WEIGHTED APPROXIMATION BY VIDENSKII AND LUPAS OPERATORS

by

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APPROVAL PAGE

This is to certify that I have read this thesis written by Akif Barbaros DİKMEN and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy in Mathematics.

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ABSTRACT

In this dissertation we focus on the boundedness and convergence properties of linear positive operators.

In chapter 1 we give some basic information about Bernstein polynomials, Weighted approximation of functions, Lototsky transform of Bernstein operators, Quantum calculus, and Moduli of continuity.

In chapter 2 we pay attention to weighted boundedness and weigthed approximation by classical polynomial operators and to construction of their weighted modifications, because usual operators are not always suitable for approximating functions with singularities in weighted spaces. We investigate approximation properties of Videnskii operators in the weighted norm under some restrictions.

In chapter 3 we constructed Videnskii type generalization of Baskakov operators and compare it with Swetits-Wood's results.

In chapter 4 in the first section we state a new q-analogue of Durrmeyer operators which preserves the linear function and their convergence properties. In the second section we state a new Durrmeyer type modification of generalized Baskakov operators A_n for all real valued continuous and bounded functions f on$(0, \infty]$. For the operators A_n we establish certain direct theorems in terms of the modulus of continuity of second order, and we prove the continuity of the operator in Lipschitz-type space.

Keywords: Videnskii Operator, Weighted Approximation, Durrmeyer

VİDENSKİİ VE LUPAS OPERATÖRLERİNİN AĞIRLIKLI YAKINSAKLIĞI

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ÖZ

Bu doktora tezinde genel olarak doğrusal pozitif operatörlerinin sınırlılık ve yakınsaklık özelliklerini inceledik.

Birinci bölümde Bernstein polinomları, fonksiyonların ağırlıklı yaklaşımı, Lototsky transform, quantum analizi, ve süreklilik modülü hakkında temel bazı bilgiler verdik.

İkinci bölümde klasik Bernstein polinomlarının modifikasyonlarını incelendi. Bu çalışmlar bize, bu operatörlerin rasyonel benzerlerinin, ağırlıklı modifikasyonlarını oluşturma fikrini verdi. Çünkü normal operatörler ağırlıklı uzaylarda yakınsayan fonksiyonlar için uygun olmayabiliyor. Ağrılıklı uzaylarda ve bazı özel şartlar altında Videnskii operatörlerinin yakınsaklık özellikleri incelendi.

Üçüncü bölümde Baskakov operatörlerinin, Videnskii tipindeki genellemesini elde ettik. Sonuçları Swetits - Woods' ın sonuçlarıyla kıyasladık.

Dördüncü bölümün ilk kısmında, Lineeer fonksiyonlarını aynı elde ettiğimiz yeni bir Durrmeyer operatörlerinin q benzerini elde ettik. Bu operatörlerin yakınsaklık özellikleri inceledik. Ayrıca $(0, \infty]$ de reel değerli ve sınırlı bir f fonksiyonun genellenmiş Baskakov operatörlerinin, Durrmeyer tipinde ifade edilen yeni bir operatörü tanımladık. Bu operatörü için direkt teoremleri ikinci dereceden süreklilik modülü anlamında ifade ettik. Bu A_n operatörin sürekliliğini ve Lipschitz uzayının bir elemanı ve genel süreklilik modülü şeklinde ifade ettik.

Anahtar Kelimeler: Videnskii Operatörü, Ağırlıklı Yakınsaklık, Durrmeyer

To my family

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CHAPTER 1

INTRODUCTION AND BASIC NOTATION

1.1 INTRODUCTION

The main goal of this chapter is to present to the reader known results which will be used in our study. They are taken from the books : G. M. Phillips (Phillips, 2000), R. A. Devore and G. G. Lorentz (DeVore and Lorentz, 1993), G. G. Lorentz (Lorentz, 1986), F. Altomare and M. Campiti (Altomare and Campiti, 1994), V. Kac and P. Cheung (Kac and Cheung, 2002), and papers by V. S. Videnskii (Videnskii, 2008), B. D. Vecchia and G. Mastroianni and J. Szabados (Vecchia et al., 2004), J.P. King (King, 1966)

1.2 BERNSTEIN POLYNOMIALS

This part of the introduction is concerned with sequences of polynomials named after their creater S.N. Bernstein. Given a function f on $[0,1]$, we define the Bernstein polynomial

$$B_n(f,x)=\sum_{r=0}^{n}f\left(\frac{r}{n}\right)\binom{n}{r}x^r(1-x)^{n-r} \tag{1.1}$$

for each positive integer n. Thus there is a sequence of Bernstein polynomials corresponding to each function f. As we will see later in this part , if f is continous on $[0,1]$, its sequence of Bernstein polynomials converges uniformly to f on $[0,1]$. One might wonder why Bernstein created "new" polynomials for this purpose, instead of using polynomials that were already known to mathematics. For example Taylor polynomials are not appropriate; for even setting aside questions of convergence, they are applicable only to functions that are infinitely differentiable and not to all continous functions.

It is clear from (1.1) that for all $n \geq 1$,

$$B_n(f;0) = f(0) \text{ and } B_n(f;1) = f(1), \tag{1.2}$$

so that a Bernstein polynomial for f interpolates f at both endpoints of the interval $[0,1]$.

Example 1.2.1. It follows from the binomial expansion that

$$B_n(1,x) = \sum_{r=0}^{n} \tbinom{n}{r} x^r (1-x)^{n-r} = (x + (1-x))^n = 1, \tag{1.3}$$

so that the Bernstein polynomial for the constant function 1 is also 1. Since

$$\frac{r}{n}\binom{n}{r} = \binom{n-1}{r-1}$$

for $1 \leq r \leq n$ the Bernstein polynomial for the function x is

$$B_n(x,x) = \sum_{r=0}^{n} \frac{r}{n}\binom{n}{r} x^r (1-x)^{n-r} = x\sum_{r=0}^{n} \tbinom{n-1}{r-1} x^{r-1} (1-x)^{n-r}.$$

Note that the term corresponding to $r = 0$ in the first of the above two sums is zero. One putting $s = r - 1$ in the second summation, we obtain the

$$B_n(x,x) = x\sum_{r=0}^{n} \tbinom{n-1}{s} x^s (1-x)^{n-1-s} = x, \tag{1.4}$$

last step following from (1.3) with n replaced by $n-1$. Thus the Bernstein polynomial for the function x is also x.

We call B_n the Bernstein operator; it maps a function f, defined on $[0,1]$, to $B_n f$ where the function $B_n f$ evaulated at x is denoted by $B_n(f;x)$. The Bernstein operator is obviously linear, since it follows from (1.1) that

$$B_n(\lambda f + \mu g) = \lambda B_n f + \mu B_n(g), \tag{1.5}$$

for all functions f and g defined on $[0,1]$ and real λ and μ. We now require the following definition.

Definition 1.2.1. Let L denote a linear operator that maps functions f defined on $[a,b]$ to a function Lf defined on $[c,d]$. Then L is said to be a monotone operator or, equivalently a positive operator if

$$f(x) \geq g(x), \quad x \in [a,b] \Rightarrow (Lf)(x) \geq (Lg)(x), \; x \in [c,d], \tag{1.6}$$

where we have written $(Lf)(x)$ to denote the value of the function Lf at the point $x \in [a,b]$.

We can see from (1.1) that B_n is a monotone operator. It then follows from the monotonicity of B_n and (1.3) that

$$m \leq f(x) \leq M, \; x \in [0,1] \quad \Rightarrow \quad m \leq B_n(f,x) \leq M, \; x \in [0,1]. \tag{1.7}$$

In particular, if we choose $m = 0$ in (1.7) we obtain

$$f(x) \geq 0, \; x \in [a,b] \quad \Rightarrow B_n(f,x) \geq 0, \; x \in [0,1] \tag{1.8}$$

it follows from (1.3) and (1.4) the linear property (1.5) that

$$B_n(ax+b;x) = ax+b \tag{1.9}$$

for all real a and b. We therefore say that the Bernstein operator reproduces linear polynomials. We can deduce from the following result that the Bernstein operator does not reproduce any polynomial of degree greater than one.

Theorem 1.2.1. *The Bernstein polynomial may be expressed in the form*

$$B_n(f,x) = \sum_{r=0}^{n} \binom{n}{r} \Delta^r f(0)\, x^r, \tag{1.10}$$

where Δ is the forward difference operator with the step size $h = \frac{1}{n}$.

Proof. Begining with (1.1) ,and expanding the term $(1-x)^{n-r}$, we have

$$B_n(f,x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right) \binom{n}{r} x^r \sum_{s=0}^{n-r} (-1)^s \binom{n-r}{s} x^s.$$

Let us put $t = r+s$. We may write

$$\sum_{r=0}^{n} \sum_{s=0}^{n-r} = \sum_{t=0}^{n} \sum_{r=0}^{t}, \tag{1.11}$$

since both double summations in (1.11) are over all lattice points. We also have

$$\binom{n}{r}\binom{n-r}{s}=\binom{n}{t}\binom{t}{r},$$

and so we may write the double summation as

$$\sum_{t=0}^{n}\binom{n}{t}x^t\sum_{r=0}^{t}\left(-1\right)^{t-r}\binom{t}{r}f\left(\frac{r}{n}\right)=\sum_{t=0}^{n}\binom{n}{t}\Delta^t f\left(0\right)x^t,$$

on using the expansion for a higher order forward difference.

This completes the proof. □

The relation between forward differences and derivatives is

$$\frac{\Delta^m f\left(x_0\right)}{h^m}=f^{(m)}\left(\xi\right), \tag{1.12}$$

where $\xi\in(x_0,x_m)$ and $x_m=x_m+mh$. Let us put $h=\frac{1}{n}$, $x_0=0$ and $f\left(x\right)=x^k$ where $n\geq k$. Then we have

$$n^r\Delta^r f\left(0\right)=0\quad\text{for } r>k$$

and

$$n^k\Delta^k f\left(0\right)=f^{(k)}\left(\xi\right)=k!. \tag{1.13}$$

Thus, we see from (1.10) with $f\left(x\right)=x^k$ where $n\geq k$ that

$$B_n\left(x^k;x\right)=a_0x^k+a_1x^{k-1}+...+a_{k-1}x+a_k,$$

where $a_0=1$ for $k=0$ and $k=1$, and

$$a_0=\binom{n}{k}\frac{k!}{n^k}=\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{k-1}{n}\right)$$

for $k\geq 2$. Since $a_0\neq 1$ when $n\geq k\geq 2$, this justifies our above statement that the Bernstein operators does not reproduce any polynomial of degree greater than one.

Example 1.2.2. With $f\left(x\right)=x^2$, we have

$$f\left(0\right)=0,\quad \Delta f\left(0\right)=f\left(\frac{1}{n}\right)-f\left(0\right)=\frac{1}{n^2},$$

and we see from (1.10) that $n^2\Delta^2 f\left(0\right)=2!$ for $n\geq 2$. Thus it follows from (1.10) that

$$B_n\left(x^2;x\right)=\binom{n}{1}\frac{x}{n^2}+\binom{n}{2}\frac{2x^2}{n^2}=\frac{x}{n}+\left(1-\frac{1}{n}\right)x^2,$$

which may be written in the form

$$B_n\left(x^2;x\right)=x^2+\frac{1}{n}x\left(1-x\right). \tag{1.14}$$

Thus the Bernstein polynomials for x^2 converge uniformly to x^2 like $\frac{1}{n}$, very slowly.

Theorem 1.2.2. *Given a function $f\in C\left[0,1\right]$ and any $\epsilon>0$ there exists an integer N such that*

$$\left|f\left(x\right)-B_n\left(f,x\right)\right|\leq\epsilon,\quad 0\leq x\leq 1,$$

for all $n\geq N$.

Proof. In the other words the above statement says that the Bernstein polynomials for a function f that is continous on $[0,1]$ converge uniformly to f on $[0,1]$.

We begin with the identity

$$\left(\frac{r}{n}-x\right)^2=\left(\frac{r}{n}\right)^2-2\left(\frac{r}{n}\right)x+x^2,$$

multiply each term by $\binom{n}{r}x^r\left(1-x\right)^{n-r}$, and then sum up from $r=0$ to n to give

$$\sum_{r=0}^{n}\left(\frac{r}{n}-x\right)^2\binom{n}{r}x^r\left(1-x\right)^{n-r}=B_n\left(x^2;x\right)-2xB_n\left(x;x\right)+x^2B_n\left(1;x\right).$$

It then follows from (1.3), (1.4) and (1.14) that

$$\sum_{r=0}^{n}\left(\frac{r}{n}-x\right)^2\binom{n}{r}x^r\left(1-x\right)^{n-r}=\frac{1}{n}x\left(1-x\right). \tag{1.15}$$

For any fixed $x\in[0,1]$, let us assume the sum of the polynomials $p_{n,r}\left(x\right)$ over all values of r for which $\frac{r}{n}$ is not close to x. To make this notion precise, we choose a number $\delta>0$ and let S_δ denote the set of all values of r satisfying $\left|\frac{r}{n}-x\right|>\delta$. We now consider the sum of the polynomials $p_{n,r}\left(x\right)$ over all $r\in S_\delta$. Note that $\left|\frac{r}{n}-x\right|>\delta$ implies that

$$\frac{1}{\delta^2}\left(\frac{r}{n}-x\right)^2\geq 1. \tag{1.16}$$

Then using (1.16), we have

$$\sum_{r\in S_\delta}\binom{n}{r}x^r\left(1-x\right)^{n-r}\leq\frac{1}{\delta^2}\sum_{r\in S_\delta}\left(\frac{r}{n}-x\right)^2\binom{n}{r}x^r\left(1-x\right)^{n-r}.$$

The latter sum is not greater that the sum of the same expression over all r, and using (1.15) we have

$$\frac{1}{\delta^2}\sum_{r=0}\left(\frac{r}{n}-x\right)^2\binom{n}{r}x^r(1-x)^{n-r}=\frac{x(1-x)}{n\delta^2}.$$

Since $0\leq x(1-x)\leq\frac{1}{4}$ on $[0,1]$, we have

$$\sum_{r\in S_\delta}\binom{n}{r}x^r(1-x)^{n-r}\leq\frac{1}{4n\delta^2}. \tag{1.17}$$

Let us write

$$\sum_{r=0}^{n}=\sum_{r\in S_\delta}+\sum_{r\notin S_\delta},$$

where the latter sum is therefore over all r such that $\left|\frac{r}{n}-x\right|<\delta$. Having split the summation into these two parts which depend on a choice of δ that we still have to make, we are now ready to estimate the difference between $f(x)$ and its Bernstein polynomial. Using (1.3) we have

$$f(x)-B_n(f,x)=\sum_{r=0}^{n}\left(f(x)-f\left(\frac{r}{n}\right)\right)\binom{n}{r}x^r(1-x)^{n-r},$$

and hence

$$\begin{aligned}f(x)-B_n(f,x)=&\sum_{r\in S_\delta}\left(f(x)-f\left(\frac{r}{n}\right)\right)\binom{n}{r}x^r(1-x)^{n-r}\\&+\sum_{r\notin S_\delta}\left(f(x)-f\left(\frac{r}{n}\right)\right)\binom{n}{r}x^r(1-x)^{n-r}.\end{aligned}$$

We thus obtain the inequality

$$\begin{aligned}|f(x)-B_n(f,x)|=&\sum_{r\in S_\delta}\left|f(x)-f\left(\frac{r}{n}\right)\right|\binom{n}{r}x^r(1-x)^{n-r}\\&+\sum_{r\notin S_\delta}\left|f(x)-f\left(\frac{r}{n}\right)\right|\binom{n}{r}x^r(1-x)^{n-r}.\end{aligned}$$

Since $f(x)\in[0,1]$, it is bounded on $[0,1]$ and we have $|f(x)|\leq M$, for some $M>0$. We can therefore write

$$\left|f(x)-f\left(\frac{r}{n}\right)\right|\leq 2M$$

for all r and all $x\in[0,1]$ and so

$$\sum_{r\in S_\delta}\left|f(x)-f\left(\frac{r}{n}\right)\right|\binom{n}{r}x^r(1-x)^{n-r}\leq 2M\sum_{r\in S_\delta}\binom{n}{r}x^r(1-x)^{n-r}.$$

On using (1.17) we obtain

$$\sum_{r\in S_\delta}\left|f(x)-f\left(\frac{r}{n}\right)\right|\binom{n}{r}x^r(1-x)^{n-r}\le\frac{M}{2n\delta^2}. \tag{1.18}$$

Since f is continous, it is also uniformly continous, on $[0,1]$. Thus, corresponding to any choice of $\epsilon>0$ there is a number $\delta>0$ depending on ϵ and f, such that

$$|x-x'|<\delta\Rightarrow|f(x)-f(x')|<\frac{\epsilon}{2},$$

for all x, $x'\in[0,1]$. Thus, for the sum over $r\notin S_\delta$ we have

$$\begin{aligned}\sum_{r\notin S_\delta}\left|f(x)-f\left(\frac{r}{n}\right)\right|\binom{n}{r}x^r(1-x)^{n-r} &\le\frac{\epsilon}{2}\sum_{r\notin S_\delta}\binom{n}{r}x^r(1-x)^{n-r}\\ &\le\frac{\epsilon}{2}\sum_{r=0}^{n}\binom{n}{r}x^r(1-x)^{n-r}, \end{aligned} \tag{1.19}$$

and hence again using (1.3) we find that

$$\sum_{r\notin S_\delta}\left|f(x)-f\left(\frac{r}{n}\right)\right|\binom{n}{r}x^r(1-x)^{n-r}<\frac{\epsilon}{2}.$$

On combining (1.18) and (1.19) we obtain

$$|f(x)-B_n(f,x)|<\frac{M}{2n\delta^2}+\frac{\epsilon}{2}.$$

It follows from the line above that if we choose $N>\frac{M}{2\epsilon\delta^2}$, then

$$|f(x)-B_n(f,x)|<\epsilon$$

for all $n\ge N$ and this completes the proof. □

Remark 1.2.1. The proof follows P.P. Korovkin and was used by him to obtain essentially more general results (see for example F. Altomare, M. Campiti (Altomare and Campiti, 1994)).

1.3 QUANTUM CALCULUS

Let $q>0$. For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are respectively defined by

$$[k] = \begin{Bmatrix} \frac{1-q^k}{1-q}, & q \neq 1 \\ k, & q = 1. \end{Bmatrix}$$

$$[k]! := \begin{Bmatrix} [k]\,[k-1] \dots [1]\,, & k \geq 1 \\ 1, & k = 0 \end{Bmatrix}.$$

For the integers n, k satisfying $n \geq k \geq 0$ the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\,[n-k]!}.$$

If $k < 0$ or $k > n$ then $\begin{bmatrix} n \\ k \end{bmatrix} = 0$. Therefore, the equalities

$$[k] + q^k\,[n-k] = [n]!, \quad \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}$$

and

$$\frac{[k]}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad \frac{[n-k]}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix}$$

hold true.

We may also define

$$(1+x)_q^n = \prod_{j=0}^{n-1} \left(1 + q^j x\right) = (1+x)\,(1+qx) \dots \left(1 + q^{j-1}x\right).$$

Definition 1.3.1. Consider an arbitrary function $f(x)$. It's q-differential is

$$d_q f(x) = f(qx) - f(x).$$

Definition 1.3.2. The following expression

$$D_q f(x)\,\frac{d_q f(x)}{d_q x}$$

are called the q-derivative.

Example 1.3.1. Compute the q-derivative of $f(x) = x^n$, where n is positive integer. By the definition

$$D_q x^n = \frac{(qx)^n - x^n}{(q-1)\,x} = \frac{q^n - 1}{q-1} x^{n-1}. \tag{1.20}$$

Since the fraction $\frac{q^n-1}{q-1}$ appears quite frequently, then (1.20) becomes $D_q x^n = nx^{n-1}$ which resembles the ordinary derivative of x^n.

Definition 1.3.3. The q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)_q^n = \begin{Bmatrix} 1, & \text{if } n=0 \\ (x-a)(x-qa)\dots(x-q^{n-1}a) & \text{if } n\geq 1 \end{Bmatrix}. \tag{1.21}$$

Proposition 1.3.1. *For $n \geq 1$*

$$D_q (x-a)_q^n = [n](x-a)_q^{n-1}.$$

Proof. The formula is obviously true when $n = 1$. Let us assume $D_q (x-a)_q^k = [k](x-a)_q^k$ for some integer k. According to the definition $(x-a)_q^{k+1} = (x-a)_q^k \left(x-q^k a\right)$. Using the product rule

$$\begin{aligned} D_q (x-a)_q^{k+1} &= (x-a)_q^k + \left(qx - q^k a\right) D_q (x-a)_q^k \\ &= (x-a)_q^k + q\left(x-q^{k-1}a\right)[k](x-a)_q^{k-1} \\ &= (1+q[k])(x-a)_q^k = [k+1](x-a)_q^k. \end{aligned}$$

Hence, the proposition is proved by induction on k. □

Thus, $D_q P_n = P_{n-1}$ is an immediate result of the above proposition. Now let us explore some other properties of the polynomial $(x-a)_q^n$. In general, $(x-a)_q^{m+n} = (x-a)_q^m (x-a)_q^n$. Instead ,

$$\begin{aligned} (x-a)_q^{m+n} &= (x-a)(x-qa)\dots\left(x-q^{m-1}a\right)(x-q^m a)\left(x-q^{m+1}a\right) \\ &\times \dots \left(x-q^{m+n-1}a\right) \\ &= \left((x-a)(x-qa)\dots\left(x-q^{m-1}a\right)(x-q^m a)\left(x-q^{m-1}a\right)\right) \\ &\times \dots (x-q^m a)(x-q(q^m a)) .. \left(x-q^{n-1}(q^m a)\right), \end{aligned}$$

which gives

$$(x-a)_q^{m+n} = (x-a)_q^m (x-q^m a)_q^n.$$

Substituting m by $-n$ we can thus extend the definition in (1.21) to all integers by defining for any positive integer n

$$(x-a)_q^{-n} = \frac{1}{(x-q^{-n}a)_q^n}.$$

The q-analogue of the integration in the interval $[0,a]$ is defined by

$$\int_0^a f(t)\, d_q t := (1-q)\sum_{n=1}^{\infty} f(aq^n)\, q^n, \quad 0<q<1.$$

We set

$$p_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k}, \qquad p_{\infty k}(q;x) := \frac{x^k}{(1-q)_q^k [k]!} (1-x)_q^{\infty}.$$

1.4 VIDENSKII AND LUPAS OPERATORS

Videnskii constructed linear positive operators (LPO) which generalize Bernstein polynomials and are rational functions, namely

$$U_n(f,x) = \sum_{k=0}^{n} f(\tau_{nk}) u_{nk}(x) \tag{1.22}$$

$$u_{nk}(x) = \alpha_{nk} \frac{x^k (1-x)^{n-k}}{\prod_{i=1}^{n} |x_{ni} - x|}, \quad \alpha_{nk} > 0, \tag{1.23}$$

where$\{x_{ni}\}_{i=0}^{n}$ is a given arbitrary matrix of real poles which lie outside of the interval $(0,1)$. Necessary and sufficient condition for the matrix $\{x_{ni}\}$ were given, for the sake of possibility of approximation by the sequence of LPO $\{U_n\}$.

Also Lupas considered rational LPO which generalize Bernstein polynomials. He chooses poles and nodes in some dependence of geometrical progression with quotient $0 < q < 1$. Videnskii (Videnskii, 2008) observed that basic functions for Lupas operators can be considered as a particular case of $u_{nk}(x)$ from (1.22).

Let the functions $a_{nk} \in C[0,1]$, $a_{nk} \geq 0$ $(k = 0, ..., n)$, $n \in N$.

$$\sum_{k=0}^{n} a_{nk}(x) = 1 \tag{1.24}$$

for $f \in C[0,1]$ we construct a sequence of LPO

$$A_n^{\xi}(f,x) = \sum_{k=0}^{n} f(\xi_{nk}) a_{nk}(x),$$

where $0 = \xi_{n0} < \xi_{n1} < ... < \xi_{nn} = 1$; points ξ_{nk} we call nodes of LPO A_n^{ξ} the matrix $\{a_{nk}\}_{k=0}^{n}$ $n \in N$ is called a base of LPO A_n^{ξ}. The sequence of LPO is callled approximating if for only $f \in C[0,1]$

$$\lim_{n\to\infty} A_n^{\xi}(f,x) = f(x)$$

uniformly on $[0,1]$. The set of such approximation sequences is denoted by A and respectively $\left\{A_n^\xi\right\} \in A$. Denote by

$$S_v\left(A_n^\xi, x\right) = \sum_{k=0}^{n} \left(\xi_{nk} - x\right)^v a_{nk}\left(x\right)$$

$$S_v^*\left(A_n^\xi, x\right) = \sum_{k=0}^{n} \left|\xi_{nk} - x\right|^v a_{nk}\left(x\right)$$

we call $S_v\left(A_n^\xi, x\right)$ a moment of order v LPO A_n^ξ. Clearly, $S_0\left(A_n^\xi, x\right) = 1$.

Put

$$dA_n^\xi = \max_x S_2\left(A_n^\xi, x\right),\ \sigma A_n^\xi = \max_x S_1^*\left(A_n^\xi, x\right).$$

Since $\left(\xi_{nk} - x\right)^2 \le \left|\xi_{nk} - x\right| \le 1$ and because of (1.24) and Cauchy Schwarz

$$S_1^*\left(A_n^\xi, x\right) \le \sqrt{S_2\left(A_n^\xi, x\right)}$$

$$dA_n^\xi \le \sigma A_n^\xi \le \sqrt{dA_n^\xi}$$

and hence the equations

$$\lim_{n\to\infty} dA_n^\xi = 0, \text{ and } \lim_{n\to\infty} \sigma A_n^\xi = 0$$

hold simultaneously.

Now we can give some basic facts about Videnskii's operators.

Put

$$x_{ni} = 1 + \rho_{ni}, \quad \rho_{ni} > 0 \tag{1.25}$$

where x_{ni} lie to the right of the interval and all $x_{ni} > 1$.

$$P_n\left(x\right) = \prod_{i=1}^{n}\left(1 + \rho_{ni} - x\right) = \prod_{i=1}^{n}\left(\rho_{ni}x + \left(1 + \rho_{ni}\right)\left(1 - x\right)\right) = \sum_{k=0}^{n} \alpha_{nk} x^k \left(1 - x\right)^{n-k} \tag{1.26}$$

then we can obtain basic functions $u_{nk}\left(x\right)$ (1.23) from (1.26).

It is clear that u_{nk} satisfy the equality

$$\sum_{k=0}^{n} u_{nk}\left(x\right) = 1. \tag{1.27}$$

It is known that polynomial p_{nk} from (1.1) is connected with generating function

$$G_n(x,y) = (xy + 1 - x)^n = \sum_{k=0}^{n} p_{nk}(x)\, y^k. \tag{1.28}$$

Analoguesly, we construct a generating function for u_{nk}. Put

$$h_{ni}(x) = \frac{\rho_{ni} x}{1 + \rho_{ni} - x}$$

and

$$\phi_n(x) = \frac{1}{n}\sum_{i=1}^{n} h_{ni}(x) \tag{1.29}$$

then

$$g_n(x,y) = \prod_{i=1}^{n} \left(h_{ni}(x)\, y + 1 - h_{ni}(x)\right) = \sum_{k=0}^{n} u_{nk}(x)\, y^k.$$

Differentiate in y

$$\frac{\partial\left(g_n(x,y)\right)}{\partial y} = g_n(x,y)\sum_{i=0}^{n} \frac{h_{ni}(x)}{h_{ni}(x)\, y + 1 - h_{ni}(x)} = \sum_{k=0}^{n} k u_{nk}(x)\, y^{k-1}$$

and putting $y = 1$ we obtain

$$\phi_n(x) = \sum_{k=0}^{n} \frac{k}{n} u_{nk}(x) \tag{1.30}$$

Observing that $\phi_n(x)$ is strictly increasing from 0 to 1 on the int $[0,1]$ define τ_{nk} in the formula (1.22) by relations

$$\phi_n(\tau_{nk}) = \frac{k}{n}, \quad k = 0, 1, .., n.$$

Note that the functions 1, $\phi_n(x)$ play role of the fixed functions f_0, f_1 for generalized Bernstein operators in the sense of (Aldaz and Render, 2010) for the system of rational functions of degree n with denominator $P_n(x)$.

Rewrite (1.30) in the form

$$\sum_{k=0}^{n} \left(\phi_n(\tau_{nk}) - \phi_n(x)\right) u_{nk}(x) = 0. \tag{1.31}$$

Differentiation of $\ln u_{nk}(x)$ $(0 < x < 1)$ gives

$$u'_{nk}(x) = \frac{n}{x(1-x)} \left(\phi_n(\tau_{nk}) - \phi_n(x)\right) u_{nk}(x). \tag{1.32}$$

Formula (1.32) shows by the way that the point τ_{nk} $(0 \le k < n-1)$ is the unique point of maximum of the function u_{nk} in the interval $[0,1]$. That is a reason why the Videnskii operators can be considered as a natural analogue of the Bernstein operators for rational functions. Similarly to the fact that the point is a point of maximum of the function p_{nk}. Derivative of (1.31) with taking into account (1.32) gives

$$\sum_{k=0}^{n} \left(\phi_n\left(\tau_{nk}\right)-\phi_n\left(x\right)\right)^2 u_{nk}\left(x\right)=\frac{x\left(1-x\right)}{n}\phi_n'\left(x\right). \tag{1.33}$$

It is easy to see

$$x\left(1-x\right)h_{ni}'=x\left(1-x\right)\frac{\rho_{ni}\left(1+\rho_{ni}\right)}{\left(1+\rho_{ni}-x\right)^2}=h_{ni}\left(x\right)\left(1-h_{ni}\left(x\right)\right).$$

Introducing notations for generated moments

$$\sigma_{nv}\left(x\right)=\sum_{k=0}^{n}\left(\phi_n\left(\tau_{nk}\right)-\phi_n\left(x\right)\right)^v u_{nk}\left(x\right). \tag{1.34}$$

Then we may rewrite $(1.27)\,,(1.31)$ and (1.33) as

$$\sigma_{n0}\left(x\right)=1,\quad \sigma_{n1}\left(x\right)=0,\qquad \sigma_{n2}\left(x\right)=\frac{1}{n^2}\sum_{i=1}^{n}h_{ni}\left(x\right)\left(1-h_{ni}\left(x\right)\right). \tag{1.35}$$

The conditions for $\{U_n\}\in A$ are given in terms of ρ_{ni} in other words they depend on the rate of approximation of poles to the ends of $[0,1]$ as $n\to\infty$.

Denote by

$$S_n=\sum_{i=1}^{n}\frac{\rho_{ni}}{\left(1+\rho_{ni}\right)}. \tag{1.36}$$

Theorem 1.4.1. *((Videnskii, 2008)). The following inequalities hold*

$$\max\left(\frac{1}{4n^2},\frac{1}{2}e^{-3S_n}\right)\le dU_n\le\frac{1}{S_n}. \tag{1.37}$$

Hence, for $\{U_n\}\in A$ it is neccessary and sufficient that

$$\lim_{n\to\infty}S_n=\infty.$$

Left hand side of the inequality (1.37) is not connected in fact with nodes τ_{nk} and is valid for arbitrary nodes $0=\xi_{n0}<\xi_{n1}<...<\xi_{nn}=1$ for LPO

$$U_n^{\xi}\left(f,x\right)=\sum_{k=0}^{n}f\left(\xi_{nk}\right)u_{nk}\left(x\right). \tag{1.38}$$

Rational LPO of Lupas (see for instance S. Ostrovska's paper (Ostrovska, 2006))are constructed for fixed n and q with poles

$$1+\rho_i=x_i=\frac{1}{1-q^i}.$$

Functions u_{nk} are defined by the same scheme as in (1.23), but with computations of coefficients α_{nk} in (1.26). For that reason a known $q-$ generalized bynomial formulas are used which was mentioned in section 1.3 .

The identity

$$F_n(z) = \prod_{i=0}^{n-1}\left(1+q^i z\right) = \sum_{k=0}^{n}\binom{n}{k}_q q^{\frac{k(k-1)}{2}} z^k \tag{1.39}$$

is valid which is easily checked by induction with taking in to account $F_n(z) = (1+z)\,F_{n-1}(qz)$. The roots of the formulas of type (1.39) extend to Euler.

The infinite product

$$F(z) = \prod_{i=1}^{\infty}\left(1+q^i z\right) = 1 + \sum_{k=1}^{\infty} z^k \sum_{n=\frac{k(k-1)}{2}}^{\infty} A_{nk} q^n \tag{1.40}$$

is considered.

The function F is a generating function to determine amount of expansions of an integer number n into sum of k different integer numbers. The coefficient A_{nk} is the searched number. Explicit formulas for A_{nk} are not given but a recurrent way is indicated to determine them. Later many famous mathematicians in particular Gauss, Cauchy, Stieljes paid an interest to F_n function.

We will use here partly the exposition of rational Lupas operator. Their denominator we' ll write in a form analogues to (1.26) and then putting

$$\frac{x}{1-x} = z$$

we' ll use $q-$binomial identity (1.39):

$$P_{n-1}(x) = \prod_{i=1}^{n-1}\left(1-x+q^i x\right) = (1-x)^{n-1}\prod_{i=1}^{n-1}\left(1+q^i z\right) = (1-x)^{n-1} F_{n-1}(qz)\,. \tag{1.41}$$

Taking into account $x+1-x=1$ we may write (1.41) as

$$P_{n-1}(x) = \prod_{i=1}^{n-1}\left(1-x+q^i x\right) = (1-x)^n F_n(x)\,,$$

hence we have

$$P_{n-1}(x) = (1-x)^n \sum_{k=0}^{n}\binom{n}{k}_q q^{\frac{k(k-1)}{2}} z^k = \sum_{k=0}^{n}\binom{n}{k}_q q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}\,. \tag{1.42}$$

Denote by

$$u_{nk}(q;x)=\frac{1}{P_{n-1}(x)}\binom{n}{k}_q q^{\frac{k(k-1)}{2}}x^k(1-x)^{n-k} \tag{1.43}$$

so that

$$\sum_{k=0}^{n}u_{nk}(q;x)=1. \tag{1.44}$$

We'll choose $\frac{[k]}{[n]}$ as nodes and LPO are defined by the formula

$$U_n(f,x,q)=\sum_{k=0}^{n}f\left(\frac{[k]}{[n]}\right)u_{nk}(q;x).$$

1.5 WEIGHTED APPROXIMATION OF FUNCTIONS WITH END-POINT BY BERNSTEIN OPERATORS

Let B_n denote the $n-th$ Bernstein operator (see Section 1.1). The problem of wegihted approximation by Bernstein type operators of functions with endpoints or inner singularities of algebraic type is natural question.

In this part of the chapter we give the operators of Bernstein type from B. D. Vecchia and G. Mastroianni and J. Szabados (Vecchia et al., 2004) for the weighted approximation of functions with singularities at the endpoints and we give convergence results involving the weighted modulus of smoothness of second order.

For smoother functions we introduce the Sobolev type space W_w^2 defined as

$$W_w^2:=\left\{f\in C_w : f'\in AC((0,1)),\left\|f''w\varphi^2\right\|\right\}, \tag{1.45}$$

where $\varphi(x)=\sqrt{x(1-x)}$ and $AC(I)$ is the set of all absolutely continous functions ın I.

Now for every $f\in C_w$ introduce the Bernstein type operator

$$\begin{aligned}B_n^*(f,x) &:= (1-x)^n\left[2f\left(\frac{1}{n}\right)-f\left(\frac{2}{n}\right)\right]\\ &+\sum_{k=1}^{n-1}p_{n,k}(x)f\left(\frac{k}{n}\right)+x^n\left[2f\left(1-\frac{1}{n}\right)-f\left(1-\frac{2}{n}\right)\right].\end{aligned} \tag{1.46}$$

From the defininition it follows that $B_n^*(f,x)$ is a polynomial of degree at most n and B_n^* is a linear operator which reproduces linear functions. We have the following approximation result which proves the boundedness of the operator B_n^* in $C_{w,}$ and deduce the weighted convergence of $B_n^*(f,x)$ to f respectively.

Theorem 1.5.1. *For every* C, α, $\beta > 0$

a)

$$\|wB_n^*(f)\| \le C\|wf\|, \text{ for all } f \in C_{w,} \tag{1.47}$$

b)

$$w[f - B_n^*(f)] \le \frac{C}{n}\left\|w\varphi^2 f''\right\|, \quad \text{if } f \in C_w. \tag{1.48}$$

The proof of the Theorem 1.5.1 is based on some lemmas.

Lemma 1.5.2. *If* $\alpha, \beta > 0$, $0 \le x \le 1$, *then*

$$D_n(x) = w(x) \sum_{\left|\frac{k}{n}-x\right| \ge \frac{\varphi(x)}{2}} p_{nk}(x) \left| \int_{\frac{k}{n}}^{x} \frac{t-\frac{k}{n}}{\varphi^2(t)\,w(t)} dt \right| \le \frac{C}{n}.$$

Proof. By symmetry we may assume that $0 \le x \le \frac{1}{2}$. We write

$$\begin{aligned} D_n(x) &\le w(x)\left\{\sum_{0\le k\le \frac{nx}{2}} + \sum_{\frac{3nx}{2}\le k\le n}\right\} p_{nk}(x) \left|\int_{\frac{k}{n}}^{x} \frac{\left|t-\frac{k}{n}\right|}{t^{1+\alpha}} dt\right| \\ &:= D_{n1}(x) + D_{n2}(x). \end{aligned}$$

Then

$$D_{n1}(x) \le Cx^{\alpha} \sum_{0\le k\le \frac{nx}{2}} p_{nk}(x) \int_{\frac{k}{n}}^{x} t^{-\alpha} dt \le Cx^{\alpha+2} n^{\alpha+1} p_{nk_n}(x),$$

since for a fixed x, $p_{n,k}(x)$ attains its maximum in $0 \le k \le \frac{nx}{2}$ for $k_n = \left[\frac{nx}{2}\right]$. Now by Stirling's formula

$$\begin{aligned} p_{n,k_n} &\le C \frac{\left(\frac{n}{e}\right)^n \sqrt{n} x^{\frac{nx}{2}} (1-x)^{n\left(1-\frac{x}{2}\right)}}{\left(\frac{nx}{2e}\right)^{\frac{nx}{2}} \sqrt{nx} \left(\frac{n\left(1-\frac{x}{2}\right)}{e}\right)^{n\left(1-\frac{x}{2}\right)} \sqrt{n}} \\ &\le \frac{C}{\sqrt{nx}} 2^n \left(\frac{1-x}{2-x}\right)^{n\left(1-\frac{x}{2}\right)} = \frac{C}{\sqrt{nx}} \left(1 - \frac{x}{2-x}\right)^{n\left(1-\frac{x}{2}\right)} 2^{\frac{nx}{2}} \end{aligned}$$

and by $1-u \le e^{-u}, u \ge 0$,

$$p_{n,k_n}(x) \le \frac{C}{\sqrt{nx}} e^{-\frac{nx}{2}} 2^{\frac{nx}{2}}. \tag{1.49}$$

Hence,

$$D_{n1}(x) \le Cx^{\alpha+\frac{3}{2}} n^{\alpha+\frac{1}{2}} \left(\frac{2}{e}\right)^{\frac{nx}{2}} \le Cx^{-\alpha-\frac{3}{2}} n^{\alpha+\frac{1}{2}} \le \frac{C}{n}. \tag{1.50}$$

Now we estimate $D_{n2}(x)$. We have

$$\begin{aligned} D_n(x) &\le Cx^{\alpha} \sum_{k \ge \frac{3nx}{2}} p_{n,k}(x) \int_x^{\frac{k}{n}} \frac{\frac{k}{n}-t}{t^{1+\alpha}} dt \\ &\le \frac{C}{x} \sum_{k=0}^{n} p_{nk}(x) \left(\frac{k}{n}-x\right)^2 \le \frac{C}{n}, \end{aligned}$$

since

$$\sum_{k=0}^{n} p_{nk}(x) |k-xn|^{\gamma} \le Cn^{\frac{\gamma}{2}} \varphi^{\gamma}(x), \quad 0 \le x \le 1,\ \gamma \ge 0. \tag{1.51}$$

$\square$

Let $\psi(x)$ is defined as

$$\psi(x) = \begin{pmatrix} 10x^3 - 15x^4 + 6x^5, & \text{if } 0 \le x \le 1 \\ 0, & \text{if } 0 \le x \\ 1, & \text{if } x \ge 1 \end{pmatrix}$$

and $P_1(f)$ and $P_2(f)$ be the linear functions interpolating f at the points $\frac{1}{n}, \frac{2}{n}$ and $1-\frac{1}{n}, 1-\frac{2}{n}$ respectively, i.e.

$$P_1(f,x) = P_1(x) = (2-nx) f \frac{1}{n} + (nx-1) f\left(\frac{2}{n}\right),$$

$$P_2(f,x) := P_2(x) = [2-(1-x)n] f\left(1-\frac{1}{n}\right) + (n(1-x)-1) f\left(1-\frac{2}{n}\right).$$

Now define,

$$
\begin{aligned}
F_n(f,x) &:= F_n(x) \\
&= (1-\psi(nx-1))\,P_1(x) + (1-\psi(nx-n+2))\,\psi(nx-1)\,f(x) \\
&+ \psi(nx-n+2)\,P_2(x) \\
&= \begin{cases}
P_1(x), & \text{if } x \in \left[0, \frac{1}{n}\right] \\
(1-\psi(nx-1))\,P_1(x) & \\
+\psi(nx-1)\,f(x) & \text{if } x \in \left[\frac{1}{n}, \frac{2}{n}\right], \\
f(x), & \text{if } x \in \left[\frac{2}{n}, 1-\frac{2}{n}\right] \\
(1-\psi(nx-n+2))\,f(x) & \\
(\psi(nx-n+2))\,P_2(x), & \text{if } x \in \left[1-\frac{2}{n}, 1-\frac{1}{n}\right], \\
P_2(x), & \text{if } x \in \left[1-\frac{1}{n}, 1\right]
\end{cases}
\end{aligned}
$$

Note that the linear operator F_n reproduces constant and linear functions.

Lemma 1.5.3. *If $f \in W_w^2$, then for $F_n = F_n(f)$ and for all $\alpha, \beta \geq 0$*

$$
\|[F_n - B_n(F_n)]\,w\| \leq \frac{C}{n}\left\|F_n''\varphi^2 w\right\|.
$$

Proof. Again by symetry it is sufficient to estimate here for $0 \leq x \leq \frac{1}{2}$. Since B_n preserves linear functions, we get

$$
\begin{aligned}
|F_n(x) - B_n(F_n,x)|\,w(x) &= \left|\sum_{k=0}^{n} w(x)\,p_{n,k}(x)\int_x^{\frac{k}{n}}\left(t-\frac{k}{n}\right)F_n''(t)\,dt\right| \\
&\leq \sum_{\left|\frac{k}{n}-x\right| \leq \frac{x}{2}} + \sum_{\left|\frac{k}{n}-x\right| \geq \frac{x}{2}} := E_1(x) + E_2(x).
\end{aligned}
$$

From the definition $F_n''(t) = 0$, in $\left[0, \frac{1}{n}\right]$, whence $E_1(x) = 0$ for $0 \leq x \leq \frac{1}{n}$. Now if $\frac{1}{n} \leq x \leq \frac{1}{2}$, then $1 \leq k \leq n-1$, and thus

$$
E_1(x) \leq C\frac{\|F_n''\varphi^2 w\|}{x(1-x)}\sum_{\left|\frac{k}{n}-x\right| \leq \frac{x}{2}} p_{nk}(x)\left(\frac{k}{n}-x\right)^2 \leq \frac{C}{n}\left\|F_n''\varphi^2 w\right\|
$$

by using (1.51) with $\gamma = 2$. On the other hand , by Lemma 1.5.2

$$
E_2(x) \leq C\left\|F_n''\varphi^2 w\right\| D_n(x) \leq \frac{C}{n}\left\|F_n''\varphi^2 w\right\|.
$$

Hence, the assertion is proved. □

Lemma 1.5.4. *If $f \in W_w^2$, then*

$$\|w[f - P_1(f)]\|_{[0,\frac{2}{n}]} \le \frac{C}{n}\left\|f''\varphi^2 w\right\|_{[0,\frac{2}{n}]} \tag{1.52}$$

and

$$\|w[f - P_2(f)]\|_{[1-\frac{2}{n},1]} \le \frac{C}{n}\left\|f''\varphi^2 w\right\|_{[1-\frac{2}{n},1]}. \tag{1.53}$$

Proof. By symmetry it is sufficient to prove (1.52). If $f \in W_w^2$, then

$$f(x) = f\left(\frac{2}{n}\right) - f'\left(\frac{2}{n}\right)\left(\frac{2}{n} - x\right) + \int_x^{\frac{2}{n}} f''(t)(t-x)\,dt := P_1^*(x) + G_n(f,x)$$

with P_1^* a linear function. Evidently,

$$|f(x) - P_1(f,x)| = |G_n(f,x) - P_1(G_n(f),x)|.$$

Now if $x \in \left[0, \frac{2}{n}\right]$ then

$$\begin{aligned} x^\alpha |G_n(f,x)| &\le x^\alpha \int_x^{\frac{2}{n}} |f''(t)(t-x)|\,dt \le C\int_x^{\frac{2}{n}} \left|\left(f''\varphi^2 w\right)(t)\right| dt \\ &\le \frac{C}{n}\left\|f''\varphi^2 w\right\|_{[0,\frac{2}{n}]}. \end{aligned}$$

Moreover,

$$\begin{aligned} x^\alpha |P_1(G_n(f),x)| &\le x^\alpha(2-nx)\int_{\frac{1}{n}}^{\frac{2}{n}} f''(t)\left(t - \frac{1}{n}\right) dt \le Cx^\alpha \int_{\frac{1}{n}}^{\frac{2}{n}} \left|f''(t)\varphi^2(t)w(t)\right| dt \\ &\le C\int_{\frac{1}{n}}^{\frac{2}{n}} \left|f''(t)\varphi^2(t)w(t)\right| dt \le \frac{C}{n}\left\|f''\varphi^2 w\right\|_{[0,\frac{2}{n}]}, \end{aligned}$$

and (1.52) is proved. □

Lemma 1.5.5. *For every $f \in W_w^2$ we have*

$$\left\|F_n''\varphi^2 w\right\| \le C\left\|f''\varphi^2 w\right\|.$$

Proof. Again, it is sufficient to estimate $(F_n''\varphi^2 w)(x)$ for $0 \le x \le \frac{1}{2}$. For $0 \le x \le \frac{1}{n}$, $F_n''(x) = 0$, while for $\frac{2}{n} \le x \le \frac{1}{2}$, $F_n = f$. Thus, let $x \in \left[\frac{1}{n}, \frac{2}{n}\right]$. $F_n(x) = P_1(x) + \psi(nx-1)(f(x) - P_1(x))$ and

$$
\begin{aligned}
F_n''(x) &= n^2\psi''(nx-1)(f(x)-P_1(x)) \\
&\quad + 2n\psi'(nx-1)(f(x)-P_1(x))' \\
&\quad + \psi(nx-1)f''(x), && (1.54)
\end{aligned}
$$

whence for these x's

$$
\begin{aligned}
\left|\left(F_n''\varphi^2 w\right)(x)\right| \le C \Big[& n\left\|w(f-P_1)\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]} \\
& + w(x)\left\|(f-P_1)'\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]} + \left\|f''\varphi^2 w\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]}\Big].
\end{aligned}
$$

By the inequality

$$
\left\|h'\right\|_{[c,d]} \le C\left[(d-c)^{-1}\left\|h\right\|_{[c,d]} + (d-c)\left\|h''\right\|_{[c,d]}\right]
$$

from (1.54) we get

$$
\begin{aligned}
\left|\left(F_n''\varphi^2 w\right)(x)\right| &\le C\Big[n\left\|w(f-P_1)\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]} + nw(x)\left\|[f-P_1]'\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]} \\
&\quad + \frac{w(x)}{n}\left\|f''\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]} + \left\|f''\varphi^2 w\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]}\Big] \\
&\le C\left[n\left\|w[f-P_1]\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]} + \left\|f''\varphi^2 w\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]}\right]
\end{aligned}
$$

and by Lemma 1.5.4

$$
\left\|F_n''\varphi^2 w\right\|_{\left[\frac{1}{n},\frac{2}{n}\right]} \le C\left\|f''\varphi^2 w\right\|_{\left[0,\frac{2}{n}\right]}.
$$

□

Proof. (Theorem 1.5.1)

In proving Theorem 1.5.1 we may always estimate the left hand side norms in the interval by symmetry.

First we prove (1.47). We estimate :

$$
w(x)\left|\sum_{k=1}^{n-1} p_{n,k}(x) f\left(\frac{k}{n}\right)\right| \le \|wf\| \sum_{k=1}^{n-1} p_{nk}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)} := \|wf\|\,\sigma, \qquad (1.55)
$$

with

$$
\sigma \le \left\{\sum_{1\le k\le \frac{nx}{2}} + \sum_{\frac{nx}{2}\le k\le \frac{2n}{3}} + \sum_{\frac{2n}{3}\le k\le n-1}\right\} p_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)} := \sum_{i=1}^{3}\sigma_i(x) \qquad (1.56)
$$

For a fixed x using the monotone increasing property of $p_{n,k}(x)$ as well as (1.49) we obtain since the latter function attains its maximum for $x = \frac{c}{n}$.

The estimate of $\sigma_2(x)$ is simpler since in this case $\frac{w(x)}{w\left(\frac{k}{n}\right)} \leq C$ which implies the boundedness. For $\sigma_3(x)$we again use that for a fixed x the maximum of $p_{n,k}(x)$ in $\frac{2n}{3} - 1 \leq k \leq n$ is attained for $k = \left[\frac{2n}{3}\right]$. Since by Stirling's formula $p_{n,\frac{2n}{3}}(x) \leq \left(\frac{3}{4}\right)^{\frac{2n}{3}}$, we obtain

$$\sigma_3(x) \leq \left(\frac{3}{4}\right)^{\frac{2n}{3}} n^{\beta+1} \leq C.$$

Moreover

$$\left| w(x)(1-x)^n \left(2f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right)\right| \tag{1.57}$$

$$\leq 3\|wf\|(xn)^{\alpha}(1-x)^n \leq 3\|wf\|(x_0 n)^{\alpha}(1-x_0)^n \leq C\|wf\|, \tag{1.58}$$

with $x_0 = \frac{1}{1+\frac{n}{\alpha}}$. Analogously,

$$\left| w(x)x^n \left[2f\left(1-\frac{1}{n}\right) - f\left(1-\frac{2}{n}\right)\right]\right| \leq C\|wf\|.$$

Hence, from the definition of the operator B_n^* by (1.55) and (1.58) (1.47) follows.

Now we prove (1.48) by the Lemma 1.5.3 and 1.5.5 we deduce

$$\begin{aligned}\|w[f - B_n^*(f)]\| &\leq \|w[f - F_n(f)]\| + \|w[F_n(f) - B_n(F_n(f))]\| \\ &\leq \|w[f - F_n(f)]\| + \frac{C}{n}\left\|f''\varphi^2 w\right\|\end{aligned}$$

$$\leq \frac{C}{n}\left\|f''\varphi^2 w\right\| + \|w[f - F_n(f)]\|_{\left[0,\frac{2}{n}\right]} + \|w[f - F_n(f)]\|_{\left[1-\frac{2}{n},1\right]}.$$

Since

$$\|w[f - F_n(f)]\|_{\left[0,\frac{2}{n}\right]} \leq \|w[f - P_1]\|_{\left[0,\frac{2}{n}\right]}$$

and

$$\|w[f - F_n(f)]\|_{\left[1-\frac{2}{n},1\right]} \leq \|w[f - P_2]\|_{\left[1-\frac{2}{n},1\right]}$$

by 1.5.4 we get (1.48). $\square$

1.6 LOTOTSKY TRANSFORM AND BERNSTEIN POLYNOMIALS

The Bernstein polynomials (1.1) associated with a function f defined on $[0,1]$ have been the subject of much recent research and have been generalized in several directions. The generalized Lototsky or $[F, d_n]$ matrix has also been the subject of extensive research. The elements a_{nk} of the matrix defined by

$$a_{00} = 1, \quad a_{0k} = 0\,(k \neq 0)\,,$$

$$\prod_{i=1}^{n} \frac{y + d_i}{1 + d_i} = \sum_{k=0}^{n} a_{nk} y^k \tag{1.59}$$

where $\{d_i\}$ is a sequence of complex numbers with $d_i \neq -1$ $(i = 1, 2, \ldots)$. It is the purpose of this section to point out following J. Kings (King, 1966) a connection between the Lototsky matrix and the Bernstein polynomials which gives yet another extension of the latter.

It is convenient to make a change of notation. If we let $h_i = \frac{1}{1+d_i}$ equation (1.59) has the form

$$\prod_{i=1}^{n} (h_i y + 1 - h_i) = \sum_{k=0}^{n} a_{nk} y^k. \tag{1.60}$$

Now let $\{h_i(x)\}$ be a sequence of functions defined on $[0,1]$. Let $a_{nk} = a_{nk}(x)$ be the elements of the Lototsky matrix given (1.60) by corresponding to the sequence $\{h_i(x)\}$. For each f defined on $[0,1]$, let

$$L_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) a_{nk}(x)\,. \tag{1.61}$$

It is easy to see that if $h_i(x) = x$ $(i = 1, 2, \ldots)$ then $L_n(f, x) = B_n(f, x)$. Therefore in this sense the functions $L_n(f, x)$ provide an extension of the Bernstein polynomials. The following theorem gives sufficient condition on the sequence $\{h_i(x)\}$ to insure that $L_n(f, x) \to f(x)$.

Theorem 1.6.1. *For $f \in C[0,1]$ let $L_n(f, x)$ be defined by* (1.61) *and let $\{s_i(x)\}$ denote the $(C, 1)$ transform of the sequence $\{h_i(x)\}$. If $0 \le h_i(x) \le 1$ $(i = 1, 2, \ldots)$ and if $\{s_i(x)\}$ converges uniformly to x on $[0,1]$, then*

$$\lim_{n \to \infty} L_n(f, x) = f(x)$$

uniformly on $[0,1]$.

Proof. According Korovkin theorem see for instance F. Altomare and M. Campiti (Altomare and Campiti, 1994), it is sufficient to show that

$$L_n(1,x) \to 1, \qquad L_n(t,x) \to x, \qquad L_n\left(t^2,x\right) \to x^2,$$

uniformly on $[0,1]$ and that L_n is a positive linear operator. It is clear that L_n is linear. Furthermore $f \geq 0$ implies that $L_n \geq 0$ since $a_{nk}(x) \geq 0$ whenever $0 \leq h_i(x) \leq 1$.

We have

$$L_n(1,x) = 1, \quad (n = 1,2,...),$$
$$L_n(t,x) = \sum_{k=0}^{n} \frac{k}{n} a_{nk}(x),$$
$$L_n\left(t^2,x\right) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 a_{nk}(x).$$

If we let

$$P_n = \prod_{i=1}^{n} \left(y h_i(x) + 1 - h_i(x)\right)$$

and

$$r_i(x,y) = \frac{h_i(x)}{y h_i(x) + 1 - h_i(x)}$$

we have

$$P_n' = \sum_{i=0}^{n} r_i(x,y) P_n, \tag{1.62}$$

and

$$P_n'' = \left\{ \left[\sum_{i=0}^{n} r_i(x,y)\right]^2 - \sum_{i=0}^{n} r_i^2(x,y) \right\} P_n, \tag{1.63}$$

where the differentiation is with respect to y. Also

$$P_n' = \sum_{k=0}^{n} k a_{nk}(x) y^{k-1} \tag{1.64}$$

and

$$P_n'' = \sum_{k=0}^{n} k\left(k-1\right) a_{nk}\left(x\right) y^{k-2}. \tag{1.65}$$

If we set $y = 1$ in (1.62) and (1.64) we obtain

$$\frac{1}{n}\sum_{k=0}^{n} k a_{nk}\left(x\right) = s_n\left(x\right). \tag{1.66}$$

Similarly, it follows from (1.63) and (1.65) and (1.66) that:

$$\frac{1}{n^2}\sum_{k=0}^{n} k^2 a_{nk}\left(x\right) = \frac{1}{n}\left\{s_n\left(x\right) - t_n\left(x\right)\right\} + s_n^2\left(x\right), \tag{1.67}$$

where $\{t_n\left(x\right)\}$ is the $(C,1)$ transform of the sequence $\{h_n^2\left(x\right)\}$.

It is easy to see that $0 \leq h_i\left(x\right) \leq 1$ implies $t_n\left(x\right) = O\left(1\right)$ so that $\frac{t_n(x)}{n} \to 0$ uniformly on $[0,1]$. This proves the theorem. □

1.7 MODULII OF CONTINUITY

Measuring the smoothness of a function by differentiability is to crude for many purposes in approximation. More subtle measurements are provided by the modulii of continuity.

The modulus of continuity $\omega\left(f,t\right) =: \omega\left(t\right)$ of a function f can be defined when f is given on any metric space A. But we shall restrict it $A = R$, R_+, or $[a,b]$. In that case

$$\omega\left(f,t\right) =: \omega\left(t\right) := \sup_{\substack{|x-y|\leq t \\ x,y\in A}} \left|f\left(x\right) - f\left(y\right)\right|, \quad t \geq 0. \tag{1.68}$$

Clearly, $\omega\left(t\right)$ is a constant for $t > diam\ A$ if A is bounded. The function ω is continous at $t = 0$, if and only if f is uniformly continous on A . We shall assume that $f \in \widetilde{C}\left(A\right)$ that f belongs to the space of uniformly continous on A then $\omega\left(f,t\right)$ is finite for each t. For each fixed t, ω is a semi norm that is it is subadditive in f and positive homogenous .

A modulus of continouty has the following simple properties :

$$
\begin{aligned}
&a) \quad \omega(t) \to \omega(0) = 0, \text{ for } t \to 0; \\
&b) \quad \omega \text{ is non-negative and non-decreasing on } R_+; \\
&c) \quad \omega \text{ is subadditive } \omega(t_1 + t_2) \le \omega(t_1) + \omega(t_2) \\
&d) \quad \omega \text{ is continous on } R_+.
\end{aligned} \tag{1.69}
$$

Properties (a) and (b) are clear. For (c) if there is a point $z \in A$ for which $|x - z| \le t_1, \;\; |y - z| \le t_2$and (c) follows from

$$|f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)| \le \omega(t_1) + \omega(t_2).$$

Moreover,

$$\omega(t_1 + t_2) - +\omega(t_1) \le \omega(t_2). \tag{1.70}$$

Thus, (a), (b), (c) imply that ω is continous at each $t \ge 0$.

A function ω defined on R_+ and satisfying (1.69) is called modulus of continuity. This is justified since by (1.70) any such function is its own modulus of contin outy.

It follows from(1.69) (c) by induction that

$$\omega(t_1 + ... + t_n) \le \omega(t_1) + ... + \omega(t_n).$$

For $t = t_1 = ... = t_n$, we obtain

$$\omega(nt) \le n\omega(t). \tag{1.71}$$

A similar inequality holds for a nonintegral factor λ :

$$\omega(\lambda t) \le (\lambda + 1)\,\omega(t), \qquad \lambda \ge 0. \tag{1.72}$$

In fact, taking an integer n for which we see that $n \le \lambda \le n + 1$, we see that

$$\omega(\lambda t) \le \omega((n+1)\,t) \le (n+1)\,\omega(t) \le (\lambda + 1)\,\omega(t).$$

A modulus of contintuity cannot be too small. If $\frac{\omega(f,t)}{t} \to 0$ for $t \to 0$, then $f'(x) \equiv 0$, f is constant.

For concave function f on $[a,b]$, $\alpha f(x)+\beta f(y)\leq f(\alpha x+\beta y)$ for $x,y\in[a,b]$ and $\alpha\geq 0$, $\beta\geq 0$, $\alpha+\beta=1$. A concave function f on $[0,1]$ which satisfies $f(0)=0$, has the property that $\frac{f(x)}{x}$decreases for $x<y$, then

$$\frac{x}{y}f(y)=\frac{y-x}{y}f(0)+\frac{x}{y}f(y)\leq f(x).$$

A continous, increasing function ω on R_+, which satisfies $\omega(0)=0$, is a modulus of continuity if it is concave (or more generally if $\frac{\omega(t)}{t}$ is decreasing). It is necessary only to show that ω satisfies (1.69) (c) This is obtained by multiplying the inequalities

$$\frac{\omega(t_1+t_2)}{t_1+t_2}\leq\frac{\omega(t_1)}{t_1}\text{ and }\frac{\omega(t_1+t_2)}{t_1+t_2}\leq\frac{\omega(t_2)}{t_2}$$

by $t_1$an t_2 respectively, and adding.

CHAPTER 2

WEIGHTED APPROXIMATION BY ANALOGUES OF BERNSTEIN OPERATORS FOR RATIONAL FUNCTIONS

2.1 INTRODUCTION

The Bernstein polynomials

$$B_n\left(f,x\right)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\tbinom{n}{k}x^k\left(1-x\right)^{n-k} \tag{2.1}$$

associated with a function f defined on $[0,1]$ have been the subject of much recent research and have been generalized in many directions see for instance ((Phillips, 2000), (DeVore and Lorentz, 1993), (Tachev, 2012)).

In 1966 J. P. King (King, 1966) introduced the following generalization of the Bernstein polynomials

$$L_n\left(f,x\right)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)u_{nk}\left(x\right) \tag{2.2}$$

where $u_{nk}\left(x\right)$ are given by the generating function

$$g_n\left(x,y\right)=\prod_{i=1}^{n}\left(h_{ni}\left(x\right)y+\left(1-h_{ni}\left(x\right)\right)\right)=\sum_{k=0}^{n}u_{nk}\left(x\right)y^k, \tag{2.3}$$

and $h_{ni}(x)=h_i(x)$ is a sequence of continuous functions defined on $[0,1]$, $0\leq h_i\left(x\right)\leq 1$.

King's (or Bernstein-King as they are mentioned in (Altomare and Campiti, 1994)) operators converge to the approximated function if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}h_i\left(x\right)=x. \tag{2.4}$$

Now let x_{ni} be fixed poles $x_{ni}=1+\rho_{ni}$, $\rho_{ni}>0$ and

$$h_{ni}(x) = \frac{\rho_{ni} x}{1 + \rho_{ni} - x}. \tag{2.5}$$

Put

$$\phi_n(x) = \frac{1}{n} \sum_{k=1}^{n} h_{nk}(x).$$

Observe that $\phi_n(x)$ is strictly increasing from 0 to 1 on the interval $[0, 1]$. The nodes τ_{nk} are well-defined by $\phi_n(\tau_{nk}) = \frac{k}{n}, \ \ (k = 0, 1, ..., n)$.

In 1979 V. S. Videnskii (Videnskii, 1979) introduced another generalization of Bernstein operators for approximation by rational functions with fixed poles

$$U_n(f, x) = \sum_{k=0}^{n} f(\tau_{nk}) \, u_{nk}(x). \tag{2.6}$$

In 1981 V. S. Videnskii (Videnskii, 1981) considered more general case of the operators (2.6), where u_{nk} are defined for arbitrary increasing functions $h_{ni}(x)$. The main difference between those families of operators is in nodes. In fact, like for Bernstein basic functions k/n is the maximum point of $p_{n,k}$ on $[0, 1]$, the maximum point of $u_{n,k}$ on $[0, 1]$ is $\tau_{n,k}$. Hence, Videnskii operators (2.6) can be considered as the most natural generalization of Bernstein operators for approximation by rational functions. Also the advantage of Videnskii's operators can be easily seen from the conditions for their convergence. Namely V. S Videnskii ((Videnskii and Mencher, 1994). th. 3.1)) proved that sequence $V_n(f, x)$ uniformly converges to arbitrary $f \in C[0, 1]$ if and only if

$$\lim_{n \to \infty} S_n = \infty, \tag{2.7}$$

where $S_n = \sum_{i=1}^{n} \frac{\rho_{ni}}{1+\rho_{ni}}$. A simple example $(\rho_{ni} = \rho_n = 1)$ shows that condition (2.4) essentialy more restrictive than (2.7). Later V. S. Videnskii (Videnskii, 1990) considered arbitrary matrices of nodes ξ_{nk} instead of τ_{nk} and proved the convergence results for those operators $V_n^{\xi}(f, x)$. Note that for $\xi_{nk} = \frac{k}{n}$ we recover King's operators for $h_{ni}(x) = \frac{\rho_{ni} x}{1+\rho_{ni}-x}$. Moreover, as it is explained in Section 1.4 Lupas operators can be considered as a particular case of the operators $V^{\xi}(f, x)$, too.

Recently many authors pay attention to weighted approximation by classical polynomial operators and to construction of their weighted modifications. The reason is that usual operators are not always suitable for approximating functions with singularities in weighted spaces. For instance, the sequence of classical Bernstein operators (2.1) is not bounded in the space

$$C_w = \left\{ f \in C\left((0,1)\right) : \lim_{x\to 0} \left(wf\right)(x) = \lim_{x\to 1} \left(wf\right)(x) = 0 \right\},$$

where

$$\|f\|_{C_w} := \|wf\| = \sup_{x\in[0,1]} \left(wf\right)(x)$$

and $\alpha, \beta \geq 0, w(x) = x^{\alpha}(1-x)^{\beta}, 1 \geq \alpha, \beta \geq 0, \alpha + \beta > 0, 0 \leq x \leq 1$,but it's slight modification $B_n^*(f,x)$ (1.46)is bounded. One can consult papers (Vecchia et al., 2004), (Vecchia and Mastroianni, 2004)and (Guo et al., 2003) containing these and other deep results in this direction.

We consider the Sobolev type space W_ω^2 in (1.45). Observe also that modification (1.46) is not a positive operator, so general results about weighted approximation by linear positive operators on a real interval (see, for instance, (Altomare, 2013) and references therein) are not applicable here.

The main goal of the paper is to investigate approximation properties of Videnskii operators in the norm of C_w under some restrictions on the sequence of denominators.

In the following C denotes a positive constant which may assume different values in different formulas. Moreover, we write $v \sim u$ for two quantities v and u depending on some parameters, if $\left|\frac{v}{u}\right|^{\pm 1} \leqq C$ with C independent of the parameters.

Note that operators (2.6) as well as the Bernstein operators are not bounded (in fact even not defined) in C_w.

Here, we consider modifications of the Videnskii operators similar to (1.46):

$$V_n^*(f,x) = \sum_{k=1}^{n-1} f(\tau_{nk})\, u_{nk}(x) + u_{n0}(x)\left[2f(\tau_{n1}) - f(\tau_{n2})\right] \tag{2.8}$$
$$+ u_{nn}(x)\left[2f(\tau_{n\,n-1}) - f(\tau_{n\,n-2})\right].$$

The main result of the chapter is following theorem.

Theorem 2.1.1. *Suppose that ρ_{ni} satisfy $\rho_{ni} > C > 0$ and $\sum\limits_{i=1}^{n}\frac{1}{\rho_{ni}} \leq C$. Then*

a)

$$\left\| V_n^* \left(f\right)\right\|_{C_w} \leq C \left\| f\right\|_{C_w}$$

b)

$$\left\| \left[f - V_n^* \left(f\right)\right]\right\|_{C_w} \leq \frac{C}{n} \left\| \varphi^2 f''\right\|_{C_w} \;\; if \;\;\; f \in W_\omega^2.$$

Here we present several auxilary assertions.

Lemma 2.1.2. *If $n, k \in \mathbb{N}$, $f_i \in C^k\left(\left[a,b\right]\right)$, $i = 1, ..., n$ then the following equality holds*

$$\frac{d^n}{dy^n}\left(\prod_{i=1}^{m} f_i\left(y\right)\right) = \sum_{\substack{j_1+j_2+...+j_m=n \\ j_1\geq 0,...,\ j_m \geq 0}} \frac{n!}{j_1!j_2!...j_m!}\frac{d^{j_1}}{dy^{j_1}}\left(f_1\left(y\right)\right)...\frac{d^{j_m}}{dy^{j_m}}\left(f_m\left(y\right)\right). \tag{2.9}$$

Proof. We use mathematical induction on n in the proof. For $n = 1$, we get

$$f_1^{(n)}\left(y\right) = \sum_{j_1=n}\binom{n}{j_1} f_1^{(j_1)}\left(y\right) \tag{2.10}$$

which is true. So equality (2.10) holds for $m = 1$. b) Assume that the equation (2.9) holds for $m = k$. Then

$$\frac{d^n}{dy^n}\left(\prod_{i=1}^{k} f_i\left(y\right)\right) = \sum_{j_1+j_2+...+j_k=n} \frac{n!}{j_1!j_2!...j_k!}\frac{d^{j_1}}{dy^{j_1}}\left(f_1\left(y\right)\right)...\frac{d^{j_k}}{dy^{j_k}}\left(f_k\left(y\right)\right)$$

.

We show that eqution (2.10) holds for $m = k + 1$. Now

$$\frac{d^n}{dy^n}\left(\prod_{i=1}^{k+1} f_i\left(y\right)\right) = \sum_{j_i=0}^{n}\binom{n}{r}\frac{d^{n-r}}{dy^{n-r}}\left(\prod_{i=1}^{k} f_i\left(y\right)\right)\frac{d^{(r)}f_{k+1}}{dy^{(r)}}\left(y\right)$$

$$= \sum_{j_i=0}^{n} \binom{n}{r} \sum_{j_1+j_2+...+j_k=n-r} \frac{(n-r)!}{j_1!j_2!...j_k!} \frac{d^{j_1}}{dy^{j_1}} (f_1(y)) ... \frac{d^{j_k}}{dy^{j_k}} (f_k(y)) \frac{d^{(r)} f_{k+1}}{dy^{(r)}} (y) \quad (2.11)$$

setting $r = j_{k+1}$, and cancelling$(n-r)!$ in the product $\binom{n}{r}\binom{n-r}{j_{1,...,}j_k}$ and the equality (2.11) becomes

$$\frac{d^n}{dy^n} \left(\prod_{i=1}^{k+1} f_i(y) \right) = \sum_{j_1+j_2+...+j_{k+1}=n} \frac{n!}{j_1!j_2!...j_{k+1}!} \frac{d^{j_1}}{dy^{j_1}} (f_1(y)) ... \frac{d^{j_k}}{dy^{j_k}} (f_k(y)) \frac{d^{j_{k+1}}}{dy^{j_{k+1}}} (f_{k+1})(y)$$

The last expression is the right-hand side of equation (2.9) for $m = k+1$. By the induction principle, we conclude that the equation (2.9) holds for all m. □

Corollary 2.1.3. *If h_{ni} is defined as in* (2.3) *then*

$$u_{nk}(x) = \frac{1}{k!} \sum_{\substack{j_1+j_2+...+j_n=k \\ 0 \le j_i \le 1}} \frac{k!}{j_1!j_2!...j_n!} \prod_{i=1}^{n} \left(1 - j_i - (-1)^{j_i} h_{ni}(x) \right). \quad (2.12)$$

Proof. Differentiate (2.3) and use Lemma 2.1.2. □

Lemma 2.1.4. *Under suppositions of Theorem 2.1.1, for any $x \in (0,1)$*

$$C \le \frac{\phi_n(x)}{x} \le 1 \quad (2.13)$$

and

$$1 \le \frac{1-\phi_n(x)}{1-x} \le C. \quad (2.14)$$

Proof. Firstly

$$\phi_n(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{\rho_{ni} x}{1+\rho_{ni}-x} \ge x \frac{1}{n} \sum_{i=1}^{n} \frac{\rho_{ni}}{1+\rho_{ni}} \ge Cx$$

and

$$\phi_n(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{\rho_{ni} x}{1+\rho_{ni}-x} \le x.$$

Combining the inequalities we get (2.13).

For the inequality (2.14) we have

$$1-\phi_n(x) = \frac{1}{n} \sum_{i=1}^{n} \left(1 - \frac{\rho_{ni} x}{1+\rho_{ni}-x} \right) = \frac{1}{n} \sum_{i=1}^{n} \frac{(1-x)(1+\rho_{ni})}{1+\rho_{ni}-x} \le \frac{C}{2} (1-x)$$

and

$$1-\phi_n(x) \ge (1-x).$$

Combining these inequalities we get (2.14). □

Corollary 2.1.5. *Suppose that ρ_{ni} satisfy suppositions of Theorem 2.1.1 then $w(x) \sim w(\phi_n^{-1}(x))$ and $\varphi(x) \sim \varphi(\phi_n^{-1}(x))$.*

Observe also that from definition of $u_{nk}(x)$ it follows immediately that $0 \leq u_{nk}(x) \leq 1$ $k = 0, ..., n$; $n = 1, ...$

Using (2.4) and (2.12) we get

$$u_{nk}(x) = \frac{\sum\limits_{\substack{j_1+j_2+...+j_n=k \\ 0\leq j_i\leq 1}} x^k (1-x)^{n-k} \prod\limits_{i=1}^{n}(1+\rho_{ni}-j_i)}{\prod\limits_{i=1}^{n}(1+\rho_{ni}-x)}$$

and we can write $u_{nk}(x)$ as

$$u_{nk}(x) = \frac{\alpha_{nk} x^k (1-x)^{n-k}}{\prod\limits_{i=1}^{n}(1+\rho_{ni}-x)} = \sum_{\substack{j_1+j_2+...+j_n=k \\ j_i\in\{0,1\}}}^{n} \prod_{i=1}^{n} \frac{(1+\rho_{ni}-j_i)}{1+\rho_{ni}-x} x^k (1-x)^{n-k}$$

and we can write as down an explicit formula for the coefficients α_{nk} from (1.23) :

$$\alpha_{nk} = \sum_{\substack{j_1+j_2+...+j_n=k \\ j_i\in\{0,1\}}}^{n} \prod_{i=1}^{n} (\rho_{ni}+1-j_i). \tag{2.15}$$

Lemma 2.1.6. *If ρ_{ni} satisfy the suppositions of Theorem 2.1.1 then*

$$\frac{\alpha_{nk}}{\binom{n}{k} P_n(x)} \leq C$$

and

$$\left| \frac{w(\phi_n^{-1}(x))}{w(x)} \frac{w\left(\frac{k}{n}\right)}{w\left(\phi_n^{-1}\left(\frac{k}{n}\right)\right)} \right| \leq C.$$

Proof. Firstly $\ln \prod\limits_{i=1}^{n} \left(1+\frac{1}{\rho_{ni}}\right) \sum\limits_{i=0}^{n} \frac{1}{\rho_{ni}} \leq C$ then

$$\begin{aligned}
\frac{\alpha_{nk}}{\binom{n}{k} P_n(x)} &= \frac{1}{\binom{n}{k}} \sum_{\substack{j_1+j_2+...+j_n=k \\ j_i\in\{0,1\}}}^{n} \prod_{i=1}^{n} \frac{\rho_{ni}+1-j_i}{\rho_{ni}+1-x} \\
&\leq \frac{1}{\binom{n}{k}} \sum_{\substack{j_1+j_2+...+j_n=k \\ j_i\in\{0,1\}}}^{n} \prod_{i=1}^{n} \frac{\rho_{ni}+1-j_i}{\rho_{ni}} \\
&\leq \prod_{i=1}^{n} \frac{\rho_{ni}+1}{\rho_{ni}} \leq C.
\end{aligned} \tag{2.16}$$

and by Corollary 2.1.5

$$\left|\frac{w\left(\phi_n^{-1}(x)\right)}{w(x)}\frac{w\left(\frac{k}{n}\right)}{w\left(\phi_n^{-1}\left(\frac{k}{n}\right)\right)}\right|\leq C.$$

Analogous reasons give

$$|w(x)(1-x)^n[2f(\tau_{n1})-f(\tau_{n2})]|\leq|2w(x)(1-x)^n f(\tau_{n1})-w(x)(1-x)^n f(\tau_{n2})|$$

$$=\left|\frac{2w(\tau_{n1})f(\tau_{n1})w_n\left(\phi_n^{-1}(x)\right)(1-x)^n}{P_n(x)w_n(\tau_{n1})}+\frac{2w(\tau_{n2})f(\tau_{n2})w_n\left(\phi_n^{-1}(x)\right)(1-x)^n}{P_n(x)w_n(\tau_{n2})}\right|$$

$$\leq\frac{2\|wf\|\left(\phi_n^{-1}(x)\right)^\alpha\left(1-\phi_n^{-1}(x)\right)^\beta(1-x)^n}{P_n(x)\left(\phi_n^{-1}\left(\frac{1}{n}\right)\right)^\alpha\left(1-\phi_n^{-1}\left(\frac{1}{n}\right)\right)^\beta}+\frac{2\|wf\|\left(\phi_n^{-1}(x)\right)^\alpha\left(1-\phi_n^{-1}(x)\right)^\beta(1-x)^n}{P_n(x)\left(\phi_n^{-1}\left(\frac{2}{n}\right)\right)^\alpha\left(1-\phi_n^{-1}\left(\frac{2}{n}\right)\right)^\beta}$$

$$\leq C\|wf\|.$$

□

2.2 PROOF OF THEOREM 2.1.1

Proof. a)

We estimate

$$w(x)\left|\sum_{k=1}^{n-1}f(\tau_{nk})u_{nk}(x)\right|=w(x)\left|\sum_{k=1}^{n-1}\frac{\alpha_{nk}\binom{n}{k}x^k(1-x)^{n-k}}{\binom{n}{k}P_n(x)}f(\tau_{nk})\right|$$

$$=w\left(\phi_n^{-1}(x)\right)\left|\sum_{k=1}^{n-1}\frac{\alpha_{nk}\cdot p_{nk}(x)f(\tau_{nk})w(\tau_{nk})}{\binom{n}{k}P_n(x)w(\tau_{nk})}\right|$$

$$\leq\|wf\|\sum_{k=1}^{n-1}\frac{\alpha_{nk}\cdot p_{nk}(x)w\left(\phi_n^{-1}(x)\right)}{\binom{n}{k}P_n(x)w\left(\phi_n^{-1}\left(\frac{k}{n}\right)\right)}:=\|wf\|\sigma,$$

with

$$\sigma\leq\left\{\sum_{1\leq k\leq\frac{nx}{2}}+\sum_{\frac{nx}{2}\leq k\leq\frac{2n}{3}}\sum_{\frac{2n}{3}\leq k\leq n-1}\right\}p_{nk}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}:=\sum_{i=1}^{n}\sigma_i(x).$$

For a fixed x using the monotone increasing property of $p_{nk}(x)$ as well as (1.49) we obtain :

$$\sigma_1(x)\leq C(nx)^{\alpha+\frac{1}{2}}\left(\frac{2}{e}\right)^{\frac{nx}{2}}$$

since the latter function attains its maximum for $x = \frac{C}{n}$.

The estimate of $\sigma_2(x)$ is simpler, since in this case $\frac{w(x)}{w\left(\frac{k}{n}\right)} \le C$ which implies the boundedness.

For $\sigma_3(x)$we again use that for a fixed x the maximum of $p_{nk}(x)$ in $\frac{2n}{3} < k \le n$ is attained for $k = \left[\frac{2n}{3}\right]$. Since by Stirling's formula $p_{n,\frac{2n}{3}}(x) \le \left(\frac{3}{4}\right)^{\frac{2n}{3}}$ we obtain

$$\sigma_3(x) \le C\left(\frac{3}{4}\right)^{\frac{2n}{3}} n^{\beta+1}.$$

Analogously other terms in $V_n^*(f,x)$ are considered.

The Corollary 2.1.5 and Lemma 2.1.6 finish the proof. □

Lemma 2.2.1. *Under suppositions of Theorem 2.1.1 the inequalities* $\phi_n'(x) \sim 1$ *and* $\left\|\left(\phi_n^{-1}\right)''\right\| \le C$ *hold.*

Proof. We start with

$$\begin{aligned}\phi_n'(x) &= \frac{1}{n}\sum_{k=1}^{n}\frac{\rho_{nk}(1+\rho_{nk}-x)+\rho_{nk}x}{(1+\rho_{nk}-x)^2}\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{\rho_{nk}(1+\rho_{nk})}{(1+\rho_{nk}-x)^2} \ge \frac{1}{n}\sum_{k=1}^{n}\frac{\rho_{nk}(1+\rho_{nk})}{(1+\rho_{nk})^2}\\ &\ge \frac{1}{n}\sum_{k=1}^{n}\frac{\rho_{nk}}{(1+\rho_{nk})} \ge C,\end{aligned}$$

on the other hand

$$\begin{aligned}\phi_n'(x) &= \frac{1}{n}\sum_{k=1}^{n}\frac{\rho_{nk}(1+\rho_{nk}-x)+\rho_{nk}x}{(1+\rho_{nk}-x)^2}\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{\rho_{nk}(1+\rho_{nk})}{(1+\rho_{nk}-x)^2} \le \frac{1}{n}\sum_{k=1}^{n}\frac{\rho_{nk}(1+\rho_{nk})}{\rho_{nk}^2}\\ &\le \frac{1}{n}\sum_{k=1}^{n}\frac{1+\rho_{nk}}{\rho_{nk}} \le C.\end{aligned}$$

Put $t = \phi_n(x)$. Then $\left(\phi_n^{-1}\right)'(t) = \frac{1}{\phi_n'(x)}$. Then

$$\begin{aligned}\left|\left(\phi_n^{-1}\right)''(t)\right| &= \left|\left(\frac{1}{\phi_n'\left(\phi_n^{-1}(t)\right)}\right)'\right|\\ &= \frac{1}{\left(\phi_n'\left(\phi_n^{-1}(t)\right)\right)^2}\left|\phi_n''\left(\phi_n^{-1}(t)\right)\left(\phi_n^{-1}\right)'(t)\right|.\end{aligned}$$

Hence, using

$$\phi_n''(x) = \frac{2}{n}\sum_{k=1}^{n}\frac{\rho_{nk}(1+\rho_{nk})}{(1+\rho_{nk}-x)^3} \le \frac{2}{n}\sum_{k=1}^{n}\frac{1+\rho_{nk}}{\rho_{nk}^2} \le C$$

we prove the lemma. □

Lemma 2.2.2. *If $f \in W_w^2$ then $f \circ \phi_n^{-1} \in W_w^2$.*

Proof. We start with

$$\begin{aligned}\left(f \circ \phi_n^{-1}\right)''(t) &= \left(f\left(\phi_n^{-1}(t)\right)\right)'' = \left(f'\left(\phi_n^{-1}(t)\right)\left(\phi_n^{-1}(t)\right)'\right)' \\ &= f''\left(\phi_n^{-1}(t)\right)\left(\left(\phi_n^{-1}(t)\right)'\right)^2 + f'\left(\phi_n^{-1}(t)\right)\left(\phi_n^{-1}(t)\right)''.\end{aligned}$$

Consider firstly $0 \le t \le \frac{1}{2}$, then

$$f'(t)\varphi^2(t)w(t) = \int_{\frac{1}{2}}^{t} f''(x)\,dx\varphi(t)^2 w(t) + f'\left(\frac{1}{2}\right)\varphi^2(t)w(t)$$

and

$$\begin{aligned}\left|\int_{\frac{1}{2}}^{t} f''(x)\,dx\varphi(t)^2 w(t)\right| &= \left|\int_{\frac{1}{2}}^{t} f''(x)\frac{\varphi^2(x)w(x)}{\varphi^2(x)w(x)}dx\varphi^2(t)w(t)\right| \\ &\le \left\|f''\varphi^2 w\right\| \int_t^{\frac{1}{2}} \frac{dx}{\varphi^2(x)w(x)}\varphi^2(t)w(t) \\ &\le C\left\|f''\varphi^2 w\right\| \left[x^{-\alpha}\right]_t^{\frac{1}{2}}\varphi^2(t)w(t) \le C\left\|f''\varphi^2 w\right\|.\end{aligned}$$

The case $\frac{1}{2} \le t \le 1$ is analogous. Hence, by Corollary 2.1.5 and Lemma 2.2.1, the lemma is proved. □

Lemma 2.2.3. *If $\alpha, \beta > 0$, $0 \le x \le 1$ then*

$$D_n(x) = w(x) \sum_{\left|\frac{k}{n} - \phi_n(x)\right| \ge \frac{\phi_n(x)}{2}} u_{nk}(x) \left|\int_{\frac{k}{n}}^{\phi_n(x)} \frac{\left|\xi - \frac{k}{n}\right|}{\varphi^2(\xi)w(\xi)}d\xi\right| \le \frac{C}{n}.$$

Proof. Firstly, let us assume that $0 \le x \le \frac{1}{2}$. Then the restriction $\left|\frac{k}{n} - \phi_n(x)\right| \ge \frac{\phi_n(x)}{2}$ splits into either

$$\frac{k}{n} - \phi_n(x) \le -\frac{\phi_n(x)}{2}$$

i.e

$$\frac{k}{n} \le \frac{\phi_n(x)}{2}$$

or

$$\frac{k}{n} - \phi_n(x) \ge \frac{\phi_n(x)}{2}$$

i.e.

$$\frac{k}{n} \geq \frac{3\phi_n(x)}{2}.$$

So

$$D_n(x) \leq w(x) \left\{ \sum_{\frac{k}{n} \leq \frac{\phi_n(x)}{2}} + \sum_{\frac{k}{n} \geq \frac{3\phi_n(x)}{2}} \right\} u_{nk}(x) \left| \int_{\frac{k}{n}}^{\phi_n(x)} \frac{\left|\xi - \frac{k}{n}\right|}{\varphi^2(\xi)\, w(\xi)} d\xi \right| = D_{n1}(x) + D_{n2}(x)$$

and

$$\begin{aligned} D_{n1}(x) &= Cx^{\alpha} \sum_{\frac{k}{n} \leq \frac{\phi_n(x)}{2}} u_{nk}(x) \int_{\frac{k}{n}}^{\phi_n(x)} \frac{\left|\xi - \frac{k}{n}\right|}{\varphi^2(\xi)\, w(\xi)} d(\xi) \\ &\leq Cx^{\alpha} \sum_{\frac{k}{n} \leq \frac{\phi_n(x)}{2}} u_{nk}(x) \int_{\frac{k}{n}}^{\phi_n(x)} \xi^{-\alpha} d(\xi) \end{aligned}$$

$$\begin{aligned} &\leq cx^{\alpha} \sum_{\frac{k}{n} \leq \frac{\phi_n(x)}{2}} p_{nk}(x) \left(\frac{k}{n}\right)^{-\alpha} \phi_n(x) \leq cx^{\alpha} \sum_{\frac{k}{n} \leq \frac{x}{2}} p_{nk_n}(x) \left(\frac{k}{n}\right)^{-\alpha} x \\ &\leq Cx^{\alpha+2} n^{\alpha+1} p_{nk_n}(x). \end{aligned}$$

Since, for fixed x $p_{nk}(x)$ attains its maximum in $0 \leq k \leq \frac{nx}{2}$ for $k_n = \left[\frac{nx}{2}\right]$. Now by Stirling's formula

$$\begin{aligned} p_{nk_n}(x) &\leq C \frac{\left(\frac{n}{e}\right)^{e} \sqrt{n} x^{\frac{nx}{2}} (1-x)^{n\left(1-\frac{x}{2}\right)}}{\left(\frac{nx}{2e}\right)^{\frac{nx}{2}} \sqrt{nx} \left(\frac{n\left(1-\frac{x}{2}\right)}{e}\right)^{n\left(1-\frac{x}{2}\right)} \sqrt{n}} \\ &\leq \frac{C}{\sqrt{nx}} 2^{n} \left(\frac{1-x}{2-x}\right)^{n\left(1-\frac{x}{2}\right)} \\ &= \frac{C}{\sqrt{nx}} \left(1 - \frac{x}{2-x}\right)^{n\left(1-\frac{x}{2}\right)} 2^{\frac{nx}{2}} \end{aligned} \tag{2.17}$$

and by $1 - u \leq e^{-u}$, $u \geq 0$,

$$p_{nk_n}(x) \leq \frac{C}{\sqrt{nx}} e^{-\frac{nx}{2}} 2^{\frac{nx}{2}}. \tag{2.18}$$

Hence,

$$D_{n1}(x) \leq Cx^{\alpha+\frac{3}{2}} n^{\alpha+\frac{1}{2}} \left(\frac{2}{e}\right)^{\frac{nx}{2}} \leq Cn^{-\alpha-\frac{3}{2}} n^{\alpha+\frac{1}{2}} \leq \frac{C}{n}.$$

For estimating $D_{n2}(x)$

$$
\begin{aligned}
D_{n2}(x) &\le w(x) \sum_{\frac{3\phi_n(x)}{2} \le \frac{k}{n}} u_{nk}(x) \int_{\phi_n(x)}^{\frac{k}{n}} \frac{\frac{k}{n}-\xi}{\varphi^2(\xi)\, w(\xi)} d\xi \\
&= w(x) \left\{ \sum_{\frac{3\phi_n(x)}{2} \le \frac{k}{n} \le \frac{2}{3}} + \sum_{\frac{k}{n} \ge \frac{2}{3}} \right\} u_{nk}(x) \int_{\phi_n(x)}^{\frac{k}{n}} \frac{\frac{k}{n}-\xi}{\varphi^2(\xi)\, w(\xi)} d\xi \\
&= D_{n2}^{(1)} + D_{n2}^{(2)}.
\end{aligned}
$$

Note that one of $D_{n2}^{(1)}$ or $D_{n2}^{(2)}$ may be absent. For $D_{n2}^{(1)}$ we can write using Lemma 2.2.1, (1.33) and (2.13).

$$
\begin{aligned}
D_{n2}^{(1)} &\le C x^{\alpha} \sum_{\frac{3\phi_n(x)}{2} \le \frac{k}{n} \le \frac{2}{3}} u_{nk}(x) \int_{\phi_n(x)}^{\frac{k}{n}} \frac{\frac{k}{n}-\xi}{\xi^{\alpha+1}} d\xi \\
&\le \frac{C}{\phi_n(x)} \sum_{k=0} u_{nk}(x) \left(\frac{k}{n} - \phi_n(x) \right)^2 \le \frac{C}{n}.
\end{aligned}
$$

For $D_{n2}^{(2)}$ we have

$$
\begin{aligned}
D_{n2}^{(2)} &= w(x) \sum_{\frac{k}{n} \ge \frac{2}{3}} u_{nk}(x) \left(\int_{\phi_n(x)}^{\frac{2}{3}} \frac{\frac{k}{n}-\xi}{\xi^{\alpha+1}(1-\xi)^{\beta+1}} d\xi \right. \\
&\quad \left. + \int_{\frac{2}{3}}^{\frac{k}{n}} \frac{\frac{k}{n}-\xi}{\xi^{\alpha+1}(1-\xi)^{\beta+1}} d\xi \right) \\
&\le w(x) \sum_{\frac{k}{n} \ge \frac{2}{3}} u_{nk}(x) \left(\frac{C}{x^{\alpha}} + \int_0^{\frac{k}{n}} \frac{d\xi}{(1-\xi)^{\beta}} \right) \\
&\le C x^{-1} \sum_{\frac{k}{n} \ge \frac{2}{3}} p_{nk}(x) \int_0^{\frac{k}{n}} \frac{d(\xi)}{(1-\xi)^{\beta}} \le C x^{-1} p_{n,\left[\frac{2n}{3}\right]}(x)\, n \\
&= C x^{\left[\frac{2n}{3}\right]-1} (1-x)^{n-\left[\frac{2n}{3}\right]} \binom{n}{\left[\frac{2n}{3}\right]} n \le C \left(\frac{3\sqrt{3}}{4\sqrt{2}} \right)^{\frac{2n}{3}}
\end{aligned}
$$

where we use estimates

$$
\binom{n}{\frac{2n}{3}} \le \left(\frac{3}{2}\sqrt{3} \right)^{\frac{2n}{3}}
$$

$$
\begin{aligned}
x^{\left[\frac{2n}{3}\right]-1} (1-x)^{n-\left[\frac{2n}{3}\right]} &\le x^{\frac{n}{3}-1} x^{\frac{n}{3}} (1-x)^{\frac{n}{3}} \\
&\le \left(\frac{1}{2} \right)^{\frac{n}{3}-1} \left(\frac{1}{4} \right)^{\frac{n}{3}}.
\end{aligned}
$$

The case $\frac{1}{2} \le x \le 1$ is considered analogously. Now assume that $\frac{1}{2} \le x \le 1$. Put $\frac{k}{n} = 1 - \frac{m}{n}$ and $\zeta = 1 - \xi$ in $P_n(x)$

$$D_n(x) = w(x) \sum_{\left|\frac{m}{n}-(1-\phi_n(x))\right| \geq \frac{\phi_n(x)}{2}} u_{nk}(x) \left| \int_{1-\frac{m}{n}}^{\phi_n(x)} \frac{\left|\xi - \left(1-\frac{m}{n}\right)\right|}{\varphi^2(\xi)\, w(\xi)} d(\xi) \right|$$

$$w(x) \sum_{\left|\frac{m}{n}-(1-\phi_n(x))\right| \geq \frac{\phi_n(x)}{2}} u_{n,n-m}(x) \left| \int_{1-\phi_n(x)}^{\frac{m}{n}} \frac{\left|\frac{m}{n} - \zeta\right|}{\varphi^2(\zeta)\, \zeta^\beta (1-\zeta)^\alpha} d(\zeta) \right|.$$

Then the restriction $\left|\frac{m}{n} - (1-\phi_n(x))\right| \geq \frac{\phi_n(x)}{2}$ splits in to either

$$\frac{m}{n} - (1-\phi_n(x)) > \frac{\phi_n(x)}{2}$$

that is

$$\frac{m}{n} > 1 - \frac{\phi_n(x)}{2}$$

or

$$\frac{m}{n} - (1-\phi_n(x)) \leq -\frac{\phi_n(x)}{2}$$

i.e.

$$\frac{m}{n} \leq 1 - \frac{3\phi_n(x)}{2}.$$

$$D_n(x) \leq w(x) \left(\sum_{\frac{m}{n} \leq 1 - \frac{3\phi_n(x)}{2}} + \sum_{\frac{m}{n} > 1 - \frac{\phi_n(x)}{2}} \right) u_{n,n-m}(x) \left| \int_{1-\phi_n(x)}^{\frac{m}{n}} \frac{\left|\frac{m}{n} - \zeta\right|}{\varphi^2(\zeta)\, \zeta^\beta (1-\zeta)^\alpha} d(\zeta) \right|$$

$$= D_{n1}(x) + D_{n2}(x).$$

$$\begin{aligned}
D_n(x) &\leq c(1-x)^\beta \sum_{\frac{m}{n} \leq 1 - \frac{3\phi_n(x)}{2}} u_{n,n-m}(x) \int_{\frac{m}{n}}^{1-\phi_n(x)} \frac{\left|\zeta - \frac{m}{n}\right|}{\varphi^2(\zeta)\, \zeta^\beta (1-\zeta)^\alpha} d(\zeta) \\
&\leq c(1-x)^\beta \sum_{\frac{m}{n} \leq 1 - \frac{3\phi_n(x)}{2}} u_{n,n-m}(x) \int_{\frac{m}{n}}^{1-\phi_n(x)} \zeta^{-\beta} d(\zeta) \leq c(1-x)^\beta \\
&\leq \sum_{\frac{m}{n} \leq 1 - \frac{3\phi_n(x)}{2}} p_{n,n-m}(x) \left(\frac{m}{n}\right)^{-\beta} (1-\phi_n(x)) \\
&\leq c(1-x)^{\beta+1} \sum_{\frac{m}{n} \leq 1 - \frac{3\phi_n(x)}{2}} p_{n,m}(1-x) \left(\frac{m}{n}\right)^{-\beta} \\
&\leq c(1-x)^{\beta+2} n^{\beta+1} p_{n,m}(1-x).
\end{aligned}$$

Now apply the reasoning from (2.17), we obtain $D_{n1}(x) \leq \frac{C}{n}$. $\square$

Lemma 2.2.4. *If $f \in W_\omega^2$ then for $F_n = F_n\left(f \circ \phi_n^{-1}\right)$ and for all $\alpha, \beta \geq 0$*

$$\left\|\left(F_n \circ \phi_n - V_n^*\left(f\right)\right) w\right\| \leq \frac{C}{n}\left\|F_n'' \varphi^2 w\right\|.$$

Proof. First consider

$$\begin{aligned}
&\sum_{k=0}^{n} u_{nk}\left(x\right) \int_x^{\tau_{nk}} \left(\phi_n\left(t\right) - \phi_n\left(\tau_{nk}\right)\right) F_n''\left(\phi_n\left(t\right)\right) \phi_n'\left(t\right) d\left(t\right) \\
&= -\sum_{k=0}^{n} F_n\left(\frac{k}{n}\right) u_{nk}\left(x\right) + \sum_{k=0}^{n} F_n\left(\phi_n\left(x\right)\right) u_{nk}\left(x\right) \\
&= -\sum_{k=1}^{n-1} f \circ \phi_n^{-1}\left(\frac{k}{n}\right) u_{nk}\left(x\right) - \left[2f \circ \phi_n^{-1}\left(\frac{1}{n}\right) - f \circ \phi_n^{-1}\left(\frac{2}{n}\right)\right] u_{n,0}\left(x\right) \\
&- \left[2f \circ \phi_n^{-1}\left(\frac{n-1}{n}\right) - f \circ \phi_n^{-1}\left(\frac{n-2}{n}\right)\right] u_{n,n}\left(x\right) + F_n\left(\phi_n\left(x\right)\right) \\
&= F_n\left(\phi_n\left(x\right)\right) - V_n^*\left(f, x\right).
\end{aligned}$$

Hence

$$\left|F_n\left(\phi_n\left(x\right)\right) - V_n^*\left(f, x\right)\right| w\left(x\right) \leq w\left(x\right) \sum_{k=0}^{n} u_{nk}\left(x\right) \int_{\phi_n(x)}^{\frac{k}{n}} \left|\xi - \frac{k}{n}\right| \left|F_n''\left(\xi\right)\right| d\left(\xi\right)$$

$$\sum_{\left|\frac{k}{n} - \phi_n(x)\right| \leq \frac{\phi_n(x)}{2}} + \sum_{\left|\frac{k}{n} - \phi_n(x)\right| \geq \frac{\phi_n(x)}{2}} := E_1\left(x\right) + E_2\left(x\right).$$

Firstly suppose that $0 \leq x \leq \frac{1}{2}$. For $E_1\left(x\right)$ as in the proof of Lemma 1.5.3, $E_1\left(x\right) = 0$ for $0 \leq x \leq \frac{1}{n}$. Now if $\frac{1}{n} \leq x \leq \frac{1}{2}$ then $1 \leq k \leq n-1$ and

$$E_1\left(x\right) \leq \sum_{\left|\frac{k}{n} - \phi_n(x)\right| \leq \frac{\phi_n(x)}{2}} w\left(x\right) u_{nk}\left(x\right) \int_{\phi_n(x)}^{\frac{k}{n}} \frac{\left|\xi - \frac{k}{n}\right| \left|F_n''\left(\xi\right) w\left(\xi\right) \varphi^2\left(\xi\right)\right|}{w\left(\xi\right) \varphi^2\left(\xi\right)} d\left(\xi\right)$$

$$\begin{aligned}
&\leq C \frac{\left\|F_n'' \varphi^2 w\right\|}{\left(1-x\right) x} \sum_{\left|\frac{k}{n} - \phi_n(x)\right| \leq \frac{\phi_n(x)}{2}} u_{nk}\left(x\right) \left(\frac{k}{n} - \phi_n\left(x\right)\right)^2 \\
&\leq \frac{c}{n} \left\|\phi_n'\left(x\right)\right\| \left\|F_n'' \varphi^2 w\right\| \leq \frac{c}{n} \left\|F_n'' \varphi^2 w\right\|.
\end{aligned}$$

On the other hand by Lemma 2.2.3

$$E_2\left(x\right) \leq C \left\|F_n'' \varphi^2 w\right\| D_n\left(x\right) \leq \frac{C}{n} \left\|F_n'' \varphi^2 w\right\|$$

□

Lemma 2.2.5. *If $f \in W_\omega^2$ then for $P_1 := P_1\left(f \circ \phi_n^{-1}\right)$ and $P_2 := P_2\left(f \circ \phi_n^{-1}\right)$ and we have*

$$\left\|w\left[f - P_1 \circ \phi_n\right]\right\|_{[0,\tau_{2n}]} \leq \frac{C}{n}\left\|f''\varphi^2 w\right\|,$$

$$\left\|w\left[f - P_2 \circ \phi_n\right]\right\|_{[\tau_{n-2,n},0]} \leq \frac{C}{n}\left\|f''\varphi^2 w\right\|.$$

Proof. By Lemma 2.2.2 that $f \circ \phi_n^{-1} \in W_\omega^2$, then the proof of Lemma 1.5.4 gives

$$\max_{t \in \left[0, \frac{2}{n}\right]} \left|w\left(\phi_n^{-1}\right)\left|f\left(\phi_n^{-1}(t)\right) - P_1\left(f \circ \phi_n^{-1}, t\right)\right|\right| \leq \frac{C}{n}\left\|\left(f \circ \phi_n^{-1}\right)'' \varphi^2 w\right\|_{\left[0, \frac{2}{n}\right]}$$

and

$$\max_{t \in \left[1-\frac{2}{n}, 1\right]} \left|w\left(\phi_n^{-1}\right)\left|f\left(\phi_n^{-1}(t)\right) - P_2\left(f \circ \phi_n^{-1}, t\right)\right|\right| \leq \frac{C}{n}\left\|\left(f \circ \phi_n^{-1}\right)'' \varphi^2 w\right\|_{\left[1-\frac{2}{n}, 1\right]}.$$

Now the proof of Lemma 2.2.2 gives the desired result. □

Lemma 2.2.6. *Let $F_n := F_n\left(f \circ \phi_n^{-1}\right)$. If $f \in W_\omega^2$ then we have*

$$\left\|F_n''\varphi^2 w\right\| \leq C\left\|f''\varphi^2 w\right\|.$$

.

Proof. Apply the proofs of Lemma 1.5.5 and of Lemma 2.2.2

$$\left\|F_n''\varphi^2 w\right\| \leq C\left\|\left(f \circ \phi_n^{-1}\right)'' \varphi^2 w\right\|.$$

□

Proof. (*Theorem b*) We know that for $\phi_n(x) \in \left[\frac{2}{n}, 1 - \frac{2}{n}\right]$

$$F_n \circ \phi_n(x) = F_n\left(f \circ \phi_n^{-1}, \phi_n(x)\right) = f \circ \phi_n^{-1}\left(\phi_n(x)\right) = f(x).$$

Then by Lemma 2.2.4 we deduce

$$\begin{aligned}
\left\|w\left[f - V_n^*(f)\right]\right\| &\leq \left\|w\left[f - F_n \circ \phi_n\right] + w\left[F_n \circ \phi_n - V_n^*(f)\right]\right\| \\
&\leq \frac{C}{n}\left\|F_n''\varphi^2 w\right\| + w\left[f - F_n \circ \phi_n\right] \\
&= \frac{C}{n}\left\|F_n''\varphi^2 w\right\| \\
&+ \max\left(\left\|w\left[f - F_n \circ \phi_n\right]\right\|_{\phi_n^{-1}\left[0, \frac{2}{n}\right]}, \left\|w\left[f - F_n \circ \phi_n\right]\right\|_{\phi_n^{-1}\left[1-\frac{2}{n}, \frac{2}{n}\right]}\right).
\end{aligned}$$

Now

$$\max_{x\in\ \phi_n^{-1}\left[0,\frac{2}{n}\right]} w\left(x\right)\left|f\left(x\right)-F_n\left(f\circ\phi_n^{-1},\phi_n\left(x\right)\right)\right|$$
$$=\max_{t\in\left[0,\frac{2}{n}\right]} w\left(\phi_n^{-1}\left(t\right)\right)\left|f\left(\phi_n^{-1}\left(t\right)\right)-F_n\left(f\circ\phi_n^{-1},t\right)\right|$$

and Lemma 2.2.4, Lemma 2.2.5 we finish the proof. □

CHAPTER 3

CONVERGENCE OF VIDENSKII-BASKAKOV OPERATORS IN RATIONAL FUNCTIONS

3.1 INTRODUCTION

The classical Baskakov operators are defined as

$$B_n\left(x,y\right)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)b_{nk}\left(x\right) \tag{3.1}$$

$$b_{nk}\left(x\right)=\binom{n+k-1}{k}x^k\left(1+x\right)^{-n-k} \tag{3.2}$$

J. Swetits and B. Wood constructed linear positive operator (Swetits and Wood, 1973) which generalizes polynomial of (3.1) as follows

$$S_n\left(f,x\right)=\sum_{k=1}^{\infty}f\left(\frac{k}{n}\right)v_{nk}\left(x\right) \tag{3.3}$$

$$v_{nk}\left(x\right)=\frac{1}{k!}\frac{\partial^k\left(g_n\left(x,y\right)\right)}{\partial y^k} \tag{3.4}$$

where $g_n\left(x,y\right)=\left(\prod_{i=1}^{n}1+h_{ni}\left(x\right)-h_{ni}\left(x\right)y\right)^{-1}$.

(Swetits and Wood, 1973). Suppose $\left\{h_j\left(x\right)\right\}_{j=0}^{\infty}$ is a sequence of continous, nonnegative real valued functions defined on $[0,\infty)$. Suppose that each interval $[0,a]$ there is a constant M which depends only on a such that $h_j\left(x\right)\leq M$ for $j=0,1,2,...,x\in[0,a]$. Let f be continous on $[0,\infty)$ and satisfy $\left|f\left(x\right)\right|\leq e^{Ax}$ for some constant $A\geq 0$. The sequences $\left\{S_n\left(f\right)\right\}_{n=0}^{\infty}$ defined by (3.3) converges to f uniformly on $[0,a]$. if $\left\{h_j\left(x\right)\right\}_{j=0}^{\infty}$ is uniformly $(C,1)$ summable to x on $[0,a]$.

We begin with expressing the generating function of Baskakov operator (Baskakov, 1957).

$$\sum_{k=0}^{\infty}\frac{(n+k-1)!}{k!\,(n-1)!}\frac{x^k}{(1+x)^{n+k}}=1$$

and

$$\sum_{k=0}^{\infty}\frac{(n+k-1)!}{k!\,(n-1)!}\left(\frac{x}{1+x}\right)^k=(1+x)^n=\left(1-\frac{x}{1+x}\right)^{-n}.$$

We can express $(1+x)^n$ as

$$\begin{aligned}(1+x)^n&=\left(1-\frac{x}{1+x}\right)^{-n}\\&=1+\frac{(-n)}{1!}\left(-\frac{x}{1+x}\right)+\frac{-n\,(-n-1)}{2!}\left(-\frac{x}{1+x}\right)^2+...\end{aligned}$$

similar manner

$$\left(1-\frac{yx}{1+x}\right)^{-n}=1+\frac{(-n)}{1!}\left(-\frac{xy}{1+x}\right)+\frac{-n\,(-n-1)}{2!}\left(-\frac{xy}{1+x}\right)^2+...$$

therefore we can conclude that

$$\sum_{k=0}^{\infty}\frac{(n+k-1)!}{k!\,(n-1)!}\frac{(xy)^k}{(1+x)^k}=\left(1-\frac{yx}{1+x}\right)^{-n}=\frac{(1+x)^n}{(1+x-xy)^n}.$$

We can define the generating function of Baskakov operator as

$$\begin{aligned}B_n\,(x,y)&=\sum y^k b_{nk}\,(x)=\sum_{k=0}^{\infty}\tbinom{n+k-1}{k}x^k\,(1+x)^{-n-k}\,y^k\\&=\sum_{k=0}^{\infty}\frac{(n+k-1)!}{k!\,(n-1)!}\,(xy)^k\,\frac{1}{(1+x)^{n+k}}=\frac{1}{(1+x-xy)^n}.\end{aligned}$$

Now we describe a construction of generating function for v_{nk}. This was done by J. Swetits - B. Woods (Swetits and Wood, 1973) firstly and it was repeated independently by A. E. Mencher (Mencher, 1985) and by the author. Let $h_{ni}\,(x)$ be defined as $h_{ni}\,(x)=\frac{x\rho_{ni}}{x+\rho_{ni}}$, where $i=m_n+1,...,n$, $h_{ni}\,(x)=x$ where $i=1,..m_n$; $0\leq m_n\leq n$ and $\phi_n\,(x)=\frac{1}{n}\sum_{i=1}^{n}h_{ni}\,(x)$, where $\phi_n\,(x)$ is increasing 0 to ∞. The generating function of our operators is defined as

$$g_n\,(x,y)=\frac{1}{\prod_{i=1}^{n}(1+h_{ni}\,(x)-h_{ni}\,(x)\,y)}=\sum_{k=0}^{\infty}v_{nk}\,(x)\,y^k.$$

Differentiating in y

$$\frac{\partial g_n(x,y)}{\partial y} = g_n(x,y)\sum_{i=1}^{n}\frac{h_{ni}(x)}{1+h_{ni}(x)-h_{ni}(x)y}$$
$$= \sum_{k=1}^{\infty} k v_{nk}(x)\, y^{k-1},$$

and we have

$$v_{nk}(x) = \frac{1}{k!}\frac{\partial^k g_n(x,y)}{\partial y^k}\bigg|_{y=0}.$$

The $g_n(x,y)$ is an analytic function as a function of y for $|y| < \min\limits_{1\le i\le n}\frac{1+h_{ni}(x)}{h_{ni}(x)}$.

For $y=1$, $g_n(x,1)=1$ then $\sum\limits_{k=1}^{\infty} k v_{nk}(x) = \sum\limits_{i=1}^{n} h_{ni}(x)$ and $\phi_n(x) = \sum\limits_{k=0}^{\infty}\frac{k}{n} v_{nk}(x)$.

Because of

$$h'_{ni}(x) = \frac{\rho_{ni}(x+\rho_{ni}) - x\rho_{ni}}{(x+\rho_{ni})^2} = \frac{\rho_{ni}^2}{(x+\rho_{ni})^2} > 0$$

the function $\phi_n(x)$ increases from 0 to ∞.

Let's differentiate $g_n(x,y)$ twice

$$\frac{\partial^2 g(x,y)}{\partial y^2} = \sum_{i=1}^{\infty} k(k-1)\, v_{nk}(x)\, y^{k-2}$$
$$= \frac{\partial}{\partial y}\left(g_n(x,y)\sum_{i=1}^{n}\frac{h_{ni}(x)}{1+h_{ni}(x)-h_{ni}(x)y}\right)$$
$$= \frac{\partial g_n(x,y)}{\partial y}\sum_{i=1}^{n}\frac{h_{ni}(x)}{1+h_{ni}(x)-h_{ni}(x)y}$$
$$+ g_n(x,y)\sum_{i=1}^{n}\frac{h_{ni}^2(x)}{(1+h_{ni}(x)-h_{ni}(x)y)^2}$$
$$= g_n(x,y)\left(\sum_{i=1}^{n}\frac{h_{ni}(x)}{1+h_{ni}(x)-h_{ni}(x)y}\right)^2$$
$$+ g_n(x,y)\sum_{i=1}^{n}\frac{h_{ni}^2(x)}{(1+h_{ni}(x)-h_{ni}(x)y)^2}$$

and for y=1

$$\frac{\partial^2 g_n(x,y)}{\partial y^2} = \sum_{k=2}^{\infty} k(k-1) v_{nk}(x) = \left(\sum_{i=1}^{n} h_{ni}(x)\right)^2 + \sum_{i=1}^{n} (h_{ni}(x))^2$$
$$= (n\phi_n(x))^2 + \psi_n(x)$$

where $\psi_n(x) = \sum_{i=1}^{n} (h_{ni}(x))^2$. Videnskii type generalization of Baskakov operators defined as

$$V_n(f,x) = \sum_{k=1}^{\infty} f(\tau_{nk}) v_{nk}(x) \tag{3.5}$$

where $\phi_n(\tau_{nk}) = \frac{k}{n}$.

A. Mencher (Mencher, 1985) defined Videnskii type generalizations of Baskakov operators with $h_{ni}(x) = \frac{x(x_{ni}-1)}{x_{ni}+x}$ in $x_{ni} = \infty$, $i = 1, ..., m_n$; $x_{ni} < \infty$, $i = m_n+1, ..., n$; $x_{ni} > 1$ we denote these operators as $\widetilde{V}_n(f,x)$.

He proved the following results for $\widetilde{V}_n(f,x)$.

Theorem 3.1.1. *If* $\lim_{n\to\infty} \frac{S_n}{m_n^2} = 0$ *then for every* $f \in BC[0,\infty)$ *and* $a > 0$

$$\lim_{n\to\infty} \left\| \widetilde{V}_n(f,\cdot) - f \right\|_{C[0,a]} = 0. \tag{3.6}$$

Theorem 3.1.2. *If for any* $f \in BC[0,\infty)$ *and any* $a > 0$, $\lim_{n\to\infty} \left\| \widetilde{V}_n(f,\cdot) - f \right\|_{C[0,a]} = 0$ *then* $\lim_{n\to\infty} S_n = \infty$. *Here*

$$S_n = \sum_{k=m_n+1}^{n} \frac{(x_{ni}-1)}{x_{ni}} + m_n. \tag{3.7}$$

3.2 CONVERGENCE

Lemma 3.2.1. *If* V_n *is defined as in* (3.5) *then*

$$V_n(1,x) = 1,$$

$$\sum_{k=0}^{\infty} (\phi_n(\tau_{nk}) - \phi_n(x)) v_{nk}(x) = 0,$$

$$\sum_{k=0}^{\infty} (\phi_n(\tau_{nk}) - \phi_n(x))^2 v_{nk}(x) = \frac{1}{n^2}\psi_n(x) + \frac{1}{n}\phi_n(x).$$

Proof. First two equalities are immediate. Prove the third relation. Consider

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\phi_n\left(\tau_{nk}\right)-\phi_n\left(x\right)\right)^2 v_{nk}\left(x\right) &= \sum_{k=0}^{\infty}\left(\frac{k}{n}\right)^2 v_{nk}\left(x\right)-\frac{2}{n}\phi_n\left(x\right)\sum_{k=0}^{\infty}kv_{nk}\left(x\right)\\ &+\phi_n^2\left(x\right)\sum_{k=0}^{\infty}v_{nk}\left(x\right)\\ &=\sum_{k=1}^{\infty}\frac{k\left(k-1\right)}{n^2}v_{nk}\left(x\right)+\frac{1}{n^2}\sum_{k=1}^{\infty}kv_{nk}\left(x\right)-2\phi_n^2\left(x\right)+\phi_n^2\left(x\right)\\ &=\phi_n^2\left(x\right)+\frac{1}{n^2}\psi_n\left(x\right)+\frac{1}{n}\phi_n\left(x\right)-\phi_n^2\left(x\right)\\ &=\frac{1}{n^2}\psi_n\left(x\right)+\frac{1}{n}\phi_n\left(x\right).\end{aligned}$$

□

Theorem 3.2.2. *If* $f\in C\left[0,\infty\right),\rho_{ni}>c>0$ *and* $\lim\limits_{n\to\infty}\frac{1}{n^2}\left(\sum\limits_{k=m_n+1}^{n}\rho_{nk}+\sum\limits_{k=m_n+1}^{n}\left(\rho_{nk}\right)^2\right)=0$ *then* $V_n\left(f,x\right)\rightrightarrows f$ *on compacts in* $\left[0,\infty\right)$.

Proof. Fix $a>0$. At first let us note that $\phi_n^{-1}\geq\frac{m_n}{n}x+\frac{(n-m_n)cx}{a+c}\geq c_1x$ for $0\leq x\leq a$.

Next

$$-2M\leq\left(f\circ\phi_n^{-1}\right)\left(x\right)-\left(f\circ\phi_n^{-1}\right)\left(y\right)\leq 2M \tag{3.8}$$

for all $x,y\in\left[0,a\right]$ where $M=\left\|f\right\|_{C\left[0,\frac{a}{c_1}\right]}$, and for arbitrary $\varepsilon>0$, there exists $\delta>0$, such that for all x, $y\in[0,a]$, $|x-y|<\delta$

$$-\varepsilon\leq\left(f\circ\phi_n^{-1}\right)\left(x\right)-\left(f\circ\phi_n^{-1}\right)\left(y\right)\leq\varepsilon \tag{3.9}$$

(3.8) and (3.9) imply

$$-\varepsilon-\frac{2M}{\delta^2}\left(x-y\right)^2\leq\left(f\circ\phi_n^{-1}\right)\left(x\right)-\left(f\circ\phi_n^{-1}\right)\left(y\right)\leq\varepsilon+\frac{2M}{\delta^2}\left(x-y\right)^2$$

for all $x,y\in\left[0,a\right]$.

Observe also that $\phi\left(x\right)\leq x$ for all $x\in\left[0,\infty\right)$.

Now for any $x\in[0,a-\delta]$ we have $\phi_n\left(x\right)\in[0,a-\delta]$ and

$$V_n\left(f,x\right)-f\left(x\right)=\sum_{k=0}^{\infty}\left(\left(f\circ\phi_n^{-1}\right)\left(\frac{k}{n}\right)-\left(f\circ\phi_n^{-1}\right)\left(\phi_n\left(x\right)\right)\right)v_{nk}\left(x\right),$$

so

$$\left|V_n\left(f,x\right)-f\left(x\right)\right|\leq\varepsilon+\frac{2M}{\delta^2}\sum_{k=0}^{\infty}\left(\frac{k}{n}-\phi_n\left(x\right)\right)^2 v_{nk}\left(x\right),$$

or

$$\left|V_n(f,x)-f(x)\right| \leq \varepsilon + \frac{2M}{\delta^2}\frac{1}{n^2}\left(\sum_{k=m_n+1}^{n}\rho_{nk+}\sum_{k=m_n+1}^{n}(\rho_{nk})^2\right).$$

$\square$

Remark 3.2.1. Note that Theorem 3.1.1 does not guarantee the convergence of the sequence $\left(\widetilde{V}_n\right)$ to f even in the case of equal finite poles (i.e. $m_n = 0,\ x_{ni} = x_0)$, but our Theorem 3.2 gives the uniform convergence for the sequence (V_n) with equal finite poles (i.e. $m_n = 0,\ \rho_{ni} = \rho_0)$. Moreover the conditions of Theorem 3.2 are essetially less restrictive then the conditions of Theorem 3.1 of (Swetits and Wood, 1973).

CHAPTER 4

ON THE CONVERGENCE OF Q ANALOGUE OF DURRMEYER TYPE OPERATORS AND BASKAKOV-DURRMEYER OPERATORS

4.1 ON THE CONVERGENCE OF Q-ANALOGUE OF DURRMEYER TYPE OPERATORS

4.1.1 Introduction

In 1997 Philips (Phillips, 1997) proposed the following q-analogue of the well-known Bernstein polynomials,which for each positive integer n and $f \in C\left[0,1\right]$,are defined as,

$$B_{n,q}\left(f;x\right)=\sum_{k=0}^{n}f\left(\frac{[k]}{[n]}\right)p_{nk}\left(q;x\right);$$

where

$$p_{nk}\left(q;x\right)=\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{\frac{k(k-1)}{2}}x^k\left(1-x\right)^{n-k}}{\left(1-x+qx\right)...\left(1-x+q^{n-1}x\right)}.$$

After Philips several researchers have studied convergence properties of $q-$Bernstein polynomials $B_{n,q}\left(f;x\right)$.The Bernstein-type operator discussed in (Parvanov and Popov, 1994) is

$$U_n\left(f;x\right)=\left(n-1\right)\sum_{k=0}^{n}p_{nk}\left(x\right)\int_0^1 f\left(t\right)p_{n-2,k-1}\left(t\right)dt$$
$$+f\left(0\right)P_{n,0}+f\left(1\right)P_{n,n} \tag{4.1}$$

which is Durrmeyer-type modification of Bernstein polynomials where

$$[n-1]\int_0^1 f\left(t\right)p_{n-2,k-1}\left(t\right)dt$$

for $1 \leq k \leq n-1$ in the operators $U_n(f;x)$ takes place of $f\left(\frac{k}{n}\right)$ in $B_n(f;x)$ the Bernstein polynomials.

Starting with the operators(4.1) J.L Durrmeyer (see (Durrmeyer, 1967)) introduced in 1967 the operators $D_n : L_1([0,1]) \to C([0,1])$, which are integral modifications of the Bernstein polynomials in order to approximate Lebesgue integrable functions on the interval [0, 1], defined as

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{nk}(x)\int_0^1 f(t)\, p_{n,k}(t)\, dt. \tag{4.2}$$

Very recently Derriennic (Derriennic, 2005) introduced some q-analogue of the Durrmeyer operators and established some approximation properties of those $q-$Durrmeyer operators.

As Durrmeyer operators approximate integrable functions on the interval $[0,1]$, this inspired us to introduce new q analogue of the Durrmeyer-type operators of (4.1) which reproduce linear functions.

For $f \in C[0,1]$, we introduce the following q-Durrmeyer type operators as

$$\begin{aligned} K_{n,q}(f;x) &= [n-1]\sum_{k=1}^{n} q^{1-k} p_{nk}(q;x)\int_0^1 f(t)\, p_{n-2,k-1}(q;qt)\, d_q t \\ &\quad + f(0)\, p_{n,0}(q,x) \\ &=: \sum_{k=0}^{n} A_{nk}(f)\, p_{n,k}(q;x), \quad 0 \leq x \leq 1. \end{aligned} \tag{4.3}$$

It can be easily verified that in the case $q=1$ the operators defined by (4.3) reduce to the Durrmeyer-type operators recently introduced and studied by Parvanov and Popov. The advantage of the operators we defined is reproducing linear functions.

In the present chapter we study some approximation properties of q-Durrmeyer-type operators $K_{n,q}(f;x)$ defined by (4.3) for $0 < q < 1$. First we estimate the moments for the $q-$Durrmeyer-type operators. We also study the rate of convergence for these operators $K_{n,q}(f;x)$. We establish direct results in terms of $\omega(f,\cdot)$. Throughout chapter the expression $g_n(x) \Longrightarrow g(x)$ means uniform convergence of a sequence $\{g_n(x)\}$ to $g(x)$.

4.1.2 Moments

In this section we shall obtain $K_{n,q}(t^i,x), i = 0,1,2$. Note that for $s = 0,1,...$and by the definition of q-Beta function (see (Kac and Cheung, 2002)), we have,

$$\begin{aligned}\int_0^1 t^s p_{n,k}(q;qt)\,d_qt &= \begin{bmatrix} n \\ k \end{bmatrix} q^k \int_0^1 t^{k+s}(1-qt)_q^{n-k}\,d_qt \\ &= \frac{q^k[n]!}{[k]!\,[n-k]!}\frac{[k+s]!\,[n-k]!}{[k+s+n-k+1]!\,[n]!} \\ &= \frac{q^k[k+s]!\,[n]!}{[n+s+1]!\,[k]!}. \qquad (4.4)\end{aligned}$$

Theorem 4.1.1. *We have*

$$K_{n,q}(1;x) = 1, \qquad K_{n,q}(t;x) = x$$

and

$$K_{n,q}\left(t^2;x\right) = x^2 + \frac{(1+q)\,x\,(1-x)}{[n+1]}. \qquad (4.5)$$

Proof. In order to prove the theorem we shall use the following identities

$$\sum_{k=0}^{n} p_{nk}(q;x) = 1, \qquad \sum_{k=0}^{n}\frac{[k]}{[n]}p_{nk}(q;x) = x,$$

$$\sum_{k=0}^{n}\left(\frac{[k]}{[n]}\right)^2 p_{nk}(q;x) = x^2 + \frac{x\,(1-x)}{[n]}.$$

We will evaluate $K_{n,q}(t^s;x)$, $s = 0,1,2$.The result can easily be verified for $s = 0$.

$$\begin{aligned}K_{n,q}(1;x) &= \sum_{k=1}^{n} q^{1-k}p_{nk}(q;x)\int_0^1 [n-1]\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}(qt)^{k-1}(1-qt)^{n-k-1}\,d_qt \\ &= \sum_{k=1}^{n} p_{nk}(q;x)\frac{[n-1]\,[n-2]!}{[k-1]!\,[n-k-1]!}\frac{[k-1]!\,[n-k-1]!}{[k-1+n-k-1+1]!} \\ &= 1.\end{aligned}$$

Using the definition of $K_{n,q}(f;x)$ (4.3)and (4.4) for $s=1$ we have

$$
\begin{aligned}
K_{n,q}(t;x) &= \sum_{k=1}^{n} q^{1-k} p_{nk}(q;x) \int_0^1 [n-1] \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} t\,(qt)^{k-1}(1-qt)^{n-k-1}\, d_q t \\
&= \sum_{k=1}^{n} p_{nk}(q;x) \frac{[n-1]\,[n-2]!}{[k-1]!\,[n-k-1]!} \frac{[k]!\,[n-k-1]!}{[k+n-k-1+1]!} \\
&= \sum_{k=1}^{n} \frac{[k]}{[n]} p_{nk}(q;x) = x.
\end{aligned}
$$

and for $s=2$ using $[t+1]=1+q[t]$, and $[k]^2=[k](q[k-1]+1)$, $0<q<1$, we have

$$
\begin{aligned}
K_{n,q}\left(t^2;x\right) &= \sum_{k=1}^{n} q^{1-k} p_{nk}(q;x) \int_0^1 [n-1] \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} t^2 (qt)^{k-1}(1-qt)^{n-k-1}\, d_q t \\
&= \sum_{k=1}^{n} p_{nk}(q;x) \frac{[n-1]\,[n-2]!}{[k-1]!\,[n-k-1]!} \frac{[k+1]!\,[n-k-1]!}{[k+1+n-k-1+1]!} \\
&= \sum_{k=1}^{n} p_{nk}(q;x) \frac{[k]\,[k+1]}{[n]\,[n+1]} \\
&= \frac{1}{[n]\,[n+1]} \sum_{k=1}^{n} [k]\,[k+1]\, p_{nk}(q;x) \\
&= \frac{1}{[n]\,[n+1]} \sum_{k=1}^{n} [k]\,(1+q[k])\, p_{nk}(q;x) \\
&= \frac{1}{[n]\,[n+1]} \sum_{k=1}^{n} \left([k]+q[k]^2\right) p_{nk}(q;x) \\
&= \frac{x}{[n+1]} + q\frac{[n]}{[n+1]} \sum_{k=1}^{n} \frac{[k]^2}{[n]^2} p_{nk}(q;x) \\
&= \frac{x}{[n+1]} + q\frac{[n]}{[n+1]} \left(x^2 + \frac{x(1-x)}{[n]}\right) \\
&= q\frac{[n]}{[n+1]} x^2 + q\frac{x(1-x)}{[n+1]} + \frac{x}{[n+1]} \\
&= x^2 + \frac{(1+q)\,x(1-x)}{[n+1]}.
\end{aligned}
$$

□

Theorem 4.1.2. *If $q>1$ be fixed and $f \in C[0,1]$, then $K_{n,q}(f;x) \rightrightarrows f(x)$ for all $x \in [0,1]$.*

Proof. The theorem follows from the Korovkin theorem and the Theorem 4.1.2. □

Remark 4.1.1. It is observed that the operators $K_{n,q}(f;x)$ reproduce linear functions.

Remark 4.1.2. Let $x \in [0,1]$ then for every $q \in (0,1)$ we have the following ;

$$K_{n,q}((t-x);x) = 0, \qquad K_{n,q}\left((t-x)^2;x\right) = \frac{(1+q)\,x\,(1-x)}{n+1}.$$

4.1.3 Convergence of q-Durrmeyer Type Operators

Definition 4.1.1. Let $q \in (0,1)$ be fixed. We define $K_{\infty,q}(f;1) = f(1)$ for $x \in [0,1)$

$$\begin{aligned} K_{\infty,q}(f;x) &= \frac{1}{1-q}\sum_{k=0}^{n} q^{1-k} p_{\infty k}(q;x)\int_0^1 f(t)\,p_{\infty,k-1}(q;qt)\,d_qt + f(0)\,p_{\infty,0}(q,x) \\ &=: \sum_{k=0}^{n} A_{\infty k}(f)\,p_{\infty,k}(q;x) \end{aligned} \tag{4.6}$$

where

$$p_{\infty,k}(q;x) = \frac{x^k}{(1-q)^k\,[k]!}(1-x)_q^{\infty}.$$

Using the fact (see (II'inskii and Ostrovska, 2002)) , we have

$$\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1, \quad \sum_{k=0}^{\infty}\left(1-q^k\right)p_{\infty,k}(q;x) = x$$

and

$$\sum_{k=0}^{\infty}\left(1-q^k\right)^2 p_{\infty,k}(q;x) = x^2 + (1-q)\,x\,(1-x),$$

so

$$\begin{aligned} \int_0^1 t^s p_{\infty,k}(q;qt)\,d_qt &= \frac{q^k}{(1-q)^k\,[k]!}\int_0^1 t^{k+s}(1-qt)_q^{\infty}\,d_qt \\ &= \frac{q^k}{(1-q)^k\,[k]!}[k+s]!\,(1-q)^{k+s+1} \\ &= (1-q)^{s+1}\frac{q^k\,[k+s]!}{[k]!}. \end{aligned} \tag{4.7}$$

By using (4.7) and (4.6), it is easy to prove that

$$K_{\infty,q}(1;x) = 1,\ K_{\infty,q}(t;x) = x, \quad K_{\infty,q}\left(t^2;x\right) = x^2 + \left(1-q^2\right)x\,(1-x).$$

For $f \in C[0,1]$, $t > 0$, we define the modulus of continuity $\omega(f,t)$ as follows:

$$\omega(f,t) = \sup_{\substack{|x-y|\le 1 \\ x,y\,\in\,[0,1]}} |f(x) - f(y)|.$$

Lemma 4.1.3. *Let $f \in C[0,1]$ and $f(1) = 0$. Then we have*

$$|A_{nk}(f)| \leq A_{nk}(|f|) \leq w(f, q^n)\left(1 + q^{k-n}\right) \qquad (0 \leq k \leq n)$$

and for any n, k,

$$|A_{\infty k}(f)| \leq A_{\infty k}(|f|) \leq w(f, q^n)\left(1 + q^{k-n}\right).$$

Proof. By the well-known property of modulus of continuity (1.72)

$$w(f, \lambda t) \leq (1 + \lambda) w(f, t), \ \ \lambda > 0$$

we get

$$|f(t)| = |(f(t) - f(1))| \leq w(f, 1-t) \leq w(f, q^n)\left(1 + \frac{1-t}{q^n}\right).$$

Thus

$$\begin{aligned}
|A_{nk}(f(t) - f(1))| &= \left|[n-1]\int_0^1 q^{1-k}(f(t) - f(1))\, p_{n-2,k-1}(q; qt)\, d_q t\right| \\
&\leq [n-1]\int_0^1 q^{1-k}(f(t) - f(1))\, p_{n-2,k-1}(q; qt)\, d_q t \\
&\leq [n-1]\int_0^1 q^{1-k} w(f, q^n)\left(1 + \frac{1-t}{q^n}\right) p_{n-2,k-1}(q; qt)\, d_q t \\
&= w(f, q^n)\left(1 + q^{-n}\left(1 - \frac{[k]}{[n]}\right)\right) \\
&= w(f, q^n)\left(1 + \frac{q^k\left(1 - q^{n-k}\right)}{q^n(1 - q^n)}\right) \leq w(f, q^n)\left(1 + q^{k-n}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
|A_{\infty k}(f(t) - f(1))| &= \frac{q^{1-k}}{1-q}\left|\int_0^1 (f(t) - f(1))\, p_{\infty,k-1}(q; qt)\, d_q t\right| \\
&\leq w(f, q^n)\frac{q^{1-k}}{1-q}\left|\int_0^1 \left(1 + \frac{1-t}{q^n}\right) p_{\infty,k-1}(q; qt)\, d_q t\right| \\
&= w(f, q^n)\left(1 + \frac{\left(1 - \left(1 - q^k\right)\right)}{q^n}\right) \leq w(f, q^n)\left(1 + q^{k-n}\right).
\end{aligned}$$

□

Theorem 4.1.4. *Let $0 < q < 1$, then for each $f \in C[0,1]$ the sequence $\{K_{n,q}(f; x)\}$ converges to $K_{\infty,q}(f; x)$ uniformly on $[0,1]$. Furthermore*

$$|K_{n,q}(f) - K_{\infty,q}(f)| \leq C_q \omega(f, q^n). \tag{4.8}$$

Proof. $K_{\infty,q}(f;x)$ and $K_{n,q}(f;x)$ reproduce linear functions that is

$$K_{\infty,q}(at+b;x) = K_{n,q}(at+b;x) = ax+b.$$

Hence for all $x \in C[0,1)$ by the definitions of $K_{\infty,q}(f;x)$ and $K_{n,q}(f;x)$, we know that

$$\begin{aligned}
|K_{n,q}(f;x) - K_{\infty,q}(f;x)| &= \left| \sum_{k=0}^{n} A_{nk}(f)\, p_{n,k}(q;x) - \sum_{k=0}^{n} A_{\infty k}(f)\, p_{\infty,k}(q;x) \right| \\
&= \left| \sum_{k=0}^{n} A_{nk}(f - f(1))\, p_{n,k}(q;x) - \sum_{k=0}^{n} A_{\infty k}(f - f(1))\, p_{\infty,k}(q;x) \right| \\
&\le \sum_{k=0}^{n} |A_{nk}(f - f(1)) - A_{\infty k}(f - f(1))|\, p_{n,k}(q;x) \\
&+ \sum_{k=0}^{n} |A_{\infty k}(f - f(1))|\, |p_{n,k}(q;x) - p_{\infty,k}(q;x)| \\
&+ \sum_{k=n+1}^{\infty} |A_{\infty k}(f - f(1))|\, p_{\infty,k}(q;x) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

First we have

$$\begin{aligned}
&|p_{n,k}(q;x) - p_{\infty,k}(q;x)| \\
&:= \left| \begin{bmatrix} n \\ k \end{bmatrix} x^{k-1} \prod_{s=0}^{n-k-1} (1 - q^s x) - \frac{x^k}{(1-q)^k [k]!} \prod_{s=0}^{\infty} (1 - q^s x) \right| \\
&= \left| \begin{bmatrix} n \\ k \end{bmatrix} x^k \left(\prod_{s=0}^{n-k-1} (1 - q^s x) - \prod_{s=0}^{\infty} (1 - q^s x) \right) \right. \\
&\left. + x^k \prod_{s=0}^{\infty} (1 - q^s x) \left(\begin{bmatrix} n \\ k \end{bmatrix} - \frac{1}{(1-q)^k [k]!} \right) \right| \\
&\le p_{n,k}(q;x) \left| 1 - \prod_{s=n-k+1}^{\infty} (1 - q^s x) \right| \\
&+ p_{\infty,k}(q;x) \left| \prod_{s=n-k}^{n-2} (1 - q^s) - 1 \right| \\
&\le \frac{q^{n-k}}{1-q} \left(p_{n,k}(q;x) + p_{\infty,k}(q;x) \right)
\end{aligned} \tag{4.9}$$

where in the last formula we use the inequality

$$1 - \prod_{s=1}^{n} (1 - a_s) \le \sum_{s=1}^{n} a_s \;\; (a_1, ..., a_n \in (0,1),\; n = 1, 2, ...).$$

Hence by using (4.9) we have

$$\begin{aligned}
&\left|A_{nk}\left(f-f\left(1\right)\right)-A_{\infty k}\left(f-f\left(1\right)\right)\right| \\
&\leq \int_0^1 q^{1-k}\left|f\left(t\right)-f\left(1\right)\right|\left[n-1\right]\left|p_{n-2,k-1}\left(q;qt\right)-\frac{1}{1-q}p_{\infty,k-1}\left(q;qt\right)\right|d_qt \\
&\leq \int_0^1 q^{1-k}\left|f\left(t\right)-f\left(1\right)\right|\left|\left[n-1\right]-\frac{1}{1-q}\right|p_{\infty,k-1}\left(q;qt\right)d_qt \\
&+\int_0^1 q^{1-k}\left|f\left(t\right)-f\left(1\right)\right|\left[n-1\right]\left|p_{n-2,k-1}\left(q;qt\right)-p_{\infty,k-1}\left(q;qt\right)\right|d_qt \\
&\leq \frac{q^{n-1}}{1-q}\int_0^1 q^{1-k}\left|f\left(t\right)-f\left(1\right)\right|p_{\infty,k-1}\left(q;qt\right)d_qt \\
&+\frac{q^{n-k-1}}{1-q}\int_0^1 q^{1-k}\left|f\left(t\right)-f\left(1\right)\right|\left[n-1\right]\left(p_{n-2,k-1}\left(q;qt\right)-p_{\infty,k-1}\left(q;qt\right)\right)d_qt \\
&\leq q^{n-1}w\left(f,q^n\right)\left(1+q^{k-1-n}\right)+2\frac{q^{n-k-1}}{1-q}w\left(f,q^n\right)\left(1+q^{k-1-n}\right)\leq\frac{5w\left(f,q^n\right)}{1-q}.
\end{aligned}$$

Now we estimate I_1 and I_3. We have

$$I_1 \leq \frac{5w\left(f,q^n\right)}{1-q}\sum_{k=0}^{n}p_{n,k}\left(q;x\right)\leq\frac{5w\left(f,q^n\right)}{1-q}$$

and

$$I_3 \leq w\left(f,q^n\right)\sum_{k=0}^{n}\left(1+q^{k-n+2}\right)p_{\infty,k}\left(q;x\right)\leq 2w\left(f,q^n\right)\sum_{k=0}^{n}p_{\infty,k}\left(q;x\right)\leq 2w\left(f,q^n\right).$$

Finally we estimate

$$\begin{aligned}
I_2 &\leq \sum_{k=0}^{n}w\left(f,q^n\right)\left(1+q^{k-n}\right)\frac{q^{n-k}}{1-q}\left(p_{n,k}\left(q;x\right)+p_{\infty,k}\left(q;x\right)\right) \\
&\leq \frac{2w\left(f,q^n\right)}{1-q}\sum_{k=0}^{n}\left(p_{n,k}\left(q;x\right)+p_{\infty,k}\left(q;x\right)\right)\leq\frac{4w\left(f,q^n\right)}{1-q}.
\end{aligned}$$

We conclude that for x$\in[0,1)$,

$$\left|K_{n,q}\left(f\right)-K_{\infty,q}\left(f\right)\right|\leq C_q\omega\left(f,q^n\right).$$

This completes the proof of the theorem. □

4.2 ON THE CONVERGENCE OF BASKAKOV-DURRMEYER OPERATORS

4.2.1 Introduction and Notation

In (Mihesan, 1998) the following generalized Baskakov operators were introduced with non-negative constant $a \geq 0$ independent of n

$$B_n^a(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x,a) f\left(\frac{k}{n}\right), x \geq 0,\ k = 0,1,2,..., \quad n = 1,2,..., \tag{4.10}$$

where

$$p_{n,k}(x,a) = e^{\frac{-ax}{1+x}} \frac{p_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}}$$

such that

$$\sum_{k=0}^{\infty} p_{n,k}(x,a) = 1$$

and

$$p_k(n,a) = \sum_{t=0}^{k} \binom{k}{t} (n)_t\, a^{k-t}$$

with $(n)_0 = 1, (n)_t = n(n+1)...(n+t-1)$, for $t \geq 1$, defined for $f \in C[0,\infty)$.

In (Wafi and Khatoon, 2004c), for the operators B_n^a the rate of convergence via modulus of continuity of f was evaulated by Wafi and Khatoon. In (Wafi and Khatoon, 2004a) they studied some approximation properties of the operators B_n^a. In (Wafi and Khatoon, 2004b); (Wafi and Khatoon, 2005) they established direct and inverse results for the generalized Baskakov operators.

In this chapter we stated in (4.13) a new Durrmeyer type modification of generalized Baskakov operators (Mihesan, 1998) for all real valued continous and bounded functions f on $(0,\infty]$. Another Durrmeyer type modification was given by A. Erencin (Erencin, 2011).

For the operators A_n we establish certain direct theorems in terms of the modulus of continuity of second order, and prove the continuity in Lipschitz-type space.

As a special case, when a = 0, the operators given by (4.13) turn out to be the following Durrmeyer type Baskakov operators,

$$B_n(f,x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)\, f(t)\, dt, \quad x \geq 0,$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \tag{4.11}$$

and

$$B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y > 0.$$

and for all $n \in N$

$$\Gamma(n+1) = n!.$$

Lemma 4.2.1. *(Mihesan, 1998). For a, $x \geq 0$, $n = 1, 2....$ We have*

$$B_n^a(1;x) = 1,$$

$$B_n^a(t;x) = x + \frac{ax}{n(1+x)},$$

$$B_n^a(t^2;x) = \frac{x^2}{n} + \frac{x}{n} + x^2 + \frac{a^2x^2}{n^2(1+x)^2} + \frac{2ax^2}{n(1+x)} + \frac{ax}{n^2(1+x)}. \tag{4.12}$$

The main goal of this section is to try to use a similar idea which helped in former section to simplify formulas for the moments.

4.2.2 Construction of Operators

For every $n \in N$, the positive operators A_n is defined by

$$A_n(f;x) = (n-1)\sum_{k=0}^{\infty} S_{n,k}^a(x)\int_0^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}} f(t)\, d(t) \tag{4.13}$$

for $x \in [0,\infty)$ and for every real valued continous and bounded function f on $[0,\infty)$ where $n \in N$,

$$S_{n,k}^a(x) = e^{\frac{-ax}{1+x}}\frac{p_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}},$$

in which $p_k(n,a)$ is defined as in (4.11). These operators satisfy linearity property.

Lemma 4.2.2. *The following equalities hold:*

$$A_n(1;x) = 1,$$

$$A_n(t;x) = \frac{1}{(n-2)}\left(nx + \frac{ax}{(1+x)} + 1\right), \ \ n > 2, \tag{4.14}$$

$$A_n\left(t^2;x\right) = \frac{1}{(n-2)(n-3)}\left(\left(n^2+n\right)x^2 + 4nx + \frac{a^2x^2}{(1+x)^2}\right.$$

$$\left. + \frac{2nax^2}{1+x} + \frac{4ax}{1+x} + 2\right), \ \ n > 3. \tag{4.15}$$

Proof. From (4.12), we have

$$A_n(1;x) = (n-1)\sum_{k=0}^{\infty} S_{n,k}^a(x)\binom{n+k-1}{k}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k}}dt$$

$$= (n-1)\sum_{k=0}^{\infty} S_{n,k}^a(x)\frac{(n+k-1)!}{(n-1)!k!}B(k+1,n-1)$$

$$= (n-1)\sum_{k=0}^{\infty} S_{n,k}^a(x)\frac{(n+k-1)!}{(n-1)!k!}\frac{(n-2)!k!}{(n+k-1)!}$$

$$= (n-1)\sum_{k=0}^{\infty} S_{n,k}^a(x)\frac{1}{n-1} = \sum_{k=0}^{\infty} S_{n,k}^a(x)$$

$$= B_n^a(1;x) = 1,$$

where $B_n^a(f,x)$ is defined by (4.12). Similarly

$$A_n(t;x) = (n-1)\sum_{k=0}^{\infty} S_{n,k}^a(x)\binom{n+k-1}{k}\int_0^{\infty}\frac{t^{k+1}}{(1+t)^{n+k}}dt$$

$$= (n-1)\sum_{k=0}^{\infty} S_{n,k}^a(x)\frac{(n+k-1)!}{(n-1)!k!}B(k+2,n-2)$$

$$= (n-1)\sum_{k=0}^{\infty} S_{n,k}^a(x)\frac{(n+k-1)!}{(n-1)!k!}\frac{(n-3)!(k+1)!}{(n+k-1)!}$$

$$= \sum_{k=0}^{\infty} S_{n,k}^a(x)\frac{k+1}{n-2}$$

$$= \frac{1}{(n-2)}\left(n\sum_{k=0}^{\infty} S_{n,k}^a(x)\frac{k}{n} + \sum_{k=0}^{\infty} S_{n,k}^a(x)\right)$$

$$= \frac{1}{n-2}\left(nB_n^a\left(t,x\right)+B_n^a\left(1,x\right)\right)$$

$$= \frac{1}{n-2}\left(nx+\frac{ax}{1+x}+1\right).$$

Finally

$$A_n\left(t^2;x\right) = (n-1)\sum_{k=0}^{\infty}S_{n,k}^a\left(x\right)\binom{n+k-1}{k}\int_0^{\infty}\frac{t^{k+2}}{\left(1+t\right)^{n+k}}dt$$

$$= (n-1)\sum_{k=0}^{\infty}S_{n,k}^a\left(x\right)\frac{(n+k-1)!}{(n-1)!k!}B\left(k+3,n-3\right)$$

$$= (n-1)\sum_{k=0}^{\infty}S_{n,k}^a\left(x\right)\frac{(n+k-1)!}{(n-1)!k!}\frac{(n-4)!\left(k+2\right)!}{(n+k-1)!}$$

$$= \sum_{k=0}^{\infty}S_{n,k}^a\left(x\right)\frac{\left(k+1\right)\left(\left(k+2\right)\right)}{\left(n-2\right)\left(n-3\right)}$$

$$= \frac{1}{\left(n-2\right)\left(n-3\right)}\left(n^2\sum_{k=0}^{\infty}S_{n,k}^a\left(x\right)\frac{k^2}{n^2}+3n\sum_{k=0}^{\infty}S_{n,k}^a\left(x\right)\frac{k}{n}+2\sum_{k=0}^{\infty}S_{n,k}^a\left(x\right)\right)$$

$$= \frac{1}{\left(n-2\right)\left(n-3\right)}\left(n^2B_n^a\left(t^2,x\right)+3nB_n^a\left(t,x\right)+2B_n^a\left(1,x\right)\right)$$

$$= \frac{1}{\left(n-2\right)\left(n-3\right)}\left[n^2\left(\frac{x^2}{n}+\frac{x}{n}+x^2+\frac{a^2x^2}{n^2\left(1+x\right)^2}+\frac{2ax^2}{n\left(1+x\right)}+\frac{ax}{n^2\left(1+x\right)}\right)\right.$$

$$\left.+3n\left(x+\frac{ax}{n\left(1+x\right)}\right)+2\right]$$

$$= \frac{1}{\left(n-2\right)\left(n-3\right)}\left(\left(n^2+n\right)x^2+4nx+\frac{a^2x^2}{\left(1+x\right)^2}+\frac{2nax^2}{1+x}+\frac{4ax}{1+x}+2\right)$$

so the proof is completed. □

Lemma 4.2.3. *For the operators A_n we have*

$$A_n\left(\left(t-x\right)^2;x\right) = \frac{1}{\left(n-2\right)\left(n-3\right)}\left(\left(2n+6\right)x^2+\left(2n+6\right)x+\frac{a^2x^2}{\left(1+x\right)^2}\right.$$

$$\left.+\frac{6ax^2}{1+x}+\frac{4ax}{1+x}+2\right),\ \ n>3.$$

4.2.3 Local Approximation

In this section we establish direct and local approximation theorems in connection with the operators A_n. Let $C_B[0,\infty)$ be the space of all real valued continous bounded functions f on endowed with the norm. Further let us consider the following K-functional:

$$K_2(f;\delta) = \inf_{g\in W^2}\{\|f-g\| + \delta\|g''\|\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.

By [(Sahai and Prasad, 1985), $p.177$, $Theorem$ 2.4], there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right) \tag{4.16}$$

where

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+2h) - f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. By

$$\omega(f;\delta) = \sup_{0<h\le\delta}\ \sup_{x\in C_B[0,\infty)} |f(x+h) - f(x)|$$

we denote the usual modulus of continuity of $f \in C_B[0,\infty)$. Now, express the auxiliary operators

$$\breve{A}_n(f;x) = A_n(f;x) - f\left(\frac{1}{n-2}\left(nx + \frac{ax}{1+x} + 1\right)\right) + f(x)$$

for $f \in C_B[0,\infty)$, $\psi(x) \ge 0$ and $n > 2$.

Theorem 4.2.4. *Let $n > 3$. We have*

$$|A_n(f;x) - f(x)| = C\omega_2\left(f;\sqrt{\psi_n^a(x)}\right) + \omega\left(f;\frac{1}{n-2}\left(2x + \frac{ax}{1+x} + 1\right)\right).$$

for every $x \in [0,\infty)$ and $f \in C_B[0,\infty)$, where C is a positive constant and where

$$\psi_n^a(x) = \frac{1}{(n-2)(n-3)}\left((2n+10)x^2 + (2n+10)x\right.$$

$$+\frac{2a^2x^2}{(1+x)^2}+\frac{10ax^2}{1+x}+\frac{6ax}{1+x}+3\Big).$$

Proof. By the definition of $\breve{A}_n$ it is known that

$$\breve{A}_n(t-x,x)=0.$$

Let $g\in C_B^2[0,\infty)$. From the Taylor expansion of g

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t(t-u)g''(u)\,du$$

we can write

$$\begin{aligned}\breve{A}_n(g;x)-g(x)&=g'(x)\breve{A}_n(t-x,x)+\breve{A}_n\left(\int_x^t(t-u)g''(u)\,du;x\right)\\&=\breve{A}_n\left(\int_x^t(t-u)g''(u)\,du;x\right)\end{aligned}$$

$$=A_n\left(\int_x^t(t-u)g''(u)\,du;x\right)-\int_x^{\frac{1}{n-2}\left(nx+\frac{ax}{1+x}+1\right)}\left(\frac{1}{n-2}\right.$$

$$\left.\times\left(nx+\frac{ax}{1+x}+1\right)-u\right)g''(u)\,du$$

and therefore

$$\left|\breve{A}_n(g;x)-g(x)\right|\leq A_n\left(\left|\int_x^t(t-u)g''(u)\,du\right|;x\right)$$

$$+\left|\int_x^{\frac{1}{n-2}\left(nx+\frac{ax}{1+x}+1\right)}\left(\frac{1}{n-2}\left(nx+\frac{ax}{1+x}+1\right)-u\right)g''(u)\,du\right|. \tag{4.17}$$

Since

$$\left|\int_x^t(t-u)g''(u)\,du\right|\leq(t-x)^2\|g''\|$$

and

$$\left|\int_x^{\frac{1}{n-2}\left(nx+\frac{ax}{1+x}+1\right)}\left(\frac{1}{n-2}\left(nx+\frac{ax}{1+x}+1\right)-u\right)g''(u)\,du\right|$$

$$\leq\frac{1}{(n-2)^2}\left(2x+\frac{ax}{1+x}+1\right)^2\|g''\|,$$

it follows form (4.17) that

$$\left|\breve{A}_n(g;x)-g(x)\right|\leq\Big\{A_n\left((t-x)^2,x\right)$$

$$+\frac{1}{(n-2)^2}\left(2x+\frac{ax}{1+x}+1\right)^2\Big\}\|g''\|.$$

So by means of Lemma 4.2.1, we may conclude that

$$\left|\breve{A}_n(g,x)-g(x)\right| \leq \frac{1}{(n-2)(n-3)}\left\{\left((2n+6)x^2+(2n+6)x\right.\right.$$

$$\left.\left.+\frac{a^2x^2}{(1+x)^2}+\frac{6ax^2}{1+x}+\frac{4ax}{1+x}+2\right)+\frac{1}{(n-2)^2}\left(2x+\frac{ax}{1+x}+1\right)^2\right\}\|g''\|.$$

Using the fact $\frac{1}{(n-2)^2} \leq \frac{1}{(n-2)(n-3)}$ for $n>3$, we can obtain

$$\left|\breve{A}_n(g;x)-g(x)\right| \leq \left\{\frac{1}{(n-2)(n-3)}(2n+10)x^2\right.$$

$$\left.+(2n+10)x+\frac{2a^2x^2}{(1+x)^2}+\frac{10ax^2}{1+x}+\frac{6ax}{1+x}+3\right\}\|g''\|$$

and for $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$ by the definition of the operators $\breve{A}_n$, we have

$$|A_n(f;x)-f(x)| \leq \left|\breve{A}_n(f-g;x)\right|+|(f-g)(x)|$$

$$+\left|\breve{A}_n(g,x)-g(x)\right|+\left|f\left(\frac{1}{n-2}\left(nx+\frac{ax}{1+x}+1\right)-f(x)\right)\right|,$$

and

$$\left|\breve{A}_n(f;x)\right| \leq \|f\|A_n(1,x)+2\|f\| = 3\|f\|.$$

Thus, we can obtain

$$|A_n(f;x)-f(x)| \leq 4\|f-g\|+\left|\breve{A}_n(g;x)-g(x)\right|$$

$$+\omega\left(f;\frac{1}{n-2}\left(x+\frac{ax}{1+x}+1\right)\right),$$

$$|A_n(f;x)-f(x)| \leq 4\|f-g\|+\frac{1}{(n-2)(n-3)}$$

$$\times\left((2n+10)x^2+(2n+10)x+\frac{2a^2x^2}{(1+x)^2}+\frac{10ax^2}{1+x}+\frac{6ax}{1+x}+3\right)\|g''\|$$

$$+\omega\left(f;\frac{1}{n-2}\left(2x+\frac{ax}{1+x}+1\right)\right),$$

therefore by taking

$$\psi_n^a(x) = \frac{1}{(n-2)(n-3)}\left((2n+10)x^2\right.$$

$$\left.+(2n+10)x+\frac{2a^2x^2}{(1+x)^2}+\frac{10ax^2}{1+x}+\frac{6ax}{1+x}+3\right)$$

we get

$$\begin{aligned}|A_n(f;x)-f(x)| &\leq 4\|f-g\|+\|g''\|\,\psi_n^a(x)\\ &+\omega\left(f;\frac{1}{n-2}\left(x+\frac{ax}{1+x}+1\right)\right).\end{aligned}$$

Thus, taking infimum over all $g\in C_B^2[0,\infty)$ on the right hand side of the last inequality and considering (4.16) the desired result is reached. □

4.2.4 Rate of Convergence

Now consider the Lipschitz- type space

$$Lip_M^*(r)=\left\{f\in C_B[0,\infty]:|f(t)-f(x)|\leq M\frac{|t-x|^r}{(t+x)^{\frac{r}{2}}};x,t\in[0,\infty]\right\}$$

where M is a positive constant and $r\in(0,1]$.

We firstly prove the following lemma which will be used in the proof of the next theorem.

Lemma 4.2.5. *For all $x\geq 0$ and $n>2,$ we have*

$$A_n(|t-x|;x)\leq\sqrt{\delta_n(x)},$$

where

$$\delta_n(x)=A_n\left((t-x)^2;x\right).$$

Proof. By (4.13) we get

$$A_n(|t-x|;x)=(n-1)\sum_{k=0}^{\infty}S_{n,k}^a(x)\int_0^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}|t-x|\,dt.$$

If we apply the Cauchy-Schwarz inequality to the integral in the right hand side of the above inequality, then we find

$$\begin{aligned}A_n(|t-x|;x) &\leq \sqrt{(n-1)\sum_{k=0}^{\infty}S_{n,k}^a(x)}\\ &\times\left\{\left(\int_0^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}(t-x)^2\,dt\right)^2\right\}^{\frac{1}{2}}\end{aligned}\tag{4.18}$$

and again applying the Cauchy-Schwarz inequality to the series in the right hand side of (4.18) we have

$$A_n\left(|t-x|,x\right) \le \left\{(n-1)\sum_{k=0}^{\infty} S_{n,k}^{a}(x)\right.$$

$$\left.\times \int_0^\infty \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}} (t-x)^2\,dt\right\}^{\frac{1}{2}}$$

$$= \sqrt{A_n\left((t-x)^2;x\right)} = \sqrt{\delta_n(x)}.$$

□

Theorem 4.2.6. *Let $f \in Lip_M^*(r)$ then for all $x \ge 0$ and $n > 2$, we have*

$$|A_n(f;x) - f(x)| \le M\left(\frac{\delta_n(x)}{x}\right)^{\frac{r}{2}}$$

where $\delta_n(x)$ is defined as in lemma above.

Proof. At first consider the case $r = 1$

$$|A_n(f;x) - f(x)| \le \left((n-1)\sum_{k=0}^{\infty} S_{n,k}^{a}(x)\int_0^\infty \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}\right.$$

$$\left.\times |f(t) - f(x)|\,dt\right)$$

$$\le M(n-1)\sum_{k=0}^{\infty} S_{n,k}^{a}(x)\int_0^\infty \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}\frac{|t-x|}{\sqrt{t+x}}dt.$$

From the fact that $\frac{1}{\sqrt{t+x}} < \frac{1}{\sqrt{x}}$ and lemma 4, the last inequality implies that

$$|A_n(f;x) - f(x)| \le \left(\frac{M}{\sqrt{x}}(n-1)\sum_{k=0}^{\infty} S_{n,k}^{a}(x)\int_0^\infty \binom{n+k-1}{k}\right.$$

$$\left.\times\frac{t^k}{(1+t)^{n+k}}|t-x|\,dt\right)$$

$$\frac{M}{\sqrt{x}}A_n\left(|t-x|;x\right) \le M\sqrt{\frac{\delta_n(x)}{x}}.$$

This is the desired result for $r = 1$. Now, let $r \in (0,1)$. Then application of the Hölder inequality two times with $p = \frac{1}{r}$ and $q = \frac{1}{1-r}$ gives

$$|A_n(f;x) - f(x)| \le \left((n-1)\sum_{k=0}^{\infty} S_{n,k}^{a}(x)\int_0^\infty \binom{n+k-1}{k}\right.$$

$$\times \frac{t^k}{(1+t)^{n+k}} |f(t) - f(x)| \, dt \Bigg)$$

$$\leq \left\{ (n-1) \sum_{k=0}^{\infty} S_{n,k}^a(x) \left(\int_0^\infty \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}} |f(t) - f(x)| \, dt \right)^{\frac{1}{r}} \right\}^r$$

$$\leq \left\{ (n-1) \sum_{k=0}^{\infty} S_{n,k}^a(x) \int_0^\infty \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}} |f(t) - f(x)|^{\frac{1}{r}} \, dt \right\}^r.$$

Since $f \in Lip_M^*(r)$, this inequality leads to

$$|A_n(f;x) - f(x)| \leq M \left\{ (n-1) \sum_{k=0}^{\infty} S_{n,k}^a(x) \int_0^\infty \binom{n+k-1}{k} \right.$$

$$\left. \times \frac{t^k}{(1+t)^{n+k}} \frac{|t-x|}{\sqrt{t+x}} dt \right\}^r$$

$$\leq \frac{M}{x^{\frac{r}{2}}} \left\{ (n-1) \sum_{k=0}^{\infty} S_{n,k}^a(x) \int_0^\infty \binom{n+k-1}{k} \right.$$

$$\left. \times \frac{t^k}{(1+t)^{n+k}} |t-x| \, dt \right\}^r = \frac{M}{x^{\frac{r}{2}}} \left(A_n(|t-x|;x) \right)^r.$$

Therefore by lemma 4 we may conclude that

$$|A_n(f;x) - f(x)| \leq M \left(\frac{\delta_n(x)}{x} \right)^{\frac{r}{2}}$$

which completes the proof. $\square$

We consider the following class of functions:

Let $C_{x^2}[0,\infty]$ be the set of all functions defined on $[0,\infty]$ satisfying the condition $|f(x) \leq M_f(1+x^2)|$, where M_f is a constant depending only on f. By $C_{x^2}[0,\infty]$,we donote the subspace of all continous functions belonging to $H_{x^2}[0,\infty]$. Also let $H_{x^2}[0,\infty]$ be the subspace of all functions $f \in C_{x^2}[0,\infty]$, for which $\|f(x)\| = \lim\limits_{x\to\infty} \frac{f(x)}{1+x^2}$is finite. We denote the modulus of continuity of f on closed interval $[0,a]$, as $a > 0$ by

$$\omega_a(f,\delta) = \sup_{t-x\leq\delta} \sup_{t,x\in[0,a]} |f(t) - f(x)|.$$

We observe that for the function $f \in C_{x^2}[0,\infty]$ the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Theorem 4.2.7. *$f \in H_{x^2}[0,\infty]$, $\omega_{a+1}(f,\delta)$ be its modulus of continuity on finite interval $[0,a+1] \subset [0,\infty]$, where $a > 0$. Then for every $n \geq 3$*

$$\left|A_n(f;x) - f(x)\right|_{[0,a]} \leq 6M_f\left(1+a^2\right)\delta_n(a) + 2\omega_{a+1}\left(f;\sqrt{\delta_n(a)}\right).$$

Proof. For $x \in [0,a]$ and $t \geq a+1$, since $t-x > 1$, we have

$$\begin{aligned}|f(t) - f(x)| &\leq M_f\left(2 + x^2 + t^2\right)\\ &\leq M_f\left(2 + 3x^2 + 2(t-x)^2\right)\\ &\leq 6M_f\left(1+a^2\right)(t-x)^2. \qquad (4.19)\end{aligned}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f,t-x) \leq \left(1 + \frac{t-x}{\delta}\right)\omega_{a+1}(f;\delta) \qquad (4.20)$$

with $\delta > 0$. From (4.19) and (4.20) we can write

$$|f(t) - f(x)| \leq 6M_f\left(1+a^2\right)(t-x)^2 + \left(1 + \frac{t-x}{\delta}\right)\omega_{a+1}(f,\delta)$$

which yields

$$\begin{aligned}|A_n(f;x) - f(x)| &\leq 6M_f\left(1+a^2\right)A_n\left((t-x)^2;x\right)\\ &+\omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}A_n(|t-x;x|)\right)\end{aligned}$$

for $x \in [0,a]$ and $t \geq 0$. Thus by using Lemma 4.2.4., we can get

$$\begin{aligned}|A_n(f;x) - f(x)| &\leq 6M_f\left(1+a^2\right)\delta_n(x) + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\delta_n(x)}\right)\\ &\leq 6M_f\left(1+a^2\right)\delta_n(a) + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\delta_n(a)}\right)\end{aligned}$$

finally, by choosing $\delta = \sqrt{\delta_n(a)}$, we achieve the result. □

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APPENDIX A

DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS,AND FURTHER STUDIES

A.1 DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS

I hereby declare that this thesis comprises my original work. No material in this thesis has been previously published and written by another person, except where due reference is made in the text of the thesis. I further declare that this thesis contains no material which has been submitted for a degree or diploma or other qualifications at any other university.

Signature:
Date: April 4, 2013

A.1 FURTHER STUDIES

1. Study of convergence of weighted modifications of q analogue of Durrmeyer type operators.

2. Study of convergence of weighted modifications of q analogue of Baskakov operators.

3. Study of convergence of Videnskii-Baskakov operators in rational functions.

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- Akif Barbaros Dikmen, "Approximation Properties of Baskakov- Durmeyer Operators", International Conference on Applied Analysis and Algebra, ICAAA 2012, Yildiz Technical University, Istanbul / Turkey, Jul. 2012.
- Akif Barbaros Dikmen, "The Q- Analogue of the Limit Case of Bernstein Type Operators", First International Conference on Analysis and Applied Mathematics, Gümüşhane University, Gümüşhane / Turkey, October. 2012.