= \left|\frac{\rho'(t)-w_t(t,x^*)}{q(x^*)}\right| \le M(x^*,q)\left(|\rho'(t)|+|w_t(t,x^*)|\right) \\ &\le M(x^*,q)\left(\max_{0\le t\le T}|\rho'(t)|+\max_{0\le t\le T}\max_{0\le x\le\pi}|w_t(t,x)|\right) \\ &\le M(x^*,q)\left(\max_{0\le t\le T}|\rho'(t)|+\max_{0\le t\le T}\|w_t(t)\|_{\overset{\circ}{C}{}^{2\alpha}[0,\pi]}\right) \end{aligned} \tag{2.5}$$

for any $t, t\in[0,T]$. Here, $w(t,x)$ is the solution of the following problem

$$\begin{cases} \frac{\partial w(t,x)}{\partial t}-a\frac{\partial^2 w(t,x)}{\partial x^2}-a\frac{\rho(t)-w(t,x^*)}{q(x^*)}\frac{d^2q(x)}{dx^2}-D_t^{1/2}w(t,x) \\ -\frac{D_t^{1/2}\rho(t)-D_t^{1/2}w(t,x^*)}{q(x^*)}q(x)+\sigma\frac{\rho(t)-w(t,x^*)}{q(x^*)}q(x) \\ +\sigma w(t,x)=f(t,x),0<x<\pi,\ 0<t\le T, \\ \\ w(t,0)=w(t,\pi)=0,0\le t\le T, \\ w(0,x)=\varphi(x),\ 0\le x\le\pi. \end{cases} \tag{2.6}$$

For simplicity, we assign

$$F(t,x)=\frac{a\rho(t)}{q(x^*)}\frac{d^2q(x)}{dx^2}-\frac{\sigma\rho(t)}{q(x^*)}q(x)+\frac{D_t^{1/2}\rho(t)}{q(x^*)}q(x)+f(t,x),$$

$$\begin{aligned} G(t,x) &= Q_1(q,\rho,x,x^*,t)\,w(t,x^*)+Q_2(q,x,x^*)\,D_t^{1/2}w(t,x^*) \\ &\quad +D_t^{1/2}w(t,x), \end{aligned}$$

where

$$Q_1(q,\rho,x,x^*,t)=\frac{1}{q(x^*)}\left(-a\frac{d^2q(x)}{dx^2}+\sigma\rho(t)\right)$$

and

$$Q_2\left(q,x,x^*\right)=-\frac{q\left(x\right)}{q\left(x^*\right)}.$$

Note that, functions $F\left(t,x\right),Q_1\left(q,\rho,x,x^*,t\right)$ and $Q_2\left(q,x,x^*\right)$ only contain given functions. Then, we can rewrite problem (2.6) as,

$$\begin{cases} \frac{\partial w(t,x)}{\partial t}-a\frac{\partial^2 w(t,x)}{\partial x^2}+\sigma w\left(t,x\right)=F\left(t,x\right)+G\left(t,x\right), \\ 0<x<\pi,\ 0<t\leq T, \\ \\ w\left(t,0\right)=w\left(t,\pi\right)=0, 0\leq t\leq T, \\ w\left(0,x\right)=\varphi\left(x\right),\ 0\leq x\leq\pi. \end{cases} \tag{2.7}$$

So, the end of proof of Theorem 2.1 is based on estimate (2.5) and the following theorem.

Theorem 2.2. For the solution of problem (2.6), the following coercive stability estimate

$$\|w_t\|_{\overset{\circ}{C}^{2\alpha}[0,\pi]}\leq M\left(a,\delta,\sigma,\alpha,x^*,q,T\right)$$
$$\times\left(\|\varphi\|_{\overset{\circ}{C}^{2\alpha+2}[0,\pi]}+\|F\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,\pi]\right)}+\|\rho\|_{C[0,T]}\right)$$

holds.

Proof. In a Banach space $E=\overset{\circ}{C}[0,\pi]$, with the help of the positive operator A defined by

$$Au=-a(x)\frac{\partial^2 u\left(t,x\right)}{\partial x^2}+\sigma u,$$

with

$$D\left(A\right)=\left\{u\left(x\right):u,u',u''\in C[0,\pi],u\left(0\right)=u\left(\pi\right)=0\right\},$$

where σ is a positive constant, the problem (2.6) can be written in the abstract form as an initial-value problem

$$\begin{cases} w_t+Aw=F\left(t\right)+G\left(t\right), 0<t\leq T, \\ w\left(0\right)=\varphi. \end{cases} \tag{2.8}$$

By the Cauchy formula, the solution can be written as

$$w\left(t\right)=e^{-tA}\varphi-\int_0^t e^{-(t-s)A}\left(F\left(s\right)+G\left(s\right)\right)ds.$$

Applying the formula

$$D_t^{1/2}u\left(t\right)=\int_0^t\frac{u'\left(\zeta\right)d\xi}{\sqrt{\pi}\left(t-\xi\right)^{1/2}},$$

we get the following presentation of the solution of the abstract problem (2.8)

$$D^{1/2}w(t) = -\int_0^t \frac{Ae^{-\xi A}\varphi}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi - \int_0^t \frac{F(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi$$

$$-\int_0^t \frac{G(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi + \int_0^t\int_0^\xi \frac{Ae^{-(\xi-s)A}F(s)}{\sqrt{\pi}(t-\xi)^{1/2}}dsd\xi$$

$$+\int_0^t\int_0^\xi \frac{Ae^{-(\xi-s)A}G(s)}{\sqrt{\pi}(t-\xi)^{1/2}}dsd\xi.$$

Changing the order of integration, we obtain that

$$D^{1/2}w(t) = -\int_0^t \frac{Ae^{-\xi A}\varphi}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi - \int_0^t \frac{F(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi$$

$$+\int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi F(s)\,ds - \int_0^t \frac{G(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi$$

$$+\int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi G(s)\,ds = \sum_{k=1}^5 J_k,$$

where

$$J_1(t) = -\int_0^t \frac{Ae^{-\xi A}\varphi}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi,$$

$$J_2(t) = -\int_0^t \frac{F(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi,$$

$$J_3(t) = \int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi F(s)\,ds,$$

$$J_4(t) = -\int_0^t \frac{G(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi,$$

$$J_5(t) = \int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi G(s)\,ds.$$

Now, we estimate $J_k(t)$ for any $k = 1,2,3,4,5$ separately. It is known that (Ashyralyev and Sobolevskii, 1994),

$$\left\|A^\alpha e^{-tA}\right\|_{E\to E} \leq Mt^{-\alpha}, 0 \leq \alpha \leq 1. \tag{2.9}$$

Since operator A and $\exp(-tA)$ commute,

$$\left\|Ae^{-tA}\varphi\right\|_{E_\alpha} \leq \left\|e^{-tA}\right\|_{E_\alpha\to E_\alpha}\|A\varphi\|_{E_\alpha} \leq \left\|e^{-tA}\right\|_{E\to E}\|A\varphi\|_{E_\alpha}. \tag{2.10}$$

Applying the definition of norm of the spaces E_α and equations (2.9) and (2.10), we get

$$\|J_1(t)\|_{E_\alpha} = \left\|\int_0^t \frac{Ae^{-\xi A}\varphi}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi\right\|_{E_\alpha} \leq M_1\|A\varphi\|_{E_\alpha} \tag{2.11}$$

for any t, $t\in[0,T]$. Estimation of $J_2(t)$ is as follows:

$$\|J_2(t)\|_{E_\alpha} = \left\|\int_0^t \frac{F(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi\right\|_{E_\alpha}$$

$$\leq \|F\|_{C(E_\alpha)}\int_0^t \frac{1}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi \leq M_2\|F\|_{C(E_\alpha)}. \tag{2.12}$$

Let us estimate $J_3(t)$.

$$\begin{aligned}\|J_3(t)\|_{E_\alpha} &= \left\|\int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi F(s)\,ds\right\|_{E_\alpha}\\ &\leq \left\|\int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi ds\right\|_{E_\alpha\longrightarrow E_\alpha}\|F\|_{C(E_\alpha)}.\end{aligned}$$

It is proven that (see (Ashyralyev, 2011))

$$\left\|\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi\right\|_{E\to E} \leq \frac{M}{\sqrt{t-s}}. \tag{2.13}$$

Using the definition of norm of the spaces E_α, we can obtain that

$$\left\|\int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi ds\right\|_{E_\alpha\longrightarrow E_\alpha} = \int_0^t\left\|\int_s^t \frac{Ae^{-(t-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi\right\|_{E\to E} ds$$

$$+\sup_{\lambda>0}\int_0^t\left\|\int_s^t \lambda^{1-\alpha}Ae^{-\lambda A}\frac{Ae^{-(t-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi\right\|_{E\to E} ds$$

Using estimates (2.9) and (2.13), we get

$$\begin{aligned}\|J_3(t)\|_{E_\alpha} &\leq \left\|\int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}d\xi ds\right\|_{E_\alpha\longrightarrow E_\alpha}\|F\|_{C(E_\alpha)}\\ &\leq M_3\|F\|_{C(E_\alpha)}.\end{aligned} \tag{2.14}$$

Expanding $G(s)$ and using formulas , estimation of $J_4(t)$ is as follows:

$$\|J_4(t)\|_{E_\alpha} \le \int\limits_0^t \left\| \frac{Q_1(q,\rho,x,x^*,t)\, w(\xi,x^*)}{\sqrt{\pi}\,(t-\xi)^{1/2}} \right\|_{E_\alpha} d\xi$$

$$+\int\limits_0^t \left\| \frac{Q_2(q,x,x^*)\, D_t^{1/2} w(\xi,x^*)}{\sqrt{\pi}\,(t-\xi)^{1/2}} \right\|_{E_\alpha} d\xi + \int\limits_0^t \left\| \frac{D_t^{1/2} w(\xi,x)}{\sqrt{\pi}\,(t-\xi)^{1/2}} \right\|_{E_\alpha} d\xi.$$

It is known that (see (Ashyralyev, 2009))

$$\|w(t)\|_{E_\alpha} \le M \left\| D_t^{1/2} w(t) \right\|_{E_\alpha}. \tag{2.15}$$

Since $Q_1(q,\rho,x,x^*,T)$ and $Q_2(q,x,x^*)$ are known functions, it is easy to see that

$$\|J_4(t)\|_{E_\alpha} \le M_4(q,\rho,x,x^*,T)_0^t \frac{1}{\sqrt{\pi}(t-\xi)^{1/2}} \left\| D_t^{1/2} w(\xi) \right\|_{E_\alpha} d\xi. \tag{2.16}$$

Estimation of $J_5(t)$ can be given similar to the estimation of $J_4(t)$. By equations(2.9) and(2.15)

$$\begin{aligned} \|J_5(t)\|_{E_\alpha} &\le \left\| {}_{0s}^{t\,t} \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}} d\xi G(s) ds \right\|_{E_\alpha} \\ &\le M_5(q,\rho,x,x^*,T)_0^t \left\| D_t^{1/2} w(s) \right\|_{E_\alpha} ds. \end{aligned} \tag{2.17}$$

Finally, combining the estimates (2.11), (2.12), (2.14), (2.16) and (3.4), we get

$$\left\| D_t^{1/2} w \right\|_{E_\alpha} \le M_1 \|A\varphi\|_{E_\alpha} + (M_2 + M_3)\|F\|_{C(E_\alpha)}$$

$$+\int\limits_0^t \left(\frac{M_4}{\sqrt{\pi}(t-\xi)^{1/2}} + M_5 \right) \left\| D_t^{1/2} w(s) \right\|_{E_\alpha} ds.$$

Using the Gronwall's inequality, we can write

$$\left\| D_t^{1/2} w \right\|_{E_\alpha} \le e^{M_6} \left(M_1 \|A\varphi\|_{E_\alpha} + M_7 \|F\|_{C[E_\alpha]} \right).$$

From the last estimate, we can obtain the estimate for $w_t(t)$ by using problem (2.8) and well-posedness of the Cauchy problem in $C(E_\alpha)$ (see (Ashyralyev and Sobolevskii, 1994)). So, the following theorem finishes the proof of Theorem 2.2.

Theorem 2.3. (Ashyralyev, 2007) The spaces $E_\alpha\left(C[0,\pi], A\right)$ and $C^{2\alpha}[0,\pi]$ coincide for any $0 < \alpha < \frac{1}{2}$ and their norms are equivalent.

CHAPTER 3

NUMERICAL RESULTS

For numerical analysis, we consider the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u\left(t,x\right)}{\partial t}=\dfrac{\partial^{2}u\left(t,x\right)}{\partial x^{2}}-u\left(t,x\right)+D_{t}^{\frac{1}{2}}u(t,x)+p(t)\sin x+f\left(t,x\right), \\ f\left(t,x\right)=\left(-3t-\dfrac{1}{\sqrt{\pi}}t^{-\frac{1}{2}}+\dfrac{2}{\sqrt{\pi}}t^{\frac{1}{2}}\right)\sin x,\ \ x\in\left(0,\pi\right),\ \ t\in\left(0,1\right], \\ u(0,x)=\sin x,\ \ x\in[0,\pi], \\ u(t,0)=u(t,\pi)=0,\ \ t\in[0,1], \\ u(t,\frac{\pi}{2})=1-t. \end{cases} \tag{3.1}$$

The exact solution of the given problem is $u\left(t,x\right)=\left(1-t\right)\sin x$ and for the control parameter $p\left(t\right)$ is $1+t$.

3.1 THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

For the approximate solution of problem (3.1) , the Rothe difference scheme

$$\begin{cases} \dfrac{u_n^k - u_n^{k-1}}{\tau} = \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} - u_n^k + D_\tau^{\frac{1}{2}} u_n^k + p^k q_n + f\left(t_k, x_n\right), \\ f\left(t_k, x_n\right) = \left(-3t_k - \dfrac{1}{\sqrt{\pi}} t_k^{-\frac{1}{2}} + \dfrac{2}{\sqrt{\pi}} t_k^{\frac{1}{2}}\right) \sin x_n, \\ p^k = p\left(t_k\right), q_n = \sin x_n, x_n = nh,\ t_k = k\tau, \\ 1 \le k \le N, 1 \le n \le M-1, Mh = \pi, N\tau = 1, \\ u_n^0 = \sin x_n,\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N, \\ u_s^k = \rho\left(t_k\right), \rho\left(t_k\right) = 1 - t_k, 0 \le k \le N, s = \left\lfloor \frac{\pi}{2h} \right\rfloor \end{cases} \tag{3.2}$$

where $\lfloor x \rfloor$ denotes greatest integer less than x is constructed. Here, and throughout the thesis,

$$\rho(t_k) = 1 - t_k, q_n = \sin x_n,$$

$$t_k = \left\{t_k = k\tau, 0 \le k \le N, N\tau = 1\right\},$$

$$x_n = \left\{x_n = nh, 0 \le n \le M-1, Mh = \pi\right\},$$

$$f(t_k, x_n) = \left(-3t_k - \frac{1}{\sqrt{\pi}} t_k^{-\frac{1}{2}} + \frac{2}{\sqrt{\pi}} t_k^{\frac{1}{2}}\right) \sin x_n.$$

We search the solution of (3.2) in the following form

$$u_n^k = \eta^k q_n + w_n^k, \tag{3.3}$$

where

$$\eta^k = \sum_{i=1}^{k} p^i \tau, 1 \le k \le N, \eta^0 = 0. \tag{3.4}$$

Moreover, for the interior grid point u_s^k, we have that

$$u_s^k = \eta^k q_s + w_s^k = \rho\left(t_k\right).$$

From equations (3.3) , (3.4) and the condition $u_s^k = \rho(t_k)$, it follows that

$$\eta^k = \frac{\rho(t_k) - w_s^k}{q_s}, \tag{3.5}$$

$$p^k = \frac{\eta^k - \eta^{k-1}}{\tau}, 1 \leq k \leq N, \tag{3.6}$$

$$u_n^k = \frac{\rho(t_k) - w_s^k}{q_s} q_n + w_n^k, 0 \leq k \leq N, 0 \leq n \leq M, \tag{3.7}$$

where $w_n^k, 0 \leq k \leq N, 0 \leq n \leq M$ is the solution of the difference scheme

$$\begin{cases} \dfrac{w_n^k - w_n^{k-1}}{\tau} = \dfrac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} - w_n^k \\ -\dfrac{w_s^k}{\sin(x_s)}\left(\dfrac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{h^2} - \sin(x_n)\right) \\ -\dfrac{1}{\sqrt{\pi}}\sum\limits_{m=1}^{k}\dfrac{\Gamma\left(k-m+\frac{1}{2}\right)}{(k-m)!}\left(\dfrac{w_s^m - w_s^{m-1}}{\sin(x_s)\tau^{\frac{1}{2}}}\sin(x_n) - \dfrac{w_n^m - w_n^{m-1}}{\tau^{\frac{1}{2}}}\right) \\ +F(t_k, x_n), \\ F(t_k, x_n) = \dfrac{\rho(t_k)}{\sin(x_s)}\left(\dfrac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{h^2} - \sin(x_n)\right) \\ -\dfrac{1}{\sqrt{\pi}}\sum\limits_{m=1}^{k}\dfrac{\Gamma\left(k-m+\frac{1}{2}\right)}{(k-m)!}\dfrac{\tau^{\frac{1}{2}}}{\sin(x_s)}\sin(x_n) + f(t_k, x_n). \\ w_0^k = w_M^k = 0,\ 0 \leq k \leq N, \\ w_n^0 = \sin(x_n),\ 0 \leq n \leq M. \end{cases} \tag{3.8}$$

The difference scheme (3.8) can be arranged as

$$\begin{cases} \left(-\dfrac{1}{h^2}\right) w_{n+1}^k + \left(\dfrac{1}{\tau} + \dfrac{2}{h^2} + 1\right) w_n^k + \left(-\dfrac{1}{h^2}\right) w_{n-1}^k + \left(-\dfrac{1}{\tau}\right) w_n^{k-1} \\ +\left(\dfrac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{\sin(x_s)h^2} - \dfrac{\sin(x_n)}{\sin(x_s)}\right) w_s^k \\ +\dfrac{1}{\sqrt{\pi}}\sum\limits_{m=1}^{k}\dfrac{\Gamma\left(k-m+\frac{1}{2}\right)}{(k-m)!}\left(\dfrac{w_s^m - w_s^{m-1}}{\sin(x_s)\tau^{\frac{1}{2}}}\sin(x_n) - \dfrac{w_n^m - w_n^{m-1}}{\tau^{\frac{1}{2}}}\right) \\ = F(t_k, x_n), \\ F(t_k, x_n) = \dfrac{\rho(t_k)}{\sin(x_s)}\left(\dfrac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{h^2} - \sin(x_n)\right) \\ -\dfrac{1}{\sqrt{\pi}}\sum\limits_{m=1}^{k}\dfrac{\Gamma\left(k-m+\frac{1}{2}\right)}{(k-m)!}\dfrac{\tau^{\frac{1}{2}}}{\sin(x_s)}\sin(x_n) + f(t_k, x_n). \\ w_0^k = w_M^k = 0,\ 0 \leq k \leq N, \\ w_n^0 = \sin(x_n),\ 0 \leq n \leq M. \end{cases}$$

First, applying the first order of accuracy difference scheme (3.8), we obtain $(M+1) \times (M+1)$ system of linear equations and we write them in the matrix form

$$Aw^k + \sum_{j=0}^{k-1} B_j w^j = I_{M+1}\varphi^k, 1 \le k \le N, w^0 = \{\sin(x_n)\}_{n=0}^{M}, \tag{3.9}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & . & 0 & 0 & 0 \\ x & y & x & 0 & . & z_1 & . & 0 & 0 & 0 \\ 0 & x & y & x & . & z_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & y+z_s & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & z_{M-1} & . & x & y & x \\ 0 & 0 & 0 & 0 & . & 0 & . & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_0 = \begin{bmatrix} 0 & 0 & 0 & . & 0 & . & 0 & 0 \\ 0 & a & 0 & . & f_1 & . & 0 & 0 \\ 0 & 0 & a & . & f_2 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & f_s+a & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & f_{M-1} & . & 0 & a \\ 0 & 0 & 0 & . & 0 & . & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_j = \begin{bmatrix} 0 & 0 & 0 & . & 0 & . & 0 & 0 \\ 0 & c & 0 & . & d_1 & . & 0 & 0 \\ 0 & 0 & c & . & d_2 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & d_s+c & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & d_{M-1} & . & 0 & c \\ 0 & 0 & 0 & . & 0 & . & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

for any $j = 1, 2, \cdots, k-2$ and

$$B_{k-1} = \begin{bmatrix} 0 & 0 & 0 & . & 0 & . & 0 & 0 \\ 0 & v & 0 & . & c_1 & . & 0 & 0 \\ 0 & 0 & v & . & c_2 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & c_s + v & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & c_{M-1} & . & 0 & v \\ 0 & 0 & 0 & . & 0 & . & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)} .$$

Here, for any $n = 1, 2, \cdots, M-1$,

$$\begin{aligned} x &= -\frac{1}{h^2},\ y = \frac{1}{\tau} + \frac{2}{h^2} + 1 - \frac{1}{\sqrt{\tau}}, \\ z_n &= \frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{\sin(x_s)h^2} - \frac{\sin(x_n)}{\sin(x_s)} \\ &\quad + \frac{\sin(x_n)}{\sqrt{\tau}\sin(x_s)} - \frac{\sin(x_n)}{\sin(x_s)\sqrt{\tau}},\ \text{in } (s+1)^{th} \text{ column}, \\ a &= \frac{\Gamma(k-\frac{1}{2})}{\sqrt{\tau\pi}(k-1)!}\ , f_n = -\frac{\sin(x_n)\Gamma(k-\frac{1}{2})}{\sqrt{\tau\pi}\sin(x_s)(k-1)!}, \\ c &= \frac{1}{\sqrt{\tau\pi}} \left(\frac{\Gamma\left(k-j-\frac{1}{2}\right)}{(k-j-1)!} - \frac{\Gamma\left(k-j+\frac{1}{2}\right)}{(k-j)!} \right), \\ d_n &= \frac{\sin(x_n)}{\sqrt{\tau\pi}\sin(x_s)} \left(\frac{\Gamma\left(k-j-\frac{1}{2}\right)}{(k-j-1)!} - \frac{\Gamma\left(k-j+\frac{1}{2}\right)}{(k-j)!} \right), \\ v &= \frac{1}{\sqrt{\tau}} - \frac{1}{\tau}, \\ c_n &= \frac{\sin(x_n)}{\sqrt{\tau}\sin(x_s)}, \end{aligned}$$

$$w^r = \begin{bmatrix} w_0^r \\ \vdots \\ w_M^r \end{bmatrix}_{(M+1)\times 1} \qquad \text{for any } r = 0, 1, 2, \cdots, k,$$

$$\varphi^k = \begin{bmatrix} 0 \\ \phi_1^k \\ \vdots \\ \phi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\phi_n^k = \left(\frac{\rho(t_k)}{\sin(x_s)} \frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{h^2} - \frac{\rho(t_k)}{\sin(x_s)} \sin(x_n) \right)$$
$$-\frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma\left(k-m+\frac{1}{2}\right)}{(k-m)!} \frac{\tau^{\frac{1}{2}}}{\sin(x_s)} \sin(x_n) + f(t_k, x_n),$$

and I_{M+1} is the identity matrix of size $M+1$. Using (3.9), we can obtain that

$$w^k = A^{-1}\left(I_{M+1}\varphi^k - \sum_{j=0}^{k-1} B_j w^j \right), k = 1, 2, \cdots, N, w^0 = \{\sin x_n\}_{n=0}^{M}. \qquad (3.10)$$

To solve the resulting difference equations, we apply the method given in (3.11) step by step. Then, the solution pairs (u, p) are obtained by using equations (3.7) and (3.6).

3.2 THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME

For the approximate solution of problem (3.1), the Crank-Nicholson difference scheme

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2h^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{2h^2} \\ -\frac{u_n^k + u_n^{k-1}}{2} + \frac{p^k + p^{k-1}}{2} q_n + D_\tau^{\frac{1}{2}} u\left(t_k - \frac{\tau}{2}, x_n\right) + f\left(t_k - \frac{\tau}{2}, x_n\right), \\ p^k = p(t_k), \\ u_n^0 = \sin(x_n), \ 0 \le n \le M, \\ u_0^k = u_M^k = 0, 0 \ \le k \le N, \\ u_s^k + \frac{u_{s+1}^k - u_s^k}{h}(x^* - sh) = \rho(t_k), \\ 0 \le k \le N, s = \left\lfloor \frac{\pi}{2h} \right\rfloor \end{cases} \qquad (3.11)$$

is constructed.

Here,

$$\Gamma(k - r + 1/2) = \int_0^\infty t^{k-r+1/2} e^{-t}\, dt.$$

Moreover, applying the second order of approximation formula (see (Cakir, 2012)) for

$$D_t^{1/2}u(t_k-\tau/2)=\frac{1}{\Gamma(1/2)}\int_0^{t_k-\tau/2}(t_k-\tau/2-s)^{-1/2}u'(s)\,ds,$$

we get

$$D_\tau^{1/2}u=\begin{cases}\frac{-d\sqrt{2}}{3}u_0+\frac{d\sqrt{2}}{3}u_1+\frac{d\tau}{3\sqrt{2}}\sin(x_n), & k=1,\\ -\frac{2d\sqrt{6}}{5}u_0+\frac{d\sqrt{6}}{5}u_1+\frac{d\sqrt{6}}{5}u_2\\ \quad-\frac{d\tau\sqrt{6}}{10}\sin(x_n), & k=2,\\ d\sum\limits_{m=2}^{k-1}\{[(k-m)\,b_1+b_2]\,u_{m-2}\\ +[(2m-2k-1)\,b_1-2b_2]\,u_{m-1} & 3\le k\le N.\\ +[(k-m+1)\,b_1+b_2]\,u_m\}\\ +\frac{d}{6\sqrt{2}}\left[-u_{k-2}-4u_{k-1}+5u_k\right],\end{cases}$$

Here,

$$D_t^{1/2}u=D_\tau^{1/2}u\left(t_k-\frac{\tau}{2},x_n\right),$$

$$\begin{aligned}d&=\frac{2}{\sqrt{\pi\tau}},\ b_1=\sqrt{k-m+1/2}-\sqrt{k-m-1/2},\\ b_2&=-\frac{1}{3}\left((k-m+1/2)^{3/2}-(k-m-1/2)^{3/2}\right).\end{aligned}$$

We search the solution of (3.11) in the following form

$$u_n^k=\eta^k q_n+w_n^k, \tag{3.12}$$

where

$$\eta^k=\sum_{i=1}^{k}\frac{p^i+p^{i-1}}{2}\tau,1\le k\le N,\eta^0=0.$$

We have

$$\begin{aligned}u_s^k+\frac{u_{s+1}^k-u_s^k}{h}(x^*-sh)&=\eta^k\left(\left(1-\frac{x^*-sh}{h}\right)q_s+\frac{x^*-sh}{h}q_{s+1}\right)\\ &+\left(1-\frac{x^*-sh}{h}\right)w_s^k+\frac{x^*-sh}{h}w_{s+1}^k=\rho\left(t_k\right).\end{aligned}$$

Let us denote

$$y=\frac{x^*-sh}{h}=\frac{x^*}{h}-\left\lfloor\frac{x^*}{h}\right\rfloor h,$$

where $0 \leq y < 1$. Then, one can write

$$\eta^k = \frac{\rho(t_k) - (1-y)\, w_s^k - y w_{s+1}^k}{(1-y)\, q_s + y q_{s+1}}. \tag{3.13}$$

So, the values of $\dfrac{p(t_k) + p(t_{k-1})}{2}, 1 \leq k \leq N$ can be obtained by the following formula

$$\frac{p^k + p^{k-1}}{2} = \frac{\dfrac{\rho(t_k) - \rho(t_{k-1})}{\tau} - (1-y)\,\dfrac{w_s^k - w_s^{k-1}}{\tau} - y\dfrac{w_{s+1}^k - w_{s+1}^{k-1}}{\tau}}{(1-y)\, q_s + y q_{s+1}}. \tag{3.14}$$

Let w^r denotes

$$w^r = \begin{bmatrix} w_0^r \\ \vdots \\ w_M^r \end{bmatrix}_{(M+1)\times 1} \quad \text{for } r = 0, 1, \cdots, N.$$

For $k = 1$, one can show that w^1 is the solution of the difference scheme

$$\begin{cases} \dfrac{w_n^1 - w_n^0}{\tau} = \dfrac{w_{n+1}^1 - 2w_n^1 + w_{n-1}^1}{2h^2} + \dfrac{w_{n+1}^0 - 2w_n^0 + w_{n-1}^0}{2h^2} \\ -\dfrac{w_n^1 + w_n^0}{2} + \left(\dfrac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \dfrac{q_n}{2}\right) \\ \times \left(\dfrac{\rho(t_1) - (1-y)\, w_s^1 - y w_{s+1}^1}{(1-y)\, q_s + y q_{s+1}} + \dfrac{\rho(t_0) - (1-y)\, w_s^0 - y w_{s+1}^0}{(1-y)\, q_s + y q_{s+1}}\right) \\ +\dfrac{d\sqrt{2}}{3}\left(\dfrac{\rho(t_1) - (1-y)w_s^1 - y w_{s+1}^1}{(1-y)q_s + y q_{s+1}} q(n) + w_n^1\right) \\ -\dfrac{d\sqrt{2}}{3}\left(\dfrac{\rho(t_0) - (1-y)w_s^0 - y w_{s+1}^0}{(1-y)q_s + y q_{s+1}} q(n) + w_n^0\right) \\ +\dfrac{d\tau}{3\sqrt{2}} q(n) + f\left(t_1 - \dfrac{\tau}{2}, x_n\right), \\ w_0^1 = w_M^1 = 0, \\ w_n^0 = \sin(x_n),\ 0 \leq n \leq M. \end{cases} \tag{3.15}$$

The difference scheme (3.15) can be arranged as

$$
\begin{cases}
\left(-\frac{1}{2h^2}\right) w_{n+1}^1 + \left(\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - \frac{d\sqrt{2}}{3}\right) w_n^1 \\
+\left(-\frac{1}{2h^2}\right) w_{n+1}^0 + \left(-\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} + \frac{d\sqrt{2}}{3}\right) w_n^0 \\
+\left(-\frac{1}{2h^2}\right) w_{n-1}^1 + \left(-\frac{1}{2h^2}\right) w_{n-1}^0 \\
+\left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2} + \frac{d\sqrt{2}}{3} q_n\right) \\
\times\left(\frac{(1-y)}{(1-y)q_s + yq_{s+1}} w_s^1 + \frac{y}{(1-y)q_s + yq_{s+1}} w_{s+1}^1\right) \\
+\left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2} - \frac{d\sqrt{2}}{3} q_n\right) \\
\times\left(\frac{(1-y)}{(1-y)q_s + yq_{s+1}} w_s^0 + \frac{y}{(1-y)q_s + yq_{s+1}} w_{s+1}^0\right) \\
=\left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2}\right) \frac{\rho\left(t_1\right) + \rho\left(t_0\right)}{\left(1-y\right) q_s + yq_{s+1}} \\
+\frac{d\tau q_n}{3\sqrt{2}} + \frac{d\sqrt{2}q_n}{3} \frac{\rho(t_1) - \rho(t_0)}{(1-y)q_s + yq_{s+1}} + f\left(t_1 - \frac{\tau}{2}, x_n\right), \\
w_0^1 = w_M^1 = 0, \\
w_n^0 = \sin(x_n),\ 0 \le n \le M.
\end{cases}
$$

We have the system of linear equations and we write them in the matrix form

$$
A_1 w^1 + B_1 w^0 = I_{M+1} \varphi^1, \tag{3.16}
$$

where

$$
A_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\
a & y_1 & a & 0 & . & l_1 & c_1 & . & 0 & 0 & 0 \\
0 & a & y_1 & a & . & l_2 & c_2 & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & a & . & l_s + a & c_s & . & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & . & l_{s+1} + y_1 & c_{s+1} + a & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & . & l_{M-1} & c_{M-1} & . & a & y_1 & a \\
0 & 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 1
\end{bmatrix}_{(M+1)\times(M+1)},
$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ a & v_1 & a & 0 & . & d_1 & e_1 & . & 0 & 0 & 0 \\ 0 & a & v_1 & a & . & d_2 & e_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & a & . & d_s + a & e_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & d_{s+1} + v_1 & e_{s+1} + a & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & d_{M-1} & e_{M-1} & . & a & v_1 & a \\ 0 & 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}_{(.M+1)\times(M+1)} .$$

Here, for any $n = 1, 2, 3, \cdots, M-1$,

$$\begin{aligned} a &= \left(-\frac{1}{2h^2}\right), y_1 = \left(\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - d\frac{\sqrt{2}}{3}\right), \\ v_1 &= \left(-\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} + d\frac{\sqrt{2}}{3}\right), \end{aligned}$$

$$l_n = \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} + d\frac{\sqrt{2}}{3}q_n,$$

$$c_n = \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} + d\frac{\sqrt{2}}{3}q_n,$$

$$d_n = \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} - d\frac{\sqrt{2}}{3}q_n,$$

$$e_n = \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} - d\frac{\sqrt{2}}{3}q_n,$$

$$\varphi^1 = \begin{bmatrix} 0 \\ \phi_1^1 \\ \vdots \\ \phi_{M-1}^1 \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\begin{aligned} \phi_n^1 = &\left(\frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{2h^2} - \frac{\sin(x_n)}{2}\right)\frac{\rho(t_1) + \rho(t_0)}{(1-y)q_s + yq_{s+1}} \\ &+ \frac{d\sqrt{2}q_n}{3}\frac{\rho(t_1) - \rho(t_0)}{(1-y)q_s + yq_{s+1}} + \frac{d\tau}{3\sqrt{2}}q_n + f(t_1 - \frac{\tau}{2}, x_n). \end{aligned}$$

Using (3.16), we can obtain that

$$w^1 = A_1^{-1}\left(I_{M+1}\varphi^1 - B_1 w^0\right),\ w^0 = \{\sin x_n\}_{n=0}^{M}.$$

For $k = 2$, w^2 is the solution of the difference scheme

$$\begin{cases} \dfrac{w_n^2 - w_n^1}{\tau} = \dfrac{w_{n+1}^2 - 2w_n^2 + w_{n-1}^2}{2h^2} \\ -\dfrac{w_n^2 + w_n^1}{2} + \dfrac{w_{n+1}^1 - 2w_n^1 + w_{n-1}^1}{2h^2} \\ + \left(\dfrac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \dfrac{q_n}{2}\right) \\ \times \left(\dfrac{\rho(t_2) - (1-y)\, w_s^2 - y w_{s+1}^2}{(1-y)\, q_s + y q_{s+1}} \right. \\ \left. + \dfrac{\rho(t_1) - (1-y)\, w_s^1 - y w_{s+1}^1}{(1-y)\, q_s + y q_{s+1}}\right) \\ + \dfrac{d\sqrt{6}}{5}\left(\dfrac{\rho(t_2) - (1-y) w_s^2 - y w_{s+1}^2}{(1-y) q_s + y q_{s+1}} q(n) + w_n^2\right) \\ + \dfrac{d\sqrt{6}}{5}\left(\dfrac{\rho(t_1) - (1-y) w_s^1 - y w_{s+1}^1}{(1-y) q_s + y q_{s+1}} q(n) + w_n^1\right) \\ - \dfrac{2d\sqrt{6}}{5}\left(\dfrac{\rho(t_0) - (1-y) w_s^0 - y w_{s+1}^0}{(1-y) q_s + y q_{s+1}} q(n) + w_n^0\right) \\ - \dfrac{d\tau\sqrt{6}}{10} q(n) + + f\left(t_k - \dfrac{\tau}{2}, x_n\right), \\ w_0^2 = w_M^2 = 0, \\ w_n^0 = \sin(x_n),\ 0 \le n \le M. \end{cases}$$

The difference scheme can be arrange as

$$\begin{cases}
\left(-\frac{1}{2h^2}\right) w_{n+1}^2 + \left(\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - d\frac{\sqrt{6}}{5}\right) w_n^2 \\
+\left(-\dfrac{1}{2h^2}\right) w_{n+1}^1 + \left(-\dfrac{1}{\tau} + \dfrac{1}{h^2} + \dfrac{1}{2} - d\dfrac{\sqrt{6}}{5}\right) w_n^1 \\
+\left(-\frac{1}{2h^2}\right) w_{n-1}^2 + \left(-\dfrac{1}{2h^2}\right) w_{n-1}^1 + d\dfrac{2\sqrt{6}}{5} w_n^0 \\
+\left(\dfrac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \dfrac{q_n}{2} + \dfrac{d\sqrt{6}q_n}{5}\right) \\
\times\left(\dfrac{(1-y)w_s^2}{(1-y)q_s + yq_{s+1}} + \dfrac{yw_{s+1}^2}{(1-y)q_s + yq_{s+1}}\right) \\
+\left(\dfrac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \dfrac{q_n}{2} + \dfrac{d\sqrt{6}q_n}{5}\right) \\
\times\left(\dfrac{(1-y)w_s^1}{(1-y)q_s + yq_{s+1}} + \dfrac{yw_{s+1}^1}{(1-y)q_s + yq_{s+1}}\right) \\
-\dfrac{2d\sqrt{6}q_n}{5}\left(\dfrac{(1-y)w_s^0}{(1-y)q_s + yq_{s+1}} + \dfrac{yw_{s+1}^0}{(1-y)q_s + yq_{s+1}}\right) \\
= \left(\dfrac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \dfrac{q_n}{2} + \dfrac{d\sqrt{6}}{5}\right) \\
\times\left(\dfrac{\rho(t_2) + \rho(t_1)}{(1-y)\,q_s + yq_{s+1}}\right) - \dfrac{2d\sqrt{6}q_n\rho(t_0)}{5\left((1-y)q_s + yq_{s+1}\right)} \\
-\dfrac{d\tau\sqrt{6}q_n}{10} + f\left(t_k - \frac{\tau}{2}, x_n\right), \\
w_0^2 = w_M^2 = 0, \\
w_n^0 = \sin(x_n),\ 0 \le n \le M.
\end{cases}$$

The system of linear equations given above can be written in the matrix form

$$A_2 w^2 + B_2 w^1 + C_2 w^0 = I_{M+1}\varphi^2, \tag{3.17}$$

where

$$A_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\
a & y_2 & a & 0 & . & g_1 & h_1 & . & 0 & 0 & 0 \\
0 & a & y_2 & a & . & g_2 & h_2 & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & . & g_s + a & h_s & . & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & . & g_{s+1} + y_2 & h_{s+1} + a & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & . & g_{M-1} & h_{M-1} & . & a & y_2 & a \\
0 & 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 1
\end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ a & v_2 & a & 0 & . & g_1 & h_1 & . & 0 & 0 & 0 \\ 0 & a & v_2 & a & . & g_2 & h_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & g_s + a & h_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & g_{s+1} + v_2 & h_{s+1} + a & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & g_{M-1} & h_{M-1} & . & a & v_2 & a \\ 0 & 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ 0 & z & 0 & 0 & . & i_1 & j_1 & . & 0 & 0 & 0 \\ 0 & 0 & z & 0 & . & i_2 & j_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & i_s + z & j_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & i_{s+1} & j_{s+1} + z & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & i_{M-1} & j_{M-1} & . & 0 & z & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

Here, for any $n = 1, 2, 3, \cdots, M-1$,

$$\begin{aligned} a &= -\frac{1}{2h^2}, y_2 = \frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - \frac{d\sqrt{6}}{5}, \\ v_2 &= -\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - \frac{d\sqrt{6}}{5}, \end{aligned}$$

$$\begin{aligned} g_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} \\ &\quad + \frac{d\sqrt{6}q_n(1-y)}{5((1-y)q_s + yq_{s+1})}, \text{ in } (s+1)^{th} \text{ column}, \end{aligned}$$

$$\begin{aligned} h_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})\,y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} \\ &\quad + \frac{d\sqrt{6}q_n y}{5((1-y)q_s + yq_{s+1})}, \text{ in } (s+2)^{th} \text{ column}, \end{aligned}$$

$$i_n = -\frac{2d\sqrt{6}q_n(1-y)}{5((1-y)q_s + yq_{s+1})}, j_n = -\frac{2\sqrt{6}q_n y}{5((1-y)q_s + yq_{s+1})}, z = \frac{2d\sqrt{6}}{5},$$

$$\varphi^2 = \begin{bmatrix} 0 \\ \phi_1^2 \\ \vdots \\ \phi_{M-1}^2 \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\phi_n^2 = \left(\frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{2h^2} - \frac{\sin(x_n)}{2} + \frac{d\sqrt{6}q_n}{5} \right)$$
$$\times \frac{\rho(t_2) + \rho(t_1)}{(1-y)q_s + yq_{s+1}} - \frac{2d\sqrt{6}q_n}{5} \frac{\rho(t_0)}{(1-y)q_s + yq_{s+1}}$$
$$-\frac{d\tau\sqrt{6}}{10} q_n + f(t_2 - \frac{\tau}{2}, x_n).$$

Using formula (3.17) ,

$$w^2 = A_2^{-1} \left(I_{M+1}\varphi^2 - B_2 w^1 - C_2 w^0 \right), \; w^0 = \{\sin x_n\}_{n=0}^{M}.$$

For $3 \leq k \leq N$, we obtain the following difference scheme

$$\begin{cases} \frac{w_n^k - w_n^{k-1}}{\tau} = \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{2h^2} \\ + \frac{w_{n+1}^{k-1} - 2w_n^{k-1} + w_{n-1}^{k-1}}{2h^2} - \frac{w_n^k + w_n^{k-1}}{2} \\ + \left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2} \right) \\ \times \left(\frac{\rho\left(t_k\right) - \left(1-y\right) w_s^k - y w_{s+1}^k}{\left(1-y\right) q_s + y q_{s+1}} \right) \\ + \left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2} \right) \\ \times \left(\frac{\rho\left(t_{k-1}\right) - \left(1-y\right) w_s^{k-1} - y w_{s+1}^{k-1}}{\left(1-y\right) q_s + y q_{s+1}} \right) \\ \quad d \sum\limits_{m=2}^{k-1} \{ ((k-m)b_1 + b_2) \\ \quad \times \left(\frac{\rho(t_{m-2}) - (1-y)w_s^{m-2} - y w_{s+1}^{m-2}}{(1-y)q_s + y q_{s+1}} q(n) \right) \\ \quad + \left(\left(2m - 2k - 1\right) b_1 - 2b_2 \right) \\ \quad \times \left(\frac{\rho(t_{m-1}) - (1-y)w_s^{m-1} - y w_{s+1}^{m-1}}{(1-y)q_s + y q_{s+1}} q(n) \right) \\ \quad + \left(\left(2m - 2k - 1\right) b_1 - 2b_2 \right) w_n^{m-1} \\ \quad \times \left(\frac{\rho(t_m) - (1-y)w_s^m - y w_{s+1}^m}{(1-y)q_s + y q_{s+1}} q(n) \right) \\ \quad + ((k-m)b_1 + b_2) w_n^{m-2}) \\ \quad + \left((k-m-1)b_1 + b_2 \right) w_n^m \\ \quad + \left(\left(2m - 2k - 1\right) b_1 - 2b_2 \right) w_n^{m-1} \} \\ - \frac{d}{6\sqrt{2}} \left(\frac{\rho(t_{k-2}) - (1-y)w_s^{k-2} - y w_{s+1}^{k-2}}{(1-y)q_s + y q_{s+1}} q(n) + w_n^{k-2} \right) \\ - \frac{4d}{6\sqrt{2}} \left(\frac{\rho(t_{k-1}) - (1-y)w_s^{k-1} - y w_{s+1}^{k-1}}{(1-y)q_s + y q_{s+1}} q(n) + w_n^{k-1} \right) \\ + \frac{5d}{6\sqrt{2}} \left(\frac{\rho(t_k) - (1-y)w_s^k - y w_{s+1}^k}{(1-y)q_s + y q_{s+1}} q(n) + w_n^k \right) \\ + f\left(t_k - \frac{\tau}{2}, x_n\right), \\ w_0^k = w_M^k = 0, 3 \leq k \leq N, \\ w_n^0 = \sin(x_n),\ 0 \leq n \leq M. \end{cases} \tag{3.18}$$

The difference scheme (3.18) can be arranged as

$$
\begin{cases}
\left(-\frac{1}{2h^2}\right) w_{n+1}^k + \left(\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - \frac{5d}{6\sqrt{2}}\right) w_n^k \\
+\left(-\frac{1}{2h^2}\right) w_{n-1}^k + \left(-\frac{1}{2h^2}\right) w_{n+1}^{k-1} + \frac{d}{6\sqrt{2}} w_n^{k-2} \\
+\left(-\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} + \frac{4d}{6\sqrt{2}}\right) w_n^{k-1} + \left(-\frac{1}{2h^2}\right) w_{n-1}^{k-1} \\
-\frac{dq(n)(1-y)w_s^{k-2}}{6\sqrt{2}\left((1-y)q_s + yq_{s+1}\right)} - \frac{dq(n)yw_{s+1}^{k-2}}{6\sqrt{2}\left((1-y)q_s + yq_{s+1}\right)} \\
+\left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} + \frac{q_n}{2} + \frac{5dq(n)}{6\sqrt{2}}\right) \\
\times\left(\frac{(1-y)w_s^k}{(1-y)q_s + yq_{s+1}} + \frac{yw_{s+1}^k}{(1-y)q_s + yq_{s+1}}\right) \\
+\left(\frac{(q_{n+1} - 2q_n + q_{n-1})}{2h^2} + \frac{q_n}{2} - \frac{4dq(n)}{6\sqrt{2}}\right) \\
\times\left(\frac{(1-y)w_s^{k-1}}{(1-y)q_s + yq_{s+1}} + \frac{yw_{s+1}^{k-1}}{(1-y)q_s + yq_{s+1}}\right) \\
+d\sum_{m=2}^{k-1} \{((k-m)b_1 + b_2) \\
\times\left(\frac{(1-y)w_s^{m-2} + yw_{s+1}^{m-2}}{(1-y)q_s + yq_{s+1}} q(n)\right) \\
+\left((2m-2k-1)b_1 - 2b_2\right) \\
\times\left(\frac{(1-y)w_s^{m-1} + yw_{s+1}^{m-1}}{(1-y)q_s + yq_{s+1}} q(n)\right) \\
+\left((k-m+1)b_1 - 2b_2\right) \\
\times\left(\frac{(1-y)w_s^{m} + yw_{s+1}^{m}}{(1-y)q_s + yq_{s+1}} q(n)\right) \\
-\left((k-m)b_1 + b_2\right) w_n^{m-2} \\
-\left((2m-2k-1)b_1 - 2b_2\right) w_n^{m-1} \\
-\left((k-m+1)b_1 - 2b_2\right) w_n^{m} \} \\
= \frac{dq(n)(5\rho(t_k) - 4\rho(t_{k-1}) - \rho(t_{k-2}))}{6\sqrt{2}((1-y)q_s + yq_{s+1})} \\
+\left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2}\right) \frac{\rho(t_k) + \rho(t_{k-1})}{(1-y)q_s + yq_{s+1}} \\
+d\sum_{m=2}^{k-1} \left\{ ((k-m)b_1 + b_2) \frac{\rho(t_{m-2})q(n)}{(1-y)q_s + yq_{s+1}} \right. \\
+\left((2m-2k-1)b_1 - 2b_2\right) \frac{\rho(t_{m-1})q(n)}{(1-y)q_s + yq_{s+1}} \\
\left. +\left((k-m+1)b_1 + b_2\right) \frac{\rho(t_m)q(n)}{(1-y)q_s + yq_{s+1}} \right\} \\
+f\left(t_k - \frac{\tau}{2}, x_n\right), \\
w_0^k = w_M^k = 0, 3 \le k \le N, \\
w_n^0 = \sin(x_n),\ 0 \le n \le M.
\end{cases}
$$

This system can be written in matrix form as

$$A_3 w^k + B_3 w^{k-1} + C_3 w^{k-2} + \sum_{j=0}^{k-3} E_j w^j = I_{M+1}\varphi^k, \tag{3.19}$$

where,

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ a & y_3 & a & 0 & . & r_1 & m_1 & . & 0 & 0 & 0 \\ 0 & a & y_3 & a & . & r_2 & m_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & r_s + a & m_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & r_{s+1} + y_3 & m_{s+1} + a & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & r_{M-1} & s_{M-1} & . & a & y_3 & a \\ 0 & 0 & 0 & 0 & . & 0 & 0 & . & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ a & X & a & 0 & . & Y_1 & Z_1 & . & 0 & 0 & 0 \\ 0 & a & X & a & . & Y_2 & Z_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & Y_s + a & Z_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & Y_{s+1} + X & Z_{s+1} + a & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & Y_{M-1} & Z_{M-1} & . & a & X & a \\ 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ 0 & S & 0 & 0 & . & R_1 & T_1 & . & 0 & 0 & 0 \\ 0 & 0 & S & 0 & . & R_2 & T_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & R_s + S & T_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & R_{s+1} & T_{s+1} + S & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & R_{M-1} & T_{M-1} & . & 0 & S & 0 \\ 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$E_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ 0 & G & 0 & 0 & . & H_1 & I_1 & . & 0 & 0 & 0 \\ 0 & 0 & G & 0 & . & H_2 & I_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & H_s + G & I_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & H_{s+1} & I_{s+1} + G & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & H_{n-1} & I_{n-1} & . & 0 & G & 0 \\ 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

and

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ 0 & G1 & 0 & 0 & . & J_1 & K_1 & . & 0 & 0 & 0 \\ 0 & 0 & G1 & 0 & . & J_2 & K_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & J_s + G1 & K_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & J_{s+1} & K_{s+1} + G1 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & J_{M-1} & K_{M-1} & . & 0 & G1 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$E_j = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \\ 0 & G2 & 0 & 0 & . & P_1 & L_1 & . & 0 & 0 & 0 \\ 0 & 0 & G2 & 0 & . & P_2 & L_2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & P_s + G2 & L_s & . & 0 & 0 & 0 \\ 0 & 0 & 0 & . & . & P_{s+1} & L_{s+1} + G2 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & P_{M-1} & L_{M-1} & . & 0 & G2 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & . & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

for $j = 2, 3, \cdots, k-3$.

Here, for any $n = 1, 2, 3, \cdots, M-1$,

$$\begin{aligned} a &= -\frac{1}{2h^2}, d = \frac{2}{\sqrt{\pi\tau}}, \\ y_3 &= \frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - \frac{5d}{6\sqrt{2}}, \end{aligned}$$

$$v_3 = -\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} + \frac{4d}{6\sqrt{2}},$$

$$\alpha_n = \frac{dq(n)(1-y)}{6\sqrt{2}\left((1-y)q_s + yq_{s+1}\right)}, \beta_n = \frac{dq(n)y}{6\sqrt{2}\left((1-y)q_s + yq_{s+1}\right)},$$

$$\begin{aligned} r_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2\left((1-y)q_s + yq_{s+1}\right)} + \frac{q_n(1-y)}{2\left((1-y)q_s + yq_{s+1}\right)} \\ &+5\frac{dq(n)(1-y)}{6\sqrt{2}\left((1-y)q_s + yq_{s+1}\right)}, \end{aligned}$$

$$\begin{aligned} s_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2\left((1-y)q_s + yq_{s+1}\right)} + \frac{q_n y}{2\left((1-y)q_s + yq_{s+1}\right)} \\ &+5\frac{dq(n)y}{6\sqrt{2}\left((1-y)q_s + yq_{s+1}\right)}, \end{aligned}$$

$$X = v_3 - pp \times d, Y_{n-1} = \gamma_{n-1} + 6\sqrt{2}pp \times \alpha_{n-1},$$

$$Z_{n-1} = \delta_{n-1} + 6\sqrt{2}pp \times \beta_{n-1}, S = d\left(\frac{1}{6\sqrt{2}} - pp - nn\right),$$

$$\begin{aligned} R_{n-1} &= -\alpha_{n-1} + 6\sqrt{2}(nn + pp)\alpha_{n-1}, \\ T_{n-1} &= -\beta_{n-1} + 6\sqrt{2}(nn + pp)\beta_{n-1}, \end{aligned}$$

$$\begin{aligned} \gamma_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2\left((1-y)q_s + yq_{s+1}\right)} + \frac{q_n(1-y)}{2\left((1-y)q_s + yq_{s+1}\right)} \\ &-4\frac{dq(n)(1-y)}{6\sqrt{2}\left((1-y)q_s + yq_{s+1}\right)}, \end{aligned}$$

$$\begin{aligned} \delta_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2\left((1-y)q_s + yq_{s+1}\right)} + \frac{q_n y}{2\left((1-y)q_s + yq_{s+1}\right)} \\ &-4\frac{dq(n)y}{6\sqrt{2}\left((1-y)q_s + yq_{s+1}\right)}, \end{aligned}$$

$$\begin{aligned} mm &= (k-m)b_1 + b_2, nn = (2m - 2k - 1)b_1 - 2b_2, \\ pp &= (k - m + 1)b_1 - 2b_2, \end{aligned}$$

$$\begin{aligned} nn1 &= (2(m+1) - 2k - 1)b_1 - 2b_2, \\ pp1 &= (k - (m+2) + 1)b_1 + b_2, \end{aligned}$$

$$\begin{aligned} G &= -d\times mm, G1=-d\times(mm+nn1),\\ G2 &= -d\times(mm+nn1+pp1), \end{aligned}$$

$$H_{n-1}=6\sqrt{2}\times\alpha_1\times mm, I_{n-1}=6\sqrt{2}\times\beta_1\times mm,$$

$$\begin{aligned} J_{n-1} &= 6\sqrt{2}\times\alpha_{n-1}\times(mm+nn1),\\ K_{n-1} &= 6\sqrt{2}\times\beta_{n-1}\times(mm+nn1), \end{aligned}$$

$$\begin{aligned} P_{n-1} &= 6\sqrt{2}\times\alpha_{n-1}\times(mm+nn1+pp1),\\ L_{n-1} &= 6\sqrt{2}\times\beta_{n-1}\times(mm+nn1+pp1). \end{aligned}$$

$$\begin{aligned} \varphi^k &= \begin{bmatrix} 0\\ \phi_1^k\\ \vdots\\ \phi_{M-1}^k\\ 0 \end{bmatrix}_{(M+1)\times 1},\\ \phi_n^k &= \left(\frac{\sin(x_{n+1})-2\sin(x_n)+\sin(x_{n-1})}{2h^2}-\frac{\sin(x_n)}{2}\right)\frac{\rho(t_k)+\rho(t_{k-1})}{(1-y)q_s+yq_{s+1}}\\ &+\frac{5dq_n\rho(t_k)}{6\sqrt{2}((1-y)q_s+yq_{s+1})}-\frac{4dq_n\rho(t_{k-1})}{6\sqrt{2}((1-y)q_s+yq_{s+1})}\\ &-\frac{dq_n\rho(t_{k-2})}{6\sqrt{2}((1-y)q_s+yq_{s+1})}+f(t_k-\frac{\tau}{2},x_n)\\ &+\frac{dq_n}{(1-y)q_s+yq_{s+1}}\sum_{m=2}^{k-1}\{((k-m)b_1+b_2)\rho(t_{m-2})\\ &+((2m-2k-1)b_1-2b_2)\rho(t_{m-1})+((k-m+1)b_1+b_2)\rho(t_m)\}. \end{aligned}$$

Finally, from (3.19) , it follows that

$$w^k=(A_3)^{-1}\left(I_{M+1}\varphi^k-B_3w^{k-1}-C_3w^{k-2}-\sum_{j=0}^{k-3}E_jw^j\right), 3\le k\le N.$$

By using formulas (3.12) and (3.14), we get

$$u_n^k=\frac{\rho(t_k)-(1-y)w_s^k-yw_{s+1}^k}{(1-y)q_s+yq_{s+1}}q_n+w_n^k, 0\le k\le N, 0\le n\le M.$$

Therefore, we reach to the approximate solutions of $u(t,x)$ and $\frac{p(t_k)+p(t_{k-1})}{2}$.

3.3 ERROR ANALYSIS

Now, we will give the results of the numerical analysis. The numerical solutions are recorded for different values of N and M and u_n^k which represents the numerical solutions of these difference schemes at (t_k, x_n). Table 1 gives the error analysis between the exact solution and the solutions derived by difference schemes. It is constructed for $N = M = 15$, 45 and 75 respectively. For their comparisons, the errors computed by

$$E = \max_{\substack{1 \leq k \leq N \\ 1 \leq n \leq M}} \left| u(t_k, x_n) - u_n^k \right| .$$

Table 1. Comparison of exact solution and approximate solutions.

Method	N=M=15	N=M=45	N=M=75
1^{st} order of accuracy	0.1190	0.0126	0.0045
2^{nd} order of accuracy	0.0055	6.0917×10^{-4}	2.1932×10^{-4}

Thus, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

CHAPTER 4

CONCLUSION

This thesis is devoted to the analysis of fractional inverse problems with an unknown source function.

- Theorems on the stability estimates for the solution of the fractional inverse problem in a Banach space are established.
- First and second order of accuracy stable difference schemes for the approximate solution of the fractional inverse problem are presented.
- Theoretical statements for the solution of these difference schemes are supported by results of numerical examples.
- The Matlab implementation of these differences schemes are generated.

APPENDIX A

APPENDIX

A.1 MATLAB PROGRAM FOR FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

```
function firstorder(K,N)
%% initialize
if nargin¡1, K=10; N=10; end;
tau=1/K;
h=pi/N;
s=floor(pi/2/h);
%% calculate x and q values
for n=1:N+1
x=(n-1)*h;
q(n)=sin(x);
end
qs=sin(s*h); %calculate qs
%% calculate the right side of the equation
Fr=ones(N+1,K+1) * NaN;
Fr(1,:)=0; Fr(end,:)=0;
%for n=2:K
% Fr(n,1)=1/qs*(((q(n+1) - 2*q(n) + q(n-1))/(h*h))-q(n));
%end;
%Fr(:,1)=0;
for k=2:K+1
for n=2:N
t=(k-1)*tau;
x=(n-1)*h;
f = (-3*t - 1/sqrt(pi*t) + 2*sqrt(t)/sqrt(pi))*sin(x);
```

```
summm=0; t0=0; % summing part in the equatio
for m=1:k
t1=m*tau;
summ = summ + (gamma(k-m+.5)*(tau)*q(n))/(factorial(k-m)*sqrt(tau)*qs);
t0=t1;
end
Fr(n,k)=((1-t)/qs) * (((q(n+1) - 2*q(n) + q(n-1))/(h*h))-q(n))...
- (1/sqrt(pi)) * summ + f;
end
end
%% calculate the left side of the equation
w=ones(N+1,K+1) * NaN;
w(1,:)=0; w(end,:)=0; %when n=0 or n=N, w=0;
for n=1:N+1 %when k=0, w=sin(x)
x=(n-1)*h;
w(n,1)=sin(x);
end
a=-1/h/h; %coefficient of w(n+1,k)
b=1/tau+2/(h*h)+1-1/sqrt(tau); %coefficient of w(n,k)
c=a; %coefficient of w(n-1,k)
for k=2:K+1 %for each k
A=zeros(N+1,N+1); % Coefficient matrix: Aw+E=B=Fr(:,k);
B=Fr(:,k);
E=zeros(N+1,1);
for n=2:N % for each n
d=( (q(n+1)-2*q(n)+q(n-1))/(h*h*qs))...
-q(n)/qs+q(n)/(qs*sqrt(tau)); %first coefficient of w(s,k)
E(n)=-w(n,k-1) / tau; %constant number in Aw+E=B;
for m=2:k %summing part in the equation
if m==2 %summing part in the equation
ff = (1/sqrt(pi))*(gamma(k-m+.5)*q(n))/(factorial(k-m)*sqrt(tau)*qs);
gg = (1/sqrt(pi))*(gamma(k-m+.5))/(factorial(k-m)*sqrt(tau));
E(n)=E(n)-ff*w(s,m-1); %second coefficient is added to 'E'
E(n)=E(n)+gg*w(n,m-1); %third coefficient is added to 'E'
else
ff = (1/sqrt(pi))*(gamma(k-m+1.5)*q(n))/(factorial(k-m+1)*sqrt(tau)*qs)...
-(1/sqrt(pi))*(gamma(k-m+.5)*q(n))/(factorial(k-m)*sqrt(tau)*qs);
```

```
gg = (1/sqrt(pi))*(gamma(k-m+1.5))/(factorial(k-m+1)*sqrt(tau))...
- (1/sqrt(pi))*(gamma(k-m+.5))/(factorial(k-m)*sqrt(tau));
E(n)=E(n)-ff*w(s,m-1); %second coefficient is added to 'E'
E(n)=E(n)+gg*w(n,m-1); %third coefficient is added to 'E'
end
end
A(n,n+1)=A(n,n+1)+a;
A(n,n)=A(n,n)+b;
A(n,n-1)=A(n,n-1)+c;
A(n,s)=A(n,s)+d;
end
A(1)=1; A(end)=1;
w(:,k)=inv(A)*(B-E);
end
[t,x]=meshgrid(0:K,0:N);
surf(x,t,w);
xlabel('x');ylabel('t');zlabel('w');
axis auto
for k=1:K+1;
for n=1:N+1
u(n,k)=((1-(k-1)*tau)-w(s,k))/qs*sin((n-1)*h)+w(n,k);
end
end
figure;
surf(u)
for k=1:K+1;
for n=1:N+1
es(n,k)=(1-(k-1)*tau)*sin((n-1)*h);
end
end
figure; surf(es) maxes=max(max(es)) ; maxapp=max(max(u)) ; maxerror=abs(maxes-maxapp); relativeerror=maxerror/maxapp; cevap = [maxes,maxapp,maxerror,relativeerror] abserr=max(max(abs(es-u)))
```

A.2 MATLAB PROGRAM FOR SECOND ORDER OF ACCURACY DIFFERENCE SCHEME

```
function secondordernew(K,N)
%% initialize
if nargin¡1, K=10; N=10; end;
tau=1/K;
h=pi/N;
s=floor(pi/2/h);
y=((pi/2)-s*h)/h;
%% calculate x and q values
for n=1:N+1;
x=(n-1)*h;
q(n)=sin(x);
end
qs=sin(s*h); %calculate q(s)
qs1=sin((s+1)*h);%calculate q(s+1)
%% calculate the right side of the equation
Fr1=zeros(N+1,1);
Fr1(1,:)=0; Fr1(end,:)=0;
for n=2:N;
x=(n-1)*h;
f1 = (-3*(tau/2)-(1/sqrt(pi*(tau/2)))...
+((2*sqrt(tau/2))/sqrt(pi)))*sin(x); % 'f' function
Fr1(n,2)=((sqrt(2*tau)*q(n))/(3*sqrt(pi)))...
+(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2))*((2-tau)/((1-y)*qs+y*qs1))...
+((2*sqrt(2)*q(n))/(3*sqrt(pi*(tau))))*((-tau)/((1-y)*qs+y*qs1))+f1;
end
%% calculate the left side of the equation
w=zeros(N+1,N+1);
w(1,:)=0; w(end,:)=0; %when n=0 or n=N, w=0;
for n=1:N+1 %when k=0, w=sin(x)
x=(n-1)*h;
w(n,1)=sin(x);
end
%for n=2:N
% x=(n-1)*h;
```

```
% w(n,1)=sin(x);
%end
aa=-1/(2*h*h);
A1=zeros(N+1,N+1);
B1=zeros(N+1,N+1);
E1=Fr1(:,2);
for n=2:N;
b=(1/tau)+(1/(h*h))+1/2-((2*sqrt(2))/(3*sqrt(pi*tau)));
c=(-1/tau)+(1/(h*h))+1/2+((2*sqrt(2))/(3*sqrt(pi*tau)));
g=(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2)...
+((2*sqrt(2)*q(n))/(3*sqrt(pi*tau))))*((1-y)/((1-y)*qs+y*qs1));
e=(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2)...
-((2*sqrt(2)*q(n))/(3*sqrt(pi*tau))))*((1-y)/((1-y)*qs+y*qs1));
z1=(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2)...
+((2*sqrt(2)*q(n))/(3*sqrt(pi*tau))))*(y/((1-y)*qs+y*qs1));
j1= (((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2)...
-((2*sqrt(2)*q(n))/(3*sqrt(pi*tau))))*(y/((1-y)*qs+y*qs1));
A1(n,n+1)=A1(n,n+1)+aa;
A1(n,n)=A1(n,n)+b;
A1(n,n-1)=A1(n,n-1)+aa;
A1(n,s+1)=A1(n,s+1)+g;
A1(n,s+2)=A1(n,s+2)+z1;
B1(n,n+1)=B1(n,n+1)+aa;
B1(n,n)=B1(n,n)+c;
B1(n,n-1)=B1(n,n-1)+aa;
B1(n,s+1)=B1(n,s+1)+e;
B1(n,s+2)=B1(n,s+2)+j1;
A1(1)=1; A1(end)=1;
end
w(:,2)=inv(A1)*(E1-B1*w(:,1));
w(1,:)=0; w(1:end)=0;
Fr2=zeros(N+1,1);
Fr2(1,:)=0; Fr2(end,:)=0;
for n=2:N;
x=(n-1)*h;
f2= ((-3*(3*tau/2)) - (1/sqrt(pi*(3*tau/2)))...
+((2*sqrt(3*tau/2))/sqrt(pi)))*sin(x); % 'f' function
```

```
Fr2(n,3)=(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2)...
+((2*sqrt(6)*q(n))/(5*sqrt(pi*tau))))*((2-(3*tau))/((1-y)*qs+y*qs1))...
-((4*sqrt(6)*q(n))/(5*sqrt(pi*tau)))*(1/((1-y)*qs+y*qs1))...
-((sqrt(6*tau)*q(n))/(5*sqrt(pi)))+f2;
end
aa=-1/(2*h*h);
nu=(4*sqrt(6))/(5*sqrt(pi*tau));
A2=zeros(N+1,N+1);
B2=zeros(N+1,N+1);
C2=zeros(N+1,N+1);
E2=Fr2(:,3);
for n=2:N
bb=(1/tau)+(1/(h*h))+1/2-((2*sqrt(6))/(5*sqrt(pi*tau)));
cc=(-1/tau)+(1/(h*h))+1/2-((2*sqrt(6))/(5*sqrt(pi*tau)));
dd=(((q(n+1)-2*q(n)+q(n-1))/(2*(h*h)))-(q(n)/2)...
+((2*sqrt(6)*q(n))/(5*sqrt(pi*tau))))*((1-y)/((1-y)*qs+y*qs1));
ee=(((q(n+1)-2*q(n)+q(n-1))/(2*(h*h)))-(q(n)/2)...
+((2*sqrt(6)*q(n))/(5*sqrt(pi*tau))))*(y/((1-y)*qs+y*qs1));
zz=((-4*sqrt(6)*q(n))/(5*sqrt(pi*tau)))*((1-y)/((1-y)*qs+y*qs1));
jj=((-4*sqrt(6)*q(n))/(5*sqrt(pi*tau)))*(y/((1-y)*qs+y*qs1));
A2(n,n+1)=A2(n,n+1)+aa;
A2(n,n)=A2(n,n)+bb;
A2(n,n-1)=A2(n,n-1)+aa;
A2(n,s+1)=A2(n,s+1)+dd;
A2(n,s+2)=A2(n,s+2)+ee;
B2(n,n+1)=B2(n,n+1)+aa;
B2(n,n)=B2(n,n)+cc;
B2(n,n-1)=B2(n,n-1)+aa;
B2(n,s+1)=B2(n,s+1)+dd;
B2(n,s+2)=B2(n,s+2)+ee;
C2(n,n)=C2(n,n)+nu;
C2(n,s+1)=C2(n,s+1)+zz;
C2(n,s+2)=C2(n,s+2)+jj;
A2(1)=1; A2(end)=1;
end
w(:,3)=inv(A2)*(E2-B2*w(:,2)-C2*w(:,1));
%% calculate the right side of the equation
```

```
Fr=zeros(N+1,1);
Fr(1,:)=0; Fr(end,:)=0;
for k=4:K+1
for n=2:N
t=(k-1)*tau-(tau/2);
x=(n-1)*h;
f = (-3*t - (1/sqrt(pi*t))...
+ (2*sqrt(t)/sqrt(pi)))*sin(x); % 'f' function
summ=0; % summing part in the equation
d=2/(sqrt(pi*(tau)));
d1=sqrt(2)/(6*sqrt(pi*(tau)));
for m=2:k-2
b1=sqrt(k-1-m+1/2)-sqrt(k-1-m-1/2);
b2=(-1/3)*((k-1-m+1/2)^(3/2)-(k-1-m-1/2)^(3/2));
summ = summ + d*(((((k-1-m)*b1+b2)*(1-(m-2)*tau)*q(n))/((1-y)*qs+y*qs1))...
+((((2*m-2*(k-1)-1)*b1-2*b2)*(1-(m-1)*tau)*q(n))/((1-y)*qs+y*qs1))...
+((((k-1-m+1)*b1+b2)*(1-m*tau)*q(n))/((1-y)*qs+y*qs1)));
end
Fr(n,k)=((5*d1*q(n)*(1-(k-1)*tau))/((1-y)*qs+y*qs1))...
-((4*d1*q(n)*(1-(k-2)*tau))/((1-y)*qs+y*qs1))...
-((d1*q(n)*(1-(k-3)*tau))/((1-y)*qs+y*qs1))...
+(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))...
-(q(n)/2))*((2-(2*k-3)*tau)/((1-y)*qs+y*qs1))...
+summ+f;
end
end
%% calculate the left side of the equation
%w=ones(N+1,1);
%w(1,:)=0; w(end,:)=0; %when n=0 or n=N, w=0;
%for n=1:N+1 %when k=0, w=sin(x)
%x=(n-1)*h;
% w(n,1)=sin(x);
%end
for k=4:K+1
A=zeros(N+1,N+1);
B=zeros(N+1,N+1);
C=zeros(N+1,N+1);
```

```
D=zeros(N+1,1);
E=Fr(:,k);
for n=2:N % for each n
bn=(1/tau)+(1/(h*h))+1/2-(5*d1);
cn=(-1/tau)+(1/(h*h))+1/2+(4*d1);
dn=(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2...
+(5*d1*q(n)))*((1-y)/((1-y)*qs+y*qs1));
en=(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2)...
-(4*d1*q(n)))*((1-y)/((1-y)*qs+y*qs1));
fn=(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2)...
+(5*d1*q(n)))*(y/((1-y)*qs+y*qs1));
gn =(((q(n+1)-2*q(n)+q(n-1))/(2*h*h))-(q(n)/2)...
-(4*d1*q(n)))*(y/((1-y)*qs+y*qs1));
j=(d1*q(n)*(1-y))/((1-y)*qs+y*qs1);
z=(d1*q(n)*y)/((1-y)*qs+y*qs1);
r1=2*(sqrt(3/2)-sqrt(1/2))+((-1/3)*((3/2)^(3/2)-(1/2)^(3/2)));
r2=(-3)*(sqrt(3/2)-sqrt(1/2))-2*((-1/3)*((3/2)^(3/2)-(1/2)^(3/2)));
r3=(3)*(sqrt(5/2)-sqrt(3/2))+((-1/3)*((5/2)^(3/2)-(3/2)^(3/2)));
for m=2:k-2 %summing part in the equation
if m==2 %summing part in the equation
D(n)=sqrt(2)*6*nucal(k,m)*(j*w(s+1,m-1)...
+z*w(s+2,m-1)-d1*w(n,m-1));
elseif m==3
nn=sqrt(2)*6*(j*(nucal(k,m)+mucal(k,2))*w(s+1,m-1)...
+z*(nucal(k,m)+mucal(k,2))*w(s+2,m-1)...
-d1*(nucal(k,m)+mucal(k,2))*w(n,m-1));
D(n)=D(n)+nn;
else
suma=0;
suma= suma +sqrt(2)*6*(j*(mucal(k,m-1)+nucal(k,m)...
+ppcal(k,m-2))*w(s+1,m-1)...
+z*(mucal(k,m-1)+nucal(k,m)+ppcal(k,m-2))*w(s+2,m-1)...
-d1*(mucal(k,m-1)+nucal(k,m)+ppcal(k,m-2))*w(n,m-1));
D(n)=D(n)+suma;
end
end
A(n,n+1)=A(n,n+1)+aa;
```

```
A(n,n)=A(n,n)+bn;
A(n,n-1)=A(n,n-1)+aa;
A(n,s+1)=A(n,s+1)+dn;
A(n,s+2)=A(n,s+2)+fn;
B(n,n+1)=B(n,n+1)+aa;
B(n,n)=B(n,n)+cn-sqrt(2)*6*r1*d1;
B(n,n-1)=B(n,n-1)+aa;
B(n,s+1)=B(n,s+1)+en+sqrt(2)*6*r1*j;
B(n,s+2)=B(n,s+2)+gn+sqrt(2)*6*r1*z;
C(n,n)=C(n,n)+d1-sqrt(2)*6*d1*(r3+r2);
C(n,s+1)=C(n,s+1)-j+sqrt(2)*6*j*(r3+r2);
C(n,s+2)=C(n,s+2)-z+sqrt(2)*6*z*(r3+r2);
end
A(1)=1; A(end)=1;
w(:,k)=inv(A)*(E-B*w(:,k-1)-C*w(:,k-2)-D);
end
[t,x]=meshgrid(0:K,0:N);
surf(x,t,w);
xlabel('x');ylabel('t');zlabel('w');
axis auto
for k=1:K+1;
for n=1:N+1
u1(n,k)=((1-(k-1)*tau)-w(s+1,k))/qs*sin((n-1)*h)+w(n,k);
end
end
figure;
surf(u1)
for k=1:K+1;
for n=1:N+1
es(n,k)=(1-(k-1)*tau)*sin((n-1)*h);
end
end
figure;
surf(es)
maxes=max(max(es)) ;
maxapp=max(max(u1)) ;
maxerror=max(max(abs(es-u1)));
```

```
relativeerror=maxerror/maxapp;
cevap = [maxes,maxapp,maxerror,relativeerror]
abserr=max(max(abs(es-u1)))
%subfunctions
function mu=mucal(k,m);
mu=(2*m-2*(k-1)-1)*(sqrt(k-1-m+1/2)-sqrt(k-1-m-1/2))...
-2*((-1/3)*((k-1-m+1/2)^(3/2)-(k-1-m-1/2)^(3/2)));
function pp=ppcal(k,m) ;
pp=(k-1-m+1)*(sqrt(k-1-m+1/2)-sqrt(k-1-m-1/2))...
+((-1/3)*((k-1-m+1/2)^(3/2)-(k-1-m-1/2)^(3/2))) ;
function nu=nucal(k,m);
nu=(k-1-m)*(sqrt(k-1-m+1/2)-sqrt(k-1-m-1/2))...
+((-1/3)*((k-1-m+1/2)^(3/2)-(k-1-m-1/2)^(3/2))) ;
```

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