ON THE FINE SPECTRUM OF THE GENERALIZED DIFFERENCE OPERATOR DEFINED BY A DOUBLE SEQUENTIAL BAND MATRIX OVER SOME SEQUENCE SPACES

by

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APPROVAL PAGE

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ABSTRACT

In this dissertation the spectrum and the fine spectrum of the generalized difference operator $B(\widetilde{r},\widetilde{s})$ defined by a double sequential band matrix, the generalized difference operator $A(\widetilde{r},\widetilde{s})$ defined by an upper double sequential band matrix and the operator generated by the triple band matrix $A(r,s,t)$ acting on the sequence spaces ℓ_∞, c_0, c, ℓ_p with respect to the Goldberg's classification are determined, where $1 \leq p < \infty$.

In chapter 1, the required definitions and basic properties of metric space, normed space and linear operator introduced by spectral theory are discussed. In this chapter, some basic concepts related to the subject of spectrum are given by taking that subject into consideration. Also, the definition of some sequence spaces is introduced and definitions and theorems related to matrix transformations are given in the first chapter.

In chapter 2, we determine the spectra of the operator $B(\widetilde{r},\widetilde{s})$ defined by a double sequential band matrix acting on the sequence space ℓ_p with respect to the Goldberg's classification. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $B(\widetilde{r},\widetilde{s})$ over the space ℓ_p where $1 < p < \infty$.

In chapter 3, we study the fine spectrum of the generalized difference operator $A(\widetilde{r},\widetilde{s})$ defined by an upper double sequential band matrix acting on the sequence spaces c_0, c and ℓ_p with respect to Goldberg's classification. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $A(\widetilde{r},\widetilde{s})$ over the spaces c_0, c and ℓ_p together with a Mercerian Theorem, where $0 < p < \infty$.

In chapter 4, we determine the fine spectra of upper triangular triple-band matrix over the sequence spaces μ. The operator $A(r,s,t)$ on the sequence space μ is defined $A(r,s,t)x = (rx_k + sx_{k+1} + tx_{k+2})_{k=0}^{\infty}$, where $x = (x_k) \in \mu \in \{\ell_p, c, c_0\}$ with $0 < p < \infty$. In this chapter, we obtain the results on the spectrum and point spectrum for the operator $A(r,s,t)$ on the sequence space μ. Further, the results on continuous spectrum, residual spectrum and fine spectrum of the operator $A(r,s,t)$ on the sequence space μ also derive. Further, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $A(r,s,t)$ over the space μ and give some applications.

Keywords: Spectrum of an operator, double sequential band matrix, spectral mapping theorem, the sequence spaces ℓ_p, c_0, c, Goldberg's classification.

İKİLİ DİZİSEL BAND MATRİSİYLE TANIMLANAN GENELLEŞTİRİLMİŞ FARK OPERATÖRÜNÜN BAZI DİZİ UZAYLARI ÜZERİNDEKİ İNCE SPEKTRUMU ÜZERİNE

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ÖZ

Bu çalışmada; alt ve üst üçgen matrisleriyle temsil edilen operatörlerin bazı dizi uzayları üzerindeki spektrumları, Goldberg sınıflandırmasına göre incelenmiştir. Birinci bölümde; metrik uzaylar, normlu uzaylar ve lineer dönüşümlerin tanımı verilerek özelliklerinden bahsedilmiştir. Ayrıca, bölümün ikinci kısmında spektrum konusu ele alınarak, konuyla ilgili bazı temel tanım ve kavramlar verilmiştir. İkinci bölümde; ℓ_p dizi uzayı üzerinde $B(\widetilde{r},\widetilde{s})$ ikili dizi band matrisi ile tanımlanan operatörün ince spektrumu incelenmiştir. Ayrıca bu operatörün ℓ_p dizi uzayı üzerindeki spektrumu Goldberg sınıflandırmasına göre verilmiştir. Üçüncü bölümde; asli köşegeninde $\widetilde{r}=(r_k)$ ve ona paralel ikinci köşegeninde $\widetilde{s}=(s_k)$ dizilerinin terimlerini ihtiva eden $A(\widetilde{r},\widetilde{s})$ üst üçgen matrisin c_0, c, ℓ_∞ ve ℓ_p dizi uzayları üzerindeki ince spektrumu incelenmiştir. Dördüncü bölümde; $A(r,s,t)$ üst üçgen üçlü-band matrisinin ℓ_p, c_0 ve c dizi uzayları üzerindeki spektrumu Goldberg sınıfladırmasına göre incelenmiştir. Ayrıca, Teopliz matrisleriyle ilgili bazı uygulamalara yer verilmiştir. Bu çalışmalara ilave olarak artık spektrum, nokta spektrum, sürekli spektrumdan farklı olan ve ayrık olmak zorunda olmayan diğer alt spektrum sınıflarının tanımları verilerek, spektrumu verilen matrislerin alt spektrum sınıfları incelenmiştir.

Anahtar Kelimeler: ℓ_p, c_0, c dizi uzayları, Goldberg sınıflandırması, spektral teori, band matrisleri.

To My Family

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TABLE OF CONTENTS

LIST OF TABLES

TABLE

CHAPTER 1

INTRODUCTION

Spectral theory is one of the main branches of modern functional analysis and its applications. Roughly speaking it is concerned with certain inverse operator, their general properties and their relation the original operator. Such inverse operator arise naturally in connection with the problem of solving equation (system of linear algebraic equations, differential equations, integral equation) for instance, the investigation of boundary value problem by Sturm and Liouville, and Fredholm's famous theory of integral equations were important to development of the field.

1.1 BACKGROUND

In this section, following (Başar, 2011) we give some required definitions related with the spectrum.

Definition 1.1.1. (Metric space) Let X be a non-empty set and d be a distance function from $X \times X$ to the set $\mathbb{R}^+$ of non-negative real numbers. Then the pair (X, d) is called a *metric space* and d is a metric for X, if the following metric axioms are satisfied for all elements $x, y, z \in X$:

(M.1) $d(x, y) = 0$ if and only if $x = y$.

(M.2) $d(x, y) = d(y, x)$, (the symmetry property).

(M.3) $d(x, z) \leq d(x, y) + d(y, z)$, (the triangle inequality).

Definition 1.1.2. (The space w) By w, we mean the set of all sequences with complex terms, i.e., $w = \{x = (x_k) : x_k \in \mathbb{C} \text{ for all } k \in \mathbb{N}\}$. The most popular metric on the space w is defined by

$$d_w(x,y)=\sum_k \frac{|x_k-y_k|}{2^k(1+|x_k-y_k|)};\quad x=(x_k),y=(y_k)\in w.$$

Here and after, for short we use $\sum_k$ instead of $\sum_{k=0}^{\infty}$.

Definition 1.1.3. (The space ℓ_∞) The space ℓ_∞ of bounded sequences is defined by

$$\ell_\infty=\left\{x=(x_k)\in w:\sup_k |x_k|<\infty\right\}.$$

The natural metric on the space ℓ_∞ is defined by

$$d_\infty(x,y)=\sup_{k\in\mathbb{N}}|x_k-y_k|;\quad x=(x_k),y=(y_k)\in \ell_\infty.$$

Definition 1.1.4. (The spaces c and c_0) The spaces c and c_0 of convergent and null sequences are defined by

$$\begin{aligned} c &= \left\{x=(x_k)\in w:\exists l\in\mathbb{C}\ni \lim_{k\to\infty}|x_k-l|=0\right\}, \\ c_0 &= \left\{x=(x_k)\in w:\lim_{k\to\infty}|x_k|=0\right\}. \end{aligned}$$

The metric d_∞ is also a metric for the spaces c_0 and c. It is trivial that since the concept supremum and maximum are equivalent on the space c_0, the metric d_∞ is reduced to the metric d_0 defined by

$$d_0(x,y)=\max_{k\in\mathbb{N}}|x_k-y_k|;\quad x=(x_k),y=(y_k)\in c_0.$$

Definition 1.1.5. (The space ℓ_p) The space ℓ_p of absolutely $p-$summmable sequences is defined as

$$\ell_p=\left\{x=(x_k)\in w:\sum_k |x_k|^p<\infty\right\},\ 0<p<\infty).$$

In the case $1\leq p<\infty$, the metric d_p on the space ℓ_p is given by

$$d_p(x,y)=\left(\sum_k |x_k-y_k|^p\right)^{1/p};\quad x=(x_k),y=(y_k)\in \ell_p.$$

Also in the case $0<p<1$, the metric $\widetilde{d_p}$ on the space ℓ_p is given by

$$\widetilde{d_p}(x,y)=\sum_k |x_k-y_k|^p;\ \ x=(x_k),y=(y_k)\in \ell_p.$$

Definition 1.1.6. (Normed space) Let X be a real or complex linear space and $\|\cdot\|$ be a function from X to the set $\mathbb{R}^+$ of non-negative real numbers. Then the pair $(X, \|\cdot\|)$ is called a *normed space* and $\|\cdot\|$ is a *norm* for X, if the following norm axioms are satisfied for all elements $x, y \in X$ and for all scalars α:

(N.1) $\|x\| = 0$ if and only if $x = \theta$.

(N.2) $\|\alpha x\| = |\alpha|\|x\|$, (the absolute homogenity property).

(N.3) $\|x + y\| \leq \|x\| + \|y\|$, (the triangle inequality).

Definition 1.1.7. (Banach space) A Banach space X is complete normed linear space. Completeness means that if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$, where $x_n \in X$, then there exist $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Definition 1.1.8. (Linear operator) In calculus we consider real line $\mathbb{R}$ and real-valued functions on $\mathbb{R}$ (or on a subset $\mathbb{R}$). Obviously, any such function is a mapping of its domain into $\mathbb{R}$. In functional analysis, we consider more general space, such metric space and normed space and mapping of these space in the case of vector space and in particular, normed space a mapping is called an operator.

A linear operator T is an operator such that

(i) the domain $D(T)$ of T is a vector space and range $R(T)$ lies in a vector space over the same field

(ii) for all $x, y \in D(T)$ and scalars α,

$$\begin{aligned} T(x+y) &= Tx + Ty \\ T(\alpha x) &= \alpha Tx \end{aligned}$$

Observe the notation; we write Tx instead of $T(x)$; this simplification is standard in functional analysis.

Definition 1.1.9. (Bounded linear operator) Let X and Y be the normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded operator if there is a positive real number c such that

$$\|Tx\| = c\|x\|$$

for all $x \in D(T)$. Let X and Y be linear spaces. By $L(X,Y)$ and $B(X,Y)$, we denote the set of all linear operators and the set of all bounded linear operators from X into Y.

Definition 1.1.10. (Norm of a bounded operator) Let $T \in B(X,Y)$. Then, the norm of T is defined as

$$\|T\| = \sup_{x \neq \theta} \frac{\|Tx\|}{\|x\|} < \infty. \tag{1.1}$$

The supremum on the right side of (1.1) is finite which follows from the fact that $\|Tx\| = c\|x\|$ when $T \in B(X,Y)$.

1.2 SPECTRUM AND FINE SPECTRUM

Let X and Y be Banach spaces, and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Given an operator $T \in B(X)$, the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T_\lambda = \lambda I - T \text{ is a bijection}\}$$

is called the *resolvent set* of T and its complement with respect to the complex plain

$$\sigma(T) := \mathbb{C} \backslash \rho(T)$$

is called the *spectrum* of T. By the closed graph theorem, the inverse operator

$$R(\lambda; T) := (\lambda I - T)^{-1},\ (\lambda \in \rho(T)) \tag{1.2}$$

is always bounded and is usually called *resolvent operator* of T at λ.

1.3 SUBDIVISIONS OF THE SPECTRUM

In this section, we define the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

1.3.1 The point spectrum, continuous spectrum and residual spectrum

The name *resolvent* is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ's in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects. A *regular value* λ of T is a complex number such that T_λ^{-1} exists and bounded and whose domain is dense in X. For our investigation of T, T_λ and T_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see (Kreyszig, 1978, pp. 370-371)):

The *resolvent set* $\rho(T, X)$ of T is the set of all regular values λ of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. An $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and is unbounded and the domain of T_λ^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but the domain of T_λ^{-1} is not dense in X.

Therefore, these three parts form a disjoint subdivisions such that

$$\sigma(T, X) = \sigma_p(T, X) \cup \sigma_c(T, X) \cup \sigma_r(T, X). \tag{1.3}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss.

Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite dimensional case.

1.3.2 The Approximate Point Spectrum, Defect Spectrum and Compression Spectrum

In this subsection, following Appell et al. (Appell et al., 2004), we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T,X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - T\} \tag{1.4}$$

the *approximate point spectrum of* T. Moreover, the subspectrum

$$\sigma_\delta(T,X) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\} \tag{1.5}$$

is the called *defect spectrum* of T.

The two subspectra given by (1.4) and (1.5) form a (not necessarily disjoint) subdivisions

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T,X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X\}$$

which is often called the *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$$

of the spectrum. Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$.

Moreover, comparing these parts with those in (1.3) we note that

$$\begin{aligned}
\sigma_r(T,X) &= \sigma_{co}(T,X) \backslash \sigma_p(T,X), \\
\sigma_c(T,X) &= \sigma(T,X) \backslash [\sigma_p(T,X) \cup \sigma_{co}(T,X)].
\end{aligned}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints are also useful.

Proposition 1.3.1. *(Appell et al., 2004, Proposition 1.3, p. 28) Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:*

(a) $\sigma(T^*, X^*) = \sigma(T, X)$.

(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$.

(c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$.

(d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$.

(e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$.

(f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$.

(g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on the Hilbert spaces are most similar to matrices in finite dimensional spaces (see (Appell et al., 2004)).

1.3.3 Goldberg's Classification of Spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

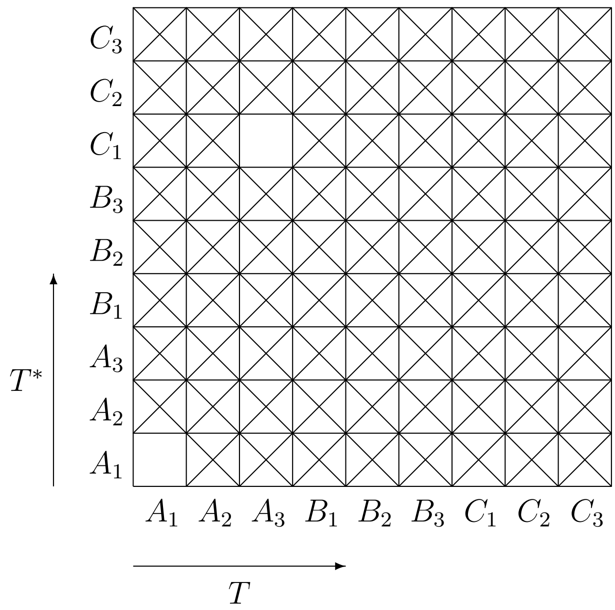
(A) $R(T) = X$.

(B) $R(T) \neq \overline{R(T)} = X$.

(C) $\overline{R(T)} \neq X$.

and

Table 1.1 State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X



(1) T^{-1} exists and is continuous.

(2) T^{-1} exists but is discontinuous.

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3. If an operator is in state C_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see (Goldberg, 1985)).

If λ is a complex number such that $T_\lambda = \lambda I - T \in A_1$ or $T_\lambda = \lambda I - T \in B_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $A_2\sigma(T, X) = \emptyset$, $A_3\sigma(T, X)$, $B_2\sigma(T, X)$, $B_3\sigma(T, X)$, $C_1\sigma(T, X)$, $C_2\sigma(T, X)$, $C_3\sigma(T, X)$. For example, if $T_\lambda = \lambda I - T$ is in a given state, C_2 (say), then we write $\lambda \in C_2\sigma(T, X)$.

By the definitions given above, we can illustrate the subdivisions in Table 1.2.

Observe that the case in the first row and second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, i.e., if λ

Table 1.2 Subdivisions of spectrum of a linear operator

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is unbounded	T_λ^{-1} does not exist
A	$R(\lambda I - T) = X$	$\lambda \in \rho(T, X)$	–	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$
B	$\overline{R(\lambda I - T)} = X$	$\lambda \in \rho(T, X)$	$\lambda \in \sigma_c(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$
C	$\overline{R(\lambda I - T)} \neq X$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$

is not an eigenvalue of T, we may always consider the resolvent operator T_λ^{-1} (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I - T$.

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_1}$ of all complex sequences which contains ϕ, the set of all finitely non–zero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and $\|x\|_{bv} = \sum_{k=0}^{\infty} |x_k - x_{k+1}|$ while ϕ is not a Banach space with respect to any norm, respectively, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Also by ℓ_p, we denote the space of all p-absolutely summable sequences which is a Banach space with the norm $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$, where $1 \leq p < \infty$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$, and write

$$(Ax)_n = \sum_k a_{nk} x_k \ ; \quad (n \in \mathbb{N},\ x \in D_{00}(A)), \tag{1.6}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x = (x_k) \in w$ for which the sum exists as a finite sum. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ and we shall use the convention that any term with negative subscript is equal to zero. More generally if μ is a normed sequence space, we can write $D_\mu(A)$ for the $x \in w$ for which the sum in (1.6) converges in the norm of μ. We write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\}$$

for the space of those matrices which send the whole of the sequence space λ into μ in this sense.

We give a short survey concerning with the spectrum and the fine spectrum of the linear operators defined by some triangle matrices over certain sequence spaces. The fine spectrum of the Cesàro operator of order one on the sequence space ℓ_p were studied by Gonzàlez (Gonzàlez, 1985), where $1 < p < \infty$. Also, weighted mean matrices of operators on ℓ_p investigated by Cartlidge (Cartlidge, 1978). The spectrum of the Cesàro operator of order one on the sequence spaces bv_0 and bv investigated by Okutoyi (Okutoyi, 1992); (Okutoyi, 1990). The spectrum and fine spectrum of the Rhally operators on the sequence spaces c_0, c, ℓ_p, bv and bv_0 examined by Yıldırım (Yıldırım, 1996); (Yıldırım, 1998); (Yıldırım, 2001); (Yıldırım, 2002a); (Yıldırım, 2002b); (Yıldırım, 2003); (Yıldırım, 2004a); (Yıldırım, 2004b). The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c studied by Altay and Başar (Altay and Başar, 2004). The same authors have also worked the fine spectrum of the generalized difference operator $B(r, s)$ over c_0 and c, in (Altay and Başar, 2005). The fine spectrum of Δ over ℓ_1 and bv is studied by Kayaduman and Furkan (Furkan et al., 2006a). Recently, the fine spectrum of the difference operator Δ over the sequence spaces ℓ_p and bv_p studied by Akhmedov and Başar (Başar and Akhmedov, 2004); (Başar and Akhmedov, 2007), where bv_p is the space of p-bounded variation sequences and introduced by Başar and Altay (Altay and Başar, 2003) with $1 \leq p < \infty$. Also, the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_1 and bv is determined by Furkan et al. (Furkan et al., 2006b). Recently, the fine spectrum of $B(r, s, t)$ over the sequence spaces c_0 and c is studied by Furkan et al. (Furkan et al., 2007). Quite recently, de Malafosse (de Malafosse, 2002); Altay and Başar (Başar and Altay, 2004) study the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c; where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k\in\mathbb{N}} \frac{|x_k|}{r^k}$, $(r > 0)$. Altay and Karakuş (Altay and Karakuş, 2005) have determined the fine spectrum of the Zweier matrix which is a band matrix as an operator over the sequence spaces ℓ_1 and bv. de Malafosse and Farés (de Malafosse and Farés, 2008) studied the spectrum of the difference operator on the sequence space $\ell_p(\alpha)$, where (α_n) denotes the sequence of positive reals and $\ell_p(\alpha)$ is the Banach space of all sequences $x = (x_n)$ normed by $\|x\|_{\ell_p(\alpha)} = \left[\sum_{n=1}^{\infty} (|x_n|/\alpha_n)^p\right]^{1/p}$ with $p \geq 1$. Also the fine spectrum of the same operator over ℓ_1 and bv is studied by Bilgiç and Furkan (Bilgiç and Furkan, 2007). More recently the fine spectrum of the operator $B(r,s)$ over ℓ_p and bv_p has been studied by Bilgiç and Furkan (Bilgiç and Furkan, 2008). In 2010, Srivastava and Kumar (Srivastava and Kumar, 2010a) have determined the spectrum and the fine spectrum of the generalized difference operator Δ_ν on ℓ_1, where Δ_ν is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$, under certain conditions on the sequence $\nu = (\nu_n)$ and they have just generalized these results by the generalized difference operator Δ_{uv} defined by $\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$, (see (Srivastava and Kumar, 2010b)). Karakaya and Altun computed respectively the fine spectrums of the upper triangular double-band matrices and the lacunary matrices as an operator over the sequence spaces c_0 and c, (Karakaya and Altun, 2010); (Karakaya and Altun, 2009) Later, Altun (Altun, 2011) has studied the fine spectra of the Toeplitz operators, which are represented by upper and lower triangular n-band infinite matrices, over the sequence spaces c_0 and c. Quite recently, Akhmedov and El-Shabrawy (Akhmedov and El-Shabrawy, 2011) have obtained the fine spectrum of the generalized difference operator $\Delta_{a,b}$, defined as a double band matrix with the convergent sequences $\widetilde{a} = (a_k)$ and $\widetilde{b} = (b_k)$ having certain properties, over the sequence space c. Finally, the fine spectrum with respect to the Goldberg's classification of the operator $B(r,s,t)$ defined by a triple band matrix over the sequence spaces ℓ_p and bv_p with $1 < p < \infty$ has recently been studied by Furkan et al. (Bilgiç and Furkan, 2010).

Now, let us briefly describe the contents of the various sections of the thesis. It consists of five chapters.

First chapter is the introduction.

Second chapter we study the fine spectrum of the generalized difference operator defined by a double sequential band matrix $B(\widetilde{r},\widetilde{s})$ acting on the sequence space ℓ_p where$(1 < p < \infty)$ with respect to the Goldberg's classification .

Third chapter we study the fine spectrum of the generalized difference operator defined by an upper double sequential band matrix $A(\widetilde{r}, \widetilde{s})$ acting on the sequence spaces c_0, c and ℓ_p with respect to Goldberg's classification. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $A(\widetilde{r}, \widetilde{s})$ over the spaces c_0, c and ℓ_p, together with a Mercerian Theorem, where $0 < p \leq \infty$.

Fourth chapter we determine the fine spectra of upper triangular triple-band matrices over the sequence spaces μ, where μ denotes any of the spaces of ℓ_p, c or c_0. The operator $A(r, s, t)$ on sequence space on μ is defined $A(r, s, t)x = (rx_k + sx_{k+1} + tx_{k+2})_{k=0}^{\infty}$, where $x = (x_k) \in \mu$, with $\mu \in \{\ell_p, c, c_0\}$ with $0 < p < \infty$. In this chapter, we obtain the results on the spectrum and point spectrum for the operator $A(r, s, t)$ on the sequence space μ. Further, we also derive the results on continuous spectrum, residual spectrum and fine spectrum of the operator $A(r, s, t)$ on the sequence space ℓ_p. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $A(r, s, t)$ over the space μ with applications.

Fifth chapter contains conclusions.

CHAPTER 2

SPECTRUM OF LOWER DOUBLE SEQUENTIAL BAND MATRIX OVER THE SEQUENCE SPACE ℓ_p

In this chapter, we study the fine spectrum of the generalized difference operator $B(\widetilde{r}, \widetilde{s})$ defined by a double sequential band matrix acting on the sequence spaces ℓ_p, where $1 < p < \infty$ with respect to the Goldberg's classification.

Let $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ be sequences whose entries either constants or distinct non-zero real numbers satisfying the following conditions:

$$\begin{aligned}
&\lim_{k\to\infty} r_k = r, \\
&\lim_{k\to\infty} s_k = s \neq 0, \\
&|r_k - r| \neq |s|.
\end{aligned}$$

Then, we define the double sequential band matrix $B(\widetilde{r}, \widetilde{s})$ by

$$B(\widetilde{r}, \widetilde{s}) = \begin{bmatrix} r_0 & 0 & 0 & 0 & \dots \\ s_0 & r_1 & 0 & 0 & \dots \\ 0 & s_1 & r_2 & 0 & \dots \\ 0 & 0 & s_2 & r_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, we introduce the operator $B(\widetilde{r}, \widetilde{s})$ from ℓ_p to itself by

$$B(\widetilde{r}, \widetilde{s})x = (r_k x_k + s_{k-1} x_{k-1})_{k=0}^{\infty} \quad \text{with} \quad x_{-1} = 0, \quad \text{where} \quad x = (x_k) \in \ell_p.$$

2.1 SPECTRUM OF LOWER DOUBLE SEQUENTIAL BAND MATRIX OVER SEQUENCE SPACE ℓ_p

In this section, our purpose is to determine the spectrum of the operator $B(\widetilde{r},\widetilde{s})$ defined by a double sequential band matrix acting on the sequence space ℓ_p with respect to the Goldberg's classification, where $1 < p < \infty$. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $B(\widetilde{r},\widetilde{s})$ over the space ℓ_p.

We quote some lemmas which are needed in proving the theorems given in this section.

Lemma 2.1.1. *(Choudhary and Nanda, 1989, p. 253, Theorem 34.16) The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 2.1.2. *(Choudhary and Nanda, 1989, p. 245, Theorem 34.3) The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the rows of A is bounded.*

Lemma 2.1.3. *(Choudhary and Nanda, 1989, p. 254, Theorem 34.18) Let $1 < p < \infty$ and $A \in (\ell_\infty : \ell_\infty) \cap (\ell_1 : \ell_1)$. Then, $A \in (\ell_p : \ell_p)$.*

Theorem 2.1.4. *The operator $B(\widetilde{r},\widetilde{s}) : \ell_p \to \ell_p$ is a bounded linear operator and*

$$(|r_0|^p + |s_0|^p)^{1/p} \leq \|B(\widetilde{r},\widetilde{s})\| \leq \|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty. \tag{2.1}$$

Proof. Since the linearity of the operator $B(\widetilde{r},\widetilde{s})$ is not hard, we omit the details. Now, we prove that (2.1) holds for the operator $B(\widetilde{r},\widetilde{s})$ on the space ℓ_p. It is trivial that $B(\widetilde{r},\widetilde{s})e^{(0)} = (r_0, s_0, 0, \ldots, 0, \ldots)$ for $e^{(0)} \in \ell_p$. Therefore, we have

$$\frac{\|B(\widetilde{r},\widetilde{s})e^{(0)}\|_p}{\|e^{(0)}\|_p} = (|r_0|^p + |s_0|^p)^{1/p}$$

which implies that

$$(|r_0|^p + |s_0|^p)^{1/p} \leq \|B(\widetilde{r},\widetilde{s})\|. \tag{2.2}$$

Let $x = (x_k) \in \ell_p$, where $p > 1$. Then, since $(s_{k-1}x_{k-1}), (r_k x_k) \in \ell_p$ it is easy to see by Minkowski's inequality that

$$\begin{aligned}\|B(\widetilde{r},\widetilde{s})x\|_p &= \left(\sum_k |s_{k-1}x_{k-1} + r_k x_k|^p\right)^{1/p} \\ &\leq \left(\sum_k |s_{k-1}x_{k-1}|^p\right)^{1/p} + \left(\sum_k |r_k x_k|^p\right)^{1/p} \\ &\leq (\|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty)\|x\|_p\end{aligned}$$

which leads us to the the result that

$$\|B(\widetilde{r},\widetilde{s})\| \leq \|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty. \tag{2.3}$$

Therefore, by combining the inequalities in (2.2) and (2.3) we have (2.1), as desired. □

Theorem 2.1.5. *Let* $\mathcal{A} = \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\}$ *and* $\mathcal{B} = \{r_k : k \in \mathbb{N}, |r - r_k| > |s|\}$*. Then, the set* $\mathcal{B}$ *is finite and* $\sigma[B(\widetilde{r},\widetilde{s}), \ell_p] = \mathcal{A} \cup \mathcal{B}$*.*

Proof. We firstly prove that

$$\sigma[B(\widetilde{r},\widetilde{s}), \ell_p] \subseteq \mathcal{A} \cup \mathcal{B} \tag{2.4}$$

which is equivalent to show that $\alpha \in \mathbb{C}$ such that $|r - \alpha| > |s|$ and $\alpha \neq r_k$ for all $k \in \mathbb{N}$ implies $\alpha \notin \sigma[B(\widetilde{r},\widetilde{s}), \ell_p]$. Since $r_k \to r$ as $k \to \infty$, $\mathcal{B}$ is finite and $\{r_k \in \mathbb{R} : k \in \mathbb{N}\} \subseteq \mathcal{A} \cup \mathcal{B}$.

It is immediate that $B(\widetilde{r},\widetilde{s}) - \alpha I$ is a triangle and so has an inverse. Let $y = (y_k) \in \ell_1$. Then, by solving the equation

$$\begin{aligned}[B(\widetilde{r},\widetilde{s}) - \alpha I]x &= \begin{bmatrix} r_0-\alpha & 0 & 0 & \ldots \\ s_0 & r_1-\alpha & 0 & \ldots \\ 0 & s_1 & r_2-\alpha & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} (r_0-\alpha)x_0 \\ s_0x_0 + (r_1-\alpha)x_1 \\ s_1x_1 + (r_2-\alpha)x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}\end{aligned}$$

for $x=(x_k)$ in terms of y, we obtain

$$\begin{aligned}
x_0 &= \frac{y_0}{r_0-\alpha},\\
x_1 &= \frac{y_1}{r_1-\alpha}+\frac{-s_0y_0}{(r_1-\alpha)(r_0-\alpha)},\\
x_2 &= \frac{y_2}{r_2-\alpha}+\frac{-s_1y_1}{(r_2-\alpha)(r_1-\alpha)}+\frac{s_0s_1y_0}{(r_2-\alpha)(r_1-\alpha)(r_0-\alpha)},\\
&\vdots\\
x_k &= \frac{(-1)^k s_0s_1s_2\cdots s_{k-1}y_0}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\cdots(r_k-\alpha)}+\cdots-\frac{s_{k-1}y_{k-1}}{(r_k-\alpha)(r_{k-1}-\alpha)}+\frac{y_k}{r_k-\alpha}\\
&\vdots
\end{aligned}$$

Therefore, we obtain $B=(b_{nk})=[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}$ as follows:

$$(b_{nk})=\begin{bmatrix}
\frac{1}{r_0-\alpha} & 0 & 0 & \cdots\\
\frac{-s_0}{(r_1-\alpha)(r_0-\alpha)} & \frac{1}{r_1-\alpha} & 0 & \cdots\\
\frac{s_0s_1}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)} & \frac{-s_1}{(r_2-\alpha)(r_1-\alpha)} & \frac{1}{r_2-\alpha} & \cdots\\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

Then, $\sum_k |x_k|\leq \sum_k S^k|y_k|$, where

$$S^k = \left|\frac{1}{r_k-\alpha}\right|+\left|\frac{s_k}{(r_k-\alpha)(r_{k+1}-\alpha)}\right|+\left|\frac{s_ks_{k+1}}{(r_k-\alpha)(r_{k+1}-\alpha)(r_{k+2}-\alpha)}\right|+\cdots.$$

Since $|s_k/(r_{k+1}-\alpha)|\longrightarrow|s/(r-\alpha)|<1$, as $k\longrightarrow\infty$, then there exists $k_0\in\mathbb{N}$ and a real number q_0 such that $|s_k/(r_k-\alpha)|<q_0$ for all $k\geq k_0$. Then, for all $k\geq k_0+1$,

$$S^k\leq\frac{1}{|r_k-\alpha|}\left(1+q_0+q_0^2+\cdots\right).$$

But, there exists $k_1\in\mathbb{N}$ and a real number q_1 such that $|1/(r_k-\alpha)|<q_1$ for all $k\geq k_1$. Then, $S^k\leq q_1/(1-q_0)$, for all $k>\max\{k_0,k_1\}$. Thus, $\sup_{k\in\mathbb{N}}S^k<\infty$. Therefore,

$$\sum_k|x_k|\leq\sum_k S^k|y_k|\leq\|(S^k)\|_\infty\sum_k|y_k|<\infty,$$

since $y\in\ell_1$. This shows that $[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}\in(\ell_1:\ell_1)$.

Suppose that $y=(y_k)\in\ell_\infty$. By solving the equation $[B(\widetilde{r},\widetilde{s})-\alpha I]x=y$, for $x=(x_k)$ in terms of y, we get

$$|x_k|\leq S_k\left(\sup_{k\in\mathbb{N}}|y_k|\right),$$

where;

$$
\begin{aligned}
S_k &= \left|\frac{1}{r_k-\alpha}\right| + \left|\frac{s_{k-1}}{(r_{k-1}-\alpha)(r_k-\alpha)}\right| + \left|\frac{s_{k-1}s_{k-2}}{(r_{k-2}-\alpha)(r_{k-1}-\alpha)(r_k-\alpha)}\right| + \\
&\quad + \cdots + \left|\frac{s_0 s_1 \ldots s_{k-1}}{(r_0-\alpha)(r_1-\alpha)\cdots(r_k-\alpha)}\right|.
\end{aligned}
$$

Now, we prove that $(S_k) \in \ell_\infty$. Since $|s_k/(r_k-\alpha)| \longrightarrow |s/(r-\alpha)| = p < 1$, as $k \longrightarrow \infty$ then there exists $k_0 \in \mathbb{N}$ such that $|s_k/(r_k-\alpha)| < p_0$ with $p_0 < 1$ for all $k \geq k_0 + 1$,

$$
\begin{aligned}
S_k &= \frac{1}{|r_k-\alpha|}\left[1 + \left|\frac{s_{k-1}}{r_{k-1}-\alpha}\right| + \left|\frac{s_{k-1}s_{k-2}}{(r_{k-1}-\alpha)(r_{k-2}-\alpha)}\right| + \right. \\
&\quad \left. + \cdots + \left|\frac{s_{k-1}s_{k-2}\ldots s_{k_0+1}s_{k_0}\ldots s_0}{(r_{k-1}-\alpha)(r_{k-2}-\alpha)\cdots(r_{k_0+1}-\alpha)(r_{k_0}-\alpha)\cdots(r_0-\alpha)}\right|\right] \\
&\leq \frac{1}{|r_k-\alpha|}\left[1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0}\frac{|s_{k_0-1}|}{|r_{k_0-1}-\alpha|} + \right. \\
&\quad \left. + \cdots + p_0^{k-k_0}\left|\frac{s_{k_0-1}s_{k_0-2}\ldots s_0}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)\cdots(r_0-\alpha)}\right|\right].
\end{aligned}
$$

Therefore;

$$
S_k \leq \frac{1}{|r_k-\alpha|}\left(1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0} M k_0\right),
$$

where

$$
Mk_0 = 1 + \left|\frac{s_{k_0-1}}{r_{k_0-1}-\alpha}\right| + \left|\frac{s_{k_0-1}s_{k_0-2}}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)}\right| + \cdots + \left|\frac{s_{k_0-1}s_{k_0-2}\ldots s_0}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)\cdots(r_0-\alpha)}\right|.
$$

Then, $Mk_0 \geq 1$ and so

$$
S_k \leq \frac{Mk_0}{|r_k-\alpha|}\left(1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0}\right).
$$

But there exists $k_1 \in \mathbb{N}$ and a real number p_1 such that $1/(|r_k-\alpha|) < p_1$ for all $k \geq k_1$. Then, $S_k \leq (Mk_0 p_1)/(1-p_0)$ for all $k > \max\{k_0, k_1\}$. Hence, $\sup_{k\in\mathbb{N}} S_k < \infty$. This shows that $\|x\|_\infty \leq \|(S_k)\|_\infty \|y\|_\infty < \infty$ which means $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_\infty : \ell_\infty)$. By Lemma 2.1.2, we have

$$
[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_p : \ell_p) \quad \text{for} \quad \alpha \in \mathbb{C} \quad \text{with} \quad |r-\alpha| > |s| \quad \text{and} \quad \alpha \neq r_k. \tag{2.5}
$$

Hence,

$$
\sigma[B(\widetilde{r},\widetilde{s}), \ell_p] \subseteq \mathcal{A} \cup \mathcal{B}. \tag{2.6}
$$

Now, we will show that $\mathcal{A} \cup \mathcal{B} \subseteq \sigma[B(\widetilde{r},\widetilde{s}), \ell_p]$.

Conversely, assume that $\alpha \notin \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Then, $[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1} \in B(\ell_p)$. Since $e^{(0)} = (1,0,0,0\ldots) \in \ell_p$, $[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}$ transform of the unite sequence $e^{(0)}$ in ℓ_p. The calculation $[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}e^{(0)}$ gives that;

$$S_k^0 = \left(\frac{1}{r_0-\alpha}, \frac{-s_0}{(r_1-\alpha)(r_0-\alpha)}, \cdots, \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha), \cdots (r_k-\alpha)}, \cdots\right).$$

By ratio test;

$$\lim_{k\to\infty}\left|\frac{S_{k+1}^0}{S_k^0}\right|^p = \lim_{k\to\infty}\left|\frac{s_k}{r_{k+1}-\alpha}\right|^p = \left|\frac{s}{r-\alpha}\right|^p \leq 1 \text{ for all } k \in \mathbb{N},\ r_k \neq \alpha.$$

Hence, $\{\alpha \in \mathbb{C} : |r-\alpha| < |s|\} \subseteq \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Since the spectrum any bounded operator is closed, we have $\{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\} \subseteq \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$.

If $r_k = \alpha$ for some k, then we have either $\alpha = r$ or $\alpha = r_k \neq r$ for some k. We have

$$\begin{aligned}
[B(\widetilde{r},\widetilde{s}) - r_k I]x &= \begin{bmatrix} r_0 - r_k & 0 & 0 & \ldots \\ s_0 & r_1 - r_k & 0 & \ldots \\ 0 & s_1 & r_2 - r_k & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \\
&= \begin{bmatrix} (r_0 - r_k)x_0 \\ s_0 x_0 + (r_1 - r_k)x_1 \\ s_1 x_1 + (r_2 - r_k)x_2 \\ \vdots \\ s_{k-2}x_{k-2} + (r_{k-1} - r_k)x_{k-1} \\ s_{k-1}x_{k-1} + (r_k - r_k)x_k \\ s_k x_k + (r_{k+1} - r_k)x_{k+1} \\ \vdots \end{bmatrix}.
\end{aligned}$$

Let $\alpha = r_k = r$ for all k and solving the equation $[B(\widetilde{r},\widetilde{s})-\alpha I]x = \theta$ we obtain $x_0 = x_1 = x_2 = \cdots = 0$ which shows that $B(\widetilde{r},\widetilde{s}) - \alpha I$ is one to one but its range $R[B(\widetilde{r},\widetilde{s})-\alpha I] = \{y = (y_k) \in \omega : y \in \ell_p,\ y_1 = 0\}$ is not dense in ℓ_p and $\alpha = r \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Now let $\alpha = r_k$ for some k. Then the equation $[B(\widetilde{r},\widetilde{s})-\alpha I]x = \theta$ yields

$$x_0 = x_1 = x_2 = \cdots = x_{k-1} = 0 \text{ and } x_n = \frac{s_{n-1}}{r_k - r_n}x_{n-1} \text{ for all } n \geq k+1.$$

This shows that $B(\widetilde{r},\widetilde{s}) - \alpha I$ is not injective for $\alpha = r_k$ such that $|\alpha - r| > |s|$. Therefore $[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}$ does not exist. So $r_k \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$ for all $k \in \mathbb{N}$. Thus,

$$\mathcal{A} \cup \mathcal{B} \subseteq \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]. \tag{2.7}$$

Combining the inclusions (2.6) and (2.7), we get $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p] = \mathcal{A}\cup\mathcal{B}$.
This completes the proof. $\square$

Throughout the text, by $\mathcal{C}$ and $\mathcal{SD}$ we denote the set of constant sequences and the set of sequences of distinct none-zero real numbers, respectively.

Theorem 2.1.6. $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] = \begin{cases} \emptyset & , \quad \widetilde{r},\widetilde{s}\in\mathcal{C}, \\ \mathcal{B} & , \quad \widetilde{r},\widetilde{s}\in\mathcal{SD}, \end{cases}$

Proof. We prove the theorem by dividing into two parts.
Part 1. Assume that $\widetilde{r},\widetilde{s}\in\mathcal{C}$. Consider $B(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p. Now, we solve the system of linear equations

$$\begin{aligned} rx_0 &= \alpha x_0 \\ sx_0 + rx_1 &= \alpha x_1 \\ sx_1 + rx_2 &= \alpha x_2 \\ &\vdots \\ sx_{k-1} + rx_k &= \alpha x_k \\ &\vdots \end{aligned}$$

Case $\alpha = r$. Let x_{n_0} is the first non zero entry of the sequence $x = (x_n)$ and $\alpha = r$, then we get $sx_{n_0} + rx_{n_0+1} = \alpha x_{n_0+1}$. This implies $x_{n_0} = 0$ which contradicts the assumption $x_{n_0} \neq 0$. Hence, the equation $B(\widetilde{r},\widetilde{s})x = \alpha x$ has no solution $x \neq \theta$.
Part 2. Assume that $\widetilde{r},\widetilde{s}\in\mathcal{SD}$. Then, by solving the equation $B(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p we obtain $(r_0-\alpha)x_0 = 0$ and $(r_{k+1}-\alpha)x_{k+1} + s_k x_k = 0$ for all $k\in\mathbb{N}$. Hence, for all $\alpha \notin \{r_k : k\in\mathbb{N}\}$, we have $x_k = 0$ for all $k\in\mathbb{N}$, which contradicts our assumption. So, $\alpha \notin \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$. This shows that $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] \subseteq \{r_k : k\in\mathbb{N}\}\backslash\{r\}$. Now, we prove that

$$\alpha \in \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] \text{ if and only if } \alpha\in\mathcal{B}.$$

Let $\alpha \in \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$. We consider the case $\alpha = r_0$ and $\alpha = r_k$ for some $k \geq 1$. Then, by solving the equation $B(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p with $\alpha = r_0$ we final that

$$x_k = \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0-r_k)(r_0-r_{k-1})(r_0-r_{k-2})\cdots(r_0-r_1)} x_0 \quad \text{for all} \quad k \geq 1$$

which can be expressed by the recursion relation

$$x_k = \frac{s_{k-1}}{r_0 - r_k} x_{k-1} \quad \text{for all} \quad k \in \mathbb{N}_1.$$

Therefore,

$$\lim_{k\to\infty} \left|\frac{x_k}{x_{k-1}}\right|^p = \lim_{k\to\infty} \left|\frac{s_{k-1}}{r_k - r_0}\right|^p = \left|\frac{s}{r - r_0}\right|^p \leq 1.$$

But, $\left|\frac{s}{r-r_0}\right|^p \neq 1$. Then, $\alpha = r_0 \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$.
If we choose $\alpha = r_k \neq r$ for all $k \in \mathbb{N}_1$, then we get $x_0 = x_1 = x_2 = \cdots = x_{k-1} = 0$ and

$$x_{n+1} = \frac{s_n s_{n-1} s_{n-2} \ldots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} x_k \quad \text{for all} \quad n \geq k$$

which can also be expressed by the recursion relation

$$x_{n+1} = \frac{s_n}{r_k - r_{n+1}} x_n \quad \text{for all} \quad n \geq k.$$

Therefore, we have

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|^p = \lim_{n\to\infty} \left|\frac{s_n}{r_{n+1} - r_k}\right|^p = \left|\frac{s}{r - r_k}\right|^p \leq 1.$$

But $\left|\frac{s}{r-r_k}\right| \neq 1$. Then $\alpha = r_k \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. Thus $\sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \mathcal{B}$.
Conversely, let $\alpha \in \mathcal{B}$. Then, there exists $k \in \mathbb{N}$, $\alpha = r_k \neq r$ and

$$\lim_{n\to\infty} \left|\frac{s_n}{r_{n+1} - r_k}\right| = \left|\frac{s}{r - r_k}\right| < 1,$$

so we have $x \in \ell_p$. Thus $\mathcal{B} \subseteq \sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p]$. This completes the proof. □

If $T : \ell_p \to \ell_p$ is a bounded linear operator with the matrix A, then it is known that the adjoint operator $T^* : \ell_p^* \to \ell_p^*$ is defined by the transpose of the matrix A. It is known that the dual space ℓ_p^* of ℓ_p is isomorphic to ℓ_q, where $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$.

Theorem 2.1.7. $\sigma_p[B(\widetilde{r}, \widetilde{s})^*, \ell_p^*] = \begin{cases} \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} & , \quad \widetilde{r}, \widetilde{s} \in \mathcal{C}, \\ \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\} \cup \mathcal{B} & , \quad \widetilde{r}, \widetilde{s} \in \mathcal{SD}. \end{cases}$

Proof. By solving the equation $B(\widetilde{r},\widetilde{s})^*f = \alpha f$ for $\theta \neq f \in \ell_p^* \cong \ell_q$, we derive the system of linear equations

$$\begin{aligned} r_0 f_0 + s_0 f_1 &= \alpha f_0 \\ r_1 f_1 + s_1 f_2 &= \alpha f_1 \\ r_2 f_2 + s_2 f_3 &= \alpha f_2 \\ &\vdots \\ r_{k-1} f_{k-1} + s_{k-1} f_k &= \alpha f_{k-1} \\ &\vdots \end{aligned}$$

This gives $f_k = \left(\frac{\alpha - r_{k-1}}{s_{k-1}}\right) f_{k-1}$ for all $k \geq 1$. Therefore, we have

$$|f_k| = \left|\frac{\alpha - r_{k-1}}{s_{k-1}}\right| |f_{k-1}| \quad \text{for all} \quad k \in \mathbb{N}_1. \tag{2.8}$$

We also prove this theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r},\widetilde{s} \in \mathcal{C}$ with $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. Using (2.8), we get

$$f_k = \left(\frac{\alpha - r}{s}\right)^k f_0 \quad \text{for all} \quad k \in \mathbb{N}_1.$$

Then, since

$$\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right|^q = \left|\frac{\alpha - r}{s}\right|^q < 1 \ \text{ provided } \ \left|\frac{r-\alpha}{s}\right| < 1$$

the series $\sum_{k=1}^{\infty} |f_k|^q = \sum_{k=1}^{\infty} |(\alpha - r)/s|^{q(k-1)}|f_0|$ converges by the ratio test, i.e., $f \in \ell_q$.

If $\alpha \in \mathbb{C}$ with $|\alpha - r| = |s|$, then the ratio test fails. But, since $|f_k| \longrightarrow |f_0| \neq 0$ as $k \longrightarrow \infty$ the series $\sum_{k=0}^{\infty} |f_k|^q$ is divergent. This means that $f \in \ell_q$ if and only if $f_0 \neq 0$ and $|r - \alpha| < |s|$. Hence, $\sigma_p[B(\widetilde{r},\widetilde{s})^*, \ell_p^*] = \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\}$.

Part 2. Let $\widetilde{r},\widetilde{s} \in \mathcal{SD}$. It is clear that for all $k \in \mathbb{N}$, the vector $f = (f_0, f_1, \ldots, f_k, 0, 0, \ldots)$ is an eigenvector of the operator $B(\widetilde{r},\widetilde{s})^*$ corresponding to the eigenvalue $\alpha = r_k$, where $f_0 \neq 0$ and $f_n = \left(\frac{\alpha - r_{n-1}}{s_{n-1}}\right) f_{n-1}$ for all $k \in \{1, 2, 3, \ldots, n\}$. Thus $\mathcal{B} \subseteq \sigma_p[B(\widetilde{r},\widetilde{s})^*, \ell_p^*]$. If $|r - \alpha| < |s|$ and $\alpha \neq r_k$, by taking into account (2.8), since

$$\lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right|^q = \lim_{k\to\infty} \left|\frac{\alpha - r_{k-1}}{s_{k-1}}\right|^q = \left|\frac{r-\alpha}{s}\right|^q < 1,$$

the ratio test gives that $f \in \ell_q$. If $\alpha \in \mathbb{C}$ with $|r-\alpha| = |s|$, the ratio test fails. But one can easily find a decreasing sequence of positive real numbers $f = (f_k) \in \ell_q$ such that $|f_k/f_{k-1}| \longrightarrow 1$ as $k \longrightarrow \infty$, for example $f = (f_k) = (1/k^2)$. Hence, $|r-\alpha| \leq |s|$ implies $f \in \ell_q$.

Conversely, we have to show that $f \in \ell_q$ implies $|r-\alpha| \leq |s|$. If the condition $|r-\alpha| \leq |s|$ does not hold, then $|r-\alpha| > |s|$ which implies that $\sum_{k=0}^{\infty} |f_k|^q$ is divergent. This means that $f \in \ell_q$ if and only if $f_0 \neq 0$ and $|r-\alpha| \leq |s|$. Hence,

$$\sigma_p[B(\widetilde{r},\widetilde{s})^*, \ell_p^*] = \{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\} \cup \mathcal{B}.$$

This completes the proof. □

Lemma 2.1.8. *(Goldberg, 1985, p. 59) T has a dense range if and only if T^* is one to one.*

Lemma 2.1.9. *(Goldberg, 1985, p. 60) The adjoint operator T^* of T is onto if and only if T is a bounded operator.*

Theorem 2.1.10. $\sigma_r[B(\widetilde{r},\widetilde{s}), \ell_p] = \begin{cases} \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{C}, \\ \{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{SD}. \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Let $\widetilde{r},\widetilde{s} \in \mathcal{C}$. We show that the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ has an inverse and $\overline{R[B(\widetilde{r},\widetilde{s}) - \alpha I]} \neq \ell_p$ for α satisfying $|r-\alpha| < |s|$. For $\alpha \neq r$, $B(\widetilde{r},\widetilde{s}) - \alpha I$ is a triangle so has an inverse. For $\alpha = r$, the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is one to one by Theorem 2.1.6. So it has an inverse. By Theorem 2.1.7, the operator $[B(\widetilde{r},\widetilde{s}) - \alpha I)]^* = B(\widetilde{r},\widetilde{s})^* - \alpha I$ is not one to one for $\alpha \in \mathbb{C}$ such that $|r-\alpha| < |s|$. Hence the range of the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is not dense in ℓ_p by Lemma 2.1.8. So, $\sigma_r[B(\widetilde{r},\widetilde{s}), \ell_p] = \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\}$.

Part 2. Let $\widetilde{r},\widetilde{s} \in \mathcal{SD}$ with $r_k \longrightarrow r$ and $s_k \longrightarrow s$ as $k \longrightarrow \infty$ for $\alpha \in \mathbb{C}$ such that $|r-\alpha| \leq |s|$. Then, the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is triangle with $\alpha \neq r_k$ for all $k \in \mathbb{N}$. So, the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ has an inverse. By Theorem 2.1.6 the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is one to one for $\alpha = r_k$ for all $k \in \mathbb{N}$. Thus, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1}$ exists. But by Theorem 2.1.7, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^* = B(\widetilde{r},\widetilde{s})^* - \alpha I$ is not one to one with $\alpha \in \mathbb{C}$ such that $|r-\alpha| \leq |s|$. Hence, the range of the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is not dense in ℓ_p, by Lemma 2.1.8. So, $\sigma_r[B(\widetilde{r},\widetilde{s}), \ell_p] = \{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\}$.

This completes the proof. □

Theorem 2.1.11. $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \begin{cases} \{\alpha\in\mathbb{C} : |r-\alpha| = |s|\} & , \quad \widetilde{r},\widetilde{s}\in\mathcal{C}, \\ \emptyset & , \quad \widetilde{r},\widetilde{s}\in\mathcal{SD}. \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Let $\widetilde{r},\widetilde{s}\in\mathcal{C}$ for $\alpha\in\mathbb{C}$ such that $|r-\alpha| = |s|$. Since $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$ is the disjoint union of the parts $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ and $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p]$, we must have $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha\in\mathbb{C} : |r-\alpha| = |s|\}$.

Part 2. Let $\widetilde{r},\widetilde{s}\in\mathcal{SD}$. It is known that $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ and $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p]$ are mutually disjoint sets and their union is $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Therefore, it is immediate from Theorems 2.1.5, 2.1.6 and 2.1.10 that
$\sigma[B(\widetilde{r},\widetilde{s}),\ell_p] = \sigma_p[B(\widetilde{r},\widetilde{s})),\ell_p]\cup\sigma_r[B(\widetilde{r},\widetilde{s})),\ell_p]$ and hence $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \emptyset$.
This completes the proof. □

Theorem 2.1.12. *When* $|r-\alpha| > |s|$ *for* $\alpha\neq r_k$, $[B(\widetilde{r},\widetilde{s})-\alpha I]\in A_1$.

Proof. We show that the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ is bijective and has a continuous inverse for $\alpha\in\mathbb{C}$ such that $|r-\alpha| > |s|$. Since $\alpha\neq r_k$, then $B(\widetilde{r},\widetilde{s})-\alpha I$ is a triangle. So, it has an inverse. The inverse of the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ is continuous for $\alpha\in\mathbb{C}$ such that $|r-\alpha| > |s|$, by equation (2.5). Thus for every $y\in\ell_p$, we can find that $x\in\ell_p$ such that

$$[B(\widetilde{r},\widetilde{s})-\alpha I]x = y, \quad \text{since} \quad [B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}\in(\ell_p:\ell_p).$$

This shows that the operator $B(\widetilde{r},\widetilde{s})-\alpha I$ is onto and so $B(\widetilde{r},\widetilde{s})-\alpha I\in A_1$. □

Theorem 2.1.13. *Let* $\widetilde{r},\widetilde{s}\in\mathcal{C}$ *with* $r_k = r$ *and* $s_k = s$ *for all* $k\in\mathbb{N}$. *Then,* $r\in\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_1$.

Proof. We have $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha\in\mathbb{C} : |r-\alpha| < |s|\}$, by Theorem 2.1.10. Clearly, $r\in\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$. It is sufficient to show that the operator $[B(\widetilde{r},\widetilde{s})-rI]^{-1}$ is continuous. By Lemma 2.1.9, it is enough to show that $[B(\widetilde{r},\widetilde{s})-Ir]^*$ is onto and for given $y = (y_k)\in\ell_p^* = \ell_q$, we have to find $x = (x_k)\in\ell_q$ such that

$[B(\widetilde{r},\widetilde{s})-Ir]^*x=y$. Solving the system of linear equations

$$\begin{aligned} s_0x_1 &= y_0\\ s_1x_2 &= y_1\\ s_2x_3 &= y_2\\ &\vdots\\ s_{k-1}x_k &= y_{k-1}\\ &\vdots \end{aligned}$$

one can easily observe that $sx_k=y_{k-1}$ for all $k\geq 1$ which implies that $(x_k)\in\ell_q$, since $y=(y_k)\in\ell_q$. This shows that $[B(\widetilde{r},\widetilde{s})-Ir]^*$ is onto. Hence, $r\in\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_1$. □

Theorem 2.1.14. *Let $\widetilde{r},\widetilde{s}\in\mathcal{SD}$ and $\alpha\in\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}$. Then, $\alpha\in\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_1$.*

Proof. Let $\alpha\in\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}$. Then, by Theorem 2.1.10 $\alpha\in\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$. So we have $\overline{R[B(\widetilde{r},\widetilde{s})-\alpha I]}\neq\ell_p$. Since $B(\widetilde{r},\widetilde{s})-\alpha I$ is triangle, it has an inverse. It is sufficient to show that the operator $[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}$ is continuous. By Lemma 2.1.9, it is enough to show that $[B(\widetilde{r},\widetilde{s})-\alpha I]^*$ is onto and for given $y=(y_k)\in\ell_p^*=\ell_q$, we have to find $x=(x_k)\in\ell_q$ such that $[B(\widetilde{r},\widetilde{s})-Ir]^*x=y$. Let us solve the matrix equation $[B(\widetilde{r},\widetilde{s})-Ir]^*x=y$. Let $x_0=0$. Therefore, we obtain

$$\begin{aligned} x_1 &= \frac{y_0}{s_0},\\ x_2 &= \frac{(\alpha-r_1)y_0}{s_1s_0}+\frac{y_1}{s_1},\\ &\vdots\\ x_k &= \frac{(\alpha-r_1)(\alpha-r_2)\cdots(\alpha-r_{k-1})y_0}{s_0s_1\cdots s_{k-1}}+\cdots+\frac{(r_{k-2}-\alpha)y_{k-2}}{s_{k-1}s_{k-2}}+\frac{y_{k-1}}{s_{k-1}}. \end{aligned}$$

Then, $\sum_k|x_k|^p\leq\sup_{k\in\mathbb{N}}(R_k)^p\sum_k|y_k|^p$, where

$$R_k = \left|\frac{1}{s_k}\right|+\left|\frac{(r_{k+1}-\alpha)}{s_ks_{k+1}}\right|+\left|\frac{(r_{k+1}-\alpha)(r_{k+2}-\alpha)}{s_ks_{k+1}s_{k+2}}\right|+\cdots$$

for all $k\in\mathbb{N}$. Since $|(r_{k+1}-\alpha)/s_{k+1}|\longrightarrow|s/(r-\alpha)|<1$, as $k\longrightarrow\infty$, then there exists $k_0\in\mathbb{N}$ and a real number z_0 such that $|s_{k+1}/(r_{k+1}-\alpha)|<z_0$ for all $k\geq k_0$. Then, for all $k\geq k_0+1$,

$$R^k\leq\frac{1}{|s_k|}\left(1+z_0+z_0^2+\cdots\right).$$

But, there exists $k_1 \in \mathbb{N}$ and a real number z_1 such that $|1/s_k| < z_1$ for all $k \geq k_1$. Then, $R^k \leq z_1/(1-z_0)$, for all $k > \max\{k_0, k_1\}$. Thus, $\sup_{k\in\mathbb{N}} R^k < \infty$. Therefore,

$$\sum_k |x_k| \leq \sup_{k\in\mathbb{N}} (R_k)^p \sum_k |y_k|^p < \infty.$$

This shows that $[B(\widetilde{r}, \widetilde{s}) - I\alpha]^*$ is onto for $\alpha \in \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\}$. This completes the proof.

Theorem 2.1.15. *Let $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ with $r_k = r$, $s_k = s$ for all $k \in \mathbb{N}$. Then, the following statements hold:*

(i) $\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \backslash \{r\}$.

(ii) $\sigma_{\delta}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A}$.

(iii) $\sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A}^{\circ}$.

Proof. (i) Since from Table 1.2,

$$\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] \backslash \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_1$$

we have by Theorem 2.1.13 and Theorem 2.1.5 that

$$\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \backslash \{r\}.$$

(ii) Since the following equality

$$\sigma_{\delta}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma[B(\widetilde{r}, \widetilde{s}), \ell_p] \backslash \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] A_3$$

holds from Table 1.2, we derive by Theorem 2.1.5 and Theorem 2.1.6 that $\sigma_{\delta}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A}$.

(iii) From Table 1.2, we have

$$\sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_1 \cup \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_2 \cup \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_3$$

and since $\sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_3 = \emptyset$ by Theorem 2.1.6 it is immediate that $\sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma_r\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] = \mathcal{A}^{\circ}$. □

Theorem 2.1.16. *Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. Then*

$$\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma_{\delta}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \cup \mathcal{B}.$$

Proof. We have by Theorem 2.1.7 and Part (e) of Proposition 1.3.1 that

$$\sigma_p[B^*(\widetilde{r},\widetilde{s}),\ell_p^*] = \sigma_{co}[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha\in\mathbb{C} \ : \ |r-\alpha|\leq|s|\}.$$

Furthermore, because of $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] = \{r_k\}$ by Theorem 2.1.6 and the subdivisions in Goldberg's classification are disjoint, we must have

$$\sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\,A_3 = \sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\,B_3 = \emptyset.$$

Hence, $\sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\,C_3 = \{r_k\}$. Additionally, by Theorem 2.1.14 $\sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\,C_1 = \{\alpha\in\mathbb{C} : |r-\alpha|<|s|\}$. Therefore, we derive from Table 1.2 that

$$\begin{aligned}\sigma_{ap}[B(\widetilde{r},\widetilde{s}),\ell_p] &= \sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\setminus\sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\,C_1 = \{\alpha\in\mathbb{C} : |r-\alpha| = |s|\},\\ \sigma_\delta[B(\widetilde{r},\widetilde{s}),\ell_p] &= \sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\setminus\sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\,A_3 = \sigma\,[B(\widetilde{r},\widetilde{s}),\ell_p]\,.\end{aligned}$$

□

CHAPTER 3

SPECTRUM OF UPPER DOUBLE SEQUENTIAL BAND MATRIX OVER SOME SEQUENCES SPACES

In this chapter, we study the fine spectrum of the generalized difference operator $A(\widetilde{r},\widetilde{s})$ defined by an upper double sequential band matrix acting on the sequence spaces c_0, c and ℓ_p with respect to Goldberg's classification. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $A(\widetilde{r},\widetilde{s})$ over the spaces c_0, c and ℓ_p, together with a Mercerian Theorem, where $0 < p \leq \infty$.

Lemma 3.0.17. (Akhmedov and El-Shabrawy, 2011) Let (c_n), $(d_n) \in \omega$ such that $\lim_{n\to\infty} c_n = c$ with $|c| < 1$. Define the sequence $(z_n) \in \omega$ such that $z_{n+1} = z_n c_{n+1} + d_{n+1}$ for all $n \in \mathbb{N}$. Then we have;

(i) If $(d_n) \in \ell_\infty$, then $(z_n) \in \ell_\infty$.

(ii) If $(d_n) \in c$, then $(z_n) \in c$.

(iii) If $(d_n) \in c_0$, then $(z_n) \in c_0$.

Let $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ be sequences whose entries either constants or distinct non-zero real numbers satisfying the following conditions:

$$\lim_{k\to\infty} r_k = r > 0,$$
$$\lim_{k\to\infty} s_k = s; |s| = r,$$
$$\sup_{k\in\mathbb{N}} |r_k| \leq r,\ r_k^2 > s_k^2.$$

Then, we define the upper double sequential band matrix $A(\widetilde{r},\widetilde{s})$ by

$$A(\widetilde{r},\widetilde{s}) = \begin{bmatrix} r_0 & s_0 & 0 & 0 & \dots \\ 0 & r_1 & s_1 & 0 & \dots \\ 0 & 0 & r_2 & s_2 & \dots \\ 0 & 0 & 0 & r_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let λ denote any of the spaces c_0, c or ℓ_p. Now, we introduce the operator $A(\widetilde{r},\widetilde{s})$ from λ to itself by

$$\begin{aligned} A(\widetilde{r},\widetilde{s}) \quad : \qquad \lambda \qquad &\longrightarrow \ \lambda \\ x=(x_k) \ &\longmapsto \ A(\widetilde{r},\widetilde{s})x = (r_k x_k + s_k x_{k+1})_{k=0}^{\infty}. \end{aligned}$$

3.1 THE FINE SPECTRUM OF THE OPERATOR $A(\widetilde{r},\widetilde{s})$ ON THE SEQUENCE SPACE c_0

In this section, we examine the spectrum, the point spectrum, the continuous spectrum, the residual spectrum, the fine spectrum, the approximate point spectrum, the defect spectrum and the compression spectrum of the operator defined by the upper double sequential band matrix $A(\widetilde{r},\widetilde{s})$ on the sequence space c_0.

Theorem 3.1.1. $A(\widetilde{r},\widetilde{s}) : c_0 \to c_0$ *is a bounded linear operator and*

$$\|A(\widetilde{r},\widetilde{s})\|_{(c_0:c_0)} = \sup_{k\in\mathbb{N}}\{|r_k| + |s_k|\}. \tag{3.1}$$

Proof. Since the linearity of the operator $A(\widetilde{r},\widetilde{s})$ is clear. Now we prove that (3.1) holds on the space c_0. Let $x=(x_k)\in c_0$. Then, since $(s_k x_{k+1}), (r_k x_k) \in c_0$ it is easy to see by triangle inequality that

$$\begin{aligned} \|A(\widetilde{r},\widetilde{s})x\|_\infty &= \sup_{k\in\mathbb{N}} |r_k x_k + s_k x_{k+1}| \\ &\leq \sup_{k\in\mathbb{N}} (|r_k x_k| + |s_k x_{k+1}|) \\ &\leq \sup_{k\in\mathbb{N}} (|r_k| + |s_k|)\|x\|_\infty. \end{aligned}$$

Hence; $\|A(\widetilde{r},\widetilde{s})x\|_\infty \leq \|\widetilde{r}\|_\infty + \|\widetilde{s}\|_\infty$ which leads us

$$\|A(\widetilde{r},\widetilde{s})\|_{(c_0:c_0)} = \sup_{x\in c_0\setminus\{\theta\}} \frac{\|A(\widetilde{r},\widetilde{s})x\|_\infty}{\|x\|_\infty} \leq \|\widetilde{r}\|_\infty + \|\widetilde{s}\|_\infty. \tag{3.2}$$

Conversely, define $y = (y_n) = (0, 0, 0, \ldots, 0, 1, 1, 0, \ldots) \in c_0$, where 1 stand on k^{th} and $(k+1)^{th}$ places. Then, we have
$A(\widetilde{r}, \widetilde{s})y = (0, 0, 0, \ldots, s_{k-1}, r_k + s_k, r_{k+1}, 0, \ldots)$. Therefore, we see that

$$\|A(\widetilde{r}, \widetilde{s})\|_{(c_0:c_0)} \geq \|A(\widetilde{r}, \widetilde{s})x\|_\infty = \max_{k\in\mathbb{N}}\{|s_{k-1}|, |r_k + s_k|, |r_{k+1}|\} \geq \max_{k\in\mathbb{N}} |r_k + s_k|. \quad (3.3)$$

Combining the inclusions (3.2) and (3.3), we derive (3.1), as desired. □

If $T : c_0 \to c_0$ is a bounded matrix operator with the matrix A, then it is known that the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose of the matrix A. The dual space c_0^* of c_0 is isomorphic to ℓ_1.

Theorem 3.1.2. $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, c_0^*] = \begin{cases} \emptyset & , \quad \widetilde{s}, \widetilde{r} \in \mathcal{C}, \\ \mathcal{B} & , \quad \widetilde{s}, \widetilde{r} \in \mathcal{SD}. \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{s}, \widetilde{r} \in \mathcal{C}$. Consider $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ with $f \neq \theta = (0, 0, 0, \ldots)$ in $c_0^* = \ell_1$. Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} rf_0 & = & \alpha f_0 \\ sf_0 + rf_1 & = & \alpha f_1 \\ sf_1 + rf & = & \alpha f_2 \\ & \vdots & \\ sf_{k-1} + rf_k & = & \alpha f_k \\ & \vdots & \end{array}\right\}$$

we find that $f_0 = 0$ if $\alpha \neq r = r_k$ and $f_1 = f_2 = \cdots = 0$ if $f_0 = 0$ which contradicts $f \neq \theta$. If f_{n_0} is the first non zero entry of the sequence $f = (f_n)$ and $\alpha = r$, then we get $sf_{n_0} + rf_{n_0+1} = \alpha f_{n_0+1}$ which implies $f_{n_0} = 0$ which contradicts the assumption $f_{n_0} \neq 0$. Hence, the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ has no solution $f \neq \theta$.

Part 2. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. Then, by solving the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $f \neq \theta = (0, 0, 0, \ldots)$ in ℓ_1 we obtain $(r_0 - \alpha)f_0 = 0$ and $(r_{k+1} - \alpha)f_{k+1} + s_k f_k = 0$ for all $k \in \mathbb{N}$. Hence, for all $\alpha \notin \{r_k : k \in \mathbb{N}\}$, we have $f_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. So, $\alpha \notin \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1]$. This shows that $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1] \subseteq \{r_k : k \in \mathbb{N}\} \backslash \{r\}$. Now, we prove that $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1]$ if and only if $\alpha \in \mathcal{B}$. Let $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1]$. Then, by solving the equation

$A(\widetilde{r},\widetilde{s})^*f = \alpha f$ for $f \neq \theta$ in ℓ_1 with $\alpha = r_0$ we obtain that

$$f_k = \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_k)(r_0 - r_{k-1})(r_0 - r_{k-2}) \cdots (r_0 - r_1)} f_0 \quad \text{for all} \quad k \in \mathbb{N}_1$$

which can be expressed by the recursion relation

$$|f_k| = \left| \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_1)(r_0 - r_2) \cdots (r_0 - r_k)} \right| |f_0|.$$

Here and after $\mathbb{N}_1$ denotes the set of positive integers. Using the ratio test, we have

$$\lim_{k\to\infty} \left| \frac{f_k}{f_{k-1}} \right| = \lim_{k\to\infty} \left| \frac{s_{k-1}}{r_k - r_0} \right| = \left| \frac{s}{r - r_0} \right| \leq 1.$$

Since $|s/(r - r_0)| \neq 1$ by the hypothesis, we have

$\alpha = r_0 \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. If we choose $\alpha = r_k \neq r$ for all $k \in \mathbb{N}_1$,

then we get $f_0 = f_1 = f_2 = \cdots = f_{k-1} = 0$ and

$$f_{n+1} = \frac{s_n s_{n-1} s_{n-2} \ldots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} f_k \text{ for all } n \geq k$$

which can be expressed by the recursion relation

$$|f_{n+1}| = \left| \frac{s_n s_{n-1} s_{n-2} \ldots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} \right| |f_k|.$$

Using the ratio test, one can see that

$$\lim_{n\to\infty} \left| \frac{f_{n+1}}{f_n} \right| = \lim_{n\to\infty} \left| \frac{s_n}{r_{n+1} - r_k} \right| = \left| \frac{s}{r - r_k} \right| \leq 1.$$

Nevertheless $|s/(r - r_k)| \neq 1$, by our assumption. So, we have

$\alpha = r_k \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. Hence, $\sigma_p[A(\widetilde{r},\widetilde{s})^*, \ell_1] \subseteq \mathcal{B}$.

Conversely, let $\alpha \in \mathcal{B}$. Then, there exists $k \in \mathbb{N}$ with $\alpha = r_k \neq r$ and

$$\lim_{n\to\infty} \left| \frac{f_n}{f_{n-1}} \right| = \lim_{n\to\infty} \left| \frac{s_{n-1}}{r_n - r_k} \right| = \left| \frac{s}{r - r_k} \right| < 1.$$

That is, $f \in \ell_1$. So, we have $\mathcal{B} \subseteq \sigma_p[A(\widetilde{r},\widetilde{s})^*, \ell_1]$. This completes the proof. □

Theorem 3.1.3. $\sigma_p[A(\widetilde{r},\widetilde{s}), c_0] = \begin{cases} \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{C}, \\ \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \cup \mathcal{B} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{SD}. \end{cases}$

Proof. Let $A(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \in c_0 \setminus \{\theta\}$. Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} r_0 x_0 + s_0 x_1 & = & \alpha x_0 \\ r_1 x_1 + s_1 x_2 & = & \alpha x_1 \\ r_2 x_2 + s_2 x_3 & = & \alpha x_2 \\ & \vdots & \\ r_{k-1} x_{k-1} + s_{k-1} x_k & = & \alpha x_{k-1} \\ & \vdots & \end{array}\right\}$$

we obtain that $x_k = [(\alpha - r_{k-1})/s_{k-1}]x_{k-1}$ for all $k \in \mathbb{N}_1$ and

$$x_k = \prod_{j=0}^{k-1} \frac{r_j - \alpha}{s_j} x_0.$$

Part 1. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ with $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. Therefore, we observe that $x_k = [(\alpha - r)/s]^k x_0$ for all $k \in \mathbb{N}_1$. This shows that $x \in c_0$ if and only if $|\alpha - r| < |s|$, as asserted.

Part 2. Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. Now, firstly we show that $\{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \cup \mathcal{B} \subseteq \sigma_p[A(\widetilde{r}, \widetilde{s}), c_0]$. Let $|\alpha - r| < |s|$. Since $|x_k/x_{k-1}| = |(r_{k-1} - \alpha)/s_{k-1}| \longrightarrow |(r - \alpha)/s| < 1$ as $k \longrightarrow \infty$, $x \in \ell_1$. Since $\ell_1 \subseteq c_0$, $x \in c_0$. It is clear that $x = (x_0, x_1, \ldots, x_k, 0, 0, \ldots)$ is an eigenvector of the operator $A(\widetilde{r}, \widetilde{s})$ corresponding to the eigenvalue $\alpha = r_k$ for all $k \in \mathbb{N}$, where $x_0 \neq 0$ and $x_n = \big[(\alpha - r_{n-1})/s_{n-1}\big]x_{n-1}$ for $1 \leq n \leq k$. Thus, $\{r_k : k \in \mathbb{N}\} \subseteq \sigma_p[A(\widetilde{r}, \widetilde{s}), c_0]$. This shows that $\{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \cup \mathcal{B} \subseteq \sigma_p[A(\widetilde{r}, \widetilde{s}), c_0]$.

Conversely, let $x = (x_k) \in c_0$. Since $\ell_1 \subset c_0$, we can apply the ratio test that

$$\lim_{k\to\infty} \left| \frac{x_k}{x_{k-1}} \right| = \lim_{k\to\infty} \left| \frac{r_{k-1} - \alpha}{s_{k-1}} \right| = \left| \frac{r - \alpha}{s} \right| < 1.$$

In the case $|(r - \alpha)/s| = 1$, the ratio test fails. Now, we prove that $x = (x_k) \notin c_0$ if $|(r - \alpha)/s| = 1$. Let $\alpha = \alpha_1 + i\alpha_2$ such that $|r - \alpha| = |s|$. Then,

$$\begin{aligned} &|r - \alpha| = |s|, \\ &r^2 - 2r\alpha_1 + \alpha_1^2 + \alpha_2^2 = s^2, \\ &\alpha_1^2 + \alpha_2^2 = 2r\alpha_1. \end{aligned}$$

For all $k \in \mathbb{N}$, we have

$$\begin{aligned} |r_k - \alpha|^2 &= r_k^2 - 2r_k\alpha_1 + \alpha_1^2 + \alpha_2^2 \\ &= r_k^2 + 2\alpha_1(r - r_k) > s_k^2. \end{aligned}$$

Hence, $\big|(r_k - \alpha)/s_k\big| > 1$ for all $k \in \mathbb{N}$. This shows that $x \notin c_0$ if $|r - \alpha| = |s|$. Hence, $\sigma_p[A(\widetilde{r}, \widetilde{s}), c_0] \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \cup \mathcal{B}$. This completes the proof. □

Theorem 3.1.4. $\sigma_r[A(\widetilde{r}, \widetilde{s}), c_0] = \sigma_p[A(\widetilde{r}, \widetilde{s})^*, c_0^*] \backslash \sigma_p[A(\widetilde{r}, \widetilde{s}), c_0]$.

Proof. The proof is obvious so is omitted. □

Theorem 3.1.5. *Let $\widetilde{r}$, $\widetilde{s} \in \mathcal{SD}$ or $\mathcal{C}$. Then,* $\sigma_r[A(\widetilde{r}, \widetilde{s}), c_0] = \emptyset$.

Proof. Theorem 3.1.2 and Theorem 3.1.4 imply that $\sigma_r[A(\widetilde{r}, \widetilde{s}), c_0] = \emptyset$, as asserted. □

Theorem 3.1.6. $\sigma[A(\widetilde{r}, \widetilde{s}), c_0] = \mathcal{A} \cup \mathcal{B}$.

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ and $y = (y_k) \in \ell_1$. Then, by solving the equation $A_\alpha x = A[(\widetilde{r}, \widetilde{s}) - I\alpha]^* x = y$ for $x = (x_k)$ in terms of y, we obtain

$$\begin{cases} x_0 &= \frac{y_0}{r-\alpha}, \\ x_1 &= \frac{y_1}{r-\alpha} + \frac{-sy_0}{(r-\alpha)^2}, \\ x_2 &= \frac{y_2}{r-\alpha} + \frac{-sy_1}{(r-\alpha)^2} + \frac{s^2 y_0}{(r-\alpha)^3}, \\ &\vdots \\ x_k &= \frac{s^k y_0}{(r-\alpha)^{k+1}} + \cdots - \frac{sy_{k-1}}{(r-\alpha)^2} + \frac{y_k}{r-\alpha}, \\ &\vdots \end{cases}$$

which yields that

$$x_k = \frac{1}{r-\alpha} \sum_{i=0}^{k} \left(\frac{-s}{r-\alpha} \right)^{k-i} y_i$$

for all $k \in \mathbb{N}$. Therefore, we can observe with $|s| < |r - \alpha|$ that

$$|x_0| + |x_1| + \cdots + |x_k| \leq \frac{|y_0| + |y_1| + \cdots + |y_k|}{|r-\alpha| - |s|} \tag{3.4}$$

for $k \in \mathbb{N}$. Therefore, by letting $k \to \infty$ in (3.4), we derive that

$$\|(x_k)\|_1 \leq \frac{1}{|r-\alpha| - |s|} \|(y_k)\|_1.$$

Thus, $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is onto for $|s| < |r - \alpha|$ and $A_\alpha(\widetilde{r}, \widetilde{s})$ has a bounded inverse by Lemma 2.1.9. This means that

$$\sigma_c[A(\widetilde{r}, \widetilde{s}), c_0] \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\}.$$

Combining this fact with Theorem 3.1.3 and Theorem 3.1.5, we get

$$\{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), c_0] \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\}.$$

Since the spectrum is closed, we have $\sigma[A(\widetilde{r}, \widetilde{s}), c_0] = \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\}$.

Part 2. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ and $y = (y_k) \in \ell_1$. Then, by solving the equation $A_\alpha(\widetilde{r}, \widetilde{s})^* x = y$ in terms of y, we obtain that

$$\begin{cases} x_0 &= \frac{y_0}{r_0-\alpha}, \\ x_1 &= \frac{y_1}{r_1-\alpha} + \frac{-s_0 y_0}{(r_1-\alpha)(r_0-\alpha)}, \\ x_2 &= \frac{y_2}{r_2-\alpha} + \frac{-s_1 y_1}{(r_2-\alpha)(r_1-\alpha)} + \frac{s_0 s_1 y_0}{(r_2-\alpha)(r_1-\alpha)(r_0-\alpha)}, \\ \vdots & \\ x_k &= \frac{(-1)^k s_0 s_1 s_2 \cdots s_{k-1} y_0}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\cdots(r_k-\alpha)} + \cdots - \frac{s_{k-1} y_{k-1}}{(r_k-\alpha)(r_{k-1}-\alpha)} + \frac{y_k}{r_k-\alpha}, \\ \vdots & \end{cases}$$

Then, $\sum_k |x_k| \leq \sum_k S^k |y_k|$, where

$$S^k = \left|\frac{1}{r_k-\alpha}\right| + \left|\frac{s_k}{(r_k-\alpha)(r_{k+1}-\alpha)}\right| + \left|\frac{s_k s_{k+1}}{(r_k-\alpha)(r_{k+1}-\alpha)(r_{k+2}-\alpha)}\right| + \cdots$$

for all $k \in \mathbb{N}$. Since $|s_k/(r_{k+1}-\alpha)| \longrightarrow |s/(r-\alpha)| < 1$, as $k \longrightarrow \infty$, then there exists $k_0 \in \mathbb{N}$ and a real number q_0 such that $|s_k/(r_k-\alpha)| < q_0$ for all $k \geq k_0$. Then, for all $k \geq k_0 + 1$,

$$S^k \leq \frac{1}{|r_k-\alpha|}\left(1 + q_0 + q_0^2 + \cdots\right).$$

But, there exists $k_1 \in \mathbb{N}$ and a real number q_1 such that $|1/(r_k-\alpha)| < q_1$ for all $k \geq k_1$. Then, $S^k \leq q_1/(1-q_0)$, for all $k > \max\{k_0, k_1\}$. Thus, $\sup_{k\in\mathbb{N}} S^k < \infty$. Therefore,

$$\sum_k |x_k| \leq \sum_k S^k |y_k| \leq \|(S^k)\|_\infty \sum_k |y_k| < \infty,$$

since $y \in \ell_1$. Thus for $|s| < |r-\alpha|$, $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is onto and by Lemma 2.1.9, $A_\alpha(\widetilde{r}, \widetilde{s})$ has a bounded inverse. This means that

$$\sigma_c[A(\widetilde{r}, \widetilde{s}), c_0] \subseteq \{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\}.$$

Combining this fact with Theorem 3.1.3 and Theorem 3.1.5, we get

$$\sigma[A(\widetilde{r}, \widetilde{s}), c_0] \subseteq \{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\} \cup \mathcal{B}. \tag{3.5}$$

Again from Theorem 3.1.3, $\{\alpha \in \mathbb{C} : |r-\alpha| < |s|\} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), c_0]$ and $\mathcal{B} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), c_0]$. Since the spectrum is closed, thus we have

$$\{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\} \cup \mathcal{B} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), c_0]. \tag{3.6}$$

Combining the relations (3.5) and (3.6), we get $\sigma[A(\widetilde{r}, \widetilde{s}), c_0] = \mathcal{A} \cup \mathcal{B}$.
This completes the proof. $\square$

Theorem 3.1.7. *Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ or $\mathcal{C}$. Then,*
$\sigma_c[A(\widetilde{r}, \widetilde{s}), c_0] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.

Proof. Since the union of disjoint parts $\sigma_c[A(\widetilde{r}, \widetilde{s}), c_0]$, $\sigma_r[A(\widetilde{r}, \widetilde{s}), c_0]$ and $\sigma_p[A(\widetilde{r}, \widetilde{s}), c_0]$ is $\sigma[A(\widetilde{r}, \widetilde{s}), c_0]$, the proof immediately follows from Theorem 3.1.6, Theorem 3.1.5 and Theorem 3.1.3. □

Theorem 3.1.8. *Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. If $|\alpha - r| < |s|$, $\alpha \in \sigma[A(\widetilde{r}, \widetilde{s}), c_0]A_3$.*

Proof. From Theorem 3.1.3, $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s}), c_0]$. Thus, $[A(\widetilde{r}, \widetilde{s}) - \alpha I]^{-1}$ does not exist. It is sufficient to show that the operator $A(\widetilde{r}, \widetilde{s}) - \alpha I$ is onto, i.e., for given $y = (y_k) \in c_0$, we have to find $x = (x_k) \in c_0$ such that $[A(\widetilde{r}, \widetilde{s}) - \alpha I]x = y$. Therefore, solving the matrix equation $[A(\widetilde{r}, \widetilde{s}) - \alpha I]x = y$, we derive

$$\begin{aligned} x_1 &= \frac{y_0}{s_0}, \\ x_2 &= \frac{(\alpha - r_1)y_0}{s_1 s_0} + \frac{y_1}{s_1}, \\ &\vdots \\ x_k &= \frac{(\alpha - r_1)(\alpha - r_2)\cdots(\alpha - r_{k-1})y_0}{s_0 s_1 \cdots s_{k-1}} + \cdots + \frac{(r_{k-2} - \alpha)y_{k-2}}{s_{k-1}s_{k-2}} + \frac{y_{k-1}}{s_{k-1}} \end{aligned} \tag{3.7}$$

for all $k \in \mathbb{N}_1$, if $x_0 = 0$. Thus, (3.7) gives for all $k \in \mathbb{N}_1$ that

$$x_k = \frac{\alpha - r_{k-1}}{s_{k-1}} x_{k-1} + \frac{y_{k-1}}{s_{k-1}}. \tag{3.8}$$

Since $|(\alpha - r_{k-1})/s_{k-1}| \longrightarrow |(\alpha - r)/s| < 1$ as $k \longrightarrow \infty$ and $(y_{k-1}/s_{k-1}) \in c_0$, $x = (x_k) \in c_0$ by Lemma 3.0.17. Hence, $A(\widetilde{r}, \widetilde{s}) - \alpha I$ is onto. So, we have $\alpha \in \sigma[A(\widetilde{r}, \widetilde{s}), c_0]A_3$. □

Theorem 3.1.9. *Let $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ with $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. If $|\alpha - r| < |s|$, then $\alpha \in \sigma[A(\widetilde{r}, \widetilde{s}), c_0]A_3$.*

Proof. From Theorem 3.1.3, $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s}), c_0]$. Thus, $[A(\widetilde{r}, \widetilde{s}) - \alpha I]^{-1}$ does not exist. It is sufficient to show that the operator $A(\widetilde{r}, \widetilde{s}) - \alpha I$ is onto, i.e., for given $y = (y_k) \in c_0$, we have to find $x = (x_k) \in c_0$. Let $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. The relation (3.8) yields for all $k \in \mathbb{N}_1$ that

$$x_k = \frac{\alpha - r}{s} x_{k-1} + \frac{y_{k-1}}{s}.$$

By Lemma 3.0.17, $x = (x_k) \in c_0$. Hence, the operator $A(\widetilde{r}, \widetilde{s}) - \alpha I$ is onto. So, we have $\alpha \in \sigma[A(\widetilde{r}, \widetilde{s}), c_0]A_3$. □

Theorem 3.1.10. *Let $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ with $r_k = r$, $s_k = s$ for all $k \in \mathbb{N}$. Then, the following statements hold:*

(i) $\sigma_{ap}[A(\widetilde{r}, \widetilde{s}), c_0] = \sigma[A(\widetilde{r}, \widetilde{s}), c_0]$.

(ii) $\sigma_{\delta}[A(\widetilde{r}, \widetilde{s}), c_0] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.

(iii) $\sigma_{co}[A(\widetilde{r}, \widetilde{s}), c_0] = \emptyset$.

Proof. (i) Since $\sigma_{ap}[A(\widetilde{r}, \widetilde{s}), c_0] = \sigma[A(\widetilde{r}, \widetilde{s}), c_0] \backslash \sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_1$ from Table 1.2, we have by Theorem 3.1.2 that $\sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_1 = \sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_2 = \emptyset$. Hence, $\sigma_{ap}[A(\widetilde{r}, \widetilde{s}), c_0] = \mathcal{A}$.

(ii) Since the equality $\sigma_{\delta}[A(\widetilde{r}, \widetilde{s}), c_0] = \sigma[A(\widetilde{r}, \widetilde{s}), c_0] \backslash \sigma[A(\widetilde{r}, \widetilde{s}), c_0]A_3$ holds from Table 1.2, we derive by Theorem 3.1.6 and Theorem 3.1.9 that $\sigma_{\delta}[A(\widetilde{r}, \widetilde{s}), c_0] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.

(iii) From Table 1.2, we have

$$\sigma_{co}[A(\widetilde{r}, \widetilde{s}), c_0] = \sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_1 \cup \sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_2 \cup \sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_3$$

by Theorem 3.1.2 it is immediate that $\sigma_{co}[A(\widetilde{r}, \widetilde{s}), c_0] = \emptyset$. □

Theorem 3.1.11. *Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. Then, the following statements holds:*

$$\begin{aligned} (i) \qquad & \sigma_{ap}[A(\widetilde{r}, \widetilde{s}), c_0] = \mathcal{A} \cup \mathcal{B}, \\ (ii) \qquad & \sigma_{\delta}[A(\widetilde{r}, \widetilde{s}), c_0] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\} \cup \mathcal{B}, \\ (iii) \qquad & \sigma_{co}[A(\widetilde{r}, \widetilde{s}), c_0] = \mathcal{B}. \end{aligned}$$

Proof. We have by Theorem 3.1.2 and Part (e) of Proposition 1.3.1 that $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, c_0^*] = \sigma_{co}[A(\widetilde{r}, \widetilde{s}), c_0] = \mathcal{B}$. By Theorem 3.1.5 and Theorem 3.1.3, we must have $\sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_1 = \sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_2 = \emptyset$. Hence, $\sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_3 = \{r_k\}$. Additionally, since $\sigma[A(\widetilde{r}, \widetilde{s}), c_0]C_1 = \emptyset$.

Therefore, we deduce from Table 1.2 that the statements $(i) - (iii)$ are satisfied. □

3.2 THE FINE SPECTRUM OF THE OPERATOR $A(\widetilde{r},\widetilde{s})$ ON THE SEQUENCE SPACES c AND ℓ_∞

Theorem 3.2.1. *The operator $A(\widetilde{r},\widetilde{s}):\mu\to\mu$ is a bounded linear operator with the norm $\|A(\widetilde{r},\widetilde{s})\|_{(\mu:\mu)}=\|A(\widetilde{r},\widetilde{s})\|_{(c_0:c_0)}$, where $\mu\in\{c,\ell_\infty\}$.*

Proof. This proof can be obtained by proceeding as in the proof of Theorem 3.1.1. So, we omit the details. □

Theorem 3.2.2. *The following statements hold:*

(i) If $\widetilde{r},\widetilde{s}\in\mathcal{C}$, $\quad\sigma_p[A(\widetilde{r},\widetilde{s}),c]=\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}\cup\{r+s\}$.

(ii) If $\widetilde{r},\widetilde{s}\in\mathcal{SD}$, $\quad\sigma_p[A(\widetilde{r},\widetilde{s}),c]=\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}\cup\mathcal{B}$.

Proof. (i) Assume that $\widetilde{r},\widetilde{s}\in\mathcal{C}$. Let $r_k=r$ and $s_k=s$ for all $k\in\mathbb{N}$. Consider $A(\widetilde{r},\widetilde{s})x=\alpha x$ for $f\neq\theta$ in c. Then, by solving the system of linear equations, we observe that $x_k=[(\alpha-r)/s]^k x_0$. This shows that $x\in c$ if and only if $|\alpha-r|<|s|$ and $\alpha=s+r$, as asserted.

(ii) This part is similar to the proof of the second part of Theorem 3.1.3. □

If $T:c\to c$ is a bounded matrix operator with the matrix A, then $T^*:c^*\to c^*$ acting on $\mathcal{C}\oplus\ell_1$ has a matrix representation of the form $\begin{bmatrix}\chi & 0\\ b & A^t\end{bmatrix}$, where χ denotes the characteristic of the matrix A and b is the column vector whose k^{th} entry is the limit of the column of A for each $k\in\mathbb{N}$. For $A(\widetilde{r},\widetilde{s}):c\to c$, the matrix $A(\widetilde{r},\widetilde{s})^*\in B(\ell_1)$ is of the form

$$A(\widetilde{r},\widetilde{s})^*=\begin{bmatrix}r+s & 0\\ 0 & A(\widetilde{r},\widetilde{s})\end{bmatrix}.$$

Theorem 3.2.3. $\sigma_p[A(\widetilde{r},\widetilde{s})^*,c^*]=\begin{cases} r+s & ,\quad \widetilde{s},\widetilde{r}\in\mathcal{C},\\ \mathcal{B}\cup\{r+s\} & ,\quad \widetilde{s},\widetilde{r}\in\mathcal{SD}.\end{cases}$

Proof. Let $f \in c^* = \ell_1$ with $f \neq \theta$. Consider the system of linear equations:

$$\left.\begin{array}{rcl} (r+s)f_0 & = & \alpha f_0 \\ r_0 f_1 & = & \alpha f_1 \\ s_0 f_1 + r_1 f_2 & = & \alpha f_2 \\ & \vdots & \\ s_{k-2} f_{k-1} + r_{k-1} f_k & = & \alpha f_k \\ & \vdots & \end{array}\right\}$$

We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ with $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. If $f_0 \neq 0$, then $\alpha = r + s$. So, $\alpha = r + s$ is an eigenvalue with corresponding eigenvector $(f_0, 0, 0, \ldots)$, that is, $\alpha = r + s \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, c^*]$. If $\alpha \neq r + s$, then $f_0 = 0$. On the other hand, if $\alpha \neq r$ we find that $f_1 = 0$ and $f_1 = f_2 = \cdots = 0$ which contradicts $f \neq \theta$. If f_{n_0} is the first non zero entry of the sequence $f = (f_n)$ and $\alpha = r$, then we get $s_{n_0-1} f_{n_0} + r_{n_0} f_{n_0+1} = \alpha f_{n_0+1}$ which implies $f_{n_0} = 0$ which contradicts the assumption $f_{n_0} \neq 0$. Hence, the equation $A(\widetilde{r}, \widetilde{s})f = \alpha f$ has no solution $f \neq \theta$.

Part 2. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. Then, by solving the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $f \neq \theta$ in ℓ_1 we obtain $[(r+s) - \alpha]f_0 = 0$, $(r_0 - \alpha)f_1 = 0$ and $(r_{k+1} - \alpha)f_{k+1} + s_k f_k = 0$ for all $k \in \mathbb{N}$. If $f_0 \neq 0$, then $\alpha = r + s$. So, $\alpha = r + s$ is an eigenvalue with corresponding eigenvector $(f_0, 0, 0, \ldots)$, that is, $\alpha = r + s \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, c^*]$. If $\alpha \neq r + s$, then $f_0 = 0$. On the other hand, for all $\alpha \notin \{r_k : k \in \mathbb{N}\}$, we have $f_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. So, $\alpha \notin \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1]$. This shows that $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1] \subseteq \{r_k : k \in \mathbb{N}\}$. Now, we prove that $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1]$ if and only if $\alpha \in \mathcal{B}$. Let $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1]$. Then, by solving the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $f \neq \theta$ in ℓ_1 with $\alpha = r_0$ that

$$f_k = \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_k)(r_0 - r_{k-1})(r_0 - r_{k-2}) \cdots (r_0 - r_1)} f_0 \quad \text{for all} \quad k \in \mathbb{N}_1$$

which can be expressed by the following recursion relation

$$|f_k| = \left| \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_1)(r_0 - r_2) \cdots (r_0 - r_k)} \right| |f_0| \quad \text{for all} \quad k \in \mathbb{N}_1.$$

Using the ratio test, one can see that

$$\lim_{k \to \infty} \left| \frac{f_k}{f_{k-1}} \right| = \lim_{k \to \infty} \left| \frac{s_{k-1}}{r_k - r_0} \right| = \left| \frac{s}{r - r_0} \right| \leq 1.$$

Since $|s/(r-r_0)| \neq 1$, we have $\alpha = r_0 \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$.

If we choose $\alpha = r_k \neq r$ for all $k \in \mathbb{N}_1$, then we get $f_0 = f_1 = f_2 = \cdots = f_{k-1} = 0$ and

$$f_{n+1} = \frac{s_n s_{n-1} s_{n-2} \dots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} f_k \text{ for all } n \geq k$$

which can be expressed by the following recursion relation

$$|f_{n+1}| = \left| \frac{s_n s_{n-1} s_{n-2} \dots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} \right| |f_k|.$$

Therefore, we obtain by applying the ratio test that

$$\lim_{n\to\infty} \left| \frac{f_{n+1}}{f_n} \right| = \lim_{n\to\infty} \left| \frac{s_n}{r_{n+1} - r_k} \right| = \left| \frac{s}{r - r_k} \right| \leq 1.$$

Since $|s/(r - r_k)| \neq 1$ by the assumption, we have
$\alpha = r_k \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. Hence, $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1] \subseteq \mathcal{B}$. Conversely, let $\alpha \in \mathcal{B}$. Then there exists a $k \in \mathbb{N}$ such that $\alpha = r_k \neq r$ and

$$\lim_{n\to\infty} \left| \frac{f_n}{f_{n-1}} \right| = \lim_{n\to\infty} \left| \frac{s_{n-1}}{r_n - r_k} \right| = \left| \frac{s}{r - r_k} \right| < 1.$$

That is, $f = (f_n) \in \ell_1$. So, we have $\mathcal{B} \subseteq \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_1]$. This completes the proof. □

Theorem 3.2.4. *The following statements hold:*

(i) If $\widetilde{r}, \widetilde{s} \in \mathcal{C}$, $\sigma_r[A(\widetilde{r}, \widetilde{s}), c] = \emptyset$.

(ii) If $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$, $\sigma_r[A(\widetilde{r}, \widetilde{s}), c] = \{r + s\}$.

Proof. This is immediate by Theorem 3.1.4 and Theorem 3.2.3. □

Theorem 3.2.5. $\sigma[A(\widetilde{r}, \widetilde{s}), c] = \mathcal{A} \cup \mathcal{B}$.

Proof. The proof is similar to the proof of Theorem 3.1.6. □

Theorem 3.2.6. *Let* $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ *or* $\mathcal{C}$. *Then,*

$$\sigma_c[A(\widetilde{r}, \widetilde{s}), c] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\} \setminus \{r + s\}.$$

Proof. This immediately follows from Theorem 3.2.2, Theorem 3.2.4 and Theorem 3.2.5 because of the union of disjoint parts $\sigma_c[A(\widetilde{r}, \widetilde{s}), c]$, $\sigma_r[A(\widetilde{r}, \widetilde{s}), c]$ and $\sigma_p[A(\widetilde{r}, \widetilde{s}), c]$ is $\sigma[A(\widetilde{r}, \widetilde{s}), c]$. □

Theorem 3.2.7. *The following statements hold:*

(i) Let $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ *with* $r_k = r$ *and* $s_k = s$ *for all* $k \in \mathbb{N}$. *If* $|\alpha - r| < |s|$, *then* $\alpha \in \sigma[A(\widetilde{r}, \widetilde{s}), c]A_3$.

(ii) Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. *If* $|\alpha - r| < |s|$, *then* $\alpha \in \sigma[A(\widetilde{r}, \widetilde{s}), c]A_3$.

Proof. This is obtained by proceeding as in Theorems 3.1.8 and 3.1.9. □

Theorem 3.2.8. *Let* $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ *with* $r_k = r$ *and* $s_k = s$ *for all* $k \in \mathbb{N}$. *Then, the following statements hold:*

(i) $\sigma_{ap}[A(\widetilde{r}, \widetilde{s}), c] = \sigma[A(\widetilde{r}, \widetilde{s}), c]$.

(ii) $\sigma_{\delta}[A(\widetilde{r}, \widetilde{s}), c] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.

(iii) $\sigma_{co}[A(\widetilde{r}, \widetilde{s}), c] = \{r + s\}$.

Proof. (i) Since $\sigma_{ap}[A(\widetilde{r}, \widetilde{s}), c] = \sigma[A(\widetilde{r}, \widetilde{s}), c] \backslash \sigma[A(\widetilde{r}, \widetilde{s}), c]C_1$ from Table 1.2, we have by part (i) Theorem 3.2.4 that $\sigma[A(\widetilde{r}, \widetilde{s}), c]C_1 = \sigma[A(\widetilde{r}, \widetilde{s}), c]C_2 = \emptyset$. Hence, $\sigma_{ap}[A(\widetilde{r}, \widetilde{s}), c] = \mathcal{A}$.
(ii) Since Table 1.2 gives the equality $\sigma_{\delta}[A(\widetilde{r}, \widetilde{s}), c] = \sigma[A(\widetilde{r}, \widetilde{s}), c] \backslash \sigma[A(\widetilde{r}, \widetilde{s}), c]A_3$, we derive by Theorem 3.2.5 and Theorem 3.2.7 that
$\sigma_{\delta}[A(\widetilde{r}, \widetilde{s}), c] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.
(iii) Combining Theorem 3.2.3 with Table 1.2, we have

$$\sigma_{co}[A(\widetilde{r}, \widetilde{s}), c] = \sigma[A(\widetilde{r}, \widetilde{s}), c]C_1 \cup \sigma[A(\widetilde{r}, \widetilde{s}), c]C_2 \cup \sigma[A(\widetilde{r}, \widetilde{s}), c]C_3$$

which immediately gives that $\sigma_{co}[A(\widetilde{r}, \widetilde{s}), c] = \{r + s\}$. □

Theorem 3.2.9. *Let* $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. *Then, the following statements hold:*

(i) $\sigma_{ap}[A(\widetilde{r}, \widetilde{s}), c] = \sigma[A(\widetilde{r}, \widetilde{s}), c]$ *or* $\sigma[A(\widetilde{r}, \widetilde{s}), c] \backslash \{r + s\}$.

(ii) $\sigma_{\delta}[A(\widetilde{r}, \widetilde{s}), c] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\} \cup \mathcal{B}$.

(iii) $\sigma_{co}[A(\widetilde{r}, \widetilde{s}), c] = \mathcal{B} \cup \{r + s\}$.

Proof. We have by Theorem 3.2.3 and Part (e) of Proposition 1.3.1 that
$\sigma_p[A(\widetilde{r}, \widetilde{s})^*, c^*] = \sigma_{co}[A(\widetilde{r}, \widetilde{s}), c] = \mathcal{B} \cup \{r + s\}$. Therefore, (iii) holds. By Theorem

3.2.4 and Theorem 3.2.5, we also have $\sigma[A(\widetilde{r},\widetilde{s}),c]C_1 \cup \sigma[A(\widetilde{r},\widetilde{s}),c]C_2 = \{r+s\}$. If $(r+s) \in \sigma[A(\widetilde{r},\widetilde{s}),c]C_1$, we derive from Table 1.2 that

$$\sigma_{ap}[A(\widetilde{r},\widetilde{s}),c] = \sigma[A(\widetilde{r},\widetilde{s}),c] \backslash \sigma[A(\widetilde{r},\widetilde{s}),c]C_1 = \sigma[A(\widetilde{r},\widetilde{s}),c] \setminus \{r+s\}.$$

If $(r+s) \notin \sigma[A(\widetilde{r},\widetilde{s}),c]C_1$, $\sigma[A(\widetilde{r},\widetilde{s}),c]C_1 = \emptyset$. Therefore,

$$\sigma_{ap}[A(\widetilde{r},\widetilde{s}),c] = \sigma[A(\widetilde{r},\widetilde{s}),c] \backslash \sigma[A(\widetilde{r},\widetilde{s}),c]C_1 = \sigma[A(\widetilde{r},\widetilde{s}),c].$$

Hence, (i) holds. By Theorem 3.2.5 and Theorem 3.2.7, we see that (ii) holds. □

It is known from Cartlidge (Cartlidge, 1978) that if a matrix operator A is bounded on c, then $\sigma(A,c) = \sigma(A,\ell_\infty)$. So, we have the following.

Corollary 3.2.10. $\sigma[A(\widetilde{r},\widetilde{s}),\ell_\infty] = \mathcal{A} \cup \mathcal{B}$.

Theorem 3.2.11. *If* $\widetilde{r},\widetilde{s} \in \mathcal{C}$, $\sigma_p[A(\widetilde{r},\widetilde{s}),\ell_\infty] = \{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\}$.

Proof. Assume that $\widetilde{r},\widetilde{s} \in \mathcal{C}$. Let $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. Consider $A(\widetilde{r},\widetilde{s})x = \alpha x$ for $f \neq \theta$ in ℓ_∞. Then, by solving the matrix equation we observe that $x_k = [(\alpha - r)/s]^k x_0$. This shows that $x = (x_k) \in \ell_\infty$ if and if only $|\alpha - r| \leq |s|$, as asserted. □

Theorem 3.2.12. *If* $\widetilde{r},\widetilde{s} \in \mathcal{C}$, $\sigma_c[A(\widetilde{r},\widetilde{s}),\ell_\infty] = \emptyset$ *and* $\sigma_r[A(\widetilde{r},\widetilde{s}),\ell_\infty] = \emptyset$.

Proof. Because of the parts $\sigma_c[A(\widetilde{r},\widetilde{s}),\ell_\infty]$, $\sigma_r[A(\widetilde{r},\widetilde{s}),\ell_\infty]$ and $\sigma_p[A(\widetilde{r},\widetilde{s}),\ell_\infty]$ are pairwise disjoint sets and their union is $\sigma[A(\widetilde{r},\widetilde{s}),\ell_\infty]$, the proof immediately follows from Corollary 3.2.10 and Theorem 3.2.11. □

To avoid the repetition of the similar statements we give the results in the following theorem without proof.

Theorem 3.2.13. *Let* $(r_k),(s_k) \in \mathcal{C}$ *with* $r_k = r$, $s_k = s$ *for all* $k \in \mathbb{N}$. *Then, the following statements hold:*

(i) *If* $|\alpha - r| < |s|$, *then* $\alpha \in \sigma[A(\widetilde{r},\widetilde{s}),\ell_\infty]A_3$.

(ii) $\sigma_{ap}[A(\widetilde{r},\widetilde{s}),\ell_\infty] = \sigma[A(\widetilde{r},\widetilde{s}),\ell_\infty]$.

(iii) $\sigma_\delta[A(\widetilde{r},\widetilde{s}),\ell_\infty] = \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\}$.

Let A be an infinite matrix and the set c_A denotes the convergence domain of that matrix A. A theorem which proves that $c_A = c$ is called a *Mercerian theorem*, after Mercer, who proved a significant theorem of this type.

Theorem 3.2.14. *Suppose that α satisfies the inequality $|\alpha(1-r)+r| > |s(1-\alpha)|$. Then the convergence field of $B = \alpha I(1-\alpha)A(\widetilde{r},\widetilde{s})$ is c.*

Proof. $B \in B(c)$, since

(i) $\sup_{n\in\mathbb{N}} \sum_k |b_{nk}| = \sup_{n\in\mathbb{N}}[|\alpha(1-r_n)+r_n| + |s_n(1-\alpha)|] < \infty$,

(ii) $\lim_{n\to\infty} b_{nk} = 0$,

(iii) $\lim_{n\to\infty} \sum_k b_{nk} = \alpha(1-r)+r+s(1-\alpha)$.

Now, we show that B^{-1} exists and belongs to $B(c)$ for $|\alpha(1-r)+r| > |s(1-\alpha)|$. By matrix multiplication, one can see that B^{-1} is both a right inverse and left inverse of the matrix B.

$$B^{-1} = (a_{kj}) = \begin{bmatrix} \frac{1}{r_0(1-\alpha)+\alpha} & \frac{-s_0(1-\alpha)}{(r_0(1-\alpha)+\alpha)(r_1(1-\alpha)+\alpha)} & \frac{s_0 s_1(1-\alpha)^2}{(r_0(1-\alpha)+\alpha)(r_1(1-\alpha)+\alpha)(r_2(1-\alpha)+\alpha)} & \cdots \\ 0 & \frac{1}{r_1(1-\alpha)+\alpha} & \frac{-s_1(1-\alpha)}{(r_1(1-\alpha)+\alpha)(r_2(1-\alpha)+\alpha)} & \cdots \\ 0 & 0 & \frac{1}{r_2(1-\alpha)+\alpha} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$B^{-1} \in B(c)$ if and only if

(i) $\sup_{k\in\mathbb{N}} S^k < \infty$, where $S^k = \sum_j |a_{kj}|$ for each $k \in \mathbb{N}$.

(ii) For each $j \in \mathbb{N}$ the sequence $(a_{0j}, a_{1j}, a_{2j}, \ldots)$ is convergent.

(iii) The sum of row sequences (a_{kj}) is finite.

$$\begin{aligned} S^k &= \left|\frac{1}{r_k(1-\alpha)+\alpha}\right| + \left|\frac{s_{k-1}(1-\alpha)}{(r_{k-1}(1-\alpha)+\alpha))(r_k(1-\alpha)+\alpha))}\right| \\ &\quad + \cdots + \left|\frac{s_0 s_1 \ldots s_{k-1}(1-\alpha)^k}{(r_0(1-\alpha)+\alpha))(r_1-(1-\alpha)+\alpha))\cdots(r_k(1-\alpha)+\alpha)}\right|. \end{aligned}$$

Then, we have

$$S^k = \left|\frac{s_{k-1}(1-\alpha)}{r_k(1-\alpha)+\alpha)}\right| S^{k-1} + \left|\frac{1}{r_k(1-\alpha)+\alpha)}\right|.$$

By applying Lemma 3.0.17, we see that (S^k) is convergent. Hence, $\sup_{k\in\mathbb{N}} S^k < \infty$. Also it is easy to see that $\lim_{k\to\infty} |a_{kj}| = 0$ for each $j \in \mathbb{N}$, since

$$\lim_{k\to\infty}\left|\frac{a_{k,j}}{a_{k-1,j}}\right| = \lim_{k\to\infty}\left|\frac{s_{k-1}(1-\alpha)}{r_k(1-\alpha)+\alpha)}\right| = \left|\frac{s(1-\alpha)}{r(1-\alpha)+\alpha)}\right| < 1.$$

So, the sequence $(a_{0j}, a_{1j}, a_{2j}, \ldots)$ is convergent for each fixed $j \in \mathbb{N}$. Now, let the general term in sum of the row sequences B^{-1} is S_k and

$$S_0 = \sum_{j=0}^{\infty} a_{0j} = \frac{1}{r_0(1-\alpha)+\alpha)} + \sum_{j=0}^{\infty}\prod_{i=1}^{j}\frac{s_{i-1}(1-\alpha)}{r_i(1-\alpha)+\alpha)}.$$

Since,

$$\lim_{j\to\infty}\left|\frac{a_{0,j}}{a_{0,j-1}}\right| = \lim_{j\to\infty}\left|\frac{s_{j-1}(1-\alpha)}{r_j(1-\alpha)+\alpha)}\right| = \left|\frac{s(1-\alpha)}{r(1-\alpha)+\alpha)}\right| < 1,$$

$\sum_j a_{0j}$ is convergent. Similarly, we can say that $\sum_j a_{kj}$ is also convergent for all $k \in \mathbb{N}_1$. Hence, $B^{-1} \in B(c)$. Since both B and B^{-1} are in $B(c)$, $c_B = c$. □

3.3 FINE SPECTRA OF UPPER TRIANGULAR DOUBLE-BAND MATRIX OVER THE SEQUENCE SPACE ℓ_p, $(1 < p < \infty)$

The fine spectra of lower triangular double-band matrix have been examined by several authors. Here we determine the fine spectra of upper triangular double-band matrix over the sequence spaces ℓ_p, where $0 < p < \infty$. The main purpose of this paper is to determine the fine spectrum of $A(\widetilde{r}, \widetilde{s})$ in the space of ℓ_p with respect to the Goldberg's classification, where $p > 1$.

Theorem 3.3.1. *The operator $A(\widetilde{r}, \widetilde{s}) : \ell_p \to \ell_p$ is a bounded linear operator and*

$$\sup_{k\in\mathbb{N}}(|r_k|^p + |s_k|^p)^{1/p} \le \|A(\widetilde{r}, \widetilde{s})\|_{(\ell_p:\ell_p)} \le \sup_{k\in\mathbb{N}}|r_k| + \sup_{k\in\mathbb{N}}|s_k|. \tag{3.9}$$

Proof. Since the linearity of the operator $A(\widetilde{r}, \widetilde{s})$ clear. Now we prove that (3.9) holds on the space ℓ_p. It is trivial that $A(\widetilde{r}, \widetilde{s})e^{(k)} = (0, 0, \ldots, s_{k-1}, r_k, 0, \ldots, 0, \ldots)$ for $e^{(k)} \in \ell_p$. Therefore, we have

$$\|A(\widetilde{r}, \widetilde{s}\|_{(\ell_p:\ell_p)} \ge \frac{\|A(\widetilde{r}, \widetilde{s})e^{(k)}\|_{\ell_p}}{\|e^{(k)}\|_{\ell_p}} = (|r_k|^p + |s_{k-1}|^p)^{1/p}$$

which implies that

$$\|A(\widetilde{r}, \widetilde{s}\|_{(\ell_p:\ell_p)} \ge \sup_{k\in\mathbb{N}}(|r_k|^p + |s_k|^p)^{1/p} \tag{3.10}$$

Let $x = (x_k) \in \ell_p$, where $p > 1$. Then, since $(s_k x_{k+1}), (r_k x_k) \in \ell_p$ it is easy to see by Minkowski's inequality that

$$
\begin{aligned}
\|A(\widetilde{r},\widetilde{s})x\|_{(\ell_p:\ell_p)} &= \left(\sum_k |s_k x_{k+1} + r_k x_k|^p\right)^{1/p} \\
&\le \left(\sum_k |s_k x_{k+1}|^p\right)^{1/p} + \left(\sum_k |r_k x_k|^p\right)^{1/p} \\
&\le \sup_{k\in\mathbb{N}} |r_k| \left(\sum_k |x_k|^p\right)^{1/p} + \sup_{k\in\mathbb{N}} |s_k| \left(\sum_k |x_{k+1}|^p\right)^{1/p} \\
&= \left(\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|\right) \|x\|_{\ell_p}
\end{aligned}
$$

which leads us to the the result that

$$
\|A(\widetilde{r},\widetilde{s})\|_{(\ell_p:\ell_p)} \le \sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|. \tag{3.11}
$$

Therefore, by combining the inequalities in (3.10) and (3.11) we have (3.9), as desired. □

Lemma 3.3.2. *(R.El-Shabrawy, 2012, p. 115, Lemma 3.1) Let* $1 < p < \infty$. *If*

$$
\alpha \in \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\},
$$

then the series

$$
\sum_{k=1}^{\infty} \left| \frac{(r_{k-1}-\alpha)(r_{k-2}-\alpha)\cdots(r_1-\alpha)(r_0-\alpha)}{s_{k-1}s_{k-2}\dots s_1 s_0} \right|^p
$$

is not convergent.

Theorem 3.3.3.

$$
\sigma_p[A(\widetilde{r},\widetilde{s}),\ell_p] = \begin{cases} \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\} & , \ \widetilde{r},\widetilde{s} \in \mathcal{C}, \\ \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\} \cup \{(r_k)_{k\in\mathbb{N}}\} & , \ \widetilde{r},\widetilde{s} \in \mathcal{SD}, \end{cases}
$$

Proof. Let $A(\widetilde{r},\widetilde{s})x = \alpha x$ for $\theta \ne x \in \ell_p$ Then, by solving the system of linear equations

$$
\left.
\begin{aligned}
r_0 x_0 + s_0 x_1 &= \alpha x_0 \\
r_1 x_1 + s_1 x_2 &= \alpha x_1 \\
r_2 x_2 + s_2 x_3 &= \alpha x_2 \\
&\vdots \\
r_{k-1} x_{k-1} + s_{k-1} x_k &= \alpha x_{k-1} \\
&\vdots
\end{aligned}
\right\}
$$

we find that $x_k = \left(\frac{\alpha - r_k}{s_{k-1}}\right) x_{k-1}$ for all $k \geq 1$ and

$$x_k = \left[\frac{(r_{k-1} - \alpha)(r_{k-2} - \alpha) \cdots (r_1 - \alpha)(r_0 - \alpha)}{s_{k-1} s_{k-2} \ldots s_1 s_0}\right] x_0.$$

Part 1. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{C}$. Let $r_k = r$ and $s_k = s$ For all $k \in \mathbb{N}$. We observe that $x_k = \left(\frac{\alpha - r}{s}\right)^k x_0$. This shows that $x = (x_k) \in \ell_p$ if and only if $|\alpha - r| < |s|$, as asserted.

Part 2. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. We can take $x_0 \neq 0$, since $x \neq 0$. It is clear that, for all $k \in \mathbb{N}$, the vector $x = (x_0, x_1, \ldots, x_k, 0, 0, \ldots)$ is an eigenvector of the operator $A(\widetilde{r}, \widetilde{s})$ corresponding to the eigenvalue $\alpha = r_k$, where $x_0 \neq 0$ and $x_n = \left(\frac{\alpha - r_n}{s_{n-1}}\right) x_{n-1}$ for $1 \leq n \leq k$. Thus, $\{r_k : k \in \mathbb{N}\} \subseteq \sigma_p[A(\widetilde{r}, \widetilde{s}), \ell_p]$. If $r_k \neq \alpha$, for all $k \in \mathbb{N}$, then $x_k \neq 0$. If we take $|\alpha - r| < |s|$, since $\left|\frac{x_{k+1}}{x_k}\right|^p = \left|\frac{r_k - \alpha}{s_k}\right|^p \longrightarrow \left|\frac{r - \alpha}{s}\right|^p < 1$ as $k \longrightarrow \infty$, $x = (x_k) \in \ell_p$. Hence, $\{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \subseteq \sigma_p[A(\widetilde{r}, \widetilde{s}), \ell_p]$. Conversely, let $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s}), \ell_p]$. Then, there exists $x = (x_0, x_1, x_2, \ldots)$ in ℓ_p and we have $x_k = \left(\frac{\alpha - r_k}{s_{k-1}}\right) x_{k-1}$ for all $k \geq 1$. Since $x \in \ell_p$, we can use ratio test. Therefore, $\left|\frac{x_{k+1}}{x_k}\right|^p = \left|\frac{r_k - \alpha}{s_k}\right|^p \longrightarrow \left|\frac{r - \alpha}{s}\right|^p < 1$ as $k \longrightarrow \infty$ or $\alpha \in \{r_k : k \in \mathbb{C}\}$. If $|\alpha - r| = |s|$, by Lemma 3.3.2; $x \notin \ell_p$. This completes the proof. □

Theorem 3.3.4. $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_p^*] = \begin{cases} \emptyset & , \quad \widetilde{r}, \widetilde{s} \in \mathcal{C}, \\ \mathcal{B} & , \quad \widetilde{r}, \widetilde{s} \in \mathcal{SD}, \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{C}$. Consider $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $f \neq \theta = (0, 0, 0, \ldots)$ in $\ell_p^* = \ell_q$. Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} r_0 f_0 & = & \alpha f_0 \\ s_0 f_0 + r_1 f_1 & = & \alpha f_1 \\ s_1 f_1 + r_2 f_2 & = & \alpha f_2 \\ & \vdots & \\ s_{k-1} f_{k-1} + r_k f_k & = & \alpha f_k \\ & \vdots & \end{array}\right\}$$

we find that $f_0 = 0$ if $\alpha \neq r = r_k$ and $f_1 = f_2 = \cdots = 0$ if $f_0 = 0$ which contradicts $f \neq \theta$. If f_{n_0} is the first non zero entry of the sequence $f = (f_n)$ and $\alpha = r$, then we get $s_{n_0} f_{n_0} + r f_{n_0+1} = \alpha f_{n_0+1}$ which implies $f_{n_0} = 0$ which contradicts the assumption $f_{n_0} \neq 0$. Hence, the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ has no solution $f \neq \theta$.

Part 2. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. Then, by solving the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $f \neq \theta = (0,0,0,\ldots)$ in ℓ_q we obtain $(r_0 - \alpha)f_0 = 0$ and $(r_{k+1} - \alpha)f_{k+1} + s_k f_k = 0$ for all $k \in \mathbb{N}$. Hence, for all $\alpha \notin \{r_k : k \in \mathbb{N}\}$, we have $f_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. So, $\alpha \notin \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_q]$. This shows that $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_q] \subseteq \{r_k : k \in \mathbb{N}\} \setminus \{r\}$. Now, we prove that $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_q]$ if and only if $\alpha \in \mathcal{B}$. If $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_q]$, Then, by solving the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $f \neq \theta = (0,0,0,\ldots)$ in ℓ_q with $\alpha = r_0$

$$f_k = \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_k)(r_0 - r_{k-1})(r_0 - r_{k-2}) \cdots (r_0 - r_1)} f_0 \quad \text{for all} \quad k \geq 1$$

which can be expressed by the recursion relation

$$|f_k| = \left| \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_1)(r_0 - r_2) \cdots (r_0 - r_k)} \right| |f_0|.$$

Using ratio test,

$$\lim_{k\to\infty} \left| \frac{f_k}{f_{k-1}} \right|^q = \lim_{k\to\infty} \left| \frac{s_{k-1}}{r_k - r_0} \right|^q = \left| \frac{s}{r - r_0} \right|^q \leq 1.$$

But $\left| \frac{s}{r - r_0} \right| \neq 1$. Hence, $\alpha = r_0 \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. If we choose $\alpha = r_k \neq r$ for all $k \in \mathbb{N}_1$, then we get $f_0 = f_1 = f_2 = \cdots = f_{k-1} = 0$ and

$$f_{n+1} = \frac{s_n s_{n-1} s_{n-2} \ldots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} f_k \text{ for all } n \geq k$$

which can be expressed by the recursion relation

$$|f_{n+1}| = \left| \frac{s_{n-1} s_{n-2} s_{n-2} \ldots s_k}{(r_k - r_{n+1})(r_k - r_{n-1})(r_k - r_{n-2}) \cdots (r_k - r_{k+1})} \right| |f_k|.$$

Using ratio test;

$$\lim_{n\to\infty} \left| \frac{f_{n+1}}{f_n} \right|^q = \lim_{n\to\infty} \left| \frac{s_n}{r_{n+1} - r_k} \right|^q = \left| \frac{s}{r - r_k} \right|^q \leq 1.$$

But $\left| \frac{s}{r - r_k} \right| \neq 1$. So we have, $\alpha = r_k \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. Hence, $\sigma_p(A(\widetilde{r}, \widetilde{s})^*, \ell_q) \subseteq \mathcal{B}$. Conversely, Let $\alpha \in \mathcal{B}$. Then, there exists $k \in \mathbb{N}$ such that $\alpha = r_k \neq r$ and

$$\lim_{n\to\infty} \left| \frac{f_n}{f_{n-1}} \right|^q = \lim_{n\to\infty} \left| \frac{s_n}{r_{n+1} - r_k} \right|^q = \left| \frac{s}{r - r_k} \right|^q < 1.$$

That is $f = (f_n) \in \ell_q$. So we have $\mathcal{B} \subseteq \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_q]$. This completes the proof. $\square$

Theorem 3.3.5. *If $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ and $\mathcal{C}$, then $\sigma_r[A(\widetilde{r}, \widetilde{s}), \ell_p] = \emptyset$.*

Proof. By Theorem 3.3.4 and Theorem 3.1.4 $\sigma_r[A(\widetilde{r}, \widetilde{s}), \ell_p] = \emptyset$. □

Theorem 3.3.6. *$\sigma[A(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \cup \mathcal{B}$, where the set $\mathcal{B}$ is finite.*

Proof. We will show that $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is onto, for $|r - \alpha| > |s|$. Thus, for every $y \in \ell_q$, we find $x \in \ell_q$. $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is triangle so it has an inverse. Also the equation $A_\alpha(\widetilde{r}, \widetilde{s})^* x = y$ gives $[A_\alpha(\widetilde{r}, \widetilde{s})^*]^{-1} y = x$. It is sufficient to show that $[A_\alpha(\widetilde{r}, \widetilde{s})^*]^{-1} \in (\ell_q : \ell_q)$. By equation (2.5), we have

$$[(A(\widetilde{r}, \widetilde{s}) - \alpha I)^*]^{-1} \in (\ell_q : \ell_q) \text{ for } \alpha \in \mathbb{C} \text{ with } |r - \alpha| > |s|\,.$$

Hence, $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is onto. By Lemma 2.1.9, $A_\alpha(\widetilde{r}, \widetilde{s})$ has bounded inverse. This means that $\sigma_c[A(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\}$.
Combining this with Theorem 3.3.3 and Theorem 3.3.5, we get

$$\sigma[A(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\} \cup \mathcal{B} \tag{3.12}$$

Again from Theorem 3.3.3 $\{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), \ell_p]$ and $\mathcal{B} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), \ell_p]$. Since the spectrum of any bounded operator is closed, we have

$$\{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\} \cup \mathcal{B} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), \ell_p]. \tag{3.13}$$

Combining (3.12) and (3.13), we get

$$\sigma[A(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \cup \mathcal{B}.$$

□

Theorem 3.3.7. *If $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ or $\mathcal{C}$, then $\sigma_c[A(\widetilde{r}, \widetilde{s}), \ell_p] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.*

Proof. The proof immediately follows from Theorem 3.3.3, Theorem 3.3.5 and Theorem 3.3.6 because the parts $\sigma_c[A(\widetilde{r}, \widetilde{s}), \ell_p]$, $\sigma_r[A(\widetilde{r}, \widetilde{s}), \ell_p]$ and $\sigma_p[A(\widetilde{r}, \widetilde{s}), \ell_p]$ are pairwise disjoint sets and union of these sets is $\sigma[A(\widetilde{r}, \widetilde{s}), \ell_p]$. □

Theorem 3.3.8. *If $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ or $\mathcal{C}$ and $|\alpha - r| < |s|$, then $\alpha \in \sigma[A(\widetilde{r}, \widetilde{s}), \ell_p]A_3$.*

Proof. From Theorem 3.3.3, $\alpha \in \sigma_p[A(\widetilde{r},\widetilde{s}),\ell_p]$. Thus, $[A(\widetilde{r},\widetilde{s})-\alpha I]^{-1}$ does not exist. It is sufficient to show that the operator $A(\widetilde{r},\widetilde{s})-\alpha I$ is onto, i.e., for given $y=(y_k)\in\ell_p$, we have to find $x=(x_k)\in\ell_p$ such that $[A(\widetilde{r},\widetilde{s})-\alpha I]x=y$. Let us solve the matrix equation $[A(\widetilde{r},\widetilde{s})-\alpha I]x=y$. Let $x_0=0$. Therefore, we obtain

$$\begin{aligned}
x_1 &= \frac{y_0}{s_0},\\
x_2 &= \frac{(\alpha-r_1)y_0}{s_1s_0}+\frac{y_1}{s_1},\\
&\vdots\\
x_k &= \frac{(\alpha-r_1)(\alpha-r_2)\cdots(\alpha-r_{k-1})y_0}{s_0s_1\cdots s_{k-1}}+\cdots+\frac{(r_{k-2}-\alpha)y_{k-2}}{s_{k-1}s_{k-2}}+\frac{y_{k-1}}{s_{k-1}}.
\end{aligned}$$

Then, $\sum_k |x_k|^p \le \sup_{k\in\mathbb{N}}(R_k)^p\sum_k|y_k|^p$, where

$$R_k = \left|\frac{1}{s_k}\right|+\left|\frac{(r_{k+1}-\alpha)}{s_ks_{k+1}}\right|+\left|\frac{(r_{k+1}-\alpha)(r_{k+2}-\alpha)}{s_ks_{k+1}s_{k+2}}\right|+\cdots$$

for all $k\in\mathbb{N}$. By Theorem 2.1.14, we have $\sup_{k\in\mathbb{N}}(R_k)^p<\infty$. Therefore,

$$\sum_k |x_k|^p \le \sup_{k\in\mathbb{N}}(R_k)^p\sum_k |y_k|^p<\infty.$$

This shows that $x=(x_k)\in\ell_p$. Thus $A(\widetilde{r},\widetilde{s})-\alpha I$ is onto. So, we have $\alpha\in\sigma[A(\widetilde{r},\widetilde{s}),\ell_p]A_3$. □

Theorem 3.3.9. *Let $\widetilde{r},\widetilde{s}\in\mathcal{C}$ with $r_k=r$, $s_k=s$ for all $k\in\mathbb{N}$. Then, the following statements hold:*

(i) $\sigma_{ap}[A(\widetilde{r},\widetilde{s}).\ell_p]=\sigma[A(\widetilde{r},\widetilde{s}),\ell_p]$.

(ii) $\sigma_\delta[A(\widetilde{r},\widetilde{s}),\ell_p]=\{\alpha\in\mathbb{C}:|r-\alpha|=|s|\}$.

(iii) $\sigma_{co}[A(\widetilde{r},\widetilde{s}),\ell_p]=\emptyset$.

Proof. (i) Since from Table 1.2,

$$\sigma_{ap}[A(\widetilde{r},\widetilde{s}),\ell_p]=\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_1\right]\backslash\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_1$$

we have by Theorem 3.3.5 that $\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_1=\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_2=\emptyset$. Hence; $\sigma_{ap}[A(\widetilde{r},\widetilde{s}),\ell_p]=\mathcal{A}$.

(ii) Since the following equality

$$\sigma_\delta[A(\widetilde{r},\widetilde{s}),\ell_p]=\sigma[A(\widetilde{r},\widetilde{s}),\ell_p]\backslash\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]A_3$$

holds from Table 1.2, we derive by Theorem 3.3.6 and Theorem 3.3.8 that
$\sigma_\delta[A(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\}$.

(iii) From Table 1.2, we have

$$\sigma_{co}[A(\widetilde{r},\widetilde{s}),\ell_p] = \sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_1 \cup \sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_2 \cup \sigma\left[A(\widetilde{r},\widetilde{s}),c_0\right]C_3$$

by Theorem 3.3.4 it is immediate that $\sigma_{co}[A(\widetilde{r},\widetilde{s}),\ell_p] = \emptyset$. □

Theorem 3.3.10. *Let* $\widetilde{r},\widetilde{s} \in \mathcal{SD}$. *Then*

$$\sigma_{ap}[A(\widetilde{r},\widetilde{s}),\ell_p] = \mathcal{A}\cup\mathcal{B}, \sigma_\delta[A(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha\in\mathbb{C} : |r-\alpha| = |s|\}\cup\mathcal{B}, \sigma_{co}[A(\widetilde{r},\widetilde{s}),\ell_p] = \mathcal{B}.$$

Proof. We have by Theorem 3.3.4 and Part (e) of Proposition 1.3.1 that

$$\sigma_p[A(\widetilde{r},\widetilde{s})^*,\ell_p^*] = \sigma_{co}[A(\widetilde{r},\widetilde{s}),\ell_p] = \mathcal{B}.$$

By Theorem 3.3.5 and Theorem 3.3.4, we must have

$$\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_1 = \sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_2 = \emptyset.$$

Hence, $\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_3 = \{r_k\}$. Additionally, since $\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_1 = \emptyset$.
Therefore, we derive from Table 1.2, Theorem 3.3.6 and Theorem 3.3.8 that

$$\begin{aligned}
\sigma_{ap}[A(\widetilde{r},\widetilde{s}),\ell_p] &= \sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]\backslash\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]C_1 = \sigma\left[A(\widetilde{r},\widetilde{s}),\ell_1\right].\\
\sigma_\delta[A(\widetilde{r},\widetilde{s}),\ell_p] &= \sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]\backslash\sigma\left[A(\widetilde{r},\widetilde{s}),\ell_p\right]A_3 = \{\alpha\in\mathbb{C} : |r-\alpha| = |s|\}\cup\mathcal{B}.
\end{aligned}$$

□

3.4 FINE SPECTRA OF UPPER TRIANGULAR DOUBLE-BAND MATRIX OVER THE SEQUENCE SPACE ℓ_p, $(0 < p < 1)$

Theorem 3.4.1. *The operator* $A(\widetilde{r},\widetilde{s}) : \ell_p \to \ell_p$ *is a bounded linear operator and*

$$\|A(\widetilde{r},\widetilde{s}\|_{(\ell_p:\ell_p)} = \sup_{k\in\mathbb{N}}|r_k|^p + \sup_{k\in\mathbb{N}}|s_k|^p. \tag{3.14}$$

Proof. It is obvious that the operator $A(\widetilde{r},\widetilde{s})$ is linear. Now, we prove that (3.14) holds for the operator $A(\widetilde{r},\widetilde{s})$ on the space ℓ_p. It is trivial that
$A(\widetilde{r},\widetilde{s})e^{(k)} = (0,0,\ldots,s_{k-1},r_k,0,\ldots,0,\ldots)$ for $e^{(k)}\in\ell_p$. Therefore, we have

$$\|A(\widetilde{r},\widetilde{s}\|_{(\ell_p:\ell_p)} \geq \frac{\|A(\widetilde{r},\widetilde{s})e^{(k)}\|_{\ell_p}}{\|e^{(k)}\|_{\ell_p}} = |r_k|^p + |s_{k-1}|^p$$

which implies that

$$\|A(\widetilde{r},\widetilde{s}\|_{(\ell_p:\ell_p)} \geq \sup_{k\in\mathbb{N}} |r_k|^p + \sup_{k\in\mathbb{N}} |s_k|^p. \tag{3.15}$$

Let $x = (x_k) \in \ell_p$, where $0 < p < 1$. Then, since $(s_k x_{k+1}), (r_k x_k) \in \ell_p$ it is easy to see by triangle inequality that

$$\begin{aligned}
\|A(\widetilde{r},\widetilde{s})x\|_{\ell_p} &= \sum_k |s_k x_{k+1} + r_k x_k|^p \\
&\leq \sum_k |s_k x_{k+1}|^p + \sum_k |r_k x_k|^p \\
&\leq \sup_{k\in\mathbb{N}} |r_k|^p \sum_k x_k|^p + \sup_{k\in\mathbb{N}} |s_k|^p \sum_k |x_{k+1}|^p \\
&= (\sup_{k\in\mathbb{N}} |r_k|^p + \sup_{k\in\mathbb{N}} |s_k|^p)\|x\|_{\ell_p}
\end{aligned}$$

which leads us to the the result that

$$\|A(\widetilde{r},\widetilde{s})\|_{(\ell_p:\ell_p)} = \sup_{k\in\mathbb{N}} |r_k|^p + \sup_{k\in\mathbb{N}} |s_k|^p. \tag{3.16}$$

Therefore, by combining the inequalities in (3.15) and (3.16) we have (3.14), as desired. □

Theorem 3.4.2. $\sigma_p[A(\widetilde{r},\widetilde{s}),\ell_p] = \begin{cases} \{\alpha\in\mathbb{C} : |r-\alpha| < |s|\} & , \quad \widetilde{r},\widetilde{s}\in\mathcal{C}, \\ \{\alpha\in\mathbb{C} : |r-\alpha| < |s|\}\cup\mathcal{B} & , \quad \widetilde{r},\widetilde{s}\in\mathcal{SD}, \end{cases}$

Proof. This may be obtained in the similar way used in the proof of Theorem 3.3.3. So, we omit the details. □

If $T : \ell_p \longrightarrow \ell_p$ is a bounded matrix operator with the matrix A, then it is known that the adjoint operator $T^* : \ell_p^* \longrightarrow \ell_p^*$ is defined by the transpose of the matrix A. It is known that the dual space ℓ_p^* of ℓ_p is isomorphic to ℓ_∞, where $0 < p < 1$.

Theorem 3.4.3. $\sigma_p[A(\widetilde{r},\widetilde{s})^*,\ell_p^*] = \begin{cases} \emptyset & , \quad \widetilde{r},\widetilde{s}\in\mathcal{C}, \\ \mathcal{B} & , \quad \widetilde{r},\widetilde{s}\in\mathcal{SD}, \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r},\widetilde{s}\in\mathcal{C}$. Consider $A(\widetilde{r},\widetilde{s})^* f = \alpha f$ for $f \neq \theta = (0,0,0,\ldots)$ in

$\ell_p^* = \ell_\infty$. Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} r_0 f_0 &=& \alpha f_0 \\ s_0 f_0 + r_1 f_1 &=& \alpha f_1 \\ s_1 f_1 + r_2 f_2 &=& \alpha f_2 \\ &\vdots& \\ s_{k-1} f_{k-1} + r_k f_k &=& \alpha f_k \\ &\vdots& \end{array}\right\}$$

we find that $f_0 = 0$ if $\alpha \neq r = r_k$ and $f_1 = f_2 = \cdots = 0$ if $f_0 = 0$ which contradicts $f \neq \theta$. If f_{n_0} is the first non zero entry of the sequence $f = (f_n)$ and $\alpha = r$, then we get $s_{n_0} f_{n_0} + r f_{n_0+1} = \alpha f_{n_0+1}$ which implies $f_{n_0} = 0$ which contradicts the assumption $f_{n_0} \neq 0$. Hence, the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ has no solution $f \neq \theta$.

Part 2. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. Then, by solving the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $f \neq \theta = (0, 0, 0, \ldots)$ in ℓ_∞ we obtain $(r_0 - \alpha) f_0 = 0$ and $(r_{k+1} - \alpha) f_{k+1} + s_k f_k = 0$ for all $k \in \mathbb{N}$. Hence, for all $\alpha \notin \{r_k : k \in \mathbb{N}\}$, we have $f_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. So, $\alpha \notin \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_\infty]$. This shows that $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_\infty] \subseteq \{r_k : k \in \mathbb{N}\} \setminus \{r\}$. Now, we prove that $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_\infty]$ if and only if $\alpha \in \mathcal{B}$. If $\alpha \in \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_\infty]$, then, by solving the equation $A(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $f \neq \theta = (0, 0, 0, \ldots)$ in ℓ_∞ with $\alpha = r_0$ we find

$$f_k = \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_k)(r_0 - r_{k-1})(r_0 - r_{k-2}) \cdots (r_0 - r_1)} f_0 \quad \text{for all} \quad k \in \mathbb{N}$$

which can be expressed by the recursion relation

$$|f_k| = \left| \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_1)(r_0 - r_2) \cdots (r_0 - r_k)} \right| |f_0|.$$

Since $\ell_1 \subseteq \ell_\infty$, we can apply the ratio test,

$$\lim_{k \to \infty} \left| \frac{f_k}{f_{k-1}} \right| = \lim_{k \to \infty} \left| \frac{s_{k-1}}{r_k - r_0} \right| = \left| \frac{s}{r - r_0} \right| \leq 1.$$

But $\left| \frac{s}{r - r_0} \right| \neq 1$. Hence, $\alpha = r_0 \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. Similarly we can prove that $\alpha = r_k \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$, for $\alpha = r_k \neq r$ for all $k \in \mathbb{N}_1$. Hence, $\sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_\infty] \subseteq \mathcal{B}$. Conversely, Let $\alpha \in \mathcal{B}$. Then exists $k \in \mathbb{N}$, $\alpha = r_k \neq r$ and

$$\lim_{n \to \infty} \left| \frac{f_n}{f_{n-1}} \right| = \lim_{n \to \infty} \left| \frac{s_{n-1}}{r_n - r_k} \right| = \left| \frac{s}{r - r_k} \right| < 1.$$

This shows that $f = (f_n) \in \ell_\infty$, since $\ell_1 \subseteq \ell_\infty$. So, we have $\mathcal{B} \subseteq \sigma_p[A(\widetilde{r}, \widetilde{s})^*, \ell_\infty]$. This completes the proof. □

Theorem 3.4.4. *If $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ or $\mathcal{C}$, then $\sigma_r[A(\widetilde{r}, \widetilde{s}), \ell_p] = \emptyset$.*

Proof. Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ or $\mathcal{C}$. By Theorem 3.4.3 and Theorem 3.1.4 it immediate that $\sigma_r[A(\widetilde{r}, \widetilde{s}), \ell_p] = \emptyset$. □

Theorem 3.4.5. $\sigma[A(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \cup \mathcal{B}$.

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ and $y = (y_k) \in \ell_\infty$. Then, by solving the equation $A_\alpha(\widetilde{r}, \widetilde{s})^* x = y$ for $x = (x_k)$ in terms of $y = (y_k)$, we obtain

$$\begin{aligned}
x_0 &= \frac{y_0}{r-\alpha}, \\
x_1 &= \frac{y_1}{r-\alpha} + \frac{-sy_0}{(r-\alpha)^2}, \\
x_2 &= \frac{y_2}{r-\alpha} + \frac{-sy_1}{(r-\alpha)^2} + \frac{s^2 y_0}{(r-\alpha)^3}, \\
&\vdots \\
x_k &= \frac{s^{k-1} y_0}{(r-\alpha)^k} + \cdots - \frac{s y_{k-1}}{(r-\alpha)^2} + \frac{y_k}{r-\alpha}, \\
&\vdots
\end{aligned}$$

which gives that,

$$x_k = \frac{1}{r-\alpha} \sum_{i=0}^{k} \left(\frac{s}{r-\alpha} \right)^{k-i} y_i$$

for all $k \in \mathbb{N}$. Hence,

$$|x_k| \leq \frac{1}{|r-\alpha|} \sum_{i=0}^{\infty} \left| \frac{s}{r-\alpha} \right|^i \|y\|_\infty.$$

For $|s| < |r-\alpha|$, we can observe that

$$\|x\|_\infty \leq \frac{1}{|r-\alpha| - |s|} \|y\|_\infty.$$

Thus, $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is onto for $|s| < |r-\alpha|$ and by Lemma 2.1.9, $A_\alpha(\widetilde{r}, \widetilde{s})$ has a bounded inverse. This means that

$$\sigma_c[A(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \{\alpha \in \mathbb{C} : |r-\alpha| \leq |s|\}.$$

Combining this with Theorem 3.4.2 and Theorem 3.4.4, we get

$$\{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \le |s|\}.$$

Since the spectrum any given operator closed. Thus, we have

$$\sigma[A(\widetilde{r}, \widetilde{s}), \ell_p] = \{\alpha \in \mathbb{C} : |r - \alpha| \le |s|\}.$$

Part 2. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ and $y = (y_k) \in \ell_\infty$. Then, by solving the equation $A_\alpha(\widetilde{r}, \widetilde{s})^* x = y$ in terms of y, we obtain

$$\begin{aligned}
x_0 &= \frac{y_0}{r_0 - \alpha}, \\
x_1 &= \frac{y_1}{r_1 - \alpha} + \frac{-s_0 y_0}{(r_1 - \alpha)(r_0 - \alpha)}, \\
x_2 &= \frac{y_2}{r_2 - \alpha} + \frac{-s_1 y_1}{(r_2 - \alpha)(r_1 - \alpha)} + \frac{s_0 s_1 y_0}{(r_2 - \alpha)(r_1 - \alpha)(r_0 - \alpha)}, \\
&\vdots \\
x_k &= \frac{(-1)^k s_0 s_1 s_2 \cdots s_{k-1} y_0}{(r_0 - \alpha)(r_1 - \alpha)(r_2 - \alpha) \cdots (r_k - \alpha)} + \cdots - \frac{s_{k-1} y_{k-1}}{(r_k - \alpha)(r_{k-1} - \alpha)} + \frac{y_k}{r_k - \alpha}, \\
&\vdots
\end{aligned}$$

Then, $|x_k| \le S_k \|y\|_\infty$, where

$$\begin{aligned}
S_k &= \left|\frac{1}{r_k - \alpha}\right| + \left|\frac{s_{k-1}}{(r_{k-1} - \alpha)(r_k - \alpha)}\right| + \left|\frac{s_{k-1} s_{k-2}}{(r_{k-2} - \alpha)(r_{k-1} - \alpha)(r_k - \alpha)}\right| + \\
&\quad + \cdots + \left|\frac{s_0 s_1 \ldots s_{k-1}}{(r_0 - \alpha)(r_1 - \alpha) \cdots (r_k - \alpha)}\right|.
\end{aligned}$$

By Theorem 2.1.5, we have $\sup_{k \in \mathbb{N}} S_k < \infty$. This shows that $|x\|_\infty \le \|(S_k)\|_\infty \|y\|_\infty < \infty$, since $(y_k) \in \ell_\infty$. Thus for $|s| < |r - \alpha|$, $A_\alpha(\widetilde{r}, \widetilde{s})^*$ is onto and by Lemma 2.1.9, $A_\alpha(\widetilde{r}, \widetilde{s})$ has a bounded inverse. This means that

$$\sigma_c[A(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \le |s|\}.$$

Combining this with Theorem 3.4.2 and Theorem 3.4.4, we get

$$\mathcal{B} \cup \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} \subseteq \sigma[A(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \{\alpha \in \mathbb{C} : |r - \alpha| \le |s|\} \cup \mathcal{B}$$

Since the spectrum compact operator so it has closed. Thus, we have

$$\sigma[A(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \cup \mathcal{B}.$$

This completes the proof. □

In the case $(0 < p \leq 1)$, since the spectrum and fine spectrum of the matrix $A(\widetilde{r}, \widetilde{s})$ as an operator on the sequence space ℓ_p are similar to that of the $(1 < p < \infty)$, to avoid the repetition of similar statements we give the results by the following theorem without proof:

Theorem 3.4.6. *The following statements hold:*

(i) *If* $\widetilde{r}, \widetilde{s}$ *in* $\mathcal{SD}$ *or* $\mathcal{C}$, *then* $\sigma_c[A(\widetilde{r}, \widetilde{s}), \ell_p] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.

(ii) *If* $\widetilde{r}, \widetilde{s}$ *in* $\mathcal{SD}$ *or* $\mathcal{C}$ *and* $|\alpha - r| < |s|$, *then* $\alpha \in \sigma[A(\widetilde{r}, \widetilde{s}), \ell_p]A_3$.

(iii) *If* $\widetilde{r}, \widetilde{s}$ *in* $\mathcal{SD}$ *or* $\mathcal{C}$, *then* $\sigma_{ap}[A(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma[A(\widetilde{r}, \widetilde{s}, \ell_p]$.

(iv) *If* $\widetilde{r}, \widetilde{s}$ *in* $\mathcal{SD}$ *or* $\mathcal{C}$. $\sigma_\delta[A(\widetilde{r}, \widetilde{s}), \ell_p] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.

(v) *If* $\widetilde{r} \in \mathcal{SD}$, *then* $\sigma_{co}[A(\widetilde{r}, \widetilde{s}), \ell_p] = \emptyset$.

(vi) *If* $\widetilde{r} \in \mathcal{SD}$, $\sigma_{co}[A(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{B}$.

CHAPTER 4

FINE SPECTRUM OF UPPER TRIANGULAR TRIPLE-BAND MATRIX OVER SOME SEQUENCE SPACES

In chapter 4, we determine the fine spectrum of the upper triangular triple-band matrix $A(r, s, t)$ over the sequence space μ where $\mu \in \{\ell_p, c, c_0\}$ with $(0 < p < \infty)$. The operator $A(r, s, t)$ on sequence space μ is defined by $A(r, s, t)x = (rx_k + sx_{k+1} + tx_{k+2})_{k=0}^{\infty}$, where $x = (x_k) \in \mu$. In this chapter, we obtain the results on the spectrum and point spectrum for the operator $A(r, s, t)$ on the sequence space μ. Further, the results on continuous spectrum, residual spectrum and fine spectrum of the operator $A(r, s, t)$ on the sequence space μ is also derived. Further, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $A(r, s, t)$ over the space μ and we give some applications.

Our main focus in this chapter is on the triple-band matrix $A(r, s, t)$, where

$$A(r, s, t) = \begin{bmatrix} r & s & t & 0 & \dots \\ 0 & r & s & t & \dots \\ 0 & 0 & r & s & \dots \\ 0 & 0 & 0 & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We assume here and after that s and t are complex parameters which do not simultaneously vanish. Let we introduce the operator $A(r, s, t)$ from μ to itself by

$$A(r, s, t)x = (rx_k + sx_{k+1} + tx_{k+2})_{k=0}^{\infty}, \text{ where } x = (x_k) \in \mu.$$

4.1 FINE SPECTRUM OF UPPER TRIANGULAR TRIPLE-BAND MATRIX $A(r,s,t)$ OVER THE SEQUENCE SPACE ℓ_p, $(0<p\le 1)$

Theorem 4.1.1. *The operator* $A(r,s,t):\ell_p\to\ell_p$ *is a bounded linear operator and*

$$\|A(r,s,t)\|_{(\ell_p:\ell_p)} = |r|^p+|s|^p+|t|^p. \tag{4.1}$$

Proof. The linearity of the operator $A(r,s,t)$ is trivial. Let us take $e^{(2)}\in\ell_p$. Then, $A(r,s,t)e^{(2)}=(t,s,r,0,\ldots)$ and observe that

$$\|A(r,s,t)\|_{(\ell_p:\ell_p)} \ge \frac{\|A(r,s,t)e^{(2)}\|_p}{\|e^{(2)}\|_p} = |r|^p+|s|^p+|t|^p$$

which gives the fact that

$$\|A(r,s,t)\|_{(\ell_p:\ell_p)} \ge |r|^p+|s|^p+|t|^p. \tag{4.2}$$

Let $x=(x_k)\in\ell_p$, where $0<p\le 1$. Then, since $(tx_{k+2}),(rx_k)$ and $(sx_{k+1})\in\ell_p$ it is easy to see by triangle inequality that

$$\begin{aligned}
\|A(r,s,t)x\|_p &= \sum_k |rx_k+sx_{k+1}+tx_{k+2}|^p \\
&\le \sum_k |rx_k|^p+\sum_k |sx_{k+1}|^p+\sum_k |tx_{k+2}|^p \\
&= |r|^p\sum_k |x_k|^p+|s|^p\sum_k |x_{k+1}|^p+|t|^p\sum_k |x_{k+2}|^p \\
&= (|r|^p+|s|^p+|t|^p)\|x\|_p
\end{aligned}$$

which leads us to the result that

$$\|A(r,s,t))\|_{(\ell_p:\ell_p)} \le |r|^p+|s|^p+|t|^p. \tag{4.3}$$

Therefore, by combining the inequalities (4.2) and (4.3) we see that (4.1) holds which completes the proof. □

Before giving the main theorem of this section, we should note the following remark. In this work, here and in what follows, if z is a complex number then by $\sqrt{z}$ we always mean the square root of z with a nonnegative real part. If $Re(\sqrt{z})=0$ then $\sqrt{z}$ represents the square root of z with $Im(\sqrt{z})>0$. The same results are obtained if $\sqrt{z}$ represents the other square root.

Theorem 4.1.2. *Let s be a complex number such that $\sqrt{s^2} = -s$ and define the set D_1 by*

$$D_1 = \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \big|-s+\sqrt{s^2-4t(r-\alpha)}\big|\right\}.$$

Then, $\sigma_c[A(r,s,t),\ell_p] \subseteq D_1$.

Proof. Let $y = (y_k) \in \ell_\infty$. Then, by solving the equation $A_\alpha(r,s,t)^* x = y$ for $x = (x_k)$ in terms of y, we obtain

$$\begin{aligned} x_0 &= \frac{y_0}{r-\alpha}, \\ x_1 &= \frac{y_1}{r-\alpha} + \frac{-sy_0}{(r-\alpha)^2}, \\ x_2 &= \frac{y_2}{r-\alpha} + \frac{-sy_1}{(r-\alpha)^2} + \frac{[s^2-t(r-\alpha)]y_0}{(r-\alpha)^3} \\ &\vdots \end{aligned}$$

If we denote $a_1 = \frac{1}{r-\alpha}$, $a_2 = \frac{-s}{(r-\alpha)^2}$, $a_3 = \frac{s^2-t(r-\alpha)}{(r-\alpha)^3}$, then we have

$$\begin{aligned} x_0 &= a_1y_0, \\ x_1 &= a_1y_1 + a_2y_0, \\ x_2 &= a_1y_2 + a_2y_1 + a_3y_0, \\ &\vdots \\ x_n &= a_1y_n + a_2y_{n-1} + \cdots + a_{n+1}y_0 = \sum_{k=0}^{n} a_{n+1-k}y_k. \end{aligned} \tag{4.4}$$

Now, we must find a_n. We have $y_n = tx_{n-2} + sx_{n-1} + (r-\alpha)x_n$. If we use relation (4.4), then we obtain that

$$\begin{aligned} y_n &= t\sum_{k=0}^{n-2} a_{n-1-k}y_k + s\sum_{k=0}^{n-1} a_{n-k}y_k + (r-\alpha)\sum_{k=0}^{n} a_{n+1-k}y_k \\ &= y_0(ta_{n-1} + sa_n + (r-\alpha)a_{n+1}) + y_1(ta_{n-2} + sa_{n-1} + (r-\alpha)a_n) + \cdots + y_na_1(r-\alpha). \end{aligned}$$

This implies that

$$ta_{n-1} + sa_n + (r-\alpha)a_{n+1} = 0, \quad ta_{n-2} + sa_{n-1} + (r-\alpha)a_n = 0, \ldots, \quad a_1(r-\alpha) = 1.$$

In fact, this sequence is obtained recursively by letting

$$a_1 = \frac{1}{r-\alpha}, \quad a_2 = \frac{-s}{(r-\alpha)^2} \quad \text{and} \quad ta_{n-2} + sa_{n-1} + (r-\alpha)a_n = 0 \quad \text{for all} \quad n \geq 3.$$

The characteristic polynomial of the recurrence relation is $(r-\alpha)\lambda^2+s\lambda+t=0$. There are two cases:

Case 1. If $\Delta=s^2-4t(r-\alpha)\neq 0$ whose roots are

$$\lambda_1=\frac{-s+\sqrt{\Delta}}{2(r-\alpha)},\quad \lambda_2=\frac{-s-\sqrt{\Delta}}{2(r-\alpha)}.$$

Elementary calculation on recurrent sequence gives that

$$a_n=\frac{\lambda_1^n-\lambda_2^n}{\sqrt{s^2-4t(r-\alpha)}}\quad \text{for all}\quad n\geq 1. \tag{4.5}$$

In this case $x_k=\frac{1}{\sqrt{\Delta}}\sum_{k=0}^{n}(\lambda_1^{n+1-k}-\lambda_2^{n+1-k})y_k$. Assume that $|\lambda_1|<1$. So we have

$$\left|1+\sqrt{\frac{4t(r-\alpha)}{s^2}}\right|<\left|\frac{2(r-\alpha)}{-s}\right|.$$

Since $|1-\sqrt{z}|\leq|1+\sqrt{z}|$ for any $z\in\mathbb{C}$, we must have

$$\left|1-\sqrt{\frac{4t(r-\alpha)}{s^2}}\right|<\left|\frac{2(r-\alpha)}{-s}\right|.$$

It follows that $|\lambda_2|<1$. Now, for $|\lambda_1|<1$ we can see that

$$|x_n|\leq\frac{1}{|\sqrt{\Delta}|}\sum_{k=0}^{n}|\lambda_1^{n+1-k}||y_k|+\sum_{k=0}^{n}|\lambda_2^{n+1-k}||y_k| \tag{4.6}$$

for all $n\in\mathbb{N}$. Taking limit on the inequality (4.6) as $n\to\infty$, we get

$$\|x\|_\infty\leq\frac{1-(|\lambda_2|+|\lambda_2|)}{|(1-|\lambda_2||1-\lambda_2|)\sqrt{\Delta}|}\|y\|_\infty.$$

Thus for $|\lambda_1|<1$, $A_\alpha(r,s,t)^*$ is onto and by Lemma 2.1.9, $A_\alpha(r,s,t)$ has a bounded inverse. This means that

$$\sigma_c[A(r,s,t),\ell_p]\subseteq\left\{\alpha\in\mathbb{C}:2\left|r-\alpha\right|\leq\left|-s+\sqrt{s^2-4t(r-\alpha)}\right|\right\}=D_1.$$

Case 2. If $\Delta=s^2-4t(r-\alpha)=0$, a calculation on recurrent sequence gives that

$$a_n=\left(\frac{2n}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^n\quad \text{for all}\ \ n\geq 1.$$

Now, for $|-s|<2|r-\alpha|$ we can see that

$$|x_n|\leq\sum_{k=0}^{n}|a_{n-k}y_k| \tag{4.7}$$

for all $n \in \mathbb{N}$. Taking limit on the inequality (4.7) as $n \to \infty$, we obtain that

$$\|x\|_\infty \leq \|y\|_\infty \sum_{k=0}^{\infty} |a_k|.$$

$\sum_k |a_k|$ is convergent, since $|-s| < 2|r-\alpha|$. Thus, for $|-s| < 2|r-\alpha|$, $A_\alpha(r,s,t)^*$ is onto and by Lemma 2.1.9, $A_\alpha(r,s,t)$ has a bounded inverse. This means that

$$\sigma_c[A(r,s,t),\ell_p] \subseteq \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \big|-s\big|\right\} \subseteq D_1.$$

□

Theorem 4.1.3. $\sigma_p[A(r,s,t)^*,\ell_p^*] = \emptyset$.

Proof. Consider $A(r,s,t)^* f = \alpha f$ with $f \neq \theta = (0,0,0,\ldots)$ in $\ell_p^* = \ell_\infty$. Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} rf_0 &=& \alpha f_0 \\ sf_0 + rf_1 &=& \alpha f_1 \\ tf_0 + sf_1 + rf_2 &=& \alpha f_2 \\ tf_1 + sf_2 + rf_3 &=& \alpha f_3 \\ &\vdots& \\ tf_{k-2} + sf_{k-1} + rf_k &=& \alpha f_k \\ &\vdots& \end{array}\right\}$$

we find that $f_0 = 0$ if $\alpha \neq r$ and $f_1 = f_2 = \cdots = 0$ if $f_0 = 0$ which contradicts $f \neq \theta$. If f_{n_0} is the first non zero entry of the sequence $f = (f_n)$ and $\alpha = r$, then we get $tf_{n_0-2} + sf_{n_0-1} + rf_{n_0} = \alpha f_{n_0}$ which implies $f_{n_0} = 0$ which contradicts the assumption $f_{n_0} \neq 0$. Hence, the equation $A(r,s,t)^* f = \alpha f$ has no solution $f \neq \theta$. □

Theorem 4.1.4. $\sigma_p[A(r,s,t),\ell_p] = D_2$, *where*

$$D_2 = \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| < \big|-s+\sqrt{s^2-4t(r-\alpha)}\big|\right\}.$$

Proof. Let $A(r,s,t)x = \alpha x$ for $\theta \neq x \in \ell_p$ Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} rx_0 + sx_1 + tx_2 &=& \alpha x_0 \\ rx_1 + sx_2 + tx_3 &=& \alpha x_1 \\ rx_2 + sx_3 + tx_4 &=& \alpha x_2 \\ &\vdots& \\ rx_{k-2} + sx_{k-1} + tx_k &=& \alpha x_k \\ &\vdots& \end{array}\right\}$$

we have

$$\begin{aligned} x_2 &= \frac{-s}{t}x_1 - \frac{r-\alpha}{t}x_0 \\ x_3 &= \frac{s^2 - t(r-\alpha)}{t^2}x_1 + \frac{s(r-\alpha)}{t^2}x_0 \\ &\vdots \\ x_n &= \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0 \quad \text{for all} \quad n \geq 2. \end{aligned} \tag{4.8}$$

Assume that $\alpha \in D_2$. Then, we choose $x_0 = 1$ and $x_1 = \frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}$. We show that $x_n = x_1^n$ for all $n \geq 2$. Since λ_1, λ_2 are roots of the characteristic equation $(r-\alpha)\lambda^2 + s\lambda + t = 0$ we must have

$$\lambda_1\lambda_2 = \frac{t}{r-\alpha} \quad \text{and} \quad \lambda_1 - \lambda_2 = \frac{\sqrt{\Delta}}{r-\alpha}$$

combining the fact $x_1 = 1/\lambda_1$ with relation (4.8) we can see that

$$\begin{aligned} x_n &= \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0 \\ &= \left(\frac{r-\alpha}{t}\right)^{n-1}(r-\alpha)(-a_{n-1}x_0 + a_nx_1) \\ &= \frac{1}{(\lambda_1\lambda_2)^{n-1}}\frac{r-\alpha}{\sqrt{\Delta}}(-\lambda_1^{n-1} + \lambda_2^{n-1} + \lambda_1^{n-1} - \lambda_2^n\lambda_1^{-1}) \\ &= \frac{1}{\lambda_1^{n-1}\lambda_2^{n-1}}\left(\frac{1}{\lambda_1 - \lambda_2}\right)\lambda_2^{n-1}\left(\frac{\lambda_1 - \lambda_2}{\lambda_1}\right) \\ &= \frac{1}{\lambda_1^n} \\ &= x_1^n. \end{aligned}$$

The same result is obtained in case $\Delta = 0$. Now $x = (x_k) \in \ell_p$, since $|x_1| < 1$. This shows that $D_2 \subseteq \sigma_p[A(r,s,t), \ell_p]$.

Now, we assume that $\alpha \notin D_2$, i.e, $|\lambda_1| \leq 1$. We show that $\alpha \notin \sigma_p[A(r,s,t),\ell_p]$. Therefore we obtain from the relation (4.8) that

$$\frac{x_{n+1}}{x_n} = \left(\frac{r-\alpha}{t}\right)\frac{a_{n-1}}{a_{n-2}}\left(\frac{-x_0+\frac{a_n}{a_{n-1}}x_1}{-x_0+\frac{a_{n-1}}{a_{n-2}}x_1}\right).$$

Now, we examine three cases.

Case 1. $|\lambda_2| < |\lambda_1| < 1$. In this case we have $s^2 \neq 4t(r-\alpha)$ and

$$\frac{a_n}{a_{n-1}} = \frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1^n-\lambda_2^n} = \frac{\lambda_1\left[1-\left(\frac{\lambda_2}{\lambda_1}\right)^{n+1}\right]}{\left[1-\left(\frac{\lambda_2}{\lambda_1}\right)^{n}\right]}.$$

Then, we have

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right|^p = \lim_{n\to\infty}\left|\frac{a_{n-1}}{a_{n-2}}\right|^p = \lim_{n\to\infty}\frac{|\lambda_1|^p\left|1-\left(\frac{\lambda_2}{\lambda_1}\right)^{n+1}\right|^p}{\left|1-\left(\frac{\lambda_2}{\lambda_1}\right)^{n}\right|^p} = |\lambda_1|^p.$$

Now, if $-x_0+\lambda_1 x_1 = 0$; then we have $(x_n) = (x_0/\lambda_1^n)$ which is not in ℓ_p. Otherwise

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|^p = \frac{1}{|\lambda_1|^p|\lambda_2|^p}|\lambda_1|^p = \frac{1}{|\lambda_2|^p} > 1.$$

Case 2. $|\lambda_2| = |\lambda_1| < 1$. In this case we have $s^2 = 4t(r-\alpha)$ and using the formula

$$a_n = \left(\frac{2n}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^n \quad \text{for all} \quad n \geq 1$$

we obtain that

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n-1}}\right|^p = \left|\frac{-s}{2(r-\alpha)}\right|^p = |\lambda_1|^p$$

which leads to

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|^p = \frac{1}{|\lambda_1|^p|\lambda_2|^p}|\lambda_1|^p = \frac{1}{|\lambda_2|^p} > 1.$$

Case 3. $|\lambda_2| = |\lambda_1| = 1$. In this case we have $s^2 = 4t(r-\alpha)$ and so we have $|-s/2t| = 1$. Assume that $\alpha \in \sigma_p[A(r,s,t),\ell_p]$. This implies that $x \in \ell_p$ and $x \neq \theta$ Thus we again derive (4.8)

$$x_n = \left(\frac{-s}{2t}\right)^{n-1}\left[-(n-1)\frac{-s}{2t}x_0 + nx_1\right].$$

Since $x_n \longrightarrow 0$ as $n \longrightarrow \infty$, we must have $x_0 = x_1 = 0$ which implies that $x = \theta$, a contradiction. This means that $\alpha \notin \sigma_p[A(r,s,t),\ell_p]$. Thus $\sigma_p[A(r,s,t),\ell_p] \subseteq D_2$. This completes the proof. $\square$

Theorem 4.1.5. $\sigma_r[A(r,s,t),\ell_p] = \emptyset$.

Proof. By Proposition 1.3.1, $\sigma_r[A(r,s,t),\ell_p] = \sigma_p[A(r,s,t)^*,\ell_p^*]\backslash\sigma_p[A(r,s,t),\ell_p]$. Since by Theorem 4.1.3, $\sigma_p[A(r,s,t)^*,\ell_p^*] = \emptyset$ and $\sigma_r[A(r,s,t),\ell_p] = \emptyset$. This completes the proof. □

Theorem 4.1.6. *Let s be a complex number such that $\sqrt{s^2} = -s$. Then,* $\sigma[A(r,s,t),\ell_p] = D_1$.

Proof. By Theorem 4.1.4,

$$\left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| < \Big|-s+\sqrt{s^2-4t(r-\alpha)}\Big|\right\} \subseteq \sigma[A(r,s,t),\ell_p].$$

Since the spectrum of any bounded operator is closed, we have

$$\left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \big|-s+\sqrt{s^2-4t(r-\alpha)}\big|\right\} \subseteq \sigma[A(r,s,t),\ell_p] \tag{4.9}$$

and again from Theorem 4.1.2, Theorem 4.1.4 and Theorem 4.1.5

$$\sigma[A(r,s,t),\ell_p] \subseteq \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \big|-s+\sqrt{s^2-4t(r-\alpha)}\big|\right\}. \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\sigma[A(r,s,t),\ell_p] = \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \big|-s+\sqrt{s^2-4t(r-\alpha)}\big|\right\} = D_1$$

□

Theorem 4.1.7. $\sigma_c[A(r,s,t),\ell_p] = D_3$, *where*

$$D_3 = \left\{\alpha \in \mathbb{C} : 2\big|r-\alpha\big| = \big|-s+\sqrt{s^2-4t(r-\alpha)}\big|\right\}.$$

Proof. Because of the parts $\sigma_c[A(r,s,t),\ell_p]$, $\sigma_r[A(r,s,t),\ell_p]$ and $\sigma_p[A(r,s,t),\ell_p]$ are pairwise disjoint sets and the union of these sets is $\sigma[A(r,s,t),\ell_p]$, the proof immediately follows from Theorem 4.1.4, Theorem 4.1.5 and Theorem 4.1.6. □

Theorem 4.1.8. *If* $\alpha \in D_2$, $\alpha \in \sigma[A(r,s,t),\ell_p]A_3$.

Proof. From Theorem 4.1.4, $\alpha \in \sigma_p[A(r,s,t),\ell_p]$. Thus, $[A(r,s,t)-\alpha I]^{-1}$ does not exist. By Theorem 4.1.3 $A(r,s,t)^* - \alpha I$ is one to one, so $A(r,s,t) - \alpha I$ has a dense range in ℓ_p by Lemma 2.1.8. □

Theorem 4.1.9. *The following statements hold:*

(i) $\sigma_{ap}[A(r,s,t).\ell_p] = D_1$,

(ii) $\sigma_\delta[A(r,s,t),\ell_p] = D_3$.

(iii) $\sigma_{co}[A(r,s,t),\ell_p] = \emptyset$.

Proof. (i) Since from Table 1.2,

$$\sigma_{ap}[A(r,s,t),\ell_p] = \sigma\left[A(r,s,t),\ell_p\right] \backslash \sigma\left[A(r,s,t),\ell_p\right] C_1$$

we have by Theorem 4.3.6 $\sigma\left[A(r,s,t),\ell_p\right] C_1 = \sigma\left[A(r,s,t),\ell_p\right] C_2 = \emptyset$. Hence; $\sigma_{ap}[A(r,s,t),\ell_p] = D_1$ (ii) Since the following equality

$$\sigma_\delta[A(r,s,t),\ell_p] = \sigma[A(r,s,t),\ell_p] \backslash \sigma\left[A(r,s,t),\ell_p\right] A_3$$

holds from Table 1.2, we derive by Theorem 4.1.5 and Theorem 4.1.8 that $\sigma_\delta[A(r,s,t),\ell_p] = D_2$.

(iii) From Table 1, we have

$$\sigma_{co}[A(r,s,t),\ell_p] = \sigma\left[A(r,s,t),\ell_p\right] C_1 \cup \sigma\left[A(r,s,t),\ell_p\right] C_2 \cup \sigma\left[A(r,s,t),c_0\right] C_3$$

by Theorem 4.1.3 it is immediate that $\sigma_{co}[A(r,s,t),\ell_p] = \emptyset$. □

4.2 FINE SPECTRUM OF UPPER TRIANGULAR TRIPLE-BAND MATRIX $A(r,s,t)$ OVER THE SEQUENCE SPACE ℓ_p, $(1 < p < \infty)$

In the present section, we determine the fine spectrum of the operator $A(r,s,t) : \ell_p \to \ell_p$ in the case $1 < p < \infty$.

Theorem 4.2.1. *The operator $A(r,s,t) : \ell_p \to \ell_p$ is a bounded linear operator and*

$$(|r|^p + |s|^p + |t|^p)^{1/p} \leq \|A(r,s,t)\|_{(\ell_p:\ell_p)} \leq |r| + |s| + |t|. \tag{4.11}$$

Proof. Since the linearity of the operator $A(r,s,t)$ is trivial. Now, we prove that (4.11) holds for the operator $A(r,s,t)$ on the space ℓ_p. It is trivial that $A(r,s,t)e^{(2)} = (t,s,r,0,\ldots)$ for $e^{(2)} \in \ell_p$. Therefore, we have

$$\|A(r,s,t)\|_{(\ell_p:\ell_p)} \geq \frac{\|A(r,s,t)e^{(2)}\|_p}{\|e^{(2)}\|_p} = (|r|^p + |s|^p + |t|^p)^{1/p}.$$

which implies that

$$\|A(r,s,t)\|_{(\ell_p:\ell_p)} \geq (|r|^p + |s|^p + |t|^p)^{1/p}. \tag{4.12}$$

Let $x = (x_k) \in \ell_p$, where $1 < p < \infty$. Then, since $(tx_{k+2}), (rx_k)$ and $(sx_{k+1}) \in \ell_p$ it is easy to see by Minkowsky's inequality that

$$\begin{aligned}
\|A(r,s,t)x\|_p &= \left(\sum_k |rx_k + sx_{k+1} + tx_{k+2}|^p\right)^{1/p} \\
&\leq \left(\sum_k |rx_k|^p\right)^{1/p} + \left(\sum_k |sx_{k+1}|^p\right)^{1/p} + \left(\sum_k |tx_{k+2}|^p\right)^{1/p} \\
&= |r|\left(\sum_k |x_k|^p\right)^{1/p} + |s|\left(\sum_k |x_{k+1}|^p\right)^{1/p} + |t|\left(\sum_k |x_{k+2}|^p\right)^{1/p} \\
&= (|r| + |s| + |t|)\|x\|_p
\end{aligned}$$

which leads us to the the result that

$$\|A(r,s,t)\|_{(\ell_p:\ell_p)} \leq |r| + |s| + |t|. \tag{4.13}$$

Therefore, by combining the inequalities in (4.12) and (4.13) we have (4.11), as desired. □

Theorem 4.2.2. *Let s be a complex number such that $\sqrt{s^2} = -s$. Then, $\sigma_c[A(r,s,t), \ell_p] \subseteq D_1$.*

Proof. We show that $A_\alpha(r,s,t)^*$ is onto, for $2\big|r-\alpha\big| > \big|-s + \sqrt{s^2 - 4t(r-\alpha)}\big|$. Thus, for every $y \in \ell_q$, we find $x \in \ell_q$. $A_\alpha(r,s,t)^*$ is triangle so it has an inverse. Also the equation $A_\alpha(r,s,t)^* x = y$ gives $[A_\alpha(r,s,t)^*]^{-1} y = x$. It is sufficient to show that $[A_\alpha(r,s,t)^*]^{-1} \in (\ell_q : \ell_q)$. We calculate that $A = (a_{nk}) = [A_\alpha(r,s,t)^*]^{-1}$ as follows:

$$A = (a_{nk}) = \begin{bmatrix} a_1 & 0 & 0 & \dots \\ a_2 & a_1 & 0 & \dots \\ a_3 & a_2 & a_1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$\begin{aligned} a_1 &= \frac{1}{r-\alpha} \\ a_2 &= \frac{-s}{(r-\alpha)^2} \\ a_3 &= \frac{s^2 - t(r-\alpha)}{(r-\alpha)^3} \\ &\vdots \end{aligned}$$

It is known from Theorem 4.1.2 that

$$a_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{s^2 - 4t(r-\alpha)}} \quad \text{for all} \quad n \geq 1, \quad \text{where} \quad \lambda_1 = \frac{-s + \sqrt{\Delta}}{r-\alpha}, \quad \lambda_2 = \frac{-s - \sqrt{\Delta}}{r-\alpha}.$$

Now, we show that $[A_\alpha(r,s,t)^*]^{-1} \in (\ell_1 : \ell_1)$ for $|\lambda_1| < 1$. Since $|\lambda_1| < 1$, Theorem 4.1.2 gives that $|\lambda_2| < 1$. We assume that $s^2 - 4t(r-\alpha) \neq 0$ and $|\lambda_1| < 1$. Therefore,

$$\begin{aligned} \|[A_\alpha(r,s,t)^*]^{-1}\|_{(\ell_1:\ell_1)} = \sup_{n\in\mathbb{N}} \sum_{k=n}^{\infty} |a_k| &= \sum_{k=1}^{\infty} |a_k| \\ &\leq \frac{1}{|\Delta|} \left(\sum_{k=1}^{\infty} |\lambda_1|^k + \sum_{k=1}^{\infty} |\lambda_2|^k \right) < \infty. \end{aligned}$$

This shows that $[A_\alpha(r,s,t)^*]^{-1} \in (\ell_1 : \ell_1)$. Similarly we can show that $[A_\alpha(r,s,t)^*]^{-1} \in (\ell_\infty : \ell_\infty)$.
Now assume that $s^2 - 4t(r-\alpha) = 0$. Then,

$$a_n = \left(\frac{2n}{-s}\right) \left[\frac{-s}{2(r-\alpha)}\right]^n$$

and simple calculation gives that $(a_n) \in \ell_q$ if and only if $|-s| < 2|r-\alpha|$.

$$[(A(r,s,t) - \alpha I)^*]^{-1} \in (\ell_q : \ell_q) \quad \text{for} \quad \alpha \in \mathbb{C} \quad \text{with} \quad 2\left|r-\alpha\right| > \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right|$$

Hence, $A_\alpha(r,s,t)^*$ is onto. By Lemma 3.3.2, $A_\alpha(r,s,t)$ has a bounded inverse. This means that

$$\sigma_c[A(r,s,t), \ell_p] \subseteq \left\{\alpha \in \mathbb{C} : 2\left|r-\alpha\right| \leq \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right|\right\} = D_1.$$

□

Theorem 4.2.3. $\sigma_p[A(r,s,t)^*, \ell_p^*] = \emptyset$.

Proof. Consider $A(r,s,t)^*f = \alpha f$ with $f \neq \theta = (0,0,0,\ldots)$ in $\ell_p^* = \ell_q$. Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} rf_0 & = & \alpha f_0 \\ sf_0 + rf_1 & = & \alpha f_1 \\ tf_0 + sf_1 + rf_2 & = & \alpha f_2 \\ tf_1 + sf_2 + rf_3 & = & \alpha f_3 \\ & \vdots & \\ tf_{k-2} + sf_{k-1} + rf_k & = & \alpha f_k \\ & \vdots & \end{array}\right\}$$

we find that $f_0 = 0$ if $\alpha \neq r$ and $f_1 = f_2 = \cdots = 0$ if $f_0 = 0$ which contradicts $f \neq \theta$. If f_{n_0} is the first non zero entry of the sequence $f = (f_n)$ and $\alpha = r$, then we get $tf_{n_0-2} + sf_{n_0-1} + rf_{n_0} = \alpha f_{n_0}$ which implies $f_{n_0} = 0$ which contradicts the assumption $f_{n_0} \neq 0$. Hence, the equation $A(r,s,t)^*f = \alpha f$ has no solution $f \neq \theta$. □

In the case $0 < p \leq 1$, since the spectrum and fine spectrum of the matrix $A(r,s,t)$ as an operator on the sequence space ℓ_p are similar to to case $1 < p < \infty$, to avoid the repetition of similar statements we give the results by the following theorem without proof:

Theorem 4.2.4. *The following statements hold:*

(i) $\sigma[A(r,s,t),\ell_p] = D_1$.

(ii) $\sigma_r[A(r,s,t),\ell_p] = \emptyset$.

(iii) $\sigma_p[A(r,s,t),\ell_p] = D_2$.

(iv) $\sigma_c[A(r,s,t),\ell_p] = D_3$.

(v) $\sigma_{ap}[A(r,s,t),\ell_p] = D_1$.

(vi) $\sigma_{co}[A(r,s,t),\ell_p] = \emptyset$.

(vii) $\sigma_\delta[A(r,s,t),\ell_p] = D_3$.

4.3 FINE SPECTRUM OF UPPER TRIANGULAR TRIPLE-BAND MATRIX $A(r, s, t)$ OVER THE SPACE OF NULL SEQUENCES

In the present section, we determine the fine spectrum of $A(r, s, t)$ in the space of null sequences.

Theorem 4.3.1. *(Wilansky, 1984) Let T be an operator with the associated matrix $A = (a_{nk})$. Then, the following statements hold:*

(i) $T \in B(c)$ *if and only if*

$$\|A\| := \sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty, \tag{4.14}$$

$$a_k := \lim_{n\to\infty} a_{nk} \quad \text{exists for each fixed } k \in \mathbb{N}, \tag{4.15}$$

$$a := \lim_{n\to\infty} \sum_k a_{nk} \quad \text{exists.}$$

(ii) $T \in B(c_0)$ *if and only if (4.14) holds and (4.15) are also hold with* $a_k = 0$ *for each* k.

(iii) $T \in B(\ell_\infty)$ *if and only if (4.14) holds.*

In these cases, the operator norm of T *is* $\|T\|_{(c:c)} = \|T\|_{(c_0:c_0)} = \|T\|_{(\ell_\infty:\ell_\infty)} = \|A\|$.

Corollary 4.3.2. *Let* $\lambda \in \{\ell_\infty, c_0, c\}$. $A(r, s, t) : \lambda \to \lambda$ *is a bounded linear operator and*
$\|A(r, s, t)\|_{(\lambda:\lambda)} = |r| + |s| + |t|$.

Proof. The linearity of $A(r, s, t)$ is trivial and so it is omitted. By Theorem 4.3.1, it is immediate that $\|A(r, s, t)\|_{(\lambda:\lambda)} = \sup_{n\in\mathbb{N}} \sum_k |a_{nk}| = |r| + |s| + |t|$. □

Theorem 4.3.3. $\sigma_c[A(r, s, t), c_0] \subseteq D_1$.

Proof. Let $y = (y_k) \in \ell_1$. Then, by solving the equation $A_\alpha(r, s, t)^* x = y$ we find the matrix in the proof of Theorem 4.1.2. Then, we have

$$x_n = a_1 y_n + a_2 y_{n-1} + \cdots + a_{n+1} y_0 = \sum_{k=0}^{n} a_{n+1-k} y_k \tag{4.16}$$

where,

$$a_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{s^2 - 4t(r-\alpha)}} \quad \text{for all } n \geq 1. \tag{4.17}$$

If we use the relation (4.16), then we get

$$|x_n| \ \leq \ \sum_{k=0}^{n} |a_{n+1-k}||y_k|$$

and so we have

$$\begin{aligned} |x_0| + |x_1| + \cdots + |x_n| &\leq \sum_{k=0}^{0} |a_{1-k}||y_k| + \sum_{k=0}^{1} |a_{2-k}||y_k| + \cdots + \sum_{k=0}^{n} |a_{n+1-k}||y_k| \\ &= \sum_{j=1}^{n+1} |a_j||y_0| + \sum_{j=1}^{n} |a_j||y_1| + \cdots + \sum_{j=1}^{1} |a_j||y_n| \\ &\leq \sum_{j=1}^{n+1} |a_j|(|y_0| + |y_1| + \cdots + |y_n|) \end{aligned}$$

for all $n \in \mathbb{N}$. By letting $n \to \infty$, we get

$$\|x\|_1 \leq \||y\|_1 \sum_{j=1}^{\infty} |a_j|.$$

We must show that $\sum_j |a_j| < \infty$. There are two cases here:

Case 1. If $\Delta = s^2 - 4t(r-\alpha) \neq 0$, the relation (4.17) holds for all $k \in \mathbb{N}_1$. Since $|\lambda_1| < 1$, Theorem 4.1.2 gives that $|\lambda_2| < 1$. Now, for $|\lambda_1| < 1$ we can see that

$$\sum_j |a_j| \leq \frac{1}{|\sqrt{\Delta}|} \left(\sum_j |\lambda_1|^j + \sum_j |\lambda_2|^j \right).$$

Thus, for $|\lambda_1| < 1$, $A_\alpha(r,s,t)^*$ is onto and by Lemma 2.1.9, $A_\alpha(r,s,t)$ has bounded inverse. This means that

$$\sigma_c[A(r,s,t), c_0] \subseteq \left\{ \alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \left|-s + \sqrt{s^2 - 4t(r-\alpha)}\right| \right\} = D_1.$$

Case 2. If $\Delta = s^2 - 4t(r-\alpha) = 0$, calculation on recurrent sequence give

$$a_n = \left(\frac{2n}{-s} \right) \left[\frac{-s}{2(r-\alpha)} \right]^n \qquad \text{for all } \ n \geq 1.$$

Now, for $|-s| < 2|r-\alpha|$ we can see that

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n-1}} \right| = \left| \frac{-s}{2(r-\alpha)} \right| < 1.$$

Therefore, $\sum_k |a_k|$ is convergent. $A_\alpha(r,s,t)^*$ is onto by Lemma 2.1.9, $A_\alpha(r,s,t)$ has bounded inverse. This means that

$$\sigma_c[A(r,s,t), c_0] \subseteq \big\{ \alpha \in \mathbb{C} : 2\big|r-\alpha\big| \leq \big|-s\big| \big\} \subseteq D_1.$$

□

Theorem 4.3.4. $\sigma_p[A(r,s,t)^*, \ell_1] = \emptyset$.

Proof. Consider $A(r,s,t)^* f = \alpha f$ with $f \neq \theta = (0,0,0,\ldots)$ in $c_0^* = \ell_1$. Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} rf_0 &=& \alpha f_0 \\ sf_0 + rf_1 &=& \alpha f_1 \\ tf_0 + sf_1 + rf_2 &=& \alpha f_2 \\ tf_1 + sf_2 + rf_3 &=& \alpha f_3 \\ &\vdots& \\ tf_{k-2} + sf_{k-1} + rf_k &=& \alpha f_k \\ &\vdots& \end{array}\right\}$$

we find that $f_0 = 0$ if $\alpha \neq r$ and $f_1 = f_2 = \cdots = 0$ if $f_0 = 0$ which contradicts $f \neq \theta$. If f_{n_0} is the first non zero entry of the sequence $f = (f_n)$ and $\alpha = r$, then we get $tf_{n_0-2} + sf_{n_0-1} + rf_{n_0} = \alpha f_{n_0}$ which implies $f_{n_0} = 0$ which contradicts the assumption $f_{n_0} \neq 0$. Hence, the equation $A(r,s,t)^* f = \alpha f$ has no solution $f \neq \theta$. □

Theorem 4.3.5. $\sigma_p[A(r,s,t), c_0] = D_2$.

Proof. Let $A(r,s,t)x = \alpha x$ for $\theta \neq x \in c_0$. Then, by solving the system of linear equations

$$\left.\begin{array}{rcl} rx_0 + sx_1 + tx_2 &=& \alpha x_0 \\ rx_1 + sx_2 + tx_3 &=& \alpha x_1 \\ rx_2 + sx_3 + tx_4 &=& \alpha x_2 \\ &\vdots& \\ rx_{k-2} + sx_{k-1} + tx_k &=& \alpha x_k \\ &\vdots& \end{array}\right\}$$

we have

$$\left.\begin{array}{rcl} x_2 &=& \frac{-s}{t}x_1 - \frac{r-\alpha}{t}x_0 \\ x_3 &=& \frac{s^2-t(r-\alpha)}{t^2}x_1 + \frac{s(r-\alpha)}{t^2}x_0 \\ \vdots && \\ x_n &=& \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0 \end{array}\right\} \tag{4.18}$$

for all $n \geq 2$. Assume that $\alpha \in D_2$. Then, we choose $x_0 = 1$ and $x_1 = 2(r-\alpha)/[-s+\sqrt{s^2-4t(r-\alpha)}]$. By Theorem 4.1.4, $x_n = x_1^n$ for all $n \geq 2$. Now, $x = (x_k) \in c_0$, since $|x_1| < 1$. This shows that $D_2 \subseteq \sigma_p(A(r,s,t), c_0)$. Now, we assume that $\alpha \notin D_2$, i.e, $|\lambda_1| \leq 1$. We must show that $\alpha \notin \sigma_p(A(r,s,t), c_0)$. In this situation, we examine three cases.

Case 1. $|\lambda_2| < |\lambda_1| < 1$. In this case we have $s^2 \neq 4t(r-\alpha)$ and we obtain from relation (4.18) that

$$\begin{aligned} x_n &= \frac{a_n(r-\alpha)^n}{t^{n-1}}x_1 - \frac{a_{n-1}(r-\alpha)^n}{t^{n-1}}x_0 \\ &= \left(\frac{r-\alpha}{t}\right)^{n-1}(r-\alpha)(-a_{n-1}x_0 + a_n x_1) \\ &= \frac{r-\alpha}{\sqrt{\Delta}(\lambda_1\lambda_2)^{n-1}}(-\lambda_1^{n-1}x_0 + \lambda_2^{n-1}x_0 + \lambda_1^{n-1}x_1 - \lambda_2^n x_1) \\ &= \frac{r-\alpha}{\sqrt{\Delta}}\left[\left(\frac{1}{\lambda_1^{n-1}} - \frac{1}{\lambda_2^{n-1}}\right)x_0 + \left(\frac{\lambda_1}{\lambda_2^{n-1}} - \frac{\lambda_2}{\lambda_1^{n-1}}\right)x_1\right] \\ &= \frac{r-\alpha}{\sqrt{\Delta}}\left[\frac{1}{\lambda_1^{n-1}}(x_0 - \lambda_2 x_1) + \frac{1}{\lambda_2^{n-1}}(-x_0 + \lambda_1 x_1)\right]. \end{aligned} \tag{4.19}$$

If $-x_0 + \lambda_1 x_1 = 0$ and $x_0 - \lambda_2 x_1 = 0$ in (4.19), then we have $\lambda_1 = \lambda_2$ which is a contradiction. Otherwise, $x = (x_k) \notin c_0$.

Case 2. $|\lambda_2| = |\lambda_1| < 1$. In this case we have $s^2 = 4t(r-\alpha)$ and using the formula

$$a_n = \left(\frac{2n}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^n \quad \text{for all } n \geq 1.$$

We again derive (4.18)

$$x_n = \frac{2(r-\alpha)}{s\lambda_1^{n-1}}\left[x_0(n-1) - nx_1\lambda_1\right].$$

If $x_0 = x_1 = 0$, then $x = \theta$, a contradiction. Otherwise $x = (x_k) \notin c_0$, since $1/\lambda_1 > 1$.

Case 3. $|\lambda_2| = |\lambda_1| = 1$. In this case, we have $s^2 = 4t(r-\alpha)$ and so we have $|-s/2t| = 1$. Assume that $\alpha \in \sigma_p[A(r,s,t), c_0]$. This implies that $x \in c_0 - \{\theta\}$. Thus, we again derive (4.18)

$$x_n = \left(\frac{-s}{2t}\right)^{n-1}\left[-(n-1)\frac{-s}{2t}x_0 + nx_1\right].$$

Since $x_n \longrightarrow 0$ as $k \longrightarrow \infty$ we must have $x_0 = x_1 = 0$. This yields that $x = \theta$, a contradiction which means $\alpha \notin \sigma_p[A(r,s,t), c_0]$. Thus $\sigma_p[A(r,s,t), c_0] \subseteq D_2$. This completes the proof. □

Theorem 4.3.6. $\sigma_r[A(r,s,t),c_0] = \emptyset$.

Proof. This may be obtained in the similar way as mentioned in the proof of Theorem 4.1.5. So we omit details. □

Theorem 4.3.7. *Let s be a complex number such that $\sqrt{s^2} = -s$. Then, $\sigma[A(r,s,t),c_0] = D_1$.*

Proof. The inclusion

$$\left\{\alpha \in \mathbb{C} : 2\left|r-\alpha\right| < \left|-s+\sqrt{s^2-4t(r-\alpha)}\right|\right\} \subseteq \sigma[A(r,s,t),c_0]$$

holds by Theorem 4.3.5, since the spectrum of any bounded operator is closed, we have

$$\left\{\alpha \in \mathbb{C} : 2\left|r-\alpha\right| \leq \left|-s+\sqrt{s^2-4t(r-\alpha)}\right|\right\} \subseteq \sigma[A(r,s,t),c_0]. \tag{4.20}$$

Again, Theorem 4.3.3, Theorem 4.3.5 and Theorem 4.3.6 give that

$$\sigma[A(r,s,t),c_0] \subseteq \left\{\alpha \in \mathbb{C} : 2\left|r-\alpha\right| \leq \left|-s+\sqrt{s^2-4t(r-\alpha)}\right|\right\}. \tag{4.21}$$

By combining (4.20) and (4.21), one can observe that $\sigma[A(r,s,t),c_0] = D_1$, as desired. □

Theorem 4.3.8. $\sigma_c[A(r,s,t),c_0] = D_3$.

Proof. Since the parts $\sigma_c[A(r,s,t),c_0]$, $\sigma_r[A(r,s,t),c_0]$ and $\sigma_p[A(r,s,t),c_0]$ are pairwise disjoint and their union is $\sigma[A(r,s,t),c_0]$, the proof is immediate, from Theorem 4.3.5, Theorem 4.3.6 and Theorem 4.3.7. □

Theorem 4.3.9. *Let s be a complex number such that $\sqrt{s^2} = -s$. If $\alpha \in D_2$, then $\alpha \in \sigma[A(r,s,t),c_0]A_3$.*

Proof. From Theorem 4.3.5, $\alpha \in \sigma_p[A(r,s,t),c_0]$. Thus, $[A(r,s,t)-\alpha I]^{-1}$ does not exist. By Theorem 4.3.4 $A(r,s,t)^* - \alpha I$ is one to one, so $A(r,s,t)-\alpha I$ has a dense range in c_0 by Lemma 2.1.8. This completes the proof. □

Theorem 4.3.10. *The following statements hold:*

(i) $\sigma_{ap}[A(r,s,t)c_0] = D_1$.

(ii) $\sigma_\delta[A(r,s,t),c_0] = D_3$.

(iii) $\sigma_{co}[A(r,s,t),c_0] = \emptyset$.

Proof. (i) Since from Table 1.2,
$\sigma_{ap}[A(r,s,t),c_0] = \sigma\,[A(r,s,t),c_0]\,\backslash\sigma\,[A(r,s,t),c_0]\,C_1$, we have by Theorem 4.3.6
$\sigma\,[A(r,s,t),c_0]\,C_1 = \sigma\,[A(\widetilde{r},\widetilde{s}),c_0]\,C_2 = \emptyset$. Hence, $\sigma_{ap}[A(r,s,t),c_0] = D_1$.

(ii) Since the following equality

$$\sigma_\delta[A(r,s,t),c_0] = \sigma[A(r,s,t),c_0]\backslash\sigma\,[A(r,s,t),c_0]\,A_3$$

holds from Table 1.2, we derive by Theorem 4.3.6 and Theorem 4.3.9 that
$\sigma_\delta[A(r,s,t),c_0] = D_3$.

(iii) From Table 1.2, we have

$$\sigma_{co}[A(r,s,t),c_0] = \sigma\,[A(r,s,t),c_0]\,C_1 \cup \sigma\,[A(r,s,t),c_0]\,C_2 \cup \sigma\,[A(r,s,t),c_0]\,C_3$$

by Theorem 4.3.4 it is immediate that $\sigma_{co}[A(r,s,t),c_0] = \emptyset$. □

4.4 FINE SPECTRA OF UPPER TRIANGULAR TRIPLE-BAND MATRICES OVER THE SPACE OF CONVERGENT SEQUENCES

In the this section, we determine the fine spectrum of the operator
$A(r,s,t) : c \to c$.

For $A(r,s,t) : c \to c$, the matrix $A(r,s,t)^* \in B(\ell_1)$ is of the form

$$A(r,s,t)^* = \begin{bmatrix} r+s+t & 0 \\ 0 & A^t(r,s,t) \end{bmatrix}.$$

Theorem 4.4.1. *$A_\alpha(r,s,t) : c \to c$ has a dense range if and only if $\alpha \neq r+s+t$.*

Proof. First let us show that $\sigma_p[A(r,s,t)^*, \mathbb{C}\oplus\ell_1] = \{r+s+t\}$. Suppose that α is an eigenvalue of the operator $A(r,s,t)^* : \mathbb{C}\oplus\ell_1 \to \mathbb{C}\oplus\ell_1$. Then there exists $f \in \ell_1$

satisfying the system of equations

$$\left.\begin{array}{rcl}(r+s+t)f_0 & = & \alpha f_0 \\ rf_1 & = & \alpha f_1 \\ sf_1 + rf_2 & = & \alpha f_2 \\ tf_1 + sf_2 + rf_3 & = & \alpha f_3 \\ & \vdots & \end{array}\right\}$$

From above one can see that $\alpha = r+s+t$ is an eigenvalue corresponding to the eigenvector $(1,0,0,0,\ldots)$. Now, suppose that $\alpha \neq r+s+t$. Then we find that $f_1 = f_2 = \cdots = 0$ if $f_0 = 0$ which contradicts $f \neq \theta$. If f_{n_0} is the first non zero entry of the sequence $f = (f_n)$ and $\alpha = r$, then we get
$tf_{n_0-2} + sf_{n_0-1} + rf_{n_0} = \alpha f_{n_0}$ which implies $f_{n_0} = 0$ which contradicts the assumption $f_{n_0} \neq 0$. Hence, the equation $A(r,s,t)^* f = \alpha f$ has no solution $f \neq \theta$. So, $\sigma_p[A(r,s,t)^*, \mathbb{C} \oplus \ell_1] = \{r+s+t\}$. □

Since the operator on the sequence space c is similar to that of on the space c_0, to avoid the repetition of similar statements we give the results by the following theorem without proof:

Theorem 4.4.2. *Following statements hold:*

(i) $\sigma[A(r,s,t), c] = D_1$.

(ii) $\sigma_p[A(r,s,t), c] = D_2 \cup \{r+s+t\}$.

(iii) $\sigma_r[A(r,s,t), c] = \emptyset$.

(iv) $\sigma_c[A(r,s,t), c] = D_3 \setminus \{r+s+t\}$.

(v) $\sigma_\delta(A(r,s,t), c] = D_3$.

(vi) $\sigma_{co}[A(r,s,t), c] = \{r+s+t\}$.

(vii) $\sigma_{ap}[A(r,s,t), c] = D_1$.

It is known from Cartlidge (Cartlidge, 1978) that, if a matrix operator A is bounded on c, then $\sigma(A, c) = \sigma(A, \ell_\infty)$. So we have the following.

Corollary 4.4.3. $\sigma[A(r,s,t), \ell_\infty] = D_1$.

Theorem 4.4.4. $\sigma_p[A(r,s,t), \ell_\infty] = D_1$.

Proof. Let $A(r,s,t)x = \alpha x$ for $x \in \ell_\infty$ with $x \neq \theta$. Then, by solving the matrix equation $A(r,s,t)x = \alpha x$, we obtain the relation (4.18). Combining the fact $x_0 = 1$ and $x_1 = 1/\lambda_1$ with the relation (4.18), one can see that
$x_2 = x_1^2, x_3 = x_1^3, \ldots, x_n = x_1^n, \ldots$ for all $n \geq 2$, and so $x \in \ell_\infty$ since $|x_1| \leq 1$. This shows that $D_1 \subseteq \sigma_p[A(r,s,t), \ell_\infty]$. Conversely, we prove that $\sigma_p[A(r,s,t), \ell_\infty] \subseteq D_1$ which is similar to the proof of Theorem 4.3.3. □

Theorem 4.4.5. *$\sigma_c[A(r,s,t), \ell_\infty] = \emptyset$ and $\sigma_r[A(r,s,t), \ell_\infty] = \emptyset$.*

Proof. Because of the parts $\sigma_c[A(r,s,t), \ell_\infty]$, $\sigma_r[A(r,s,t), \ell_\infty]$ and $\sigma_p[A(r,s,t), \ell_\infty]$ are pairwise disjoint and their union is $\sigma[A(r,s,t), \ell_\infty]$, the proof immediately follows from Corollary 4.4.3 and Theorem 4.4.4. □

Theorem 4.4.6. *Let s be a complex number such that $\sqrt{s^2} = -s$. If $\alpha \in D_2, \alpha \in \sigma[A(r,s,t), \ell_\infty]A_3$.*

Proof. This is similar to the proof of Theorem 4.3.9. So, we omit the detail. □

4.5 SOME APPLICATIONS

In this section, we give two theorems related to Toeplitz matrix.

Theorem 4.5.1. *Let P be a polynomial that corresponds to the n-tuple a and $z_1, z_2, z_3, \ldots, z_{n-1}$ also be the roots of P. Define T as a Toeplitz matrix associated with P, that is,*

$$T = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_n & 0 & 0 & 0 & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots & a_n & 0 & 0 & \ldots \\ 0 & 0 & a_0 & a_1 & a_2 & \ldots & a_n & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The resolvent operator T over ℓ_p with $1 < p < \infty$, where the domain of the resolvent operator is the whole space ℓ_p, exists if and only if all the roots of the polynomial are outside the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. That is $T^{-1} \in (\ell_p : \ell_p)$ if and if only $|z_i| > 1$, $1 \leq i \leq n-1$. In this case the resolvent operator is *represented by the*

$$T^{-1} = \frac{1}{a_{n-1}} A^{-1}(-z_1,1)A^{-1}(-z_2,1)\cdots A^{-1}(-z_{n-1},1), \quad \textit{where}$$

$$A^{-1}(-z_i,1) = -\begin{bmatrix} 1/z_i & 1/z_i^2 & 1/z_i^3 & 1/z_i^4 & 1/z_i^5 & \dots \\ 0 & 1/z_i & 1/z_i^2 & 1/z_i^3 & 1/z_i^4 & \dots \\ 0 & 0 & 1/z_i & 1/z_i^2 & 1/z_i^3 & \dots \\ 0 & 0 & 0 & 1/z_i & 1/z_i^2 & \dots \\ 0 & 0 & 0 & 0 & 1/z_i & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proof. Suppose all the roots of the polynomial
$P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} = a_n(z-z_1)(z-z_2)\cdots(z-z_{n-1})$ are outside of the unit disc. The Toeplitz matrix associated with P can be written as the product

$$T = a_n A(-z_1,1)A(-z_2,1)\cdots A(-z_{n-1},1).$$

Since multiplication of upper triangular Toeplitz matrices is commutative, one can see that

$$T^{-1} = \frac{1}{a_{n-1}} A^{-1}(-z_1,1)A^{-1}(-z_2,1)\cdots A^{-1}(-z_{n-1},1)$$

is left inverse of T. Since all roots polynomial are out side of the unit disc,

$$\|T^{-1}(-z_i,1)\|_{(\ell_\infty:\ell_\infty)} = \sup_n \sum_{k=n}^{\infty} \frac{1}{|z_i|^{k+1-n}} = \sum_{k=1}^{\infty} \frac{1}{|z_i|^k} < \infty$$

Therefore, each $T^{-1}(-z_i,1) \in (\ell_\infty : \ell_\infty)$, for $1 \le i \le n-1$. Similarly we can say that $T^{-1}(-z_i,1) \in (\ell_1 : \ell_1)$. So we have $T^{-1}(-z_i,1) \in (\ell_p : \ell_p)$. □

Theorem 4.5.2. *The resolvent operator of $A(r,s,t)$ over ℓ_p with $1 < p < \infty$, where the domain of the resolvent operator is the space ℓ_p, exists if and only if $2|r| > |-s+\sqrt{s^2-4tr}|$. In this case, the resolvent operator is represented by the infinite band Toeplitz matrix*

$$A(r,s,t)^{-1} = \frac{1}{t}\begin{bmatrix} u_1 & u_1^2 & u_1^3 & u_1^4 & \dots \\ 0 & u_1 & u_1^2 & u_1^3 & \dots \\ 0 & 0 & u_1 & u_1^2 & \dots \\ 0 & 0 & 0 & u_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} u_2 & u_2^2 & u_2^3 & u_2^4 & \dots \\ 0 & u_2 & u_2^2 & u_2^3 & \dots \\ 0 & 0 & u_2 & u_2^2 & \dots \\ 0 & 0 & 0 & u_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{4.22}$$

$$\textit{where } u_1 = \frac{-s+\sqrt{s^2-4tr}}{2r}, \quad u_2 = \frac{-s-\sqrt{s^2-4tr}}{2r}. \tag{4.23}$$

Proof. By Theorem 4.5.3, we can see that $A(r,s,t)^{-1}$ is inverse of the matrix $A(r,s,t)$. But this is not enough to say it is resolvent operator. By Lemma 2.1.1, 2.1.2 and 2.1.3, $A(r,s,t)^{-1} \in (\ell_p : \ell_p)$, when $2|r| > |-s+\sqrt{s^2-4tr}|$. That is for $2|r| > |-s+\sqrt{s^2-4tr}|$, $A(r,s,t)^{-1}$ is resolvent operator. □

Theorem 4.5.3. *Let λ denotes any of the spaces ℓ_∞, c or c_0. The resolvent operator T over λ, where the domain of the resolvent operator is the whole space λ, exists if and only if all the roots of the polynomial are outside of the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. That is $T^{-1} \in (\lambda : \lambda)$ if and only if $|z_i| > 1$, $1 \leq i \leq n-1$. In this case the resolvent operator is presented by*

$$T^{-1} = \frac{1}{a_{n-1}} A^{-1}(-z_1, 1) A^{-1}(-z_2, 1) \cdots A^{-1}(-z_{n-1}, 1).$$

Proof. Suppose all the roots of the polynomial $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} = a_n (z - z_1)(z - z_2) \cdots (z - z_{n-1})$ are outside of the unit disc. The Toeplitz matrix associated with P can be written as the product

$$T = a_n A(-z_1, 1) A(-z_2, 1) \cdots A(-z_{n-1}, 1).$$

Since multiplication of upper triangular Toeplitz matrices is commutative, we can see that

$$T^{-1} = \frac{1}{a_{n-1}} A^{-1}(-z_1, 1) A^{-1}(-z_2, 1) \cdots A^{-1}(-z_{n-1}, 1)$$

is the left inverse of T. If we apply Theorem 4.3.1,

$$\|T^{-1}(-z_i, 1)\| = \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} \frac{1}{|z_i|^{k+1+n}} = \sum_{k=1}^{\infty} \frac{1}{|z_i|^k} < \infty,$$

$$\lim_{n \to \infty} \frac{1}{|z_i|^{k+1+n}} = 0 \quad \text{for each} \quad k,$$

$$\lim_{n \to \infty} \sum_k \frac{1}{|z_i|^{k+1+n}} \quad \text{exists, since} \quad z_i > 1.$$

Since all the roots of the polynomial are out side the unit disc, each $T^{-1}(-z_i, 1) \in (\lambda : \lambda)$, for $1 \leq i \leq n-1$.

□

Theorem 4.5.4. *Let $\lambda \in \{c_0, c\}$. The resolvent operator $A(r,s,t)$ over λ, where the domain of the resolvent operator is the space λ, exists if and only if $2|r| > |-s+\sqrt{s^2-4tr}|$. In this case, the resolvent operator is represented by the infinite band Toeplitz matrix defined (4.22) which are continuous.*

Proof. By Theorem 4.5.3, we can see that $[A(r,s,t)]^{-1}$ is inverse of the matrix of $A(r,s,t)$. But this is not enough to say it is resolvent operator. By Theorem 4.3.1, $[A(r,s,t)]^{-1} \in (\lambda : \lambda)$, when $2|r| > |-s+\sqrt{s^2-4tr}|$. That is for $2|r| > |-s+\sqrt{s^2-4tr}|$, $A(r,s,t)^{-1}$ is resolvent operator. If $2|r| \leq |-s+\sqrt{s^2-4tr}|$, we know by Theorem 4.3.5, 4.3.9 and 4.4.2 that the resolvent operator whose domain is the whole space λ does not exist. For $2|r| > |-s+\sqrt{s^2-4tr}|$ the continuity of the resolvent operator follows from Theorem 4.3.3. □

Theorem 4.5.5. *Suppose that satisfies the inequality* $2|\alpha(1-r)+r| > |-s(1-\alpha)+\sqrt{s^2(1-\alpha)^2-4t(1-\alpha)(\alpha(1-r)+r)}|$. *Then the convergence field of* $B = \alpha I + (1-\alpha)A(r,s,t)$ *is* c.

Proof. Since B is an upper triangle triple-band Toeplitz matrix, the polynomial P that corresponds to a upper triangular matrix. So we have $P(z) = (1-\alpha)tz^2 + (1-\alpha)sz + (\alpha(1-r)+r)z$ and whose roots z_1, z_2 such that

$$\frac{1}{z_1} = \frac{-(1-\alpha)s + \sqrt{(1-\alpha)s - 4t(1-\alpha)\alpha(1-r)+r}}{2[\alpha(1-r)+r]},$$
$$\frac{1}{z_2} = \frac{-(1-\alpha)s - \sqrt{(1-\alpha)s - 4t(1-\alpha)\alpha(1-r)+r}}{2[\alpha(1-r)+r]}.$$

We know from Theorem 4.3.3 that if $|1/z_1| < 1$, then $|1/z_2| < 1$. Since B is an upper triangle triple-band Toeplitz matrix and $|1/z_i| < 1$ for $i = 1, 2$, by Theorem 4.5.3 B has an inverse and $B^{-1} \in B(c)$.

Now, we show that $B \in B(c)$.

$B \in B(c)$ if and only if

(i) $\sup_k \sum_j |b_{kj}| = |\alpha(1-r)+r| + |(1-\alpha)s| + |(1-\alpha)t|$.

(ii) For each $j \in \mathbb{N}$, $\quad b_{kj} \longrightarrow 0 \ \text{as}\, k \longrightarrow \infty$.

(iii) $\sum_j^k b_{kj} \longrightarrow \alpha(1-r) + r + (1-\alpha)s + (1-\alpha)t, \text{as}\, k \longrightarrow \infty.$

Hence, $B^{-1} \in B(c)$. Since both B and B^{-1} are in $B(c)$, $c_B = c$. □

CHAPTER 5

CONCLUSION

In the present work, as a natural continuation of Yıldırım (Yıldırım, 1998), Altay and Başar (Altay and Başar, 2005); (Başar and Akhmedov, 2007); (Başar and Altay, 2004) and, Akhmedov and El-Shabrawy (Akhmedov and El-Shabrawy, 2011), we have determined the spectrum and the fine spectrum of the upper double sequential band matrix $A(\widetilde{r}, \widetilde{s})$ on the spaces ℓ_p, c_0 and c, the lower the double sequential band matrix $B(\widetilde{r}, \widetilde{s})$ on the spaces bv_p, ℓ_p; and determined the spectrum and the fine spectrum of upper triangular triple-band matrix $A(r, s, t)$ over the sequence spaces ℓ_p, c_0 and c. Since Akhmedov and El-Shabrawy (Akhmedov and El-Shabrawy, 2011), and Srivastava and Kumar (Srivastava and Kumar, 2010a); (Srivastava and Kumar, 2010b) are interested in the fine spectrum of the operator defined by a lower double sequential band matrix over the spaces c and ℓ_1, respectively, our work is a natural continuation of them. In addition to this, we add the definition of some new divisions of spectrum called approximate point spectrum, defect spectrum and compression spectrum of the matrix operator and give the related results for the matrix operator $A(\widetilde{r}, \widetilde{s})$, $B(\widetilde{r}, \widetilde{s})$ and $A(r, s, t)$ on the spaces ℓ_p, bv_p, c_0 and c which is a new development for this type works giving the fine spectrum of a matrix operator on a sequence space with respect to Goldberg's classification.

Finally, we should note that in the case $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$ since the operator $A(\widetilde{r}, \widetilde{s})$ defined by an upper double sequential band matrix reduces to the operator $U(r, s)$ defined by the upper triangular double-band matrix, our results are more general and more comprehensive than the corresponding results obtained by Karakaya and Altun (Karakaya and Altun, 2010) and since the operator $B(\widetilde{r}, \widetilde{s})$ defined by a double sequential band matrix is reduce to the operator $B(r, s)$

defined by the generalized difference matrix, our results are more general and more comprehensive than the corresponding results obtained by Furkan et al. (Bilgiç and Furkan, 2008).

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APPENDIX A

DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS, FURTHER STUDIES AND PUBLICATIONS FROM THESIS WORK

A.1 DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS

I hereby declare that this thesis comprises my original work. No material in this thesis has been previously published and written by another person, except where due reference is made in the text of the thesis. I further declare that this thesis contains no material which has been submitted for a degree or diploma or other qualifications at any other university.

Signature:
Date: December 28, 2013

A.2 FURTHER STUDIES

1. on almost convergence and difference sequence spaces of order m with core theorems.
2. almost difference sequence space derived by using a generalized weighted mean.
3. some new generalized difference spaces of non-absolute type derived from the spaces ℓ_p and ℓ_∞.
4. some new paranormed lambda sequence space with core theorems .

A.3 PUBLICATIONS FROM THESIS WORK

- Feyzi Başar, Ali Karaisa, "Fine spectra of upper triangle triple band matrices over the sequence spaces ℓ_p $(0 < p \leq \infty)$", AIP Conference Proceedings, Vol. 1470, Oct. 2012, pp. 134-137.
- Feyzi Başar, Ali Karaisa, "Fine spectra of upper triangle triple band matrices over the sequence spaces ℓ_p $(0 < p \leq \infty)$", Abstract and Applied Analysis, Vol. 2013, No. ID 342682, December. 2013, pp. 1-10.
- Ali Karaisa, "Fine spectra of upper triangular double-band matrices over the sequence space ", Discrete Dynamics in Nature and Society, Vol. 2012, No. ID 381069, July. 2012, pp. 1-19.

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Dissertation: "On the Fine Spectrum of the Generalized difference operator Defined by a Double sequential Band Matrix over Some Sequence Spaces"
M.S., Mathematics, Fatih University, Istanbul, Turkey, June 2010 .
Thesis: "On the Spectrum of Some Triangle Operators Over the Classical Sequence Spaces".
B.S., Mathematics, Erciyes University , Kayseri, Turkey, January 2002

PROFESSIONAL EXPERIENCE

- Instructor, Necmettin Erbakan University, December 2011-

PUBLICATIONS

Academic Journals

- Feyzi Başar, Ali Karaisa, "Fine spectra of upper triangle triple band matrices over the sequence spaces ℓ_p $(0 < p < \infty)$", AIP Conference Proceedings, Vol. 1470, Oct. 2012, pp. 134-13.
- Feyzi Başar, Ali Karaisa, "Fine spectra of upper triangle triple band matrices over the sequence spaces ℓ_p $(0 < p < \infty)$", Abstract and Applied Analysis, Vol. 2013, No. ID 342682, December. 2013, pp. 1-10.

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- Feyzi Başar, Ali Karaisa, "On the fine spectrum of the generalized difference operator defined by a double sequential band matrix over the sequence space ℓ_p, $(1 < p < \infty)$", Hacettepe Journal of Mathematics and Statistics, Oct. 2012 (under review).
- Feyzi Başar, Ali Karaisa, "Spectrum of upper double sequential band matrix over the spaces of null, convergent and bounded sequences", Mathematical Acta Mathematica Universitatis Comenianae, Submission: Jan. 2012 (under review).
- Feyzi Başar, Ali Karaisa, "Spectrum and fine spectrum of the upper triangular triple-band matrix over some sequence spaces", Journal of Inequalities and Applications, Submission: December. 2012 (under review).
- Feyzi Başar, Ali Karaisa, "Some new generalized difference spaces of non-absolute type derived from the spaces ℓ_p and ℓ_∞",The Scientific World Journal, Vol. 2013, ID 349346, pp. 1-15.

Conference Proceedings

- Feyzi Başar, Ali Karaisa "On the fine spectrum of the generalized difference operator defined by a double sequential band matrix over the sequence space ℓ_1", International Conference on Applied Analysis and Algebra, ICAAA 2011, Yıldız Technical University, Istanbul, Turkey, Jul. 2011.
- Feyzi Başar, Ali Karaisa, "On the fine spectrum of the generalized difference operator defined by a double sequential band matrix over the sequence space c_0 and c", XXIII. Matematik Sempozyumu, Kayseri, 2010.
- Ali Karaisa, "On the fine spectrum of the generalized difference operator defined by a double sequential band matrix over the sequence space ℓ_p, $(0 < p < 1)$", International Conference on Applied Analysis and Algebra, ICAAA 2012, Yıldız Technical University, Istanbul, Turkey, Jul.
- Ali Karaisa, "On the fine spectrum of the generalized difference operator defined by a trible band matrix over the sequence space ℓ_p, $(0 < p < 1)$", First International Conference on Analysis and Applied Mathematics, 2012, Gümüşhane University, Gümüşhane, Turkey.
- Ali Karaisa, "Fine spectra of upper triangular double-band matrices over the sequence space ℓ_p, $(1 < p < \infty)$ ", 25. Ulusal Matematik Sempozyumu, 2012, Niğde, Turkey.
- Ümit Karabıyık, Ali Karaisa, "Almost sequence spaces derived by the domain of the matrix A^r", The Algerian-Turkish International days on Mathematics 2013, 12-14 September 2013, Fatih University, İstanbul, Turkey.
- Ali Karaisa, "Some topological and geometrical properties of the generalized difference Euler sequence space", The Algerian-Turkish International days on Mathematics 2013, 12-14 September 2013, Fatih University, İstanbul, Turkey.