WELL-POSEDNESS OF TELEGRAPH DIFFERENTIAL AND DIFFERENCE EQUATIONS

by

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APPROVAL PAGE

This is to certify that I have read this thesis written by Mahmut MODANLI and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy in Mathematics.

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ABSTRACT

Cauchy and nonlocal boundary value problems for telegraph equations in a Hilbert space H with the self-adjoint positive definite operator A are analyzed. Stability estimates for the solution of these problems are formed. A first and a second order of accuracy difference schemes for the approximate solution of these problems are presented. Stability estimates for the solutions of these difference schemes are installed. In implementations, two mixed problems for telegraph partial differential equations are investigated. The methods are tested by numerical examples by MATLAB programming.

Keywords: Telegraph Equations, Cauchy Problem, Hilbert Space, Difference Schemes, Stability, Initial Boundary Value Problem.

TELEGRAF DİFERENSİYEL VE FARK DENKLEMLERİNİN İYİ TANIMLILIĞI

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ÖZ

H Hilbert uzayında bir self-adjoint pozitif tanımlı A operatörü için telegraf diferensiyel denklemlerde Cauchy ve lokal olmayan bir sınır değer problemi çalışıldı. Bu denklemin çözümü için kararlılık kestirimleri bulundu. Bu problemler için birinci ve ikinci dereceden doğruluk fark şemaları verildi. Bu doğruluk fark denklemleri için kararlılık kestirimleri inşa edildi. Uygulamalarda telegraf kısmi diferansiyel denklemleri için karışık problem incelendi. Metotlar, MATLAB programı yardımıyla nümerik örneklerle gösterildi.

Anahtar Kelimeler: Telegraf Denklemler, Hilbert Uzayı, Fark Şeması, Kararlılık, Başlangıç Sınır Değer Problemleri.

To my family

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOL/ABBREVIATION

Ω	Open set
Ω^{+}	Positive open set
L	Laplace transform
F	Fourier transform
W_2^1, W_2^2	Sobolev spaces
A	Operator
M	Constant coefficient
$L_2(\overline{\Omega})$	Hilbert space

CHAPTER 1

INTRODUCTION

It is well-known that hyperbolic partial differential equations arise in many branches of science and engineering *e.g.*, electromagnetic, electrodynamics, thermodynamics, hydrodynamics, elasticity, fluid dynamics, wave propagation, materials science. In numerical methods for solving these equations, the problem of stability has received a great deal of importance and attention. Specially, a suitable model for analyzing the stability is provided by a proper unconditionally absolutely stable difference scheme with an unbounded operator (Ashyralyev and Yildirim, 2013). The role played by positivity property of differential and difference operators in Hilbert and Banach spaces in the study of various properties of boundary value problems for partial differential equations, of stability of difference schemes for partial differential equations, and of summation of Fourier series is well-known (see, (Fattorini, 1985); (Goldstein, 1985); (Ashyralyev and Koksal, 2009); (Ashyralyev and Sobolevskii, 2004); (Ashyralyev and Sobolevskii, 1994); (la Sen, 2013); (la Sen, 2011); (Achour and Belacel, 2014); (Ghorbanalizadeh and Sawano, 2014)). The method of operators as a instrument for the examination of the solution of local and nonlocal problems for hyperbolic differential equations in Hilbert and Banach spaces, has been systematically developed by several authors (see, e.g., (Fattorini, 1985); (Goldstein, 1985); (Ashyralyev and Sobolevskii, 2004); (Krein, 1971); (Vasilev et al., 1990)).

The telegraph hyperbolic partial differential equation is significant for modeling a few suitable relevant problems such as signal analysis, wave propagation, random walk theory (Jordan and Puri, 1999); (Weston and He, 1993); (Banasiak and Mika, 1998). To deal with the equation, various mathematical methods have been proposed for obtaining exact and approximate analytic solutions. For instance, Dehghan and Shokri proposed a new numerical scheme based on radial based function method (Kansa's method) (Dehghan and Shokri, 2008). Gao and Chi developed a numerical algorithm for the solution of nonlinear telegraph equations (Gao and Chi, 2007). Biazar applied the variational iteration method to obtain the approximate solutions to the telegraph equations (Biazar et al., 2009). Saadatmandi and Dehghan used the Chebyshev Tau method for numerically solving the telegraph equation (Saadatmandi and Dehghan, 2010). Twizell used the explicit difference method for the wave equation with extended stability range (Twizell, 1979). Finally, Ashyralyev and Akat applied the difference method for the approximate solution of stochastic hyperbolic and stochastic telegraph equations (Ashyralyev and Akat, 2011); (Ashyralyev and Akat, 2013); (Ashyralyev and Akat, 2012). Koksal computed numerical solutions of the telegraph equations arising in transmission lines (Koksal, 2011).

In this thesis, we consider a Cauchy problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) + \beta u(t) = f(t) \ (0 \leq t \leq T), \\ \\ u(0) = \varphi, u'(0) = \psi \end{cases}$$

and a nonlocal boundary value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) + \beta u(t) = f(t) \ (0 \leq t \leq T), \\ \\ u(0) = \lambda u(T) + \varphi, u'(0) = \mu u'(T) + \psi \end{cases}$$

for a telegraph equation in a Hilbert space H with a self-adjoint positive definite operator A and $A \geq \delta I$. Here $\delta > 0$, $\alpha > 0$ and

$$\beta + \delta \geq \frac{\alpha^2}{4}.$$

In this thesis, we research the stability of solutions of Cauchy and nonlocal boundary value problems for telegraph equations.

A problem is called well-posed if for each set of data there exists exactly one solution and dependence of the solution on the data continuous (Jiwari et al., 2012). Our goal in this work is to show that various types of the boundary value problems for equations of telegraph type are stable (Jiwari et al., 2012). Also, we will consider the stability of difference schemes for solving these problems for telegraph equations.

It is known that local and nonlocal boundary value problems for telegraph equations can be solved analytically by Fourier series, Fourier transform and Laplace transform methods. Now, let us illustrate these three different analytical methods by examples.

Example 1.1. *"Obtain the Fourier series solution of the initial-boundary value problem for a telegraph equation*

$$\begin{cases} u_{tt}(t,x)+u_t(t,x)-u_{xx}(t,x)+u(t,x)=2\exp(-t)\sin x, \\ t>0, 0<x<\pi, \\ u(0,x)=\sin x, u_t(0,x)=-\sin x, 0\le x\le \pi, \\ u(t,0)=u(t,\pi)=0, t\ge 0. \end{cases} \tag{1.1}$$

Solution. In order to solve problem (1.1), we will use following transformation and to get

$$u\left(t,x\right)=v\left(t,x\right)+w\left(t,x\right), \tag{1.2}$$

$v\left(t,x\right)$ is the solution of the initial-boundary value problem

$$\begin{cases} \frac{\partial^2 v}{\partial t^2}+\frac{\partial v}{\partial t}+v=\frac{\partial^2 v}{\partial x^2}, t>0, 0<x<\pi, \\ v(0,x)=\sin x, v_t(0,x)=-\sin x, 0\le x\le \pi, \\ v(t,0)=v(t,\pi)=0, t\ge 0, \end{cases} \tag{1.3}$$

and $w\left(t,x\right)$ is the solution of the initial-boundary value problem

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}+\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}+w=2\exp(-t)\sin x, t>0, 0<x<\pi, \\ w(0,x)=0, w_t(0,x)=0, 0\le x\le \pi, \\ w(t,0)=w(t,\pi)=0, t\ge 0. \end{cases} \tag{1.4}$$

First, let us obtain the solution of (1.3) by the method of separation of variables. To do this a solution of the form

$$v(t,x) = T(t)X(x) \neq 0$$

is suggested. Taking the partial derivatives and substituting the result in (1.3), we obtain

$$\frac{T^{''}(t) + T^{'}(t) + T(t)}{T(t)} - \frac{X^{''}(x)}{X(x)} = 0,$$

or

$$\frac{T^{''}(t) + T^{'}(t) + T(t)}{T(t)} = \frac{X^{''}(x)}{X(x)} = \lambda. \tag{1.5}$$

The boundary conditions presented in (1.3), require $X(0) = X(\pi) = 0$. Hence, from (1.5) we have the boundary value problem

$$X^{''}(x) = \lambda X(x), 0 < x < \pi, X(0) = X(\pi) = 0. \tag{1.6}$$

If $\lambda \geq 0$, then boundary value problem (1.6) has only trivial solution $X(x) = 0$. For $\lambda < 0$, the nontrivial solutions of the initial value problem (1.6) are

$$X_k(x) = \sin(kx), k = 1, 2, ..., \lambda_k = -k^2.$$

The other ordinary differential equation presented in (1.5) is

$$T^{''}(t) + T^{'}(t) + T(t) = \lambda T(t), 0 < t < T$$

and applying formula $\lambda = -k^2$, we get

$$T_k^{''}(t) + T_k^{'}(t) + \left(k^2 + 1\right) T_k(t) = 0. \tag{1.7}$$

The auxiliary equation is

$$m^2 + m + \left(k^2 + 1\right) = 0.$$

We have two roots

$$m_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{4k^2+3}}{2}.$$

Then, the general solution of equation (1.7) is

$$T_k(t) = \exp(-\frac{t}{2}) \left[A_k \cos\left(\frac{\sqrt{4k^2+3}}{2}t\right) + B_k \sin\left(\frac{\sqrt{4k^2+3}}{2}t\right)\right].$$

Therefore, using the superposition principle, we get the formula

$$v(t,x) = \sum_{k=1}^{\infty} T_k(t)\sin kx = \sum_{k=1}^{\infty} \exp\left(-\frac{t}{2}\right)$$

$$\times \left[A_k \cos\left(\frac{\sqrt{4k^2+3}}{2}t\right) + B_k \sin\left(\frac{\sqrt{4k^2+3}}{2}t\right)\right] \sin kx$$

for the general solution of problem (1.3). Applying initial condition $v(0,x) = \sin x$, we get

$$\sum_{k=1}^{\infty} A_k \sin kx = \sin x.$$

From that it follows that

$$A_1 = 1, A_k = 0, k \geq 2. \tag{1.8}$$

Taking the derivative, we get

$$v_t(t,x) = -\frac{1}{2}\exp\left(-\frac{t}{2}\right)\sum_{k=1}^{\infty}\left[A_k \cos\left(\frac{\sqrt{4k^2+3}}{2}t\right)\right.$$

$$\left. + B_k \sin\left(\frac{\sqrt{4k^2+3}}{2}t\right)\right] \sin kx$$

$$+ \exp\left(-\frac{t}{2}\right)\sum_{k=1}^{\infty}\left[-A_k\frac{\sqrt{4k^2+3}}{2}\sin\left(\frac{\sqrt{4k^2+3}}{2}t\right)\right.$$

$$\left. + B_k\frac{\sqrt{4k^2+3}}{2}\cos\left(\frac{\sqrt{4k^2+3}}{2}t\right)\right] \sin kx.$$

Applying initial condition $v_t(0,x) = -\sin x$, we get

$$-\frac{1}{2}\sum_{k=1}^{\infty} A_k \sin kx + \sum_{k=1}^{\infty} B_k \frac{\sqrt{4k^2+3}}{2}\sin kx = -\sin x.$$

From that it follows that

$$-\frac{1}{2}A_1 + B_1\frac{\sqrt{7}}{2} = -1, -\frac{1}{2}A_k + B_k = 0, k \geq 2.$$

Then, using (1.8), we get

$$B_k = 0, k \geq 2, B_1 = -\frac{1}{\sqrt{7}}.$$

Thus,

$$v\left(t,x\right) = \exp\left(-\frac{t}{2}\right)\left[\cos\frac{\sqrt{7}}{2}t - \frac{1}{\sqrt{7}}\sin\frac{\sqrt{7}}{2}t\right]\sin x \tag{1.9}$$

is the solution of problem (1.3).

Second, we will obtain the solution of problem (1.4). We seek a solution of problem (1.4) by the Fourier series method

$$w\,(t,x)=\sum_{k=1}^{\infty}A_k(t)\sin kx.$$

Then, taking derivatives and using equation in (1.4) and initial conditions, we can write

$$\sum_{k=1}^{\infty}A_k^{''}(t)\sin kx+\sum_{k=1}^{\infty}A_k^{'}(t)\sin kx$$

$$+\sum_{k=1}^{\infty}A_k(t)k^2\sin kx+\sum_{k=1}^{\infty}A_k(t)\sin kx=2\exp(-t)\sin x,$$

$$\sum_{k=1}^{\infty}A_k(0)\sin kx=0,\sum_{k=1}^{\infty}A_k'(0)\sin kx=0.$$

From that it follows that

$$A_1^{''}(t)+A_1^{'}(t)+2A_1(t)=2\exp(-t),$$

$$A_k^{''}(t)+A_k^{'}(t)+(k^2+1)A_k(t)=0,k\geq 2,$$

$$A_k(0)=0,A_k^{'}(0)=0.$$

It is easy to see that $A_k(t)=0,k\geq 2$. Therefore, we will obtain $A_1(t)$ as the solution of the following initial value problem

$$A_1^{''}(t)+A_1^{'}(t)+2A_1(t)=2\exp(-t),A_1(0)=0,A_1^{'}(0)=0.$$

We have that

$$A_1(t)=A_1^c(t)+A_1^p(t),$$

where

$$A_1^c(t)=\exp(-\frac{t}{2})\left[c_1\cos\frac{\sqrt{7}}{2}t+c_2\sin\frac{\sqrt{7}}{2}t\right],$$

$$A_1^p(t)=A\exp(-t).$$

It is easy to see that $A=1$. Therefore,

$$A_1^p(t)=\exp(-t)$$

and

$$A_1(t) = \exp(-\frac{t}{2})\left[c_1 \cos\frac{\sqrt{7}}{2}t + c_2 \sin\frac{\sqrt{7}}{2}t\right] + \exp(-t).$$

Using initial conditions $A_1(0) = 0, A_1^{'}(0) = 0$, we get

$$A_1(0) = c_1 + 1 = 0,$$

$$A_1^{'}(0) = \frac{1}{2} - 1 + c_2\frac{\sqrt{7}}{2} = 0.$$

From that it follows that $c_1 = -1$ and $c_2 = \frac{1}{\sqrt{7}}$. Then,

$$A_1(t) = \exp(-\frac{t}{2})\left[-\cos\frac{\sqrt{7}}{2}t + \frac{1}{\sqrt{7}}\sin\frac{\sqrt{7}}{2}t\right] + \exp(-t)$$

and

$$w(t,x) = A_1(t)\sin x \tag{1.10}$$

$$= -\exp(-\frac{t}{2})\cos\frac{\sqrt{7}}{2}t\sin x + \exp(-\frac{t}{2})\sin\frac{\sqrt{7}}{2}t\frac{1}{\sqrt{7}}\sin x + \exp(-t)\sin x$$

is the solution of problem (1.4). Applying formulas (1.9) and (1.10), we can write

$$u(t,x) = v(t,x) + w(t,x) = \exp(-t)\sin x.$$

Note that using similar procedure one can obtain the solution of the following initial-boundary value problem for a multidimensional telegraph equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}, 0 < t < T, \\ u(0,x) = \phi(x), u_t(0,x) = \psi(x), x \in \overline{\Omega}, \\ \frac{\partial u}{\partial n}(t,x) = 0, \frac{\partial u(t,x)}{\partial x_r} = 0, x \in S, \end{cases}$$

where $\alpha_r > \alpha > 0$ and $f(t,x)$, $(t \in [0,T], x \in \overline{\Omega})$, $\phi(x)$, $\psi(x)$, $(x \in \overline{\Omega})$ are given smooth functions. Here Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1,\ 1 \le k \le n)$ with the boundary S,

$$\overline{\Omega} = \Omega \cup S.$$

Here $\frac{\partial}{\partial n}$ indicates differentiation in the direction of the exterior normal to S. However, the method of separation of variables described in solving (1.3) can be used only in the case when (1.1) has constant coefficients."

Example 1.2. *"Solve the mixed problem*

$$\begin{cases} u_{tt}(t,x)+u_t(t,x)-u_{xx}(t,x)+2u(t,x)=\exp(-t-x), \\ t>0, x>0, \\ u(0,x)=\exp(-x), u_t(0,x)=-\exp(-x), x\geq 0, \\ u(t,0)=\exp(-t), u_x(t,0)=-\exp(-t), t\geq 0 \end{cases} \tag{1.11}$$

using the Laplace transform method.

Solution. We denote

$$u(s,x)=L\left\{u(t,x)\right\}.$$

Then, taking Laplace transform of both sides of the differential equation (1.11) and given conditions, we get

$$s^2L\left\{u(t,x)\right\}-su(0,x)-u_t(0,x)+sL\left\{u(t,x)\right\}-u(0,x)$$

$$=\frac{\partial^2}{\partial x^2}L\left\{u(t,x)\right\}-2L\left\{u(t,x)\right\}+L\{\exp(-t-x)\},$$

$$L\left\{u(t,0)\right\}=\frac{1}{s+1}, L\left\{u_x(t,0)\right\}=-\frac{1}{s+1}$$

or

$$s^2u(s,x)-s\exp(-x)+\exp(-x)+su(s,x)-\exp(-x)+2u(s,x)$$

$$=u_{xx}(s,x)+\frac{\exp(-x)}{s+1}, x>0, u(s,0)=\frac{1}{s+1}, u_x(s,0)=-\frac{1}{s+1}.$$

Therefore,

$$-u_{xx}(s,x)+\left(s^2+s+2\right)u(s,x)=\frac{s^2+s+1}{s+1}\exp(-x), x>0,$$

$$u(s,0)=\frac{1}{s+1}, u_x(s,0)=-\frac{1}{s+1}.$$

In order to solve the problem, we need to separate $u(t,x)$ into two parts

$$u(s,x)=u^c(s,x)+u^p(s,x),$$

where $u^c(s,x)$ is the solution of homogeneous equation

$$-u_{xx}(s,x)+\left(s^2+s+2\right)u(s,x)=0,$$

and $u^p(s,x)=A(s)\exp(-x)$ is the solution of nonhomogeneous equation

$$-u_{xx}(s,x)+\left(s^2+s+2\right)u(s,x)=\frac{s^2+s+1}{s+1}\exp(-x).$$

Now, we will obtain $u^c(s,x)$. The auxiliary equation is

$$-m^2+(s^2+s+2)=0.$$

We have two roots

$$m_{1,2}=\pm\sqrt{s^2+s+2}.$$

Then, the general solution of homogeneous equation is

$$u^c(s,x)=c_1\exp(x\sqrt{s^2+s+2})+c_2\exp(-x\sqrt{s^2+s+2}). \qquad (1.12)$$

It is easy to see that

$$A(s)=\frac{1}{s+1}. \qquad (1.13)$$

Using formulas (1.12) and (1.13), we obtain

$$u\left(s,x\right)=c_1\exp(x\sqrt{s^2+s+2})+c_2\exp(-x\sqrt{s^2+s+2})+\frac{1}{s+1}\exp(-x).$$

Using initial conditions $u(s,0)=\frac{1}{s+1}, u_x(s,0)=-\frac{1}{s+1}$, we get

$$c_1=0, c_2=0.$$

Then,

$$u\left(s,x\right)=\frac{1}{s+1}\exp(-x). \qquad (1.14)$$

Applying inverse Laplace transform in formula (1.14), we get the exact solution of the problem (1.11)"

$$u\left(t,x\right)=\exp(-x-t).$$

Note that using same procedure one can obtain the solution of the following initial boundary value problem for the multidimensional telegraph equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} \alpha_r \frac{\partial^2 u(t,x)}{\partial x_r^2} = f(t,x), \\ x = (x_1, ..., x_n) \in \overline{\Omega}^+, 0 \le t \le T, \\ u(0,x) = \phi(x), u_t(0,x) = \psi(x), x \in \overline{\Omega}^+, \\ u(t,x) = 0, x \in S^+, \end{cases}$$

where $\alpha_r > \alpha > 0$ and $f(t,x)$, $,\phi(x), \psi(x)$, are given smooth functions. $(t \in [0,T], x \in \overline{\Omega}^+)$. Here Ω^+ is the open set in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < \infty, 1 \le k \le n$ with the boundary S^+,

$$\overline{\Omega}^+ = \Omega^+ \cup S^+.$$

However, as the method of separation of variables, Laplace transform method can be used only in the case when the differential equation has constant coefficients.

Example 1.3. *Obtain the Fourier transform solution of the following Cauchy problem*

$$\begin{cases} u_{tt}(t,x) + u_t(t,x) - u_{xx}(t,x) + u(t,x) = (3 - 4x^2)\exp(-t - x^2), \\ t > 0, x \in (-\infty, \infty), \\ u(0,x) = \exp(-x^2), u_t(0,x) = -\exp(-x^2), x \in (-\infty, \infty). \end{cases} \tag{1.15}$$

Solution. Let us denote

$$u(t,s) = F\{u(t,x)\}.$$

Then, taking Fourier transform of both sides of the differential equation (1.15) and given conditions, we get

$$u_{tt}(t,s) + u_t(t,s) + \left(s^2 + 1\right) u(t,s) = \exp(-t) F\left\{\left(-4x^2 + 2\right) \exp\left(-x^2\right)\right\}$$

$$+\exp(-t)\,\mathcal{F}\left\{\exp\left(-x^2\right)\right\},t>0,$$

$$u(0,s)=\mathcal{F}\left\{\exp(-x^2)\right\},u_t(0,s)=-\mathcal{F}\left\{\exp(-x^2)\right\}.$$

Then, in order to solve the problem we need to separate $u(t,x)$ into two parts

$$u(t,s)=u^c(t,s)+u^p(t,s),$$

where $u^c(t,s)$ is the solution of homogeneous equation

$$u_{tt}(t,s)+u_t(t,s)+\left(s^2+1\right)u(t,s)=0,$$

and $u^p(t,s)=A(s)\exp(-t)$ is the solution of nonhomogeneous equation

$$u_{tt}(t,s)+u_t(t,s)+\left(s^2+1\right)u(t,s)=\exp(-t)\left(s^2+1\right)\mathcal{F}\left\{\exp\left(-x^2\right)\right\}.$$

Now, we will obtain $u^c(t,s)$. The auxiliary equation is

$$m^2+m+\left(s^2+1\right)=0.$$

We have two roots

$$m_{1,2}=\frac{-1\mp i\sqrt{4s^2+3}}{2}.$$

Then, we obtain

$$u^c(t,s)=\exp\left(-\frac{t}{2}\right)\left[c_1\cos\frac{\sqrt{4s^2+3}}{2}t+c_2\sin\frac{\sqrt{4s^2+3}}{2}t\right]. \tag{1.16}$$

It is easy to see that

$$A(s)=\mathcal{F}\left\{\exp\left(-x^2\right)\right\}$$

and

$$u^p(t,s)=\mathcal{F}\left\{\exp\left(-x^2\right)\right\}\exp(-t). \tag{1.17}$$

Using formulas (1.16) and (1.17), we get

$$u(t,s)=\exp\left(-\frac{t}{2}\right)\left[c_1\cos\frac{\sqrt{4s^2+3}}{2}t+c_2\sin\frac{\sqrt{4s^2+3}}{2}t\right]$$

$$+\exp(-t)\,\mathcal{F}\left\{\exp\left(-x^2\right)\right\}.$$

Applying initial conditions $u(0,s)=\mathcal{F}\{\exp(-x^2)\},u_t(0,s)=-\mathcal{F}\{\exp(-x^2)\}$, we get

$$c_1+\mathcal{F}\left\{\exp\left(-x^2\right)\right\}=\mathcal{F}\left\{\exp\left(-x^2\right)\right\},$$

$$-F\left\{\exp\left(-x^2\right)\right\}+c_2\frac{\sqrt{4s^2+3}}{2}=-F\left\{\exp\left(-x^2\right)\right\}.$$

From that it follows that

$$c_1=0,c_2=0.$$

Then,

$$u\left(t,s\right)=\exp\left(-t\right)F\left\{\exp\left(-x^2\right)\right\}$$

and

$$u\left(t,x\right)=F^{-1}\left\{u\left(t,s\right)\right\}=\exp\left(-t\right)\exp\left(-x^2\right)=\exp\left(-t-x^2\right)$$

is the solution of problem (1.15).

Note that using the same manner one obtains the solution of the following initial value problem for the $2m-th$ order multidimensional telegraph equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}-\sum\limits_{|r|=2m}\alpha_r\frac{\partial^{|r|}u}{\partial x_1^{r_1}...\partial x_n^{r_n}}=f(t,x), \\ 0\le t\le T, x,r\in\mathbb{R}^n, |r|=r_1+...+r_n, \\ u(0,x)=\phi(x), u_t(0,x)=\psi(x), x\in\mathbb{R}^n, \end{cases}$$

where $\alpha_r\ge\alpha\ge 0$, $f(t,x)$, $\phi(x)$, $\psi(x)$ are given smooth functions. $(t\in[0,T]$, $x\in\mathbb{R}^n)$.

However, all analytical methods defined above, that is to say the Fourier series method, the Laplace transform method and the Fourier transform method can be used only when the differential equation has constant coefficients. It is well-known that the most general method for solving PDEs with dependent coefficients in t and in the space variables is difference method, which is basically achieved by digital computers and known to be numerical method. However the stability of different difference schemes used in numerical methods need to be proved or justified theoretically.

Let us give a brief description of the contents of the various sections. It consists of five chapters.

First Chapter is the introduction.

Second Chapter analyzes the Cauchy problem for telegraph differential equations in a Hilbert space with a self-adjoint operator. Stability estimates for the solution of this problem are formed. The first and second order of accuracy difference schemes for the approximate solution of the Cauchy problem are constructed. Stability estimates for the solution of these difference schemes are established. In applications, two mixed problems for telegraph partial differential equations are formed. The methods are tested by numerical examples.

Third Chapter investigated the nonlocal boundary value problem for telegraph differential equations in a Hilbert space with a self-adjoint operator. Stability estimates for the solution of this problem are established. The first and second order of accuracy difference schemes for the approximate solution of this problem are given. Stability estimates for the solution of these difference schemes are established. In applications, two mixed problems for telegraph partial differential equations are researched. The methods are showed by numerical examples.

Fourth Chapter is conclusion.

Fifth Chapter is the algorithm and programming for the given applications.

CHAPTER 2

THE CAUCHY PROBLEM FOR A TELEGRAPH DIFFERENTIAL EQUATION

We search a Cauchy problem for a telegraph equation

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) + \beta u(t) = f(t) \ (0 \leq t \leq T), \\ \\ u(0) = \varphi, u'(0) = \psi \end{cases} \tag{2.1}$$

in a Hilbert space H with a self-adjoint positive definite operator A and $A \geq \delta I$. Here $\delta > 0$, $\alpha > 0$ and

$$\beta + \delta \geq \frac{\alpha^2}{4}. \tag{2.2}$$

"A function $u(t)$ is called a solution of the problem (2.1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the interval $[0, T]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0, T]$ and the function $Au(t)$ is continuous on the segment $[0, T]$.

(iii) $u(t)$ satisfies the equation and initial conditions (2.1)."

"Let $\{c(t), t \geq 0\}$ be a strongly continuous cosine operator-function defined by the formula

$$c(t) = \frac{e^{itB^{1/2}} + e^{-itB^{1/2}}}{2}.$$

Then, from the definition of the sine operator-function $s\,(t)$

$$s(t)u = \int_0^t c(s)u\ ds$$

it follows that

$$s\,(t) = B^{-1/2}\frac{e^{itB^{1/2}} - e^{-itB^{1/2}}}{2i}. \quad ''$$

Here $B = A + \left(\beta - \frac{\alpha^2}{4}\right) I$. It is easy to check under the assumption (2.2) that the problem (2.1) for a telegraph equation has a unique mild solution given by the formula

$$u(t) = e^{-\frac{\alpha}{2}t}c\,(t)\,\varphi + \frac{\alpha}{2}e^{-\frac{\alpha}{2}t}s(t)\varphi + e^{-\frac{\alpha}{2}t}s(t)\psi + \int_0^t e^{-\frac{\alpha}{2}(t-z)}s(t-z)f(z)dz. \qquad (2.3)$$

It is clear that (2.1) can be rewritten as the equivalent initial-value problem for a system of first-order differential equations (see (Ashyralyev and Sobolevskii, 2005))

$$\begin{cases} u'(t) + \frac{\alpha}{2}u(t) + iB^{\frac{1}{2}}u(t) = z(t), (0 \le t \le T), u(0) = u_0, u'(0) = u_0', \\ \\ z'(t) + \frac{\alpha}{2}z(t) - iB^{\frac{1}{2}}z(t) = f(t). \end{cases} \qquad (2.4)$$

Integrating these, now we get

$$\begin{cases} u(t) = e^{-(\frac{\alpha}{2}+iB^{\frac{1}{2}})t}u(0) + \int_0^t e^{-(\frac{\alpha}{2}+iB^{\frac{1}{2}})(t-s)}z(s)ds, \\ \\ z(t) = e^{-(\frac{\alpha}{2}-iB^{\frac{1}{2}})t}z(0) + \int_0^t e^{-(\frac{\alpha}{2}-iB^{\frac{1}{2}})(t-s)}f(s)ds. \end{cases}$$

Applying the initial condition $z(0) = u'(0) + (\frac{\alpha}{2} + iB^{\frac{1}{2}})u(0)$, we get

$$u(t) = e^{-(\frac{\alpha}{2}+iB^{\frac{1}{2}})t}u(0) + \int_0^t e^{-(\frac{\alpha}{2}+iB^{\frac{1}{2}})(t-s)} \int_0^s e^{-(\frac{\alpha}{2}-iB^{\frac{1}{2}})(s-p)}f(p)dpds$$

$$+ \int_0^t e^{-(\frac{\alpha}{2}+iB^{\frac{1}{2}})(t-s)}e^{-(\frac{\alpha}{2}-iB^{\frac{1}{2}})s}ds\left(u'(0) + \left(\frac{\alpha}{2} + iB^{\frac{1}{2}}\right)u(0)\right).$$

By an interchange of the order of integration, we can write

$$u(t) = \left[e^{-(\frac{\alpha}{2}+iB^{\frac{1}{2}})t} + \left(\frac{\alpha}{2} + iB^{\frac{1}{2}}\right)\int_0^t e^{-(\frac{\alpha}{2}+iB^{\frac{1}{2}})(t-s)}e^{-(\frac{\alpha}{2}-iB^{\frac{1}{2}})s}ds\right]u(0)$$

$$+\int_0^t e^{-(\frac{\alpha}{2}+iB^{\frac{1}{2}})(t-s)} e^{-(\frac{\alpha}{2}-iB^{\frac{1}{2}})s} ds u'(0)$$

$$+\int_0^t e^{-\frac{\alpha}{2}(t-s)} B^{-\frac{1}{2}} \frac{e^{i(t-s)B^{\frac{1}{2}}} - e^{-i(t-s)B^{\frac{1}{2}}}}{2i} f(s)ds$$

$$= e^{-\frac{\alpha}{2}t}\left[\frac{e^{itB^{\frac{1}{2}}} + e^{-itB^{\frac{1}{2}}}}{2} + \frac{\alpha}{2}B^{-\frac{1}{2}}\frac{e^{tiB^{\frac{1}{2}}} - e^{-itB^{\frac{1}{2}}}}{2i}\right]u(0)$$

$$+e^{-\frac{\alpha}{2}t}\left[B^{-\frac{1}{2}}\frac{e^{itB^{\frac{1}{2}}} - e^{-itB^{\frac{1}{2}}}}{2i}\right]u'(0) + \int_0^t e^{-\frac{\alpha}{2}(t-s)} B^{-\frac{1}{2}} \frac{e^{i(t-s)B^{\frac{1}{2}}} - e^{-i(t-s)B^{\frac{1}{2}}}}{2i} f(s)ds.$$

Thus, by the definitions of $B^{\frac{1}{2}}$, $c(t)$, and $s(t)$ we obtain the formula (2.3). We will prove the following main theorem on continuous dependence of the solution on the given data.

Theorem 2.1. *Suppose that $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$ and $f(t)$ is a continuously differentiable function on $[0,T]$ and the assumption (2.2) holds. Then, there is a unique solution of problem (2.1) and the stability inequalities*

$$\max_{0\le t\le T} \|u(t)\|_H \tag{2.5}$$

$$\le M(\alpha,\beta,\delta)\left[\|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H + \max_{0\le t\le T}\left\|A^{-1/2}f(t)\right\|_H\right],$$

$$\max_{0\le t\le T}\left\|\frac{du(t)}{dt}\right\|_H + \max_{0\le t\le T}\left\|A^{1/2}u(t)\right\|_H \tag{2.6}$$

$$\le M(\alpha,\beta,\delta)\left[\left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \max_{0\le t\le T}\|f(t)\|_H\right],$$

$$\max_{0\le t\le T}\left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{0\le t\le T}\|Au(t)\|_H \tag{2.7}$$

$$\le \quad M(\alpha,\beta,\delta)\left[\|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H\right.$$

$$\left. + \|f(0)\|_H + \max_{0\le t\le T}\|f'(t)\|_H dt\right]$$

hold, where $M(\alpha,\beta,\delta)$ does not depend on φ, ψ and $f(t)$.

Proof. Using the formula (2.3), $A \geq \delta I$ and the following estimates

$$\begin{cases} \| c(t) \|_{H\to H} \leq 1, \| B^{\frac{1}{2}} s(t) \|_{H\to H} \leq 1, \left| e^{-\frac{\alpha}{2}t} \right| \leq 1, \\ \left\| A^{-1/2}\varphi \right\|_H \leq \frac{1}{\sqrt{\delta}} \left\| \varphi \right\|_H, \left\| A^{1/2}B^{-\frac{1}{2}} \right\|_{H\to H} \leq M(\delta), \end{cases} \tag{2.8}$$

we can write the following inequalities

$$\|u(t)\|_H \leq \|c(t)\|_{H\to H} e^{-\frac{\alpha}{2}t} \|\varphi\|_H + \left\| B^{\frac{1}{2}} s(t) \right\|_{H\to H} \left\| A^{1/2}B^{-\frac{1}{2}} \right\|_{H\to H}$$

$$\left| \frac{\alpha}{2e^{\frac{\alpha}{2}t}} \right| \| A^{-1/2}\varphi \|_H + \left\| B^{\frac{1}{2}} s(t) \right\|_{H\to H} \left\| A^{1/2}B^{-\frac{1}{2}} \right\|_{H\to H} e^{-\frac{\alpha}{2}t} \left\| A^{-1/2}\psi \right\|_H$$

$$+ \int_0^t \left\| B^{\frac{1}{2}} s(t-s) \right\|_{H\to H} \left\| A^{1/2}B^{-\frac{1}{2}} \right\|_{H\to H} \left\| A^{-1/2} f(s) \right\|_H ds$$

$$\leq M_1(\alpha,\beta,\delta) \left[\|\varphi\|_H + \| A^{-1/2}\psi \|_H + \max_{0\leq t\leq T} \| A^{-1/2} f(t) \|_H \right]$$

for any $t \in [0,T]$. Then, we obtain

$$\max_{0\leq t\leq T} \| u(t) \|_H \leq M_1(\alpha,\beta,\delta) \left[\| \varphi \|_H + \| A^{-1/2}\psi \|_H + \max_{0\leq t\leq T} \| A^{-1/2} f(t) \|_H \right].$$

Applying $A^{\frac{1}{2}}$ to the formula (2.3) and using estimate for (2.8), in a similar manner, we get

$$\left\| A^{\frac{1}{2}} u(t) \right\|_H \leq \|c(t)\|_{H\to H} e^{-\frac{\alpha}{2}t} \left\| A^{\frac{1}{2}}\varphi \right\|_H$$

$$+ \| B^{\frac{1}{2}} s(t) \|_{H\to H} \left\| A^{1/2}B^{-\frac{1}{2}} \right\|_{H\to H} \left| \frac{\alpha}{2e^{\frac{\alpha}{2}t}} \right| \| \varphi \|_H$$

$$+ \| B^{\frac{1}{2}} s(t) \|_{H\to H} \left\| A^{1/2}B^{-\frac{1}{2}} \right\|_{H\to H} e^{-\frac{\alpha}{2}t} \| \psi \|_H$$

$$+ \int_0^t \left\| A^{1/2}B^{-\frac{1}{2}} \right\|_{H\to H} \| B^{\frac{1}{2}} s(t-s) \|_{H\to H} \| f(s) \|_H ds$$

$$\leq M_2(\alpha,\beta,\delta) \left[\left\| A^{\frac{1}{2}}\varphi \right\|_H + \| \psi \|_H + \max_{0\leq t\leq T} \| f(t) \|_H \right]$$

for any $t \in [0,T]$. Then, we get

$$\max_{0\leq t\leq T} \| A^{\frac{1}{2}} u(t) \|_H \leq M_2(\alpha,\beta,\delta) \left[\| A^{\frac{1}{2}}\varphi \|_H + \| \psi \|_H + \max_{0\leq t\leq T} \| f(t) \|_H \right].$$

First, we obtain an estimate for $\|Au(t)\|_H$. Applying A to the formula (2.3) and using an integration by parts, we can write the formula

$$Au(t)e^{\frac{\alpha}{2}t} = c(t)A\varphi + \frac{\alpha}{2} A^{\frac{1}{2}} s(t) A^{\frac{1}{2}}\varphi + A^{\frac{1}{2}} s(t) A^{\frac{1}{2}}\psi$$

$$+AB^{-1}\left[e^{\frac{\alpha}{2}t}f(t)-c(t)f(0)-\int\limits_0^t e^{\frac{\alpha}{2}s}c(t-z)\left[\frac{\alpha}{2}f(z)+f'(z)\right]dz\right].$$

Using the latest formula and estimates (2.8), we obtain

$$\|Au(t)\|_H \le \|c(t)\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\|A\varphi\|_H$$

$$+\parallel B^{\frac{1}{2}}s(t)\parallel_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|\frac{\alpha}{2e^{\frac{\alpha}{2}t}}\right|\parallel A^{\frac{1}{2}}\varphi\parallel_H$$

$$+\parallel B^{\frac{1}{2}}s(t)\parallel_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\parallel A^{\frac{1}{2}}\psi\parallel_H$$

$$+\left\|AB^{-1}\right\|_{H\to H}\left[\|f(t)\|_H+\|c(t)\|_{H\to H}\|f(0)\|_H\right]$$

$$+\left\|AB^{-1}\right\|_{H\to H}\int\limits_0^t e^{-\frac{\alpha}{2}(t-z)}\parallel c(t-z)\parallel_{H\to H}\left[\frac{\alpha}{2}\parallel f(z)\parallel_H+\parallel f'(z)\parallel_H\right]dz$$

$$\le M_3(\alpha,\beta,\delta)\left[\|A\varphi\|_H+\parallel A^{\frac{1}{2}}\psi\parallel_H+\|f(0)\|_H+\max_{0\le t\le T}\parallel f'(t)\parallel_H\right]$$

for any $t\in[0,T]$. Then, we get

$$\max_{0\le t\le T}\parallel Au(t)\parallel_H\le M_3(\alpha,\beta,\delta)\left[\parallel A\varphi\parallel_H+\parallel A^{\frac{1}{2}}\psi\parallel_H+\|f(0)\|_H+\max_{0\le t\le T}\parallel f'(t)\parallel_H\right].$$

The estimate for $\max\limits_{0\le t\le T}\left\|\frac{d^2u}{dt^2}\right\|_H$ follows from the final estimate and the triangle inequality. Theorem 2.1. is proved. □

Remark 2.1. *All statements of Theorem 2.1. hold in an arbitrary Banach space E under the assumptions (see (Ashyralyev and Sobolevskii, 2005)):*

$$\begin{cases}\parallel c(t)\parallel_{E\to E}\le M,\parallel B^{\frac{1}{2}}s(t)\parallel_{E\to E}\le M,\\ \left\|B^{-1/2}\varphi\right\|_E\le M(\delta)\|\varphi\|_E,\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{E\to E}\le M(\delta).\end{cases}\tag{2.9}$$

Now, we assume the implementation of abstract Theorem 2.1. First, we take into account the boundary value problem for telegraph equations

$$\begin{cases}u_{tt}(t,x)+\alpha u_t(t,x)-(a(x)u_x)_x+\delta u(t,x)+\beta u(t,x)=f(t,x),\\ 0<t<T,0<x<l,\\ u(0,x)=\varphi(x),u_t(0,x)=\psi(x),0\le x\le l,\\ u(t,0)=u(t,l),\ u_x(t,0)=u_x(t,l),\ 0\le t\le T.\end{cases}\tag{2.10}$$

Problem (2.10) has a unique smooth solution $u(t,x)$ for smooth $a(x) \geq a > 0$, $x \in (0,l)$, $\delta > 0$, $a(l) = a(0)$, $\varphi(x)$, $\psi(x)$ $(x \in [0,l])$ and $f(t,x)$ $(t \in (0,T), x \in (0,l))$ functions. This permits us to reduce the problem (2.10) to the initial value (2.1) in a Hilbert space $H = L_2[0,l]$ with a self-adjoint positive definite operator A^x defined by the formula (2.10). Let us show a number of corollaries of abstract Theorem 2.1.

Theorem 2.2. *For solutions of the problem (2.10) the stability inequalities*

$$\max_{0\leq t\leq T} \|u(t,.)\|_{W_2^1[0,l]} \tag{2.11}$$
$$\leq \quad M_1(\alpha,\beta,\delta)\left[\max_{0\leq t\leq T}\|f(t,.)\|_{L_2[0,l]} + \|\varphi\|_{W_2^1[0,l]} + \|\psi\|_{L_2[0,l]}\right],$$

$$\max_{0\leq t\leq T} \|u(t,.)\|_{W_2^2[0,l]} + \max_{0\leq t\leq T}\|u_{tt}(t,.)\|_{L_2[0,l]} \tag{2.12}$$
$$\leq \quad M_1(\alpha,\beta,\delta)\left[\max_{0\leq t\leq T}\|f_t(t,.)\|_{L_2[0,l]} + \|f(0,.)\|_{L_2[0,l]} + \|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]}\right]$$

hold,where $M_1(\alpha,\beta,\delta)$ does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$.

Proof. Problem (2.10) can be written in abstract form

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) + \beta u(t) = f(t) \ (0 \leq t \leq T), \\ \\ u(0) = \varphi, u'(0) = \psi \end{cases} \tag{2.13}$$

in a Hilbert space $L_2[0,l]$ of all square integrable functions defined on $[0,l]$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -\left(a(x)u_x\right)_x + \sigma u(x) \tag{2.14}$$

with the domain

$$D(A^x) = \left\{u(x) : u, u_x, (a(x)u_x)_x \in L_2[0,l], u(0) = u(l), u'(0) = u'(l)\right\}.$$

"Here, $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions defined on $[0,l]$ with the values in $H = L_2[0,l]$. Therefore, estimates (2.11) and (2.12) follow from estimates (2.5), (2.6) and (2.7). Thus, Theorem 2.2 is proved." □

Second, "let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T]\times\Omega$ we consider the boundary value problem for telegraph equations

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-\sum\limits_{r=1}^{n}(a_r(x)u_{x^r})x_r+\beta u(t,x)=f(t,x), \\ \\ x=(x_1,...,x_n)\in\Omega, 0<t<T, \\ \\ u(0,x)=\varphi(x), \frac{\partial u(0,x)}{\partial t}=\psi(x), x\in\overline{\Omega}, \\ \\ u(t,x)=0, x\in S, 0\leq t\leq T, \end{cases} \tag{2.15}$$

where $a_r(x)$, $(x\in\Omega)$, $\varphi(x)$, $\psi(x)$ $(x\in\overline{\Omega})$ and $f(t,x)$, $t\in(0,T)$, $x\in\Omega$ are given smooth functions and $a_r(x)>0$. We introduce the Hilbert space $L_2(\overline{\Omega})$, the space of all integrable functions defined on $\overline{\Omega}$, equipped with the norm"

$$\|f\|_{L_2(\overline{\Omega})}=\left\{\int\cdots\int\limits_{x\in\overline{\Omega}}|f(x)|^2\,dx_1...dx_n\right\}^{\frac{1}{2}}.$$

Theorem 2.3. *For solutions of problem (2.15) the stability inequalities*

$$\max_{0\leq t\leq T}\|u(t,.)\|_{W_2^1(\overline{\Omega})} \tag{2.16}$$

$$\leq M(\alpha,\beta,\delta)\left[\max_{0\leq t\leq T}\|f(t,.)\|_{L_2(\overline{\Omega})}+\|\ \varphi\ \|_{W_2^1(\overline{\Omega})}+\|\ \psi\ \|_{L_2(\overline{\Omega})}\right],$$

$$\max_{0\leq t\leq T}\|u(t,.)\|_{W_2^2(\overline{\Omega})}+\max_{0\leq t\leq T}\|u_{tt}(t,.)\|_{L_2(\overline{\Omega})} \tag{2.17}$$

$$\leq M_1(\alpha,\beta,\delta)\left[\max_{0\leq t\leq T}\|f_t(t,.)\|_{L_2(\overline{\Omega})}+\|f(0,.)\|_{L_2(\overline{\Omega})}+\|\varphi\|_{W_2^2(\overline{\Omega})}+\|\psi\|_{W_2^1(\overline{\Omega})}\right]$$

hold, where $M(\alpha,\beta,\delta)$ and $M_1(\alpha,\beta,\delta)$ do not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$.

Proof. Problem (2.15) can be written in abstract form (2.13) in Hilbert space $L_2(\overline{\Omega})$ with self-adjoint positive definite operator $A=A^x$ defined by formula

$$A^x u(x)=-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\sigma u(x) \tag{2.18}$$

with domain

$$D(A^x)=\left\{u(x):u(x),u_{x_r}(x),(a_r(x)u_{x_r})_{x_r}\in L_2(\overline{\Omega}),1\leq r\leq n,u(x)=0,x\in S\right\}.$$

Here, $f(t)=f(t,x)$ and $u(t)=u(t,x)$ are known and unknown abstract functions defined on $\overline{\Omega}$ with the values in $H=L_2(\overline{\Omega})$. So, estimates (2.16) and (2.17) follow from estimates (2.5), (2.6) and (2.7) and the following theorem. □

Theorem 2.4. *For the solutions of the elliptic differential problem (see (Sobolevskii, 1975))*

$$\begin{cases} A^x u(x) = \omega(x), x \in \Omega, \\ u(x) = 0, x \in S, \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M_1 ||\omega||_{L_2(\overline{\Omega})}.$$

Here M_1 does not depend on $\omega(x)$.

In the next section, the first and second order of accuracy difference schemes for the approximate solution of problem (2.1) are investigated. Stability estimates for the solution of the first and second order of accuracy difference schemes are established. In applications, difference schemes for the approximate solution of two mixed problems (2.10) and (2.15) are presented. Stability estimates for the solution of two mixed problems (2.10) and (2.15) difference schemes are established.

2.1 STABLE TWO-STEP DIFFERENCE SCHEMES

First, we consider the first order of approximation in t two-step difference scheme

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \alpha \frac{u_{k+1}-u_k}{\tau} + Au_{k+1} + \beta u_{k+1} = f_k, \\ f_k = f(t_{k+1}), 1 \leq k \leq N-1, N\tau = T, \\ u_0 = \varphi, (1+\alpha\tau) \frac{u_1-u_0}{\tau} + (A+\beta I)\, \tau u_1 = \psi \end{cases} \tag{2.19}$$

for the numerical solution of the initial value problem (2.1). Now, let us give some lemmas that will be needed below.

Lemma 2.1. *The estimates hold:*

$$\begin{cases} \|R\|_{H\to H} \le \frac{1}{1+\frac{\alpha\tau}{2}}, \left\|\widetilde{R}\right\|_{H\to H} \le \frac{1}{1+\frac{\alpha\tau}{2}}, \\ \left\|\tau B^{\frac{1}{2}}R\right\|_{H\to H} \le 1, \left\|\tau B^{\frac{1}{2}}\widetilde{R}\right\|_{H\to H} \le 1, \\ \left\|\left(\pm\frac{\alpha}{2} + iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H} \le 1 + \frac{\alpha}{2\sqrt{\delta+\left(\beta-\frac{\alpha^2}{4}\right)}} \end{cases} \quad (2.20)$$

Here

$$R = \left((1+\frac{\alpha\tau}{2})I - i\tau B^{\frac{1}{2}}\right)^{-1}, \widetilde{R} = \left((1+\frac{\alpha\tau}{2})I + i\tau B^{\frac{1}{2}}\right)^{-1}.$$

Theorem 2.5. *Suppose that the assumption (2.2) holds and $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$. Then, for the solution of difference scheme (2.19) the following stability estimates*

$$\max_{1\le k\le N}\|u_k\|_H \le M(\alpha,\beta,\delta)\left\{\max_{1\le k\le N-1}\|A^{-1/2}f_k\|_H + \|A^{-1/2}\psi\|_H + \|\varphi\|_H\right\}, \quad (2.21)$$

$$\max_{1\le k\le N}\|A^{1/2}u_k\|_H \le M(\alpha,\beta,\delta)\left\{\max_{1\le k\le N-1}\|f_k\|_H + \|\psi\|_H + \|A^{1/2}\varphi\|_H\right\}, \quad (2.22)$$

$$\max_{1\le k\le N}\|Au_k\|_H \le M(\alpha,\beta,\delta)\left\{\max_{2\le k\le N-1}\left\|\frac{1}{\tau}(f_k - f_{k-1})\right\|_H \right.$$
$$\left. + \|f_1\|_H + \|A^{1/2}\psi\|_H + \|A\varphi\|_H\right\} \quad (2.23)$$

hold, where $M(\alpha,\beta,\delta)$ does not depend on τ, φ, ψ and f_k, $1 \le s \le N-1$.

Proof. We will obtain the formula for the solution of problem (2.1). We can rewrite (2.1) into the following difference problem

$$\begin{cases} u_{k-1} - (2+\alpha\tau)Iu_k + \left((1+\alpha\tau)I + \tau^2\left(B + \frac{\alpha^2}{4}I\right)\right)u_{k+1} = \tau^2 f_k, \\ 1 \le k \le N-1, \\ u_0 = \varphi, u_1 = (1+\alpha\tau)R\widetilde{R}\varphi + \tau R\widetilde{R}\psi. \end{cases} \quad (2.24)$$

It is clear that there exist a unique solution of this initial value problem

$$\begin{cases} u_{k-1} - (2+\alpha\tau)Iu_k + \left((1+\alpha\tau)I + \tau^2\left(B + \frac{\alpha^2}{4}I\right)\right)u_{k+1} \\ = \tau^2 f_k, 1 \le k \le N-1, \\ u_0, u_1 \text{ are given values} \end{cases}$$

and for the solution of this problem the following formula is satisfied (see (Ashyralyev and Sobolevskii, 2004))

$$u_k = R\widetilde{R}(\widetilde{R}-R)^{-1}[R^{k-1}-\widetilde{R}^{k-1}]u_0 + (\widetilde{R}-R)^{-1}(\widetilde{R}^k - R^k)u_1$$
$$+\sum_{s=1}^{k-1} R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right]\tau^2 f_s, 2\le k\le N. \tag{2.25}$$

Applying formula (2.25) and conditions $u_0=\varphi, u_1=(1+\alpha\tau)R\widetilde{R}\varphi+\tau R\widetilde{R}\psi$, we can obtain the following formula for the solution of (2.24)

$$u_0=\varphi, u_1=(1+\alpha\tau)R\widetilde{R}\varphi+\tau R\widetilde{R}\psi,$$

$$u_k = R\widetilde{R}(\widetilde{R}-R)^{-1}[R^{k-1}-\widetilde{R}^{k-1}]\varphi + (\widetilde{R}-R)^{-1}(\widetilde{R}^k - R^k)\left[(1+\alpha\tau)R\widetilde{R}\varphi+\tau R\widetilde{R}\psi\right]$$
$$+\sum_{s=1}^{k-1} R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right]\tau^2 f_s, 2\le k\le N. \tag{2.26}$$

Thus, we obtain estimates (2.21)-(2.23). Using the triangle inequality, formula (2.26), and estimates (2.20), we obtain

$$\|u_1\|_H \le (1+\alpha\tau)\left\|R\widetilde{R}\right\|_{H\to H}\|\varphi\|_H$$
$$+\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|\tau B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\left\|A^{-1/2}\psi\right\|_H$$
$$\le M(\alpha,\beta,\delta)\left[\| A^{-1/2}\psi\|_H + \|\varphi\|_H\right].$$

In exactly the same manner, one establishes

$$\left\|A^{\frac{1}{2}}u_1\right\|_H \le (1+\alpha\tau)\left\|R\widetilde{R}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\varphi\right\|_H$$
$$+\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|\tau B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\|\psi\|_H$$
$$\le M(\alpha,\beta,\delta)\left[\|\psi\|_H + \|A^{\frac{1}{2}}\varphi\|_H\right],$$
$$\|Au_1\|_H \le (1+\alpha\tau)\left\|R\widetilde{R}\right\|_{H\to H}\|A\varphi\|_H$$
$$+\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|\tau B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\|\psi\|_H$$
$$\le M(\alpha,\beta,\delta)\left[\|A^{\frac{1}{2}}\psi\|_H + \|A\varphi\|_H\right].$$

Now, we will establish estimates (2.21)-(2.23) for $k\ge 2$. Using formula (2.26) and identities

$$I-\widetilde{R}=\tau\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\widetilde{R}, I-R=\tau\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)R, \widetilde{R}-R=\left(-2i\tau B^{\frac{1}{2}}\right)\widetilde{R}R, \tag{2.27}$$

we can write

$$u_k = \left(2iB^{\frac{1}{2}}\right)^{-1}\left[\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^k+\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\widetilde{R}^k\right]\varphi+\left(-2iB^{\frac{1}{2}}\right)^{-1}(\widetilde{R}^k-R^k)\psi$$

$$+\sum_{s=1}^{k-1}\left(-2iB^{\frac{1}{2}}\right)^{-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right]\tau f_s, 2\le k\le N. \tag{2.28}$$

Using the triangle inequality, formula (2.28), and estimates (2.20), we get

$$\begin{aligned}\|u_k\|_H \le\ & \frac{1}{2}\left[\left\|\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H}\right.\\ &\left.+\left\|R^k\right\|_{H\to H}\left\|\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\right]\|\varphi\|_H\\ &+\frac{1}{2}\left[\left\|\widetilde{R}^k\right\|_{H\to H}+\left\|R^k\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_H\\ &+\sum_{s=1}^{k-1}\frac{1}{2}\left[\left\|\widetilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right]\tau\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_s\right\|_H\\ &\le M_3(\alpha,\beta,\delta)\left[\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\max_{1\le k\le N-1}\left\|A^{-\frac{1}{2}}f_k\right\|_H\right].\end{aligned}$$

for any $k\ge 2$. Combining the estimates $\|u_k\|_H$ for any k, we obtain (2.21). Using the triangle inequality, formula (2.28), and estimates (2.20), we obtain

$$\begin{aligned}\left\|A^{\frac{1}{2}}u_k\right\|_H \le\ & \frac{1}{2}\left[\left\|\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H}\right.\\ &\left.+\left\|R^k\right\|_{H\to H}\left\|\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}\varphi\right\|_H\\ &+\frac{1}{2}\left[\left\|\widetilde{R}^k\right\|_{H\to H}+\left\|R^k\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\|\psi\|_H\\ &+\sum_{s=1}^{k-1}\frac{1}{2}\left[\left\|\widetilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right]\tau\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\|f_s\|_H\\ &\le M_3(\alpha,\beta,\delta)\left[\left\|A^{\frac{1}{2}}\varphi\right\|_H+\|\psi\|_H+\max_{1\le k\le N-1}\|f_k\|_H\right]\end{aligned}$$

for any $k\ge 2$. Combining the estimates $\left\|A^{\frac{1}{2}}u_k\right\|_H$ for any k, we obtain (2.22). Using the Abel's formula, we can write

$$\sum_{s=1}^{k-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right]\tau^2 f_s$$

$$=\tau^2\left\{(I-\widetilde{R})^{-1}\left[\widetilde{R}f_{k-1}-\widetilde{R}^k f_1+\sum_{s=1}^{k-2}\widetilde{R}^{k-s}\left(f_s-f_{s+1}\right)\right]\right.$$

$$-(I-R)^{-1}\left[Rf_{k-1}-R^{k}f_{1}+\sum_{s=1}^{k-2}R^{k-s}\left(f_{s}-f_{s+1}\right)\right]\Bigg\}.$$

Since identities (2.27) and

$$(I-\widetilde{R})(I-R)=\widetilde{R}R\tau^{2}\left(\frac{\alpha^{2}}{4}I+B\right)=A\tau^{2}\widetilde{R}R,$$

$$\widetilde{R}(I-\widetilde{R})^{-1}-R(I-R)^{-1}=\left(\widetilde{R}-R\right)(I-\widetilde{R})^{-1}(I-R)^{-1}=\left(\widetilde{R}-R\right)\left(A\tau^{2}\widetilde{R}R\right)^{-1},$$

$$\widetilde{R}^{k}(I-\widetilde{R})^{-1}-R^{k}(I-R)^{-1}=\left(\widetilde{R}^{k}(I-R)-R^{k}(I-\widetilde{R})\right)(I-\widetilde{R})^{-1}(I-R)^{-1}$$

$$=\left\{\tau\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k}-\tau\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k}\right\}\left(A\tau^{2}\widetilde{R}R\right)^{-1},$$

we have that

$$\sum_{s=1}^{k-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right]\tau^{2}f_{s}$$

$$=A^{-1}\Bigg\{f_{k-1}-\left(\left(-2i\tau B^{\frac{1}{2}}\right)\widetilde{R}R\right)^{-1}\left(\widetilde{R}-R\right)^{-1}\left\{\left[\widetilde{R}^{k}-R^{k}\right]-\widetilde{R}R\left[\widetilde{R}^{k-1}-R^{k-1}\right]\right\}f_{1}$$

$$+\left(\left(-2i\tau B^{\frac{1}{2}}\right)\widetilde{R}R\right)^{-1}\sum_{s=1}^{k-2}\left\{\tau\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)R\widetilde{R}^{k-s}-\tau\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\widetilde{R}R^{k-s}\right\}\left(f_{s}-f_{s+1}\right)\Bigg\}$$

$$=A^{-1}\Bigg\{f_{k-1}-\left(-2iB^{\frac{1}{2}}\right)^{-1}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-1}\right\}f_{1}$$

$$+\left(-2iB^{\frac{1}{2}}\right)^{-1}\sum_{s=1}^{k-2}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-s-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-s-1}\right\}\left(f_{s}-f_{s+1}\right)\Bigg\}.$$

Using this formula and applying A to the formulas (2.28), we can write

$$Au_{k}=\left(2iB^{\frac{1}{2}}\right)^{-1}\left[\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k}+\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\widetilde{R}^{k}\right]A\varphi+A\left(-2iB^{\frac{1}{2}}\right)^{-1}(\widetilde{R}^{k}-R^{k})\psi$$

$$+f_{k-1}-\left(-2iB^{\frac{1}{2}}\right)^{-1}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-1}\right\}f_{1}+\left(-2iB^{\frac{1}{2}}\right)^{-1}$$

$$\times\sum_{s=1}^{k-2}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-s-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-s-1}\right\}\left(f_{s}-f_{s+1}\right),2\leq k\leq N.$$

Using the triangle inequality, last formula, and estimates (2.20), we obtain

$$\|Au_{k}\|_{H}\leq\frac{1}{2}\Bigg[\left\|\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^{k}\right\|_{H\to H}$$

$$+\left\|R^{k}\right\|_{H\to H}\left\|\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\Bigg]\|A\varphi\|_{H}$$

$$+\frac{1}{2}\left[\left\|\widetilde{R}^{k}\right\|_{H\to H}+\left\|R^{k}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_{H}$$

$$+\|f_{k-1}\|_{H}+\frac{1}{2}\left[\frac{\alpha}{2}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}+1\right]\left\{\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\left\|R^{k-1}\right\|_{H\to H}\right\}\|f_{1}\|_{H}$$

$$+\frac{1}{2}\left[\frac{\alpha}{2}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}+1\right]\sum_{s=1}^{k-2}\left\{\left\|\widetilde{R}^{k-s-1}\right\|_{H\to H}+\left\|R^{k-s-1}\right\|_{H\to H}\right\}\|f_s-f_{s+1}\|_H\right\}$$

$$\leq M_4(\alpha,\beta,\delta)\left[\|A\varphi\|_H+\|\varphi\|_H+\left\|A^{\frac{1}{2}}\psi\right\|_H\right.$$

$$\left.+\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H\right]$$

for any $k\geq 2$. Combining the estimates $\|Au_k\|_H$ for any k, we obtain (2.23). Theorem 2.5 is proved. □

Now, we consider two types of the second order of approximation in t two-step difference schemes for the numerical solution of the initial value problem (2.1) which are as following

$$\begin{cases}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_{k-1}}{2\tau}+\frac{A}{2}(u_{k+1}+u_{k-1})\\ +\frac{\beta}{2}\left(u_{k+1}+u_{k-1}\right)=f_k, f_k=f(t_k), 1\leq k\leq N-1,\\ u_0=\varphi,\frac{u_1-u_0}{\tau}+\frac{\tau}{4}Bu_1+\frac{1}{1+\frac{\alpha}{4}\tau}\left(\frac{1}{4}B-\frac{\alpha\tau B}{16}+\frac{\alpha^2}{8}I\right)\tau u_0\\ =\frac{1-\frac{\alpha}{4}\tau}{1+\frac{\alpha}{4}\tau}\left(\psi+\frac{\tau}{2}f_0\right), f_0=f(0),\end{cases} \quad (2.29)$$

$$\begin{cases}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_{k-1}}{2\tau}+\frac{A}{2}u_k+\frac{A}{4}(u_{k+1}+u_{k-1})\\ +\frac{\beta}{2}u_k+\frac{\beta}{4}\left(u_{k+1}+u_{k-1}\right)=f_k, f_k=f(t_k), 1\leq k\leq N-1,\\ u_0=\varphi,\frac{u_1-u_0}{\tau}+\frac{\tau}{4}Bu_1+\frac{1}{1+\frac{\alpha}{4}\tau}\left(\frac{1}{4}B-\frac{\alpha\tau B}{16}+\frac{\alpha^2}{8}I\right)\tau u_0\\ =\frac{1-\frac{\alpha}{4}\tau}{1+\frac{\alpha}{4}\tau}\left(\psi+\frac{\tau}{2}f_0\right), f_0=f(0).\end{cases} \quad (2.30)$$

Theorem 2.6. *Suppose that the assumption (2.2) holds and $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$. Then, for the solution of difference schemes (2.29) and (2.30) the following stability estimates*

$$\max_{1\leq k\leq N}\|u_k\|_H\leq M(\alpha,\beta,\delta)\left\{\max_{0\leq k\leq N-1}\|A^{-1/2}f_k\|_H+\|\ A^{-1/2}\psi\ \|_H+\|\ \varphi\ \|_H\right\},$$

$$\max_{1\leq k\leq N}\|A^{1/2}u_k\|_H\leq M(\alpha,\beta,\delta)\left\{\max_{0\leq k\leq N-1}\|f_k\|_H+\|\ \psi\ \|_H+\|\ A^{1/2}\varphi\ \|_H\right\},$$

$$\max_{1\leq k\leq N}\|Au_k\|_H \leq M(\alpha,\beta,\delta)\left\{\max_{1\leq k\leq N-1}\left\|\frac{1}{\tau}\left(f_k-f_{k-1}\right)\right\|_H\right.$$

$$\left.+\parallel f_0\parallel_H+\parallel A^{1/2}\psi\parallel_H+\parallel A\varphi\parallel_H\right\}$$

hold, where $M(\alpha,\beta,\delta)$ *does not depend on* τ, φ, ψ *and* f_k, $0\leq s\leq N-1$.

The proof of Theorem 2.6 is based on the formulas for the solution of difference schemes (2.29) and (2.30), on the estimates for the step operators and on the self-adjointness and positivity of operator A.

Now, we consider implementations of Theorem 2.5. "First, we consider the boundary value problem (2.10). The discretization of problem (2.10) is carried out in two steps. In the first step, we define the grid space

$$[0,l]_h=\{x=x_n : x_n=nh,\ 0\leq n\leq M,\ Mh=l\}.$$

Let us introduce the Hilbert space $L_{2h}=L_2([0,l]_h)$ of the grid functions $\varphi^h(x)=\{\varphi_n\}_0^M$ defined on $[0,l]_h$, equipped with the norm"

$$\|\varphi^h\|_{L_{2h}}=\left(\sum_{x\in[0,l]_h}|\varphi(x)|^2h\right)^{1/2}.$$

To the differential operator A^x defined by the formula (2.14), we appoint the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x)=\{-(a(x)\varphi_{\overline{x}})_{x,n}+\delta\varphi_n\}_1^{M-1} \tag{2.31}$$

moving in the space of grid functions $\varphi^h(x)=\{\varphi_n\}_0^M$ providing the conditions $\varphi_0=\varphi_M$, $\varphi_1-\varphi_0=\varphi_M-\varphi_{M-1}$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the aid of A_h^x, we reach the boundary value problem

$$\begin{cases} u_{tt}^h(t,x)+\alpha u_t^h(t,x)+A_h^xu^h(t,x)+\beta u^h(t,x)=f^h(t,x), \\ 0<t<T, x\in[0,l]_h, \\ u^h(0,x)=\varphi^h(x), u_t^h(0,x)=\psi^h(x), x\in[0,l]_h. \end{cases} \tag{2.32}$$

In the second step, we modify (2.32) with the difference scheme (2.19)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\alpha\frac{u_{k+1}^h(x)-u_k^h(x)}{\tau}+A_h^x u_{k+1}^h(x)+\beta u_{k+1}^h(x)=f_k^h(x), \\ f_k^h(x)=f^h(t_{k+1},x),\ t_k=k\tau,\ 1\leq k\leq N-1,\ x\in[0,l]_h, N\tau=T, \\ u_0^h(x)=\varphi^h(x), \\ (1+\alpha\tau)\,\frac{u_1^h(x)-u_0^h(x)}{\tau}+\left(A_h^x+\beta\right)\tau u_1^h(x)=\psi^h(x), x\in[0,l]_h. \end{cases} \tag{2.33}$$

Theorem 2.7. *For the solution* $\left\{u_k^h\left(x\right)\right\}_0^N$ *of problem* (2.33) *the following stability estimates*

$$\max_{1\leq k\leq N}\|u_k^h\|_{L_{2h}}\leq M_1(\alpha,\beta,\delta)\left\{\max_{1\leq k\leq N-1}\|f_k^h\|_{L_{2h}}+\|\ \psi^h\ \|_{L_{2h}}+\|\ \varphi^h\ \|_{L_{2h}}\right\},$$

$$\max_{1\leq k\leq N}\|u_k^h\|_{W_{2h}^1}\leq M_1(\alpha,\beta,\delta)\left\{\max_{1\leq k\leq N-1}\|f_k^h\|_{L_{2h}}+\|\ \psi^h\ \|_{L_{2h}}+\|\ \varphi^h\ \|_{W_{2h}^1}\right\},$$

$$\max_{1\leq k\leq N}\|u_k^h\|_{W_{2h}^2}\leq M_2(\alpha,\beta,\delta)\left\{\max_{2\leq k\leq N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right.$$
$$\left.+\|\ f_1^h\ \|_{L_{2h}}+\|\ \psi^h\ \|_{W_{2h}^1}+\|\ \varphi^h\ \|_{W_{2h}^2}\right\}$$

hold, where $M_1(\alpha,\beta,\delta)$ *and* $M_2(\alpha,\beta,\delta)$ *do not depend on* $\varphi^h(x),\psi^h(x)$ *and* $f_k^h(x), 1\leq k\leq N-1$.

Proof. Difference scheme (2.33) can be written in abstract form

$$\begin{cases} \frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}+\alpha\frac{u_{k+1}^h-u_k^h}{\tau}+A_h u_{k+1}^h+\beta u_{k+1}^h=f_k^h, \\ 1\leq k\leq N-1, N\tau=T, \\ u_0^h=\varphi^h, (1+\alpha\tau)\,\frac{u_1^h(x)-u_0^h(x)}{\tau}+\left(A_h^x+\beta\right)\tau u_1^h(x)=\psi^h \end{cases} \tag{2.34}$$

in a Hilbert space L_{2h} with self-adjoint positive definite operator $A_h=A_h^x$ by formula (2.31).

Here, $f_k^h=f_k^h(x)$ and $u_k^h=u_k^h(x)$ are known and unknown abstract mesh functions defined on $[0,l]_h$ with the values in $H=L_{2h}$. Therefore, estimates of

Theorem 2.7 follow from estimates (2.21), (2.22) and (2.23). Thus, Theorem 2.7 is proved. □

"Second, we consider the boundary value problem (2.15). The discretization of problem (2.15) is carried out in two steps. In the first step, we define the grid space

$$\overline{\Omega}_h = \{x = x_r = (h_1 j_1, \cdots, h_n j_n), j = (j_1, \cdots, j_n), \ 0 \leq j_r \leq N_r,$$
$$N_r h_r = 1, \ r = 1, \cdots, n\}, \ \Omega_h = \overline{\Omega}_h \cap \Omega, S_h = \overline{\Omega}_h \cap S$$

and introduce the Hilbert space $L_{2h} = L_2(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 j_1, \cdots, h_n j_n)\}$ defined on $\overline{\Omega}_h$ equipped with the norm"

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x \in \Omega_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{\frac{1}{2}}.$$

To the differential operator A^x defined by the formula (2.18), we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{r=1}^{n} \left(\alpha_r(x) u_{x_r}^h\right)_{x_r, j_r}, \tag{2.35}$$

where A_h^x is known as self-adjoint positive definite operator in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of the difference operator A_h^x, we arrive at the following boundary value problem

$$\begin{cases} u_{tt}^h(t,x) + \alpha u_t^h(t,x) + A_h^x u^h(t,x) + \beta u^h(t,x) = f^h(t,x), \\ 0 < t < T, x \in \Omega_h, \\ u^h(0,x) = \varphi^h(x), u_t^h(0,x) = \psi^h(x), x \in \Omega_h. \end{cases} \tag{2.36}$$

In the second step, we change (2.36) with the difference scheme (2.19)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\alpha\frac{u_{k+1}^h(x)-u_k^h(x)}{\tau}+A_h^x u_{k+1}^h(x)+\beta u_{k+1}^h(x)=f_k^h(x), \\ f_k^h(x)=f^h(t_{k+1},x),\ t_k=k\tau,\ 1\le k\le N-1,\ x\in\Omega_h, N\tau=T, \\ u_0^h(x)=\varphi^h(x),(1+\alpha\tau)\,\frac{u_1^h(x)-u_0^h(x)}{\tau}+(A_h^x+\beta)\,\tau u_1^h(x) \\ =\psi^h(x), x\in\Omega_h \end{cases} \tag{2.37}$$

for an infinite system of ordinary differential equations.

Theorem 2.8. *For the solution* $\left\{u_k^h(x)\right\}_0^N$ *of problem* (2.33) *the following stability estimates*

$$\max_{1\le k\le N}\|u_k^h\|_{L_{2h}}\le M_1(\alpha,\beta,\delta)\left\{\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\ \psi^h\ \|_{L_{2h}}+\|\ \varphi^h\ \|_{L_{2h}}\right\},$$

$$\max_{1\le k\le N}\|u_k^h\|_{W_{2h}^1}\le M_1(\alpha,\beta,\delta)\left\{\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\ \psi^h\ \|_{L_{2h}}+\|\ \varphi^h\ \|_{W_{2h}^1}\right\},$$

$$\max_{1\le k\le N}\|u_k^h\|_{W_{2h}^2}\le M_2(\alpha,\beta,\delta)\left\{\max_{2\le k\le N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right.$$

$$\left.+\|\ f_1^h\ \|_{L_{2h}}+\|\ \psi^h\ \|_{W_{2h}^1}+\|\ \varphi^h\ \|_{W_{2h}^2}\right\}$$

hold, where $M_1(\alpha,\beta,\delta)$ *and* $M_2(\alpha,\beta,\delta)$ *do not depend on* $\varphi^h(x),\psi^h(x)$ *and* $f_k^h(x), 1\le k\le N-1$.

Proof. Difference scheme (2.33) can be written in abstract form (2.34) in a Hilbert space $L_{2h}=L_2(\overline{\Omega}_h)$ with self-adjoint positive definite operator $A_h=A_h^x$ by formula (2.35).

Here, $f_k^h=f_k^h(x)$ and $u_k^h=u_k^h(x)$ are known and unknown abstract mesh functions defined on $\overline{\Omega}_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem 2.8 follow from estimates (2.21), (2.22) and (2.23) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}. □

Theorem 2.9. *For the solutions of the elliptic difference problem (Sobolevskii, 1975)*

$$\begin{cases} A_h^x u^h(x) = \omega^h(x), \ x \in \Omega_h, \\ \\ u^h(x) = 0, \ x \in S_h, \end{cases} \tag{2.38}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n} \left\| {u^h}_{x_r x_{\overline{r}}} \right\|_{L_{2h}} \leq M_3 ||\omega^h||_{L_{2h}},$$

where M_3 does not depend on h and ω^h.

Note that the difference schemes of the second order of accuracy with respect to one variable for approximate solutions of the mixed problems (2.10) and (2.15) generated by difference schemes (2.29) and (2.30) can be constructed. The abstract theorem given above and Theorem 2.8 and Theorem 2.9 permit us to establish the stability estimates for the solution of these difference schemes.

In implementations, theorems on convergence estimates can be established. The theoretical statements for the solution of difference schemes can be supported by the result of the numerical experiment. We have not been able to obtain a sharp estimate for the constants figuring in the stability inequality. Therefore we will give the conclusions of numerical examples for the initial-boundary value problem. (Ashyralyev and Altay, 2006)

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = \exp{(-t)} \sin x, \\ \\ 0 < t < 1, 0 < x < \pi, \\ \\ u(0,x) = \sin x, \frac{\partial}{\partial t} u(0,x) = -\sin x, 0 \leq x \leq \pi, \\ \\ u(t,0) = u(t,\pi) = 0, 0 \leq t \leq 1 \end{cases} \tag{2.39}$$

for the telegraph equation. The exact solution of above problem $u(t,x) = \exp{(-t)} \sin x$.

For the approximate solution of the initial-boundary value problem (2.39), we consider the set $w_{\tau,h} = [0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h. We present the following first order of accuracy in t and second order of accuracy in x difference scheme for the approximate solutions of the problem (2.39)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\frac{u_n^{k+1}-u_n^k}{\tau}-\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+u_n^{k+1}=\exp(-t_{k+1})\sin x_n, \\ x_n=nh, t_{k+1}=(k+1)\tau, \\ 1\leq k\leq N-1, 1\leq n\leq M-1, \\ u_n^0=\sin x_n, \frac{u_n^1-u_n^0}{\tau}=-\sin x_n, 0\leq n\leq M, \\ u_0^k=u_M^k=0, 0\leq k\leq N. \end{cases} \tag{2.40}$$

Now, we consider two types of second order of accuracy in t and x difference schemes for the approximate solutions of the problem (2.39)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\frac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\frac{1}{2}\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -\frac{1}{2}\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\frac{1}{2}\left(u_n^{k+1}+u_n^{k-1}\right)=\exp(-t_k)\sin(x_n), \\ x_n=nh, t_k=k\tau, 1\leq k\leq N-1, 1\leq n\leq M-1, \\ u_n^0=\sin(x_n), x_n=nh, \\ \frac{u_n^1-u_n^0}{\tau}=-\sin(x_n)+\frac{\tau}{2}\frac{u_n^2-2u_n^1+u_n^0}{\tau^2}, 0\leq n\leq M, \\ u_0^k=u_M^k=0, 0\leq k\leq N, \end{cases} \tag{2.41}$$

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} + 2\frac{u_n^{k+1}-u_n^{k-1}}{2\tau} - \frac{1}{2}\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} - \frac{1}{4}\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -\frac{1}{4}\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2} + \frac{1}{2}u_n^k + \frac{1}{4}\left(u_n^{k+1}+u_n^{k-1}\right) = \exp(-t_k)\sin(x_n), \\ x_n = nh, t_k = k\tau, 1 \le k \le N-1, 1 \le n \le M-1, \\ u_n^0 = \sin(x_n), x_n = nh, \\ \frac{u_n^1-u_n^0}{\tau} = -\sin(x_n) + \frac{\tau}{2}\frac{u_n^2-2u_n^1+u_n^0}{\tau^2}, 0 \le n \le M, \\ u_0^k = u_M^k = 0, 0 \le k \le N. \end{cases} \tag{2.42}$$

To solve these difference equations, an operation of modified Gauss elimination method is applied. Hence, we look for a solution of the matrix equation in the following form:

$$u_j = \alpha_{j+1}u_{j+1} + \beta_{j+1},\ u_M = 0,\ j = M-1, ..., 2, 1.$$

where α_j $(j = 1, 2, ..., M)$ are $(N+1)\times(N+1)$ square matrices, and β_j $(j = 1, 2, ..., M)$ are $(N+1)\times 1$ column matrices defined by

$$\alpha_{j+1} = -(B + C\alpha_j)^{-1}A,$$

$$\beta_{j+1} = (B + C\alpha_j)^{-1}(D\phi - C\beta_j),\ \ j = 1, 2, ..., M-1,$$

where $j = 1, 2, ..., M-1$, α_1 is the $(N+1)\times(N+1)$ zero matrix, and β_1 is the $(N+1)\times 1$ zero matrix. The results of computer calculations show that the second order difference schemes are more accurate than first order of accuracy difference scheme. Table 1 is established for $N = M = 20, 40$ and 80, in order of.

The errors are computed by

$$E_M^N = \max_{1\le k\le N-1, 1\le n\le M-1} \left|u(t_k, x_n) - u_n^k\right|,$$

where $u(t_k, x_n)$ symbolizes the exact solution and u_n^k seymbolizes the numerical solution at (t_k, x_n) and the results are given in Table 2.1.

Table 2.1 Error analysis

$\tau = \frac{1}{N}, h = \frac{pi}{M}$	$N = M = 20$	$N = M = 40$	$N = M = 80$
The difference scheme (2.40)	0.0046	0.0021	0.0010
The difference scheme (2.41)	2.3651×10^{-4}	6.0209×10^{-5}	1.5196×10^{-5}
The difference scheme (2.42)	1.3510×10^{-4}	3.4524×10^{-5}	8.7409×10^{-6}

CHAPTER 3

NONLOCAL BOUNDARY VALUE PROBLEMS FOR A TELEGRAPH EQUATION

We consider nonlocal boundary value problems for a telegraph equation

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) + \beta u(t) = f(t) \ (0 \leq t \leq T), \\ \\ u(0) = \lambda u(T) + \varphi, u'(0) = \mu u'(T) + \psi, \\ \\ \alpha > 0, \beta + \delta \geq \frac{\alpha^2}{4}, A \geq \delta I \end{cases} \tag{3.1}$$

in a Hilbert space H with a self-adjoint positive definite operator A.

"A function $u(t)$ is called a solution of the problem (3.1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the segment $[0, T]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0, T]$ and the function $Au(t)$ is continuous on the segment $[0, T]$.

(iii) $u(t)$ satisfies the equation and boundary conditions (3.1)."

Now, we will obtain the formula for the mild solution of problem (3.1) under the assumption (2.2). Applying formula (2.3) for the mild solution of problem (2.1),

we can write

$$u(t) = e^{-\frac{\alpha}{2}t}c\,(t)\,u(0) + \frac{\alpha}{2}e^{-\frac{\alpha}{2}t}s(t)u(0) + e^{-\frac{\alpha}{2}t}s(t)u'(0) \tag{3.2}$$

$$+\int\limits_0^t e^{-\frac{\alpha}{2}(t-z)}s(t-z)f(z)dz.$$

From that it follows that

$$u'(t) = e^{-\frac{\alpha}{2}t}\left[-\frac{\alpha}{2}c(t) - (A + \left(\beta - \frac{\alpha^2}{4}\right) I)s(t)\right]u(0) + \frac{\alpha}{2}e^{-\frac{\alpha}{2}t}\left[-\frac{\alpha}{2}s(t) + c(t)\right]u(0)$$

$$+e^{-\frac{\alpha}{2}t}\left[-\frac{\alpha}{2}s(t) + c(t)\right]u'(0) + \int\limits_0^t e^{-\frac{\alpha}{2}(t-z)}\left[-\frac{\alpha}{2}s(t-z) + c(t-z)\right]f(z)dz$$

$$= -(A+\beta I)e^{-\frac{\alpha}{2}t}s(t)u(0) + e^{-\frac{\alpha}{2}t}\left[-\frac{\alpha}{2}s(t) + c(t)\right]u'(0)$$

$$+\int\limits_0^t e^{-\frac{\alpha}{2}(t-z)}\left[-\frac{\alpha}{2}s(t-z) + c(t-z)\right]f(z)dz.$$

Appying this formula, conditions $u(0) = \lambda u(T) + \varphi, u'(0) = \mu u'(T) + \psi$, and formula (3.2), we get

$$u(0) = \lambda\left[\left(c(T) + \frac{\alpha}{2}s(T)\right)e^{-\frac{\alpha}{2}T}u(0) + e^{-\frac{\alpha}{2}T}s(T)u'(0)\right] \tag{3.3}$$

$$+\lambda\int\limits_0^T e^{-\frac{\alpha}{2}(T-z)}s(T-z)f(z)dz + \varphi,$$

$$u'(0) = \mu\left[-(A+\beta I)e^{-\frac{\alpha}{2}T}s(T)u(0) + e^{-\frac{\alpha}{2}T}\left[-\frac{\alpha}{2}s(T) + c(T)\right]u'(0)\right] \tag{3.4}$$

$$+\mu\int\limits_0^T e^{-\frac{\alpha}{2}(T-z)}\left(-\frac{\alpha}{2}s(T-z) + c(T-z)\right)f(z)dz + \psi.$$

We will obtain $u(0)$ and $u'(0)$. We have that

$$\Delta = \begin{vmatrix} I - \lambda e^{-\frac{\alpha}{2}T}\left(c(T) + \frac{\alpha}{2}s(T)\right) & -\lambda e^{-\frac{\alpha}{2}T}s(T) \\ \\ \mu(A+\beta I)e^{-\frac{\alpha}{2}T}s(T) & I - \mu e^{-\frac{\alpha}{2}T}\left(-\frac{\alpha}{2}s(T) + c(T)\right) \end{vmatrix}$$

$$= \left[I - \lambda e^{-\frac{\alpha}{2}T}\left(c(T) + \frac{\alpha}{2}s(T)\right)\right]\left[I - \mu e^{-\frac{\alpha}{2}T}\left(-\frac{\alpha}{2}s(T) + c(T)\right)\right]$$

$$+\lambda\mu(A+\beta I)e^{-\alpha T}s^2(T)$$

$$=\left\{I+\lambda\mu\left[c^2(T)+Bs^2(T)\right]e^{-\alpha T}\right\}-\left[(\lambda+\mu)c(T)+\frac{\alpha}{2}((\lambda-\mu)s(T)\right]e^{-\frac{\alpha}{2}T}.$$

Since

$$c^2(T)+Bs^2(T)=\left\{\frac{e^{iBT}+e^{-iBT}}{2}\right\}^2+B\left\{B^{-1/2}\frac{e^{iBT}-e^{-iBT}}{2i}\right\}^2=I,$$

we have that

$$\Delta=\left\{1+\lambda\mu e^{-\alpha T}\right\}I-\left[(\lambda+\mu)c(T)+\frac{\alpha}{2}(\lambda-\mu)s(T)\right]e^{-\frac{\alpha}{2}T}.$$

So, under the assumption

$$\left|1+\lambda\mu e^{-\alpha T}\right|>\left(|\lambda+\mu|+\frac{\frac{\alpha}{2}(|\lambda-\mu|}{\sqrt{\delta+\beta-\frac{\alpha^2}{4}}}\right)e^{-\frac{\alpha}{2}T}, \tag{3.5}$$

there exists of inverse of operator

$$\left(1+\lambda\mu e^{-\alpha T}\right)I-\left[(\lambda+\mu)c(T)+\frac{\alpha}{2}(\lambda-\mu)s(T)\right]e^{-\frac{\alpha}{2}T}.$$

We denote that

$$P=\left\{\left(1+\lambda\mu e^{-\alpha T}\right)I-\left[(\lambda+\mu)c(T)+\frac{\alpha}{2}(\lambda-\mu)s(T)\right]e^{-\frac{\alpha}{2}T}\right\}^{-1}.$$

Using the operator P, solving (3.3) and (3.4), we obtain

$$u(0)=P\Bigg\{\left(I-\mu e^{-\frac{\alpha}{2}T}\left(c(T)-\frac{\alpha}{2}s(T)\right)\right)\lambda\int_0^T e^{-\frac{\alpha}{2}(T-z)}s(T-z)f(z)dz$$

$$+\lambda e^{-\frac{\alpha}{2}T}s(T)\mu\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left[c(T-z)-\frac{\alpha}{2}s(T-z)\right]f(z)dz \tag{3.6}$$

$$+\left(I-\mu e^{-\frac{\alpha}{2}T}\left(c(T)-\frac{\alpha}{2}s(T)\right)\right)\varphi+\lambda e^{-\frac{\alpha}{2}T}s(T)\psi\Bigg\}.$$

$$u'(0)=P\left\{\left(I-\lambda e^{-\frac{\alpha}{2}T}\left(c(T)+\frac{\alpha}{2}s(T)\right)\right)\right.$$

$$\times\mu\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left[c(T-z)-\frac{\alpha}{2}s(T-z)\right]f(z)dz \tag{3.7}$$

$$-\mu\lambda(A+\beta I)e^{-\frac{\alpha}{2}T}s(T)\int_0^T e^{-\frac{\alpha}{2}(T-z)}s(T-z)f(z)dz$$

$$+\left(I-\lambda e^{-\frac{\alpha}{2}T}\left(c(T)+\frac{\alpha}{2}s(T)\right)\right)\psi-\mu(A+\beta I)e^{-\frac{\alpha}{2}T}s(T)\varphi\Big\}.$$

Consequently, the solution of problem (3.1) satisfy formulas (3.2), (3.6) and (3.7).

Theorem 3.1. *Suppose that $\varphi \in D(A)$, $\psi \in D(A^{1/2})$ and $f(t)$ are continuously differentiable on $[0,T]$ and assumpion (3.5) holds.Then, there is a unique solution of problem (3.1) and the stability inequalities*

$$\max_{0\le t\le T}\|u(t)\|_H$$

$$\le M(\lambda,\mu,\alpha,\beta,\delta)\left\{\|\varphi\|_H+\left\|A^{-1/2}\psi\right\|_H+\max_{0\le t\le T}\left\|A^{-1/2}f(t)\right\|_H\right\}, \tag{3.8}$$

$$\max_{0\le t\le T}\left\|A^{1/2}u(t)\right\|_H$$

$$\le M(\lambda,\mu,\alpha,\beta,\delta)\left\{\left\|A^{1/2}\varphi\right\|_H+\|\psi\|_H+\max_{0\le t\le T}\|f(t)\|_H\right\}, \tag{3.9}$$

$$\max_{0\le t\le T}\left\|\frac{d^2u(t)}{dt^2}\right\|_H+\max_{0\le t\le T}\|Au(t)\|_H \tag{3.10}$$

$$\le M(\lambda,\mu,\alpha,\beta,\delta)\left\{\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\int_0^T\|f'(t)\|_H\,dt\right\}$$

hold, where $M(\lambda,\mu,\alpha,\beta,\delta)$ does not depend on $f(t), t\in[0,T]$, φ, and ψ.

Proof. Using (3.2) and estimates (2.8), we obtain estimate

$$\|u(t)\|_H\le\|c(t)\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\|u(0)\|_H+\left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}$$

$$\left|\frac{\alpha}{2e^{\frac{\alpha}{2}t}}\right|\parallel A^{-1/2}u(0)\parallel_H+\left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\left\|A^{-1/2}u'(0)\right\|_H$$

$$+\int_0^t\left\|B^{\frac{1}{2}}s(t-s)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-1/2}f(s)\right\|_H ds$$

$$\le M_1(\lambda,\mu,\alpha,\beta,\delta)\left[\|u(0)\|_H+\left\|A^{-1/2}u'(0)\right\|_H+\max_{0\le t\le T}\left\|A^{-1/2}f(t)\right\|_H\right]$$

for any $t\in[0,T]$. Then, we obtain

$$\max_{0\le t\le T}\parallel u(t)\parallel_H \tag{3.11}$$

$$\leq M_1(\lambda,\mu,\alpha,\beta,\delta)\left[\|u(0)\|_H+\left\|A^{-1/2}u'(0)\right\|_H+\max_{0\leq t\leq T}\|\ A^{-1/2}f(t)\ \|_H\right].$$

Now, we will obtain estimate $\|u(0)\|_H$ and $\left\|A^{-1/2}u'(0)\right\|_H$. Applying the triangle inequality, formulas (3.6) and (3.7) and estimate (2.8), we get

$$\|u(0)\|_H\leq\|P\|_{H\to H} \tag{3.12}$$

$$\times\left\{\left(1+|\mu|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac12}s(T)\right\|_{H\to H}\left\|B^{-\frac12}\right\|_{H\to H}\right)\right)\right.$$

$$\times|\lambda|\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left\|B^{\frac12}s(T-z)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac12}\right\|_{H\to H}\left\|A^{-1/2}f(z)\right\|_H dz$$

$$+|\lambda|\,e^{-\frac{\alpha}{2}T}\left\|B^{\frac12}s(T)\right\|_{H\to H}|\mu|\left\|A^{1/2}B^{-\frac12}\right\|_{H\to H}$$

$$\times\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left(\|c(T-z)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac12}s(T-z)\right\|_{H\to H}\left\|B^{-\frac12}\right\|_{H\to H}\right)\left\|A^{-1/2}f(z)\right\|_H dz$$

$$+\left(1+|\mu|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac12}s(T)\right\|_{H\to H}\left\|B^{-\frac12}\right\|_{H\to H}\right)\right)\|\varphi\|_H$$

$$\left.+|\lambda|\,e^{-\frac{\alpha}{2}T}\left\|B^{\frac12}s(T)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac12}\right\|_{H\to H}\left\|A^{-1/2}\psi\right\|_H\right\}$$

$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\|\varphi\|_H+\left\|A^{-1/2}\psi\right\|_H+\max_{0\leq t\leq T}\left\|A^{-1/2}f(t)\right\|_H\right\},$$

$$\left\|A^{-1/2}u'(0)\right\|_H\leq\|P\|_{H\to H} \tag{3.13}$$

$$\times\left\{\left(1+|\lambda|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac12}s(T)\right\|_{H\to H}\left\|B^{-\frac12}\right\|_{H\to H}\right)\right)\right.$$

$$\times|\mu|\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left[\|c(T-z)\|_{H\to H}\right.$$

$$\left.+\frac{\alpha}{2}\left\|B^{\frac12}s(T-z)\right\|_{H\to H}\left\|B^{-\frac12}\right\|_{H\to H}\right]\left\|A^{-1/2}f(z)\right\|_H dz$$

$$+|\mu|\,|\lambda|\left\|(A+\beta I)B^{-1}\right\|_{H\to H}\left\|B^{\frac12}s(T)\right\|_{H\to H}$$

$$\times\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left\|B^{\frac12}s(T-z)\right\|_{H\to H}\left\|A^{-1/2}f(z)\right\|_H dz$$

$$\left(1+|\lambda|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac12}s(T)\right\|_{H\to H}\left\|B^{-\frac12}\right\|_{H\to H}\right)\right)\left\|A^{-1/2}\psi\right\|_H$$

$$\left.+|\mu|\,e^{-\frac{\alpha}{2}T}\left\|B^{\frac12}s(T)\right\|_{H\to H}\left\|(A+\beta I)B^{-\frac12}A^{-1/2}\right\|_{H\to H}\|\varphi\|_H\right\}$$

$$\leq M_1(\lambda,\mu,\alpha,\beta,\delta)\left\{\|\varphi\|_H+\left\|A^{-1/2}\psi\right\|_H+\max_{0\leq t\leq T}\left\|A^{-1/2}f(t)\right\|_H\right\}.$$

Estimate (3.8) follows from estimates (3.11), (3.12) and (3.13).

Applying $A^{\frac{1}{2}}$ to formula (3.2) and estimates (2.8), we obtain

$$\left\|A^{\frac{1}{2}}u(t)\right\|_H \leq \|c(t)\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\left\|A^{\frac{1}{2}}u(0)\right\|_H + \left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}$$

$$\left|\frac{\alpha}{2e^{\frac{\alpha}{2}t}}\right| \| u(0) \|_H + \left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right| \|u'(0)\|_H$$

$$+\int_0^t \left\|B^{\frac{1}{2}}s(t-s)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\|f(s)\|_H\,ds$$

$$\leq M_1(\lambda,\mu,\alpha,\beta,\delta)\left[\left\|A^{\frac{1}{2}}u(0)\right\|_H + \|u'(0)\|_H + \max_{0\leq t\leq T}\|f(t)\|_H\right]$$

for any $t \in [0,T]$.Then, we obtain

$$\max_{0\leq t\leq T} \| A^{\frac{1}{2}}u(t) \|_H \tag{3.14}$$

$$\leq M_1(\lambda,\mu,\alpha,\beta,\delta)\left[\left\|A^{\frac{1}{2}}u(0)\right\|_H + \|u'(0)\|_H + \max_{0\leq t\leq T}\| f(t) \|_H\right].$$

Now, we will get estimate $\left\|A^{\frac{1}{2}}u(0)\right\|_H$ and $\|u'(0)\|_H$. Implementing $A^{\frac{1}{2}}$ to formulas (3.6) and (3.7) and estimates (2.8), we get

$$\left\|A^{\frac{1}{2}}u(0)\right\|_H \leq \|P\|_{H\to H} \tag{3.15}$$

$$\times\left\{\left(1+|\mu|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H} + \frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\right)\right)\right.$$

$$\times|\lambda|\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left\|B^{\frac{1}{2}}s(T-z)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\|f(z)\|_H\,dz$$

$$+|\lambda|\,e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}|\mu|\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}$$

$$\times\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left(\|c(T-z)\|_{H\to H} + \frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T-z)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\right)\|f(z)\|_H\,dz$$

$$+\left(1+|\mu|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H} + \frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\right)\right)\left\|A^{\frac{1}{2}}\varphi\right\|_H$$

$$\left.+|\lambda|\,e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\|\psi\|_H\right\}$$

$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\left\|A^{\frac{1}{2}}\varphi\right\|_H + \|\psi\|_H + \max_{0\leq t\leq T}\|f(t)\|_H\right\},$$

$$\|u'(0)\|_H \leq \|P\|_{H\to H} \tag{3.16}$$

$$\times\left\{\left(1+|\lambda|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\right)\right)\right.$$

$$\times|\mu|\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left[\|c(T-z)\|_{H\to H}\right.$$

$$\left.+\frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T-z)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\right]\left\|A^{-1/2}f(z)\right\|_H dz$$

$$+|\mu|\,|\lambda|\left\|(A+\beta I)B^{-1}\right\|_{H\to H}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}$$

$$\times\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left\|B^{\frac{1}{2}}s(T-z)\right\|_{H\to H}\|f(z)\|_H\,dz$$

$$+\left(1+|\lambda|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\right)\right)\|\psi\|_H$$

$$\left.+|\mu|\,e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|(A+\beta I)B^{-\frac{1}{2}}A^{-1/2}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\varphi\right\|_H\right\}$$

$$\leq M_1(\lambda,\mu,\alpha,\beta,\delta)\left\{\left\|A^{\frac{1}{2}}\varphi\right\|_H+\|\psi\|_H+\max_{0\leq t\leq T}\|f(t)\|_H\right\}.$$

Estimate (3.9) follows from estimates (3.14), (3.15) and (3.16).

Now, we obtain an estimate for $\|Au(t)\|_H$. Applying A to formula (3.2) and using an integration by parts, we obtain the formula

$$Au(t)e^{\frac{\alpha}{2}t}=c(t)Au(0)+\frac{\alpha}{2}A^{\frac{1}{2}}s(t)A^{\frac{1}{2}}u(0)+A^{\frac{1}{2}}s(t)A^{\frac{1}{2}}u'(0)$$

$$+AB^{-1}\left[e^{\frac{\alpha}{2}t}f(t)-c(t)f(0)-\int_0^t e^{\frac{\alpha}{2}z}c(t-z)\left[\frac{\alpha}{2}f(z)+f'(z)\right]dz\right].$$

Using the last formula and estimates (2.8), we obtain

$$\|Au(t)\|_H\leq\|c(t)\|_{H\to H}\,e^{-\frac{\alpha}{2}t}\,\|Au(0)\|_H$$

$$+\parallel B^{\frac{1}{2}}s(t)\parallel_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|\frac{\alpha}{2e^{\frac{\alpha}{2}t}}\right|\parallel A^{\frac{1}{2}}u(0)\parallel_H$$

$$+\parallel B^{\frac{1}{2}}s(t)\parallel_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\parallel A^{\frac{1}{2}}u'(0)\parallel_H$$

$$+\left\|AB^{-1}\right\|_{H\to H}\left[\|f(t)\|_H+e^{-\frac{\alpha}{2}t}\|c(t)\|_{H\to H}\|f(0)\|_H\right]$$

$$+\left\|AB^{-1}\right\|_{H\to H}\int_0^t e^{-\frac{\alpha}{2}(t-z)}\parallel c(t-z)\parallel_{H\to H}\left[\frac{\alpha}{2}\parallel f(z)\parallel_H+\parallel f'(z)\parallel_H\right]dz$$

$$\leq M_3(\lambda,\mu,\alpha,\beta,\delta)\left[\|Au(0)\|_H+\parallel A^{\frac{1}{2}}u'(0)\parallel_H+\|f(0)\|_H+\max_{0\leq t\leq T}\parallel f'(t)\parallel_H\right]$$

for any $t \in [0,T]$. Then, we get

$$\max_{0\le t\le T} \parallel Au(t) \parallel_H \tag{3.17}$$

$$\le M_3(\lambda,\mu,\alpha,\beta,\delta)\left[\parallel Au(0) \parallel_H + \parallel A^{\frac{1}{2}}u'(0) \parallel_H + \|f(0)\|_H + \max_{0\le t\le T} \parallel f'(t) \parallel_H\right].$$

Now, we will obtain estimates $\|Au(0)\|_H$ and $\left\|A^{\frac{1}{2}}u'(0)\right\|_H$. Applying A to formula (3.6) and applying $A^{\frac{1}{2}}$ to (3.7) and estimates (2.8), we get

$$\|Au(0)\|_H \le \|P\|_{H\to H}$$

$$\times\Big\{\Big(1+|\mu|\, e^{-\frac{\alpha}{2}T}\Big(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\Big)\Big)|\lambda|$$

$$+\left\|AB^{-1}\right\|_{H\to H}\left[\|f(T)\|_H + e^{-\frac{\alpha}{2}T}\|c(T)\|_{H\to H}\|f(0)\|_H\right]$$

$$\times\left\|AB^{-1}\right\|_{H\to H}\int_0^T e^{-\frac{\alpha}{2}(T-z)} \parallel c(T-z) \parallel_{H\to H}$$

$$\times\left[\frac{\alpha}{2} \parallel f(z) \parallel_H + \parallel f'(z) \parallel_H\right]dz$$

$$+\left\|AB^{-1}\right\|_{H\to H}|\lambda|\, e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}|\mu|$$

$$\times\Big[e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\|f(0)\|_H$$

$$+\int_0^T e^{-\frac{\alpha}{2}(T-z)}\left\|B^{\frac{1}{2}}s(T-z)\right\|_{H\to H}\|f'(z)\|_H\,dz\Big]$$

$$+\Big(1+|\mu|\, e^{-\frac{\alpha}{2}T}\Big(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\Big)\Big)\|A\varphi\|_H$$

$$+\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}|\lambda|\, e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H\Big\}$$

$$\le M_1(\lambda,\mu,\alpha,\beta,\delta)\left\{\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\int_0^T\|f'(t)\|_H\,dt\right\}, \tag{3.18}$$

$$\left\|A^{\frac{1}{2}}u'(0)\right\|_H \le \|P\|_{H\to H}$$

$$\times\Big\{\Big(1+|\lambda|\, e^{-\frac{\alpha}{2}T}\Big(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\Big)\Big)$$

$$\times\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}|\mu|$$

$$\times\left[e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\|f(0)\|_H+\int_0^T\left\|B^{\frac{1}{2}}s(T-z)\right\|_{H\to H}\|f'(z)\|_H\,dz\right]$$

$$+|\mu|\,|\lambda|\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|(A+\beta I)B^{-1}\right\|_{H\to H}e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}$$

$$\times\left[\|f(T)\|_H+e^{-\frac{\alpha}{2}T}\|c(T)\|_{H\to H}\|f(0)\|_H\right]$$

$$+|\mu|\,|\lambda|\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|(A+\beta I)B^{-1}\right\|_{H\to H}e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}$$

$$\times\int\limits_0^T e^{-\frac{\alpha}{2}(T-z)}\parallel c(T-z)\parallel_{H\to H}\left[\frac{\alpha}{2}\parallel f(z)\parallel_H+\parallel f'(z)\parallel_H\right]dz$$

$$+\left(1+|\lambda|\,e^{-\frac{\alpha}{2}T}\left(\|c(T)\|_{H\to H}+\frac{\alpha}{2}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}\right)\right)\left\|A^{\frac{1}{2}}\psi\right\|_H$$

$$+|\mu|\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|(A+\beta I)A^{-1}\right\|_{H\to H}e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\|A\varphi\|_H\Big\}$$

$$\leq M_2(\lambda,\mu,\alpha,\beta,\delta)\left\{\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\int\limits_0^T\|f'(t)\|_H\,dt\right\}. \qquad (3.19)$$

Estimate

$$\max_{0\leq t\leq T}\|Au(t)\|_H$$

$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\|A\varphi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\int\limits_0^T\|f'(t)\|_H\,dt\right\}$$

follows from estimates (3.17), (3.18) and (3.19). Finally, estimate for $\max\limits_{0\leq t\leq T}\left\|\frac{d^2u}{dt^2}\right\|_H$ follows from the recent estimate and the triangle inequality. Theorem 3.1. is proved.

□

Now, we will search two implements of Theorem 3.1.

First, for implements of Theorem 3.1 we search nonlocal the boundary value problem for telegraph equations

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-(a(x)u_x)_x+\delta u(t,x)+\beta u(t,x)=f(t,x), \\ 0<t<T, 0<x<l, \\ u(0,x)=\lambda u(T,x)+\varphi(x), u_t(0,x)=\mu u_t(T,x)+\psi(x), 0\leq x\leq l, \\ u(t,0)=u(t,l), u_x(0,x)=u_x(t,l), 0\leq t\leq T. \end{cases} \qquad (3.20)$$

Problem (3.20) has a unique smooth solution $u(t,x)$ for smooth $a(x)\geq a>0$, $x\in(0,l)$, $\delta>0$, $a(l)=a(0)$, $\varphi(x),\psi(x)(x\in[0,l]$ and $f(t,x)(t\in(0,T),x\in(0,l))$

functions. This allows us to reduce the problem (3.20) to the nonlocal boundary value problem (3.1) in a Hilbert space $H = L_2[0,l]$ with a self-adjoint positive definite operator A^x defined by (3.20). Let us give a number of corollaries of abstract Theorem 3.1.

Theorem 3.2. *For solutions of the problem (3.20) the stability inequalities*

$$\max_{0\leq t\leq T}\|u(t,.)\|_{W_2^1[0,l]} \leq M_1(\lambda,\mu,\alpha,\beta,\delta)$$
$$\times[\max_{0\leq t\leq T}\|f(t,.)\|_{L_2[0,l]} + \|\varphi\|_{W_2^1[0,l]} + \|\psi\|_{L_2[0,l]}], \quad (3.21)$$

$$\max_{0\leq t\leq T}\|u(t,.)\|_{W_2^2[0,l]} + \max_{0\leq t\leq T}\|u_{tt}(t,.)\|_{L_2[0,l]} \quad (3.22)$$
$$\leq M_1(\lambda,\mu,\alpha,\beta,\delta)[\max_{0\leq t\leq T}\|f_t(t,.)\|_{L_2[0,l]} + \|f(0,.)\|_{L_2[0,l]} + \|\varphi\|_{W_2^2[0,l]} + \|\psi\|_{W_2^1[0,l]}],$$

hold, where $M_1(\lambda,\mu,\alpha,\beta,\delta)$ does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$.

Proof. Problem (3.20) can be written in abstract form

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) + \beta u(t) = f(t)\ (0 \leq t \leq T), \\ \\ u(0) = \lambda u(T) + \varphi, u'(0) = \mu u_t(T) + \psi \end{cases} \quad (3.23)$$

in a Hilbert space $L_2[0,l]$ of all square integrable functions defined on $[0,l]$ with self-adjoint positive definite operator $A = A^x$ defined by formula (2.14). Here, $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions defined on $[0,l]$ with the values in $H = L_2[0,l]$. Therefore, estimates (3.21) and (3.22) follow from estimates (3.8), (3.9) and (3.10). Thus, Theorem 3.2 is proved. □

"Second, let $\Omega \subset R^n$ be a bounded open domain with smooth boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T]\times\Omega$, we consider the nonlocal boundary value problem for the telegraph equation"

$$\begin{cases} u_{tt}(t,x) + \alpha u_t(t,x) - \sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} + \beta u(t,x) = f(t,x), \\ \\ x = (x_1,...,x_n) \in \Omega, 0 < t < T, \\ \\ u(0,x) = \lambda u(T,x) + \varphi(x), \frac{\partial u(0,x)}{\partial t} = \mu\frac{\partial u(T,x)}{\partial t} + \psi(x), x \in \overline{\Omega}, \\ \\ u(t,x) = 0, x \in S, 0 \leq t \leq T, \end{cases} \quad (3.24)$$

where $\alpha_r(x)$, $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$, $\left(x \in \overline{\Omega}\right)$ and $f(t,x), (t \in (0,T))$, $x \in \Omega$ are given smooth functions and $\alpha_r(x) > 0$.

Theorem 3.3. *For the solution of the problem (3.24) the stability inequalities*

$$\max_{0 \le t \le T} \|u(t,.)\|_{W_2^1(\overline{\Omega})} \tag{3.25}$$

$$\le M_2(\lambda,\mu,\alpha,\beta,\delta)[\max_{0 \le t \le T} \|f(t,.)\|_{L_2(\overline{\Omega})} + \|\varphi\|_{W_2^1(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})}],$$

$$\max_{0 \le t \le T} \|u(t,.)\|_{W_2^2(\overline{\Omega})} + \max_{0 \le t \le T} \|u_{tt}(t,.)\|_{L_2(\overline{\Omega})} \tag{3.26}$$

$$\le M_2(\lambda,\mu,\alpha,\beta,\delta)[\max_{0 \le t \le T} \|f_t(t,.)\|_{L_2(\overline{\Omega})} + \|f(0,.)\|_{L_2(\overline{\Omega})} + \|\varphi\|_{W_2^2(\overline{\Omega})} + \|\psi\|_{W_2^1(\overline{\Omega})}],$$

hold, where $M_2(\lambda,\mu,\alpha,\beta,\delta)$ does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$.

Proof. Problem (3.24) can be written in abstract form (3.23) in Hilbert space $L_2(\overline{\Omega})$ with self-adjoint positive definite operator $A = A^x$ defined by formula (2.18). Here, $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions defined on $\overline{\Omega}$ with the values in $H = L_2(\overline{\Omega})$. So, estimates (3.25) and (3.26) follow from estimates (3.8), (3.9) and (3.10) and Theorem 2.4 on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$. □

In the next section, the first and second order of accuracy difference schemes for the approximate solution of problem (3.1) are studied. Stability estimates for the solution of these difference schemes are established. In applications, difference schemes for the approximate solution of two nonlocal boundary value problems (3.20) and (3.24) are presented. Stability estimates for the solution of these difference schemes are established.

3.1 STABLE TWO-STEP DIFFERENCE SCHEMES

First, we search the first order of accuracy difference scheme for approximately solving the nonlocal boundary value problem (3.1)

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_k}{\tau}+Au_{k+1}+\beta u_{k+1}=f_k, \\ f_k=f(t_{k+1}), t_{k+1}=(k+1)\tau, 1\le k\le N-1, N\tau=T, \\ u_0=\lambda u_N+\varphi, \\ (1+\alpha\tau)\frac{u_1-u_0}{\tau}+(A+\beta I)\tau u_1=\mu\frac{u_N-u_{N-1}}{\tau}+\psi. \end{cases} \tag{3.27}$$

We are interested to study the stability of solutions of the difference scheme (3.27) under the presentations (3.5). We have not been able to get the discrete analogue of estimates (3.8), (3.9), and (3.10) under the presumption (3.5) for the solution of the difference scheme (3.27). Afterall, we can established the separate similiar of estimates (3.8), (3.9), and (3.10) under the more powerful presumption than (3.5).

Theorem 3.4. *Let $\varphi\in D(A), \psi\in D(A^{1/2})$ and*

$$1>|\lambda|\,|\mu|\left[1+\frac{\frac{\alpha^2}{4}}{\delta+\left(\beta-\frac{\alpha^2}{4}\right)}\right]\frac{3}{4}\frac{1}{\left(1+\frac{\alpha\tau}{2}\right)^{2N}}$$
$$+|\lambda+\mu|\frac{1}{\left(1+\frac{\alpha\tau}{2}\right)^{N}}+\frac{\frac{\alpha}{2}|\lambda-\mu|}{\sqrt{\delta+\beta-\frac{\alpha^2}{4}}}\frac{1}{\left(1+\frac{\alpha\tau}{2}\right)^{N}}. \tag{3.28}$$

Then for the solution of the difference scheme (3.27) the stability inequalities

$$\max_{0\le k\le N}\|u_k\|_H$$
$$\le M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|A^{-1/2}\psi\right\|_H+\|\varphi\|_H\right\}, \tag{3.29}$$

$$\max_{0\le k\le N}\left\|A^{1/2}u_k\right\|_H$$
$$\le M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\psi\|_H+\left\|A^{1/2}\varphi\right\|_H\right\}, \tag{3.30}$$

$$\max_{1\leq k\leq N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H+\max_{0\leq k\leq N}\|Au_k\|_H$$

$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\left\|A^{1/2}\psi\right\|_H+\|A\varphi\|_H\right\}, \quad (3.31)$$

hold, where $M(\lambda,\mu,\alpha,\beta,\delta)$ *does not depend on* $f_s, 1\leq s\leq N-1,$ *and* ψ, φ.

Proof. We will write the formula for the solution of the difference scheme (3.27). It is easy to show that (see (Ashyralyev and Sobolevskii, 2001)) there are unique solution of the problem

$$\begin{cases}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_k}{\tau}+Au_{k+1}+\beta u_{k+1}=f_k,\\ f_k=f(t_{k+1}),1\leq k\leq N-1,N\tau=T,\\ u_0=\eta,(1+\alpha\tau)\frac{u_1-u_0}{\tau}+(A+\beta I)\tau u_1=w\end{cases} \quad (3.32)$$

and for the solution of these problems the following formulas hold:

$$\begin{cases}u_0=\eta,u_1=(1+\alpha\tau)R\widetilde{R}\eta+\tau R\widetilde{R}w,\\ u_k=R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}[R^{k-1}-\widetilde{R}^{k-1}]\eta\\ +R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\tau\left(\widetilde{R}^k-R^k\right)[(1+\alpha\tau)\eta+\tau w]\\ +\sum_{s=1}^{k-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}[\widetilde{R}^{k-s}-R^{k-s}]f_s,2\leq k\leq N,\end{cases} \quad (3.33)$$

where $R=\left(1+\frac{\alpha\tau}{2}-i\tau B^{1/2}\right)^{-1},\widetilde{R}=\left(1+\frac{\alpha\tau}{2}+i\tau B^{1/2}\right)^{-1}$. Applying nonlocal boundary conditions

$$u_0=\lambda u_N+\varphi,(1+\alpha\tau)\frac{u_1-u_0}{\tau}+(A+\beta I)\tau u_1=\mu\frac{u_N-u_{N-1}}{\tau}+\psi,$$

we can write

$$\begin{cases} \eta = \lambda \left[\left(R\widetilde{R} \left(\widetilde{R} - R \right)^{-1} \left(R^{N-1} - \widetilde{R}^{N-1} \right) \right. \right. \\ \left. + R\widetilde{R} \left(\widetilde{R} - R \right)^{-1} \left(\widetilde{R}^{N} - R^{N} \right) (1 + \alpha\tau) \right) \eta \\ + \left(\widetilde{R} - R \right)^{-1} \left(\widetilde{R}^{N} - R^{N} \right) \tau R\widetilde{R} w \\ \left. + \sum\limits_{s=1}^{N-1} R\widetilde{R} \left(\widetilde{R} - R \right)^{-1} \left[\widetilde{R}^{N-s} - R^{N-s} \right] \tau^2 f_s \right] + \varphi, \\ w = \frac{\mu}{\tau} \left\{ \left[R\widetilde{R} \left(\widetilde{R} - R \right)^{-1} \left(R^{N-1} - \widetilde{R}^{N-1} - R^{N-2} + \widetilde{R}^{N-2} \right) \right. \right. \\ \left. + \left(\widetilde{R} - R \right)^{-1} \left(\widetilde{R}^{N} - R^{N} + R^{N-1} - \widetilde{R}^{N-1} \right) (1 + \alpha\tau) R\widetilde{R} \right] \eta \\ + \left(\widetilde{R} - R \right)^{-1} \left(\widetilde{R}^{N} - R^{N} + R^{N-1} - \widetilde{R}^{N-1} \right) \tau R\widetilde{R} w \\ + \sum\limits_{s=1}^{N-1} R\widetilde{R} \left(\widetilde{R} - R \right)^{-1} \left[\widetilde{R}^{N-s} - R^{N-s} \right] \tau^2 f_s \\ \left. - \sum\limits_{s=1}^{N-2} R\widetilde{R} \left(\widetilde{R} - R \right)^{-1} \left[\widetilde{R}^{N-s-1} - R^{N-s-1} \right] \tau^2 f_s \right\} + \psi. \end{cases}$$

From that it follows that

$$\begin{cases} \left\{I - \lambda R\widetilde{R}\left(\widetilde{R} - R\right)^{-1}\left[\left(R^{N-1} - \widetilde{R}^{N-1}\right) + \left(\widetilde{R}^{N} - R^{N}\right)(1+\alpha\tau)\right]\right\}\eta \\ -\lambda\left(\widetilde{R} - R\right)^{-1}\left(\widetilde{R}^{N} - R^{N}\right)\tau R\widetilde{R}w \\ = \lambda\sum\limits_{s=1}^{N-1} R\widetilde{R}\left(\widetilde{R} - R\right)^{-1}\left[\widetilde{R}^{N-s} - R^{N-s}\right]\tau^2 f_s + \varphi, \\ -\frac{\mu}{\tau}R\widetilde{R}\left(\widetilde{R} - R\right)^{-1}\left\{\left(R^{N-1} - \widetilde{R}^{N-1} - R^{N-2} + \widetilde{R}^{N-2}\right)\right. \\ \left. + \left(\widetilde{R}^{N} - R^{N} + R^{N-1} - \widetilde{R}^{N-1}\right)(1+\alpha\tau)\right\}\eta \\ + \left\{I - \frac{\mu}{\tau}\tau R\widetilde{R}(1+\alpha\tau)\left(\widetilde{R} - R\right)^{-1}\left(\widetilde{R}^{N} - R^{N} + R^{N-1} - \widetilde{R}^{N-1}\right)\right\}w \\ = \mu\sum\limits_{s=1}^{N-1} R\widetilde{R}\left(\widetilde{R} - R\right)^{-1}\left[\widetilde{R}^{N-s} - R^{N-s} - \widetilde{R}^{N-s-1} + R^{N-s-1}\right]\tau f_s \\ +\mu R\widetilde{R}\tau f_{N-1} + \psi. \end{cases}$$

Since

$$\widetilde{R} - R = -2i\tau B^{1/2}R\widetilde{R},$$

$$\begin{aligned} &\left(R^{N-1} - \widetilde{R}^{N-1}\right) + \left(\widetilde{R}^{N} - R^{N}\right)(1+\alpha\tau) \\ &= \quad R^{N-1} - (1+\alpha\tau)R^{N} - \widetilde{R}^{N-1} + (1+\alpha\tau)\widetilde{R}^{N} \end{aligned}$$

$$= \left[1 + \frac{\alpha\tau}{2} - i\tau B^{1/2} - (1+\alpha\tau)\right]R^{N} + \left[-1 - \frac{\alpha\tau}{2} - i\tau B^{1/2} + (1+\alpha\tau)\right]\widetilde{R}^{N}$$

$$= -\tau\left[\frac{\alpha}{2} + iB^{1/2}\right]R^{N} + \tau\left[\frac{\alpha}{2} - iB^{1/2}\right]\widetilde{R}^{N},$$

$$\left(R^{N-1} - \widetilde{R}^{N-1} - R^{N-2} + \widetilde{R}^{N-2}\right) + \left(\widetilde{R}^{N} - R^{N} + R^{N-1} - \widetilde{R}^{N-1}\right)(1+\alpha\tau)$$

$$= -R^{N-2}(I-R) + (1+\alpha\tau)R^{N-1}(I-R) + \widetilde{R}^{N-2}\left(I - \widetilde{R}\right) - (1+\alpha\tau)\widetilde{R}^{N-1}\left(I - \widetilde{R}\right)$$

$$\begin{aligned} = \quad &\left[-1 - \frac{\alpha\tau}{2} + i\tau B^{1/2} + (1+\alpha\tau)\right]R^{N-1}(I-R) \\ &+ \left[1 + \frac{\alpha\tau}{2} + i\tau B^{1/2} - (1+\alpha\tau)\right]\widetilde{R}^{N-1}\left(I - \widetilde{R}\right) \end{aligned}$$

$$= \tau\left[\frac{\alpha}{2} + iB^{1/2}\right]R^{N-1}(I-R) + \tau\left[-\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R}^{N-1}\left(I - \widetilde{R}\right),$$

$$\widetilde{R}^N - R^N + R^{N-1} - \widetilde{R}^{N-1} = R^{N-1}(I-R) - \widetilde{R}^{N-1}\left(I-\widetilde{R}\right),$$
$$I - R = \tau\left[\frac{\alpha}{2} - iB^{1/2}\right]R, I - \widetilde{R} = \tau\left[\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R},$$

we have that

$$\begin{cases} \left\{I - \lambda\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} + iB^{1/2}\right]R^N - \left[\frac{\alpha}{2} - iB^{1/2}\right]\widetilde{R}^N\right]\right\}\eta \\ +\lambda\left(2iB^{1/2}\right)^{-1}\left(\widetilde{R}^N - R^N\right)w \\ = -\lambda\sum\limits_{s=1}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\widetilde{R}^{N-s} - R^{N-s}\right]\tau f_s + \varphi, \\ \mu\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} + iB^{1/2}\right]\left[\frac{\alpha}{2} - iB^{1/2}\right]R^N\right. \\ \left. + \left[-\frac{\alpha}{2} + iB^{1/2}\right]\left[\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R}^N\right]\eta \\ + \left\{I - \mu\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} - iB^{1/2}\right]R^N - \left[\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R}^N\right]\right\}w \\ = \mu\sum\limits_{s=1}^{N-1}\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} - iB^{1/2}\right]R^{N-s} - \left[\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R}^{N-s}\right]\tau f_s \\ +\mu R\widetilde{R}\tau f_{N-1} + \psi. \end{cases} \tag{3.34}$$

For solving this system, we will consider the following operator

$$\Delta = \left\{I - \lambda\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} + iB^{1/2}\right]R^N - \left[\frac{\alpha}{2} - iB^{1/2}\right]\widetilde{R}^N\right]\right\}$$
$$\times\left\{I - \mu\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} - iB^{1/2}\right]R^N - \left[\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R}^N\right]\right\}$$

$$-\lambda\left(2iB^{1/2}\right)^{-1}\left(\widetilde{R}^N - R^N\right)\mu\left(2iB^{1/2}\right)^{-1}$$
$$\times\left[\left[\frac{\alpha}{2} + iB^{1/2}\right]\left[\frac{\alpha}{2} - iB^{1/2}\right]R^N + \left[-\frac{\alpha}{2} + iB^{1/2}\right]\left[\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R}^N\right]$$
$$= I - \lambda\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} + iB^{1/2}\right]R^N - \left[\frac{\alpha}{2} - iB^{1/2}\right]\widetilde{R}^N\right]$$
$$-\mu\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} - iB^{1/2}\right]R^N - \left[\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R}^N\right]$$
$$+\lambda\mu\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} + iB^{1/2}\right]R^N - \left[\frac{\alpha}{2} - iB^{1/2}\right]\widetilde{R}^N\right]$$
$$\times\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2} - iB^{1/2}\right]R^N - \left[\frac{\alpha}{2} + iB^{1/2}\right]\widetilde{R}^N\right]$$

$$-\lambda\mu\left(2iB^{1/2}\right)^{-1}\left(\widetilde{R}^N-R^N\right)\left(2iB^{1/2}\right)^{-1}$$

$$\times\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]\left[\frac{\alpha}{2}-iB^{1/2}\right]R^N+\left[-\frac{\alpha}{2}+iB^{1/2}\right]\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^N\right]$$

$$=I-(\lambda+\mu)(2)^{-1}\left[R^N+\widetilde{R}^N\right]$$

$$-(\lambda-\mu)\frac{\alpha}{4}\left(iB^{1/2}\right)^{-1}\left[R^N-\widetilde{R}^N\right]$$

$$+\lambda\mu(4B)^{-1}\left\{\left[\frac{\alpha^2}{4}+B\right]\left(\widetilde{R}^{2N}-R^{2N}\right)-\left[\frac{\alpha^2}{4}-B\right]R^N\widetilde{R}^N\right\}.$$

Using estimates

$$\begin{cases}\|R\|_{H\to H}\leq\frac{1}{1+\frac{\alpha\tau}{2}},\ \left\|\widetilde{R}\right\|_{H\to H}\leq\frac{1}{1+\frac{\alpha\tau}{2}},\\ \left\|\tau B^{\frac{1}{2}}R\right\|_{H\to H}\leq 1,\ \left\|\tau B^{\frac{1}{2}}\widetilde{R}\right\|_{H\to H}\leq 1,\\ \left\|\left(\pm\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\leq 1+\frac{\alpha}{2\sqrt{\delta+\left(\beta-\frac{\alpha^2}{4}\right)}},\\ \left\|\left[\frac{\alpha^2}{4}\pm B\right]B^{-1}\right\|_{H\to H}\leq 1+\frac{\frac{\alpha^2}{4}}{\delta+\left(\beta-\frac{\alpha^2}{4}\right)}\end{cases}\tag{3.35}$$

we get

$$\left\|(\lambda+\mu)(2)^{-1}\left[R^N+\widetilde{R}^N\right]\right\|_{H\to H}\leq|\lambda+\mu|\frac{1}{\left(1+\frac{\alpha\tau}{2}\right)^N},$$

$$\left\|(\lambda-\mu)\frac{\alpha}{4}\left(iB^{1/2}\right)^{-1}\left[R^N-\widetilde{R}^N\right]\right\|_{H\to H}\leq|\lambda-\mu|\frac{\alpha}{2\sqrt{\delta+\left(\beta-\frac{\alpha^2}{4}\right)}}\frac{1}{\left(1+\frac{\alpha\tau}{2}\right)^N},$$

$$\left\|\lambda\mu(4B)^{-1}\left\{\left[\frac{\alpha^2}{4}+B\right]\left(\widetilde{R}^{2N}-R^{2N}\right)-\left[\frac{\alpha^2}{4}-B\right]R^N\widetilde{R}^N\right\}\right\|_{H\to H}$$

$$\leq\ |\lambda||\mu|\left[1+\frac{\frac{\alpha^2}{4}}{\delta+\left(\beta-\frac{\alpha^2}{4}\right)}\right]\frac{3}{4}\frac{1}{\left(1+\frac{\alpha\tau}{2}\right)^{2N}}.$$

Therefore, under the assumption (3.28) there exists of inverse of operator Δ

$$T_\tau=\Delta^{-1}$$

$$=\left\{I-(\lambda+\mu)(2)^{-1}\left[R^N+\widetilde{R}^N\right]-(\lambda-\mu)\frac{\alpha}{4}\left(iB^{1/2}\right)^{-1}\left[R^N-\widetilde{R}^N\right]\right.$$

$$\left.+\lambda\mu(4B)^{-1}\left\{\left[\frac{\alpha^2}{4}+B\right]\left(\widetilde{R}^{2N}-R^{2N}\right)-\left[\frac{\alpha^2}{4}-B\right]R^N\widetilde{R}^N\right\}\right\}^{-1}$$

and the following estimate holds

$$\|T_\tau\|_{H\to H} \le \left\{1 - |\lambda|\,|\mu| \left[1 + \frac{\frac{\alpha^2}{4}}{\delta + \left(\beta - \frac{\alpha^2}{4}\right)}\right] \frac{3}{4} \frac{1}{\left(1 + \frac{\alpha\tau}{2}\right)^{2N}} \right. \tag{3.36}$$

$$\left. + |\lambda + \mu| \frac{1}{\left(1 + \frac{\alpha\tau}{2}\right)^N} + \frac{\frac{\alpha}{2}|\lambda - \mu|}{\sqrt{\delta + \beta - \frac{\alpha^2}{4}}} \frac{1}{\left(1 + \frac{\alpha\tau}{2}\right)^N} \right\}^{-1}.$$

Using of operator T_τ, solving (3.34), we obtain

$$\eta = T_\tau \left[\left\{ I - \mu \left(-2iB^{1/2}\right)^{-1} \left[\left[\frac{\alpha}{2} - iB^{1/2}\right] R^N - \left[\frac{\alpha}{2} + iB^{1/2}\right] \widetilde{R}^N \right] \right\} \right.$$

$$\times \left[-\lambda \sum_{s=1}^{N-1} \left(2iB^{1/2}\right)^{-1} \left[\widetilde{R}^{N-s} - R^{N-s}\right] \tau f_s + \varphi \right] - \lambda \left(2iB^{1/2}\right)^{-1} \left(\widetilde{R}^N - R^N\right)$$

$$\times \left[\mu \sum_{s=1}^{N-1} \left(-2iB^{1/2}\right)^{-1} \left[\left[\frac{\alpha}{2} - iB^{1/2}\right] R^{N-s} \right.\right.$$

$$\left.\left.\left. - \left[\frac{\alpha}{2} + iB^{1/2}\right] \widetilde{R}^{N-s} \right] \tau f_s + \mu R \widetilde{R} \tau f_{N-1} + \psi \right] \right], \tag{3.37}$$

$$w = T_\tau \left[\left\{ I - \lambda \left(2iB^{1/2}\right)^{-1} \left[\left[\frac{\alpha}{2} + iB^{1/2}\right] R^N - \left[\frac{\alpha}{2} - iB^{1/2}\right] \widetilde{R}^N \right] \right\} \right.$$

$$\times \left[\mu \sum_{s=1}^{N-1} \left(-2iB^{1/2}\right)^{-1} \left[\left[\frac{\alpha}{2} - iB^{1/2}\right] R^{N-s} \right.\right.$$

$$\left.\left. - \left[\frac{\alpha}{2} + iB^{1/2}\right] \widetilde{R}^{N-s} \right] \tau f_s + \mu R \widetilde{R} \tau f_{N-1} + \psi \right] - \mu \left(2iB^{1/2}\right)^{-1}$$

$$\times \left[\left[\frac{\alpha}{2} + iB^{1/2}\right] \left[\frac{\alpha}{2} - iB^{1/2}\right] R^N + \left[-\frac{\alpha}{2} + iB^{1/2}\right] \left[\frac{\alpha}{2} + iB^{1/2}\right] \widetilde{R}^N \right]$$

$$\left. \times \left[-\lambda \sum_{s=1}^{N-1} \left(2iB^{1/2}\right)^{-1} \left[\widetilde{R}^{N-s} - R^{N-s}\right] \tau f_s + \varphi \right] \right]. \tag{3.38}$$

Hence, for the formal solution of the nonlocal boundary value problem (3.27) can be use the formulas (3.33), (3.37), and (3.38). For substantiation of these formulas can be need to obtain the estimates (3.29), (3.30), and (3.31). Using formulas (3.37), (3.38) and estimates (3.36), (3.35), we obtain

$$\|\eta\|_H \le \|T_\tau\|_{H\to H}$$

$$\times \left[\left\| \left\{ I - \mu \left(-2iB^{1/2}\right)^{-1} \left[\left[\frac{\alpha}{2} - iB^{1/2}\right] R^N - \left[\frac{\alpha}{2} + iB^{1/2}\right] \widetilde{R}^N \right] \right\} \right\|_{H\to H} \right.$$

$$\times \left\| -\lambda \sum_{s=1}^{N-1} \left(2iB^{1/2}\right)^{-1} \left[\widetilde{R}^{N-s} - R^{N-s}\right] \tau f_s + \varphi \right\|_H + \left\| -\lambda (2)^{-1} \left(\widetilde{R}^N - R^N\right) \right\|_{H\to H}$$

$$\times \left\| \left(iB^{1/2}\right)^{-1} \left[\mu \sum_{s=1}^{N-1} \left(-2iB^{1/2}\right)^{-1} \left[\left[\frac{\alpha}{2} - iB^{1/2}\right] R^{N-s} \right] \right. \right.$$

$$
\begin{aligned}
&-\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^{N-s}\Big]\tau f_s+\mu R\widetilde{R}\tau f_{N-1}+\psi\Big]\Big\|_H \le \|T_\tau\|_{H\to H}\\
&\times\Bigg\{1+\frac{1}{2}|\mu|\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\|R^N\|_{H\to H}\right.\\
&\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\|R^{N-1}\|_{H\to H}\right]\\
&\times\Bigg[|\lambda|\,\|B^{-1/2}A^{1/2}\|_{H\to H}\sum_{s=1}^{N-1}\frac{1}{2}\left[\|R^{N-s}\|_{H\to H}+\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right]\\
&\times\left\|A^{-1/2}f_s\right\|_H\tau+\|\varphi\|_H\Bigg]\\
&+\frac{1}{2}|\lambda|\left(\|R^N\|_{H\to H}+\left\|\widetilde{R}^N\right\|_{H\to H}\right)\|A^{1/2}B^{-1/2}\|_{H\to H}\\
&\times\Bigg(|\mu|\sum_{s=1}^{N-1}\frac{1}{2}\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\|R^{N-s}\|_{H\to H}\right.\\
&\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right]\left\|A^{-1/2}f_s\right\|_H\tau\\
&+|\mu|\left\|\tau R\widetilde{R}\right\|_{H\to H}\|f_{N-1}\|_H+\left\|A^{-1/2}\psi\right\|_H\Bigg)\Bigg\}\\
&\le M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|A^{-1/2}\psi\right\|_H+\|\varphi\|_H\right\},
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
&\left\|A^{-1/2}w\right\|_H\le\|T_\tau\|_{H\to H}\\
&\times\Bigg[\Bigg\{1+\frac{1}{2}|\lambda|\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\|R^N\|_{H\to H}\right.\\
&\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^N\right\|_{H\to H}\right]\\
&\times\Bigg(|\mu|\sum_{s=1}^{N-1}\frac{1}{2}\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\|R^{N-s}\|_{H\to H}\right.\\
&\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right]\left\|A^{-1/2}f_s\right\|_H\tau\\
&+|\mu|\left\|\tau R\widetilde{R}\right\|_{H\to H}\|f_{N-1}\|_H+\left\|A^{-1/2}\psi\right\|_H\Bigg)\Bigg\}\\
&+\frac{1}{2}|\mu|\left\|\left[\frac{\alpha^2}{4}+B\right]B^{-1}\right\|_{H\to H}\left[\|R^N\|_{H\to H}+\left\|\widetilde{R}^N\right\|_{H\to H}\right]\\
&\times\Bigg[|\lambda|\sum_{s=1}^{N-1}\frac{1}{2}\left(\|R^{N-s}\|_{H\to H}+\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right)\left\|A^{-1/2}f_s\right\|_H\\
&+\|B^{1/2}A^{-1/2}\|_{H\to H}\|\varphi\|_H\Bigg]|\lambda|\sum_{s=1}^{N-1}\frac{1}{2}\left\|A^{-1/2}f_s\right\|_H\tau\\
&\times\|A^{1/2}B^{-1/2}\|_{H\to H}\left(\|R^{N-s}\|_{H\to H}+\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right)+\|\varphi\|_H\Big\}
\end{aligned}
$$

$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|A^{-1/2}\psi\right\|_H+\|\varphi\|_H\right\}. \tag{3.40}$$

Applying $A^{1/2}$ to the formulas (3.37), (3.38) and using the estimates (3.36), (3.35) in a similiar manner, we obtain

$$\left\|A^{1/2}\eta\right\|_H\leq\|T_\tau\|_{H\to H}$$

$$\times\left[\left\|\left\{I-\mu\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]R^N-\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^N\right]\right\}\right\|_{H\to H}\right.$$

$$\times\left\|-\lambda\sum_{s=1}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\widetilde{R}^{N-s}-R^{N-s}\right]\tau A^{1/2}f_s+A^{1/2}\varphi\right\|_H$$

$$+\left\|-\lambda(2)^{-1}\left(\widetilde{R}^N-R^N\right)\right\|_{H\to H}\left\|A^{1/2}\left(iB^{1/2}\right)^{-1}\left[\mu\sum_{s=1}^{N-1}\left(-2iB^{1/2}\right)^{-1}\right.\right.$$

$$\left.\left.\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^{N-s}\right]\tau f_s+\mu R\widetilde{R}\tau f_{N-1}+\psi\right]\right\|_H\right]$$

$$\leq\|T_\tau\|_{H\to H}\left\{1+\frac{1}{2}|\mu|\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^N\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}\right]$$

$$\times\left[|\lambda|\left\|B^{-1/2}A^{1/2}\right\|_{H\to H}\sum_{s=1}^{N-1}\frac{1}{2}\left[\left\|R^{N-s}\right\|_{H\to H}+\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right]\|f_s\|_H\tau\right.$$

$$\left.+\left\|A^{1/2}\varphi\right\|_H\right]+\frac{1}{2}|\lambda|\left(\left\|R^N\right\|_{H\to H}+\left\|\widetilde{R}^N\right\|_{H\to H}\right)\left\|A^{1/2}B^{-1/2}\right\|_{H\to H}$$

$$\times\left(|\mu|\sum_{s=1}^{N-1}\frac{1}{2}\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^{N-s}\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right]\|f_s\|_H\tau$$

$$\left.\left.+|\mu|\left\|A^{1/2}\tau R\widetilde{R}\right\|_{H\to H}\|f_{N-1}\|_H+\|\psi\|_H\right)\right\}$$

$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\psi\|_H+\left\|A^{1/2}\varphi\right\|_H\right\}, \tag{3.41}$$

$$\|w\|_H\leq\|T_\tau\|_{H\to H}$$

$$\times\left[\left\{1+\frac{1}{2}|\lambda|\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^N\right\|_{H\to H}\right.\right.\right.$$

$$\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^N\right\|_{H\to H}\right]$$

$$\times\left(|\mu|\sum_{s=1}^{N-1}\frac{1}{2}\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^{N-s}\right\|_{H\to H}\right.\right.$$

$$+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right]\|f_s\|_H\,\tau$$

$$+|\mu|\left\|\tau R\widetilde{R}\right\|_{H\to H}\|f_{N-1}\|_H+\|\psi\|_H\Big)\Big\}$$

$$+\frac{1}{2}|\mu|\left\|\left[\frac{\alpha^2}{4}+B\right]B^{-1}\right\|_{H\to H}\left[\|R^N\|_{H\to H}+\left\|\widetilde{R}^N\right\|_{H\to H}\right]$$

$$\times\left(|\lambda|\sum_{s=1}^{N-1}\frac{1}{2}\left(\|R^{N-s}\|_{H\to H}+\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right)\|f_s\|_H+\|B^{1/2}A^{-1/2}\|_{H\to H}\|A^{1/2}\varphi\|_H\right)$$

$$\times|\lambda|\sum_{s=1}^{N-1}\frac{1}{2}\|f_s\|_H\,\tau\,\|A^{1/2}B^{-1/2}\|_{H\to H}\left(\|R^{N-s}\|_{H\to H}+\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right)+\|A^{1/2}\varphi\|_H\Bigg\}$$

$$\le M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=1}^{N-1}\|f_s\|_H\,\tau+\|\psi\|_H+\left\|A^{1/2}\varphi\right\|_H\right\}. \tag{3.42}$$

Now, we obtain the estimates for $\|A\eta\|_H$, $\left\|A^{1/2}w\right\|_H$. Using formulas

$$I-R=\tau\left[\frac{\alpha}{2}-iB^{1/2}\right]R,\ I-\widetilde{R}=\tau\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R},$$

the Abel's formula, we can write

$$\sum_{s=1}^{N-1}\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^{N-s}\right]\tau f_s$$

$$=\sum_{s=1}^{N-1}\left(-2iB^{1/2}\right)^{-1}\left[(I-R)R^{N-s-1}-\left(I-\widetilde{R}\right)\widetilde{R}^{N-s-1}\right]f_s$$

$$=\sum_{s=1}^{N-1}\left(-2iB^{1/2}\right)^{-1}\left[(I-R)R^{N-s-1}-\left(I-\widetilde{R}\right)\widetilde{R}^{N-s-1}\right]f_s$$

$$=\sum_{s=2}^{N-1}\left(-2iB^{1/2}\right)^{-1}\left[R^{N-s}-\widetilde{R}^{N-s}\right](f_{s-1}-f_s)+\left(-2iB^{1/2}\right)^{-1}\times\left[\widetilde{R}^{N-1}-R^{N-1}\right]f_1, \tag{3.43}$$

$$\sum_{s=1}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\widetilde{R}^{N-s}-R^{N-s}\right]\tau f_s$$

$$=-\sum_{s=1}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}\right.$$

$$\left.\left[\frac{\alpha}{2}-iB^{1/2}\right]R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^{N-s}\right]\tau f_s$$

$$=-\sum_{s=1}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}(I-R)R^{N-s-1}\right.$$

$$-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\left(I-\widetilde{R}\right)\widetilde{R}^{N-s-1}\Bigg]f_s$$

$$=-\sum_{s=2}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-s}\right](f_{s-1}-f_s)$$

$$-\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\right]f_{N-1}$$

$$-\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-1}-R^{N-1}\right]\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}f_1. \tag{3.44}$$

Applying A to the formula (3.37), and applying $A^{\frac{1}{2}}$ to the formula (3.38) and using formulas (3.43), (3.44), we can write

$$A\eta=T_\tau\Bigg[\Bigg\{I-\mu\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]R^N-\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^N\right]\Bigg\}$$

$$\times A\Bigg[\lambda\sum_{s=2}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-s}\right](f_{s-1}-f_s)$$

$$-\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\right]f_{N-1}$$

$$-\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-1}-R^{N-1}\right]\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}f_1+\varphi\Bigg]$$

$$-A\lambda\left(2iB^{1/2}\right)^{-1}\left(\widetilde{R}^N-R^N\right)$$

$$\times\Bigg[\mu\sum_{s=2}^{N-1}\left(-2iB^{1/2}\right)^{-1}\left[R^{N-s}-\widetilde{R}^{N-s}\right](f_{s-1}-f_s)+\mu\left(-2iB^{1/2}\right)^{-1} \tag{3.45}$$

$$\times\left[\widetilde{R}^{N-1}-R^{N-1}\right]f_1+\mu R\widetilde{R}\tau f_{N-1}+\psi\Bigg]\Bigg],$$

$$A^{\frac{1}{2}}w=T_\tau\Bigg[\Bigg\{I-\lambda\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]R^N-\left[\frac{\alpha}{2}-iB^{1/2}\right]\widetilde{R}^N\right]\Bigg\}$$

$$\times A^{\frac{1}{2}}\Bigg[\mu\sum_{s=2}^{N-1}\left(-2iB^{1/2}\right)^{-1}\left[R^{N-s}-\widetilde{R}^{N-s}\right](f_{s-1}-f_s)+\mu\left(-2iB^{1/2}\right)^{-1}$$

$$\left[\widetilde{R}^{N-1}-R^{N-1}\right]f_1+\mu R\widetilde{R}\tau f_{N-1}+\psi-\mu\left(2iB^{1/2}\right)^{-1}$$

$$\times\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]\left[\frac{\alpha}{2}-iB^{1/2}\right]R^N+\left[-\frac{\alpha}{2}+iB^{1/2}\right]\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^N\right]$$

$$\times A^{\frac{1}{2}}\Bigg[\lambda\sum_{s=2}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-s}\right](f_{s-1}-f_s)$$

$$-\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\right]f_{N-1}$$

$$-\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-1}-R^{N-1}\right]\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}f_1+\varphi\right]\right]. \qquad (3.46)$$

Using the last two formulas and the estimates (3.41) and (3.42), we obtain

$$\|A\eta\|_H \le \|T_\tau\|_{H\to H}$$

$$\times\left[\left\|\left\{I-\mu\left(-2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]R^N-\left[\frac{\alpha}{2}+iB^{1/2}\right]\widetilde{R}^N\right]\right\}\right\|_{H\to H}\right.$$

$$\times|\lambda|\left\|A\left[\lambda\sum_{s=2}^{N-1}\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-s}\right]\right.\right.$$

$$\times\left(f_{s-1}-f_s\right)-A\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\right]f_{N-1}$$

$$\left.\left.-A\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-1}-R^{N-1}\right]\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}f_1+A\varphi\right]\right\|_H$$

$$+\left\|-\lambda\,(2)^{-1}\left(\widetilde{R}^N-R^N\right)\right\|_{H\to H}\times A\left(iB^{1/2}\right)^{-1}$$

$$\times\left[\mu\sum_{s=2}^{N-1}\left(-2iB^{1/2}\right)^{-1}\left[R^{N-s}-\widetilde{R}^{N-s}\right]\left(f_{s-1}-f_s\right)+\mu\left(-2iB^{1/2}\right)^{-1}\right.$$

$$\left.\left.\left[\widetilde{R}^{N-1}-R^{N-1}\right]f_1+\mu R\widetilde{R}\tau f_{N-1}+\psi\right]\right\|_H\right]$$

$$\le\|T_\tau\|_{H\to H}\left\{1+\frac{1}{2}|\mu|\left[\left\|\frac{\alpha}{2}-iB^{1/2}\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^N\right\|_{H\to H}\right.\right.$$

$$\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^{N-1}\right\|_{H\to H}\right]$$

$$\times|\lambda|\left[\sum_{s=2}^{N-1}\left\|A\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-s}\right]\right\|_{H\to H}\right.$$

$$\|f_{s-1}-f_s\|_H+\left\|A\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\right]\right\|_{H\to H}\|f_{N-1}\|_H$$

$$+\left\|-A\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-1}-R^{N-1}\right]\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}\right\|_{H\to H}$$

$$\left.\times\ \|f_1\|_H\right]+\|A\varphi\|_H\Big]$$

$$+\frac{1}{2}|\lambda|\left(\left\|R^N\right\|_{H\to H}+\left\|\widetilde{R}^N\right\|_{H\to H}\right)\left\|A^{1/2}B^{-1/2}\right\|_{H\to H}$$

$$\times\left[|\mu|\sum_{s=2}^{N-1}\left\|A^{1/2}\left(-2iB^{1/2}\right)^{-1}\left[R^{N-s}-\widetilde{R}^{N-s}\right]\right\|_{H\to H}\|f_{s-1}-f_s\|_H\right.$$

$$+|\mu|\left\|A^{1/2}\left(-2iB^{1/2}\right)^{-1}\left[\widetilde{R}^{N-1}-R^{N-1}\right]\right\|_{H\to H}\|f_1\|_H$$

$$\left.\left.+|\mu|\left\|A^{1/2}\tau R\widetilde{R}\right\|_{H\to H}\|f_{N-1}\|_H+\left\|A^{1/2}\psi\right\|_H\right]\right\}$$

$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\left\|A^{1/2}\psi\right\|_H+\|A\varphi\|_H\right\}, \quad (3.47)$$

$$\left\|A^{1/2}w\right\|_H\leq\|T_\tau\|_{H\to H}$$

$$\times\left[\left\{1+\frac{1}{2}|\lambda|\left[\left\|\left[\frac{\alpha}{2}-iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^N\right\|_{H\to H}\right.\right.\right.$$
$$\left.+\left\|\left[\frac{\alpha}{2}+iB^{1/2}\right]\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^N\right\|_{H\to H}\right]$$
$$\times\left(|\mu|\sum_{s=1}^{N-1}\frac{1}{2}\left[\left\|A^{\frac{1}{2}}\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{N-s}\right\|_{H\to H}+\left\|\widetilde{R}^{N-s}\right\|_{H\to H}\right]\right]\|f_{s-1}-f_s\|_H\right.$$
$$+|\mu|\left\|A^{\frac{1}{2}}\left(-2iB^{1/2}\right)^{-1}\right\|_{H\to H}\left[\left\|R^{N-1}\right\|_{H\to H}\right.$$
$$\left.\left.\left.+\left\|\widetilde{R}^{N-1}\right\|_{H\to H}\right]\|f_1\|_H+|\mu|\left\|A^{\frac{1}{2}}\tau R\widetilde{R}\right\|_{H\to H}\|f_{N-1}\|_H+\left\|A^{\frac{1}{2}}\psi\right\|_H\right)\right\}$$
$$+\frac{1}{2}|\mu|\left\|\left[\frac{\alpha^2}{4}+B\right]B^{-1}\right\|_{H\to H}\left[\left\|R^N\right\|_{H\to H}+\left\|\widetilde{R}^N\right\|_{H\to H}\right]$$
$$\times\left[|\lambda|\sum_{s=2}^{N-1}\left\|A^{\frac{1}{2}}\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}R^{N-s}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-s}\right]\right\|_{H\to H}\right.$$
$$\|f_{s-1}-f_s\|_H+\left\|A^{\frac{1}{2}}\left[\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}-\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\right]\right\|_{H\to H}\|f_{N-1}\|_H$$
$$+\left\|A^{\frac{1}{2}}\left(2iB^{1/2}\right)^{-1}\left[\left[\frac{\alpha}{2}+iB^{1/2}\right]^{-1}\widetilde{R}^{N-1}-R^{N-1}\right]\left[\frac{\alpha}{2}-iB^{1/2}\right]^{-1}\right\|_{H\to H}$$
$$\left.\times\|f_1\|_H\right]+\|A\varphi\|_H]$$
$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\left\|A^{1/2}\psi\right\|_H+\|A\varphi\|_H\right\}. \quad (3.48)$$

Now, we will prove the estimates for (3.29), (3.30), and (3.31). We have that

$$\|u_1\|_H\leq(1+\alpha\tau)\left\|R\widetilde{R}\right\|_{H\to H}\|\eta\|_H$$
$$+\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|\tau B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\left\|A^{-1/2}w\right\|_H$$
$$\leq M(\alpha,\delta)\left[\|A^{-1/2}w\|_H+\|\eta\|_H\right],$$

In exactly the same manner, one establishes

$$\left\|A^{\frac{1}{2}}u_1\right\|_H\leq(1+\alpha\tau)\left\|R\widetilde{R}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\eta\right\|_H$$
$$+\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|\tau B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\|w\|_H$$
$$\leq M(\alpha,\delta)\left[\|w\|_H+\|A^{\frac{1}{2}}\eta\|_H\right],$$

$$\|Au_1\|_H \le (1+\alpha\tau)\left\|R\widetilde{R}\right\|_{H\to H}\|A\varphi\|_H$$
$$+\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|\tau B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\|\psi\|_H$$
$$\le M(\alpha,\delta)\left[\| A^{\frac{1}{2}}w \|_H + \| A\eta \|_H\right].$$

Let $k\ge 2$. Then, using the formula (3.33) and estimates (3.39), (3.40),(3.41) and (3.42), we obtain that

$$\|u_k\|_H \le \frac{1}{2}\left[\left\|\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H}\right.$$
$$\left.+\left\|R^k\right\|_{H\to H}\left\|\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\right]\|\eta\|_H$$
$$+\frac{1}{2}\left[\left\|\widetilde{R}^k\right\|_{H\to H}+\left\|R^k\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}w\right\|_H$$
$$+\sum_{s=1}^{k-1}\frac{1}{2}\left[\left\|\widetilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right]\tau\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_s\right\|_H$$
$$\le M_3(\lambda,\mu,\alpha,\beta,\delta)\left[\|\eta\|_H+\left\|A^{-\frac{1}{2}}w\right\|_H+\max_{1\le k\le N-1}\left\|A^{-\frac{1}{2}}f_k\right\|_H\right],$$

for any $k\ge 2$. Combining the estimates $\|u_k\|_H$ for any k, we obtain (3.29). Using the triangle inequality, formula (3.33) and estimates (3.39), (3.40),(3.41) and (3.42), we obtain that

$$\left\|A^{\frac{1}{2}}u_k\right\|_H \le \frac{1}{2}\left[\left\|\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H}\right.$$
$$\left.+\left\|R^k\right\|_{H\to H}\left\|\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}\eta\right\|_H$$
$$+\frac{1}{2}\left[\left\|\widetilde{R}^k\right\|_{H\to H}+\left\|R^k\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\|w\|_H$$
$$+\sum_{s=1}^{k-1}\frac{1}{2}\left[\left\|\widetilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right]\tau\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\|f_s\|_H$$
$$\le M_3(\lambda,\mu,\alpha,\beta,\delta)\left[\left\|A^{\frac{1}{2}}\eta\right\|_H+\|w\|_H+\max_{1\le k\le N-1}\|f_k\|_H\right]$$

for any $k\ge 2$. Combining the estimates $\left\|A^{\frac{1}{2}}u_k\right\|_H$ for any k, we obtain (3.41). Using the Abel's formula, we can write

$$Au_k=\left(2iB^{\frac{1}{2}}\right)^{-1}\left[\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^k+\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\widetilde{R}^k\right]A\eta+A\left(-2iB^{\frac{1}{2}}\right)^{-1}(\widetilde{R}^k-R^k)w$$

$$+f_{k-1}-\left(-2iB^{\frac{1}{2}}\right)^{-1}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-1}\right\}f_1+\left(-2iB^{\frac{1}{2}}\right)^{-1}$$

$$\times\sum_{s=1}^{k-2}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-s-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-s-1}\right\}(f_s-f_{s+1}),2\leq k\leq N.$$

Using the triangle inequality, last formula, and estimates (3.39), (3.40),(3.41) and (3.42), we obtain that

$$\|Au_k\|_H\leq\frac{1}{2}\left[\left\|\left(-\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H}\right.$$

$$\left.+\left\|R^k\right\|_{H\to H}\left\|\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\left(iB^{\frac{1}{2}}\right)^{-1}\right\|_{H\to H}\right]\|A\eta\|_H$$

$$+\frac{1}{2}\left[\left\|\widetilde{R}^k\right\|_{H\to H}+\left\|R^k\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{\frac{1}{2}}w\right\|_H$$

$$+\|f_{k-1}\|_H+\frac{1}{2}\left[\frac{\alpha}{2}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}+1\right]\left\{\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\left\|R^{k-1}\right\|_{H\to H}\right\}\|f_1\|_H$$

$$\left.+\frac{1}{2}\left[\frac{\alpha}{2}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}+1\right]\sum_{s=1}^{k-2}\left\{\left\|\widetilde{R}^{k-s-1}\right\|_{H\to H}+\left\|R^{k-s-1}\right\|_{H\to H}\right\}\|f_s-f_{s+1}\|_H\right]$$

$$\leq M_4(\lambda,\mu,\alpha,\beta,\delta)\left[\|A\eta\|_H+\|\eta\|_H+\left\|A^{\frac{1}{2}}w\right\|_H\right.$$

$$\left.+\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H\right]$$

for any $k\geq 2$. Combining the estimates $\|Au_k\|_H$ for any k, we obtain (3.42). Theorem 3.4 is proved. □

Second, we consider the second order accuracy difference schemes for approximately solving the boundary value problem (3.1)

$$\begin{cases}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_{k-1}}{2\tau}+\frac{A}{2}(u_{k+1}+u_{k-1})\\[2ex]+\frac{\beta}{2}\left(u_{k+1}+u_{k-1}\right)=f_k,f_k=f(t_k),1\leq k\leq N-1,\\[2ex]u_0=\lambda u_N+\varphi,f_0=f(0),f_N=f(1),\\[2ex]\left(I+\frac{\alpha\tau}{2}+\frac{\tau^2}{2}(B+\frac{\alpha^2}{4})\right)\frac{u_1-u_0}{\tau}-\frac{\tau}{2}\left(f_0-(B+\frac{\alpha^2}{4})u_0-\alpha\frac{u_1-u_0}{\tau}\right)\\[2ex]=\mu[(\frac{u_N-u_{N-1}}{\tau})+\frac{\tau}{2}\left(f_N-(B+\frac{\alpha^2}{4})u_N-\alpha\frac{u_N-u_{N-1}}{\tau}\right)]+\psi,\end{cases}\tag{3.49}$$

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \alpha\frac{u_{k+1}-u_{k-1}}{2\tau} + \frac{A}{2}u_k + \frac{A}{4}(u_{k+1}+u_{k-1}) \\ \\ +\frac{\beta}{2}u_k + \frac{\beta}{4}\left(u_{k+1}+u_{k-1}\right) = f_k, f_k = f(t_k), 1 \leq k \leq N-1, \\ \\ u_0 = \lambda u_N + \varphi, f_0 = f(0), f_N = f(1), \\ \\ \left(I + \frac{\alpha\tau}{2} + \frac{\tau^2}{4}(B + \frac{\alpha^2}{4})\right)\frac{u_1-u_0}{\tau} - \frac{\tau}{2}\left(f_0 - (B+\frac{\alpha^2}{4})u_0 - \alpha\frac{u_1-u_0}{\tau}\right) \\ = \mu[(\frac{u_N-u_{N-1}}{\tau}) + \frac{\tau}{2}\left(f_N - (B+\frac{\alpha^2}{4})u_N - \alpha\frac{u_N-u_{N-1}}{\tau}\right)] + \psi. \end{cases} \tag{3.50}$$

Theorem 3.5. *Suppose that $\varphi \in D(A)$, $\psi \in D(A^{1/2})$ and assumpion (3.28) holds. Then for the solution of the difference scheme (3.49) the stability inequalities*

$$\max_{0\leq k\leq N}\|u_k\|_H \leq M(\lambda,\mu,\alpha,\beta,\delta) \times \left\{\sum_{s=0}^{N}\left\|A^{-1/2}f_s\right\|_H\tau + \left\|A^{-1/2}\psi\right\|_H + \|\varphi\|_H\right\}, \tag{3.51}$$

$$\max_{0\leq k\leq N}\left\|A^{1/2}u_k\right\|_H \leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=0}^{N}\|f_s\|_H\tau + \|\psi\|_H + \left\|A^{1/2}\varphi\right\|_H\right\}, \tag{3.52}$$

$$\max_{0\leq k\leq N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\| + \max_{0\leq k\leq N}\|Au_k\|_H$$

$$\leq M(\lambda,\mu,\alpha,\beta,\delta)\left\{\sum_{s=1}^{N}\|f_s - f_{s-1}\|_H + \|f_0\|_H + \left\|A^{1/2}\psi\right\|_H + \|A\varphi\|_H\right\}, \tag{3.53}$$

hold, where $M(\lambda,\mu,\alpha,\beta,\delta)$ does not depend on $f_s, 1 \leq s \leq N-1$ and φ, ψ.

The proof of Theorem 3.5 is based on the formulas for the solution of difference schemes (3.49) and (3.50), on the estimates for the step operators and on the self-adjointness and positivity of operator A.

Now, we consider applications of Theorems 3.4-3.5. First, we consider the nonlocal boundary value problem (3.20). The discretization of problem (3.20) is carried out in two steps. In the first step, we consider the discretization in x.To the differential operator A^x defined by the formula (2.14), we assign the difference operator A_h^x by formula (2.31). With the help of A_h^x, we reach the nonlocal boundary value problem

$$\begin{cases} u_{tt}^h(t,x) + \alpha u_t^h(t,x) + A_h^x u^h(t,x) + \beta u^h(t,x) = f^h(t,x), \\ 0 < t < T, x \in [0,l]_h, \\ u^h(0,x) = \lambda u^h(T,x) + \varphi^h(x), u_t^h(0,x) = \mu u_t^h(0,x) + \psi^h(x), x \in [0,l]_h. \end{cases} \tag{3.54}$$

In the second step, we replace (3.54) with difference scheme (3.27)

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + \alpha \frac{u_{k+1}^h(x) - u_k^h(x)}{\tau} + A_h^x u_{k+1}^h(x) + \beta u_{k+1}^h(x) = f_k^h(x), \\ f_k^h(x) = f^h(t_{k+1}, x),\ t_k = k\tau,\ 1 \le k \le N-1,\ x \in [0,l]_h, N\tau = T, \\ u_0^h(x) = \lambda u_N^h(x) + \varphi^h(x), \Big((1+\alpha\tau)\, I + \tau^2 (B_h^x + \frac{\alpha^2}{4} I) \Big) \frac{u_1^h(x) - u_0^h(x)}{\tau} \\ + (A_h^x + \beta I_h)\, \tau u_1^h(x) \\ = \mu \tau^{-1} (u_N^h(x) - u_{N-1}^h(x)) + \psi^h(x), x \in [0,l]_h. \end{cases} \tag{3.55}$$

Theorem 3.6. *For the solution* $\left\{ u_k^h(x) \right\}_0^N$ *of problem* (3.55) *the following stability estimates*

$$\max_{1 \le k \le N} \|u_k^h\|_{L_{2h}} \le M_1(\lambda, \mu, \alpha, \beta, \delta) \left\{ \max_{1 \le k \le N-1} \|f_k^h\|_{L_{2h}} + \| \psi^h \|_{L_{2h}} + \| \varphi^h \|_{L_{2h}} \right\},$$

$$\max_{1 \le k \le N} \|u_k^h\|_{W_{2h}^1} \le M_1(\lambda, \mu, \alpha, \beta, \delta) \left\{ \max_{1 \le k \le N-1} \|f_k^h\|_{L_{2h}} + \| \psi^h \|_{L_{2h}} + \| \varphi^h \|_{W_{2h}^1} \right\},$$

$$\max_{1 \le k \le N} \|u_k^h\|_{W_{2h}^2} \le M_2(\lambda, \mu, \alpha, \beta, \delta) \left\{ \max_{2 \le k \le N-1} \left\| \frac{1}{\tau} \left(f_k^h - f_{k-1}^h \right) \right\|_{L_{2h}} \right.$$

$$\left. + \| f_1^h \|_{L_{2h}} + \| \psi^h \|_{W_{2h}^1} + \| \varphi^h \|_{W_{2h}^2} \right\}$$

hold, where $M_1(\lambda, \mu, \alpha, \beta, \delta)$ *and* $M_2(\lambda, \mu, \alpha, \beta, \delta)$ *do not depend on* $\varphi^h(x), \psi^h(x)$ *and* $f_k^h(x), 1 \le k \le N-1$.

Proof. Difference scheme (3.55) can be written in abstract form

$$\begin{cases} \frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}+\alpha\frac{u_{k+1}^h-u_k^h}{\tau}+A_hu_{k+1}^h+\beta u_{k+1}^h=f_k^h, \\ 1\leq k\leq N-1, N\tau=T, \\ u_0^h=\varphi^h, \left((1+\alpha\tau)\,I+\tau^2(B_h+\frac{\alpha^2}{4}I)\right)\frac{u_1^h-u_0^h}{\tau}+ \\ (A_h+\beta I_h)\,u_1^h=\mu\tau^{-1}(u_N^h-u_{N-1}^h)+\psi^h \end{cases} \tag{3.56}$$

in a Hilbert space L_{2h} with self-adjoint positive definite operator $A_h=A_h^x$ by formula (2.31). Here, $f_k^h=f_k^h(x)$ and $u_k^h=u_k^h(x)$ are known and unknown abstract mesh functions defined on $[0,l]_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem 3.6 follow from estimates (3.51), (3.52), (3.53). Thus, Theorem 3.6 is proved. □

Second, we consider the nonlocal boundary value problem (3.24). The discretization of problem (3.24) is carried out in two steps. In the first step, we consider the discretization in x. To the differential operator A^x defined by the formula (2.18), we assign the difference operator A_h^x defined by formula (2.35). With the help of A_h^x, we reach the nonlocal boundary value problem

$$\begin{cases} u_{tt}^h(t,x)+\alpha u_t^h(t,x)+A_h^xu^h(t,x)+\beta u^h(t,x)=f^h(t,x), \\ 0<t<T, x\in\Omega_h, \\ u^h(0,x)=\varphi^h(x), u_t^h(0,x)=\psi^h(x), x\in\Omega_h. \end{cases} \tag{3.57}$$

In the second step, we replace (3.57) with the difference scheme (3.27)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\alpha\frac{u_{k+1}^h(x)-u_k^h(x)}{\tau}+A_h^xu_{k+1}^h(x)+\beta u_{k+1}^h(x)=f_k^h(x), \\ f_k^h(x)=f^h(t_{k+1},x),\ t_k=k\tau,\ 1\leq k\leq N-1,\ x\in\Omega_h, N\tau=T, \\ u_0^h(x)=\varphi^h(x), (1+\alpha\tau)\frac{u_1^h(x)-u_0^h(x)}{\tau}+(A_h^x+\beta I_h)\,\tau u_1^h(x) \\ =\psi^h(x), x\in\Omega_h, \end{cases} \tag{3.58}$$

for an infinite system of ordinary differential equations.

Theorem 3.7. *For the solution* $\left\{u_k^h(x)\right\}_0^N$ *of problem* (3.58) *the following stability estimates*

$$\max_{1\le k\le N}\|u_k^h\|_{L_{2h}} \le M_1(\lambda,\mu,\alpha,\beta,\delta)\left\{\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\parallel \psi^h \parallel_{L_{2h}}+\parallel \varphi^h \parallel_{L_{2h}}\right\},$$

$$\max_{1\le k\le N}\|u_k^h\|_{W_{2h}^1} \le M_1(\lambda,\mu,\alpha,\beta,\delta)\left\{\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\parallel \psi^h \parallel_{L_{2h}}+\parallel \varphi^h \parallel_{W_{2h}^1}\right\},$$

$$\max_{1\le k\le N}\|u_k^h\|_{W_{2h}^2} \le M_2(\lambda,\mu,\alpha,\beta,\delta)\left\{\max_{2\le k\le N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right.$$

$$\left.+\parallel f_1^h \parallel_{L_{2h}}+\parallel \psi^h \parallel_{W_{2h}^1}+\parallel \varphi^h \parallel_{W_{2h}^2}\right\}$$

hold, where $M_1(\lambda,\mu,\alpha,\beta,\delta)$ *and* $M_2(\lambda,\mu,\alpha,\beta,\delta)$ *do not depend on* $\varphi^h(x),\psi^h(x)$ *and* $f_k^h(x), 1\le k\le N-1.$

Proof. Difference scheme (3.58) can be written in abstract form (3.56) in a Hilbert space $L_{2h}=L_2(\overline{\Omega}_h)$ with self-adjoint positive definite operator $A_h=A_h^x$ by formula (2.35). Here, $f_k^h=f_k^h(x)$ and $u_k^h=u_k^h(x)$ are known and unknown abstract mesh functions defined on $\overline{\Omega}_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem 3.7 follow from estimates (3.51), (3.52), (3.53) and Theorem 2.8 on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}. □

Note that the difference schemes of the second order of accuracy with respect to one variable for approximate solutions of the mixed problems (3.20) and (3.24) generated by difference schemes (3.49) and (3.50) can be constructed. The abstract Theorems 3.4-3.6 given above and Theorem 2.8 permit us to establish the stability estimates for the solution of these difference schemes.

In applications, the theorems on convergence estimates for the solution of nonlocal problems can be established. The theoretical statements for the solution of difference schemes can be supported by the result of the numerical experiment. We have not been able to obtain a sharp estimate for the constants figuring in the stability inequality. Therefore we will give the results of numerical experiments for the nonlocal boundary value problem.

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = \exp(-t)\sin x, \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \frac{1}{2}u(1,x) + (1 - \frac{1}{2}\exp(-1))\sin x, \\ \frac{\partial}{\partial t}u(0,x) = \frac{1}{2}\frac{\partial}{\partial t}u(1,x) + (-1 + \frac{1}{2}\exp(-1))\sin x, 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, 0 \le t \le 1 \end{cases} \tag{3.59}$$

for the telegraph equation. The exact solution of this problem $u(t,x) = \exp(-t)\sin x$.

For the approximate solution of the nonlocal initial-boundary value problem (3.59), we consider the set $w_{\tau,h} = [0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h. We present the following first order of accuracy in t and second order of accuracy in x difference scheme for the approximate solutions of the problem (3.59)

$$\begin{cases} \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + 2\frac{u_n^{k+1} - u_n^k}{\tau} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} + u_n^{k+1} \\ = \exp(-t_{k+1})\sin x_n, x_n = nh, t_{k+1} = (k+1)\tau, \\ 1 \le k \le N-1, 1 \le n \le M-1, \\ u_n^0 = \frac{1}{2}u_n^N + (1 - \frac{1}{2}\exp(-1))\sin x_n, \\ \frac{u_n^1 - u_n^0}{\tau} = \frac{1}{2}\frac{u_n^N - u_n^{N-1}}{\tau} + (-1 + \frac{1}{2}\exp(-1))\sin x_n, 0 \le n \le M, \\ u_0^k = u_M^k = 0, 0 \le k \le N. \end{cases} \tag{3.60}$$

Now, we consider two types of second order of accuracy in t and x difference schemes for the approximate solutions of the problem (3.59)

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\frac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\frac{1}{2}\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -\frac{1}{2}\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\frac{1}{2}\left(u_n^{k+1}+u_n^{k-1}\right)=\exp\left(-t_k\right)\sin(x_n), \\ x_n=nh, t_k=k\tau, 1\leq k\leq N-1, 1\leq n\leq M-1, \\ u_n^0=\frac{1}{2}u_n^N+(1-\frac{1}{2}\exp\left(-1\right))\sin x_n, x_n=nh, \\ \frac{u_n^1-u_n^0}{\tau}-\frac{u_n^2-2u_n^1+u_n^0}{2\tau}= \\ \frac{1}{2}\left[\frac{u_n^N-u_n^{N-1}}{\tau}+\frac{u_n^N-2u_n^{N-1}+u_n^{N-2}}{2\tau}\right]+(-1+\frac{1}{2}\exp\left(-1\right))\sin(x_n), 0\leq n\leq M, \\ u_0^k=u_M^k=0, 0\leq k\leq N, \end{cases} \tag{3.61}$$

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\frac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\frac{1}{2}\frac{u_{n+1}^{k}-2u_n^{k}+u_{n-1}^{k}}{h^2}-\frac{1}{4}\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -\frac{1}{4}\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\frac{1}{2}u_n^k+\frac{1}{4}\left(u_n^{k+1}+u_n^{k-1}\right)=\exp\left(-t_k\right)\sin(x_n), \\ x_n=nh, t_k=k\tau, 1\leq k\leq N-1, 1\leq n\leq M-1, \\ u_n^0=\frac{1}{2}u_n^N+(1-\frac{1}{2}\exp\left(-1\right))\sin x_n, x_n=nh, \\ \frac{u_n^1-u_n^0}{\tau}-\frac{u_n^2-2u_n^1+u_n^0}{2\tau}=\frac{1}{2}\left[\frac{u_n^N-u_n^{N-1}}{\tau}+\frac{u_n^N-2u_n^{N-1}+u_n^{N-2}}{2\tau}\right] \\ +(-1+\frac{1}{2}\exp\left(-1\right))\sin(x_n), 0\leq n\leq M, \\ u_0^k=u_M^k=0, 0\leq k\leq N. \end{cases} \tag{3.62}$$

To solve these difference equations, a procedure of modified Gauss elimination method is applied. Hence, we seek a solution of the matrix equation in the following form:

$$u_j=\alpha_{j+1}u_{j+1}+\beta_{j+1},\ u_M=0,\ j=M-1,...,2,1$$

where α_j $(j = 1, 2, ..., M)$ are $(N+1) \times (N+1)$ square matrices, and β_j $(j = 1, 2, ..., M)$ are $(N+1) \times 1$ column matrices defined by

$$\alpha_{j+1} = -(B + C\alpha_j)^{-1} A,$$

$$\beta_{j+1} = (B + C\alpha_j)^{-1}(D\phi - C\beta_j), \ \ j = 1, 2, ..., M-1,$$

where $j = 1, 2, ..., M-1$, α_1 is the $(N+1) \times (N+1)$ zero matrix, and β_1 is the $(N+1) \times 1$ zero matrix. The results of computer calculations show that the second order difference schemes are more accurate than first order of accuracy difference scheme. Table 1 is constructed for $N = M = 20, 40$ and 80, respectively.

The errors are computed by

$$E_M^N = \max_{1 \le k \le N-1, 1 \le n \le M-1} \left| u(t_k, x_n) - u_n^k \right|,$$

where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n) and the results are given in Table 3.1.

Table 3.1 Error analysis

$\tau = \frac{1}{N}, h = \frac{pi}{M}$	$N = M = 20$	$N = M = 40$	$N = M = 80$
The difference scheme (3.60)	0.0047	0.0020	0.0010
The difference scheme (3.61)	2.3457×10^{-4}	6.0377×10^{-5}	1.5322×10^{-5}
The difference scheme (3.62)	1.0311×10^{-4}	2.4370×10^{-5}	5.8980×10^{-6}

CHAPTER 4

CONCLUSION

This work is dedicate to study the stability of Cauchy problem and nonlocal boundary value problems for telegraph equations. The following original results are acquired:

- The abstract theorem on the stability estimate for the solution of Cauchy problem for telegraph equations is proved.
- Stability estimates for the solution of two initial boundary value problems for telegraph equations are acquired.
- The first and second order of accuracy difference schemes for the approximate solution of Cauchy problem for telegraph equations are introduced.
- Abstract theorems on the stability estimates for the solution of difference schemes for the approximate solution of Cauchy problem for telegraph equations are proved.
- Stability estimates for the solution of difference schemes for two initial boundary value problems for telegraph equations are obtained.
- The abstract theorem on the stability estimate for the solution of nonlocal boundary value problems for telegraph equations is constructed.
- Stability estimates for the solution of two nonlocal boundary value problems for telegraph equations are acquired.

- The first and second order of accuracy difference schemes for the approximate solution of nonlocal boundary value problems for telegraph equations are presented.
- Abstract theorems on the stability estimates for the solution of difference schemes for the approximate solution of nonlocal boundary value problems for telegraph equations are proved.
- Stability estimates for the solution of difference schemes for two nonlocal boundary value problems for telegraph equations are obtained.
- The Matlab implementation of the first and second order of accuracy difference schemes for the approximate solution of initial boundary value problem and nonlocal boundary value problems for telegraph equations are presented.
- The theoretical expressions for the solution of these difference schemes are supported by the results of numerical examples.

CHAPTER 5

MATLAB PROGRAMING

In this chapter, Matlab programs for first and second order of accuracy difference schemes for test examples are given and numerical results are compared with the exact solution.

i. **The initial boundary value problem for a telegraph equation.**

"First Order of Accuracy Difference Scheme

```
function firstorderaccuracy(N,M)

tau=1/N;h=pi/M;

a=(-1/(h^2));

b=(1/(tau^2));

c=(-2/(tau^2)-2/tau);

d=(1/(tau^2)+1/(tau)+2/(h^2)+1);

for i=2:N; A(i,i+1)=a; end;

A(N+1,N+1)=0;A;

C=A;

for i=2:N; B(i,i-1)=b; end;
```

```
for i=2:N; B(i,i)=c; end;

for i=2:N; B(i,i+1)=d; end;

B(1,1)=1;B(N+1,1)=1/tau,B(N+1,2)=-1/tau;B(N+1,N+1)=0;B;

for i=1:N+1;D(i,i)=1;end;D;

'fii(j) finding';

for j=1:M+1;

x=((j)*h);

fii(1,j:j)=sin(x);

fii(N+1,j:j)=sin(x);

for k=2:N;

fii(k,j:j)=exp(-tau*(k-1))*sin(x);

end;

end;

alpha(N+1,N+1,1:1)=0;

betha(N+1,1:1)=0;

for j=1:M-1;

alpha(:,:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(-A);

betha(:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(D*fii(:,j:j)-C*betha(:,j:j));

end;

U(N+1,1,M:M)=0;

for z=M-1:-1:1;
```

```
U(:,:,z:z)=alpha(:,:,z+1:z+1)*U(:,:,z+1:z+1)+betha(:,z+1:z+1);
end;
for z=1:M;
p(:,z+1:z+1)=U(:,:,z:z);
end;
'EXACT SOLUTION OF THIS PROBLEM';
for j=1:M+1;
for k=1:N+1;
x=((j-1)*h);
es(k,j:j)=exp(-tau*(k-1))*sin(x);
end;
end;
es;
%'ERROR ANALSIS';
maxes=max(max(es));
maxapp=max(max(p));
maxerror=max(max(abs(es-p)))
relativeerror=max(max(abs(es-p)))/max(max(abs(p)));
cevap=[maxes,maxapp,maxerror,relativeerror].
```

Second Order of Accuracy Difference Scheme.

```
function secondorderaccuracy(N,M)

tau=1/N;h=pi/M;

a=(-1/(2*(h^2)));

b=(1/(tau^2)-1/tau+1/(h^2)+1/2);

c=(-2/(tau^2));

d=(1/(tau^2)+1/(tau)+1/(h^2)+1/2);

for i=2:N; A(i,i-1)=a; A(i,i+1)=a; end;

A(N+1,N+1)=0;A;

C=A;

for i=2:N; B(i,i-1)=b; end;

for i=2:N; B(i,i)=c; end;

for i=2:N; B(i,i+1)=d; end;

B(1,1)=1;B(N+1,1)=3/2,B(N+1,2)=-2;B(N+1,3)=1/2;B(N+1,N+1)=0;B;

for i=1:N+1;D(i,i)=1;end;D;

'fii(j) finding';

for j=1:M+1;

x=((j)*h);

fii(1,j:j)=sin(x);

fii(N+1,j:j)=tau*sin(x);

for k=2:N;
```

```
fii(k,j:j)=exp(-tau*(k-1))*sin(x);

end;

end;

alpha(N+1,N+1,1:1)=0;

betha(N+1,1:1)=0;

for j=1:M-1;

alpha(:,:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(-A);

betha(:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(D*fii(:,j:j)-C*betha(:,j:j));

end;

U(N+1,1,M:M)=0;

for z=M-1:-1:1;

U(:,:,z:z)=alpha(:,:,z+1:z+1)*U(:,:,z+1:z+1)+betha(:,z+1:z+1);

end;

for z=1:M;

p(:,z+1:z+1)=U(:,:,z:z);

end;

'EXACT SOLUTION OF THIS PROBLEM';

for j=1:M+1;

for k=1:N+1;

x=((j-1)*h);

es(k,j:j)=exp(-tau*(k-1))*sin(x);
```

```
end;

end;

es;

%'ERROR ANALSIS';

maxes=max(max(es));

maxapp=max(max(p));

maxerror=max(max(abs(es-p)))

relativeerror=max(max(abs(es-p)))/max(max(abs(p)));

cevap=[maxes,maxapp,maxerror,relativeerror].
```

ii. **The nonlocal boundary value problem for a telegraph equation.**"

First Order of Accuracy Difference Scheme.

```
"function firstorderaccuracy(N,M)

tau=1/N;h=pi/M;

a=(-1/(h^2));

b=(1/(tau^2));

c=(-2/(tau^2)-2/tau);

d=(1/(tau^2)+1/(tau)+2/(h^2)+1);

for i=2:N; A(i,i+1)=a; end;

A(N+1,N+1)=0;A;

C=A;

for i=2:N; B(i,i-1)=b; end;

for i=2:N; B(i,i)=c; end;

for i=2:N; B(i,i+1)=d; end;

B(1,1)=1;B(N+1,1)=-1/tau,B(N+1,2)=1/tau;B(1,N+1)=-1/2;B(N+1,N)
=1/(2*tau);B(N+1,N+1)=-1/(2*tau);B;

for i=1:N+1;D(i,i)=1;end;D;

'fii(j) finding';

for j=1:M+1;

x=((j)*h);

fii(1,j:j)=(1-1/(2*e))*sin(x);
```

```
fii(N+1,j:j)=(-1+1/(2*e))*sin(x);

for k=2:N;

fii(k,j:j)=exp(-tau*(k-1))*sin(x);

end;

end;

alpha(N+1,N+1,1:1)=0;

betha(N+1,1:1)=0;

for j=1:M-1;

alpha(:,:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(-A);

betha(:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(D*fii(:,j:j)-C*betha(:,j:j));

end;

U(N+1,1,M:M)=0;

for z=M-1:-1:1;

U(:,:,z:z)=alpha(:,:,z+1:z+1)*U(:,:,z+1:z+1)+betha(:,z+1:z+1);

end;

for z=1:M;

p(:,z+1:z+1)=U(:,:,z:z);

end;

'EXACT SOLUTION OF THIS PROBLEM';

for j=1:M+1;

for k=1:N+1;
```

```
x=((j-1)*h);

es(k,j:j)=exp(-tau*(k-1))*sin(x);

end;

end;

es;

%'ERROR ANALSIS';

maxes=max(max(es));

maxapp=max(max(p));

maxerror=max(max(abs(es-p)))

relativeerror=max(max(abs(es-p)))/max(max(abs(p)));

cevap=[maxes,maxapp,maxerror,relativeerror]."
```

Second Order of Accuracy Difference Scheme.

```
function secondorderaccuracy(N,M)

tau=1/N;h=pi/M;

a=(-1/(2*h^2));

b=(1/(tau^2)-1/tau+1/(h^2)+1/2);

c=(-2/(tau^2));

d=(1/(tau^2)+1/(tau)+1/(h^2)+1/2);

for i=2:N; A(i,i-1)=a;A(i,i+1)=a; end;

A(N+1,N+1)=0;A;

C=A;

for i=2:N; B(i,i-1)=b; end;

for i=2:N; B(i,i)=c; end;

for i=2:N; B(i,i+1)=d; end;

B(1,1)=1;B(N+1,1)=-6;B(N+1,2)=8;B(N+1,3)=-2;B(1,N+1)=-1/2;

B(N+1,N-1)=-1;B(N+1,N)=4;B(N+1,N+1)=-3;B;

for i=1:N+1;D(i,i)=1;end;D;

'fii(j) finding';

for j=1:M+1;

x=((j)*h);

fii(1,j:j)=(1-1/(2*e))*sin(x);

fii(N+1,j:j)=2*tau*(-1+1/(2*e))*sin(x);
```

```
for k=2:N;

fii(k,j:j)=exp(-tau*(k-1))*sin(x);

end;

end;

alpha(N+1,N+1,1:1)=0;

betha(N+1,1:1)=0;

for j=1:M-1;

alpha(:,:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(-A);

betha(:,j+1:j+1)=inv(B+C*alpha(:,:,j:j))*(D*fii(:,j:j)-C*betha(:,j:j));

end;

U(N+1,1,M:M)=0;

for z=M-1:-1:1;

U(:,:,z:z)=alpha(:,:,z+1:z+1)*U(:,:,z+1:z+1)+betha(:,z+1:z+1);

end;

for z=1:M;

p(:,z+1:z+1)=U(:,:,z:z);

end;

'EXACT SOLUTION OF THIS PROBLEM';

for j=1:M+1;

for k=1:N+1;

x=((j-1)*h);
```

```
es(k,j:j)=exp(-tau*(k-1))*sin(x);
end;
end;
es;
%'ERROR ANALSIS';
maxes=max(max(es));
maxapp=max(max(p));
maxerror=max(max(abs(es-p)))
relativeerror=max(max(abs(es-p)))/max(max(abs(p)));
cevap=[maxes,maxapp,maxerror,relativeerror].
```

REFERENCES

Achour, D. and Belacel, A., "Domination and factorization theorems for positive strongly p-summing operators", *Positivity*, Vol. 18, pp. 785–804, 2014.

Ashyralyev, A. and Akat, M., "An approximation of stochastic hyperbolic equations", *AIP Conf. Proc.*, Vol. 1389, pp. 625–628, 2011.

Ashyralyev, A. and Akat, M., "An approximation of stochastic telegraph equations", *AIP Conf. Proc.*, Vol. 1479, pp. 598–601, 2012.

Ashyralyev, A. and Akat, M., "An approximation of stochastic hyperbolic equations: case with Wiener process", *Math. Method. Appl. Sci.*, Vol. 36, pp. 1095–1106, 2013.

Ashyralyev, A. and Altay, N., "A note on the well-posedness of the nonlocal boundary value problem for elliptic difference equations", *Appl. Math. Comput.*, Vol. 175, pp. 49–60, 2006.

Ashyralyev, A. and Köksal, M.N., "On the numerical solution of hyperbolic PDEs with variable space operator", *Numerical Methods for Partial Differential Equations*, Vol. 25, pp. 1086–1099, 2009.

Ashyralyev, A. and Sobolevskii, P., "A note on the difference schemes for hyperbolic equations", *Abstr. Appl. Anal.*, Vol. 6, pp. 63–70, 2001.

Ashyralyev, A. and Sobolevskii, P.E., *Well-Posedness of Parabolic Difference Equations*, Birkhäuser Verlag, Basel, Boston, Berlin, 1994.

Ashyralyev, A. and Sobolevskii, P.E., *New Difference Schemes for Partial Differential Equations*, Birkhäuser Verlag, Basel, Boston, Berlin, 2004.

Ashyralyev, A. and Sobolevskii, P.E., "Two new approaches for construction of the high order of accuracy difference schemes for hyperbolic differential equations", *Discrete Dyn. Nat. Soc.*, Vol. 2005, pp. 183–213, 2005.

Ashyralyev, A. and Yıldırım, O., " On multipoint nonlocal boundary value problems for hyperbolic differential and difference equations" *Taiw. J. Math.*, Vol. 14, pp. 165–194, 2010.

Ashyralyev, A. and Yıldırım, O., "On the numerical solution of hyperbolic IBVP with high-order stable finite difference schemes" *Boundary Value Problem*, Vol. 2013, pp. 1–34, 2013.

Banasiak, J. and Mika, J., "Singularly perturved telegraph equations with applications in the random walk theory", *J. Appl. Math. Stoch. Anal.*, Vol. 11, pp. 9–28, 1998.

Biazar, J., Ebrahimi, H., and Ayati, Z., "An approximation to the solution of telegraph equation by variational iteration method", *Numer. Meth. Part. D. E.*, Vol. 25, pp. 797–801, 2009.

Dehghan, M. and Shokri, A., "A numerical methods for solving the hyperbolic telegraph equation", *Numer. Meth. Part. D. E.*, Vol. 24, pp. 1080–1093, 2008.

Fattorini, H.O., *Second Order Linear Differential Equations in Banach Spaces*, Elsevier Science Publishing Company, North-Holland, 1985.

Gao, F. and Chi, C., "Unconditionally stable difference schemes for a one-space-dimensional linear hyperbolic equation", *Appl. Math. Comput.*, Vol. 187, pp. 1272–1276, 2007.

Ghorbanalizadeh, A. and Sawano, Y., "Approximation in Banach space by linear positive operators", *Positivity*, Vol. 18, pp. 585–594, 2014.

Goldstein, J.A., *Semigroups of Linear Operators and Applications*, The Clarendon Press Oxford University Press, New York, 1985.

Jiwari, R., Pandit S., Mittal, R.C., "A Differential Quadrature Algorithm for the Numerical Solution of the Second-Order One Dimensional Hyperbolic Telegraph Equation ", *Int. J. of Nonlinear Sci.*, Vol. 13, pp. 259–266, 2012.

Jordan, P. and Puri, A., "Digital signal propagation in dispersive media", *J. Appl. Phys.*, Vol. 85, pp. 1273–1282, 1999.

Koksal, M., "An operator-difference method for telegraph equations arising in transmission lines", *Discrete Dyn. Nat. Soc.*, Vol. 2011, pp. 1–17, 2011, Article Number 561015.

Krein, S., *Linear Differential Equations in Banach Space*, American Mathematical Society, Providence, 1971, Translated from the Russian by J. M. Danskin, Translations of Mathematical Monographs, Vol. 29.

la Sen, M.D., "Positivity and stability of the solutions of Caputo fractional linear time-invariant systems of any order with internal point delays", *Abstr. Appl. Anal.*, Vol. 2011, pp. 1–25, 2011, Article ID 161246.

la Sen, M.D., "On bounded strictly positive operators of closed range and some applications to asymptotic hyperstability of dynamic systems", *Abstr. Appl. Anal.*, Vol. 2013, pp. 1–12, 2013, Article ID 639576.

Saadatmandi, A. and Dehghan, M., "Numerical solution of hyperbolic telegraph equation using the Chebyshev Tau method", *Numer. Meth. Part. D. E.*, Vol. 26, pp. 239–252, 2010.

Sobolevskii, P., *Difference Methods for the Approximate Solution of Differential Equations*, Ph.D. Thesis, Izdat. Voronezh. Gosud. Univ., 1975, in Russian.

Twizell, E., "An explicit difference method for the wave equation with extended stability range", *BIT*, Vol. 19, pp. 378–383, 1979.

Vasilev, V., Krein, S., and Piskarev, S., "Operator Semigroups, Cosine Operator Functions, and Linear Differential Equations", *Itogi Nauki i Tekhniki*, Vol. 204, pp. 87–202, 1990, Translated in J. Soviet Math. Vol. 54, 1042–1129, 1991.

Weston, V. and He, S., "Wave splitting of telegraph equation in R3 and its applications to inverse scattering", *Inverse Probl.*, Vol. 9, pp. 789–812, 1993.

APPENDIX A

DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS, FURTHER STUDIES AND PUBLICATIONS FROM THESIS WORK

A.1 DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS

"I hereby declare that this thesis comprises my original work. No material in this thesis has been previously published and written by another person, except where due reference is made in the text of the thesis. I further declare that this thesis contains no material which has been submitted for a degree or diploma or other qualifications at any other university."

Signature:
Date:

A.2 FURTHER STUDIES

A.3 PUBLICATIONS FROM THESIS WORK

Academic Journals

1. Ashyralyev A., Modanli M.: (2015). "An operator method for telegraph partial differential and difference equations". Boundary Value Problems, 2015(1), 1-17.

2. Ashyralyev A., Modanli M.: (2014). "A numerical solution for a telegraph equation". AIP Conference Proceedings, 300-304, Doi: 10.1063/1.4893851.

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B.Sc., Department of Mathematics, Harran University, Şanlıurfa, Turkey, June 2003.

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PUBLICATIONS

Academic Journals

1. Ashyralyev A., Modanli M.: (2015). "An operator method for telegraph partial differential and difference equations". Boundary Value Problems, 2015(1), 1-17.
2. Ashyralyev A., Modanli M.: (2014). "A numerical solution for a telegraph equation". AIP Conference Proceedings, 300-304, Doi: 10.1063/1.4893851.

SKILLS/INTERESTS

- Programming languages: C, Java, JavaScrip
- Analytical and statistics programs: MATLAB
- Word processing and publication: Microsoft Word
- Spreadsheets and databases: Microsoft Excel
- Operating systems: Microsoft Windows 7
- Presentation: Microsoft PowerPoint
- Email: Microsoft Outlook, Google Mail