DOMAIN OF THE DOUBLE BAND MATRIX DEFINED BY FIBONACCI NUMBERS IN THE MADDOX'S SPACE $\ell(p)$

by

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APPROVAL PAGE

This is to certify that I have read this thesis written by Hüsamettin ÇAPAN and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science in Mathematics.

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ABSTRACT

In the study, we have studied; the sequence space $\ell(F,p)$ of non-absolute type which is the domain of the double band matrix F defined by the sequence of the Fibonacci numbers in the sequence space $\ell(p)$, where $\ell(p)$ denotes the space of all sequences $x = (x_k)$ such that $\sum_k |x_k|^{p_k} < \infty$ and was defined by Maddox [1]. Furthermore, the alpha-, beta- and gamma-duals of the space $\ell(F,p)$ are determined, and the Schauder basis is given. The classes of matrix transformations from the space $\ell(F,p)$ to the spaces ℓ_∞, c and c_0 are characterized. Additionally, the characterizations of some other classes of matrix transformations from the space $\ell(F,p)$ to the Euler, Riesz, difference, etc., sequence spaces are obtained from the main results of the study.

Keywords: Paranormed sequence space, double sequential band matrix, alpha-, beta- and gamma-duals, matrix transformations of a sequence space.

FİBONACCİ SAYILARI İLE TANIMLANAN ÇİFT BANT MATRİSİNİN $\ell(p)$ MADDOX UZAYI ÜZERİNDEKİ ETKİ ALANI

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ÖZ

Yapmış olduğumuz çalışmada; Fibonacci sayılarının bir dizisi ile tanımlanan F çift bant matrisinin, Maddox [1] tarafından tanımlanan $\sum_k |x_k|^{p_k} < \infty$ olacak şekilde $x = (x_k)$ dizilerinin $\ell(p)$ uzayı üzerindeki etki alanı olan mutlak olmayan türden $\ell(F,p)$ dizi uzayı incelendi. Ayrıca, $\ell(F,p)$ uzayının alfa-, beta- ve gamma-dualleri hesaplandı ve Schauder bazı verildi. $\ell(F,p)$ uzayından ℓ_∞, c ve c_0 uzaylarına matris dönüşümlerinin sınıfları karakterize edildi. İlâveten, $\ell(F,p)$ uzayından Euler, Riesz, fark, vb. dizi uzaylarına bazı matris dönüşümlerinin karakterizasyonları çalışmanın ana sonuçlarından elde edildi.

Anahtar Kelimeler: Paranormlu dizi uzayı, çift sıralı bant matris, alfa-, beta-, gama-dualler, dizi uzayında matris dönüşümleri.

To my parents *Gülfikâr & Ali Rıza ÇAPAN...*

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOL/ABBREVIATION

$\mathbb{N}$	Set of natural numbers, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$
$\mathbb{R}^+$	Set of non-negative real numbers
$\mathbb{R}$	Set of real numbers, the real field
$\mathbb{C}$	Set of complex numbers, the complex field
$\mathcal{F}$	Collection of all finite subsets of $\mathbb{N}$
w	Set of all sequences with complex entries
θ	Zero vector in a linear space X
ϕ	Set of all finitely non-zero sequences
$\emptyset$	Empty set
ℓ_∞	Space of all bounded sequences
c	Space of convergent sequences
c_0	Space of null sequences
bs	Space of bounded series
cs	Space of convergent series
ℓ_1	Space of absolutely summable sequences
ℓ_p	Space of absolutely p-summable sequences
$\ell_\infty(p)$	Space of sequences (x_k) such that $\sup_{k\in\mathbb{N}} \lvert x_k\rvert^{p_k} < \infty$
$c(p)$	Space of sequences (x_k) such that $\lvert x_k - l\rvert^{p_k} \to 0$, as $k \to \infty$
$c_0(p)$	Space of sequences (x_k) such that $\lvert x_k\rvert^{p_k} \to 0$, as $k \to \infty$
$\ell(p)$	Space of sequences (x_k) such that $\sum_k \lvert x_k\rvert^{p_k} < \infty$
$x^{[m]}$	m^{th} section of a sequence $x = (x_k)$
$e^{(k)}$	Sequences whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$

Ax	$\{(Ax)_n^i\}_{i,n=0}^{\infty}$
$\{(Ax)_n\}$	A-transform of a sequence x
$A^{(-1)}$	Right inverse of a matrix A
(λ, μ)	Class of all matrices from a sequence space λ into a sequence space μ
$(c : c)$	Class of conservative matrices
$(c : c; p)$	Class of Teoplitz (regular) matrices
$(\ell_\infty : c)$	Class of Schur (coercive) matrices
$\mathcal{K}(A)$	Characteristic of a matrix A
λ^α	$\alpha-$dual of a sequence space λ
λ^β	$\beta-$dual of a sequence space λ
λ^γ	$\gamma-$dual of a sequence space λ
$\mathcal{L}(X, Y)$	Set of linear operators from a space X into a space Y
$\mathcal{B}(X, Y)$	Set of bounded linear operators from a space X into a space Y
X^*	Continuous dual of a sequence space X
X^f	$f-$dual of a sequence space X
λ_A	Domain of a infinite matrix A in a sequence space λ
$\sum_k$	$\sum_{k=0}^{\infty}$

CHAPTER 1

INTRODUCTION

1.1 Preliminaries

Definition 1.1.1. [2, p. 71] A **linear space** over the field $\mathbb{C}$ is a nonempty set X with the functions

$$\begin{aligned} + &: X \times X \to X, \\ \cdot &: \mathbb{C} \times X \to X \end{aligned}$$

such that for all scalars $\lambda,\ \mu \in \mathbb{C}$ and elements (vectors) $x, y, z \in X$ we have

(L1) $x + y = y + x$,

(L2) $(x + y) + z = x + (y + z)$,

(L3) there exists $\theta \in X$ such that $x + \theta = x$,

(L4) there exists $-x \in X$ such that $x + (-x) = \theta$,

(L5) $1 \cdot x = x$,

(L6) $\lambda(x + y) = \lambda x + \lambda y$,

(L7) $(\lambda + \mu)x = \lambda x + \mu x$,

(L8) $\lambda(\mu x) = (\lambda\mu)x$.

Definition 1.1.2. [2, p. 74] A **subset** M in a linear space X is a nonempty subset of X such that $\lambda x + \mu y \in M$ whenever $x, y \in M$, for all $\lambda, \mu \in \mathbb{C}$.

By ω, we denote the space of all sequences with complex entries which contains ϕ, the set of all finitely non-zero sequences, that is,

$$\omega := \{x = (x_k) : x_k \in \mathbb{C} \text{ for all } k \in \mathbb{N}\},$$

where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. It is routine verification that w is a linear space with respect to coordinatewise addition and scalar multiplication of sequences which are defined, as usual, by

$$x + y = (x_k) + (y_k) = (x_k + y_k) \text{ and } \alpha x = \alpha(x_k) = (\alpha x_k),$$

respectively; where $x = (x_k)$, $y = (y_k) \in w$ and $\alpha \in \mathbb{C}$. By a sequence space, we understand a linear subspace of the space ω.

Definition 1.1.3. [2, p. 25] A **metric space** is a pair (X, d), consisting of nonempty set X and a metric (or distance) function $d : X \times X \to \mathbb{R}$ such that for all x, y, z in X, the following conditions hold:

(M1) $d(x, y) = 0$ if and only if $x = y$,

(M2) $d(x, y) = d(y, x)$,

(M3) $d(x, z) \leq d(x, y) + d(y, z)$, (the triangle inequality).

A metric function is thus a real-valued function defined on pairs of elements of X. It is important to notice that d is necessarily non-negative.

Example 1.1.4. The most popular metric on the space w is defined by

$$d_w(x, y) \quad := \quad \sum_k \frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)},$$

where $x = (x_k)$, $y = (y_k) \in w$.

Definition 1.1.5. [2, p. 34] A sequence $(x_n) = (x_1, x_2, ...)$, where $x_n \in X$ for every n, is called a **Cauchy sequence** in a metric space (X, d) if and only if

$$\lim_{n\to\infty} d(x_n, x_m) = 0,$$

i.e. for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$d(x_n, x_m) < \varepsilon$$

for all n, $m > N$.

Definition 1.1.6. [2, p. 34] A sequence (x_n) in (X, d) is called **convergent** (to x) if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. We then write $x = \lim x_n$ or $x_n \to x$ and call x the limit of the sequence (x_n).

Now, we can give the following theorem.

Theorem 1.1.7. *[2, p. 35]*

(i) A convergent sequence has a unique limit.

(ii) Every convergent sequence is also a Cauchy sequence, but not conversely, in general.

(iii) If a Cauchy sequence has a convergent subsequence then the whole sequence is convergent.

Definition 1.1.8. [2, p. 36] A metric space (X, d) is called **complete metric space** if and only if every Cauchy sequence converges (to point of X). Explicitly, we require that if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$ then there exists $x \in X$ such that $d(x_n, x) \to 0$ $(n \to \infty)$.

Definition 1.1.9. [3, p. 16] Let X be a real or copmlex linear space and $\|\cdot\|$ be a function from X to the set $\mathbb{R}^+$ of non-negative real numbers. Then the pair $(X, \|\cdot\|)$ is called a **normed space** and $\|\cdot\|$ is a norm on X, if the following axioms are satisfied for all elements $x, y \in X$ and for all scalars λ:

(N1) $\|x\| = 0$ if and only if $x = \theta$.

(N2) $\|\lambda x\| = |\lambda|\|x\|$, (the absolute homogenity property).

(N3) $\|x + y\| \leq \|x\| + \|y\|$, (the triangle inequality).

Definition 1.1.10. [3, p. 17] Let X be a real or copmlex linear space and $\|\cdot\|$ be a function from X to the set $\mathbb{R}^+$ of non-negative real numbers and $p > 0$. Then the pair $(X, \|\cdot\|)$ is called a **p-normed space** and $\|\cdot\|$ is a p-norm on X, if the following axioms are satisfied for all elements $x, y \in X$ and for all scalars λ:

(pN1) $\|x\| = 0$ if and only if $x = \theta$,

(pN2) $\|\lambda x\| = |\lambda|^p\|x\|$,

(pN3) $\|x + y\| \leq \|x\| + \|y\|$.

Now, we can give some examples for normed and p-normed spaces.

Example 1.1.11. Let us define the relations $\|\cdot\|_\infty$ and $\|\cdot\|_p$ by

$$\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|,$$
$$\|x\|_p = \begin{cases} \sum_k |x_k|^p & , \quad 0 < p < 1, \\ (\sum_k |x_k|^p)^{1/p} & , \quad 1 \le p < \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_\infty$ satisfies the norm conditions on the space ℓ_∞. Also, $\|\cdot\|_p$ defines on the space ℓ_p p-norm and norm for $0 < p < 1$ and $1 \le p < \infty$, respectively.

Definition 1.1.12. [4, p. 67]

(i) A sequence (x_n) in a normed space X is called **convergent** if X contains an x such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Then we write $x_n \to x$ and call x the limit of (x_n).

(ii) A sequence (x_n) in a normed space X is called **Cauchy** if for every $\varepsilon > 0$ there is an N such that

$$\|x_m - x_n\| < \varepsilon$$

for all m, $n > N$.

Definition 1.1.13. [2, p. 96] A **Banach space** X is a complete normed linear space. Completeness means that if $\|x_m - x_n\| \to 0$, as $m, n \to \infty$, where $x_n, x_m \in X$, then there exists $x \in X$ such that $\|x_n - x\| \to 0$, as $n \to \infty$.

Example 1.1.14. The spaces ℓ_∞ and c are Banach spaces with the norm $\|\cdot\|_\infty$ defined in Example 1.1.11. In the cases $1 \le p < \infty$ and $0 < p < 1$, the space ℓ_p is a Banach space and a complete p-normed space with the norm $\|\cdot\|_p$ defined in Example 1.1.11, respectively.

Definition 1.1.15. A linear topological space X over the real field $\mathbb{R}$ is said to be a **paranormed space** if there is a function $g : X \to \mathbb{R}$ satisfying the following conditions for all $x, y \in X$:

(i) If $x = \theta$, $g(x) = 0$,

(ii) $g(x) = g(-x)$,

(iii) $g(x + y) \leq g(x) + g(y)$,

(iv) Scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

If g is a paranorm on X, then (X, g) is called a paranormed space. A paranorm g is called total if $g(x) = 0$ implies $x = \theta$.

Definition 1.1.16. [2, p. 87] Let (X, g) be a paranormed space. A sequence (b_k) of elements of X is called a **Schauder basis** for X if and only if, for each $x \in X$ there exists a unique sequence (λ_k) of scalars such that $x = \sum_k \lambda_k b_k$, i.e such that

$$\lim_{n\to\infty} g\left(x - \sum_{k=0}^{n} \lambda_k b_k\right) = 0.$$

Example 1.1.17. Let $e^{(n)}$ be the sequence with $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ whenever $k \neq n$ for all $n \in \mathbb{N}$. Then $(e^{(n)})_{n=0}^{\infty}$ is a Schauder basis of w. More precisely, every sequence $x = (x_k)_{k=0}^{\infty} \in w$ has a unique representation $x = \sum_k x_k e^{(k)}$ that is $x^{[m]} \to x$, as $n \to \infty$, for $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$, the m-section of x.

Definition 1.1.18. [2, p. 102] Let X, Y be linear spaces. Then, a function $T : X \to Y$ is called a **linear operator** (or map, transformation) if and only if for all $x_1, x_2 \in X$, and all scalars λ,

$$T(x_1 + x_2) = Tx_1 + Tx_2 \qquad \text{and} \qquad T(\lambda x_1) = \lambda T x_1.$$

Definition 1.1.19. [2, p. 102] f is a **linear functional** on X if $f : X \to \mathbb{C}$ is a linear operator, i.e. a linear functional is a complex-valued linear operator.

Definition 1.1.20. [2, p. 103] A linear operator $T : X \to Y$ is called **bounded** if and only if there exists a constant M such that

$$\|Tx\| \leq M\|x\| \qquad \text{for all } x \in X.$$

Note that a bounded functional f on X satisfies

$$|f(x)| \leq M\|x\|$$

for all $x \in X$.

Theorem 1.1.21. *[2, p. 104] Let X, Y be two normed spaces and $T : X \to Y$ be a linear operator. Then, T is* ***continuous*** *on X if and only if it is bounded.*

Definition 1.1.22. [2, p. 105] Let X, Y be linear spaces. Then $\mathcal{L}(X, Y)$ denotes the set of all linear operators on X into Y.

Definition 1.1.23. [2, p. 105] The set $\mathcal{L}(X, \mathbb{C})$ of all linear functionals on X is usually denoted by $X^\dagger$ and is called the algebraic dual of X, that is

$$X^\dagger \ := \ \{f \mid f : X \to \mathbb{C}, \text{linear}\}.$$

Definition 1.1.24. [2, p. 105] Let X, Y be normed spaces. Then $\mathcal{B}(X, Y)$ denotes the set of all bounded (i.e. continuous) linear operators on X into Y.

Definition 1.1.25. [2, p. 106] The set $\mathcal{B}(X, \mathbb{C})$ of all bounded linear functionals on X is called the dual (or continuous dual) of X and is denoted by X^*, that is

$$X^* \ := \ \{f \mid f : X \to \mathbb{C}, \ \text{linear and bounded}\}.$$

Definition 1.1.26. [3, p. 65] The $f-$dual X^f of a sequence space X is defined by

$$X^f \ := \ \left\{\left\{f\left(e^{(k)}\right)\right\} : \ f \in X^*\right\}.$$

Definition 1.1.27. [2, p. 106] Let $T \in \mathcal{B}(X, Y)$. Then the norm of T is defined as

$$\|T\| \ := \ \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} < \infty.$$

That the supremum is finite which follows from the fact that

$$\|Tx\| \leq M\|x\| \qquad \text{when } T \in \mathcal{B}(X, Y).$$

Definition 1.1.28. [4, p. 75] A norm $\|\cdot\|$ on a vector space X is said to be **equivalent** to a norm $\|\cdot\|_0$ on X if there are positive number a and b such that

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0$$

for all $x \in X$. This concept is motivated by equivalent norms on X define the same topology for X.

Theorem 1.1.29. *[4, p. 75] On the finite dimensional vector space X, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.*

Definition 1.1.30. A sequence space λ with a linear topology is called a K**-space**, provided each of the maps $q_i : \lambda \to \mathbb{C}$ defined by $q_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field. If sequence space λ is complete and convergence in λ requires coordinatewise convergence, then λ is called FK**-space**. An FK-space whose topology is normable is called a BK**-space**.

Definition 1.1.31. [5] Let d be a metric on a linear space X. If algebraic operations are continuous, namely (x_n) and (y_n) are two sequences in X, and (α_n) is a sequence of scalars such that

$$\begin{array}{llllll} \lim_{n\to\infty} d(x_n, x) = 0 & \text{and} & \lim_{n\to\infty} d(y_n, y) = 0 & \text{implies} & \lim_{n\to\infty} d(x_n + y_n, x + y) = 0, \\ \lim_{n\to\infty} \alpha_n = \alpha & \text{and} & \lim_{n\to\infty} d(x_n, x) = 0 & \text{implies} & \lim_{n\to\infty} d(\alpha_n x_n, \alpha x) = 0 \end{array}$$

then, (X, d) is called **linear metric space**.

Definition 1.1.32. [6] If X is a complete linear metric space then it is called **Frechet sequence space**.

Definition 1.1.33. [5] An FK space $X \supset \phi$ has **AK** if, for every sequence $x = (x_k) \in X$, $x = \sum_k x_k e^{(k)}$, that is

$$\lim_{n\to\infty} x^{[m]} = \lim_{m\to\infty} \sum_{k=0}^{m} x_k e^{(k)} = x$$

and X has **AD** if ϕ is dense in X. If an FK space has AK or AD we also say that it is an **AK** or **AD space**.

Remark 1.1.34. [5] Every AK space has AD. The converse is not true in general.

Now, let we define classical sequence spaces.

We write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively, that is

$$\begin{aligned} \ell_\infty &:= \left\{ x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k| < \infty \right\}, \\ c &:= \left\{ x = (x_k) \in w : \lim_{k\to\infty} |x_k - l| = 0 \text{ for some } l \in \mathbb{C} \right\}, \\ c_0 &:= \left\{ x = (x_k) \in w : \lim_{k\to\infty} x_k = 0 \right\}. \end{aligned}$$

Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely convergent and $p-$absolutely convergent series, respectively, that is

$$\begin{aligned} bs &:= \left\{x=(x_k)\in w : \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n} x_k\right| < \infty\right\}, \\ cs &:= \left\{x=(x_k)\in w : \lim_{n\to\infty}\left|\sum_{k=0}^{n} x_k - l\right| = 0 \text{ for some } l\in\mathbb{C}\right\}, \\ \ell_1 &:= \left\{x=(x_k)\in w : \sum_k |x_k| < \infty\right\}, \\ \ell_p &:= \left\{x=(x_k)\in w : \sum_k |x_k|^p < \infty\right\}; \end{aligned}$$

where $0 < p < \infty$.

Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $\ell(p)$ was defined by Maddox [1] (see also Simons [7] and Nakano [8]) as follows:

$$\ell(p) := \left\{x=(x_k)\in w : \sum_k |x_k|^{p_k} < \infty\right\}, \ (0 < p_k \le H < \infty)$$

which is the complete space paranormed by

$$g(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. Also other well-known paranormed spaces defined by Maddox [1] as follows:

$$\begin{aligned} \ell_\infty(p) &:= \left\{x=(x_k)\in w : \ \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\right\}, \\ c(p) &:= \left\{x=(x_k)\in w : \ \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l\in\mathbb{C}\right\}, \\ c_0(p) &:= \left\{x=(x_k)\in w : \ \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}. \end{aligned}$$

We assume throughout that $p_k^{-1} + (p_k')^{-1} = 1$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$ and use the convention that any term with negative subscript is equal to naught.

Definition 1.1.35. [3, p. 21] For the sequence spaces λ and μ, the set $\mathcal{S}(\lambda, \mu)$ defined by

$$\mathcal{S}(\lambda,\mu) := \{z=(z_k)\in w : \ xz=(x_k z_k)\in\mu \text{ for all } x=(x_k)\in\lambda\} \quad (1.1.1)$$

is called the multiplier space of λ and μ. With the notation of (1.1.1), the alpha-, beta- and gamma-duals of a sequence space λ which are denoted by λ^α, λ^β and λ^γ, respectively, are defined by

$$\lambda^\alpha = \mathcal{S}(\lambda, \ell_1), \qquad \lambda^\beta = \mathcal{S}(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = \mathcal{S}(\lambda, bs),$$

that is

$$\begin{aligned}
\lambda^\alpha &:= \left\{ x = (x_k) \in \omega : \sum_k |x_k y_k| < \infty \ \text{ for all } \ y = (y_k) \in \lambda \right\}, \\
\lambda^\beta &:= \left\{ x = (x_k) \in \omega : \left(\sum_{k=0}^{n} x_k y_k \right)_{n\in\mathbb{N}} \in c \ \text{ for all } \ y = (y_k) \in \lambda \right\}, \\
\lambda^\gamma &:= \left\{ x = (x_k) \in \omega : \left(\sum_{k=0}^{n} x_k y_k \right)_{n\in\mathbb{N}} \in \ell_\infty \ \text{ for all } \ y = (y_k) \in \lambda \right\}.
\end{aligned}$$

Theorem 1.1.36. *[9, pp. 106, 108] Let λ be an $FK-$space which contains ϕ. Then,*

(i) $\lambda^\beta \subset \lambda^\gamma \subset \lambda^f$.

(ii) If λ has AK, $\lambda^\beta = \lambda^f$.

(iii) If λ has AD, $\lambda^\beta = \lambda^\gamma$.

(iv) $\lambda^f = \lambda^*$ iff λ has AD.

Definition 1.1.37. [3, p. 31] Suppose that $A = (a_{nk})$ is an infinite matrix of real or complex numbers a_{nk} and $x = (x_k) \in w$, where $k, n \in \mathbb{N}$. Then, we obtain the sequence Ax, the **A-transform of x**, by the usual matrix product

$$\begin{aligned}
\mathbf{Ax} &= \begin{pmatrix}
a_{00} & a_{01} & a_{02} & \cdots & a_{0k} & \cdots \\
a_{10} & a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\
a_{20} & a_{21} & a_{22} & \cdots & a_{1k} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix}
a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + \cdots + a_{0k}x_k + \cdots \\
a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k + \cdots \\
\vdots \\
a_{n0}x_0 + a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k + \cdots \\
\vdots
\end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \sum_k a_{0k}x_k \\ \sum_k a_{1k}x_k \\ \vdots \\ \sum_k a_{nk}x_k \\ \vdots \end{pmatrix}.$$

Hence, in this way, we transform the sequence x into the sequence space $Ax = \{(Ax)_n\}$ with

$$(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}), \tag{1.1.2}$$

provided the series on the right hand side of (1.1.2) converges for each $n \in \mathbb{N}$. Let λ and μ be any two sequence spaces. If Ax exists and is in μ for every sequence $x = (x_k) \in \lambda$, then we say that A defines matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$. By (λ, μ), we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if Ax exists i.e. $A_n \in \lambda^\beta$ for all $n \in \mathbb{N}$ and is in μ for all $x \in \lambda$, where A_n denotes the sequence in the n-th row of A.

Definition 1.1.38. For any sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A \ := \ \{x = (x_k) \in w : Ax \in \lambda\}.$$

Definition 1.1.39. Let $A = (a_{nk})$ be an infinite matrix of complex numbers. If the A-transform of any convergent sequence of complex numbers exists and converges then, A is called **conservative matrix**. By $(c : c)$, we denote the set of conservative matrices.

Theorem 1.1.40 (Kojima-Schur)**.** *[3, p. 35] $A = (a_{nk})$ is a conservative matrix if and only if*

(i) $\|A\| = \sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} a_{nk} = \alpha_k$ *for each* $k \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \sum_k a_{nk} = \alpha$.

Definition 1.1.41. Let $A = (a_{nk})$ be an infinite matrix and $(x_k) \in w$. If A is conservative and preserves limits, i.e. $x_k \to x$, as $k \to \infty$, implies $(Ax)_n \to x$, as

$k \to \infty$, where $(Ax)_n$ is the A-transform of the convergent sequence (x_k), then A is called **regular matrix**. By $(c : c; p)$, we denote the set of all regular matrices.

Theorem 1.1.42 (Silverman-Teoplitz)**.** *[3, p. 35] $A = (a_{nk})$ is a regular matrix if and only if*

(i) $\|A\| = \sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} a_{nk} = 0$ *for each* $k \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \sum_k a_{nk} = 1$.

Theorem 1.1.43 (Schur matrix)**.** *[3, p. 36] $A = (a_{nk}) \in (\ell_\infty : c)$ if and only if*

(i) The series $\sum_k |a_{nk}|$ must be uniformly convergent with respect to n.

(ii) There exists $\alpha_k \in \mathbb{C}$ such that $a_{nk} \to \alpha_k$, as $n \to \infty$.

Definition 1.1.44. [3, p. 38] The characteristic $\mathcal{K}(A)$ of a matrix $A = (a_{nk})$ is defined by

$$\mathcal{K}(A) \quad := \quad \lim_{n\to\infty} \sum_k a_{nk} - \sum_k \left(\lim_{n\to\infty} a_{nk} \right)$$

which is a multiplicate linear functional. A matrix A is called **coregular** if $\mathcal{K}(A) \neq 0$ and is called **conull** if $\mathcal{K}(A) = 0$.

Remark 1.1.45. [3, p. 39] The Silverman-Teoplitz theorem yields for a regular matrix A that $\mathcal{K}(A) = 1$ which leads us to the fact that Toeplitz matrices form a subset of coregular matrices. One can easily see for a Schur matrix A that $\mathcal{K}(A) = 0$ which says us that coercive matrices for a subset of conull matrices.

1.2 Some Inequalities

Here, we give the inequalities which will be used in the following chapters.

(1) **Triangle inequality:** Let a, b be any two complex numbers. Then, the inequality

$$|a + b| \leq |a| + |b|$$

holds.

(2) Let $a, b \in \mathbb{C}$ and $0 < p \leq 1$. Then we have the inequality

$$|a + b|^p \leq |a|^p + |b|^p. \tag{1.2.1}$$

(3) **Minkowski's inequalty:** Let $1 \leq p < \infty$ and $x_0, x_1, ..., x_n, y_0, y_1, ..., y_n \in \mathbb{C}$. Then we have

$$\left(\sum_{k=0}^{\infty} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=0}^{\infty} |y_k|^p\right)^{1/p}.$$

Also, if $x, y \in \ell_p$ then $x + y \in \ell_p$ and we can write

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

(4) Let a, b be any camplex numbers and B be any positive number. Then, the inequality

$$|ab| \leq B\left(\left|aB^{-1}\right|^{p'} + |b|^p\right) \tag{1.2.2}$$

holds, where $p > 1$ and $p^{-1} + p'^{-1} = 1$.

CHAPTER 2

CHARACTERIZATIONS OF $F = (f_{nk})$ MATRIX TO SOME MATRIX CLASSES

Consider the sequence (f_n) of Fibonacci numbers defined by the linear recurrence relations

$$f_n := \begin{cases} 1 & , \quad n = 0, 1, \\ f_{n-1} + f_{n-2} & , \quad n \geq 2. \end{cases}$$

Let us define the double band matrix $F = (f_{nk})$ by the sequence (f_n), as follows:

$$f_{nk} := \begin{cases} -\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ \frac{f_n}{f_{n+1}} & , \quad k = n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n \end{cases} \tag{2.1}$$

for all $k, n \in \mathbb{N}$. That is to say that

$$F = (f_{nk}) = \begin{pmatrix} \frac{f_0}{f_1} & 0 & 0 & 0 & \cdots \\ -\frac{f_2}{f_1} & \frac{f_1}{f_2} & 0 & 0 & \cdots \\ 0 & -\frac{f_3}{f_2} & \frac{f_2}{f_3} & 0 & \cdots \\ 0 & 0 & -\frac{f_4}{f_3} & \frac{f_3}{f_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -2 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & -\frac{3}{2} & \frac{2}{3} & 0 & \cdots \\ 0 & 0 & -\frac{5}{3} & \frac{3}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, let us investigate the classes of our matrix $F = (f_{nk})$ belonging to. Let us consider the entries of the sequence (f_n)

$$f_0 = f_1 = 1, \quad f_2 = 2, \quad f_3 = 3, \quad f_4 = 5, \ldots \text{ and general term } f_n = f_{n-1} + f_{n-2}.$$

It is easy to see that $|-f_{n+1}/f_n| \leq 2$ and $|f_n/f_{n+1}| \leq 1$. Also, we have $|-f_{n+1}/f_n| \to 1,618\ldots$ and $|f_n/f_{n+1}| \to 0,618\ldots$, as $n \to \infty$.

(i) Firstly, let us check the norm of the $F = (f_{nk})$ matrix.

$$\begin{aligned} \| F \| &= \sup_{n\in\mathbb{N}} \sum_k |f_{nk}| = \sup_{n\in\mathbb{N}} \sum_{k=n-1}^{n} |f_{nk}| \\ &= \sup_{n\in\mathbb{N}} \left(\left| -\frac{f_{n+1}}{f_n} \right| + \left| \frac{f_n}{f_{n+1}} \right| \right) < 3 < \infty. \end{aligned}$$

(ii) Since almost all of the elements of the column vectors of the matrix $F = (f_{nk})$ are zero,

$$\lim_{n\to\infty} f_{nk} = 0 \tag{2.2}$$

for every $k \in \mathbb{N}$.

(iii) Let us compute the value of the expression $\sum_k f_{nk}$, as $n \to \infty$.

$$\begin{aligned} \lim_{n\to\infty} \sum_k f_{nk} &= \lim_{n\to\infty} \sum_{k=n-1}^{n} f_{nk} \\ &= \lim_{n\to\infty} \left(-\frac{f_{n+1}}{f_n} + \frac{f_n}{f_{n+1}} \right) \cong -1. \end{aligned}$$

(iv) Now, we show whether the series $\sum_k |f_{nk}|$ is uniformly convergent with respect to n or not. For this, it is sufficient to analyze the values of $\lim_{n\to\infty} \sum_k |f_{nk}|$ and $\sum_k \lim_{n\to\infty} |f_{nk}|$. Then, we have

$$\lim_{n\to\infty} \sum_k |f_{nk}| = \lim_{n\to\infty} \left(\left| -\frac{f_{n+1}}{f_n} \right| + \left| \frac{f_n}{f_{n+1}} \right| \right) \cong 2.2 \tag{2.3}$$

and by (2.2) that

$$\sum_k \lim_{n\to\infty} |f_{nk}| = 0. \tag{2.4}$$

Since (2.3) and (2.4) not equal to each other, the series $\sum_k |f_{nk}|$ is not uniformly convergent with respect to n.

(v) Finally, we find the characteristic $\mathcal{K}(F)$ of $F = (f_{nk})$ matrix that

$$\mathcal{K}(F) = \lim_{n\to\infty} \sum_k f_{nk} - \sum_k \left(\lim_{n\to\infty} f_{nk} \right) \cong -1.$$

By means of (i)-(iii), (iv) and (v) we can say that; $F = (f_{nk})$ is a conservative matrix but not regular matrix, it is not Schur matrix and it is coregular matrix but not conull matrix, respectively.

CHAPTER 3

THE SEQUENCE SPACE $\ell(F,p)$

We employ the Fibonacci matrix $F=(f_{nk})$ as in (2.1), where $k,n\in\mathbb{N}$. Then, we obtain the sequence Fx, the F*-transform of* x, by the usual matrix product

$$
\begin{aligned}
Fx &= \begin{pmatrix}
\frac{f_0}{f_1} & 0 & 0 & \cdots & 0 & 0 & \cdots \\
-\frac{f_2}{f_1} & \frac{f_1}{f_2} & 0 & \cdots & 0 & 0 & \cdots \\
0 & -\frac{f_3}{f_2} & \frac{f_2}{f_3} & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{f_{k+1}}{f_k} & \frac{f_k}{f_{k+1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ x_k \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix}
\frac{f_0}{f_1}x_0 \\
-\frac{f_2}{f_1}x_0+\frac{f_1}{f_2}x_1 \\
-\frac{f_3}{f_2}x_1+\frac{f_2}{f_3}x_2 \\
\vdots \\
-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k \\
\vdots
\end{pmatrix}
\end{aligned}
$$

where $x=(x_k)\in w$. Hence, we transform the sequence x into the sequence $Fx=\{(Fx)_k\}$.

We can define the sequence $y=(y_k)$ by the F-transform of the sequence $x=(x_k)$, i.e.,

$$y_k=(Fx)_k=-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k \tag{3.1}$$

for all $k\in\mathbb{N}$. At this situation we can express x in terms of y that

$$x_k=\left(F^{-1}y\right)_k=\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_jf_{j+1}}y_j \tag{3.2}$$

for all $k \in \mathbb{N}$. The inverse $F^{-1} = (c_{nk})$ of the matrix F can be expressed as follows

$$c_{nk} := \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

The main purpose of this study is to introduce the domain $\ell(F, p)$ of the double band matrix F in the sequence space $\ell(p)$, that is to say that

$$\ell(F,p) := \left\{ x = (x_k) \in \omega : \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^{p_k} < \infty \right\},$$

where $0 < p_k \leq H < \infty$. In the case $p_k = p$ for all $k \in \mathbb{N}$, the space $\ell(F, p)$ is reduced to the space $\ell_p(F)$, i.e.,

$$\ell_p(F) := \left\{ x = (x_k) \in \omega : \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^{p} < \infty \right\}, \quad (p \geq 1).$$

Furthermore, the alpha-, beta- and gamma-duals of the space $\ell(F, p)$ are determined, and the Schauder basis is constructed. The classes of matrix transformations from the space $\ell(F, p)$ to the spaces ℓ_∞, c and c_0 are characterized. Some other classes of matrix transformations are also characterized by means of a given basic lemma.

Theorem 3.1. *$\ell(F, p)$ is a linear, complete and metric space paranormed by h defined by*

$$h(x) = \left(\sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^{p_k} \right)^{1/M}, \tag{3.3}$$

where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.

Proof. To show the linearity of the space with respect to the coordinatewise addition and scalar multiplication is trivial. Firstly, we show that $\ell(F, p)$ is a paranormed space with the paranorm h defined by (3.3).

It is clear that $h(\theta) = 0$, where $\theta = (0, 0, \ldots)$ and $h(x) = h(-x)$ for all $x \in \ell(F, p)$.

Let $x = (x_k), y = (y_k) \in \ell(F, p)$. Then, by Minkowski's inequality and the inequality (1.2.1), we have

$$\begin{aligned}
h(x+y) &= \left[\sum_k \left|-\frac{f_{k+1}}{f_k}(x_{k-1}+y_{k-1})+\frac{f_k}{f_{k+1}}(x_k+y_k)\right|^{p_k}\right]^{1/M} \\
&= \left[\sum_k \left(\left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}y_{k-1}+\frac{f_k}{f_{k+1}}y_k\right|^{p_k/M}\right)^M\right]^{1/M} \\
&\leq \left[\sum_k \left(\left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k\right|^{p_k/M}+\left|-\frac{f_{k+1}}{f_k}y_{k-1}+\frac{f_k}{f_{k+1}}y_k\right|^{p_k/M}\right)^M\right]^{1/M} \\
&\leq \left[\sum_k \left(\left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k\right|^{p_k/M}\right)^M\right]^{1/M}+\left[\sum_k \left(\left|-\frac{f_{k+1}}{f_k}y_{k-1}+\frac{f_k}{f_{k+1}}y_k\right|^{p_k/M}\right)^M\right]^{1/M} \\
&= \left(\sum_k \left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k\right|^{p_k}\right)^{1/M}+\left(\sum_k \left|-\frac{f_{k+1}}{f_k}y_{k-1}+\frac{f_k}{f_{k+1}}y_k\right|^{p_k}\right)^{1/M} \\
&= h(x)+h(y).
\end{aligned}$$

Also, since the inequality $|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}$ holds for $\alpha \in \mathbb{R}$, we get

$$\begin{aligned}
h(\alpha x) &= \left[\sum_k \left|-\frac{f_{k+1}}{f_k}(\alpha x_{k-1})+\frac{f_k}{f_{k+1}}(\alpha x_k)\right|^{p_k}\right]^{1/M} \\
&= \left(\sum_k |\alpha|^{p_k}\left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k\right|^{p_k}\right)^{1/M} \\
&\leq \max\{1,|\alpha|\}h(x).
\end{aligned}$$

Let (α_n) be a sequence of scalars with $\alpha_n \to \alpha$, as $n \to \infty$ and $\left\{x^{(n)}\right\}_{n=0}^{\infty}$ be a sequence of elements $x^{(n)} \in \ell(F,p)$ with $h\left[x^{(n)}-x\right] \to 0$, as $n \to \infty$. Then, we observe that

$$\begin{aligned}
0 \leq h\left[\alpha_n x^{(n)}-\alpha x\right] &= h\left[\alpha_n x^{(n)}-\alpha x^{(n)}+\alpha x^{(n)}-\alpha x\right] \\
&= h\left[(\alpha_n-\alpha)x^{(n)}+\alpha\left(x^{(n)}-x\right)\right] \\
&\leq h\left[(\alpha_n-\alpha)x^{(n)}\right]+h\left[\alpha\left(x^{(n)}-x\right)\right] \\
&= |\alpha_n-\alpha|h\left[x^{(n)}\right]+\max\{1,|\alpha|\}h\left[x^{(n)}-x\right].
\end{aligned} \tag{3.4}$$

If we combine the facts $\alpha_n-\alpha \to 0$, as $n \to \infty$ and $h\left[x^{(n)}-x\right] \to 0$, as $n \to \infty$ with (3.4) we obtain that $h\left[\alpha_n x^{(n)}-\alpha x\right] \to 0$, as $n \to \infty$. That is to say that the scalar multiplication is continuous. This shows that h is a paranorm on $\ell(F,p)$.

Moreover, if we assume $h(x)=0$, then we get

$$\left|-\frac{f_{k+1}}{f_k}x_{k-1}+\frac{f_k}{f_{k+1}}x_k\right|=0$$

for each $k \in \mathbb{N}$. If we put $k=0$, since $x_{-1}=0$ and $f_0/f_1 \neq 0$, we have $x_0=0$. For $k=1$, since $x_0=0$ and $f_1/f_2 \neq 0$, we have $x_1=0$. Continuing in this way, we obtain $x_k = 0$ for all $k \in \mathbb{N}$. Namely, we obtain $x = \theta = (0, 0, \ldots)$. This shows that h is a total paranorm.

Now, we show that $\ell(F, p)$ is complete. Let (x^n) be any Cauchy sequence in $\ell(F, p)$; where $x^n = \left\{x_0^{(n)}, x_1^{(n)},\ x_2^{(n)}, \ldots\right\}$. Then, for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that $[h(x^n - x^m)]^M < \varepsilon^M$ for all $n, m > n_0(\varepsilon)$. Since for each fixed $k \in \mathbb{N}$

$$\begin{aligned}
|(Fx^n)_k - (Fx^m)_k|^{p_k} &\leq \sum_k |(Fx^n)_k - (Fx^m)_k|^{p_k} \\
&= \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1}^{(n)} + \frac{f_k}{f_{k+1}} x_k^{(n)} - \left[-\frac{f_{k+1}}{f_k} x_{k-1}^{(m)} + \frac{f_k}{f_{k+1}} x_k^{(m)} \right] \right|^{p_k} \\
&= \sum_k \left| -\frac{f_{k+1}}{f_k} \left[x_{k-1}^{(n)} - x_{k-1}^{(m)} \right] + \frac{f_k}{f_{k+1}} \left[x_k^{(n)} - x_k^{(m)} \right] \right|^{p_k} \\
&= [h(x^n - x^m)]^M < \varepsilon^M
\end{aligned}$$

for every $n, m > n_0(\varepsilon)$, $\{(Fx^0)_k, (Fx^1)_k, (Fx^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(Fx^n)_k \to (Fx)_k$ as $n \to \infty$. Using these infinitely many limits $(Fx)_0, (Fx)_1, (Fx)_2, \ldots$ we define the sequence $\{(Fx)_0, (Fx)_1, (Fx)_2, \ldots\}$. For each $k \in \mathbb{N}$ and $n > n_0(\varepsilon)$

$$\begin{aligned}
[h(x^n - x)]^M &= \sum_k \left| -\frac{f_{k+1}}{f_k} \left[x_{k-1}^{(n)} - x_{k-1} \right] + \frac{f_k}{f_{k+1}} \left[x_k^{(n)} - x_k \right] \right|^{p_k} \\
&= \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1}^{(n)} + \frac{f_k}{f_{k+1}} x_k^{(n)} - \left[-\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right] \right|^{p_k} \\
&= \sum_k |(Fx^n)_k - (Fx)_k|^{p_k} < \varepsilon^M.
\end{aligned}$$

This shows that $x^n - x \in \ell(F, p)$. Since $\ell(F, p)$ is a linear space, we conclude that $x \in \ell(F, p)$. It follows that $x^n \to x$, as $n \to \infty$ in $\ell(F, p)$ which means that $\ell(F, p)$ is complete.

Now, one can easily check that the absolute property does not hold on the space $\ell(F, p)$, that is

$$\begin{aligned}
h(x) = \left(\sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^{p_k} \right)^{1/M} &\neq \left(\sum_k \left| -\frac{f_{k+1}}{f_k} |x_{k-1}| + \frac{f_k}{f_{k+1}} |x_k| \right|^{p_k} \right)^{1/M} \\
&= h(|x|),
\end{aligned}$$

where $|x| = (|x_k|)$. This says that $\ell(F, p)$ is the sequence space of non-absolute type. □

Theorem 3.2. *Convergence in $\ell(F,p)$ is strictly stronger than coordinatewise convergence, but the converse is not true, in general.*

Proof. First we show that $h(x^n - x) \to 0$, as $n \to \infty$ implies $x_k^{(n)} \to x_k$, as $n \to \infty$ for all $k \in \mathbb{N}$. If we fix k, then we have

$$\begin{aligned} 0 &\leq \left| -\frac{f_{k+1}}{f_k} x_{k-1}^{(n)} + \frac{f_k}{f_{k+1}} x_k^{(n)} - \left(-\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right) \right|^{p_k} \\ &\leq \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1}^{(n)} + \frac{f_k}{f_{k+1}} x_k^{(n)} - \left(-\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right) \right|^{p_k} \\ &= \sum_k \left| -\frac{f_{k+1}}{f_k} \left(x_{k-1}^{(n)} - x_{k-1} \right) + \frac{f_k}{f_{k+1}} \left(x_k^{(n)} - x_k \right) \right|^{p_k} \\ &= \left[h\left(x^n - x\right) \right]^M . \end{aligned}$$

Hence, we have for $k = 0$

$$\lim_{n\to\infty} \left| -\frac{f_1}{f_0} x_{-1}^{(n)} + \frac{f_0}{f_1} x_0^{(n)} - \left(-\frac{f_1}{f_0} x_{-1} + \frac{f_0}{f_1} x_0 \right) \right| = 0,$$

that is, $\left| \frac{f_0}{f_1} \left[x_0^{(n)} - x_0 \right] \right| \to 0$, as $n \to \infty$ and $f_0/f_1 = 1 \neq 0$, then $\left| x_0^{(n)} - x_0 \right| \to 0$, as $n \to \infty$. Likewise, for each $k \in \mathbb{N}$, we have $\left| x_k^{(n)} - x_k \right| \to 0$, as $n \to \infty$.

Now, we show that the converse is not true in general. We assume $x_k^{(n)} \to x_k$, as $n \to \infty$. Then, there exists an $N \in \mathbb{N}$ such that $\left| x_k^{(n)} - x_k \right| < 1$ for each fixed k and for all $n \geq N$. Therefore, we see that

$$\begin{aligned} 0 &\leq h(x^n - x) = \left[\sum_k \left| -\frac{f_{k+1}}{f_k} \left(x_{k-1}^{(n)} - x_{k-1} \right) + \frac{f_k}{f_{k+1}} \left(x_k^{(n)} - x_k \right) \right|^{p_k} \right]^{1/M} \qquad (3.5) \\ &= \left\{ \sum_k \left[\left| -\frac{f_{k+1}}{f_k} \left(x_{k-1}^{(n)} - x_{k-1} \right) + \frac{f_k}{f_{k+1}} \left(x_k^{(n)} - x_k \right) \right|^{p_k/M} \right]^M \right\}^{1/M} \\ &\leq \left\{ \sum_k \left[\left| -\frac{f_{k+1}}{f_k} \left(x_{k-1}^{(n)} - x_{k-1} \right) \right|^{p_k/M} + \left| \frac{f_k}{f_{k+1}} \left(x_k^{(n)} - x_k \right) \right|^{p_k/M} \right]^M \right\}^{1/M} \\ &\leq \left[\sum_k \left| -\frac{f_{k+1}}{f_k} \left(x_{k-1}^{(n)} - x_{k-1} \right) \right|^{p_k} \right]^{1/M} + \left[\sum_k \left| \frac{f_k}{f_{k+1}} \left(x_k^{(n)} - x_k \right) \right|^{p_k} \right]^{1/M} \\ &\leq \left(\sum_k \left| -\frac{f_{k+1}}{f_k} \right|^{p_k} \left| x_{k-1}^{(n)} - x_{k-1} \right|^{p_k} \right)^{1/M} + \left(\sum_k \left| \frac{f_k}{f_{k+1}} \right|^{p_k} \left| x_k^{(n)} - x_k \right|^{p_k} \right)^{1/M} \\ &\leq \left(\sum_k \left| -\frac{f_{k+1}}{f_k} \right|^{p_k} \right)^{1/M} + \left(\sum_k \left| \frac{f_k}{f_{k+1}} \right|^{p_k} \right)^{1/M} \end{aligned}$$

for all k and $n \geq N$. Since $|-f_{k+1}/f_k| \to 1.6$ and $|f_k/f_{k+1}| \to 0.6$, as $k \to \infty$, $h(x^n - x)$ in (3.5) does not converge for each fixed $k \in \mathbb{N}$ and for all $n \geq N$. This implies that the converse is not true. Let us consider the elements of the sequence x^n be equal then we observe $h(x^n - x) = 0$, that is to say that coordinatewise convergence requires convergence. Hence, we can say that the converse is not true in general. $\square$

Theorem 3.3. *$\ell(F, p)$ is a K-space.*

Proof. Firstly, we show that $q_i(x) = x_i$ is linear for all $i \in \mathbb{N}$. Let $x, y \in \ell(F, p)$ and $\alpha \in \mathbb{C}$. Then, we get

$$q_i(x + y) = (x + y)_i = x_i + y_i = q_i(x) + q_i(y) \quad \text{and} \quad q_i(\alpha x) = (\alpha x)_i = \alpha x_i = \alpha q_i(x)$$

for all $i \in \mathbb{N}$. Hence, q_i is linear.

Now, we prove that q_i is continuous. For this, it is sufficient to show that q_i is bounded.

Let $x \in \ell(F, p)$ be any vector. Then, since $|q_i(x)| = |x_i|$ for all $i \in \mathbb{N}$ one can see that

$$\|q_i\| \quad := \quad \sup_{x \neq \theta} \frac{|q_i(x)|}{\|x\|_{\ell(F,p)}} = \sup_{x \neq \theta} \frac{|x_i|}{\|x\|_{\ell(F,p)}} \leq \sup_{x \neq \theta} \frac{\|x\|_{\ell(F,p)}}{\|x\|_{\ell(F,p)}} = 1 < \infty,$$

i.e. q_i is bounded. Hence, p_i is linear and continuous functional. That is to say that $\ell(F, p)$ is a K-space. $\square$

Theorem 3.4. *$\ell(F, p)$ is an FK-space.*

Proof. It is easy to see by Theorems 3.1 and 3.2 that $\ell(F, p)$ is complete sequence space and convergence requires coordinatewise convergence. Hence, $\ell(F, p)$ is an FK-space. $\square$

Theorem 3.5. *$\ell_p(F)$ is the linear space under the coordinatewise addition and scalar multiplication which is a BK-space with the norm*

$$\|x\| \quad = \quad \left(\sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^p \right)^{1/p},$$

where $x = (x_k) \in \ell_p(F)$ and $1 \leq p < \infty$.

Proof. Since the first part of the theorem is a routine verification, we omit the detail. Since ℓ_p is a BK-space with respect to its usual norm and F is a triangle matrix, Theorem 4.3.2 of Wilansky [9, p. 61] gives the fact that $\ell_p(F)$ is a BK-space, where $1 \leq p < \infty$. This completes the proof. □

Theorem 3.6. *$\ell_p(F)$ is a Frechet space.*

Proof. It is easy to see that $\ell_p(F)$ is a linear, complete and metric space. We only need to prove that $\ell_p(F)$ is a linear metric space. Let (x_n) and (y_n) be two sequences in $\ell_p(F)$, and (α_n) be a sequence of scalars such that $d(x_n, x) \to 0$, $d(y_n, y) \to 0$ and $\alpha_n \to \alpha$, as $n \to \infty$. Then, we get that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} d(x_n + y_n, x + y) = \lim_{n\to\infty} [\|x_n + y_n - (x + y)\|] \\ &\leq \lim_{n\to\infty} (\|x_n - x\| + \|y_n - y\|) \\ &= \lim_{n\to\infty} d(x_n, x) + \lim_{n\to\infty} d(y_n, y) = 0, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} d(\alpha_n x_n, \alpha x) = \lim_{n\to\infty} \|\alpha_n x_n - \alpha x\| \\ &= \lim_{n\to\infty} \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \\ &\leq \lim_{n\to\infty} (|\alpha_n - \alpha|\|x_n\| + |\alpha|\|x_n - x\|) \\ &= \lim_{n\to\infty} |\alpha_n - \alpha|\|x_n\| + |\alpha| \lim_{n\to\infty} d(x_n, x) = 0. \end{aligned} \tag{3.7}$$

It is easy to see from (3.6) and (3.7) that $\ell_p(F)$ is a linear metric space. Hence, $\ell_p(F)$ is a Frechet space. □

With the notation of (3.1), the transformation T defined from $\ell(F, p)$ to $\ell(p)$ by $x \mapsto y = Tx$ is linear bijection, we have the following

Theorem 3.7. *The sequence space $\ell(F, p)$ of the non-absolute type is linearly paranorm isomorphic to the space $\ell(p)$, where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.*

Proof. To prove the theorem, we should show the existence of a linear bijection between the spaces $\ell(F, p)$ and $\ell(p)$ for $0 < p_k \leq H < \infty$. Let T be a transformation from $\ell(F, p)$ to $\ell(p)$ such that

$$\begin{aligned} T \;:\; \ell(F, p) &\longrightarrow \ell(p) \\ x &\longmapsto Tx = Fx = y. \end{aligned}$$

The linearity of T is trivial. Further it is obvious that $x = \theta$ whenever $Tx = \theta$, hence T is injective. Let $y \in \ell(p)$ and define the sequence $x = (x_k)$ as in (3.2). Then we have

$$\begin{aligned}
(Fx)_k &= -\frac{f_{k+1}}{f_k}x_{k-1} + \frac{f_k}{f_{k+1}}x_k \\
&= -\frac{f_{k+1}}{f_k}\sum_{n=0}^{k-1}\frac{f_k^2}{f_n f_{n+1}}y_n + \frac{f_k}{f_{k+1}}\sum_{n=0}^{k}\frac{f_{k+1}^2}{f_n f_{n+1}}y_n \\
&= -\sum_{n=0}^{k-1}\frac{f_k f_{k+1}}{f_n f_{n+1}}y_n + \sum_{n=0}^{k}\frac{f_k f_{k+1}}{f_n f_{n+1}}y_n \\
&= y_k
\end{aligned}$$

for all $k \in \mathbb{N}$, which leads us to the fact that

$$h(x) = \left(\sum_k \left|-\frac{f_{k+1}}{f_k}x_{k-1} + \frac{f_k}{f_{k+1}}x_k\right|^{p_k}\right)^{1/M} = \left(\sum_k |y_k|^{p_k}\right)^{1/M} = h(y) < \infty.$$

Thus we deduce that $x \in \ell(F,p)$, T is surjective and paranorm preserving. Hence, T is a linear bijection and so the spaces $\ell(F,p)$ and $\ell(p)$ are paranorm isomorphic. $\square$

Theorem 3.8. *Let $0 < p_k \le H < \infty$ and $\lambda_k = (Fx)_k$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathbb{N}}$ of the elements of the spaces $\ell(F,p)$ by*

$$b_n^{(k)} = \begin{cases} \frac{f_{k+1}^2}{f_n f_{n+1}} &, \quad 0 \le n \le k, \\ 0 &, \quad n > k \end{cases} \tag{3.8}$$

for every fixed $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the space $\ell(F,p)$ and any $x \in \ell(F,p)$ has a unique representation of the form

$$x = \sum_k \lambda_k b^{(k)}. \tag{3.9}$$

Proof. It is clear that $\{b^{(k)}\}_{k\in\mathbb{N}} \subset \ell(F,p)$, since

$$Fb^{(k)} = e^{(k)} \in \ell(p),\ k \in \mathbb{N}$$

for $0 < p_k \le H < \infty$. Let $x \in \ell(F,p)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k b^{(k)}.$$

Then, we have

$$Fx^{[m]} = \sum_{k=0}^{m} \lambda_k F b^{(k)} = \sum_{k=0}^{m} \lambda_k e^{(k)} = \sum_{k=0}^{m} (Fx)_k e^{(k)}$$

and

$$\left\{F\left(x - x^{[m]}\right)\right\}_i = \begin{cases} 0 & , \quad 0 \leq i \leq m; \\ (Fx)_i & , \quad i > m \end{cases} \quad (i, m \in \mathbb{N}).$$

Given $\varepsilon > 0$, then there is an integer m_0 such that for all $m \geq m_0$

$$\left(\sum_{i=m}^{\infty} |(Fx)_i|^{p_k}\right)^{1/M} < \frac{\varepsilon}{2}.$$

Therefore,

$$h\left(x - x^{[m]}\right) = \left(\sum_{i=m}^{\infty} |(Fx)_i|^{p_k}\right)^{1/M} \leq \left(\sum_{i=m_0}^{\infty} |(Fx)_i|^{p_k}\right)^{1/M} < \varepsilon$$

for all $m \geq m_0$, which proves that $x \in \ell(F, p)$ is represented as in (3.9).

Let us show the uniqueness of the representation for $x \in \ell(F, p)$ given by (3.9). Suppose, on the contrary, that there exists a representation $x = \sum_k \mu_k b^{(k)}$. Since the linear transformation T from $\ell(F, p)$ to $\ell(p)$, used in the proof of Theorem 3.7 is continuous, we have that

$$(Fx)_n = \sum_k \lambda_k \left(Fb^{(k)}\right)_n = \sum_k \mu_k e_n^{(k)} = \mu_n$$

which contradicts the fact that $(Fx)_n = \lambda_n$ for all $n \in \mathbb{N}$. Hence, the representation in (3.9) of $x \in \ell(p)$ is unique. This completes the proof. $\square$

CHAPTER 4

THE ALPHA-, BETA- AND GAMMA-DUALS OF THE SPACE $\ell(F, p)$

Prior to giving the alpha-, beta- and gamma-duals of the space $\ell(F, p)$, we quote some required lemmas for proving our theorems.

Lemma 4.1. [10, Theorem 5.1.0] *Let $A = (a_{nk})$ be an infinite matrix over the complex field. The following statements hold:*

(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if

$$\sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} a_{nk} \right|^{p_k} < \infty.$$

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} a_{nk} B^{-1} \right|^{p_k'} < \infty. \tag{4.1}$$

Lemma 4.2. [11, (i) and (ii) of Theorem 1] *Let $A = (a_{nk})$ be an infinite matrix over the complex field. The following statements hold:*

(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if

$$\sup_{n,k \in \mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{4.2}$$

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{n \in \mathbb{N}} \sum_k \left| a_{nk} B^{-1} \right|^{p_k'} < \infty. \tag{4.3}$$

Lemma 4.3. [11, Corollary for Theorem 1] *Let $A = (a_{nk})$ be an infinite matrix over the complex field and $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : c)$ if and only if (4.2), (4.3) hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k \quad \text{for each } k \in \mathbb{N} \tag{4.4}$$

also holds.

Let us define the sets $E_1(p)$, $E_2(p)$, $E_3(p)$, $E_4(p)$ and $E_5(p)$, as follows:

$$\begin{aligned}
E_1(p) &:= \left\{ a = (a_k) \in \omega : \sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in N} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \right|^{p_k} < \infty \right\}, \\
E_2(p) &:= \left\{ a = (a_k) \in \omega : \sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n B^{-1} \right|^{p_k'} < \infty \right\}, \\
E_3(p) &:= \left\{ a = (a_k) \in \omega : \sup_{k,n\in\mathbb{N}} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right|^{p_k} < \infty \right\}, \\
E_4(p) &:= \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \text{ is convergent} \right\}, \\
E_5(p) &:= \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j B^{-1} \right|^{p_k'} < \infty \right\}.
\end{aligned}$$

Because of Part (i) can be established in a similar way to the proof of Part (ii), we give the proof only for Part (ii) in Theorems 4.4 and 4.5, below.

Theorem 4.4. *The following statements hold:*

(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $\{\ell(F,p)\}^\alpha = E_1(p)$.

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $\{\ell(F,p)\}^\alpha = E_2(p)$.

Proof. Let us take any $a = (a_n) \in \omega$. By using (3.2), we obtain that

$$a_n x_n = \sum_{k=0}^{n} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n y_k = (Ey)_n \quad \text{for all } n \in \mathbb{N}, \tag{4.5}$$

where $E = (e_{nk})$ is defined by

$$e_{nk} = \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n &, \quad 0 \leq k \leq n, \\ 0 &, \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus, we observe by combining (4.5) with the condition (4.1) of Part (ii) of Lemma 4.1 that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \ell(F, p)$ if and only if $Ey \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. This leads to $\{\ell(F, p)\}^\alpha = E_2(p)$, as asserted. □

Theorem 4.5. *The following statements hold:*

(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $\{\ell(F, p)\}^\beta = E_3(p) \cap E_4(p)$.

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $\{\ell(F, p)\}^\beta = E_4(p) \cap E_5(p)$.

Proof. Take any $a = (a_j) \in \omega$. Then, one can obtain by (3.2) that

$$\sum_{j=0}^{n} a_j x_j = \sum_{j=0}^{n} \left(\sum_{k=0}^{j} \frac{f_{j+1}^2}{f_k f_{k+1}} y_k \right) a_j = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right) y_k = (Dy)_n \tag{4.6}$$

for all $n \in \mathbb{N}$, where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases} \tag{4.7}$$

for all $n, k \in \mathbb{N}$. Thus, we deduce from Lemma 4.3 with (4.6) that $ax = (a_j x_j) \in cs$ whenever $x = (x_j) \in \ell(F, p)$ if and only if $Dy \in c$ whenever $y = (y_k) \in \ell(p)$. Therefore, we derive from (4.3) and (4.4) that

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j B^{-1} \right|^{p_k'} < \infty,$$
$$\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j < \infty.$$

This shows that $\{\ell(F, p)\}^\alpha = E_4(p) \cap E_5(p)$. □

Theorem 4.6. *The following statements hold:*

(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $\{\ell(F, p)\}^\gamma = E_3(p)$.

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $\{\ell(F, p)\}^\gamma = E_5(p)$.

Proof. From Lemma 4.2 and (4.6), we obtain that $ax = (a_j x_j) \in bs$ whenever $x = (x_j) \in \ell(F, p)$ if and only if $Dy \in \ell_\infty$ whenever $y = (y_k) \in \ell(p)$, where $D = (d_{nk})$ is defined by (4.7). Therefore we obtain from (4.2) and (4.3) that $\{\ell(F, p)\}^\gamma = \begin{cases} E_3(p) & , \quad p_k \leq 1, \\ E_5(p) & , \quad p_k > 1 \end{cases}$, as desired. □

CHAPTER 5

MATRIX TRANSFORMATONS ON THE SPACE $\ell(F,p)$

In this section, we characterize some matrix transformations on the space $\ell(F,p)$. Since the cases $0 < p_k \leq 1$ and $1 < p_k \leq H < \infty$ are combined, Theorem 5.1 gives the exact conditions of the general case $0 < p_k \leq H < \infty$. We consider only the case $1 < p_k \leq H < \infty$ and omit the proof of the case $0 < p_k \leq 1$, since it can be proven in a similar way.

Theorem 5.1. *The following statements hold:*

(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : \ell_\infty)$ if and only if

$$\sup_{k,n\in\mathbb{N}} \left| \sum_{i=k}^{\infty} \frac{f_{i+1}^2}{f_k f_{k+1}} a_{ni} \right|^{p_k} < \infty, \tag{5.1}$$

$$\sum_{i=k}^{\infty} \frac{f_{i+1}^2}{f_k f_{k+1}} a_{ni} < \infty. \tag{5.2}$$

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then $A = (a_{nk}) \in (\ell(F,p) : \ell_\infty)$ if and only if (5.2) holds and there exists an integer $B > 1$ such that

$$\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{i=k}^{\infty} \frac{f_{i+1}^2}{f_k f_{k+1}} a_{ni} B^{-1} \right|^{p_k'} < \infty. \tag{5.3}$$

Proof. Let $A \in (\ell(F,p) : \ell_\infty)$ and $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, Ax exists for every $x \in \ell(F,p)$ and this implies that $A_n \in \{\ell(F,p)\}^\beta$ for each fixed $n \in \mathbb{N}$. Therefore, the necessities of (5.2) and (5.3) are immediate.

Conversely, suppose that the conditions (5.2) and (5.3) hold, and take any $x \in \ell(F,p)$. Since $A_n \in \{\ell(F,p)\}^\beta$ for every $n \in \mathbb{N}$, the A-transform of x exists. By using (3.2), we obtain that

$$\sum_{j=0}^{m} a_{nj} x_j = \sum_{j=0}^{m} \sum_{k=0}^{j} \frac{f_{j+1}^2}{f_k f_{k+1}} y_k a_{nj} = \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k \tag{5.4}$$

for all $m, n \in \mathbb{N}$. Taking into account the hypothesis, we drive from (5.4), as $m \to \infty$ that

$$\sum_i a_{ni} x_i = \sum_k \sum_{i=k}^{\infty} \frac{f_{i+1}^2}{f_k f_{k+1}} a_{ni} y_k \quad \text{for all} \quad n \in \mathbb{N}. \tag{5.5}$$

By combining (5.5) and the inequality which holds for any complex numbers a, b and any $B > 0$

$$|ab| \leq B \left(\left| aB^{-1} \right|^{p'} + |b|^p \right),$$

where $p > 1$ and $p^{-1} + p'^{-1} = 1$, we obtain that

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \left| \sum_j a_{nj} x_j \right| &= \sup_{n\in\mathbb{N}} \left| \sum_k \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k \right| \\
&\leq \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k \right| \\
&\leq \sup_{n\in\mathbb{N}} \sum_k B \left(\left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} + |y_k|^{p_k} \right) \\
&= B \left(\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} + \sup_{n\in\mathbb{N}} \sum_k |y_k|^{p_k} \right) < \infty.
\end{aligned}$$

This shows that $Ax \in \ell_\infty$. □

Theorem 5.2. *The following statements hold:*

(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : c)$ if and only if (5.1) and (5.2) hold, and there is a sequence $\alpha = (\alpha_k)$ of scalars such that

$$\lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} = \alpha_k \quad \textit{for all} \quad k \in \mathbb{N}. \tag{5.6}$$

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : c)$ if and only if (5.2), (5.3) and (5.6) hold.

Proof. Let $A \in (\ell(F,p) : c)$ and $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, since the inclusion $c \subset \ell_\infty$ holds, the necessities of (5.2) and (5.3) are immediately obtained from Theorem 5.1.

To prove the necessity of (5.6), consider the sequence $b^{(k)}$ defined by (3.8), which is in the space $\ell(F,p)$ for every fixed $k \in \mathbb{N}$. Since the A-transform of every $x \in \ell(F,p)$ exists and is in c by the hypothesis, we have

$$Ab^{(k)} = \left(\sum_{j=0}^{\infty} a_{ij} b_j^{(k)}\right)_{i=0}^{\infty} = \left(\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{ij}\right)_{i=0}^{\infty} \in c$$

for every fixed $k \in \mathbb{N}$, which shows the necessity (5.6).

Conversely, suppose that the conditions (5.2), (5.3) and (5.6) hold, and take any $x = (x_k)$ in the space $\ell(F,p)$. Then, Ax exists.

We observe for all $m, n \in \mathbb{N}$ that

$$\sum_{k=0}^{m} \left|\sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1}\right|^{p_k'} \leq \sup_{n\in\mathbb{N}} \sum_k \left|\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1}\right|^{p_k'} < \infty$$

which gives the fact by letting $m, n \to \infty$ with (5.3) and (5.6)

$$\lim_{m,n\to\infty} \sum_{k=0}^{m} \left|\sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1}\right|^{p_k'} \leq \sup_{n\in\mathbb{N}} \sum_k \left|\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1}\right|^{p_k'} < \infty.$$

This shows that $\sum_k |\alpha_k B^{-1}|^{p_k'} < \infty$ and $(\alpha_k) \in \{\ell(F,p)\}^\beta$ which implies that the series $\sum_k \alpha_k x_k$ converges for all $x \in \ell(F,p)$.

Now, let us consider the equality obtained from (5.5) with $a_{nj} - \alpha_j$ instead of a_{nj}

$$\sum_j (a_{nj} - \alpha_j) x_j = \sum_k \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} (a_{nj} - \alpha_j) y_k = \sum_k c_{nk} y_k, \tag{5.7}$$

where $C = (c_{nk})$ defined by $c_{nk} = \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} (a_{nj} - \alpha_j)$ for all $k, n \in \mathbb{N}$. From Lemma 4.3, $c_{nk} \to 0$, as $n \to \infty$ for all $k \in \mathbb{N}$. Therefore, we see by (5.7) that $\sum_k (a_{nk} - \alpha_k) x_k \to 0$, as $n \to \infty$. This means that $Ax \in c$ whenever $x \in \ell(F,p)$ and this step completes the proof. □

Corollary 5.3. *(i) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : c_0)$ if and only if (5.1) and (5.2) hold, and (5.6) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

(ii) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : c_0)$ if and only if (5.2) and (5.3) hold, and (5.6) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

Now, we can give the following lemma which is useful for deriving the characterization of the classes of matrix transformations from the space $\ell(F,p)$ to the space λ_A, where $\lambda \in \{\ell_\infty, c, c_0\}$ and $A \in \{\Delta, E^r, C_1, R^t, \sum, F\}$.

Lemma 5.4. *[12, Lemma 5.3] Let λ, μ be any two sequence spaces, A be an infinite matrix and B be a triangle matrix. Then, $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.*

Lemma 5.4 has several consequences depending on the choice of the space μ. Indeed, combining Lemma 5.4 with Theorems 5.1, 5.2 and Corollary 5.3, one can obtain the following results:

Corollary 5.5. *Let $A = (a_{nk})$ be an infinite matrix of complex entries. Then, the following statements hold:*

(i) $E = (e_{nk}) \in (\ell(F,p) : bv_\infty)$ *if and only if (5.1), (5.3) hold with d_{nk} instead of a_{nk}; where $d_{nk} = e_{nk} - e_{n-1,k}$ for all $k, n \in \mathbb{N}$ and bv_∞ denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_\infty$, and was introduced by Başar and Altay [12].*

(ii) $E = (e_{nk}) \in (\ell(F,p) : e^r_\infty)$ *if and only if (5.1), (5.3) hold with d_{nk} instead of a_{nk}, where $d_{nk} = \sum_{j=0}^{n} \binom{n}{j}(1-r)^{n-j}r^j e_{jk}$ for all $k, n \in \mathbb{N}$ and e^r_∞ denotes the space of all sequences $x = (x_k)$ such that $E^r x \in \ell_\infty$, and was introduced by Altay, Başar and Mursaleen [13].*

(iii) $E = (e_{nk}) \in (\ell(F,p) : X_\infty)$ *if and only if (5.1), (5.3) hold with d_{nk} instead of a_{nk}; where $d_{nk} = \sum_{j=0}^{n} e_{jk}/(n+1)$ for all $k, n \in \mathbb{N}$ and X_∞ denotes the space of all sequences $x = (x_k)$ such that $C_1 x \in \ell_\infty$, and was introduced by Ng and Lee [14].*

(iv) $E = (e_{nk}) \in (\ell(F,p) : r^t_\infty)$ *if and only if (5.1), (5.3) hold with d_{nk} instead of a_{nk}; where $d_{nk} = \sum_{j=0}^{n} t_j e_{jk}/T_n$ for all $k, n \in \mathbb{N}$ and r^t_∞ denotes the space of all sequences $x = (x_k)$ such that $R^t x \in \ell_\infty$, and was introduced by Altay and Başar [15].*

(v) $E = (e_{nk}) \in (\ell(F,p) : bs)$ *if and only if (5.1), (5.3) hold with d_{nk} instead of a_{nk}, where $d_{nk} = \sum_{j=0}^{n} e_{jk}$ for all $k, n \in \mathbb{N}$.*

(vi) $E = (e_{nk}) \in (\ell(F,p) : \ell_\infty(\widehat{F}))$ *if and only if (5.1), (5.3) hold with* d_{nk} *instead of* a_{nk}*, where* $d_{nk} = -\frac{f_{n+1}}{f_n} e_{n-1,k} + \frac{f_n}{f_{n+1}} e_{nk}$ *for all* $k, n \in \mathbb{N}$ *and* $\ell_\infty(\widehat{F})$ *denotes the space of all sequences* $x = (x_k)$ *such that* $Fx \in \ell_\infty$*, and was introduced by Kara [16].*

Corollary 5.6. *Let* $A = (a_{nk})$ *be an infinite matrix of complex entries. Then, the following statements hold:*

(i) $E = (e_{nk}) \in (\ell(F,p) : c(\Delta))$ *if and only if (5.1), (5.3) and (5.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = e_{nk} - e_{n+1,k}$ *for all* $k, n \in \mathbb{N}$ *and* $c(\Delta)$ *denotes the space of all sequences* $x = (x_k)$ *such that* $(x_k - x_{k+1}) \in c$*, and was introduced by Kızmaz [17].*

(ii) $E = (e_{nk}) \in (\ell(F,p) : e_c^r)$ *if and only if (5.1), (5.3) and (5.6) hold with* d_{nk} *instead of* a_{nk}*, where* $d_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j e_{jk}$ *for all* $k, n \in \mathbb{N}$ *and* e_c^r *denotes the space of all sequences* $x = (x_k)$ *such that* $E^r x \in c$*, and was introduced by Altay and Başar [18].*

(iii) $E = (e_{nk}) \in (\ell(F,p) : \widetilde{c})$ *if and only if (5.1), (5.3) and (5.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = \sum_{j=0}^{n} e_{jk}/(n+1)$ *for all* $k, n \in \mathbb{N}$ *and* $\widetilde{c}$ *denotes the space of all sequences* $x = (x_k)$ *such that* $C_1 x \in c$*, and was introduced by Şengönül and Başar [19].*

(iv) $E = (e_{nk}) \in (\ell(F,p) : r_c^t)$ *if and only if (5.1), (5.3) and (5.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = \sum_{j=0}^{n} t_j e_{jk}/T_n$ *for all* $k, n \in \mathbb{N}$ *and* r_c^t *denotes the space of all sequences* $x = (x_k)$ *such that* $R^t x \in c$*, and was introduced by Altay and Başar [20].*

(v) $E = (e_{nk}) \in (\ell(F,p) : c(\widehat{F}))$ *if and only if (5.1), (5.3) and (5.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = -\frac{f_{n+1}}{f_n} e_{n-1,k} + \frac{f_n}{f_{n+1}} e_{nk}$ *for all* $k, n \in \mathbb{N}$ *and* $c(\widehat{F})$ *denotes the space of all sequences* $x = (x_k)$ *such that* $Fx \in c$*, and was introduced by Başarır et al. [21].*

(vi) $E = (e_{nk}) \in (\ell(F,p) : cs)$ *if and only if (5.1), (5.3) and (5.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = \sum_{j=0}^{n} e_{jk}$ *for all* $k, n \in \mathbb{N}$*.*

Corollary 5.7. *Let* $A = (a_{nk})$ *be an infinite matrix of complex entries. Then, the following statements hold:*

(i) $E = (e_{nk}) \in (\ell(F,p) : c_0(\Delta))$ *if and only if (5.1), (5.3) hold and (5.6) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = e_{nk} - e_{n+1,k}$ *for all* $k, n \in \mathbb{N}$ *and* $c_0(\Delta)$ *denotes the space of all sequences* $x = (x_k)$ *such that* $(x_k - x_{k+1}) \in c_0$*, and was introduced by Kızmaz [17].*

(ii) $E = (e_{nk}) \in (\ell(F,p) : e_0^r)$ *if and only if (5.1), (5.3) hold and (5.6) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*, where* $d_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j e_{jk}$ *for all* $k, n \in \mathbb{N}$ *and* e_0^r *denotes the space of all sequences* $x = (x_k)$ *such that* $E^r x \in c_0$*, and was introduced by Altay and Başar [18].*

(iii) $E = (e_{nk}) \in (\ell(F,p) : \widetilde{c}_0)$ *if and only if (5.1), (5.3) hold and (5.6) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = \sum_{j=0}^{n} e_{jk}/(n+1)$ *for all* $k, n \in \mathbb{N}$ *and* $\widetilde{c}_0$ *denotes the space of all sequences* $x = (x_k)$ *such that* $C_1 x \in c_0$*, and was introduced by Şengönül and Başar [19].*

(iv) $E = (e_{nk}) \in (\ell(F,p) : r_0^t)$ *if and only if (5.1), (5.3) hold and (5.6) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = \sum_{j=0}^{n} t_j e_{jk}/T_n$ *for all* $k, n \in \mathbb{N}$ *and* r_0^t *denotes the space of all sequences* $x = (x_k)$ *such that* $R^t x \in c_0$*, and was introduced by Altay and Başar [20].*

(v) $E = (e_{nk}) \in (\ell(F,p) : c_0(\widehat{F}))$ *if and only if (5.1), (5.3) hold and (5.6) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = -\frac{f_{n+1}}{f_n} e_{n-1,k} + \frac{f_n}{f_{n+1}} e_{nk}$ *for all* $k, n \in \mathbb{N}$ *and* $c_0(\widehat{F})$ *denotes the space of all sequences* $x = (x_k)$ *such that* $Fx \in c_0$*, and was introduced by Başarır et al. [21].*

(vi) $E = (e_{nk}) \in (\ell(F,p) : c_0 s)$ *if and only if (5.1), (5.3) hold and (5.6) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk} = \sum_{j=0}^{n} e_{jk}$ *for all* $k, n \in \mathbb{N}$ *and* $c_0 s$ *denotes the space of all sequences* $x = (x_k)$ *such that* $\sum_k x_k = 0$*.*

CHAPTER 6

CONCLUSION

Let $0 < r < 1$, $q = (q_k)$ be a sequence of non-negative reals with $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k$ for all $n \in \mathbb{N}$, $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ be the convergent sequences. Suppose that the sequences $u = (u_k)$ and $v = (v_k)$ consist of non-zero entries; $u, s \in \mathbb{R}$, and $\lambda = (\lambda_n)$ be the strictly increasing sequence of positive real numbers tending to infinity with $\lambda_{n+1} \geq 2\lambda_n$.

Let us define the summation matrix $S = (s_{nk})$, the matrix $A^r = (a^r_{nk})$, the generalized difference matrix $B(u,s) = \{b_{nk}(u,s)\}$, the matrix $A^u = (a^u_{nk})$, the double sequential band matrix $B(\widetilde{r},\widetilde{s}) = \{b_{nk}(r_k, s_k)\}$, the Riesz matrix $R^q = (r^q_{nk})$ with respect to the sequence $q = (q_k)$, the factorable matrix $G(u,v) = (g_{nk})$, the matrix $\widetilde{A} = \{a_{nk}(\lambda)\}$ and the Nörlund matrix $N^q = (a^q_{nk})$ with respect to the sequence $q = (q_k)$ by

$$s_{nk} := \begin{cases} 1 & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases} \qquad a^u_{nk} := \begin{cases} (-1)^{n-k} u_k & , \quad n-1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n-1 \ \text{ or } \ k > n \end{cases}$$

$$b_{nk}(u,s) := \begin{cases} u & , \quad k = n, \\ s & , \quad k = n-1, \\ 0 & , \quad 0 \leq k < n-1 \ \text{ or } \ k > n, \end{cases} \qquad a^r_{nk} := \begin{cases} \frac{1+r^k}{n+1} u_k & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

$$b_{nk}(r_k, s_k) = \begin{cases} r_k & , \quad k = n, \\ s_k & , \quad k = n-1, \\ 0 & , \quad 0 \leq k < n-1 \ \text{ or } \ k > n, \end{cases} \qquad r^q_{nk} = \begin{cases} \frac{q_k}{Q_n} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

$$g_{nk} := \begin{cases} u_n v_k & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases} \qquad a_{nk}(\lambda) := \begin{cases} \frac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_n - \lambda_{n-1}} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases}$$

$$a_{nk}^{q} = \begin{cases} \frac{q_{n-k}}{Q_n} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

For concerning literature about the domain of the infinite matrix A in the sequence space $\ell(p)$, the following table may be useful:

A	$[\ell(p)]_A$	refer to:
R^q	$r^q(p)$	[15]
S	$\overline{\ell(p)}$	[22]
A^r	$a^r(u,p)$	[23]
$B(u,s)$	$\widehat{\ell}(p)$	[24]
A^u	$bv(u,p)$	[25]
$B(\widetilde{r},\widetilde{s})$	$\ell(\widetilde{B},p)$	[26]
$G(u,v)$	$\ell(u,v;p)$	[27]
$\widetilde{A}$	$\ell(\widetilde{A},p)$	[28]
N^q	$N^q(p)$	[29]

Table 1: The domains of some triangle matrices in the space $\ell(p)$.

In first, the domains $\ell_p(\widehat{F})$ and $c_0(\widehat{F})$, $c(\widehat{F})$ of the double band matrix F defined by a sequence of Fibonacci numbers in the sequence spaces ℓ_p and c, c_0 have recently been studied by Kara [16] and Başarır et al. [21], respectively. It is natural to expect for extending the normed space $\ell_p(\widehat{F})$ to the paranormed space $\ell(F,p)$ as was the space ℓ_p extended to the space $\ell(p)$ which is the main subject of the present paper. As a continuation of Kara [16], we have introduced the space $\ell(F,p)$ and studied its algebraic and topological properties. We should record that the geometric properties of the space $\ell(F,p)$ can be investigated in a separate paper which will be the main subject of our next work.

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