BOUNDARY VALUE PROBLEMS FOR A THIRD ORDER PARTIAL DIFFERENTIAL EQUATION

by

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APPROVAL PAGE

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ABSTRACT

The two-point boundary value problem for third order partial differential equations in a Hilbert space is investigated. The main theorem on stability of the problem is established. To validate the main result, some stability estimates for solutions of the boundary value problems for third order equations are obtained.

Keywords: Boundary Value Problem, Stability, Partial Operator-Differential Equations, Difference Schemes

ÜÇÜNCÜ MERTEBEDEN KISMİ DİFERENSİYEL DENKLEM İÇİN SINIR DEĞER PROBLEMLERİ

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ÖZ

Hilbert uzayında üçüncü mertebeden kısmi diferensiyel denklemler için iki noktada sınır değer problemlemi incelenmiştir. Problemin kararlılık kestirimi üzerinde ana teorem kurulmuştur. Ana sonucu doğrulamak için, üçüncü mertebeden sınır değer problemlerinin çözümlerinin bazı kararlılık kestirimleri elde edilmiştir.

Anahtar Kelimeler: Sınır Değer Problemi, Kararlılık, Kısmi Operatör-Diferensiyel Denklemler, Fark Şemaları

To my family

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INTRODUCTION

Boundary value problems have been a major research area in many branches of science and engineering particularly in applied mathematics when it is impossible to determine the boundary values of the unknown function. In the last century, interest towards to the subject of boundary value problem for a partial differential equations with time and space variables has been substantial and growing tendency because of science and industry. Furthermore, various problems arising in thermal conductivity, microscale heat transfer and so on can be reduced to the nonlocal problems. For these reasons, we have worked on these issues in this thesis.

Local and nonlocal boundary value problems for third order ordinary differential equations and system of ordinary differential equations have been considered in the field of science and engineering such as modern physics, mathematical biology, chemical diffusions and fluid mechanics. Additionally, this type of boundary value problems has been studied widely in the literature (for instance, see (Guezane-Lakoud et al., 2012); (Sun and Ren, 2010); (Suryanarayana, 2011); (Noor et al., 2012)).

The authors (Grossinho et al., 2005) studied existence and location result for the boundary value problem for third-order nonlinear ordinary differential equation. In the papers (Guo et al., 2007); (Guezane-Lakoud and Khaldi, 2010); (Guezane-Lakoud and Frioui, 2012); (Wang et al., 2009), nonlocal boundary value problem for third-order nonlinear ordinary differential equations was considered. Existence of the problems were established by using the Leggett-Williams fixed point theorem (Guo et al., 2007) and Leray Schauder nonlinear alternative (Guezane-Lakoud and Khaldi, 2010); (Guezane-Lakoud and Frioui, 2012). Additionally, some sufficient conditions for the existence of the problem in Banach spaces were obtained by using fixed point index theory (Wang et al., 2009). Moreover, the authors (Palamides and Veloni, 2007); (Palamides and Palamides, 2008); (Liu et al., 2009); (Smirnov, 2011); (Smirnov, 2012) investigated local boundary value problem for third-order nonlinear ordinary differential equations. Existence of the problems was established by using the Krasnoselskii's fixed-point theorem of cone (Palamides and Veloni, 2007); (Palamides and Palamides, 2008); (Liu et al., 2009) and Leggett-Williams (Liu et al., 2009) also established existence of the problem. Similarly, the authors (Smirnov, 2011); (Smirnov, 2012) also established existence of the problem. Finally, local boundary value problem for system of third-order nonlinear ordinary differential equation was studied in the paper (Qu, 2010). The multiplicity and existence of the problem was also established by using the Krasnoselskii's fixed-point theorem of cone (Qu, 2010).

Local and nonlocal boundary value problems for third order partial differential equations have been studied widely in the literature (for instance, see (Latrous and Memou, 2005); (Dzhuraev and Apakov, 2010); (Apakov, 2011); (Denche and Marhoune, 2001); (Denche and Marhoune, 2003); (Ashyralyev et al., 2012)).

The authors (Denche and Memou, 2003) investigated local boundary value problem

$$\begin{cases} \frac{\partial^3 u(t)}{\partial t^3} + \frac{\partial}{\partial x}(a\left(x,t\right)\frac{\partial u}{\partial x}) = f(x,t),\ 0<t<T,\, 0<x<1, \\ u(x,0)=0,\, \frac{\partial u}{\partial t}\left(x,0\right)=0,\, \frac{\partial^2 u}{\partial t^2}(x,T)=0,\, x\in(0,1), \\ u(0,t)=0,\, t\in\left[0,T\right], \\ \int_0^1 u(x,t)dx=0,\, t\in[0,T] \end{cases}$$

for third-order partial differential equations with integral conditions. This paper was proved the existence and uniqueness of a strong solution for a linear equation. The authors were also used energy inequalities and the density of the range of the generated operator.

In the paper (Apakov and Rutkauskas, 2011), the local boundary value problem

$$\begin{cases} \frac{\partial^3 u(x)}{\partial x^3} - \frac{\partial^2 u(y)}{\partial y^2} = f(x,y),\ 0 < x < p,\, 0 < y < l, \\ u_y(x,0) = \varphi_1(x),\, u_y(x,l) = \varphi_2(x),\, p > 0,\, l > 0, \\ u(0,y) = \psi_1(y),\, u(p,y) = \psi_2(y),\, u_x(p,y) = \psi_3(y) \end{cases}$$

for third-order partial differential equations in a rectangular domain was studied. The investigation of authors of this paper is based on the fundamental solutions of corresponding nonhomogeneous equation the green function of analyzed problem.

There are several types of methods for solving partial differential equations. For instance, the method of separation of variables can be used only in the case when it has constant coefficients. In particular, a boundary value problem for third-order partial differential equation can be solved by Fourier series method, by Laplace transform method and by Fourier transform method. Now, let us illustrate these three different analytical methods by examples.

Example 1.1. *Obtain the Fourier series solution of a third-order partial differential equation*

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} + \frac{\partial^2 u(t,x)}{\partial x^2} - u\,(t,x) = -3e^{-t}\sin x,\ 0 < t < 1,\ 0 < x < \pi, \\ u\,(0,x) = \sin x,\ u_t\,(0,x) = -\sin x,\ u_{tt}(1,x) = e^{-1}\sin x,\ 0 \le x \le \pi, \\ u\,(t,0) = u\,(t,\pi) = 0,\ 0 \le t \le 1. \end{cases} \tag{1.1}$$

Solution. In order to solve the problem, first we need to define

$$u\,(t,x) = v\,(t,x) + w\,(t,x)\,,$$

where

$$\begin{cases} \frac{\partial^3 v(t,x)}{\partial t^3} + \frac{\partial^2 v(t,x)}{\partial x^2} - v\,(t,x) = 0,\ 0 < t < 1,\ 0 < x < \pi, \\ v\,(0,x) = 0,\ v_t\,(0,x) = 0,\ v_{tt}(1,x) = 0,\ 0 \le x \le \pi, \\ v\,(t,0) = v\,(t,\pi) = 0,\ 0 \le t \le 1, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \frac{\partial^3 w(t,x)}{\partial t^3} + \frac{\partial^2 w(t,x)}{\partial x^2} - w\left(t,x\right) = -3e^{-t}\sin x,\ 0<t<1,\ 0<x<\pi, \\ \\ w\left(0,x\right)=\sin x,\ w_t\left(0,x\right)=-\sin x,\ w_{tt}(1,x)=e^{-1}\sin x,\ 0\le x\le\pi, \\ \\ w\left(t,0\right)=w\left(t,\pi\right)=0,\ 0\le t\le 1. \end{cases} \tag{1.3}$$

Now, let us obtain the solution of (1.2) by the method of separation of variables. To do this a solution of the form

$$v(t,x) = T(t)X(x) \neq 0$$

is suggested. Taking the partial derivatives and substituting the result in (1.2), we obtain

$$\frac{T'''(t)-T(t)}{T(t)} + \frac{X''(x)}{X(x)} = 0,\ 0<t<1,\ 0<x<\pi. \tag{1.4}$$

It is easy to see that problem (1.4) leads to

$$\frac{T'''(t)-T(t)}{T(t)} = \lambda,\ \frac{X''(x)}{X(x)} = \lambda,\ 0<t<1,\ 0<x<\pi$$

equivalently

$$\begin{cases} T'''(t) = (\lambda+1)\,T(t),\ X''(x) = \lambda X(x), \\ X'(0) = X'(\pi) = 0,\ 0<t<1,\ 0<x<\pi. \end{cases}$$

We have that

$$\begin{cases} T_k\left(t\right) = A_k e^{-\sqrt[3]{(1-n^2)}t} + e^{-\frac{\sqrt[3]{(1-n^2)}t}{2}}\left[B_k\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right. \\ \left.+C_k\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right], \\ X_k\left(x\right) = \sin\left(kx\right), k=1,2,\cdots,\ 0<t<1,\ 0<x<\pi. \end{cases}$$

By using superposition principle, we get

$$v(t,x) = \sum_{n=1}^{\infty}\left[A_n e^{\sqrt[3]{(1-n^2)}t} + e^{-\frac{\sqrt[3]{(1-n^2)}t}{2}}\left[B_n\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right.\right.$$

$$\left.\left.+C_n\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right]\right]\sin(nx),\ 0<t<1,\ 0<x<\pi,$$

$$v_t(t,x) = \sum_{n=1}^{\infty}\left[A_n\sqrt[3]{(1-n^2)}e^{\sqrt[3]{(1-n^2)}t} - B_n\frac{\sqrt[3]{(1-n^2)}}{2}e^{-\frac{\sqrt[3]{(1-n^2)}t}{2}}\right.$$

$$\times\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right]$$

$$+C_n\frac{\sqrt[3]{(1-n^2)}}{2}e^{-\frac{\sqrt[3]{(1-n^2)}t}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right.$$

$$\left.\left.-\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right]\right]\sin(nx),$$

and

$$v_{tt}(t,x)=\sum_{n=1}^{\infty}\left[A_n\sqrt[3]{(1-n^2)^2}e^{\sqrt[3]{(1-n^2)}t}\right.$$

$$+B_n\frac{\sqrt[3]{(1-n^2)^2}}{2}e^{-\frac{\sqrt[3]{(1-n^2)}t}{2}}\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right]$$

$$+C_n\frac{\sqrt[3]{(1-n^2)^2}}{2}e^{-\frac{\sqrt[3]{(1-n^2)}t}{2}}\left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right.$$

$$\left.\left.-\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}t\right)\right]\right]\sin(nx).$$

Using given boundary conditions, we have

$$v(0,x)=\sum_{n=1}^{\infty}(A_n+B_n)\sin(nx)=0,$$

$$v_t(0,x)=\sum_{n=1}^{\infty}\left[A_n\sqrt[3]{(1-n^2)}-B_n\frac{1}{2}\sqrt[3]{(1-n^2)}+C_n\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}\right]\sin(nx)=0,$$

$$v_{tt}(1,x)=\sum_{n=1}^{\infty}\left[A_n\sqrt[3]{(1-n^2)^2}e^{\sqrt[3]{(1-n^2)}}\right.$$

$$+B_n\frac{\sqrt[3]{(1-n^2)^2}}{2}e^{-\frac{\sqrt[3]{(1-n^2)}}{2}}\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}\right)\right]$$

$$+C_n\frac{\sqrt[3]{(1-n^2)^2}}{2}e^{-\frac{\sqrt[3]{(1-n^2)}}{2}}\left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}\right)\right.$$

$$\left.\left.-\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{(1-n^2)}\right)\right]\right]\sin(nx)=0,$$

So, we get the system of equations

$$\begin{cases} A_n + B_n = 0, \\ A_n - \frac{B_n}{2} + C_n \frac{\sqrt{3}}{2} = 0, \\ A_n e^{\sqrt[3]{1-n^2}} + B_n e^{-\frac{\sqrt[3]{1-n^2}}{2}} \left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-n^2}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-n^2}\right)\right] \\ \quad + C_n e^{-\frac{\sqrt[3]{n^2-1}}{2}} \left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-n^2}\right) - \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-n^2}\right)\right] = 0. \end{cases}$$

Solving the system of equations, we obtain

$$A_n = B_n = C_n = 0.$$

That gives us

$$v(t,x) = 0.$$

Second, for the solution of (1.3), assume that

$$w(t,x) = \sum_{n=1}^{\infty} A_n(t) \sin(nx), \ \ 0 < x < \pi.$$

Putting it into the equation and using given boundary conditions, we obtain

$$w(t,x) = \sum_{n=1}^{\infty} \left[A_n^{'''}(t) - (n^2+1) A_n(t) \right] \sin(nx) = -3e^{-t} \sin x,$$

and

$$w(0,x) = \sum_{n=1}^{\infty} A_n(0) \sin(nx) = \sin x, \ A_1(0) = 1; \ A_n(0) = 0, \ n = 2,3,...,$$

$$w_t(0,x) = \sum_{n=1}^{\infty} A_n^{'}(0) \sin(nx) = -\sin x, \ A_1^{'}(0) = -1; \ A_n^{'}(0) = 0, \ n = 2,3,...,$$

$$w_{tt}(1,x) = \sum_{n=1}^{\infty} A_n^{''}(1) \sin(nx) = e^{-1} \sin x, \ A_1^{''}(1) = e^{-1}; \ A_n^{''}(1) = 0, \ n = 2,3,....$$

Equating the coefficients of $\sin(nx)$, we get

$$\begin{cases} A_1^{'''}(t) - 2A_1(t) = -3e^{-t}, \ 0 < t < 1, \\ A_1(0) = 1, \ A_1^{'}(0) = -1, \ A_1^{''}(1) = e^{-1}, \end{cases}$$

and

$$A_n(t) = 0, \ n \neq 1.$$

Here,

$$A_1(t) = A_c(t) + A_p(t).$$

The homogeneous part of the problem is

$$A_c\left(t\right) = C_1 e^{\sqrt[3]{2}t} + e^{-\frac{\sqrt[3]{2}t}{2}}\left[C_2 \cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right) + C_3 \sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right)\right].$$

For the particular solution of the problem, we assume that

$$A_p(t) = Be^{-t}.$$

Putting into the equation, we get

$$-Be^{-t} - 2Be^{-t} = -3e^{-t},$$

then B=1. We have

$$A_1(t) = C_1 e^{\sqrt[3]{2}t} + e^{-\frac{\sqrt[3]{2}t}{2}}\left[C_2 \cos\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right) + C_3 \sin\left(\frac{\sqrt{3}}{\sqrt[3]{2}}t\right)\right] + e^{-t},$$

$$A_1'(t) = C_1\sqrt[3]{2}e^{\sqrt[3]{2}t} + C_2\frac{\sqrt[3]{2}}{2}e^{-\frac{\sqrt[3]{2}t}{2}}\left[-\cos\left(\frac{\sqrt{3}}{\sqrt[3]{4}}t\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{4}}t\right)\right]$$
$$+C_3\frac{\sqrt[3]{2}}{2}e^{-\frac{\sqrt[3]{2}t}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{4}}t\right) - \sin\left(\frac{\sqrt{3}}{\sqrt[3]{4}}t\right)\right] - e^{-t},$$

$$A_1''(t) = C_1\sqrt[3]{4}e^{\sqrt[3]{2}t} + C_2\frac{e^{-\frac{\sqrt[3]{2}t}{2}}}{\sqrt[3]{2}}\left[-\cos\left(\frac{\sqrt{3}}{\sqrt[3]{4}}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{4}}t\right)\right]$$
$$+C_3\frac{e^{-\frac{\sqrt[3]{2}t}{2}}}{\sqrt[3]{2}}\left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{4}}t\right) - \sin\left(\frac{\sqrt{3}}{\sqrt[3]{4}}t\right)\right] + e^{-t}.$$

Using given boundary conditions, the following system of equations are obtained

$$\begin{cases} C_1 + C_2 = 0, \\ \\ C_1 - \frac{C_2}{2} + C_3\frac{\sqrt{3}}{2} = 0, \\ \\ C_1\sqrt[3]{4}e^{\sqrt[3]{2}} + C_2\frac{e^{-\frac{\sqrt[3]{2}}{2}}}{\sqrt[3]{2}}\left[-\cos\left(\frac{\sqrt{3}}{\sqrt[3]{4}}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{\sqrt[3]{4}}\right)\right] \\ \quad +C_3\frac{e^{-\frac{\sqrt[3]{2}}{2}}}{\sqrt[3]{2}}\left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{\sqrt[3]{4}}\right) - \sin\left(\frac{\sqrt{3}}{\sqrt[3]{4}}\right)\right] = 0. \end{cases}$$

Solving the system of equations, we get

$$C_1 = C_2 = C_3 = 0.$$

Hence, we obtain

$$w\left(t,x\right)=e^{-t}\sin(x).$$

The solution of the problem is

$$u\left(t,x\right)=v(t,x)+w(t,x)=e^{-t}\sin(x).$$

Example 1.2. *Solve the mixed problem*

$$\begin{cases}\frac{\partial^3 u(t,x)}{\partial t^3}+\frac{\partial^2 u(t,x)}{\partial x^2}-u\left(t,x\right)=-e^{-(t+x)},\ 0<t<1,\ 0<x<\infty,\\ \\ u\left(0,x\right)=e^{-x},\ u_t\left(0,x\right)=-e^{-x},\ u_{tt}(1,x)=e^{-(1+x)}\sin x,\ 0\leq x<\infty,\\ \\ u\left(t,0\right)=e^{-t},\ u_x\left(t,0\right)=-e^{-t},\ 0\leq t\leq 1\end{cases}$$

by using Laplace transform method.

Solution. Let us denote $\mathbf{L}\left\{u\left(t,x\right)\right\}=U\left(t,s\right).$ Taking the Laplace transform of both sides of the differential equation we can write that

$$\mathbf{L}\left\{u_{ttt}\left(t,x\right)\right\}+\mathbf{L}\left\{u_{xx}\left(t,x\right)\right\}-\mathbf{L}\left\{u\left(t,x\right)\right\}=-\mathbf{L}\left\{e^{-(t+x)}\right\},\ 0<x<\infty.$$

It is equivalent to

$$U_{ttt}\left(t,s\right)-\left(1-s^2\right)U\left(t,s\right)=\left(s-1-\frac{1}{1+s}\right)e^{-t}.$$

So, our problem becomes

$$\begin{cases}U_{ttt}\left(t,s\right)-\left(1-s^2\right)U\left(t,s\right)=\left(s-1-\frac{1}{1+s}\right)e^{-t},\\ \\ U(0,s)=\frac{1}{1+s},\ U_t(0,s)=-\frac{1}{1+s},\ U_{tt}(1,s)=\frac{1}{e(1+s)}.\end{cases}\tag{1.5}$$

The homogeneous part of the problem is

$$U_c\left(t,s\right)=C_1e^{\sqrt[3]{1-s^2}t}+e^{-\frac{\sqrt[3]{1-s^2}t}{2}}\left[C_2\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)+C_3\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right].$$

Second, we assume a particular solution of the form

$$U_p\left(t,s\right)=A\left(s\right)e^{-t}.$$

Putting into the equation, we get

$$-A\left(s\right)e^{-t}-\left(1-s^2\right)A\left(s\right)e^{-t}=\left(s-1-\frac{1}{1+s}\right)e^{-t},$$

then

$$A(s) = \frac{1}{1+s}.$$

We obtain

$$U(t,s) = C_1 e^{\sqrt[3]{1-s^2}t} + e^{-\frac{\sqrt[3]{1-s^2}}{2}t}\left[C_2 \cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right.$$

$$\left. + C_3 \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right] + \frac{e^{-t}}{1+s},$$

$$U_t(t,s) = C_1\sqrt[3]{1-s^2}e^{\sqrt[3]{1-s^2}t} + C_2\frac{\sqrt[3]{1-s^2}}{2}e^{-\frac{\sqrt[3]{1-s^2}t}{2}}$$

$$\times\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right]$$

$$+C_3\frac{\sqrt[3]{1-s^2}}{2}e^{-\frac{\sqrt[3]{1-s^2}t}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right.$$

$$\left. - \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right] - \frac{e^{-t}}{1+s},$$

and

$$U_{tt}(t,s) = C_1\sqrt[3]{(1-s^2)^2}e^{\sqrt[3]{1-s^2}t}$$

$$+C_2\frac{\sqrt[3]{(1-s^2)^2}}{2}e^{-\frac{\sqrt[3]{1-s^2}t}{2}}\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right]$$

$$+C_3\frac{\sqrt[3]{(1-s^2)^2}}{2}e^{-\frac{\sqrt[3]{1-s^2}t}{2}}\left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right.$$

$$\left. - \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}t\right)\right] + \frac{e^{-t}}{1+s}.$$

Using given boundary conditions, we obtain the system of equations

$$\begin{cases} C_1 + C_2 = 0, \\ C_1 - \frac{C_2}{2} + C_3\frac{\sqrt{3}}{2} = 0, \\ C_1 e^{\sqrt[3]{1-s^2}} + C_2 e^{-\frac{\sqrt[3]{1-s^2}}{2}}\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}\right)\right] \\ \quad +C_3 e^{-\frac{\sqrt[3]{s^2-1}}{2}}\left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}\right) - \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{1-s^2}\right)\right] = 0. \end{cases}$$

Therefore,

$$C_1 = C_2 = C_3 = 0.$$

Then, the general solution of the problem is

$$U(t,s) = \frac{e^{-t}}{1+s}.$$

Taking the inverse of Laplace transform, we have

$$u(t,x) = e^{-(t+x)}.$$

Example 1.3. *Obtain the Fourier transform solution*

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} + \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) = (4x^2-4)\, e^{-\left(t-x^2\right)},\ 0<t<1, \quad -\infty < x < \infty, \\ \\ u(0,x) = e^{-x^2},\ u_t(0,x) = -e^{-x^2},\ u_{tt}(1,x) = e^{-\left(x^2+1\right)}. \end{cases}$$

Solution. Let denote $\mathbf{F}\{u(t,x)\} = U(t,s)$. By taking the Fourier transform of both sides, we obtain

$$\mathbf{F}\{u_{ttt}(t,x)\} + \mathbf{F}\{u_{xx}(t,x)\} - \mathbf{F}\{u(t,x)\} = \mathbf{F}\left\{\left(4x^2-4\right) e^{-\left(t-x^2\right)}\right\}.$$

Then we have

$$\begin{cases} U_{ttt}(t,s) - (s^2+1)\, U(t,s) = -e^{-t}(s^2+2)\, \mathbf{F}\left\{e^{-x^2}\right\}, \quad -\infty < x < \infty, \\ \\ U(0,s) = \mathbf{F}\left\{e^{-x^2}\right\},\ U_t(0,s) = -\mathbf{F}\left\{e^{-x^2}\right\},\ U_{tt}(1,s) = \mathbf{F}\left\{e^{-(x^2+1)}\right\}. \end{cases}$$

We obtain the solution $U_c(t,s)$ of corresponding homogeneous problem

$$U_c(t,s) = C_1 e^{\sqrt[3]{s^2+1}t} + e^{-\frac{\sqrt[3]{s^2+1}t}{2}} \left[C_2 \cos\left(\tfrac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right) + C_3 \sin\left(\tfrac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\right].$$

By undetermined coefficients method, we have

$$U_p(t,s) = A(s)\, e^{-t}.$$

Putting into the equation, we get

$$-A(s)\, e^{-t} - A(s)\left(s^2+1\right) e^{-t} = -e^{-t}\left(s^2+2\right) \mathbf{F}\left\{e^{-x^2}\right\},$$

then

$$A(s) = \mathbf{F}\left\{e^{-x^2}\right\}.$$

We have

$$U(t,s) = C_1 e^{\sqrt[3]{s^2+1}t} + e^{-\frac{\sqrt[3]{s^2+1}t}{2}} \left[C_2 \cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\right.$$

$$+C_3 \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\Bigg] + \mathbf{F}\left\{e^{-x^2}\right\} e^{-t},$$

$$U_t(t,s) = C_1\sqrt[3]{s^2+1}e^{\sqrt[3]{s^2+1}t} + C_2\frac{\sqrt[3]{s^2+1}}{2}e^{-\frac{\sqrt[3]{s^2+1}t}{2}}$$
$$\times\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\right]$$
$$+C_3\frac{\sqrt[3]{s^2+1}}{2}e^{-\frac{\sqrt[3]{s^2+1}t}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\right.$$
$$\left.-\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\right] - \mathbf{F}\left\{e^{-x^2}\right\} e^{-t},$$

and

$$U_{tt}(t,s) = C_1\sqrt[3]{(s^2+1)^2}e^{\sqrt[3]{s^2+1}t} + C_2\frac{\sqrt[3]{(s^2+1)^2}}{2}e^{-\frac{\sqrt[3]{s^2+1}t}{2}}$$
$$\times\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\right]$$
$$+C_3\frac{\sqrt[3]{(s^2+1)^2}}{2}e^{-\frac{\sqrt[3]{s^2+1}t}{2}}\left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\right.$$
$$\left.-\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}t\right)\right] + \mathbf{F}\left\{e^{-x^2}\right\} e^{-t}.$$

Thus, we get the following system of equations by using given boundary conditions

$$\begin{cases} C_1 + C_2 = 0, \\ C_1 - \frac{C_2}{2} + C_3\frac{\sqrt{3}}{2} = 0, \\ C_1 e^{\sqrt[3]{s^2+1}} + C_2 e^{-\frac{\sqrt[3]{s^2+1}}{2}}\left[-\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}\right)\right] \\ \quad +C_3 e^{-\frac{\sqrt[3]{s^2+1}}{2}}\left[-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}\right) - \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{s^2+1}\right)\right] = 0. \end{cases}$$

The system of equations gives us

$$C_1 = C_2 = C_3 = 0.$$

Hence,

$$U(t,s) = \mathbf{F}\left\{e^{-\left(t+x^2\right)}\right\}.$$

Taking the inverse of Fourier transform, we get

$$u(t,x) = e^{-\left(t+x^2\right)}.$$

The objective of this work is to show that our problem is stable. Let us briefly describe the contents of the various sections. It consists of five chapters and a appendix.

First chapter is the introduction.

Second chapter investigates boundary value problem for third-order partial differential equations

$$\begin{cases} \frac{d^3u(t)}{dt^3} - Au(t) = f(t),\ 0 < t < 1, \\ u(0) = \varphi, u_t(0) = \psi, u_{tt}(1) = \xi \end{cases} \tag{1.6}$$

in a Hilbert space H with a self-adjoint positive definite operator A. Moreover, the main theorem is on the stability estimates for the solution of the abstract boundary value problem (1.6). Lastly, this chapter proves that following theorem is on stability analysis of problem (1.6).

Theorem 1.1. *Suppose that* $\varphi \in D(A)$, $\xi \in D(A^{2/3})$, $\psi \in D(A^{1/3})$ *and* $f(t)$ *is a continuously differentiable on* $[0, 1]$. *Then, there is a unique solution of problem (1.6) and the following stability inequalities*

$$\max_{0\le t\le 1} \|u(t)\|_H \le M \left[\|\varphi\|_H + \left\|A^{-1/3}\psi\right\|_H + \left\|A^{-1/3}\xi\right\|_H + \max_{0\le t\le 1} \left\|A^{-2/3}f(t)\right\|_H \right],$$

$$\max_{0\le t\le 1} \left\| \frac{d^3u(t)}{dt^3} \right\|_H + \max_{0\le t\le 1} \|Au(t)\|_H$$

$$\le M \left\{ \|A\varphi\|_H + \left\|A^{2/3}\psi\right\|_H + \left\|A^{1/3}\xi\right\|_H + \max_{0\le t\le 1} \left\|f'(t)\right\|_H + \|f(0)\|_H \right\}$$

are valid, where M *does not depend on* $f(t)$, $t \in [0, 1]$, φ, ξ *and* ψ.

Third chapter includes two applications of the boundary value problem for a third order partial differential equation.

First, boundary value problem for third order partial differential equation

$$\begin{cases} u_{ttt} - (a(x)u_x)_x + \sigma u = f(t, x),\ 0 < t < 1,\ 0 < x < 1, \\ u(0, x) = \varphi(x),\ \ u_t(0, x) = \psi(x),\ u_{tt}(1, x) = \xi(x),\ 0 \le x \le 1, \\ u(t, 0) = u(t, 1),\ u_x(t, 0) = u_x(t, 1),\ 0 \le t \le 1 \end{cases} \tag{1.7}$$

is considered. Problem (1.7) has a unique smooth solution $u(t,x)$, smooth functions $a(x) \geq a > 0$, (a(1)=a(0), $x \in (0,1)$), $\varphi(x)$, $\xi(x)$, $\psi(x)$ $(x \in [0,1])$ and $f(t,x)$ $(t,x \in [0,1])$, σ positive constant and under some conditions. This allows us to reduce problem (1.7) to boundary value problem (1.6) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by equation (1.7)

Theorem 1.2. *For the solution of problem (1.7), we have the following stability inequalities*

$$\max_{0\leq t\leq 1} \|u(t,\cdot)\|_{L_2[0,1]} \leq M \Bigg[\max_{0\leq t\leq 1} \|f(t,\cdot)\|_{L_2[0,1]} + \|\varphi\|_{L_2[0,1]} + \|\psi\|_{L_2[0,1]} + \|\xi\|_{L_2[0,1]} \Bigg],$$

$$\max_{0\leq t\leq 1} \|u_{xx}(t,\cdot)\|_{L_2[0,1]} + \max_{0\leq t\leq 1} \|u_{ttt}(t,\cdot)\|_{L_2[0,1]} \leq M \Bigg[\max_{0\leq t\leq 1} \|f_t(t,\cdot)\|_{L_2[0,1]} + \|f(0,\cdot)\|_{L_2[0,1]} + \|\varphi_{xx}\|_{L_2[0,1]} + \|\psi_{xx}\|_{L_2[0,1]} + \|\xi_x\|_{L_2[0,1]} \Bigg],$$

where M is independent of $\varphi(x)$, $\xi(x)$, $\psi(x)$ and $f(t,x)$.

Second, let Ω be the unit open cube in m-dimensional Euclidean space $\mathbb{R}^m$: $\{x = (x_1, \cdots, x_m) : 0 < x_j < 1, 1 \leq j \leq m\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, let us consider boundary value problem for multidimensional third order equation

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} + \sigma u(x) = f(t,x), \\ x = (x_1, \ldots, x_m) \in \Omega, \quad 0 < t < 1, \\ u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad u_{tt}(1,x) = \xi(x), \ x \in \overline{\Omega}, \\ u(t,x) = 0, \quad x \in S. \end{cases} \tag{1.8}$$

Here, $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$, $\xi(x)$, $\psi(x)$ $(x \in \overline{\Omega})$ and $f(t,x)$ $(t \in (0,1), x \in \Omega)$ are given smooth functions and $a_r(x) \geq a > 0$. Let us introduce Hilbert space $L_2(\overline{\Omega})$ of all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int_{x\in\overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_m \right\}^{\frac{1}{2}}.$$

Theorem 1.3. *For the solution of problem (1.8), the following stability inequalities hold:*

$$\max_{0\leq t\leq 1} \|u(t,\cdot)\|_{L_2(\overline{\Omega})} \leq M \Bigg[\max_{0\leq t\leq 1} \|f(t,\cdot)\|_{L_2(\overline{\Omega})} + \|\varphi\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} + \|\xi\|_{L_2(\overline{\Omega})} \Bigg],$$

$$\begin{aligned}
& \max_{0\le t\le 1}\sum_{r=1}^{m}\|u_{x_rx_r}(t,\cdot)\|_{L_2(\overline{\Omega})}+\max_{0\le t\le 1}\|u_{ttt}(t,\cdot)\|_{L_2(\overline{\Omega})} \\
\le\ & M\left[\max_{0\le t\le 1}\|f_t(t,\cdot)\|_{L_2(\overline{\Omega})}+\|f(0,\cdot)\|_{L_2(\overline{\Omega})}\right. \\
& \left.+\sum_{r=1}^{m}\|\varphi_{x_rx_r}\|_{L_2(\overline{\Omega})}+\sum_{r=1}^{m}\left\|\psi_{x_rx_r}\right\|_{L_2(\overline{\Omega})}+\sum_{r=1}^{m}\|\xi_{x_rx_r}\|_{L_2(\overline{\Omega})}\right],
\end{aligned}$$

where M does not depend on $\varphi(x)$, $\xi(x)$, $\psi(x)$ and $f(t,x)$ $(t\in(0,1),x\in\Omega)$.

Fourth chapter is the numerical analysis.

Fifth chapter is the conclusion.

Appendix-A is the algorithm and programming for the given applications.

CHAPTER 2

THE MAIN THEOREM ON STABILITY

We consider boundary value problem for third-order partial differential equations

$$\begin{cases} \frac{d^3u(t)}{dt^3} - Au(t) = f(t),\ 0 < t < 1, \\ u(0) = \varphi, u_t(0) = \psi, u_{tt}(1) = \xi \end{cases} \tag{2.1}$$

in a Hilbert space H with a self-adjoint positive definite operator A. We are interested in studying the stability of solutions of problem (2.1). A function $u(t)$ is a *solution* of problem (2.1) if the following conditions are satisfied:

i) $u(t)$ is thrice continuously differentiable on the interval $(0, 1)$ and continuously differentiable on the segment $[0, 1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0, 1]$, and function $Au(t)$ is continuous on the segment $[0, 1]$.

iii) $u(t)$ satisfies the equation and boundary conditions (2.1).

Let H be a Hilbert space, A be a self-adjoint positive definite operator with $A \geq \delta I$, where $\delta > \delta_0 > 0$.

Throughout this work, $\{\tilde{c}(t), t \geq 0\}$ is a strongly continuous cosine operator-function defined by formula

$$\tilde{c}\,(t) = \frac{e^{itA^{1/3}} + e^{-itA^{1/3}}}{2}. \tag{2.2}$$

Then, from the definition of sine operator-function $\tilde{s}(t)$

$$\tilde{s}(t)u = \int_0^t \tilde{c}(\rho)u \, d\rho$$

it follows that

$$\tilde{s}(t) = A^{-1/3}\frac{e^{itA^{1/3}} - e^{-itA^{1/3}}}{2i}. \tag{2.3}$$

For the theory of cosine operator-function, we refer to (Fattorini, 1985) and (Piskarev and Shaw, 1997). Now, let us give a lemma that will be needed below.

Lemma 2.1. *The following estimates hold*

$$\|\tilde{c}(t)\|_{H\to H} \le 1, \quad \left\|A^{1/3}\tilde{s}(t)\right\|_{H\to H} \le 1. \tag{2.4}$$

The proof of Lemma 2.1 is based on the estimate

$$\left\|e^{\pm itA^{1/3}}\right\|_{H\to H} \le 1.$$

Applying the spectral representation of unit self-adjoint positive definite operator A (Ashyralyev and Sobolevskii, 2004), we get

$$\left\|e^{\pm tA^{1/3}}\right\|_{H\to H} \le \sup_{\delta\le\lambda<\infty}\left|e^{\pm tA^{1/3}}\right| = 1.$$

It is clear that problem (2.1) has a unique solution

$$u(t) = -3A^{-2/3}\left[I + 2e^{-\frac{3A^{1/3}}{2}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} \tag{2.5}$$

$$\times\left[2\int_0^1 e^{-\left(\frac{3}{2}-t-\frac{s}{2}\right)A^{1/3}}\tilde{c}\left((s-1)\frac{\sqrt{3}}{2}\right)f(s)ds - 2\int_0^t e^{-\left(\frac{3}{2}+s-t\right)A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)f(s)ds\right.$$

$$-\left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]\int_0^t e^{-\left(\frac{t-s}{2}\right)A^{1/3}}\left[\sqrt{3}A^{1/3}\tilde{s}\left((s-t)\frac{\sqrt{3}}{2}\right)\right.$$

$$\left.-\tilde{c}\left((s-t)\frac{\sqrt{3}}{2}\right)\right]f(s)ds + \int_t^1 e^{-(s-t)A^{1/3}}f(s)ds - \left[\tilde{c}\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$\times\left[\int_0^1 e^{-\left(s+\frac{t}{2}\right)A^{1/3}}f(s)ds + 2\int_0^1 e^{-\left(\frac{3+t-s}{2}\right)A^{1/3}}\tilde{c}\left((s-1)\frac{\sqrt{3}}{2}\right)f(s)ds\right]$$

$$+3\left[I - e^{-\frac{3t}{2}A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right]\right]\left[e^{-(1-t)A^{1/3}}\xi + A^{2/3}e^{-\left(\frac{3}{2}-t\right)A^{1/3}}\right.$$

$$\times\left[\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\left(A^{-1/3}\psi+\varphi\right)+\frac{1}{\sqrt{3}}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\left(A^{-1/3}\psi-\varphi\right)\right]\Big]\Big]$$

$$+e^{-\frac{t}{2}A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\varphi+\frac{A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}}(2A^{-1/3}\psi+\varphi)\right],$$

where the function $f(t)$ is not only continuous but also continuously differentiable on [0,1], $\varphi\in D(A)$, $\xi\in D(A^{2/3})$ and $\psi\in D(A^{1/3})$.

Theorem 2.1. *Suppose that $\varphi\in D(A)$, $\xi\in D(A^{2/3})$, $\psi\in D(A^{1/3})$ and $f(t)$ is a continuously differentiable on $[0,1]$. Then, there is a unique solution of problem (2.1) and the following stability inequalities*

$$\max_{0\le t\le 1}\|u(t)\|_H\le M\Big[\|\varphi\|_H+\left\|A^{-1/3}\psi\right\|_H \tag{2.6}$$

$$+\left\|A^{-1/3}\xi\right\|_H+\max_{0\le t\le 1}\left\|A^{-2/3}f(t)\right\|_H\Big],$$

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\|Au(t)\|_H \tag{2.7}$$

$$\le M\left\{\|A\varphi\|_H+\left\|A^{2/3}\psi\right\|_H+\left\|A^{1/3}\xi\right\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H+\|f(0)\|_H\right\}$$

are valid, where M does not depend on $f(t)$, $t\in[0,1]$, φ, ξ and ψ.

Proof. Firstly, we can rewrite formula (2.5)

$$u(t)=-\Big[2A_1(t)-2\tilde{c}\left(\frac{\sqrt{3}}{2}\right)A_2(t)-A_3(t)+A_4(t)$$

$$-\left[\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right][A_5(t)+2A_6(t)]$$

$$+[A_7(t)+A_8(t)]\Big[A^{-2/3}\xi+e^{-\frac{A^{1/3}}{2}}\Big(\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\left(A^{-1/3}\psi+\varphi\right)$$

$$+\frac{1}{\sqrt{3}}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\left(A^{-1/3}\psi-\varphi\right)\Big)\Big]\Big]$$

$$+e^{-\frac{t}{2}A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\varphi+\frac{A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}}\left(2A^{-1/3}\psi+\varphi\right)\right],$$

where

$$A_1(t)=1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}$$

$$\times\int_0^1 e^{-\left(\frac{3}{2}-t-\frac{s}{2}\right)A^{1/3}}\tilde{c}\left((s-1)\frac{\sqrt{3}}{2}\right)A^{-2/3}f(s)ds,$$

$$A_2(t)=1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\int_0^t e^{-\left(\frac{3}{2}+s-t\right)A^{1/3}}A^{-2/3}f(s)ds,$$

$$A_3(t)=1/3\int_0^t e^{-\frac{(t-s)}{2}A^{1/3}}\left[\sqrt{3}A^{1/3}\tilde{s}\left((s-t)\frac{\sqrt{3}}{2}\right)-\tilde{c}\left((s-t)\frac{\sqrt{3}}{2}\right)\right]A^{-2/3}f(s)ds,$$

$$A_4(t)=1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\int_t^1 e^{-(s-t)A^{1/3}}A^{-2/3}f(s)ds,$$

$$A_5(t)=1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\int_0^1 e^{-\left(s-\frac{t}{2}\right)A^{1/3}}A^{-2/3}f(s)ds, \tag{2.8}$$

$$A_6(t)=1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} \tag{2.9}$$

$$\times\int_0^1 e^{-\left(\frac{t+3-s}{2}\right)A^{1/3}}\tilde{c}\left((s-1)\frac{\sqrt{3}}{2}\right)A^{-2/3}f(s)ds,$$

$$A_7(t)=\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}e^{-(1-t)A^{1/3}},$$

$$A_8(t)=-\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}e^{-\left(1+\frac{t}{2}\right)A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right].$$

Let us obtain estimates for $\|A_k(t)\|_H$, $k=1,2,3,...,6$. We start with $A_1(t)$. Using estimates (2.4) and triangle inequality, we get

$$\|A_1(t)\|_H\le\left\|1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.10}$$

$$\times\int_0^1\left\|e^{-\left(\frac{3}{2}-t-\frac{s}{2}\right)A^{1/3}}\right\|_H\left\|A^{-2/3}f(s)\right\|_H ds$$

$$\le 1/3\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}\int_0^1 e^{-\left(\frac{3}{2}-t-\frac{s}{2}\right)\sigma^{1/3}}\left\|A^{-2/3}f(s)\right\|_H ds.$$

So, we have that

$$\|A_1(t)\|_H\le M_1\max_{0\le t\le 1}\left\|A^{-2/3}f(t)\right\|_H$$

for all t, $t\in[0,1]$. By using the triangle inequality and estimates (2.4), we obtain

$$\|A_2(t)\|_H\le\left\|1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.11}$$

$$\times \int_0^t \left\| e^{-\left(\frac{3}{2}+s-t\right)A^{1/3}} \right\|_H \left\| A^{-2/3} f(s) \right\|_H ds$$

$$\leq 1/3 \left(1 - 2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1} \int_0^t e^{-\left(\frac{3}{2}+s-t\right)\sigma^{1/3}} \left\| A^{-2/3} f(s) \right\|_H ds.$$

Thus, we get estimate for $A_2(t)$

$$\|A_2(t)\|_H \leq M_2 \max_{0 \leq t \leq 1} \left\| A^{-2/3} f(t) \right\|_H$$

for all t, $t \in [0,1]$. In a similar manner one establishes the estimate for $A_3(t)$

$$\|A_3(t)\|_H \leq 1/3 \int_0^t \left\| e^{-\frac{(t-s)}{2} A^{1/3}} \right\|_H \left\| \sqrt{3} A^{1/3} \tilde{s} \left((s-t) \frac{\sqrt{3}}{2} \right) \right. \tag{2.12}$$

$$\left. - \tilde{c} \left((s-t) \frac{\sqrt{3}}{2} \right) \right\|_{H \to H} \left\| A^{-2/3} f(s) \right\|_H ds$$

$$\leq 1/3 \int_0^t e^{-\frac{(t-s)}{2}\sigma^{1/3}} \left\| A^{-2/3} f(s) \right\|_H ds$$

$$\leq M_3 \max_{0 \leq t \leq 1} \left\| A^{-2/3} f(t) \right\|_H$$

for all t, $t \in [0,1]$ and,

$$\|A_4(t)\|_H \leq \left\| 1/3 \left[I + 2e^{-\frac{3}{2}A^{1/3}} \tilde{c} \left(\frac{\sqrt{3}}{2} \right) \right]^{-1} \right\|_{H \to H} \tag{2.13}$$

$$\times \int_t^1 \left\| e^{-(s-t)A^{1/3}} \right\|_H \left\| A^{-2/3} f(s) \right\|_H ds$$

$$\leq 1/3 \left(1 - 2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1} \int_t^1 e^{-(s-t)\sigma^{1/3}} \left\| A^{-2/3} f(s) \right\|_H ds$$

$$\leq M_4 \max_{0 \leq t \leq 1} \left\| A^{-2/3} f(t) \right\|_H$$

for all t, $t \in [0,1]$. By using formula (2.8), estimates in (2.4) and triangle inequality, we obtain estimate

$$\|A_5(t)\|_H \leq \left\| 1/3 \left[I + 2e^{-\frac{3}{2}A^{1/3}} \tilde{c} \left(\frac{\sqrt{3}}{2} \right) \right]^{-1} \right\|_{H \to H} \tag{2.14}$$

$$\times \int_0^1 \left\| e^{-\left(s-\frac{t}{2}\right)A^{1/3}} \right\|_H \left\| A^{-2/3} f(s) \right\|_H ds$$

$$\leq 1/3 \left(1 - 2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1} \int_0^1 e^{-\left(s-\frac{t}{2}\right)\sigma^{1/3}} \left\| A^{-2/3} f(s) \right\|_H ds$$

$$\leq M_5 \max_{0 \leq t \leq 1} \left\| A^{-2/3} f(t) \right\|_H$$

for all t, $t\in[0,1]$. Using estimates (2.4) and triangle inequality for the formula (2.9), we get

$$\|A_6(t)\|_H \leq 1/3\left\|\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.15}$$

$$\times\int_0^1\left\|e^{-\left(\frac{t+3-s}{2}\right)A^{1/3}}\right\|_H\left\|\tilde{c}\left((s-1)\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\left\|A^{-2/3}f(s)\right\|_H ds$$

$$\leq 1/3\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}\int_0^1 e^{-\left(\frac{t+3-s}{2}\right)\sigma^{1/3}}\left\|A^{-2/3}f(s)\right\|_H ds$$

$$\leq M_6\max_{0\leq t\leq 1}\left\|A^{-2/3}f(t)\right\|_H$$

for all t, $t\in[0,1]$. Now, we will estimate $\|A_m(t)\|_{H\to H}$, $m=7,8$. Using the triangle inequality and estimates (2.4), we obtain

$$\|A_7(t)\|_{H\to H}\leq\left\|\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H}\left\|e^{-(1-t)A^{1/3}}\right\|_H \tag{2.16}$$

$$\leq\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}e^{-(1-t)\sigma^{1/3}}\leq M_7$$

for all t, $t\in[0,1]$, and

$$\|A_8(t)\|_{H\to H}\leq\left\|\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.17}$$

$$\times\left\|e^{-\left(1+\frac{t}{2}\right)A^{1/3}}\right\|_H\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}$$

$$\leq\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}e^{-\left(1+\frac{t}{2}\right)\sigma^{1/3}}\leq M_8$$

for all t, $t\in[0,1]$. Combining the estimates (2.10)-(2.17) and estimates (2.4), we obtain the estimate

$$\|u(t)\|_H\leq 2\|A_1(t)\|_H+2\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\|A_2(t)\|_H+\|A_3(t)\|_H+\|A_4(t)\|_H$$

$$+\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right\|_{H\to H}\left[\|A_5(t)\|_H+2\|A_6(t)\|_H\right]$$

$$+\left[\|A_7(t)\|_{H\to H}+\|A_8(t)\|_{H\to H}\right]$$

$$\times\left[\left\|A^{-2/3}\xi\right\|_H+\left\|e^{-\frac{A^{1/3}}{2}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\left[\left\|A^{-1/3}\psi\right\|_H+\|\varphi\|_H\right]\right.$$

$$+\left\|\frac{e^{-\frac{A^{1/3}}{2}}}{\sqrt{3}}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\left[\left\|A^{-1/3}\psi\right\|_H-\|\varphi\|_H\right]\right]+\left\|e^{-\frac{t}{2}A^{1/3}}\right\|_H$$

$$\times\left[\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\|\varphi\|_H+\frac{\left\|A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}}{\sqrt{3}}\left(2\left\|A^{-1/3}\psi\right\|_H+\|\varphi\|_H\right)\right].$$

It follows that

$$\max_{0\le t\le 1}\|u(t)\|_H\le M\left[\|\varphi\|_H+\left\|A^{-1/3}\psi\right\|_H+\left\|A^{-1/3}\xi\right\|_H+\max_{0\le t\le 1}\left\|A^{-2/3}f(t)\right\|_H\right].$$

Thus, the estimate (2.6) holds. Secondly, applying the operator A to the formula (2.5) and using the Abel′s formula, we obtain

$$Au(t)=-1/3\left[I+2e^{-\frac{3A^{1/3}}{2}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} \tag{2.18}$$

$$\times\left[\left(e^{-(1-t)A^{1/3}}+e^{-\left(\frac{3}{2}-t\right)A^{1/3}}\left[\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)-\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]\right)\right.$$

$$\times\left(f(1)-f(0)-\int_0^1 f'(s)ds\right)+2\left[e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)f(t)-e^{-\left(\frac{3}{2}-t\right)A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)f(0)\right.$$

$$\left.-\int_0^t e^{-\left(\frac{3}{2}+s-t\right)A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)f'(s)ds\right]-\left(I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$\times\left[-e^{-\frac{t}{2}A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right]+3\right.$$

$$\left.-\frac{\sqrt{3}}{2}e^{-\frac{t}{2}A^{1/3}}\left[\sqrt{3}\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)-A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right]\right]$$

$$\times\left(f(t)-f(0)-\int_0^t f'(s)ds\right)-e^{-(1-t)A^{1/3}}f(1)+f(t)$$

$$+\int_t^1 e^{-(s-t)A^{1/3}}f'(s)ds-\left[\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$\times\left[-e^{-\left(1+\frac{t}{2}\right)A^{1/3}}f(1)+e^{-\frac{t}{2}A^{1/3}}f(0)+\int_0^1 e^{-\left(s+\frac{t}{2}\right)A^{1/3}}f'(s)ds\right.$$

$$+\left[e^{-\left(1+\frac{t}{2}\right)A^{1/3}}+e^{-\left(\frac{3+t}{2}\right)A^{1/3}}\left[\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)-\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]\right]$$

$$\left.\left.\times\left(f(1)-f(0)-\int_0^1 f'(s)ds\right)\right]\right]+1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}$$

$$\times\left[\left(I-e^{-\frac{3t}{2}A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right]\right)\right.$$

$$\times\left[e^{-(1-t)A^{1/3}}A^{1/3}\xi+e^{-\left(\frac{3}{2}-t\right)A^{1/3}}\left(\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\left(A^{2/3}\psi+A\varphi\right)\right.\right.$$

$$\left.\left.\left.+\frac{1}{\sqrt{3}}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\left(A^{2/3}\psi-A\varphi\right)\right)\right]\right]$$

$$+e^{-\frac{t}{2}A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}\right)A\varphi+\frac{A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}}\left(2A^{2/3}\psi+A\varphi\right)\right]$$

$$=-\left[B_1(t)+2\tilde{c}\left(\frac{\sqrt{3}}{2}\right)B_2(t)+\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]\right.$$

$$\times\left[B_3(t)+B_4(t)\right]+B_5(t)-\left[\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)\right.$$

$$\left.+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right]\left[B_6(t)+B_7(t)\right]$$

$$+\left[B_8(t)-\left[\tilde{c}\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right]B_9(t)\right]$$

$$\times\left[A^{1/3}\xi+e^{-\frac{A^{1/3}}{2}}\left(\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\left[A^{2/3}\psi+A\varphi\right]\right.\right.$$

$$\left.\left.+\frac{1}{\sqrt{3}}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\left[A^{2/3}\psi-A\varphi\right]\right)\right]$$

$$+e^{-\frac{t}{2}A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}\right)A\varphi+\frac{A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}}\left[2A^{2/3}\psi+A\varphi\right]\right],$$

where

$$B_1(t)=1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}$$

$$\times\left(e^{-(1-t)A^{1/3}}+e^{-\left(\frac{3}{2}-t\right)A^{1/3}}\left[\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)-\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]\right)$$

$$\times\left(f(1)-f(0)-\int_0^1 f'(s)ds\right),$$

$$B_2(t)=1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}$$

$$\times\left[e^{-\frac{3}{2}A^{1/3}}f(t)-e^{-\left(\frac{3}{2}-t\right)A^{1/3}}f(0)-\int_0^t e^{-\left(\frac{3}{2}+s-t\right)A^{1/3}}f'(s)ds\right],$$

$$B_3(t) = 1/3\left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}$$

$$\times\left[-e^{-\frac{t}{2}A^{1/3}}\left[\tilde{c}\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right]\right]$$

$$\times\left(f(t) - f(0) - \int_0^t f'(s)ds\right),$$

$$B_4(t) = 1/3\left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}$$

$$\times\left[3 - \frac{\sqrt{3}}{2}e^{-\frac{t}{2}A^{1/3}}\left[\sqrt{3}\tilde{c}\left(\frac{\sqrt{3}}{2}t\right) - A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right]\right]$$

$$\times\left(f(t) - f(0) - \int_0^t f'(s)ds\right),$$

$$B_5(t) = 1/3\left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}$$

$$\times\left[-e^{-(1-t)A^{1/3}}f(1) + f(t) + \int_t^1 e^{-(s-t)A^{1/3}}f'(s)ds\right],$$

$$B_6(t) = 1/3\left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} \tag{2.19}$$

$$\times\left[-e^{-\left(1+\frac{t}{2}\right)A^{1/3}}f(1) + e^{-\frac{t}{2}A^{1/3}}f(0) + \int_0^1 e^{-\left(s+\frac{t}{2}\right)A^{1/3}}f'(s)ds\right],$$

$$B_7(t) = 1/3\left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} \tag{2.20}$$

$$\times\left[e^{-\left(1+\frac{t}{2}\right)A^{1/3}} + e^{-\left(\frac{3+t}{2}\right)A^{1/3}}\left[\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right) - \tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]\right]$$

$$\times\left(f(1) - f(0) - \int_0^1 f'(s)ds\right),$$

$$B_8(t) = \left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} e^{-(1-t)A^{1/3}},$$

$$B_9(t) = \left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} e^{-\left(1+\frac{t}{2}\right)A^{1/3}}.$$

Let us obtain estimates for $\|B_k(t)\|_H$, $k = 1, 2, 3, ..., 7$. We start with $B_1(t)$. Using estimates in (2.4) and the triangle inequality, we obtain

$$\|B_1(t)\|_H \leq \left\|1/3\left[I + 2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.21}$$

$$\times\left(\left\|e^{-(1-t)A^{1/3}}\right\|_H+\left\|e^{-\left(\frac{3}{2}-t\right)A^{1/3}}\right\|_H\left\|\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)-\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\right)$$

$$\times\left(\|f(1)\|_H+\|f(0)\|_H+\int_0^1\|f'(s)\|_H\,ds\right)$$

$$\leq 1/3\left(e^{-(1-t)\sigma^{1/3}}+e^{-\left(\frac{3}{2}-t\right)\sigma^{1/3}}\right)$$

$$\times\left(\|f(1)\|_H+\|f(0)\|_H+\int_0^1\|f'(s)\|_H\,ds\right)\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}.$$

Therefore, we get

$$\|B_1(t)\|_H\leq N_1\left(\|f(0)\|_H+\|f(1)\|_H+\max_{0\leq t\leq 1}\|f'(t)\|_H\right)$$

for all t, $t\in[0,1]$. By using estimates (2.4) and triangle inequality, we have

$$\|B_2(t)\|_H\leq\left\|1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.22}$$

$$\times\left[\left\|e^{-\frac{3}{2}A^{1/3}}\right\|_H\|f(t)\|_H+\left\|e^{-\left(\frac{3}{2}-t\right)A^{1/3}}\right\|_H\|f(0)\|_H\right.$$

$$\left.+\int_0^t\left\|e^{-\left(\frac{3}{2}+s-t\right)A^{1/3}}\right\|_H\|f'(s)\|_H\,ds\right]$$

$$\leq 1/3\left[e^{-\frac{3}{2}\sigma^{1/3}}\|f(t)\|_H+e^{-\left(\frac{3}{2}-t\right)\sigma^{1/3}}\|f(0)\|_H\right.$$

$$\left.+\int_0^t e^{-\left(\frac{3}{2}+s-t\right)\sigma^{1/3}}\|f'(s)\|_H\,ds\right]\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}.$$

Thus, we get estimate for $B_2(t)$

$$\|B_2(t)\|_H\leq N_2\left(\max_{0\leq t\leq 1}\|f(t)\|_H+\|f(0)\|_H+\max_{0\leq t\leq 1}\|f'(t)\|_H\right)$$

for all t, $t\in[0,1]$. In a similar manner one establishes the estimate for $B_3(t)$

$$\|B_3(t)\|_H\leq\left\|1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.23}$$

$$\times\left\|e^{-\frac{t}{2}A^{1/3}}\right\|_H\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right\|_{H\to H}$$

$$\times\left(\|f(t)\|_H+\|f(0)\|_H+\int_0^t\|f'(s)\|_H\,ds\right)$$

$$\leq 1/3e^{-\frac{t}{2}\sigma^{1/3}}\left(\|f(t)\|_H+\|f(0)\|_H+\int_0^t\|f'(s)\|_H\,ds\right)\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}$$

$$\leq N_3\left(\max_{0\leq t\leq 1}\|f(t)\|_H+\|f(0)\|_H+\max_{0\leq t\leq 1}\|f'(t)\|_H\right)$$

for all t, $t\in[0,1]$. Applying the estimates (2.4) and triangle inequality, we obtain

$$\|B_4(t)\|_H\leq\left\|1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.24}$$

$$\left[3+\frac{\sqrt{3}}{2}\left\|e^{-\frac{t}{2}A^{1/3}}\right\|_H\left\|\sqrt{3}\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)-A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right\|_{H\to H}\right]$$

$$\times\left(\|f(t)\|_H+\|f(0)\|_H+\int_0^t\|f'(s)\|_H\,ds\right)$$

$$\leq 1/3\left[3+\frac{\sqrt{3}}{2}e^{-\frac{t}{2}\sigma^{1/3}}\right]\left(\|f(t)\|_H+\|f(0)\|_H+\int_0^t\|f'(s)\|_H\,ds\right)\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}.$$

So, we have

$$\|B_4(t)\|_H\leq N_4\left(\max_{0\leq t\leq 1}\|f(t)\|_H+\|f(0)\|_H+\max_{0\leq t\leq 1}\|f'(t)\|_H\right)$$

for all t, $t\in[0,1]$. Using the triangle inequality and estimates (2.4), we can write

$$\|B_5(t)\|_H\leq\left\|1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.25}$$

$$\times\left[\left\|e^{-(1-t)A^{1/3}}\right\|_H\|f(1)\|_H+\|f(t)\|_H+\int_t^1\left\|e^{-(s-t)A^{1/3}}\right\|_H\|f'(s)\|_H\,ds\right]$$

$$\leq 1/3\left[e^{-(1-t)\sigma^{1/3}}\|f(1)\|_H+\|f(t)\|_H+\int_t^1 e^{-(s-t)\sigma^{1/3}}\|f'(s)\|_H\,ds\right]\left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}.$$

We get estimate for $B_5(t)$

$$\|B_5(t)\|_H\leq N_5\left(\max_{0\leq t\leq 1}\|f(t)\|_H+\|f(1)\|_H+\max_{0\leq t\leq 1}\|f'(t)\|_H\right)$$

for all t, $t\in[0,1]$. Let us estimate (2.19). Using estimates (2.4) and triangle inequality, we have

$$\|B_6(t)\|_H\leq\left\|1/3\left[I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1}\right\|_{H\to H} \tag{2.26}$$

$$\times\left[\left\|e^{-\left(1+\frac{t}{2}\right)A^{1/3}}\right\|_H\|f(1)\|_H+\left\|e^{-\frac{t}{2}A^{1/3}}\right\|_H\|f(0)\|_H\right.$$

$$\left.+\int_0^1\left\|e^{-\left(s+\frac{t}{2}\right)A^{1/3}}\right\|_H\|f'(s)\|_H\,ds\right]$$

$$\leq 1/3\left[e^{-\left(1+\frac{t}{2}\right)\sigma^{1/3}}\|f(1)\|_H+e^{-\frac{t}{2}\sigma^{1/3}}\|f(0)\|_H\right.$$

$$+\int_0^1 e^{-\left(s+\frac{t}{2}\right)\sigma^{1/3}} \|f'(s)\|_H \, ds\Big] \left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}$$

$$\leq N_6 \left(\|f(0)\|_H + \|f(1)\|_H + \max_{0\leq t\leq 1} \|f'(t)\|_H \right)$$

for all t, $t \in [0,1]$. In a similar manner one can establish the estimate for (2.20)

$$\|B_7(t)\|_H \leq \left\| 1/3 \left[I + 2e^{-\frac{3}{2}A^{1/3}} \tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} \right\|_{H\to H} \tag{2.27}$$

$$\times \left[\left\| e^{-\left(1+\frac{t}{2}\right)A^{1/3}} \right\|_H + \left\| e^{-\left(\frac{3+t}{2}\right)A^{1/3}} \right\|_H \left\| \sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right) - \tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H} \right]$$

$$\times \left(\|f(1)\|_H + \|f(0)\|_H + \int_0^1 \|f'(s)\|_H \, ds \right)$$

$$\leq 1/3 \left[e^{-\left(1+\frac{t}{2}\right)\sigma^{1/3}} + e^{-\left(\frac{3+t}{2}\right)\sigma^{1/3}} \right]$$

$$\times \left(\|f(1)\|_H + \|f(0)\|_H + \int_0^1 \|f'(s)\|_H \, ds \right) \left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1}.$$

Thus, we obtain estimate for $B_7(t)$

$$\|B_7(t)\|_H \leq N_7 \left(\|f(0)\|_H + \|f(1)\|_H + \max_{0\leq t\leq 1} \|f'(t)\|_H \right)$$

for all t, $t \in [0,1]$. Finally, we will estimate $\|B_m(t)\|_{H\to H}$, $m = 8, 9$. Using the triangle inequality and estimates (2.4), we get

$$\|B_8(t)\|_{H\to H} \leq \left\| \left[I + 2e^{-\frac{3}{2}A^{1/3}} \tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} \right\|_{H\to H} \left\| e^{-(1-t)A^{1/3}} \right\|_H \tag{2.28}$$

$$\leq e^{-(1-t)\sigma^{1/3}} \left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1} \leq N_8$$

for all t, $t \in [0,1]$, and

$$\|B_9(t)\|_{H\to H} \leq \left\| \left[I + 2e^{-\frac{3}{2}A^{1/3}} \tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right]^{-1} \right\|_{H\to H} \left\| e^{-\left(\frac{t+2}{2}\right)A^{1/3}} \right\|_H \tag{2.29}$$

$$\leq e^{-\left(\frac{t+2}{2}\right)\sigma^{1/3}} \left(1-2e^{-\frac{3}{2}\sigma^{1/3}}\right)^{-1} \leq N_9$$

for all t, $t \in [0,1]$. Combining estimates (2.21)-(2.29) and using estimates (2.4), we obtain

$$\|Au(t)\|_H \leq \Big[\|B_1(t)\|_H + 2 \left\| \tilde{c}\left(\frac{\sqrt{3}}{2}\right) \right\|_{H\to H} \|B_2(t)\|_H$$

$$+\left\|I+2e^{-\frac{3}{2}A^{1/3}}\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\left[\|B_3(t)\|_H+\|B_4(t)\|_H\right]+\|B_5(t)\|_H$$

$$-\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}t\right)\right\|_{H\to H}\left[\|B_6(t)\|_H+\|B_7(t)\|_H\right]$$

$$+\left(\|B_8(t)\|_{H\to H}+\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\|B_9(t)\|_{H\to H}\right)$$

$$\times\left[\left\|A^{1/3}\xi\right\|_H+\left\|e^{-\frac{A^{1/3}}{2}}\right\|_{H\to H}\left(\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\left(\left\|A^{2/3}\psi\right\|_H+\|A\varphi\|_H\right)\right.\right.$$

$$\left.\left.+\frac{1}{\sqrt{3}}\left\|A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\right)\left(\left\|A^{2/3}\psi\right\|_H+\|A\varphi\|_H\right)\right]\right]+\left\|e^{-\frac{t}{2}A^{1/3}}\right\|_H$$

$$\times\left[\left\|\tilde{c}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}\|A\varphi\|_H+\frac{\left\|A^{1/3}\tilde{s}\left(\frac{\sqrt{3}}{2}\right)\right\|_{H\to H}}{\sqrt{3}}\left(2\left\|A^{2/3}\psi\right\|_H+\|A\varphi\|_H\right)\right].$$

It follows that

$$\max_{0\le t\le 1}\left\|\frac{d^3u(t)}{dt^3}\right\|_H+\max_{0\le t\le 1}\|Au(t)\|_H$$

$$\le M\left\{\|A\varphi\|_H+\left\|A^{2/3}\psi\right\|_H+\left\|A^{1/3}\xi\right\|_H+\max_{0\le t\le 1}\left\|f'(t)\right\|_H+\|f(0)\|_H\right\}.$$

As a result, the estimate (2.7) holds. This is the end of proof of Theorem 2.1. $\square$

CHAPTER 3

APPLICATION

Now, we consider the applications of Theorem 2.1. First, boundary value problem for third order partial differential equation

$$\begin{cases} u_{ttt} - (a(x)u_x)_x + \sigma u = f(t,x),\ 0 < t < 1,\ 0 < x < 1, \\ u(0,x) = \varphi(x),\quad u_t(0,x) = \psi(x),\ u_{tt}(1,x) = \xi(x),\ 0 \le x \le 1, \\ u(t,0) = u(t,1),\ u_x(t,0) = u_x(t,1),\ 0 \le t \le 1 \end{cases} \tag{3.1}$$

is considered. Problem (3.1) has a unique smooth solution $u(t,x)$, smooth functions $a(x) \ge a > 0$, (a(1)=a(0), $x \in (0,1)$), $\varphi(x)$, $\xi(x)$, $\psi(x)$ $(x \in [0,1])$ and $f(t,x)$ $(t,x \in [0,1])$, σ positive constant and under some conditions. This allows us to reduce problem (3.1) to boundary value problem (2.1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by equation (3.1).

Theorem 3.1. *For the solution of problem (3.1), we have the following stability inequalities*

$$\max_{0\le t\le 1} \|u(t,\cdot)\|_{L_2[0,1]} \le M \left[\max_{0\le t\le 1} \|f(t,\cdot)\|_{L_2[0,1]} + \|\varphi\|_{L_2[0,1]} \right.$$

$$\left. + \|\psi\|_{L_2[0,1]} + \|\xi\|_{L_2[0,1]} \right],$$

$$\max_{0\le t\le 1} \|u_{xx}(t,\cdot)\|_{L_2[0,1]} + \max_{0\le t\le 1} \|u_{ttt}(t,\cdot)\|_{L_2[0,1]} \le M \left[\max_{0\le t\le 1} \|f_t(t,\cdot)\|_{L_2[0,1]} \right.$$

$$\left. + \|f(0,\cdot)\|_{L_2[0,1]} + \|\varphi_{xx}\|_{L_2[0,1]} + \|\psi_{xx}\|_{L_2[0,1]} + \|\xi_x\|_{L_2[0,1]} \right],$$

where M is independent of $\varphi(x)$, $\xi(x)$, $\psi(x)$ and $f(t,x)$.

The proof of Theorem 3.1 is based on the abstract Theorem 2.1 and the symmetry properties of the space operator generated by problem (2.1).

Second, let Ω be the unit open cube in m-dimensional Euclidean space $\mathbb{R}^m$: $\{x = (x_1, \cdots, x_m) : 0 < x_j < 1, 1 \le j \le m\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, let us consider boundary value problem for multidimensional third order equation

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \sum\limits_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} + \sigma u(x) = f(t,x), \\ x = (x_1, \ldots, x_m) \in \Omega, \quad 0 < t < 1, \\ u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad u_{tt}(1,x) = \xi(x), \ x \in \overline{\Omega}, \\ u(t,x) = 0, \quad x \in S. \end{cases} \tag{3.2}$$

Here, $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$, $\xi(x)$, $\psi(x)$ $(x \in \overline{\Omega})$ and $f(t,x)$ $(t \in (0,1), x \in \Omega)$ are given smooth functions and $a_r(x) \ge a > 0$.

Let us introduce Hilbert space $L_2(\overline{\Omega})$ of all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_m \right\}^{\frac{1}{2}}.$$

Theorem 3.2. *For the solution of problem (3.2), the following stability inequalities hold:*

$$\max_{0 \le t \le 1} \|u(t,\cdot)\|_{L_2(\overline{\Omega})} \le M \left[\max_{0 \le t \le 1} \|f(t,\cdot)\|_{L_2(\overline{\Omega})} + \|\varphi\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} + \|\xi\|_{L_2(\overline{\Omega})} \right],$$

$$\max_{0 \le t \le 1} \sum_{r=1}^{m} \|u_{x_r x_r}(t,\cdot)\|_{L_2(\overline{\Omega})} + \max_{0 \le t \le 1} \|u_{ttt}(t,\cdot)\|_{L_2(\overline{\Omega})}$$

$$\le M \left[\max_{0 \le t \le 1} \|f_t(t,\cdot)\|_{L_2(\overline{\Omega})} + \|f(0,\cdot)\|_{L_2(\overline{\Omega})} \right.$$

$$\left. + \sum_{r=1}^{m} \|\varphi_{x_r x_r}\|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \|\psi_{x_r x_r}\|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \|\xi_{x_r x_r}\|_{L_2(\overline{\Omega})} \right],$$

where M does not depend on $\varphi(x)$, $\xi(x)$, $\psi(x)$ and $f(t,x)$ $(t \in (0,1), x \in \Omega)$.

The proof of Theorem 3.2 is based on the abstract Theorem 2.1 and the symmetry properties of the space operator generated by problem (2.1). Note that Theorem

3.2 in case m=1 and $\varphi = 0, \xi = 0, \psi = 0$ was proved in article (Denche and Memou, 2003).

Theorem 3.3. *(Sobolevskii, 1975) For the solution of the elliptic differential problem*

$$A^x u(x) = \omega(x), x \in \Omega,$$

$$u(x) = 0, x \in S,$$

the following coercivity inequality holds:

$$\sum_{r=1}^{m} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M||\omega||_{L_2(\overline{\Omega})},$$

where M is independent of ω.

CHAPTER 4

NUMERICAL EXPERIMENTS

When the analytical methods do not work properly, the numerical methods for obtaining approximate solutions of partial differential equations play an important role in applied mathematics. We can say that there are many considerable works in the literature (for instance, see (Ashyralyev and Aggez, 2004); (Ashyralyev and Aggez, 2011); (Ashyralyev and Yildirim, 2012); (Ashyralyev and Yilmaz, 2012); (Ashyralyev and Ozdemir, 2010)).

In the present chapter for the approximate solutions of a problem, we will use the first and high orders of accuracy difference schemes. The high order of accuracy for the approximate solution of the problem will be constructed in order to get more accurate result. We will apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of first and high orders of accuracy difference schemes will be given.

4.1 THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

We consider the boundary value problem for a third order partial differential equation for numerical analysis

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \frac{\partial u^2(t,x)}{\partial x^2} = f(t,x), \\ f(t,x) = e^{-t}\left(3\cos t - 2\sin t\right)\sin x, \\ 0 < t < 1,\ 0 < x < \pi, \\ u(0,x) = \sin x,\ u_t(0,x) = -sinx, \\ u_{tt}(1,x) = 2e^{-1}\sin 1 \sin x,\ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1. \end{cases} \tag{4.1}$$

The exact solution of problem (4.1) is

$$u\left(t,x\right) = e^{-t}\sin x \cos t.$$

For the approximate solutions of boundary value problem (4.1), applying the formulas

$$\begin{cases} \frac{u(t_{k+2})-3u(t_{k+1})+3u(t_k)-u(t_{k-1})}{\tau^3} - u'''(t_k) = O(\tau), \\ \frac{u(\tau)-u(0)}{\tau} - u'(0) = O(\tau), \\ \frac{u(x_{n+1})-2u(x_n)+u(x_{n-1})}{h^2} - u''(x_n) = O(h^2), \end{cases}$$

and using the first order of accuracy difference scheme we get the system

$$\begin{cases} \frac{u_n^{k+2}-3u_n^{k+1}+3u_n^{k}-u_n^{k-1}}{\tau^3} - \frac{u_{n+1}^{k}-2u_n^{k}+u_{n-1}^{k}}{h^2} = f(t_k, x_n), \\ f(t_k, x_n) = e^{-t_k}\left(3\cos t_k - 2\sin t_k\right)\sin x_n,\ 1 \le k \le N-2\ ,\ 1 \le n \le M-1, \\ N\tau = 1,\ x_n = nh,\ 1 \le n \le M-1,\ Mh = \pi, \\ u_n^0 = \sin x_n,\ u_n^N - 2u_n^{N-1} + u_n^{N-2} = \tau^2 2e^{-1}\sin 1 \sin x_n,\ 1 \le n \le M-1, \\ \varphi(x_n) = e^{-t_k}\left(3\cos t_k - 2\sin t_k\right)\sin x_n,\ \ 1 \le k \le N-2\ ,\ 1 \le n \le M-1, \\ u_0^k = u_M^k = 0,\ u_n^1 - u_n^0 = -\tau \sin x_n. \end{cases}$$

This system can be written in the matrix form

$$\begin{cases} A\, u_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ u_0 = \vec{0},\ \ u_M = \vec{0}. \end{cases} \tag{4.2}$$

Here,

$$A = C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & 3b & -b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & c & 3b & -b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & 3b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3b & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & c & 3b & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & c & 3b & -b \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$a=-\frac{1}{h^2},\ b=-\frac{1}{\tau^3},\quad c=\frac{3}{\tau^3}+\frac{2}{h^2},$$

$$\varphi_n=\begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \cdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},\quad \begin{cases} \varphi_n^k = f(t_k,x_n)=e^{-t_k}\left(3\cos t_k-2\sin t_k\right)\sin x_n, \\ 2\le k\le N-3,\ 1\le n\le M-1, \\ \\ \varphi_n^0 = \sin x_n, \\ \varphi_n^{N-1} = -\tau\sin x_n, \\ \varphi_n^N = \tau^2 2e^{-1}\sin 1\sin x_n, \end{cases}$$

and $D=I_{N+1}$ is the identity matrix,

$$U_s=\begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \cdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1},\quad s=\ n-1,\ n,\ n+1.$$

This type of system was used by Samarskii and Nikolaev (Samarskii and Nikolaev, 1989) for difference equations. For the solution of the matrix equation (4.2), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n=\alpha_{n+1}u_{n+1}+\beta_{n+1},\quad n=M-1,\cdots,2,1,$$

where $u_M = \vec{0}$, $\alpha_j \quad (j = 1, \cdots, M-1)$ are $(N+1) \times (N+1)$ square matrices, β_j $(j = 1, \cdots, M-1)$ are $(N+1) \times 1$ column matrices, $\alpha_1,\ \beta_1$ are zero matrices, and

$$\alpha_{n+1} = -\left(B_n + C_n \alpha_n\right)^{-1} A_n,$$
$$\beta_{n+1} = \left(B_n + C_n \alpha_n\right)^{-1} \left(D_n \varphi_n - C_n \beta_n\right), n = 1, 2, 3, \cdots M-1.$$

4.1.1 Error Analysis

The errors are computed by

$$E_M^N = \max_{1 \le k \le N-1, 1 \le n \le M-1} \left| u(t_k, x_n) - u_n^k \right|$$

of the numerical solutions, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n) and the results are given in the following table

Table 4.1 Error analysis for the first order of accuracy difference scheme

Difference schemes	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
1^{st} order of A.D.S.	0.0208	0.0117	0.0062	0.0032

As it is seen in Table 4.1, we get some numerical results. If N and M are doubled, the value of errors decrease by a factor of approximately 1/2 for first order difference scheme.

4.2 THE HIGH ORDER OF ACCURACY DIFFERENCE SCHEME

For the approximate solutions of boundary value problem (4.1), applying the formulas

$$\begin{cases} \frac{u(t_{k+2})-3u(t_{k+1})+3u(t_k)-u(t_{k-1})}{\tau^3} - \frac{1}{2}\left(u'''(t_k)+u'''(t_{k+1})\right) = O(\tau^3), \\ \frac{-3u(0)+4u(t_1)-u(t_2)}{2\tau} - u'(0) + \frac{\tau^2}{12}u'''(0) + \frac{\tau^2}{4}u'''(t_1) = O(\tau^4), \\ \frac{u(1)-2u(t_{N-1})+u(t_{N-2})}{\tau^2} - u''(1) + \frac{3}{8}\tau u'''(1) + \frac{2}{3}\tau u'''(t_{N-1}) \\ -\frac{1}{24}\tau u'''(t_{N-2}) = O(\tau^4), \end{cases}$$

and using the high order of accuracy difference scheme we get the system

$$\begin{cases} \frac{u_n^{k+2}-3u_n^{k+1}+3u_n^k-u_n^{k-1}}{\tau^3} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{2h^2} = \frac{(f(t_k,x_n)+f(t_{k+1},x_n))}{2}, \\ f(t_k,x_n) = e^{-t_k}\left(3\cos t_k - 2\sin t_k\right)\sin x_n,\ 1 \le k \le N-2\ ,\ 1 \le n \le M-1, \\ N\tau = 1,\ x_n = nh,\ 1 \le n \le M-1,\ Mh = \pi, \\ u_n^0 = \sin x_n,\ \ -\frac{3}{2}u_n^0 + \frac{\tau^3}{12}\left(\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2}\right) + 2u_n^1 + \frac{\tau^3}{4}\left(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}\right) \\ -\frac{u_n^2}{2} = \tau^3\left(-\frac{1}{12}f(0,x_n) - \frac{1}{4}f(t_1,x_n)\right) - \tau\sin x_n, \\ u_n^N + \frac{3\tau^3}{8}\left(\frac{u_{n+1}^N-2u_n^N+u_{n-1}^N}{h^2}\right) - 2u_n^{N-1} + \frac{2\tau^3}{3}\left(\frac{u_{n+1}^{N-1}-2u_n^{N-1}+u_{n-1}^{N-1}}{h^2}\right) + u_n^{N-2} \\ -\frac{\tau^3}{24}\left(\frac{u_{n+1}^{N-2}-2u_n^{N-2}+u_{n-1}^{N-2}}{h^2}\right) = \tau^3\left(-\frac{3}{8}f(t_N,x_n) - \frac{2}{3}f(t_{N-1},x_n)\right. \\ \left.+\frac{1}{24}f(t_{N-2},x_n)\right) + 2\tau^2e^{-1}\sin 1\sin x_n, \\ \varphi(x_n) = e^{-t_k}\left(3\cos t_k - 2\sin t_k\right)\sin x_n,\ 1 \le k \le N-2\ ,\ 1 \le n \le M-1, \\ u_0^k = u_M^k = 0, \end{cases}$$

This system can be written in the matrix form

$$\begin{cases} A\,u_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ u_0 = \vec{0},\ \ u_M = \vec{0}. \end{cases} \tag{4.3}$$

Here,

$$A = C = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & a \\ t & f & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & x & y & z \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & d & -b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & c & d & -b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c & d & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & d & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & c & d & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & c & d & -b \\ aa & bb & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & xx & yy & zz \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$a = -\frac{1}{2h^2},\ b = -\frac{1}{\tau^3},\ c = \frac{3}{\tau^3} + \frac{1}{h^2},\ d = -\frac{3}{\tau^3} + \frac{1}{h^2},$$

$$x = -\frac{\tau^3}{24h^2},\ y = \frac{2\tau^3}{3h^2},\ z = \frac{3\tau^3}{8h^2},\ t = \frac{\tau^3}{12h^2},\ f = \frac{\tau^3}{4h^2},$$

$$xx = 1 + \frac{\tau^3}{12h^2},\ yy = -\frac{4\tau^3}{3h^2} - 2,\ zz = 1 - \frac{3\tau^3}{4h^2},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \cdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\begin{cases} \varphi_n^k = \frac{1}{2}\left(f(t_k, x_n) + f(t_{k+1}, x_n)\right) \ = \frac{1}{2}\left(e^{-t_k}\left(3\cos t_k - 2\sin t_k\right)\right. \\ \left.+e^{-t_{k+1}}\left(3\cos t_{k+1} - 2\sin t_{k+1}\right)\right)\sin x_n, \\ 2 \le k \le N-3,\ 1 \le n \le M-1, \\ \varphi_n^0 = \ \sin x_n, \\ \varphi_n^{N-1} = -\frac{\tau^3}{4}\left(1 + e^{-\tau}(3\cos\tau - 2\sin\tau)\right)\sin x_n - \tau\sin x_n, \\ \varphi_n^N = \tau^3\left(-\frac{3}{8}e^{-1}(3\cos 1 - 2\sin 1) - \frac{2}{3}e^{-(1-\tau)}\right. \\ \times\left(3\cos(1-\tau) - 2\sin(1-\tau)\right) + \frac{1}{24}e^{-(1-2\tau)} \\ \left.\times\left(3\cos(1-2\tau) - 2\sin(1-2\tau)\right)\right)\sin x_n + 2\tau^2 e^{-1}\sin 1 \sin x_n, \end{cases}$$

and $D = I_{N+1}$ is the identity matrix,

$$U_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \cdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = \ n-1,\ n,\ n+1.$$

This type of system was used by Samarskii and Nikolaev (Samarskii and Nikolaev, 1989) for difference equations. For the solution of the matrix equation (4.3), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}, \quad n = M-1, \cdot\cdot\cdot, 2, 1,$$

where $u_M = \vec{0}$, $\alpha_j \quad (j = 1, \cdot\cdot\cdot, M-1)$ are $(N+1)\times(N+1)$ square matrices, β_j $(j = 1, \cdot\cdot\cdot, M-1)$ are $(N+1)\times 1$ coloumn matrices, $\alpha_1,\ \beta_1$ are zero matrices, and

$$\alpha_{n+1} = -\left(B_n + C_n\alpha_n\right)^{-1}A_n,$$
$$\beta_{n+1} = \left(B_n + C_n\alpha_n\right)^{-1}\left(D_n\varphi_n - C_n\beta_n\right), n = 1, 2, 3, \cdots M-1.$$

4.2.1 Error Analysis

The errors are computed by

$$E_M^N = \max_{1\le k\le N-1, 1\le n\le M-1}\left|u(t_k, x_n) - u_n^k\right|$$

of the numerical solutions, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n) and the results are given in the following table

Table 4.2 Error analysis for the high order of accuracy difference scheme

Difference schemes	$N = 4, M = 100$	$N = 8, M = 400$
High order of A.D.S.	0.3757×10^{-4}	0.2345×10^{-5}

When N is doubled and M is increased by four times, the value of errors decrease by a factor of approximately 1/16 for high order difference scheme, see (Table 4.2).

CHAPTER 5

CONCLUSION

This thesis is devoted to the stability of the boundary value problem for a third order partial differential equation. The following original results are obtained:

- The main theorem on the stability estimates for the solution of boundary value problems for third order partial differential equations in a Hilbert space with self adjoint operator is established.
- Two applications of the main theorem to a third order partial differential equations are given. Theorems on stability estimates for the solutions of these partial differential equations are obtained.
- The first and high order of accuracy difference scheme are constructed for the approximate solution of the boundary value problem.
- The Matlab implementation of the first and high orders difference schemes are presented.

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APPENDIX A

MATLAB PROGRAMMING

A.1 MATLAB IMPLEMENTATION OF THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

```
function firstord2(N,M)
close; close;
%N=80 ; M=80;
tau=1/N;
h=pi/M;
a = -1/(h^2);
b =-1/(tau^3);
c = -2*a-3*b;
A=zeros(N+1,N+1);
for i=2:N-1; A(i,i)=a;
end;
C=A;
for i=2:N-1 ;
B(i,i-1)= b;
B(i,i)=c;
B(i,i+1)=3*b;
B(i,i+2)=-b ;
end;
B(1,1)=1;B(N,1)=-1;B(N,2)=1;
```

```
B(N+1,N-1)=1;B(N+1,N)=-2;B(N+1,N+1)=1;
B;
for i=1:N+1; D(i,i)=1; end ;
D;
for j=1:M+1;
for k=2:N-1;
fii(k,j:j) =exp((-k)*tau)*(3*cos((k)*tau)- 2*sin((k)*tau))*...
sin((j)*h);
end;
fii(1,j:j) =sin(j*h);
fii(N,j:j) =-tau*sin(j*h);
fii(N+1,j:j) =2*tau^2*exp(-1)*sin(1)*sin(j*h);
end;
fii;
alpha(N+1,N+1,1:1)= 0 ;
betha(N+1,1:1) = 0 ;
for j=1:M;
alpha( :, :, j+1:j+1 ) = -inv(B+C*alpha(:, :, j:j))*(A) ;
betha( :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j ) )*(D*fii(:, j:j )...
- C * betha(:, j:j ) );
end;
betha;
U( N+1,1, M:M ) = 0;
for z = M-1:-1:1 ;
U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 )...
+betha(:,z+1:z+1);
end;U;
for z = 1:M ;
p(:,z+1:z+1)=U(:,:,z:z);
end;p;
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1 ;
```

```
for k=1:N+1 ;
es(k,j:j) = exp((-k+1)*tau)*cos((k-1)*tau)*sin((j-1)*h);
end;
end;es;
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;axis tight;
title('EXACT SOLUTION');
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p) ; rotate3d ;axis tight;
title('FIRST ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxerror,relativeerror]
%mm=exp(1)
```

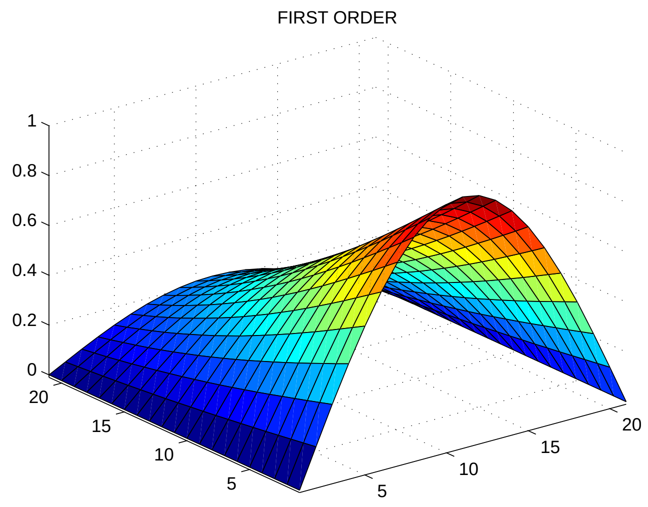

Figure A.1 Approximate solution generated by first order of accuracy difference scheme for N=M=20

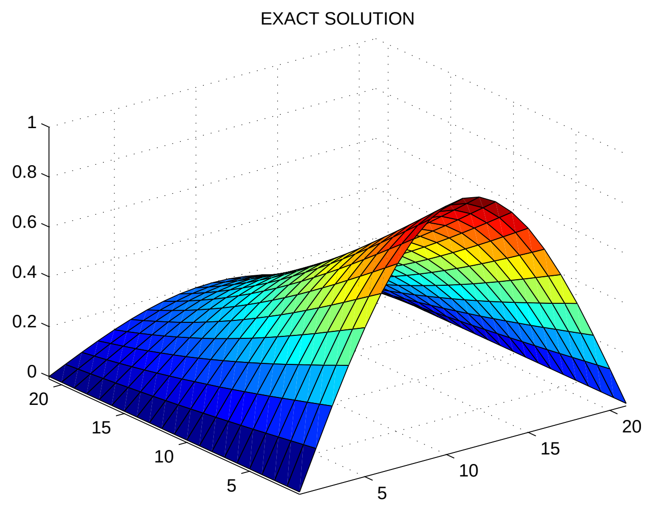

Figure A.2 Exact solution of problem (4.1) for N=M=20

A.2 MATLAB IMPLEMENTATION OF THE HIGH ORDER OF ACCURACY DIFFERENCE SCHEME

```
function firstord2(N,M)
close; close;
tau=1/N;
h=pi/M;
a = -1/(2*(h^2));
b =-1/(tau^3);
c = -2*a-3*b;
d=3*b-2*a;
t=(tau^3)/(12*h^2);
f=(tau^3)/(4*h^2);
aa=-3/2-(tau^3)/(6*h^2);
bb=2-(tau^3)/(2*h^2);
x=-(tau^3)/(24*h^2);
y=2*(tau^3)/(3*h^2);
z=3*(tau^3)/(8*h^2);
xx=1+(tau^3)/(12*h^2);
yy=-2-4*(tau^3)/(3*h^2);
zz=1-3*(tau^3)/(4*h^2);
A=zeros(N+1,N+1);
for i=2:N-1;
A(i,i)=a;
A(i,i+1)=a;
end;
A(N,1)=t;A(N,2)=f;
A(N+1,N-1)=x;A(N+1,N)=y;A(N+1,N+1)=z;
A;
C=A;
for i=2:N-1;
B(i,i-1)= b;
```

```
B(i,i)=c;
B(i,i+1)=d;
B(i,i+2)=-b;
end;
B(1,1)=1;B(N,1)=aa;B(N,2)=bb;B(N,3)=-1/2;
B(N+1,N-1)=xx;B(N+1,N)=yy;B(N+1,N+1)=zz;
B;
for i=1:N+1; D(i,i)=1; end ;
D;
for j=1:M+1;
for k=2:N-1;
fii(k,j:j) =(exp((-k)*tau)*(3*cos((k)*tau)-2*sin((k)*tau))*sin((j)*h)...
+exp((-k+1)*tau)*(3*cos((k-1)*tau)- 2*sin((k-1)*tau))*...
sin((j)*h))/2;
end;
fii(1,j:j) =sin((j)*h);
fii(N,j:j) =(tau^3)*(-(1/4)*sin((j)*h)-(1/4)*exp(-tau)*(3*cos(tau)...
-2*sin(tau))*sin((j)*h))-tau*sin((j)*h);
fii(N+1,j:j) =(tau^3)*((-3/8)*exp(-1)*(3*cos(1)-2*sin(1))-(2/3)*exp(tau-1)...
*(3*cos(1-tau)-2*sin(1-tau))+(1/24)*exp(2*tau-1)*(3*cos(1-2*tau)...
-2*sin(1-2*tau)))*sin((j)*h)+2*(tau^2)*exp(-1)*sin(1)*sin((j)*h);
end;fii;
alpha(N+1,N+1,1:1)= 0 ;
betha(N+1,1:1) = 0 ;
for j=1:M;
alpha( :, :, j+1:j+1 ) = -inv(B+C*alpha(:, :, j:j))*(A) ;
betha( :, j+1:j+1 ) = inv(B+C*alpha(:, :, j:j ) )*(D*fii(:, j:j )...
- C * betha(:, j:j ) );
end;betha;
U( N+1,1, M:M ) = 0;
for z = M-1:-1:1 ;
U(:,:, z:z ) = alpha(:,:,z+1:z+1)* U(:,:,z+1:z+1 )+betha(:,z+1:z+1);
```

```
end;U;
for z = 1:M ;
p(:,z+1:z+1)=U(:,:,z:z);
end;p;
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1 ;
for k=1:N+1 ;
es(k,j:j) = exp((-k+1)*tau)*cos((k-1)*tau)*sin((j-1)*h);
end;
end;es;
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(es) ; rotate3d ;axis tight;
title('EXACT SOLUTION');
figure ;
m(1,1)=min(min(p))-0.01;
m(2,2)=nan;
surf(m);
hold;
surf(p) ; rotate3d ;axis tight;
title('HIGH ORDER');
% .ERROR ANALYSIS.;
maxes=max(max(es)) ;
maxapp=max(max(p)) ;
maxerror=max(max(abs(es-p)));
relativeerror=max(max((abs(es-p))))/max(max(abs(p)) );
cevap = [maxerror,relativeerror]
%mm=exp(1)
```

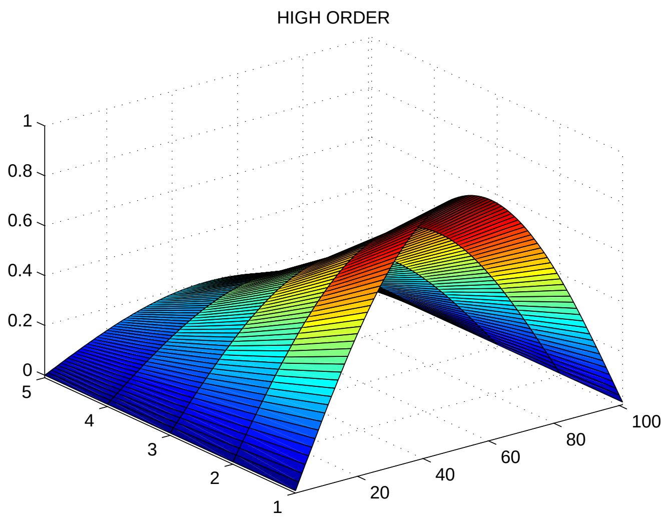

Figure A.3 Approximate solution generated by high order of accuracy difference scheme for N=4 and M=100

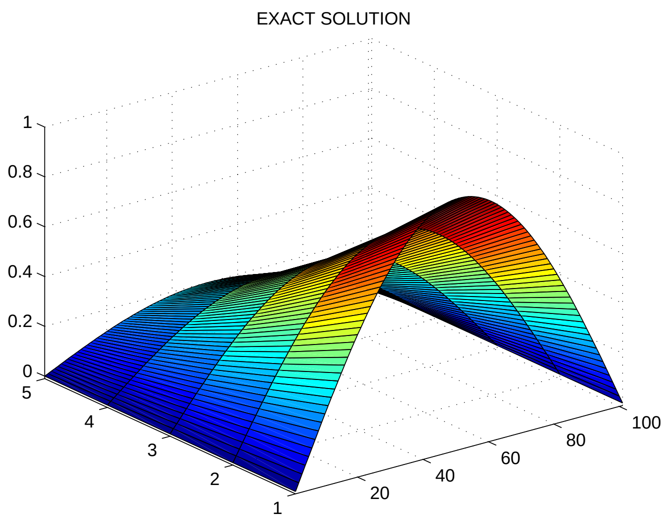

Figure A.4 Exact solution of problem (4.1) for N=4 and M=100