T.C. DOĞUŞ UNIVERSITY

INSTITUTE OF SOCIAL SCIENCES

FINANCIAL ECONOMICS

STOCHASTIC MORTALITY USING NON-LIFE METHODS

Ph.D. Thesis

Şirzat ÇETİNKAYA

2010186005

Supervisor

Prof. Dr. Kerem ŞENEL

Istanbul, June 2015

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I am also proud of our study because this is the first Ph.D thesis about the insurance in our institute. I hope the thesis about insurance continues. Insurance area will increase its importance in the future therefore it will be needed new academic researches and studies.

I would like to thank my sweet soul mate, Hale Çetinkaya for her love, patience, and understanding—she allowed me to spend most of the time on this thesis without her patience and support this road would be unbearable.

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Last and most important thanks to my deceased grandmother, Emine Akay. I dedicated this thesis to her, who could not witness the final of this achievement, but influenced me heavily with her thirst for knowledge, encouraged me always for higher studies.

Istanbul, June 2015 Şirzat Çetinkaya

SUMMARY

Although longevity and mortality risks have been studied for a long time, there has been a remarkable increase in the number and scope of these studies due to recent theoretical and practical developments. Such developments paved the way for a more detailed analysis of the asset-liability balance of life insurance and pension companies. The impact of these risks is huge albeit long-term. In addition to life insurance and pension companies, governments are also considerably affected through health expenditures and pension payments. Further, environmental impact on food and water supplies and pollution should also be considered as well as financial ramifications. Hence, the accurate modelling and forecasting of longevity and mortality risks has become more important than ever as the accelerating population increase is taking its toll on mankind both financially and environmentally. This thesis focuses on the most widely used stochastic mortality models. The results pertaining to historical death probability, force of mortality, life expectancy, and rectangularization behavior are analysed in detail. These models are applied to the data from 20 different countries. In terms of the variety of stochastic mortality models and the number of different countries, this study is singled out as the most extensive study to date. The main contribution of this thesis is the introduction of a new approach to model mortality. This approach is based on IBNR calculations in non-life insurance. The comparison of this approach with two extensively used models in practice, namely Lee and Carter and Renshaw and Haberman models, shows that the new approach outperforms the aforementioned models with UK data.

ÖZET

Uzun yaşama riski ve ölümlülük riski uzun yıllardır ele alınmasına karşın son dönemde bu alanda yapılan incelemeler ve araştırmalarda artış görülmektedir. Bunun en önemli sebeplerinden birisi riski ele alış biçimlerinde gerçekleşen gerek teorik ve gerekse uygulama yenilikleri neticesinde varlık yükümlülüklerin daha detaylı ele alınması yatmaktadır. Uzun dönemde etki eden bir risk olmasına karşın, etkisi büyük olmaktadır. Bu riski önemli kılan en önemli unsur etki alanının sigorta şirketleri ile sınırlı kalmamasıdır. Sigorta şirketlerinin varlık, yükümlülük dengeleri, hükümetlerin sağlık harcamaları ve emeklilik harcamaları, emeklilik şirketlerinin yükümlülük dengesi, finansal piyasalardaki hisse senet fiyatları sıcak para akışı gibi geniş bir etki alanına sahiptir. Buna gıda, su, çevre kirliliği gibi çevresel etkenlerde eklenebilir. Bu sebeplerden ötürü yaşama olasılıklarının modellenmesi oldukça önemli bir hal almaktadır.

Bu çalışmada literatürde sıklıkla incelenen ve kullanılan stokastik ölüm oranı modelleri incelenmiş ve açıklanmıştır. Literatürde yer alan çalışmaların genelinde model incelemeleri bir ülke üzerinden gerçekleştirilmiştir. Çalışmada 20 ülke için ilgili ülkeye en uygun stokastik model belirlenmiştir. Çalışma, bu genişlikte yapılan ilk çalışma olma özelliğini içermektedir.

Ölüm oranını modellemek için temelini hayat dışı sigortalar matematiğinde IBNR hesaplamalarının oluşturduğu yeni bir yaklaşım önerilmektedir. Bu yaklaşımdan Birleşik Krallık için elde edilen sonuçlar literatürdeki birçok karşılaştırmada kullanılan Lee ve Carter modeli ve Renshaw ve Haberman modeli ile karşılaştırılmıştır. Sonuçta yeni geliştirilen yaklaşımın sonuçlarının Lee ve Carter'ın modelinden ve Renshaw ve Haberman'ın modelinden daha iyi olduğu görülmüştür.

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ABBREVIATIONS

NMRCIR	: Non Mean Reverting Cox Ingersoll Ross
MSE	: Mean Square Error
RSSE	: Root Sum Square Error
RJMCMC	: Reversible Jump Markov Chain Monte Carlo
HMD	: Human Mortality Database
GDP	: Gross Domestic Product
UN	: United Nations
US	: United States
UK	: United Kingdom
IMF	: International Monetary Fund
MY	: Middle-aged to young
PE	: Price earnings
CBD	: Cairns, Blake and Dowd model
CBD1	: First generalisation of CBD model
CBD2	: Second generalisation of CBD model
CBD3	: Third generalisation of CBD model
PCA	: Principle component analysis
LC	: Lee and Carter model
RH	: Renshaw and Haberman model
SVD	: Singular Value Decomposition
CMI	: Continuous Mortality Investigation
BIC	: Bayesian information criterion
CIR	: Cox, Ingersoll and Ross
IBNR	: Incurred But Not Reported
SCL	: Standard Chain Ladder
MCMC	: Markov Chain Monte Carlo
GBM	: Geometric Brownian motion
LDF	: Loss development factor
CLDF	: Cumulative loss development factor
LT	: Life table

MH	: Metropolis Hastings
FIR	: Finite Impulse Response
SG	: Savitzky and Golay

CHAPTER 1: INTRODUCTION

Individuals and financial institutions are exposed to many different risks that cause undesirable financial consequences. From the insurance perspective material damages such as crashing car, transport damage, fire and disease are the examples for the individuals or entities. Conventionally, one can transfer the risks by buying an insurance contract. From the side of companies, insurance companies have to diversify the risks by pooling the individual policies. If an agent wants to use the insurance contract, company offset the loss by the premiums collected other insurers. As known there are two main parts in insurance. These topics are discussed in non-life insurance mathematics. The other important part is life insurance.

The history of life insurance mathematics is shortly summarised by Pitacco, E. (2004). The foundation of life insurance mathematics can be traced back to second half of the 17th century. Jan De Witt and Edmond Halley suggested the formula which is called as the expected present value or alternatively the actuarial value of life annuities. Calculation of insurance policies with death benefits is proposed by James Dodson in the 18th century. These approaches are deterministic actuarial models. Stochastic approaches are proposed at the end of the 18th century. The stochastic approaches to life insurance were provided in the second half of the 19th century. New approaches are still suggested and the proposed methods are improved because of the cyclical financial and demographic changes. However, deterministic approaches are still widely used. In most of the actuarial estimations, actuaries use different life tables.

Mortality rates are an integral part of the pricing and reserving of annuity portfolios and modelling for companies and it is also an integral part of the financing of public programmes such as retirement, supporting elderly population, health expenditure and also private companies aiming to provide pensions, life annuities and health products. The assumptions made by these units are very crucial because can pose financial stability and economic wealth problems on policyholders or individuals in the population. Medical advances, progressive technology, changes in lifestyle, new diseases, catastrophic events, new discoveries in genetics make the future mortality rates unpredictable. In many countries, demographic structure changes and the mortality rates decrease because of these reasons. Unpredictable movements has a stochastic behaviour affect directly and indirectly many units. Therefore to catch this trend or modelling variations is very crucial.

Mortality risk and longevity risk are two crucial risks that are the result of these variations and trend. The topic of mortality risk has been generally addressed in recent years because after the twentieth century it is experienced that there is a significant improvements in the life expectancies. Mortality rates have fallen significantly at all ages. The improvement of the rates has different characteristics at different ages and cohort years. It is referred to uncertainty in future mortality rates. Longevity risk is a risk that the uncertainty of mortality rates and its effect on the long term probability of survival of an individual. It is a risk for pension and life insurance companies that the survival probabilities changes with the minimum rates, this causes severe solvency problems for pension plans and the companies. Mortality improvements and uncertainty in the trends in other words mortality and longevity risks are an important topic for life insurance or pension companies. Because of the life insurance products long duration, the mortality rates and survival probabilities can easily affect the valuations, reserve estimations or solvency requirements. Long time horizon can cause underestimation of the liabilities if the companies do not adopt their survival probabilities to the new environment of life or future path of the mortality trends. All insurance companies have to have enough reserves to compensate the expected losses. Both non-life, life and pension companies have to calculate these reserves carefully because of the existence of solvency requirements, necessity of fair values and pricing the products linked mortality. Mortality and longevity risks are the most important risks effecting the solvency requirements for the life and pension companies. If pensioners live more than expected this causes a big problem for life companies or pension companies because their cash flow continues much longer than their expectations. The risk continuous and it must be managed and controlled because in the long run companies may face high degree of uncertainty and possible adverse financial consequences. These challenges exhibit the importance of stochastic mortality modelling. It is a fundamental prerequisite especially for correct pricing, risk measurement and reserving for the insurance companies. Recently, increasing popularity of mortality derivatives requires more advanced mathematical tools in order to have more accurate pricing formulas. In particular, stochastic behaviour of mortality rates should be incorporated to the existing mortality models to increase degree of precision of the estimated derivative prices.

The earlier studies about mortality modelling are Lee and Carter (1992), Brouhas, Denuit and Vermunt (2002), Renshaw and Haberman (2003), Currie, Durban and Eilers (2004) used similar approach to model the mortality. They used discrete time. Milevsky and Promislow (2001), Dahl (2004), Dahl and Moller (2006), Miltersen and Persson (2005), Biffis (2005), Schrager (2006) used continuous time framework. Continuous time framework has an important role especially for the pricing mortality linked securities. The mortality is modelled by affine models. Some of the papers are as follows; Biffis (2005a), Biffis et al. (2005b), Biffis and Denuit (2006), Dahl (2004), Dahl and Moller (2006), Luciano and Vigna (2005), Schrager (2006).

In this study basically have two main and two secondary objectives. The aim of the secondary objectives is to prepare the reasons and necessities of the main objectives. The first one of the secondary objectives is to give the some basic life insurance indicators for the countries investigated in the thesis. Second one is to show the literature consequences of the impacts of the longevity on financial and economic systems. The effects of the longevity and mortality risks on financial systems are brilliant area for most of the economists, sociologists and social security specialists in recent times. It is an important topic because it affects health expenditure, pension expenditure, solvency requirements, reserving, environmental factors, financial systems for instance cash flows, stock prices and risk premiums. There are lots of valuable studies in the literature. Some of the important and pioneer studies on this area are examined and summarized in this thesis.

The main problem is that mortality rates are not deterministic and needs to be modelled. There are kinds of suggested stochastic mortality models in the literature. However, there are some deficiencies of these models such as one of them is without cohort effect, the other one has identifiability problem, another one has lots of parameters and one is appropriate for just the older ages. It is important to choose correct model to precise estimations. One of the main objectives is to provide mathematical background and main critics of the mortality models as each one is a corner stone in the literature. These models are applied to 20 different countries to discover what the best model is for each one and selection criteria, residuals, model parameters are also given. The application part of the thesis is the broadest study in the literature up to the present.

The other main objective is to suggest a new approach to model the mortality which has desirable properties wanted to have any mortality model. Some of these properties are appropriate for simulations, cohort base and easily practicable. The aim is to construct a model having all desirable properties for a stochastic mortality model. The new approach is constructed by adapting the non-life methods to the mortality tables for the first time. The methodology of estimation of incurred but not reported losses is applied in the new approach and loss development factors are modelled by geometric Brownian motion for the first time in this methodology.

The findings in this study and the new approach can be used for population estimations, economic researches, mortality derivatives, longevity risk estimations, health expenditure estimations, pension expenditure estimations, environmental estimations, economic plans and life researches.

The thesis is organized as follows. In Chapter 2, life table indicators are investigated to demonstrate the historical mortality behaviour for the 22 countries. Probability of survive, probability of death, logarithm of probability of death, life expectancy are used for this purpose. There is a large literature on the impact of longevity on economy and financial system. The critical findings are given in Chapter 2. In Chapter 3, a general procedure for constructing mortality models are mentioned. The theoretical backgrounds of mostly cited mortality models are given. Desirable properties of the theoretical models and model selection criteria are presented. The base of the new approach is non-life reserving methods. Some of the well-known methods are introduced in Chapter 4. In Chapter 5, the theoretical issues of the new approach are described. In Chapter 6 is the application chapter in this thesis. The best model mentioned in Chapter 3 for the 20 countries is determined. The out sample forecast performance of the new approach is compared with the best model and the non-life models mentioned in Chapter 4. Chapter 7 presents the conclusion and ideas for further research. Appendix I presents the affine term structure mortality models.

CHAPTER 2: LIFE TABLE INDICATORS AND LONGEVITY RISK

The aim of this chapter is to demonstrate the historical mortality behaviour for the 22 countries. Probability of survive, probability of death, logarithm of probability of death, life expectancy are used for this purpose. The data is taken from the Human Mortality Database. These indicators are practical to summarize the data and show the characteristics of mortality rates of the countries. The countries are grouped according to their continent in order to investigate the differences between the continents and the countries in the same region.

Life expectancies of the countries are presented for five different years which are selected to represent the characteristics from past to now. Life expectancies of 20 countries have an increasing trend. Two countries have a contradictory behaviour, Russia and Ukraine. They do not possess a continuous increasing trend and have a decreasing trend in the 1990s.

There are lots of academic studies examining the impact of the longevity on the macroeconomic and financial system. At the end of the chapter the literature review about the impact of longevity risk on economy is given in detail. Also following ratios related with longevity are given; life premiums in percentage of GDP, private sector, government health expenditure in GDP, total investment of pension funds in OECD and non-OECD countries as a percentage of GDP, public and private expenditure on pensions as a percentage of GDP.

2.1 Some Key Indicators about Life Tables

A life table usually contains tabulations, by individual ages, of the number of living people, number of deaths and one year death probability. It is not constructed by observing for instance “N” new-borns to their entire life times. Instead, it is used estimates of probabilities of death, given survival to various ages, derived from the experience of the entire population in the years around the calculated census.

Mostly used indicators in the life insurance and life tables are as follows;

- Let $p(x,t,t+1)$ be probability of a person at age x at time t will live to the t+1:
  $p(x,t,t+1) = e^{-\int_t^{t+1} \mu(u,x-t+u)du}$, (2.1)
- Let $(q(x,t,t+1))$ be a probability of death a person at age x at time t will die before time t+1,
- Let $\mu(x,t)$ be the instantaneous force of mortality. It is the underlying force of mortality at time t and age x. There is a following relationship between q and μ:
  $q(x,t,t+1) = 1 - e^{-\int_t^{t+1} \mu(u,x-t+u)du}$, (2.2)
- Life expectancy is the expected number of years of life remaining at a given age:
  $e_x = \sum_{k=1}^{\infty} k\, p(x,k)$. (2.3)

Next pages give life expectancies, survival probabilities, logarithm of probability of deaths and probability of deaths for the selected western Europe (Belgium, France, Germany, Netherlands), northern Europe (Denmark, Ireland, Norway, Sweden, Switzerland, UK), eastern Europe (Bulgaria, Poland, Russia, Ukraine), southern Europe (Italy, Portugal, Spain), American (Canada, Chile, US) and Asian (Japan, Taiwan) countries.

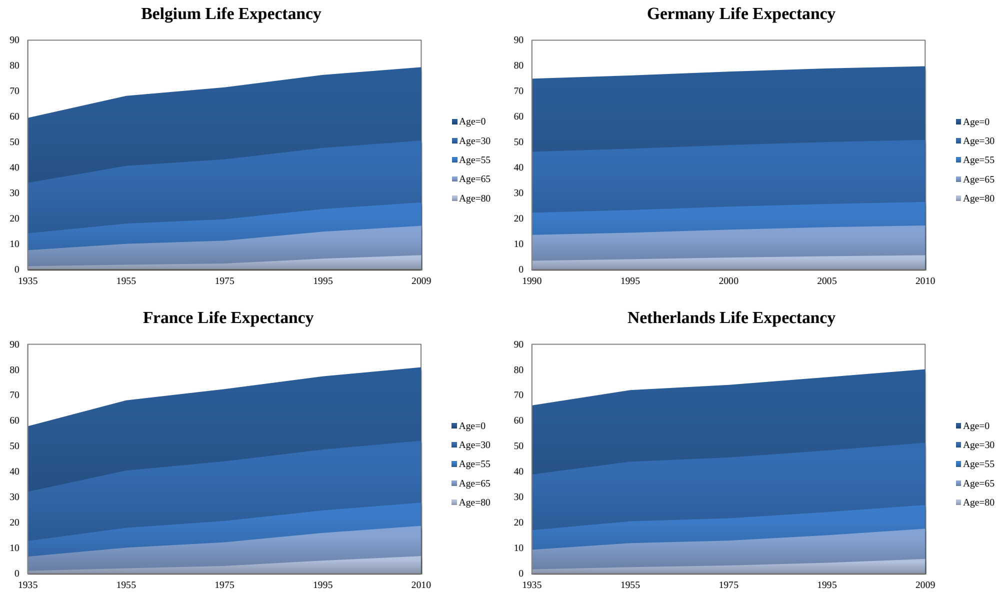


Figure 2.1 Life expectancy for Western Europe countries

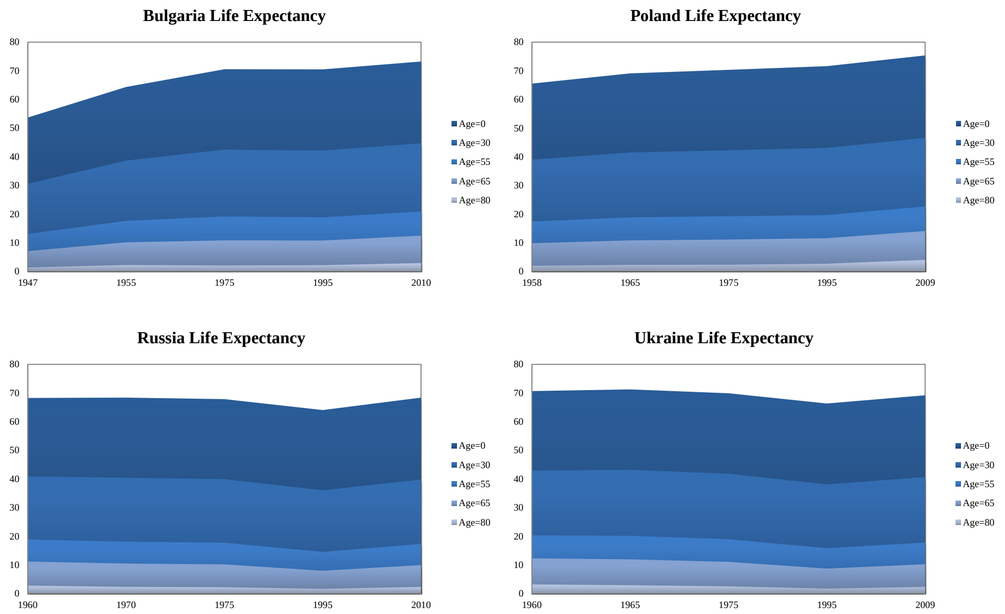


Figure 2.2 Life expectancy for Eastern Europe countries

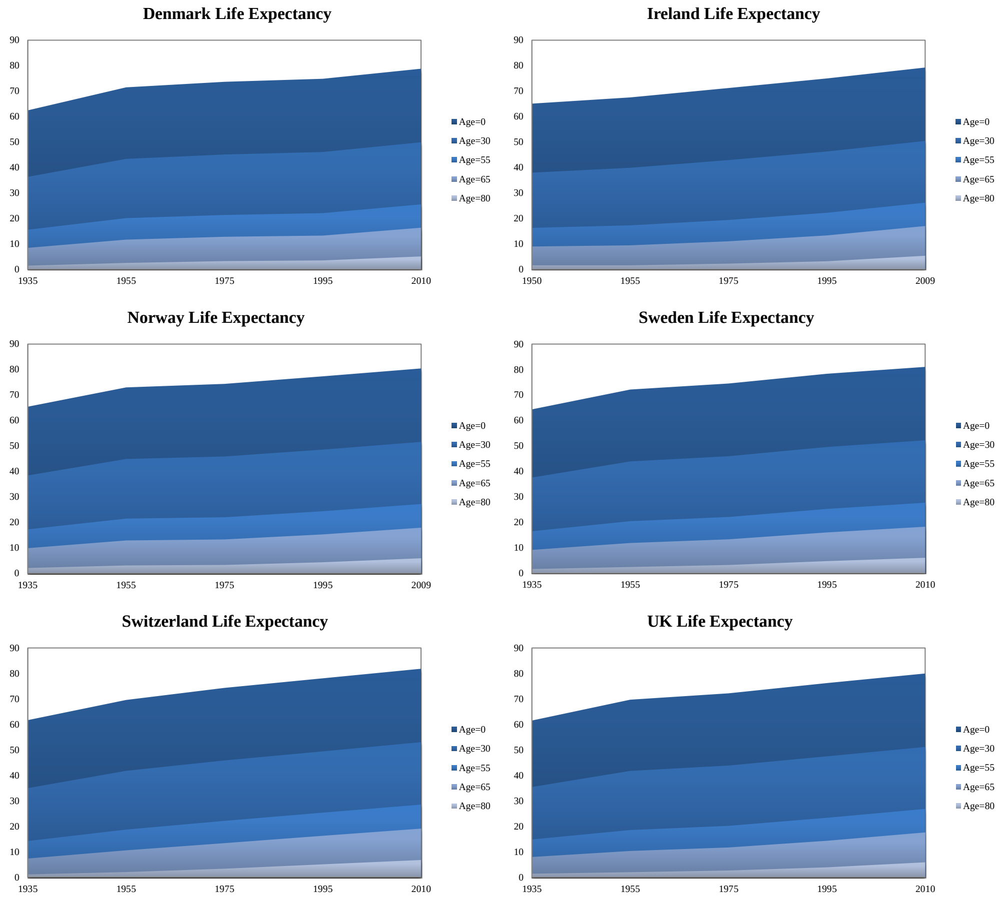


Figure 2.3 Life expectancy for Northern Europe countries

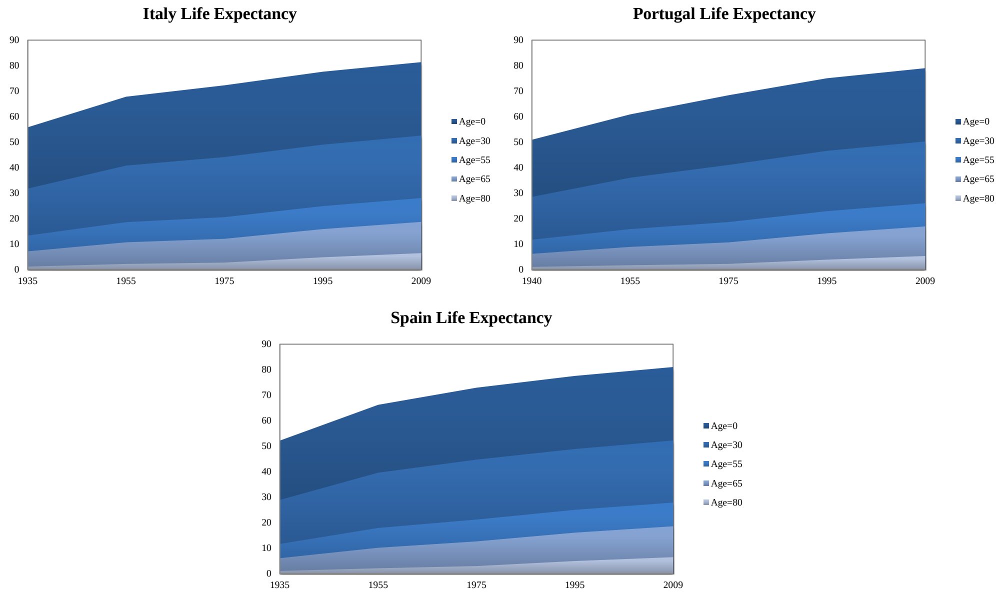


Figure 2.4 Life expectancy for Southern Europe countries

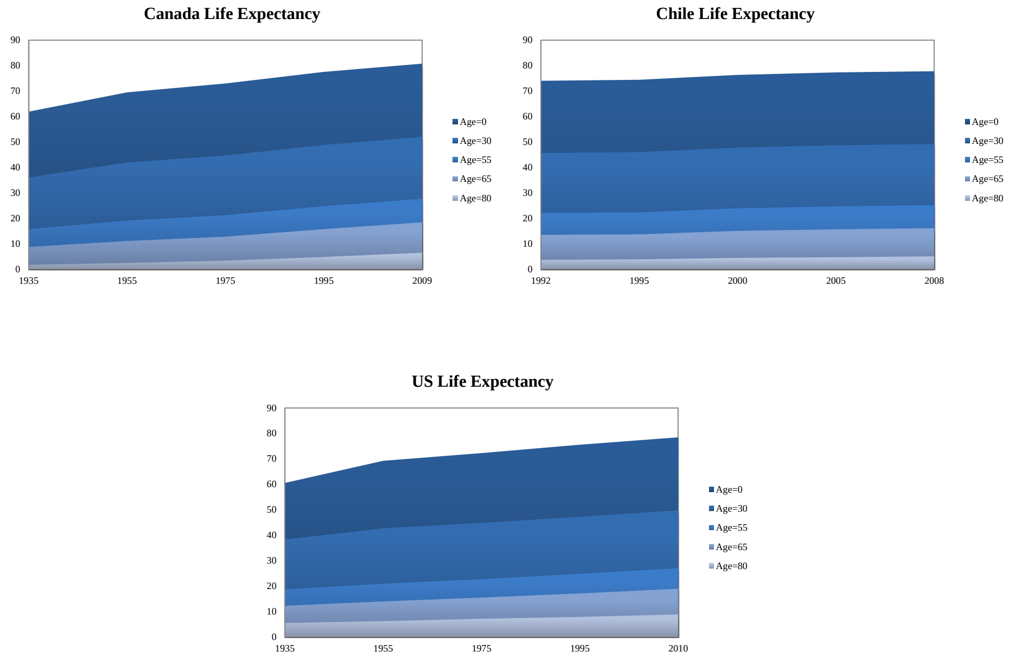


Figure 2.5 Life expectancy for American countries

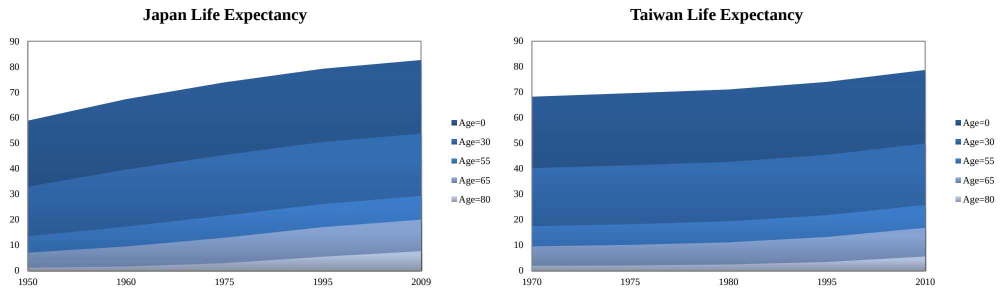


Figure 2.6 Life expectancy for Asian countries

Table 2.1 Life expectancy critics for countries

WESTERN EUROPE		
	Life Expectancy Increased	**Within**
BELGIUM	20 years	70 years
FRANCE	23 years	75 years
GERMANY	5 years	20 years
NETHERLANDS	14 years	74 years
EASTERN EUROPE		
BULGARIA	20 years	63 years
POLAND	10 years	51 years
RUSSIA	0 years	50 years
UKRAINE	-1 years	49 years
NORTHERN EUROPE		
DENMARK	16 years	75 years
IRELAND	14 years	59 years
NORWAY	15 years	74 years
SWEDEN	17 years	75 years
SWITZERLAND	20 years	75 years
UK	18 years	75 years
SOUTHERN EUROPE		
ITALY	26 years	74 years
PORTUGAL	28 years	79 years
SPAIN	29 years	74 years
ASIAN		
JAPAN	24 years	59 years
TAIWAN	10 years	40 years
AMERICAN		
CANADA	19 years	74 years
CHILE	4 years	16 years
US	18 years	75 years

Table 2.1 gives the summary of the change of the life expectancies within the giving ranges. Russia and Ukraine are the two countries they have reverse behaviour. The confusion in the early 1990s caused life expectancy in Russia to gradually decrease. The reason of confusion is the breakup of the Soviet Union. After the independence of Ukraine in 1991, economic uncertainty appears in the country. This chaos causes depression and effect the fertility and life quality. Another important event is the Chernobyl disaster in 1986 which is a catastrophic nuclear accident at the Chernobyl Nuclear Power Plant in Ukraine.

Increasing trend in the countries shows that this increase will continue in the future.

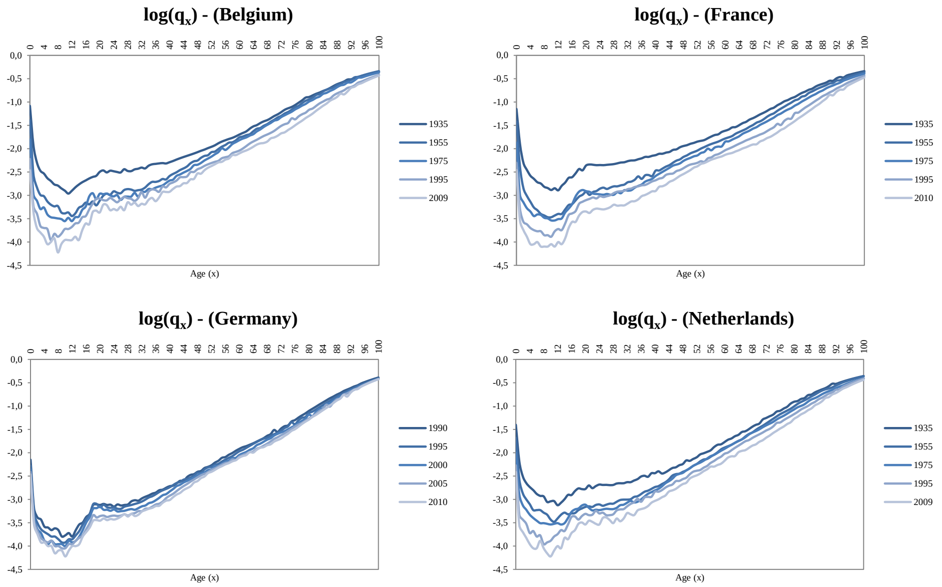


Figure 2.7 log(q_x) for Western Europe countries

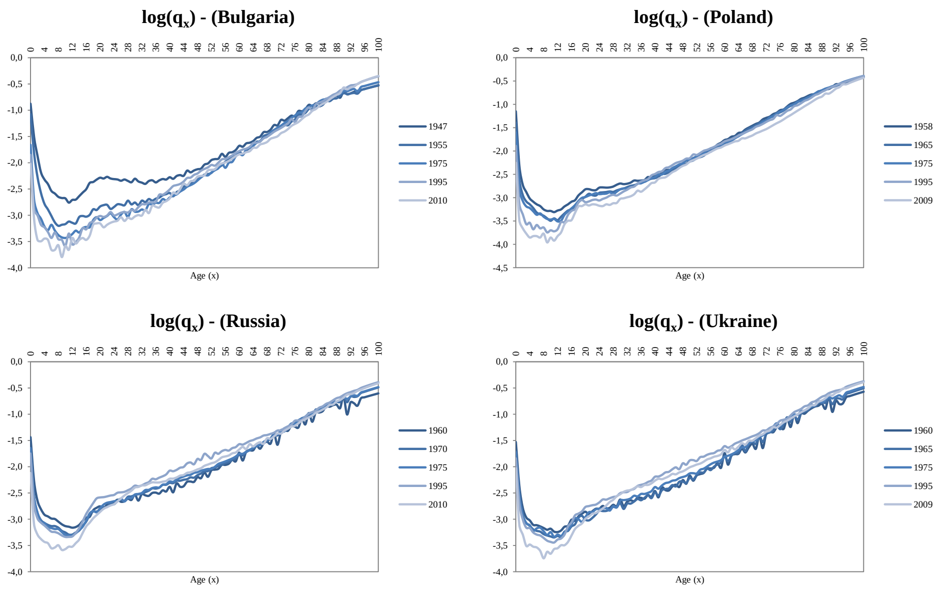


Figure 2.8 log(q_x) for Eastern Europe countries

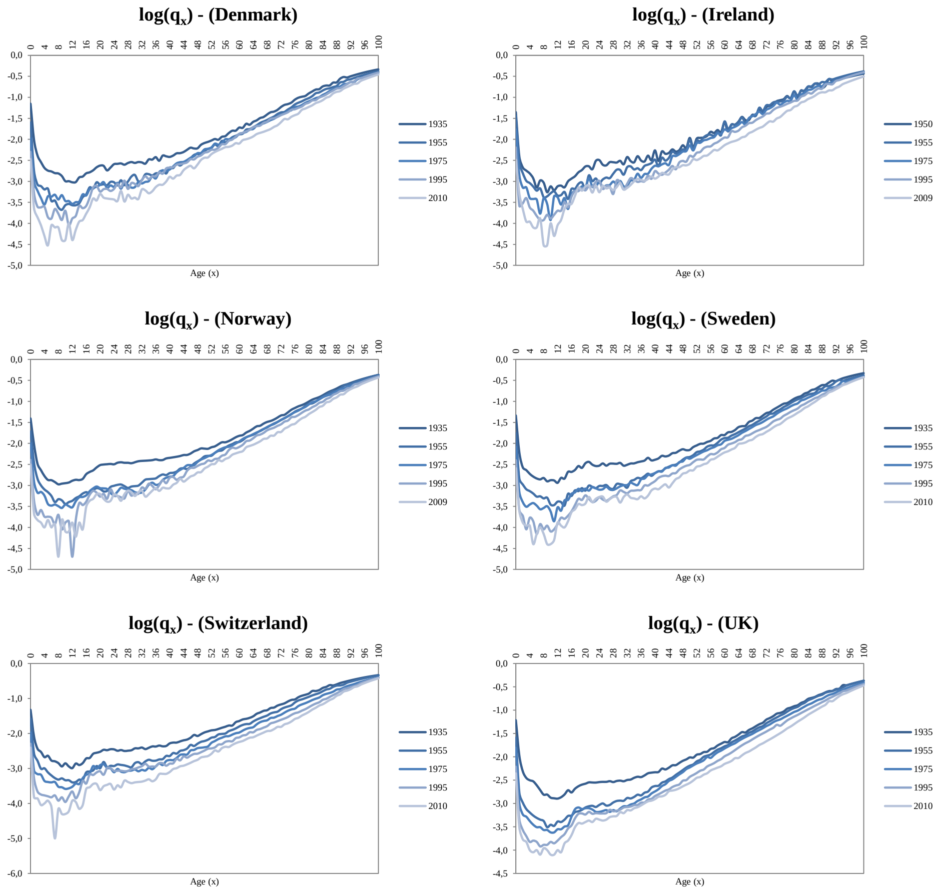


Figure 2.9 $\log(q_x)$ for Northern Europe countries

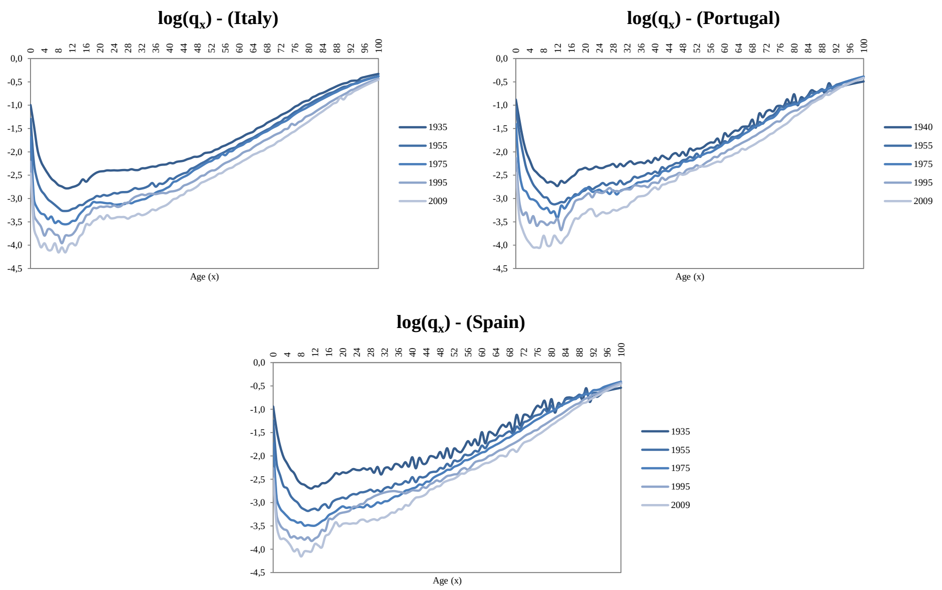


Figure 2.10 $\log(q_x)$ for Southern Europe countries

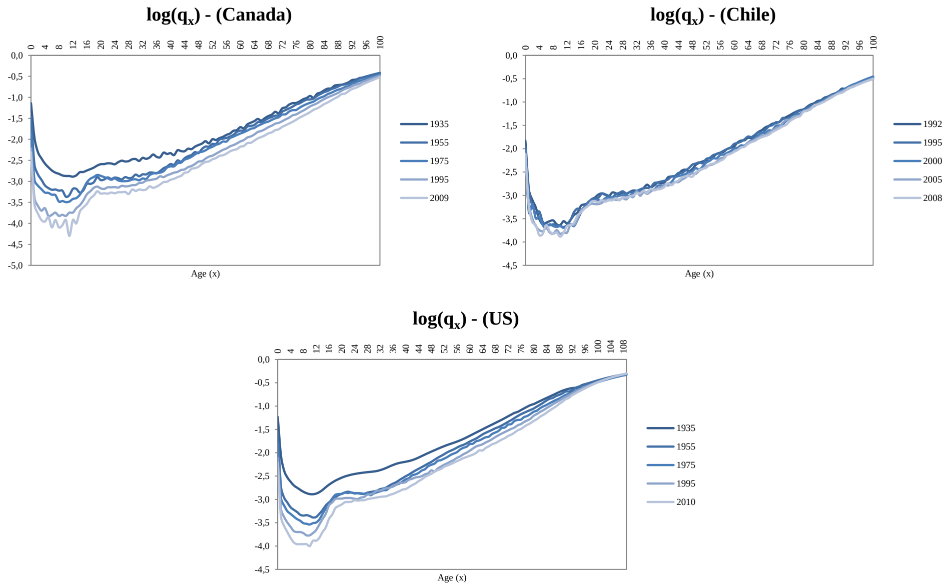


Figure 2.11 $\log(q_x)$ for American countries

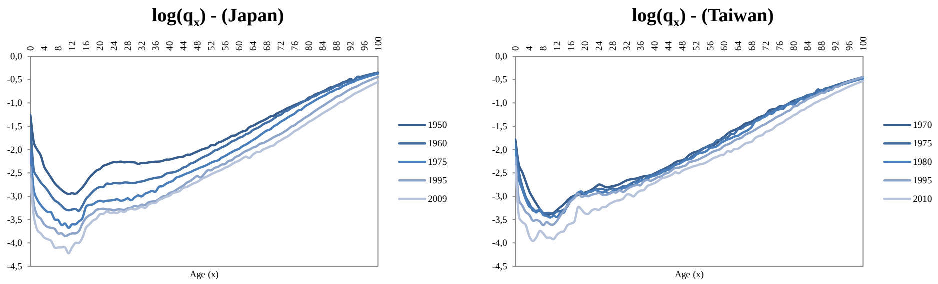


Figure 2.12 $\log(q_x)$ for Asian countries

As expected, the indicator has a decreasing trend for all ages and for all countries. The paths move down side (Figure 2.7-2.12). New born mortality rates are higher than the young's. There is a volatile structure between the 0-16 ages. The indicator has decreasing trend between the ages 0-12 and after has an increasing trend. The characteristic of Eastern Europe countries are that the rates rapidly decrease than the other countries.

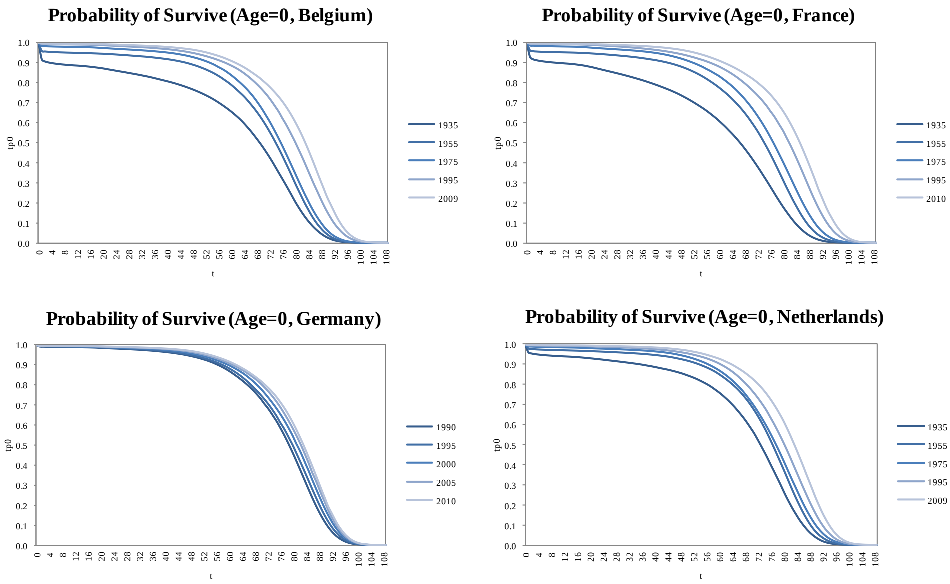


Figure 2.13 Probability of survive for Western Europe countries

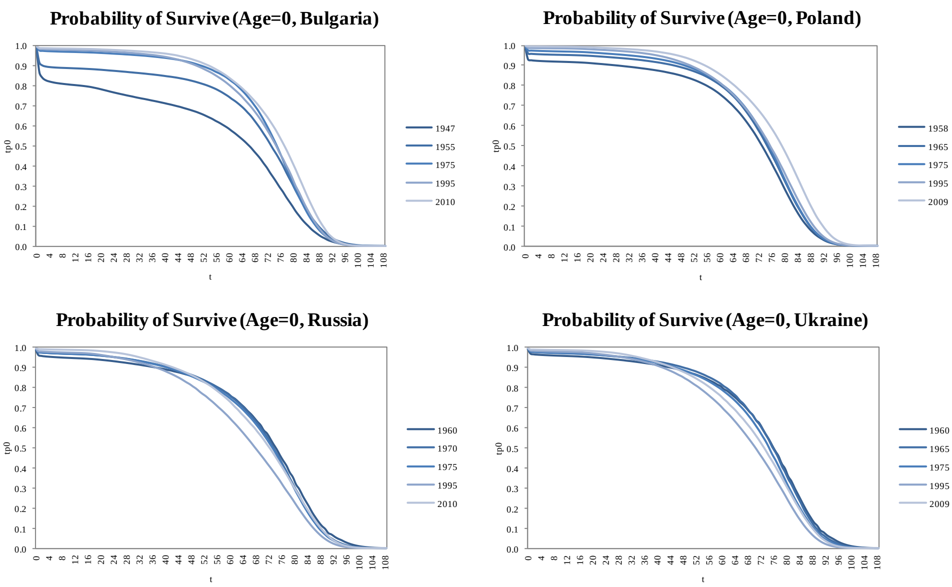


Figure 2.14 Probability of survive for Eastern Europe countries

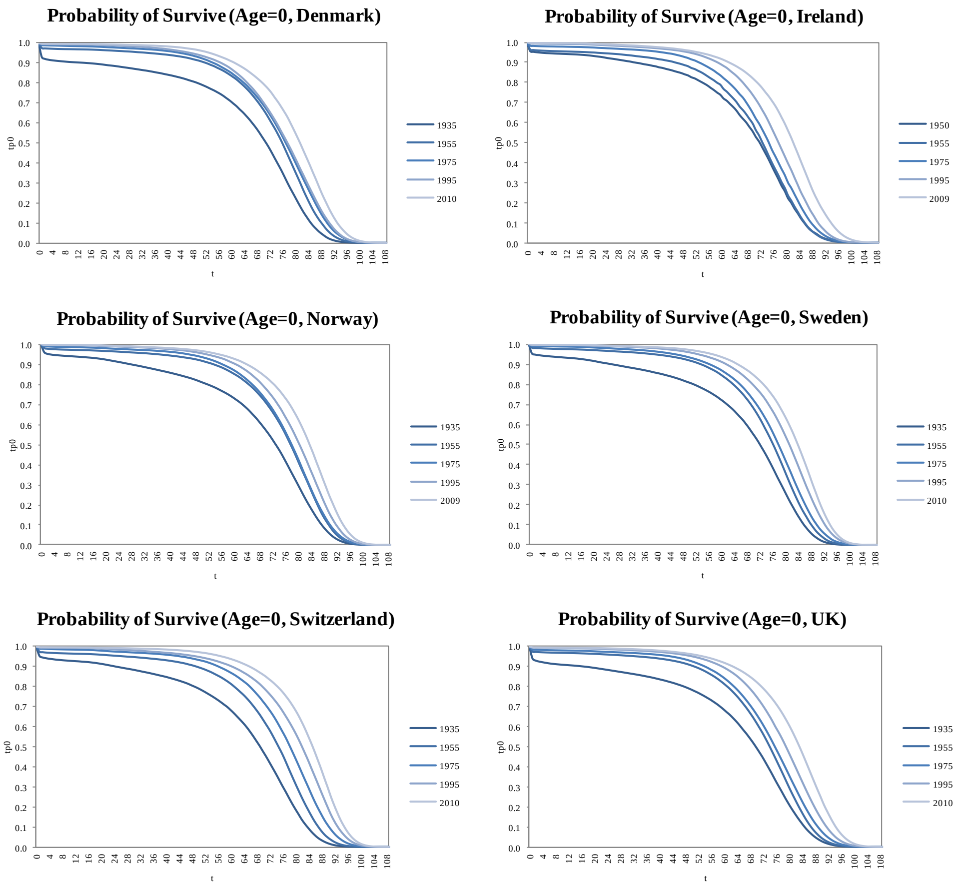


Figure 2.15 Probability of survive for Northern Europe countries

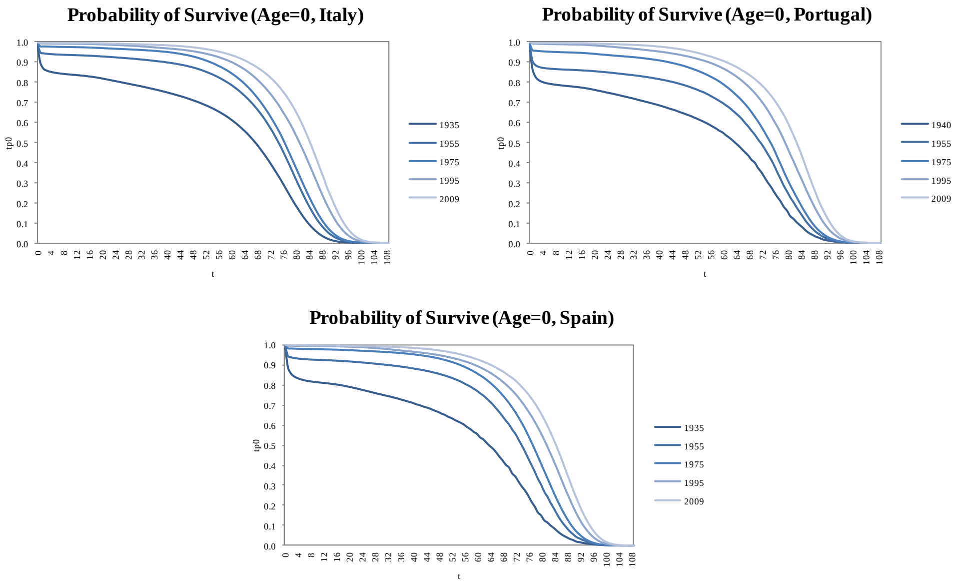


Figure 2.16 Probability of survive for Southern Europe countries

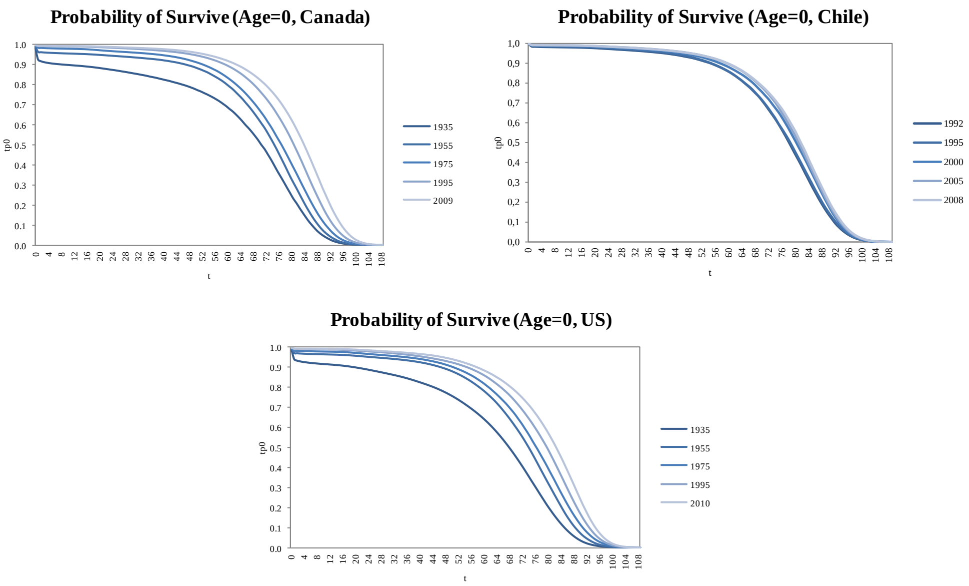


Figure 2.17 Probability of survive for American countries

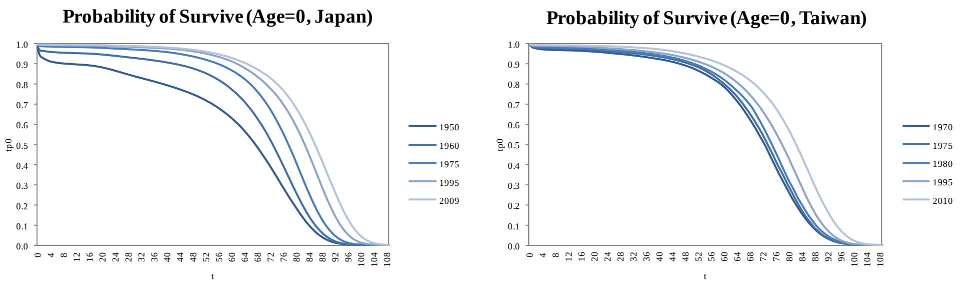


Figure 2.18 Probability of survive for Asian countries

The rectangularization of life tables is defined as a trend toward a more rectangular shape of the survival curve because of the increasing life expectancy. As it is seen from Figure 2.13 to 2.18, all countries show the rectangularization characteristics. These trends demonstrate increasing life expectancy.

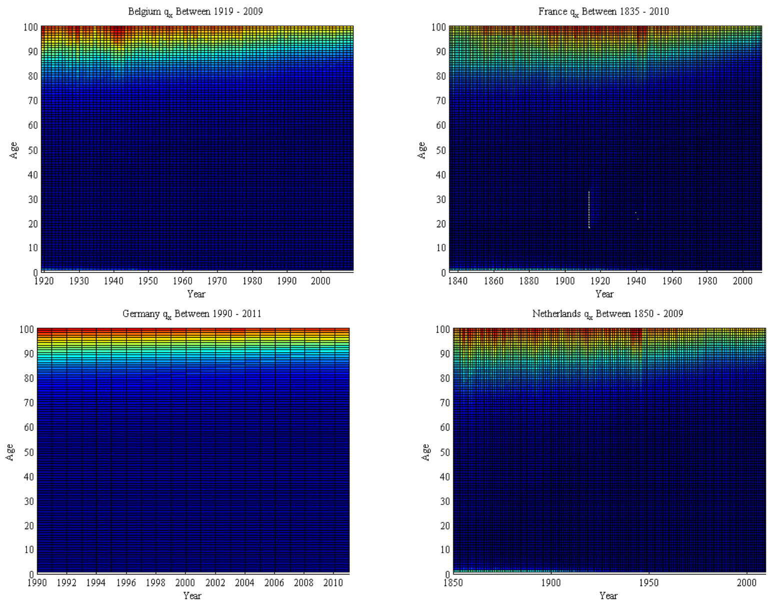


Figure 2.19 Probability of death for Western Europe countries

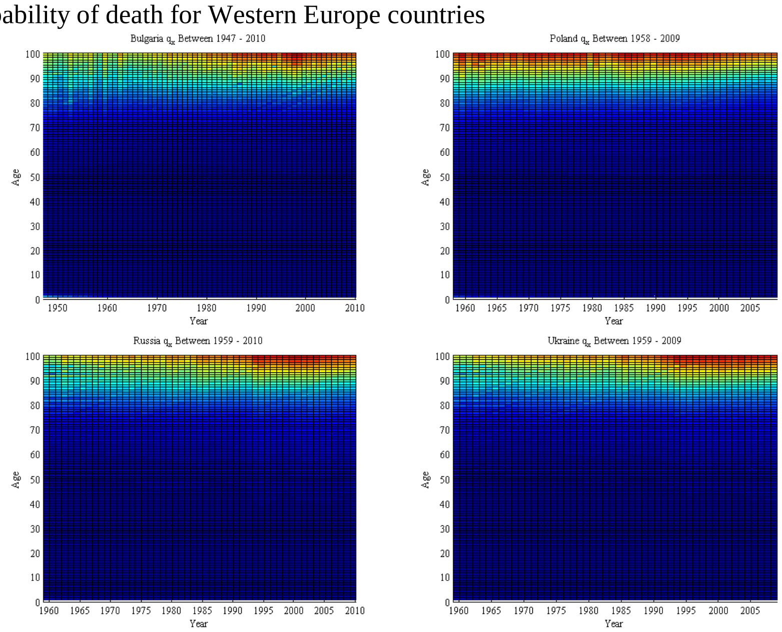


Figure 2.20 Probability of death for Eastern Europe countries

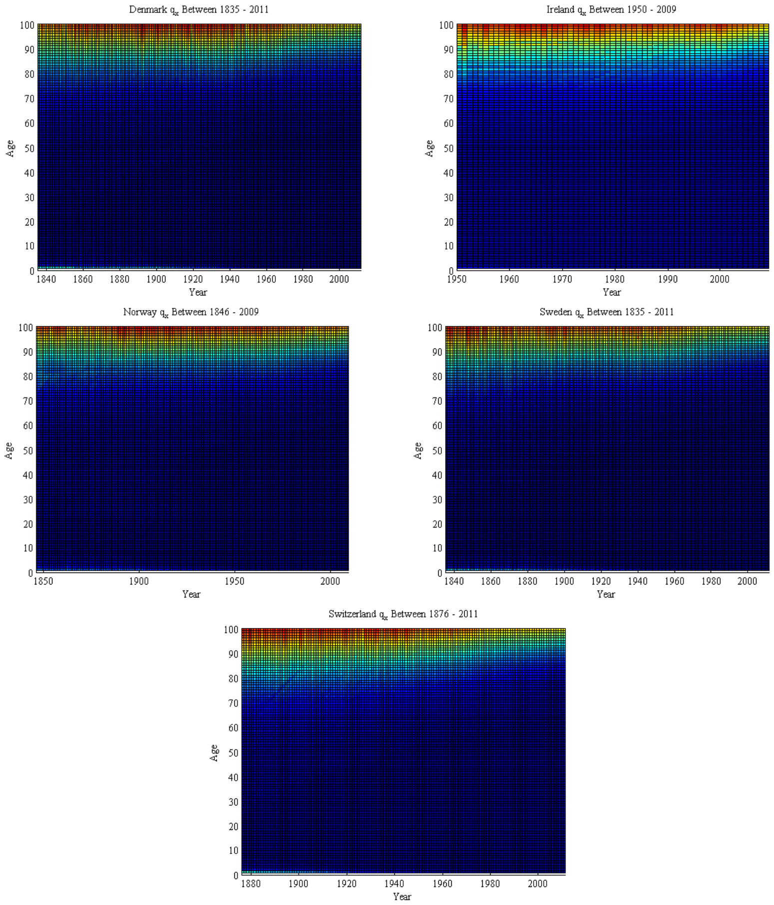


Figure 2.21 Probability of death for Northern Europe countries

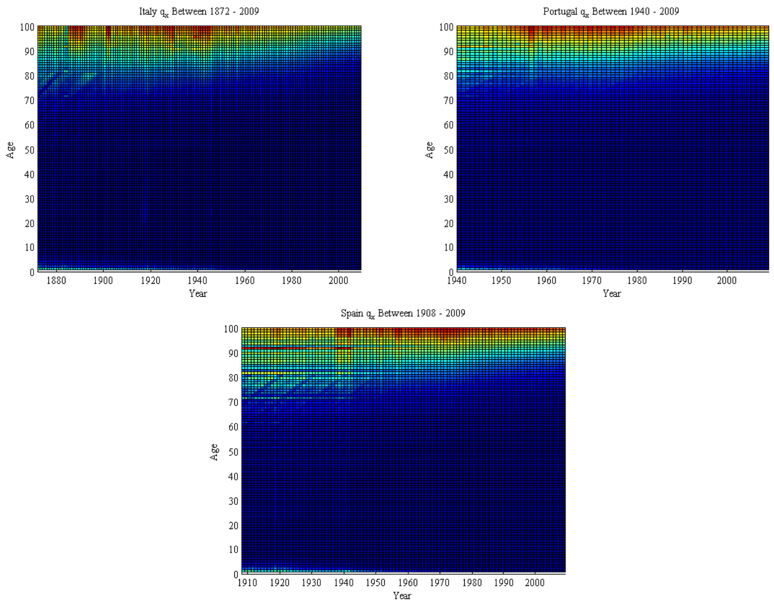


Figure 2.22 Probability of death for Southern Europe countries

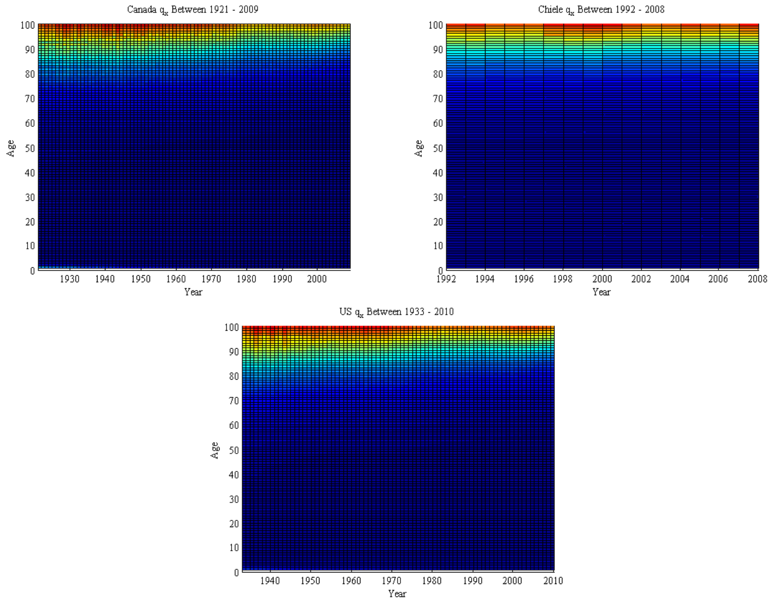


Figure 2.23 Probability of death for American countries

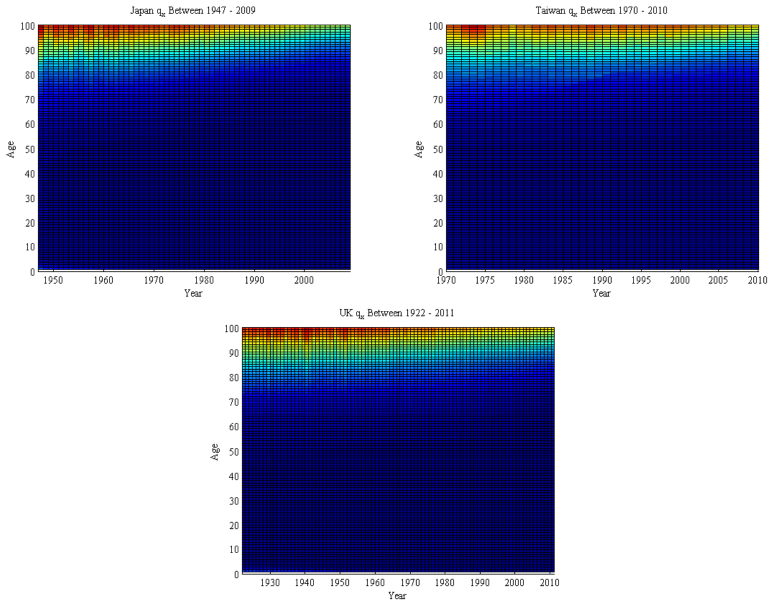


Figure 2.24 Probability of death for Asian countries

One year probability of death has a decreasing trend by the years and for all ages. As it is seen from the blue areas in Figure 2.19-2.24 increases towards to recent years.

2.2 Longevity Risk and Its Impact

There are many studies investigating the impact of the longevity on economy and financial systems. The results of mostly cited studies are given in this chapter.

Ronald Lee and Andrew Mason have many studies on the effect of aging on the macro economy. Lee and Mason (2010), Lee (2003), Lee (2014), Lee, Mason, Cotlear (2010), Lee and Mason (2011) are some of them. Some of the critical points mentioned by the authors are as follows. The proportion of elderly people rises at the most of the developed and developing countries. The main reason of this increase is longer life than the low fertility. Longer life and low fertility effect the implications of policy makers and individuals because most of the elderly people will not be the part of the labour force. Therefore their consumption must be funded by their family members, government or private pensions or their accumulated asset holdings. Nonetheless it becomes difficult to support by government or other ways since the ratio of elderly to working age people rises. Support ratio shows the number of workers divided the total population. Between 2010 and 2050 declining support ratios are expected to depress economic growth by 0.7% a year in Japan, Germany, and Spain and by 0.8% a year in Taiwan Province of China. In the United States, which is aging more slowly due to higher fertility, large net inflows of immigrants, and lower life expectancy, economic growth will be depressed by only 0.3% a year. The problem of funding the elder people in the future will be more important.

Longevity risk effects and endangers financial balances, fiscal stability and private balance sheets. The risk is a long term risk if it is not considered; in the future it may be needed large adaptation. Table 2.2 shows the impact of the longevity shock on the fiscal balances for the developed and developing countries. It is seen that many countries do not have enough financial assets. According to UN longevity assumptions, in most countries the president value of the pension income exceeds the household financial assets ((2)>(1)). As it is seen, there are differences between the results of countries because of the age structure. Column 5 shows if the longevity increases 3 years liabilities increase between 25-90% of GDP. Debt to GDP ratio is 150% for the US ((3)+(5)). If households in 2050 live 3 years longer than today, it needs to be an extra 1-2% GDP resource per year to compensate. If this shock had occurred today and the countries had aimed to allocate the required resource now, the developed countries should be needed 50% extra resources of their GDP, the developing countries should be needed 25% extra resources of their GDP (IMF, 2012).

Table 2.2 Longevity risk and fiscal challenges in selected countries

(In percent of 2010 nominal GDP)

Country	(1) Household Total Financial Assets (2010)	(2) Present Discounted Values of Needed Retirement Income	(3) General Government Gross Debt (2010)	(4) Gap: (1) – (2)	(5) Increase in Present Discounted Values Given Three-Year Increase in Longevity
United States	339	272 to 363	94	67 to –24	40 to 53
Japan	309	499 to 665	220	–190 to –356	65 to 87
United Kingdom	296	293 to 391	76	3 to –95	44 to 59
Canada	268	295 to 393	84	–27 to –125	42 to 56
Italy	234	242 to 322	119	–8 to –88	34 to 45
France	197	295 to 393	82	–97 to –196	40 to 54
Australia	190	263 to 350	21	–73 to –161	36 to 49
Germany	189	375 to 500	84	–186 to –311	55 to 74
Korea	186	267 to 357	33	–81 to –170	39 to 52
China	178	197 to 263	34	–19 to –85	34 to 45
Spain	165	277 to 370	60	–112 to –205	39 to 52
Hungary	108	190 to 254	80	–82 to –146	34 to 45
Czech Republic	89	216 to 289	39	–127 to –200	36 to 48
Poland	88	160 to 213	55	–72 to –125	27 to 35
Lithuania	80	189 to 252	39	–109 to –172	34 to 45

Sources: National flow of funds accounts; national accounts; IMF (2011c); and IMF staff estimates.

Note: Range of values in columns (2), (4), and (5) cover, at the low end, a replacement rate of 60 percent of preretirement income and, at the high end, an 80 percent replacement rate for retirees aged 65 or older to maintain preretirement standard of living during the 2010–50 period.

1For China, 2009.

Aging has some impacts on the macroeconomics. It changes consumption pattern toward non-tradables, reduces investment return, private and public saving, current account balance, growth rates, real interest rates, productivity growth and affects labour supply returns and also it has some impacts on the financial stability, it causes reallocation of saving from riskier to safe assets may lead to potential mispricing of risk, reallocation of saving, running down assets may result in negative wealth effects, changing borrowing habits may alter banks’ business model (IMF, 2012). Table 2.3 and Table 2.4 are taken from (IMF, 2012) and give the impact of aging on the macro economy and the impact of aging on the financial stability.

Table 2.3 Impact of aging on the macro economy

Framework	Variable	Impact	Channels
National account framework	Consumption	Changing consumption pattern toward nontradables	• Different consumption patterns for the elderly tend to shift demand toward services and lead to an increase in the price of nontradables compared with tradables, causing an increase in the real exchange rate.
	Investment	Reducing investment return	• If the aging population is also declining, this may lead additionally to falling rates of return on public investment. If governments do not plan for a declining population, existing public capital (e.g., schools, public infrastructure) may become underutilized to the extent that their use differs among generations.
	Savings	Reducing private and public saving	• According to the life-cycle hypothesis, older people will tend to liquidate existing savings. • Assuming no migration or fertility rise, with fewer active individuals, governments pay out more in health care and pension benefits and collect less tax revenue, leading to deteriorating fiscal conditions. • Rising fiscal deficits (negative public saving) could put the fiscal outlook on an unsustainable trajectory.
	Current account	Reducing current account balance	• The net effect of falling private and public saving on the current account depends on the relative changes in saving and investment. It is expected that the effect will apply to both current account surplus and deficit countries (see Lee and Mason, 2010). • The shrinking current account balance in some major countries, such as China and Japan, may contribute to the adjustment of global imbalances to the benefit of global financial stability.
	GDP	Reducing growth rates	• Skirbekk (2004) finds that skills that are key inputs to innovation—problem solving, learning, and speed—tend to degenerate with age, leading to a population that is less creative and entrepreneurial, thereby reducing growth rates. • Empirically, the IMF (2004) finds that per capita GDP growth is positively correlated with changes in the relative size of the working age population and negatively correlated with changes in the share of the elderly. • Empirical evidence from OECD countries shows that the complementary role of young and old workers means an optimum mix that exists may be damaged by having too many old workers (Feyrer, 2007).
Cobb–Douglas production function	Capital	Reducing real interest rates	• Aging is likely to translate into a gradual rise in the ratio of capital to labor and some concomitant decline in longer-term real interest rates (Visco, 2005). The flattened yield curve would reduce the effectiveness of monetary policy transmission and could impact institutions such as banks or pension funds that rely on a steep curve for their business model. This effect may be counterbalanced by decreasing saving, which may drive up interest rates.
	Labor	Affecting labor supply and returns	• An aging population will tend to shrink the labor force, which could lead to a lack of both unskilled and skilled workers. Countervailing factors, however, such as working longer (by raising the pension eligibility age for instance) or encouraging migration, could counteract the shrinking labor supply effect. • The higher capital-to-labor ratio would tend to lower expected returns on investment. Similarly, the same countervailing factors, such as working longer and immigration, may help buffer the decline in returns on investment.
	Productivity	Reducing productivity growth	• The elderly demand more services than the rest of the population (van Groezen, Meijdam, and Verbon, 2005), which tends to shift consumption toward services and away from durables. Given generally lower productivity growth in the service sector, this will tend to reduce productivity growth in the overall economy.

Table 2.4 Impact of aging on the financial stability

Balance Sheet Items	Impact	Channels
Assets	Reallocation of saving from riskier to safe assets may lead to potential mispricing of risk	• The rising demand for safe assets by the elderly (including through their pension funds) may lead to safe asset shortages and an overpricing of safe assets. At the same time, since risky assets such as equities are increasingly shunned, there is a possibility of an underpricing of riskier assets (Caballero, 2006). • These effects may be counterbalanced by defined-benefit funds with funding gaps in the current low interest-rate environment, which may invest in risky assets to enhance expected returns. Underpricing may also be mitigated by international investors' buying the cheaper risky assets.
	Running down assets may result in negative wealth effects	• Evidence is increasingly emerging that asset prices fall with advancing population aging (Poterba, 2004). For instance, an aging population, by requiring less housing, puts downward pressure on house prices (Takáts, 2010). The same principle applies to equity prices, although because equities are internationally tradable, they are somewhat less susceptible to supply/demand changes driven by aging (Brooks, 2006). • Negative wealth effects could have deflationary consequences (as suggested by Japan's experience), which could lead to a negative price spiral that further depresses economic activity.
Liabilities	Changing borrowing habits may alter banks' business model	• The business model of banks is closely related to the life-cycle behaviour of consumers. In their early years, consumers are net borrowers from banks, to pay for education and housing. Over their life time, consumers pay back their debt to banks. Therefore, in a consumer's later years, banks will increasingly be used for payment/transaction purposes, and less for maturity transformation. With fewer young borrowers, traditional lending activities would decline, and banks would have to enter new activities and act more like nonbanks. If not well managed (including through supervision), this transition could pose risks to financial stability. • With saving increasingly being channelled to capital markets via pension funds, the similarity of investment approaches may lead to herding, which, combined with procyclicality in the markets, could raise volatility and threaten financial stability.
	Individuals, governments, and pension providers face longevity risk	• Aging societies face heightened longevity risk—the risk of living longer than expected. Currently, there is a lack of instruments to hedge this risk. Those exposed—defined-benefit pension plan sponsors (i.e., corporations and governments), social security systems (i.e., governments), and individuals themselves—could face financial difficulties in the event of a realization of this risk. In the case of corporations, such difficulties could lead to potentially large changes in stock prices. Extreme longevity risk is likely to be borne by the sovereign, and a realization of this risk can lead to a substantial deterioration of the fiscal accounts and possible debt sustainability issues.

The simulation made by Börsch-Supan in 2006 shows that the cash flow is from rapidly aging countries to the other countries and this flow reverses when the household savings decreases. The aging characteristics differ between countries. Timing and some conditions also differs between countries these cause moving capital globally because of the aging. Econometric results show that the saving ratios, returns and international cash flow are related with the demographic changes and when the household labour force increases reaction of saving ratios, returns and the cash flow is less (Börsch-Supan, 2006)

Goyal investigates that the link between population age structure, net outflows from the stock market and stock market returns in an overlapping generation's framework in 2004. Results show that there is a positive correlation between the outflows from the stock market and the population of 65 over age and negative relationship with "45-64" middle age group. The change of population structure has a significant effect on the equity premium regression. The study dedicates that the stock returns increases when the middle age population increases and decreases when the old age population increases (Goyal, 2004).

Acemoglu et al. states that upward trend in the life expectancy increase the population and there is a positive relationship between the GDP. However, the effect size is small and this effect occurs for 40 years (Acemoglu et al., 2007).

The study of Batini et al. documents that world has a great demographic change from past to present. He is stated that the growth of population slowdowns and the demographic structure changes. The share of young population decreases, but old population conversely increases. The study investigates the impact of demographic changes to economy for Japan, US and the some European countries and the developing countries (Batini et al., 2006).

The findings of the paper are as follows; aging in industrial countries will reduce growth, demographic changes impact saving, investment and cash flow and also demographic changes increase the labour power therefore at 2050 the GDP will be greater 60% of today's GDP in developing countries and impact on the GDP growth for developing countries. According to this situation, at 2020 GDP will be greater than 2% than now. As expected, population change affects the labour power and private investments. There will be large changes in savings, investments and balance sheets because of the demographic changes in 80 years (Batini et al., 2006).

The labour force of the elderly people is less than the young. This leads to decrease the elderly people supply. However, their consumption increases. The main reason of the consumption increase is the government health expenditure so the aging is a costing situation for the economy. In the future the support ratio will decrease because of the aging nonetheless the aging can increase the investments and labour productivity and in this way productivity can balance the support ratio (Batini et al., 2006).

Paul Samuelson (1975) stated that the trade-off between the effects of demographic growth on dependency and on capital intensity is a central problem in the macroeconomics of population aging (Lee, 2012).

Geanakoplas et al. states that people choose different financial preferences during their lifetime for example: at young period they borrow, at middle age they invest and at the old age they expend their investments. Stocks are widely used as an investment tools in pension funds. If the middle age population is significantly large, the prices of the stocks increase otherwise there will be stagnation. The authors state that the equity premium is less when the saving age group is old. The result is the same whether the agents are myopic or not. The investments are directed to the stocks for the middle age group when the agents are myopic. The volume of middle age cohort and the stock price movements are correlated. The increase in share price is higher than the middle age group increase when the agents are fully informed. The real returns of assets and bonds should be the increasing function of the MY (middle aged to young adults) ratio. Price earnings (PE) ratio and MY ratio are synchronized. The effect of the demography on the PE ratio is higher than the earlier studies of the authors (Geanakoplas et al., 2004).

Ang et al. investigates the relationship between demographic changes and the equity risk premium for countries. Demographic changes have a statistically significant effect on the excess of return. Increases on the number of retire person decrease the risk premium. The old age group can affect more easily the share prices at the countries having less developed financial system than the other countries because the group prefers to hold the stocks (Ang et al., 2003)

At the end of this chapter, following critical indicators; life premiums in percentage of GDP, health expenditure in percentage of GDP, total investment of pension funds in percentage of GDP, public and private expenditure on pensions as a percentage of GDP are considered as the most crucial factors affecting by the longevity at the economy.

Table 2.5 Life premiums in % of GDP

	Life Premiums in % of GDP		Life Premiums in % of GDP
Australia	2.7	**Norway**	2.7
Belgium	4.3	**Poland**	1.6
Bulgaria	0.3	**Portugal**	6.4
Canada	2.9	**Russia**	0.1
Denmark	6.9	**Spain**	2.5
France	5.7	**Sweden**	5.6
Ireland	6.0	**Switzerland**	5.3
Italy	5.5	**UK**	8.8
Japan	8.8	**Ukraine**	0.2
Netherlands	3.2	**US**	3.2
EUROPE		**4.0**	
ASIA		**3.8**	
NORTH AMERICA		**3.1**	
WORLD		**3.5**	

Source: World Insurance in 2013, Sigma No 3/2014, Swiss Re

Life premiums in percentage of GDP for the countries in 2013 are given at the Table 2.5. The Europe average is 4%, Asia average is 3.8%, North America average is 3.1% and World average is 3.5%. Denmark, France, Ireland, Italy, Japan, Portugal, Sweden, Switzerland, UK have the average higher than 5%. Especially, the average of Japan and UK stand out.

Table 2.6 Private sector, government and total health expenditure (% of GDP)

	Private Sector Health Expenditure (% of GDP)					Government Health Expenditure (% of GDP)					Total Health Expenditure (% of GDP)				
	2000	**2005**	**2010**	**2012**	**2013**	**2000**	**2005**	**2010**	**2012**	**2013**	**2000**	**2005**	**2010**	**2012**	**2013**
Australia	2.68	2.80	2.87	..	..	5.39	5.65	6.06	..	..	8.07	8.45	8.93	..	..
Austria	2.44	2.57	2.73	2.67	..	7.58	7.85	8.40	8.42	..	10.02	10.42	11.13	11.10	..
Belgium	2.06	2.51	2.64	2.70	..	6.06	7.14	7.92	8.19	..	8.12	9.65	10.56	10.89	..
Canada	2.57	2.85	3.25	3.27	..	6.10	6.72	7.86	7.66	..	8.67	9.57	11.11	10.93	..
Chile	3.06	4.15	3.67	3.72	3.97	3.33	2.62	3.47	3.60	3.38	6.40	6.77	7.14	7.32	7.35
Cz. Republic	0.61	0.88	1.21	1.21	..	5.70	6.05	6.22	6.34	..	6.31	6.93	7.43	7.55	..
Denmark	1.40	1.52	1.65	1.56	..	7.30	8.25	9.43	9.42	..	8.70	9.77	11.08	10.98	..
Estonia	1.19	1.16	1.28	1.17	..	4.08	3.85	4.98	4.64	..	5.29	5.02	6.32	5.89	..
Finland	2.07	2.21	2.32	2.27	2.32	5.14	6.22	6.67	6.82	7.08	7.22	8.43	8.99	9.09	9.40
France	2.08	2.41	2.60	2.63	..	8.01	8.53	8.96	8.98	..	10.08	10.93	11.55	11.61	..
Germany	2.13	2.53	2.69	2.62	2.62	8.27	8.28	8.87	8.64	8.68	10.40	10.81	11.56	11.27	11.30
Greece	3.18	3.85	3.02	2.92	..	4.77	5.81	6.32	6.22	..	7.95	9.66	9.48	9.27	..
Hungary	2.10	2.53	2.84	2.98	..	5.08	5.91	5.22	4.98	..	7.18	8.45	8.06	7.97	..
Iceland	1.80	1.76	1.82	1.76	1.77	7.70	7.68	7.47	7.28	7.29	9.50	9.44	9.29	9.04	9.06
Ireland	1.60	1.82	2.79	2.88	..	4.58	5.76	6.41	6.00	..	6.18	7.58	9.21	8.87	..
Israel	2.55	2.92	2.66	2.84	..	4.54	4.48	4.49	4.39	..	7.26	7.55	7.27	7.35	..
Italy	2.03	1.93	1.98	2.08	2.00	5.84	6.82	7.42	7.10	7.09	7.87	8.74	9.41	9.19	9.09
Japan	1.46	1.51	1.72	1.84	..	6.14	6.67	7.87	8.44	..	7.60	8.18	9.59	10.28	..
Korea	2.27	2.66	3.18	3.47	3.62	2.18	3.00	4.15	4.16	4.15	4.45	5.66	7.33	7.63	7.76
Luxembourg	1.12	1.20	1.09	1.19	..	6.36	6.75	6.59	5.99	..	7.48	7.95	7.64	7.13	..
Mexico	2.66	3.42	3.28	3.04	..	2.32	2.52	2.98	3.12	..	4.98	5.94	6.26	6.16	..
Netherlands	2.94	..	..	..	..	5.02	..	..	..	..	7.96	10.88	12.15	..	..
New Zealand	1.66	1.70	1.67	..	..	5.90	6.65	8.28	..	..	7.56	8.34	9.95	..	..
Norway	1.47	1.49	1.44	1.39	1.40	6.95	7.54	7.98	7.90	8.20	8.42	9.03	9.42	9.28	9.60
Poland	1.65	1.90	1.98	2.05	..	3.87	4.31	5.00	4.67	..	5.52	6.21	7.02	6.76	..
Portugal	3.11	3.32	3.68	..	..	6.19	7.04	7.12	..	..	9.30	10.35	10.80	..	..
Slovak Republic	0.58	1.80	2.71	2.47	..	4.92	5.24	5.80	5.68	..	5.50	7.04	8.51	8.15	..
Slovenia	2.15	2.28	2.34	2.67	..	6.12	6.21	6.73	6.70	..	8.26	8.50	9.07	9.37	..
Spain	2.05	2.41	2.47	2.63	..	5.17	5.88	7.17	6.67	..	7.21	8.29	9.65	9.29	..
Sweden	1.24	1.71	1.75	1.80	..	6.94	7.35	7.72	7.78	..	8.18	9.06	9.47	9.58	..
Switzerland	4.42	4.40	3.79	3.91	..	5.49	6.46	7.12	7.52	..	9.91	10.86	10.91	11.43	..
Turkey	1.83	1.75	1.20	1.25	..	3.11	3.70	4.41	4.14	..	4.95	5.45	5.61	5.39	..
United Kingdom	1.45	1.52	1.50	1.49	..	5.49	6.61	7.87	7.79	..	6.93	8.13	9.37	9.27	..
United States	7.48	8.50	8.97	8.86	..	5.65	6.73	8.08	8.04	..	13.14	15.23	17.05	16.90	..

Source: OECD stats

As mentioned previous paragraphs, one of the critical point will be health expenditure in the future. All countries approximately have the 10% health expenditure in GDP. This ratio will increase by increasing the expected life time (Table 2.6).

Table 2.7 Total investment of pension funds in OECD and selected non-OECD countries, 2001-2013 as a % of GDP

	2001	2005	2006	2007	2008	2009	2010	2011	2012	2013
Australia	73.5	78.1	87.5	106.1	93.1	82.5	89.5	92.7	91.4	103.3
Austria	3.0	4.8	4.9	4.8	4.4	5.1	5.3	4.9	5.3	5.8
Belgium	5.5	4.4	4.2	4.4	3.3	4.1	3.7	4.2	4.6	5.2
Canada	51.2	56.7	61.6	61.0	50.1	58.7	63.1	62.2	65.6	71.3
Chile	..	55.6	57.5	61.0	49.8	62.0	62.6	58.0	59.8	62.2
Czech Rep.	2.2	4.0	4.4	4.6	5.0	5.7	6.1	6.5	7.1	7.7
Denmark	27.2	33.8	32.6	32.4	47.0	43.1	49.3	49.6	50.0	42.8
Estonia	0.0	2.7	3.6	4.4	4.5	6.8	7.5	7.0	8.5	9.6
Finland	49.7	68.6	71.9	70.6	60.7	77.2	82.8	44.2	47.1	50.8
France	..	0.0	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.4
Germany	3.5	4.1	4.2	4.6	4.8	5.3	5.4	5.7	6.3	6.2
Hungary	3.9	8.5	9.8	11.1	9.7	13.3	15.0	3.8	4.0	4.1
Iceland	84.0	119.6	129.6	131.0	112.9	119.2	124.2	128.8	140.9	148.7
Ireland	43.5	47.8	49.4	45.7	35.2	44.5	47.8	44.5	49.1	55.8
Israel	22.5	30.2	29.9	31.1	40.1	44.0	45.9	46.5	48.7	50.4
Italy	2.2	2.8	3.0	3.2	3.4	4.1	4.6	4.9	5.6	6.1
Korea	..	1.7	2.8	2.8	3.0	3.5	4.0	4.5	5.4	6.5
Luxembourg	..	1.1	1.0	1.0	1.0	2.4	2.0	2.0	2.1	2.1
Mexico	3.8	8.8	10.0	9.9	10.0	11.7	12.6	12.8	14.1	14.8
Netherlands	102.6	120.7	124.4	135.1	112.7	118.6	129.5	136.2	155.4	160.6
New Zealand	15.5	11.5	12.5	11.6	10.5	11.9	14.3	15.8	16.8	19.1
Norway	5.5	6.7	6.7	7.0	6.0	7.4	7.6	7.3	7.6	8.3
Poland	2.4	8.7	11.1	12.0	10.9	13.5	15.7	15.0	17.2	18.6
Portugal	11.0	12.3	13.2	13.2	11.8	13.0	11.4	7.7	8.8	9.1
Slovak Republic	0.0	0.5	2.4	3.7	4.7	6.3	7.4	8.4	9.6	10.0
Slovenia	..	1.3	1.6	1.8	1.9	2.6	3.1	3.3	3.7	4.0
Spain	5.8	7.2	7.5	8.2	7.2	8.1	8.0	8.0	8.4	9.0
Sweden	8.0	9.0	9.1	8.5	7.3	8.2	9.5	9.2	10.5	9.5
Switzerland	99.5	113.3	114.8	112.0	94.8	108.0	108.5	106.9	113.7	119.4
Turkey	..	0.7	0.7	1.2	1.5	2.3	2.4	4.1	3.8	4.9
United Kingdom	70.0	76.0	80.6	76.5	63.5	79.3	86.8	94.0	102.9	100.7
United States	70.7	74.5	77.1	78.0	59.5	70.0	74.6	71.7	74.3	83.0

Source: OECD Global Pension Statistics.

Total investment of pension funds as a percentage of GDP is given at Table 2.7. UK has a 100.7%, US has a 83%, Netherlands has a 160.6%. As it is seen from the ratios, especially for some countries such as UK, US pension fund investments can affect the financial dynamics easily.

Table 2.8 Public and private expenditure on pensions as a percentage of GDP

	Public expenditure						Private expenditure					
	2000	**2005**	**2006**	**2007**	**2008**	**2009**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**
Australia	3.8	3.3	3.3	3.4	3.6	3.5	3.3	5.6	4.7	4.5	4.6	4.7
Austria	12.2	12.4	12.3	12.2	12.4	13.5	0.3	0.2	0.2	0.2	..	..
Belgium	8.9	9.0	8.9	8.8	9.4	10.0	2.7	2.6	3.2	2.9	3.7	..
Canada	4.3	4.1	4.1	4.1	4.2	4.5	2.2	2.3	2.5	2.5	2.8	3.0
Chile	7.3	5.7	5.1	4.9	3.3	3.6	1.9	2.0	1.7	2.0	2.2	2.3
Czech Republic	7.2	7.0	6.9	7.1	7.4	8.3	0.3	0.3	0.4	0.5	0.5	0.6
Denmark	5.3	5.4	5.5	5.5	5.6	6.1	3.3	4.1	4.3	4.5	4.9	5.1
Estonia	6.0	5.3	5.3	5.1	6.2	7.9	..	..	0.0	0.0	0.0	0.0
Finland	7.6	8.4	8.5	8.3	8.4	9.9	0.5	0.6	0.7	0.6	0.7	0.7
France	11.8	12.4	12.4	12.5	12.9	13.7	..	..	0.4	0.4	0.4	..
Germany	11.1	11.4	11.0	10.6	10.5	11.3	0.1	0.1	0.3	0.2	0.2	0.2
Greece	10.8	11.8	11.8	12.1	12.4	13.0	0.0	0.0	0.0	0.0	0.0	0.0
Hungary	7.6	8.5	8.8	9.3	9.7	9.9	0.2	0.2	0.2	0.2	0.2	0.2
Iceland	2.2	2.0	1.8	1.9	1.8	1.7	3.6	3.8	6.4	5.5	6.3	5.7
Ireland	3.1	3.4	3.4	3.6	4.1	5.1	..	..	..	..	..	..
Israel	4.9	5.1	5.0	5.0	4.8	5.0	1.7	1.7	1.7	1.7	1.7	1.7
Italy	13.5	13.9	13.9	14.0	14.5	15.4	0.2	0.3	0.2	0.3	0.2	0.3
Japan	7.3	8.7	8.7	8.9	9.3	10.2	..	..	..	..	..	..
Korea	1.4	1.5	1.6	1.7	2.0	2.1	0.9	0.8	1.1	1.4	1.4	1.8
Luxembourg	7.5	7.2	6.8	6.5	6.6	7.7	0.1	0.1	0.1	0.1	0.1	0.1
Mexico	0.9	1.2	1.2	1.3	1.4	1.7	0.2	0.2	0.3	0.3	0.3	0.2
Netherlands	5.0	5.0	4.8	4.7	4.7	5.1	3.5	3.6	3.9	4.0	4.2	4.3
New Zealand	5.0	4.3	4.3	4.3	4.4	4.7	1.3	1.4	2.0	1.4	1.3	1.4
Norway	4.8	4.8	4.6	4.7	4.5	5.4	0.9	0.9	1.1	1.0	1.0	1.0
Poland	10.5	11.4	11.5	10.6	10.8	11.8	0.0	0.0	0.0	0.0	0.0	0.0
Portugal	7.9	10.3	10.6	10.7	11.3	12.3	0.9	1.4	1.0	0.7	0.8	0.5
Slovak Republic	6.3	6.2	6.0	5.9	5.7	7.0	..	..	..	..	..	0.1
Slovenia	10.5	9.9	10.0	9.6	9.5	10.9	0.0	0.0	0.0	0.0	0.5	0.9
Spain	8.6	8.1	8.0	8.1	8.4	9.3	0.5	0.6	0.6	0.6	0.7	0.7
Sweden	7.2	7.6	7.3	7.2	7.4	8.2	1.2	1.2	1.3	1.3	..	..
Switzerland	6.6	6.8	6.5	6.4	6.3	6.3	5.1	5.0	5.3	4.9	4.9	5.0
Turkey	4.9	5.9	5.8	6.1	5.5	6.8	0.1	0.1	0.1	0.1	0.0	..
United Kingdom	5.3	5.6	5.3	5.3	5.8	6.2	2.8	2.9	3.2	3.3	3.2	..
United States	5.9	6.0	5.9	6.0	6.2	6.8	3.2	3.0	2.9	3.2	3.1	..
OECD	6.9	7.0	7.0	7.0	7.1	7.8	1.4	1.5	1.6	1.6	1.7	1.6

Source: OECD Factbook 2014

Table 2.8 gives the old-age and survivors cash benefits. Old-age cash benefits are all cash expenditures (including lump-sum payments) on old-age pensions within the public scope. Survivors benefits are payments made to family members when a worker dies. The expenditure of Austria, Belgium, Denmark, Finland, France, Germany, Greece, Hungary, Italy, Poland, Portugal, Slovenia and Switzerland has the percentage greater than 10%. By the way, the expenditure ratio has an increasing trend.

CHAPTER 3: MORTALITY MODELS AND MODEL SELECTION

3.1 Constructing a Mortality Model

As stated Hunt and Blake (2013), the majority of existing mortality models proposed in the literature has three mean axes; age, cohort and period. Observed mortality rates are fitted the function including age factor, year factor and year of birth.

The general form of the frequently cited mortality models in the literature is as follows;

$$\eta\left(E\left(\frac{D(x,t)}{E(x,t)}\right)\right)=\alpha_x+\sum_{i=1}^{N}f^{(i)}(x;\theta_i)\kappa_t^{(i)}+\gamma_{t-x}\,. \tag{3.1}$$

Here, η is a link function transforming the observed data in to a form suitable for modelling,

- $D(x,t)$ is the death counts,
- $E(x,t)$ is the exposures for the age *x* and year *t*,
- α_x captures the general shape of mortality curve,
- $\kappa_t^{(i)}$ is the trend changes with time,
- $f^{(i)}(x;\theta_i)$ is the age function,
- γ_{t-x} is the cohort parameter which determines the generation effect.

Lee and Carter model (1992), Renshaw and Haberman (2006) propose the extension of Lee and Carter model, the Cairns-Blake-Dowd models, the models proposed by Plat (2009), Haberman and Renshaw (2012) fall within this structure with suitable link function η.

A general procedure for constructing mortality models is defined by Hunt and Blake (2013). The steps of the algorithm are as follows;

1. Start with a static life table α_x to capture the time independent shape of the mortality curve across ages,
2. Add β_x and κ_t which is a age and period function to find the age/period effect not captured by the model,
3. Observe the shape of the estimated age term β_x across ages and κ_t pattern through time,
4. Check that the addition of the new pair of terms improves the overall goodness of fit to the data,
5. Select an appropriate $f(x,\theta)$ to replace the non-parametric age term β_x. θ is the parameters in the function,
6. Check whether the fitted model with this specific functional form, a) produces similar evolution over time as the κ_t's model, b) achieves comparable improvements in the goodness of fit as the non parametric term,
7. Check whether the addition of the new companion pair of terms has significantly changed the shape of previously selected terms, in which case it might be needed to change and re-estimated the earlier terms,
8. Repeat steps 2 to 7 until model captures all significant age and period structure in the data,
9. Add a cohort term γ_{t-x} to capture the effect of generation,
10. Test the final model for goodness of fit and robustness and the residuals for the properties of normality and independence.

Hunt and Blake (2013) use this procedure and fit UK mortality data. They compare the results with the Lee and Carter model and principle component analysis (PCA) model. The results show that the general procedure produces a relatively parsimonious model that nevertheless has a good fit to the data.

3.2 Most Commonly Used Mortality Models

The mostly cited mortality models, Lee and Carter, Renshaw and Haberman, Currie, Cairns, Blake and Dowd, first generalisation of Cairns, Blake and Dowd, second generalisation of Cairns, Blake and Dowd and third generalisation of Cairns, Blake and Dowd are presented in detail in this chapter. Lee and Carter model (1992) has no cohort effect. Renshaw and Haberman (2006) generalise the Lee Carter model with a cohort effect. Currie (2006) model is an age-period-cohort (APC) model. Cairns, Blake and Dowd model (CBD) give a new model (2006) and they generalise their model with third extensions (Cairns et al. 2007).

In the literature, there are also affine term structure mortality models. It is not given in this chapter. The details can be found at Appendix I.

3.2.1 Lee and Carter Model

Lee and Carter (1992) fit the US mortality rates between 1933 and 1977 using the singular value decomposition (SVD) method. The method performs well on with in-sample forecasts. However, there is some instability for base periods of 10 or 20 years. The method proposed by Lee and Carter (1992) is extrapolative and there is no need to incorporate knowledge about medical, behavioural or social influences on mortality change.

Proposed model by Lee and Carter is as follows;

$$\log[m(x,t)] = \beta_x^{(1)} + \beta_x^{(2)} k_t + \varepsilon_{x,t}. \qquad (3.2)$$

$m(x,t)$ is the central death rate for age x in year t. There is a following relationship between $q(x,t)$ and $m(x,t)$; $q(x,t) \approx 1 - e^{-m(x,t)}$. k_t component captures the main time trend and b_x determines the trend change of x whether the change is faster or slower. The error term, $\varepsilon_{x,t}$, with mean *0* and variance σ_{ε}^{2} is for capturing the influences not explained by the model.

In order to obtain unique solution, there are some conditions such as;

$$\beta_x^{(1)} = \frac{1}{T}\sum \log m(x,t), \tag{3.3}$$

$$\sum \beta_x^{(2)} = 1, \tag{3.4}$$

$$\sum k_t = 0. \tag{3.5}$$

The model can not be fit by ordinary regression methods because there are no given regressors. The singular value decomposition (SVD) can be used to find a least squares solution. Firstly singular value decomposition (SVD) is applied to the matrix of $\{\log m(x,t) – \beta_x^{(1)}\}$ and then $\beta_x^{(2)}$ and k_t components are estimated by using this composition.

The estimation procedure is very simple after fitted $\beta_x^{(1)}$, $\beta_x^{(2)}$ and k_t. The only thing is the estimation of k_t by using a standard univariate time series model ARIMA. The model is as follows;

$$\hat{k}_t = \hat{k}_{t-1} + \beta + \varepsilon_t. \tag{3.6}$$

β is the drift term and ε is the error term.

Advantages of Lee and Carter model are robust, simple one factor model and good fit over wide age ranges.

Lee and Carter model has some basic disadvantages. There is a lack of smoothness of age effect. It can not cope with improvements at different ages and at different times. It is possible underestimation of uncertainty. $\beta_x^{(2)}$ affects both trend and uncertainty at age *x*. There is no cohort effect.

Lee and Carter model has a wide acceptance by academicians and practitioners because of its easy implementation and good fitness. It is used also as a benchmark methodology by the US Bureau of the Census.

3.2.2 Renshaw and Haberman Model

Renshaw and Haberman (2006) investigate the feasibility of extending the methodology to the modelling and projection of age–period–cohort effects. The main contribution of their study is the incorporation of cohort effects into the Lee and Carter methodology.

Their model is as follows;

$$\log[m(x,t)] = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)}. \tag{3.7}$$

There are following constraints to satisfy identifiability;

$$\sum_t \kappa_t^{(2)} = 0\,, \tag{3.8}$$

$$\sum_x \beta_x^{(2)} = 1\,, \tag{3.9}$$

$$\sum_x \beta_x^{(3)} = 1\,, \tag{3.10}$$

$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0\,. \tag{3.11}$$

Renshaw and Haberman model provides good fit to the historical data where countries have significant cohort effect. One drawback of this model is lack of robustness. The robustness causes the likelihood function possibly more than one maximum. CMI (2007) and Cairns et al. (2007, 2008b) state this situation. Cairns (2008) states that the optimises periodically jump from one local maximum to another with qualitatively quite different characteristics when they change the age or year range.

Cairns et al. (2009) compare eight stochastic models for mortality rates in England and Wales and in the United States. The age is selected between 60 and 89. According to the Bayesian Information Criterion (BIC), an extension of the Cairns-Blake-Dowd model for higher ages fits the England and Wales males data best. Renshaw and Haberman model fits the best to US males data. However, the authors identify some problems with the robustness of parameter estimates.

Cairns et al. (2009) found that parameter values converge very slowly to their maximum likelihood estimates and state this cause an identifiability problem.

3.2.3 Currie Model

Currie (2006) proposes the following model;

$$\log[m(x,t)] = \beta_x^{(1)} + \kappa_t^{(2)} + \gamma_{t-x}^{(3)}. \tag{3.12}$$

P splines are used to fit $\beta_x^{(1)}$, $\kappa_t^{(2)}$ and $\gamma_{t-x}^{(3)}$ to ensure smoothness. It is the special case of the RH model.

The model constraints are as follows;

$$\sum_t \kappa_t^{(2)} = 0\,, \tag{3.13}$$

$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0\,. \tag{3.14}$$

The revised parameters are as follows when the two constraints are satisfied;

$$\tilde{\kappa}_t^{(2)} = \kappa_t^{(2)} - \delta(t - \bar{t}), \tag{3.15}$$

$$\gamma_{t-x}^{(3)} = \gamma_t^{(3)} + \delta((t - \bar{t}) - (x - \bar{x})), \tag{3.16}$$

$$\tilde{\beta}_x^{(1)} = \beta_x^{(1)} - \delta(x - \bar{x}). \tag{3.17}$$

Renshaw and Haberman model and Currie model belongs the Lee and Carter family. They are the generalized case of the Lee and Carter model.

3.2.4 Cairns, Blake and Dowd Models

In the literature review it is seen that the papers of Cairns, Blake and Dowd (CBD) and their models are mostly used by the academicians and practitioners. Also in this thesis their models are investigated and applied to the different countries.

At first it is useful to give the notation in the CBD papers. Let $D(x,t)$ denotes the number of deaths during calendar year *t* and age *x* and $E(x,t)$ denotes average population during calendar year *t* and age *x*.

- $m(x,t)$ is the central death rate for age *x* in calendar year *t*;

$$m(x,t) = \frac{D(x,t)}{E(x,t)}, \tag{3.18}$$

- $q(x,t)$ is the mortality rate which is defined as the probability that an individual aged exactly *x* at exact time *t* will die between *t* and *t+1*,
- $\mu(x,t)$ is the force of mortality. This is the instantaneous death rate. The probability of death between *t* and *t*+*dt* is approximately $\mu(x,t)dt$,

Relationship between $m(x,t)$ and $q(x,t)$ is defined as follows;

- The death rate $m(x,t)$ and $q(x,t)$ are very close to each other.

Cairns et al. (2007) give two assumptions;

1. For integers t and x and for all $0 \leq s,\ u < 1,\ \mu(x+u, t+s) = \mu(x,t)$, the force of mortality remains constant over each year of integer age and over each calendar year,
2. They assume that there is a stationary population. The size of the population at all ages remains constant over time.

These assumptions imply the following results;

1. $m(x,t) = \mu(x,t)$: It is used generally the analysis of death rate data,
2. $q(x,t) = 1 - \exp[-\mu(x,t)] = 1 - \exp[-m(x,t)]$: It is generally used for the modelling mortality by $q(x,t)$.

Estimation procedure of the model parameters are as follows;

1. $D(x,t)$ is assumed to have a Poisson distribution, $D(x,t) \sim \text{Poisson}(\lambda(x,t))$ where $\lambda(x,t) = E(x,t)m(x,t)$ (Brouhns et al., 2002),
2. The authors use in some of their models, $m(x,t)$ and some of their models, $q(x,t)$. However to ensure a valid comparison between the different models, firstly $q(x,t)$ is calculated then transform these into death rates using the identity,

$$m(x,t) = -\log[1 - q(x,t)]. \tag{3.19}$$

For all models the log-likelihood function is as follows;

$$L(\phi; D, E) = \sum_{x,t}\{D(x,t).\log[E(x,t).m(x,t)(\phi)] - E(x,t).m(x,t)(\phi) - \log(D(x,t)!\}, \quad (3.20)$$

$$m(x,t;\phi) = -\log[1 - q(x,t;\phi)]. \quad (3.21)$$

3.2.4.1 Cairns, Blake and Dowd Model

Cairns et al. (2006) propose the following model and focus on higher ages (60 to 89);

$$\text{logit}[q(x,t)] = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)}, \quad (3.22)$$

$$\beta_x^{(1)} = 1, \quad (3.23)$$

$$\beta_x^{(2)} = (x - \bar{x}) \quad (3.24)$$

where $\bar{x} = \eta_a^{-1}\sum_i x_i$ is the mean age in the sample range.

Rewritten model is as follows;

$$logit[q(x,t)] = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}). \quad (3.25)$$

The model has following main advantages. The model is robust. There are two correlated factors, level and slope. It allows different improvements at different ages and at different times. It has simple age effect. It is easy to incorporate parameter uncertainty.

The model has also following disadvantages. There is no cohort effect. It is good for a big picture, but overall fit is not as good as Lee and Carter model. Lee and Carter model is better able to pick up small non-lineraties in mortality curve. CBD models are only designed for higher ages (Plat, 2009). The performance of fitting for full ages is very poor and the projections are unreasonable for biologically.

3.2.4.2 First Generalisation of the Cairns, Blake and Dowd Model

Cairns et al. (2007) generalise their model as the following structure;

$$\text{logit}[q(x,t)] = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)} \tag{3.26}$$

where;

$$\beta_x^{(1)} = 1, \tag{3.27}$$

$$\beta_x^{(2)} = (x - \bar{x}), \tag{3.28}$$

$$\beta_x^{(3)} = 1, \tag{3.29}$$

$$\text{logit}[q(x,t)] = \kappa_t^{(1)} + (x - \bar{x})\kappa_t^{(2)} + \gamma_{t-x}^{(3)}. \tag{3.30}$$

There is an identifiability problem. The problem can be solved by switching $\gamma_{t-x}^{(3)}$ to $\tilde{\gamma}_{t-x}^{(3)} = \gamma_{t-x}^{(3)} + \phi_1 + \phi_2(t - x - \hat{x})$. However, this needs two constraints to prevent arbitrary ϕ_1 and ϕ_2:

$$\sum_{c\in C}\gamma_c^{(3)} = 0, \tag{3.31}$$

$$\sum_{c\in C}c\gamma_c^{(3)} = 0. \tag{3.32}$$

c is the set of cohort years of birth that have been included in the analysis.

3.2.4.3 Second Generalisation of the Cairns, Blake and Dowd Model

This model is a modification of the first generalisation of CBD model with adding a quadratic term to the age effect (Cairns et al., 2007).

The proposed model is as follows;

$$\text{logit}[\,q(x,t)] = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \kappa_t^{(3)}((x-\bar{x})^2 - \sigma_x^2) + \gamma_{t-x}^{(4)} \qquad (3.33)$$

where $\bar{x} = \eta_a^{-1}\sum_i x_i$ is the mean in the range of ages from x_l to x_u.

$\sigma_x^2 = \eta_a^{-1}\sum_{x_l}^{x_u}(x-\bar{x})^2$ is the corresponding variance.

The model has a new component compared with the first generalisation of CBD model. There is an additional age-period effect, $\kappa_t^{(3)}((x-\bar{x})^2 - \sigma_x^2)$.

There is an also identifiability problem as first generalisation of CBD model. The following adjustments should be satisfied to avoid this problem:

$\tilde{\gamma}_{t-x}^{(4)} = \gamma_{t-x}^{(4)} + \phi_1 + \phi_2(t-x-\bar{x}) + \phi_3(t-x-\bar{x})^2$ and this transformation requires three constraints to avoid arbitrary choices over ϕ_1, ϕ_2 and ϕ_3; $\sum_{c\in C}\gamma_c^{(4)} = 0$, $\sum_{c\in C}c\gamma_c^{(4)} = 0$ and $\sum_{c\in C}c^2\gamma_c^{(4)} = 0$.

$\kappa_t^{(1)}$ is the level of mortality, $\kappa_t^{(2)}$ is the slope coefficient, $\kappa_t^{(3)}$ is the curvature coefficient and $\gamma_c^{(4)}$ is the cohort effect.

3.2.4.4 Third Generalisation of the Cairns, Blake and Dowd Model

Third generalisation of CBD model (Cairns et al., 2007) is as follows;

$$\text{logit}[q(x,t)] = \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\gamma_{t-x}^{(3)} \quad (3.34)$$

where;

$$\beta_x^{(1)} = 1, \quad (3.35)$$

$$\beta_x^{(2)} = (x - \bar{x}), \quad (3.36)$$

$$\beta_x^{(3)} = (x_c - x). \quad (3.37)$$

x_c is a constant parameter to be estimated. New design is as follows;

$$\text{logit}[\, q(t,x)] = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}(x_c - x) \quad (3.38)$$

to avoid identifiability problems following constraint should be satisfied,

$$\sum_{x,t} \gamma_{t-x}^{(3)} = 0\,. \quad (3.39)$$

The summary of the all mentioned models is given at Table 3.1.

Table 3.1 Summary of stochastic mortality models

Lee Carter Model (LC)	$\log[m(t,x)] = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)}$
Renshaw and Haberman Model (RH)	$\log[m(t,x)] = \beta_x^{(1)} + \beta_x^{(2)}\kappa_t^{(2)} + \beta_x^{(3)}\kappa_{t-x}^{(3)}$
Currie Model	$\log[m(t,x)] = \beta_x^{(1)} + n_a^{-1}\kappa_t^{(2)} + n_a^{-1}\gamma_{t-x}^{(3)}$
Cairns, Blake, Dowd Model (CBD)	$\text{logit}[q(t,x)] = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x})$
First Extension of CBD Model (CBD1)	$\text{logit}[q(t,x)] = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \gamma_{t-x}^{(3)}$
Second Extension of CBD Model (CBD2)	$\text{logit}[q(t,x)] = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \kappa_t^{(3)}((x-\bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{t-x}^{(4)}$
Third Extension of CBD Model (CBD3)	$\text{logit}[q(t,x)] = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \gamma_{t-x}^{(3)}(x_c - x)$

3.3 Desirable Properties of the Theoretical Models

Cairns et al. (2007) summarize the desirable properties of the models. Table 3.2 gives these properties and the models whether have or not these properties.

Table 3.2 Desirable properties of the theoretical models

MODEL	LC	RH	Currie	CBD	CBD1	CBD2	CBD3
Ease of Implementation	Y	?	Y	Y	Y	Y	?
Parsimony	Y	?	?	Y	?	?	?
Transparency	Y	Y	Y	Y	Y	Y	Y
Ability to generate sample paths	Y	Y	Y	Y	Y	Y	Y
Ability to generate percentiles	Y	Y	Y	Y	Y	Y	Y
Allowance for parameter uncertainty	Y	Y	Y	Y	Y	Y	Y
Incorporation of cohort effects	N	Y	Y	N	Y	Y	Y
Non-trivial correlation structure	N	N?	N?	Y	Y	Y	Y

Y: Yes, N: No, ?: Somewhere in the middle.

Cairns et al. (2008a) also give the details of the criteria for stochastic mortality models. The details are as follows;

i) Mortality rates should be positive,
ii) The model should be consistent with historical data,
iii) Long term dynamics under the model should be biologically reasonable,

iv) Parameter estimates should be robust relative to the period of data and range of ages employed,

v) Model forecasts should be robust relative to the period of data and range of ages employed,

vi) Forecast levels of uncertainty and central trajectories should be plausible and consistent with historical trends and variability in mortality data,

vii) The model should be straight forward to implement using analytical methods or fast numerical algorithms,

viii) The model should be relatively parsimonious,

ix) It should be possible to use the model to generate sample paths and calculate prediction intervals,

x) The structure of the model should make it possible to incorporate parameter uncertainty in simulations,

xi) At least for some countries, the model should incorporate a stochastic cohort effect,

xii) The model should have a non-trivial correlation structure.

It can be explained these bullets with the frame of Cairns (2008a);

Consistency with historical data: A good model should be consistent with historical patterns of mortality.

Biological reasonableness: Increasing the mortality rates at higher ages is reasonable. Long run mean reversion is unreasonable.

Robustness of parameter estimates and forecasts: A change in the range of years or the range of ages in the historical dataset sometimes results different parameter estimates and different forecasts of future mortality. If changes in the age or calendar year range cause different estimates and different mortality forecasts than these models are not robust.

Plausibility of forecasts: If the forecasts of central trend in mortality rates at certain range do not change dramatically or prediction intervals do not extremely wide or extremely narrow than it can be said that the forecasts are plausible.

Implementation: The model should be easily implementable using analytical methods or fast numerical algorithms.

Parsimony: The extra parameters should be added to the model if there is a significant improvement in fit. One should avoid excessive parameters.

Parameter uncertainty: Model parameters are estimated with limited amount of data so these parameters will be subjected to estimation error.

Cohort effect: Not only age and period but also year of birth affects the mortality rates. Cairns et al. (2007) demonstrate that the inclusion of cohort effect provides a statistically better fit.

Correlation term structure: Improvements at different ages might be correlated but not perfectly correlated. Perfectly correlation might cause problems in the model.

In the literature some methods are only designed for higher ages or the models are fitted only for the specific interval of ages. However, insurance portfolios consist of all ages and practitioners many times want to model whole age group. If they decide to divide the portfolio different age groups, it will be difficult since there is no theoretical or practical way to determine the age groups. Because of this reason, the model should be applicable for the large age range.

3.4 Model Selection Criteria

3.4.1 Bayesian Information Criterion (BIC)

The main drawback of maximum likelihood indicator is that it shows better likelihood ratio if there are more parameters in the model. However, more parameter sometimes means parsimony. If one model is a special case of another model, more parameters model will have a higher maximum likelihood.

BIC balances an increase in the likelihood if the number of parameters is used to achieve that increase. Another important point for BIC is that it gives the chance of compare models that are not necessarily nested. For example, Lee and Carter model and Currie model are nested within Renshaw and Haberman model, but Lee and Carter model is not nested within Currie model.

BIC is estimated as follows;

$$BIC = l(\hat{\phi}) - \frac{1}{2}\nu \log N \tag{3.40}$$

where ϕ is the parameter vector for the model, $\hat{\phi}$ is the maximum likelihood estimates of the model, $l(\hat{\phi})$ is the maximum likelihood value, N is the number of observation and ν is the estimated effective number of parameters (Cairns et al., 2009).

3.4.2 Standardized Residuals

Models having higher likelihood result have a lower variance of the standardized results. If the model is correct, standardized residuals will be approximately independent and identically distributed (i.i.d.) with standard normal random variables. Standardized residuals are defined as follows;

$$z(x,t) = \frac{D(x,t) - E(x,t).\hat{m}(x,t;\hat{\phi})}{\sqrt{E(x,t).\hat{m}(x,t;\hat{\phi})}}. \tag{3.41}$$

CBD (2009) state that the variances of the models are greater than one and their comment is that this is a general feature of mortality data in many countries. A possible reason of this over dispersion is the estimated exposure data. This over dispersion does not have a significant impact on their estimates of the future dynamics of the underlying mortality rates, *q(t, x)*. However, a Poisson model might underestimate the future variability of the actual death rates relative to the true underlying rates. If there are diagonal clusters of positive and negative residuals, this is the evidence for the existence of a cohort effect. One easily way to decide whether the residuals randomly distributed or not is looking the residuals plot and decide if there is positive and negative clustering (Cairns et al., 2009).

3.4.3 Comparison of Nested Models

If models are nested within one of the others, likelihood ratio test can be used to test that the nested or restricted model is the correct model.

The likelihood ratio test statistic is $2(\hat{l}_2 - \hat{l}_1)$. The distribution will be chi square with the degrees of freedom $\nu_2 - \nu_1$. Under null hypothesis it can be rejected if test statistic, $2(\hat{l}_2 - \hat{l}_1) > \chi^2_{\nu_2 - \nu_1, \alpha}$, at the significance level (Cairns et al., 2009).

CHAPTER 4: RESERVING METHODS IN NON-LIFE INSURANCE

Main contribution of this thesis is that first time in the literature non-life insurance method is implemented to estimate mortality rate stochastically. The background of the method is reserving methodology or incurred but not reported losses (IBNR) estimation in non-life mathematics. There are lots of IBNR estimation methods in the literature. Loss frequency, standard chain ladder, Mack method, bootstrap method, Reversible Jump Markov Chain Monte Carlo (RJMCMC), Bornheutter Ferguson, cape code, generalized cape code and generalized linear model based estimations are some of them and there are more additional methods. The concept is very important for non-life insurance mathematics therefore still new methods are investigated and suggested and also actual methods are improved. It is impossible to give details of all methods in this study therefore the following methods are selected for the aim of this thesis; standard chain ladder, Mack method, bootstrap method, Reversible Jump Markov Chain Monte Carlo approach (RJMCMC). These methods are mostly and newly used methods in the literature. In this chapter the basics of the IBNR estimation are given.

4.1 Standard Chain Ladder Approach

The amounts of reported losses are known by the companies but the estimation of unreported losses is very important to reserve correct amount for solvency requirements. Chain ladder methodology is one of the most frequently used methodologies to estimate unpaid claims. The basic of this method is that the development of the future pattern of losses will be similar for the past losses. The pattern is calculated using development triangles. Although there are lots of methodologies to estimate IBNR, chain ladder approach is the most commonly used method in the literature and by actuaries.

Run-off triangle is a kind of table that represents the cumulative or incremental losses or number of claims in non-life insurance according to the losses happened period and development period. The aim is to predict the non-observable side of the triangle. Chain ladder approach is extended and improved to predict the number of death in this thesis.

Figure 4.1 shows that the adopted form of the run-off incremental triangle to the mortality data.

YEAR OF ORIGIN	DEVELOPMENT PERIOD									
	0	1	2	…	j	…	…	…	n-1	n
1	$C_{1,0}$	$C_{1,1}$	$C_{1,2}$		$C_{1,j}$				$C_{1,n-1}$	$C_{1,n}$
2	$C_{2,0}$	$C_{2,1}$	$C_{2,2}$		$C_{1,j}$				$C_{2,n-1}$	
3										
⋮										
i	$C_{i,0}$	$C_{i,1}$	$C_{i,2}$		$C_{i,j}$					
⋮										
⋮										
⋮										
n	$C_{n,0}$									
n+1	$C_{n+1,0}$									

Figure 4.1 Incremental triangle in chain ladder

Let $C_{i,j}$ is the observed incremental number of deaths in development year j, in respect of born year i. $C_{i,j}$ is the loss amounts or number of losses in non-life mathematics. It shows the losses at the accident year *i* and development year *j*. In this approach development period shows the age.

YEAR OF ORIGIN	DEVELOPMENT PERIOD									
	0	1	2	…	j	…	…	…	n-1	n
1	$C_{1,0}$	$D_{1,1}$	$D_{1,2}$		$D_{1,j}$				$D_{1,n-1}$	$D_{1,n}$
2	$C_{2,0}$	$D_{2,1}$	$D_{2,2}$		$D_{2,j}$				$D_{2,n-1}$	
3										
⋮										
i	$C_{i,0}$	$D_{i,1}$	$D_{i,2}$		$D_{i,j}$					
⋮										
⋮										
⋮										
n	$C_{n,0}$									
n+1	$C_{n+1,0}$									

Figure 4.2 Cumulative triangle in chain ladder

$D_{i,j}$ is the observed cumulative number of deaths in development year j, in respect of born year i. At the non-life world, $D_{i,j}$ is the cumulative losses for the accident year i and development period j. The triangle is filled as follows;

$$D_{i,0} = C_{i,0}, \tag{4.1}$$

$$D_{i,j+1} = C_{i,j+1} + D_{i,j},\ \ 0 < i \leq n+1,\ \ 0 \leq j < n. \tag{4.2}$$

The lower side of triangle is predicted by the weighted (loss) development factors, λ_k;

$$\lambda_k = \frac{\sum_{i=1}^{n-k+1} D_{i,k}}{\sum_{i=1}^{n-k+1} D_{i,k-1}},\ 0 < k \leq n \tag{4.3}$$

λ_k is calculated by divided the sum of the k'th column to the sum of the *(k-1)*'st column but excluding the last entry in the run-off triangle. It is a kind of change between the columns. Figure 4.3 gives $\lambda_{(i,j)} = \frac{D_{i,j}}{D_{i,j-1}}$ which is called as the one-by-one development factor.

YEAR OF ORIGIN	**DEVELOPMENT PERIOD**										
	0	**1**	**2**	**…**	**j**	**…**	**…**	**…**	**n-1**	**n**
1		$\lambda_{(1,1)}$	$\lambda_{(1,2)}$		$\lambda_{(1,j)}$					$\lambda_{(1,n)}$
2		$\lambda_{(2,1)}$	$\lambda_{(2,2)}$							
3										
⋮										
i		$\lambda_{(i,1)}$								
⋮										
⋮										
⋮										
n		$\lambda_{(n,1)}$								
n+1										

Figure 4.3 One-by-one development factors in chain ladder

$$\widehat{D}_{i,j} = \widehat{D}_{i,j-1}\lambda_k, \qquad k>0 \tag{4.4}$$

$$\widehat{D}_{i,0} = D_{i,0} = C_{i,0} \tag{4.5}$$

Equation 4.4 gives the estimation of full triangle by weighted average loss development factors. However, the aim is only to estimate lower triangle. It is only needed to use cumulative diagonal cells at the right side of the equation. In other words, the cumulative data should be for the right side of the equation, $\widehat{D}_{i,j} = D_{i,j}, i + j \leq n + 1$.

4.2 Bootstrap Analysis

As known the bootstrap methodology is used in many fields of the science. It is a kind of sampling method with replacement from the observed data sample by creating many sets of pseudo-data selected from the same underlying distribution. It gives the answer of what confidence interval, upper and lower limit of the reserve estimation etc. is in the non-life reserve estimations.

The bootstrap procedure described by England and Verrall (2002) is as follows;

- Obtain the standard chain-ladder development factors from cumulative data,
    - $\{C_{i,j}, i = 1, \ldots, n$ and $j = 1, \ldots, n - i + 1\}$ is the incremental data,
    - $D_{i,j} = \sum_{k=1}^{j} C_{i,k}$ is the cumulative data,
    - $\hat{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} D_{i,j}}{\sum_{i=1}^{n-j+1} D_{i,j-1}}$ is the development factors.
- Obtain cumulative fitted values for the past triangle by backwards recursion, starting with the observed cumulative paid to date in the latest diagonal, using $\widehat{D}_{i,n-i+1} = D_{i,n-i+1}$ and $\widehat{D}_{i,k-1} = \widehat{D}_{i,k}\lambda_k^{-1}$ incremental fitted values, $\widehat{m}_{i,j}$, for the past triangle by differencing,
- The unscaled Pearson residuals are calculated for the past triangle:

$$r_{i,j}^{(P)} = \frac{C_{i,j} - \widehat{m}_{i,j}}{\sqrt{\widehat{m}_{i,j}}}, \tag{4.6}$$

- The Pearson scaled parameters are calculated:

$$\phi = \frac{\sum_{i,jn-i+1} \left(r_{i,j}^{(P)}\right)^2}{\frac{1}{2}n(n+1) - 2n + 1}, \tag{4.7}$$

- The Pearson residuals are adjusted;

$$r_{i,j}^{adj} = \sqrt{\frac{n}{\frac{1}{2}n(n+1) - 2n + 1}}\, r_{i,j}^{(P)}, \tag{4.8}$$

- Repeat N times the loop;
    - Resample the adjusted residuals with replacement and create a new triangle of residuals,
    - Give a set of pseudo-incremental data for the past triangle and solve for *C*,
    - Create cumulative data and fit the standard chain ladder model,
    - Forecast the future cumulative triangle,
    - Set the incremental triangle,
    - For each cell *(i,j)* in the future triangle, simulate a payment from the process distribution with mean $\widetilde{m}_{i,j}$ and variance $\phi\widetilde{m}_{i,j}$, using the value of ϕ calculed previously,
    - Store the results and return the start point.

The standard deviation of the stored results gives an estimate of the prediction error. The aim of the bootstrap analysis is to produce n different triangles and n different results as given in Figure 4.4.

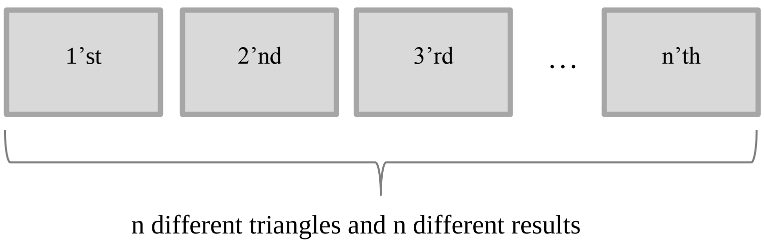


Figure 4.4 Bootstrap scheme

4.3 Mack Method and Standard Error Approach

Mack designs a new chain ladder approach to the chain ladder method. That is a distribution-free formula for the standard error of chain ladder reserve estimates.

The chain ladder method has no any assumption this is the main advantage of this method. However, the most important drawback of this method is the most recent accident years are very sensitive to variations in the data observed. Therefore it is important to know the standard error of the chain ladder reserve estimates to see the uncertainty in the data. More information can be handled from Mack (1991, 1993).

The development factors are estimated using regression. $D_{i,k}$ is the cumulative values for the development period *"k"* and $D_{i,k-1}$ is the cumulative values for the development period *"k-1"*. A link ratio is the slope of a line passing through the origin and the point $[D_{i,k}, D_{i,k-1}]$.

A link ratio average method is based on the regression,

$$D_{i,k} = \beta D_{i,k-1} + \varepsilon(i) \tag{4.9}$$

where,

$$Var\big(\varepsilon(i)\big) = \sigma^2 (D_{i,k-1})^{\delta}. \tag{4.10}$$

The parameter β represents the slope of the best line through the origin and the data points $[D_{i,k}, D_{i,k-1}]$.

Three approaches are considered to determine β. These are the $\delta = 1$ (Case 1), $\delta = 2$ (Case 2) and $\delta = 0$ (Case 3).

The variance of $D_{i,k}$ about the line depends on $D_{i,k-1}$ via the function $(D_{i,k-1})^{\delta}$ where δ is a weighting parameter and σ^2 represents an underlying level of variance.

Case 1: $\delta = 1$; the weighted least square estimator of β is

$$\hat{\beta} = \frac{\sum D_{i,k-1} D_{i,k}/D_{i,k-1}}{\sum D_{i,k-1}}. \tag{4.11}$$

Eq. (4.11) is the weighted average ratio.

Case 2: $\delta = 2$;

$$\hat{\beta} = \frac{1}{n}\sum D_{i,k}/D_{i,k-1}. \tag{4.12}$$

Eq. (4.12) is the simple arithmetic average of ratio.

Case 3: $\delta = 0$; this yields a weighted average (weighted by volume squared) corresponding to ordinary least squares regression through the origin.

$D_{i,k}$ is the accumulated total claims amount of accident year i, $1 \leq i \leq I$, either paid or incurred up to development year k, $1 \leq k \leq I$. $D_{i,k}$ is a random variable and it is observed if $i + k \leq I + 1$. The aim is to estimate ultimate claims amount $D_{i,I}$ and then the outstanding claims reserve,

$$R_i = D_{i,I} - D_{i,I+1-i} \tag{4.13}$$

for accident year *i=2, …,I.*

Development factors, $f_1, f_2, \dots, f_{I-1} > 0$, in the chain ladder have some assumptions. The assumptions in this frame are as follows;

Assumption 1: $E\left(D_{i,k+1}|D_{i,1},\dots,D_{i,k}\right) = D_{i,k}f_k, 1 \le i \le I, 1 \le k \le I-1.$

Assumption 2: $\hat{f}_k = {\sum_{j=1}^{I-k} D_{j,k+1}} \Big/ {\sum_{j=1}^{I-k} D_{j,k}}, 1 \le k \le I-1.$

The ultimate claims amount, $D_{i,I}$, is as follows,

$$\widehat{D}_{i,I} = D_{i,I+1-i}\hat{f}_{I+1-i} \cdots \hat{f}_{I-1}. \tag{4.14}$$

The chain ladder algorithm does not take into account any dependencies between accident years. There is an additionally assumption which is that $\{D_{i,1},\dots,D_{i,I}\},\{D_{j,1},\dots,D_{j,I}\}, i \neq j$, are independent.

Theorem 1: Let $\mathcal{H} = \{C_{i,k}|\ i+k \le I+1\}$ be the set of all data observed so far and

$$E\left(D_{i,I}|\mathcal{H}\right) = D_{i,I+1-i}f_{I+1-i} \cdots f_{I-1}. \tag{4.15}$$

This theorem shows that the estimator $\widehat{D}_{i,I}$ has the same form as $E\left(D_{i,I}|\mathcal{H}\right)$. This is the best forecast of $D_{i,I}$ based on the observations so far $\mathcal{H} = \{D_{i,k}|i+k \le I+1\}$. Under assumptions 1 and 2 the development factors *(f)* are unbiased and uncorrelated. The proofs can be found at Mack (1993b).

The mean square errors in the system are as follows;

$$\text{MSE}\left(\widehat{D}_{i,I}\right) = E\left(\left(\widehat{D}_{i,I} - D_{i,I}\right)^2|\mathcal{H}\right), \tag{4.16}$$

$$\text{MSE}\left(\widehat{R}_i\right) = E\left(\left(\widehat{R}_i - R_i\right)^2|\mathcal{H}\right) = E\left(\left(\widehat{D}_{i,I} - D_{i,I}\right)^2|\mathcal{H}\right) = mse\left(\widehat{D}_{i,I}\right). \tag{4.17}$$

As it is known the following equation $E(X-a)^2 = Var(X) + (E(X)-a)^2$,

$$\text{MSE}(\widehat{D}_{i,I}) = Var(D_{i,I}|\mathcal{H}) + \left(E(D_{i,I}\,|\mathcal{H}) - \widehat{D}_{i,I}\right)^2. \quad (4.18)$$

The mean square error is the sum of the stochastic error (process variance) and the sum of the estimation error.

Assumption 3: $Var(D_{i,k+1}|D_{i,1},\dots,D_{i,k}) = D_{i,k}\sigma_k^2, \quad 1 \le i \le I, \quad 1 \le k \le I-1$ with unknown parameter σ_k^2, $1 \le k \le I-1$. This is the variance assumption which is implicitly underlying the chain ladder method.

Under the assumptions (1), (2) and (3) the mean squared error, $\text{MSE}(\hat{R}_i)$, can be estimated;

$$\text{MSE}(\hat{R}_i) = \widehat{D}_{i,I}^2 \sum_{k=I+1-i}^{I-1} \frac{\widehat{\sigma}_k^2}{\hat{f}_k^2}\left(\frac{1}{\widehat{D}_{i,k}} + \frac{1}{\sum_{j=1}^{I-k} D_{j,k}}\right) \quad (4.19)$$

where $\widehat{D}_{i,k} = D_{i,I+1-i}\hat{f}_{I+1-i} \dots \hat{f}_{k-1}, k > I+1-i$, are the estimated values of the future $D_{i,k}$.

The mean square error of the overall reserve estimation, $\hat{R} = \hat{R}_2 + \cdots + \hat{R}_I$, can be estimated;

$$\widehat{\text{MSE}(\hat{R})} = \sum_{i=2}^{I}\left\{\left(s.e.(\hat{R}_i)\right)^2 + \widehat{D}_{i,I}\left(\sum_{j=i+1}^{I}\widehat{D}_{j,I}\right)\sum_{k=I+1-i}^{J-1}\frac{2\widehat{\sigma}_k^2/\hat{f}_k^2}{\sum_{n=1}^{I-k} D_{n,k}}\right\}. \quad (4.20)$$

4.4 Reversible Jump Markov Chain Monte Carlo Method

Verrall and Wüthrich (2012) introduce the reversible jump markov chain monte carlo method (RJMCMC) whose main advantages are removes the requirement for an ad hoc manual procedure, reduces the subjectivity, bias and the number of parameters required in the estimation.

In this method the incremental claims are assumed over-dispersed Poisson variables. The aim is to estimate the column and row parameters of the triangle. The column parameter estimation is divided in two parts with a determined optimal truncation column index.

The rows of the triangle is accident years $i \in \{0, \dots, I\}$ and the columns of the triangle is development years, $j \in \{0, \dots, I\}$. $X_{i,j}$ is the observed claims at time I and these claims are in the upper claims development triangle,

$$\mathcal{D}_I = \{X_{i,j}; i + j \leq I, 0 \leq i \leq I, 0 \leq j \leq I\} . \quad (4.21)$$

Predicting the lower triangle is the one of the aim;

$$\mathcal{D}_I^c = \{X_{i,j}; i + j > I, 0 \leq i \leq I, 0 \leq j \leq I\}. \quad (4.22)$$

μ_i is the row parameter and γ_j is the column parameter and then it is assumed,

$$\mathbb{E}_{\mu_i,\gamma_j}\left[X_{i,j}\right] = \mu_i \gamma_j, \forall\ i, j \in \{0, \dots, I\}. \quad (4.23)$$

The aim is to model $\gamma_0, \dots, \gamma_{k-1}$ up to $k \in \{1, \dots, I\}$ which is the truncation index and then fit an exponential decay to the remaining development parameters,

$$\gamma_j = e^{\alpha - j\beta}, j \in \{k, \dots, I\}, \alpha \epsilon \mathbb{R}, \beta \epsilon \mathbb{R}_+ . \quad (4.24)$$

One of the aims of Verrall and Wüthrich approach is to choose k with a Bayesian claim reserving model. $\mathcal{M}_k$, $k \epsilon \{1, \dots, I\}$ is the whole family of models according to different truncation index k. Preferring $\mathcal{M}_k$ is decided by solving numerically the reversible jump markov chain monte carlo method, which is a particular markov chain monte carlo (MCMC) method that can jump between different models $\mathcal{M}_k$ that have different parameter sets. As stated in Verrall and Wüthrich RJMCMC methods are developed by

Green (1995, 2003). Verrall and England (2012) consider a Bayesian model for the development pattern, using a different application of RJMCMC methods.

4.4.1 Bayesian Overdispersed Poisson Model (ODP)

The Bayesian ODP model $\mathcal{M}_k,\ k \in \{1,\ldots,I\}$ has the following model assumptions:

- Given the set of $\vartheta = (\mu_0,\ldots,\mu_I,\gamma_0,\ldots,\gamma_I,\varphi)$, $X_{i,j}$ are independent random variables with

$$\frac{X_{i,j}}{\varphi}|_{\vartheta} \sim \ Poisson\ (\frac{\mu_i\gamma_j}{\varphi}), \tag{4.25}$$

- Assume that $\varphi > 0$ is a given constant and the parameter vector is as follows:

$$\theta_k = (\alpha,\beta,\mu_0,\ldots,\mu_I,\gamma_0,\ldots,\gamma_{k-1}) \tag{4.26}$$

has prior distribution $p_k(\theta_k)$ with independent components satisfying

$$\mu_i \sim \Gamma\left(s,\frac{s}{m_i}\right),\ i = 0,\ldots,I, \tag{4.27}$$

$$\gamma_j \sim \Gamma\left(v,\frac{v}{c_j}\right),\ j = 0,\ldots,k-1, \tag{4.28}$$

$$\alpha \sim N(a,\sigma^2),\ \beta \sim N(b,\tau^2), \tag{4.29}$$

for given prior parameters $m_i, s, c_j, v, \sigma, \tau > 0,\ a, b \,\epsilon\, \mathbb{R}$ and $\gamma_j = e^{\alpha - j\beta},\ j \in \{k,\ldots,I\}$. γ_j is distributed gamma prior distributions for the development periods $j \in \{0,\ldots,k-1\}$ and there is an exponential decay for $\beta > 0$ for the development periods $j \in \{k,\ldots,I\}$.

According to the given assumptions, two conditional moments of $X_{i,j}$ are given as;

$$\mathbb{E}\left[X_{i,j}\middle|\theta_k\right] = \mu_i\gamma_j, \tag{4.30}$$

$$Var\left[X_{i,j}\middle|\theta_k\right] = \varphi\mu_i\gamma_j, \tag{4.31}$$

The joint density in model $\mathcal{M}_k$ with considering to the parameter vector θ_k is given by

$$f_k(X_{i,j},\theta_k) = f_k(X_{i,j}|\theta_k)p_k(\theta_k)$$

$$\propto \prod_{(i,j)} e^{-\mu_i\gamma_j/\varphi} \frac{\left(\frac{\mu_i\gamma_j}{\varphi}\right)^{X_{i,j}/\varphi}}{\left(\frac{X_{i,j}}{\varphi}\right)!} \prod_{i=0}^{I} \mu_i^{s-1} e^{-s/m_i\mu_i} \prod_{j=0}^{k-1} \gamma_j^{v-1} e^{-v/c_j\gamma_j}$$

$$\times \exp\left\{-\frac{1}{2\sigma^2}(\alpha - a)^2\right\}\exp\left\{-\frac{1}{2\tau^2}(\beta - b)^2\right\} \tag{4.32}$$

where the sign "∝" means the proportion.

The following titles can be separated from the joint density (Equation 4.32); the likelihood of $X_{i,j}$, prior densities of μ and γ, the regression parameters α and β.

The likelihood is as follows;

$$\prod_{(i,j)} e^{-\mu_i\gamma_j/\varphi} \frac{\left(\frac{\mu_i\gamma_j}{\varphi}\right)^{X_{i,j}/\varphi}}{\left(\frac{X_{i,j}}{\varphi}\right)!}$$

where the prior densities of $\mu_0, \dots, \mu_I$ is expressed also as:

$$\prod_{i=0}^{I} \mu_i^{s-1} e^{-s/m_i\mu_i}$$

and the prior densities of $\gamma_0, \dots, \gamma_{k-1}$ is given as:

$$\prod_{j=0}^{k-1} \gamma_j^{v-1} e^{-v/c_j\gamma_j}.$$

Then, the prior densities of the regression parameters α and β is given as:

$$\exp\left\{-\frac{1}{2\sigma^2}(\alpha - a)^2\right\}\exp\left\{-\frac{1}{2\tau^2}(\beta - b)^2\right\}.$$

A prior distribution on the model space is chosen;

$$p(\mathcal{M}_k) > 0, \sum_{k=1}^{I} p(\mathcal{M}_k). \tag{4.33}$$

The posterior distribution on the model and parameter space is given as

$$p(\mathcal{M}_k, \theta_k|y) \propto f_k(y|\theta_k)p_k(\theta_k)p(\mathcal{M}_k)$$

and $\theta_k \in \Theta_k$ is the parameter space.

The critical point is to generate a Markov Chain $(\Theta^{(t)})_{t>0} = (k^{(t)}, \theta_{k^{(t)}}^{(t)})_{t>0}$ whose limit distribution is $p(\mathcal{M}_k, \theta_k|y) \propto f_k(y|\theta_k)p_k(\theta_k)p(\mathcal{M}_k)$.

4.4.2 Algorithm of RJMCMC

The algorithm stated Verral and Wüthrich (2012) is as follows;

1. Initialize $\Theta^{(0)}$,

2. For $t \geq 0$ do

2.1 Select a model $\mathcal{M}_{k^*}$ with proposal probability $k^* \sim q(\cdot\,|k^{(t)})$,

2.2 $k^* = k^{(t)}$, update the parameters using the classical Metropolis and Hastings (MH) algorithm, which provides $\theta_{k^*}^{(t+1)}$ (from $\theta_{k^*}^{(t)}$) and set $\Theta^{(t+1)} = (k^*, \theta_{k^*}^{(t+1)})$; go to item (d),

2.3 $k^* \neq k^{(t)}$;

a. Generate $u^{(t)}$ from distribution $g_{k^{(t)} \to k^*}(\cdot\,|\theta_{k^{(t)}}^{(t)})$,

b. Set $(\theta_{k^*}^*, u^*) = T_{k^{(t)} \to k^*}(\theta_{k^{(t)}}^{(t)}, u^{(t)})$,

c. Calculate the acceptance probability,

d. According to acceptance probability set $\Theta^{(t+1)} = \Theta^{(t)} = (k^*, \theta_{k^*}^*)$ and otherwise $\Theta^{(t+1)} = (k^{(t)}, \theta_{k^{(t)}}^{(t)})$.

e. Iterate this procedure under item 2.

The general form of the RJMCMC algorithm is given above. The aim of the authors is to apply this algorithm to the Bayesian ODP model and optimize it using different truncation index.

The steps for the algorithm of RJMCMC for the Bayesian ODP model are given following paragraphs.

STEP 1 for RJMCMC: Choose the new truncation index

$$q(k^* = k-1|k) = q(k^* = k|k) = q(k^* = k|k+1) = \frac{1}{3}, k \in \{2, \dots, I-1\}, \quad (4.34)$$

$$q(1|1) = \frac{2}{3}, q(2|1) = \frac{1}{3}, q(I|I) = \frac{2}{3}, q(I-1|I) = \frac{1}{3}. \quad (4.35)$$

According to the above rules, model only jumps to its neighbour models *(k+1 or k-1)*.

STEP 2 for RJMCMC: Update the parameters if $k^* = k$

If $k^{*}=k^{(t)}$, it is set $k^{(t+1)}=k^{(t)}$ then applied Metropolis Hastings (MH) block sampler. The steps of the MH are as follows;

1. Update $(\mu_0^{(t)},\dots,\mu_I^{(t)})$ by Gibbs sampler,

$$p(\mu_0,\dots,\mu_t|\alpha^{(t)},\beta^{(t)},\gamma_0^{(t)},\dots,\gamma_{k^{(t)}-1}^{(t)},\mathcal{D}_I)$$

is independent (in μ) gamma densities with parameters;

$$s\rightarrow s_i^{post}=s+\frac{1}{\varphi}\sum_{j=0}^{I-i}X_{i,j}\text{ and }\frac{s}{m_i}\rightarrow\left(\frac{s}{m_i}\right)_i^{post}=\frac{s}{m_i}+\frac{1}{\varphi}\sum_{j=0}^{I-i}\gamma_j^{(t)},$$

$$\gamma_j^{(t)}=e^{\alpha^{(t)}-j\beta^{(t)}},j\in\left\{k^{(t)},\dots,I\right\}.$$

μ's can be generated and updated,

$$\mu_0^{(t+1)},\dots,\mu_I^{(t+1)}\sim p(\cdot\,|\alpha^{(t)},\beta^{(t)},\gamma_0^{(t)},\dots,\gamma_{k^{(t)}-1}^{(t)},\mathcal{D}_I).$$

2. Update $(\gamma_0^{(t)},\dots,\gamma_{k^{(t)}-1}^{(t)})$ by the Gibbs sampler,

$$\gamma_0^{(t+1)},\dots,\gamma_{k^{(t)}-1}^{(t+1)}\sim p(\cdot\,|\alpha^{(t)},\beta^{(t)},\mu_0^{(t+1)},\dots,\mu_I^{(t+1)},\mathcal{D}_I)$$

are independent (in γ) gamma densities with parameters;

$$v\rightarrow v_j^{post}=v+\frac{1}{\varphi}\sum_{i=0}^{I-j}X_{i,j}\text{ and }\frac{v}{c_j}\rightarrow\left(\frac{v}{c_j}\right)_j^{post}=\frac{v}{c_j}+\frac{1}{\varphi}\sum_{i=0}^{I-j}\mu_i^{(t+1)}.$$

$(\gamma_0^{(t+1)},\dots,\gamma_{k^{(t)}-1}^{(t+1)})$ can be generated and updated by the gamma densities.

3. Update $(\alpha^{(t)},\beta^{(t)})$ using by the classical MH algorithm. The relevant density in (α,β) for this update is as a function of α and β:

$$\prod_{j=k^{(t)}}^{I}\left[e^{-\exp\{\alpha-j\beta\}\sum_{i=0}^{I-j}\frac{\mu_i^{(t+1)}}{\varphi}}(\exp\{\alpha-j\beta\})^{\sum_{i=0}^{I-j}X_{i,j}/\varphi}\right]exp\left\{-\frac{1}{2\sigma^2}(\alpha-a)^2\right\}exp\left\{-\frac{1}{2\tau^2}(\beta-b)^2\right\}. \quad (4.36)$$

The updated parameter set is as follows;

$$\Theta^{(t+1)} = (k^{(t+1)}, \theta_{k^{(t+1)}}^{(t+1)}) =$$
$$(k^{(t+1)}, \left(\alpha^{(t+1)}, \beta^{(t+1)}, \mu_0^{(t+1)}, \dots, \mu_I^{(t+1)}, \gamma_0^{(t+1)}, \dots, \gamma_{k^{(t)}-1}^{(t+1)}\right)) \ . \quad (4.37)$$

STEP 3 for RJMCMC: Update the parameters if $k^* \neq k$

It is need to be chosen the proposal distributions $g_{k \to k^*}(\cdot\, |\theta_k)$ and the functions $T_{k \to k^*}(\cdot,\cdot)$.

Assume $k^{(t)} < I$ and $k^* = k \pm 1$, $\gamma_{k^{(t)}}$ is added the model. It is chosen the following u;

$$u^{(t)} \sim g_{k^{(t)} \to k^{(t)}+1}\left(\cdot \left|\theta_{k^{(t)}}^{(t)}\right.\right) \sim \Gamma(v^*, \frac{v^*}{\exp\{\alpha^{(t)} - k^{(t)}\beta^{(t)}\}}) \quad (4.38)$$

for $\theta_{k^{(t)}}^{(t)} = \left(\alpha^{(t)}, \beta^{(t)}, \mu_0^{(t)}, \dots, \mu_I^{(t)}, \gamma_0^{(t)}, \dots, \gamma_{k^{(t)}-1}^{(t)}\right)$ and given $v^* > 0$. It is set,

$$\theta_{k^*}^* = T_{k^{(t)} \to k^{(t)}+1}\left(\theta_{k^{(t)}}^{(t)}, u^{(t)}\right) = \left(\theta_{k^{(t)}}^{(t)}, u^{(t)}\right) =$$
$$(\alpha^{(t)}, \beta^{(t)}, \mu_0^{(t)}, \dots, \mu_I^{(t)}, \gamma_0^{(t)}, \dots, \gamma_{k^{(t)}-1}^{(t)}, u^{(t)}). \quad (4.39)$$

The acceptance probability will be as follows with the parameters; $k = k^{(t)}$, $\gamma_k^{(t)} = e^{\alpha^{(t)} - k^{(t)}\beta^{(t)}}$ and $\gamma_k^* = u^{(t)}$:

$$\alpha(t \to *) = min\left\{1, \prod_{i=0}^{I-k}\left[\frac{e^{-\frac{\mu_i^{(t)}\gamma_k^*}{\varphi}}(\gamma_k^*)^{X_{i,k}/\varphi}}{e^{-\frac{\mu_i^{(t)}\gamma_k^{(t)}}{\varphi}}(\gamma_k^{(t)})^{X_{i,k}/\varphi}}\right]\frac{\frac{(\frac{v}{c_k})^v}{\Gamma(v)}(\gamma_k^*)^{v-1}e^{-v/c_k\gamma_k^*}}{\frac{(\frac{v^*}{\gamma_k^{(t)}})^{v^*}}{\Gamma(v^*)}(\gamma_k^*)^{v^*-1}e^{-v^*/\gamma_k^{(t)}\gamma_k^*}}\right\}. \quad (4.40)$$

Assume $k^{(t)} > 2$ and $k^* = k^{(t)} - 1$, the parameter set is as follows;

$$(\theta^*_{k^*}, u^*) = T_{k^{(t)} \to k^{(t)}-1}\left(\theta^{(t)}_{k^{(t)}}\right) = \theta^{(t)}_{k^{(t)}} =$$

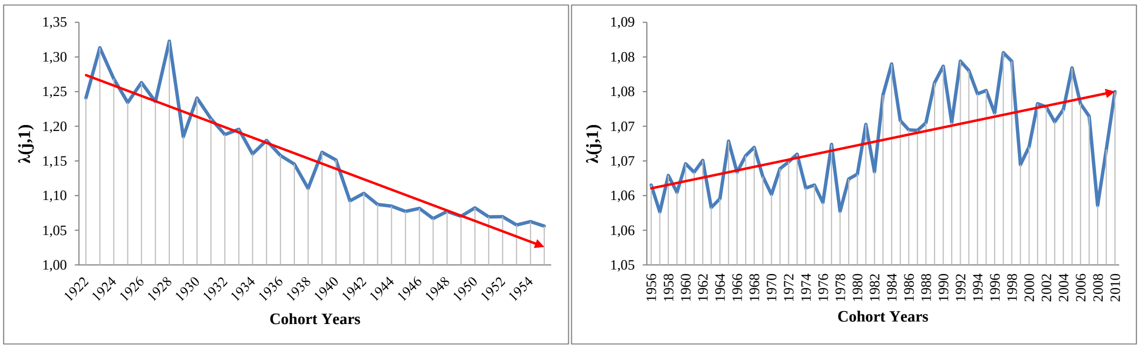
$$\left(\alpha^{(t)}, \beta^{(t)}, \mu_0^{(t)}, \dots, \mu_I^{(t)}, \gamma_0^{(t)}, \dots, \gamma^{(t)}_{k^{(t)}-2}, \gamma^{(t)}_{k^{(t)}-1}\right). \quad (4.41)$$

$u^* = \gamma^{(t)}_{k^{(t)}-1}$, $k^* = k^{(t)} - 1$ and $\gamma^*_{k^*} = \exp\{\alpha^{(t)} - k^*\beta^{(t)}\}$ and then the acceptance probability is as follows;

$$\alpha(t \to *) = min\left\{1, \prod_{i=0}^{I-k^*}\left[\frac{e^{-\frac{\mu_i^{(t)}\gamma^*_{k^*}}{\varphi}}(\gamma^*_{k^*})^{X_{i,k^*}/\varphi}}{e^{-\frac{\mu_i^{(t)}\gamma^{(t)}_{k^*}}{\varphi}}(\gamma^{(t)}_{k^*})^{X_{i,k^*}/\varphi}}\right]\frac{\frac{(\frac{v^*}{\gamma^*_{k^*}})^{v^*}}{\Gamma(v^*)}(\gamma^{(t)}_{k^*})^{v^*-1}e^{-v^*/\gamma^*_{k^*}\gamma^{(t)}_{k^*}}}{\frac{(\frac{v}{c_{k^*}})^{v}}{\Gamma(v)}(\gamma^{(t)}_{k^*})^{v-1}e^{-v/c_{k^*}\gamma^{(t)}_{k^*}}}\right\}. \quad (4.42)$$

CHAPTER 5: STOCHASTIC MORTALITY USING NON-LIFE RESERVING MODEL

Every new IBNR method in the literature is proposed to close the shortage of the previous one. The main drawback of explained models is that can not capture the trend of the development factors by rows completely. The main objective of this thesis is to propose a new approach to model mortality and the main aim of this model is to have the desirable properties of the mortality models mentioned in Chapter 3. The background of the proposed model is IBNR estimations in non-life mathematics therefore this is the first and a new perspective in the literature. It is tried to eliminate by the proposed model that the drawback which can not capture the trend of the loss development factors in chain ladder approach. Figure 5.1 gives the trends of the loss development factors ($\hat{\lambda}_{(j,1)}$) by cohort years for the number of death triangle of UK. There is a decreasing trend between the cohort years from 1922 to 1955 and increasing trend from 1956 to 2010. Therefore weighted average development factors give the wrong estimations if it is used to estimate the lower triangle of the number of death.



(a) 1922-1955 (b) 1956-2010

Figure 5.1 $\hat{\lambda}_{(j,1)}$ for the UK mortality data

The aim is to estimate and simulate the loss development factors ($\hat{\lambda}$) of the lower triangle in Figure 5.2.

YEAR OF ORIGIN	DEVELOPMENT PERIOD									
	0	1	2	…	j	…	…	…	n-1	n
1		$\lambda_{(1,1)}$	$\lambda_{(1,2)}$		$\lambda_{(1,j)}$					$\lambda_{(1,n)}$
2		$\lambda_{(2,1)}$	$\lambda_{(2,2)}$							
3										
⋮										
i										
⋮										
⋮										
⋮										
n		$\lambda_{(n,1)}$								
n+1		$\hat{\lambda}_{(n+1,1)}$	$\hat{\lambda}_{(n+1,2)}$	…	$\hat{\lambda}_{(n+1,j)}$					$\hat{\lambda}_{(n+1,n)}$

Figure 5.2 Estimated development factors

The steps and theoretical background of the proposed model are explained at Chapter 5.1.

5.1 Theoretical Background

A stochastic process $\{X(t),\ t \geq 0\}$ is said to be a Brownian motion process if;

i. *X(0) = 0,*
ii. $\{X(t), t \geq 0\}$ has stationary and independent increments,
iii. $\{X(t), t > 0\}$ is normally distributed with mean *0* and variance $t\sigma^2$.

Sometimes it is called as a Wiener process and the notation is "*w*". There are many application areas in science such as engineering, finance, insurance, physics etc.

If $\sigma = 1$, the process is called standard Brownian motion.

If the process $\{X(t), t \geq 0\}$ is a Brownian motion with drift μ and variance σ^2 if,

i. $X(0) = 0,$

ii. $\{X(t), t \geq 0\}$ has stationary and independent increments,

iii. $\{X(t), t > 0\}$ is normally distributed with mean μt and variance $t\sigma^2$.

If $\{X(t), t \geq 0\}$ is a standard Brownian motion process with drift parameter μ and variance σ^2, ($X(t) = \mu t + \sigma w(t)$), then the geometric Brownian motion is defined as follows;

$$Y(t) = e^{X(t)}. \quad (5.1)$$

The expected value of the process at time t given the history of the process up to time s is as follows;

$$E[Y(t)|Y(u),\ 0 \leq u \leq s] = Y(s)E\left[e^{X(t)-X(s)}\right]. \quad (5.2)$$

"$X(t) - X(s)$" is normal with mean $\mu(t - s)$ and variance $(t - s)\sigma^2$ since $X(t)$ process is a normal distribution. The expectation of $E\left[e^{X(t)-X(s)}\right]$ is then;

$$E\left[e^{X(t)-X(s)}\right] = e^{\mu(t-s)+(t-s)\sigma^2/2}. \quad (5.3)$$

The expectation of *Y(t)* given the history of the process up to time *s* is given by,

$$E[Y(t)|Y(u), 0 \leq u \leq s] = Y(s)e^{(t-s)(\mu+\frac{\sigma^2}{2})}. \quad (5.4)$$

Geometric Brownian motion is useful in the modelling of stock prices over time if the percentage changes are independent and identically distributed (Ross, 2006).

Finally, $Y(t)$ is a geometric Brownian motion with drift if it satisfies the following stochastic differential equation;

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dw(t). \quad (5.5)$$

For an arbitrary initial $Y(0)$ the stochastic differential equation has the following solution by Ito's interpretation;

$$Y(t) = Y(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w(t)}, \tag{5.6}$$

$$\ln\left[\frac{Y(t)}{Y(0)}\right] = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w(t). \tag{5.7}$$

Above paragraphs are mentioned some backbone explanation of the Brownian motion and geometric Brownian motion with drift.

In this study loss development factors or age to age factors of the development years (for each column in Figure 5.2) are modelled by the following equation;

$$d\lambda = \mu\lambda dt + \sigma\lambda dw(t). \tag{5.8}$$

μ is the drift term or trend term of the cohort base change of the development factors or in another words for the selected column the change of the factors by rows. σ is the standard deviation of this change. The solution of this stochastic differential equation is as follows;

$$\lambda(t) = \lambda(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w(t)}. \tag{5.9}$$

This solution can be generalized for the each development period *(j)*;

$$\lambda_j(t) = \lambda_j(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w(t)}. \tag{5.10}$$

For the simulation purpose estimation of the loss development factors for the last row is defined by;

$$ {}^{1}\lambda_{end} = {}^{1}\lambda_{0} e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)1 + (\sigma_1)(w(1))}, \tag{5.11} $$

$$ {}^{2}\lambda_{end} = {}^{2}\lambda_{0} e^{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)2 + (\sigma_2)(w(2))}, \tag{5.12} $$

$$ {}^{N}\lambda_{end} = {}^{N}\lambda_{0} e^{\left(\mu_N - \frac{\sigma_N^2}{2}\right)N + (\sigma_N)(w(N))}. \tag{5.13} $$

N is the last row of the column. ${}^{j}\lambda_0$ is the recent known development factor at the triangle (diagonal side) for column j. "end" shows the last row. $w(j)\sim N(0,j)$ is the Brownian motion for the j'th column.

It is assumed that there is no correlation between the number of deaths by columns or ages and also there is no correlation between the loss development factors. From this aspect, cumulative loss development factor for the last raw of the triangle can be derived by;

$$ \prod_{j=1}^{N} {}^{j}\lambda_{end} = \left({}^{1}\lambda_0 \, e^{\left(\mu_1 - \frac{\sigma_1{}^2}{2}\right)1 + \sigma_1 w(1)} \right) \ldots . \left({}^{N}\lambda_0 \, e^{\left(\mu_N - \frac{\sigma_N{}^2}{2}\right)N + \sigma_N w(N)} \right), $$

$$ = \prod_{j=1}^{N} \left({}^{j}\lambda_0\right) \left(e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)1 + (\sigma_1)(w(1))} \right) \ldots \left(e^{\left(\mu_N - \frac{\sigma_N^2}{2}\right)N + (\sigma_N)(w(N))} \right) \tag{5.14} $$

$$ = \prod_{j=1}^{N} \left({}^{j}\lambda_0\right) \left(e^{((\sigma_1)(w(1)) + \ldots + (\sigma_N)(w(N)))} \right) \left(e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)1 + \ldots + \left(\mu_N - \frac{\sigma_N^2}{2}\right)N} \right) $$

$$ = \prod_{j=1}^{N} \left({}^{j}\lambda_0\right) \left(e^{\sum_{j=1}^{N} (\sigma_j)(w(j))} \right) \left(e^{\sum_{j=1}^{N} j\mu_j - \sum_{j=1}^{N} j \frac{\sigma_j^2}{2}} \right) $$

$$ = \prod_{j=1}^{N} \left({}^{j}\lambda_0\right) \left(e^{\sum_{j=1}^{N} (\sigma_j)(w(j))} \right) \left(e^{j\left(\sum_{j=1}^{N} \mu_j - \frac{\sigma_j^2}{2}\right)} \right) $$

CHAPTER 6: APPLICATION OF THE MODELS

6.1 Brief about the Applications

"LifeMetrics" software package is a toolkit for measuring and managing longevity and mortality risks which is used with R Project. It gives the parameters of mortality models, residuals etc. It is used to estimate the model parameters, residuals, likelihoods, BICs in Chapter 3.

The annual age specific death rates are used to estimate the parameters of the models for the following countries; *Belgium, Bulgaria, Canada, France, Italy, Ireland, Japan, Netherlands, Norway, Poland, Russia, Spain, Sweden, Switzerland, UK, Denmark, Portugal, US, Ukraine.* The data is supplied from the Human Mortality Database (HMD, www.mortality.org). The Human Mortality Database (HMD) is created to provide detailed mortality and population data to researchers, students, journalists, policy analysts and others interested in the history of human longevity.

The data consists of numbers of deaths and survive by age. The parameter sets, log likelihoods, BIC results of Lee and Carter, Renshaw and Haberman, Currie, CBD, CBD1, CBD2, CBD3 models are analysed for the following 20 countries. The blue areas in Figure 6.1 show the modelled countries. As it is seen in Figure 6.1 and Table 6.1, most of the countries in the world are analysed in detail. In this aspect this thesis is the first academic study by its wide application range.

Figure 6.1 Countries modelled

Table 6.1 Countries modelled

1-Australia,	11-Norway,
2-Belgium,	12-Poland,
3-Bulgaria,	13-Portugal,
4-Canada,	14-Russia,
5-Denmark,	15-Spain,
6-France,	16-Sweden,
7-Ireland,	17-Switzerland,
8-Italy,	18-UK,
9-Japan,	19-Ukraine,
10-Netherlands,	20-US.

6.2 Results of the LC, RH, Currie, CBDs Models

The models are set for the “5-85” age range. The range includes young, middle and old age groups. The aim is to find the appropriate model for the large range because practitioners want to use unique model appropriate for all ages for the simplicity and prevent the time consuming. There is no any practical method to find the appropriate age range if one wants to divide the ages by range.

Table 6.2 Log likelihood results of the models

	Log Likelihood						
	LC	**RH**	**Currie**	**CBD**	**CBD1**	**CBD2**	**CBD3**
Australia	-39,532	**-30,481**	-37,506	-87,481	-49,214	-38,810	-46,184
Belgium	-49,693	**-34,451**	-52,818	-114,789	-45,698	-39,588	-45,452
Bulgaria	-42,705	**-28,452**	-44,597	-88,721	-48,630	-34,285	-46,675
Canada	-36,859	**-29,845**	-38,032	-93,508	-48,589	-39,374	-
Denmark	-42,742	**-35,902**	-47,501	-85,341	-44,988	-38,699	-
France	-49,693	**-34,451**	-52,818	-114,789	-45,698	-39,588	-45,452
Ireland	-35,283	**-27,528**	-34,623	-67,584	-50,276	-43,689	-
Italy	-50,196	**-34,651**	-69,687	-169,135	-57,813	-52,875	-
Japan	-41,433	**-22,210**	-37,517	-85,477	-34,820	-32,649	-32,122
Netherlands	-37,917	**-32,329**	-50,473	-111,199	-49,711	-37,139	-
Norway	-51,040	**-34,962**	-59,803	-142,126	-49,095	-44,193	-
Poland	-22,527	**-18,780**	-21,587	-41,170	-23,439	-21,499	-22,119
Portugal	-45,957	**-30,595**	-47,274	-154,107	-51,128	-45,255	-50,733
Russia	-25,426	**-20,693**	-28,481	-63,175	-29,991	-26,326	-27,792
Spain	-58,452	**-38,687**	-72,283	-174,991	-77,211	-72,048	-69,326
Sweden	-42,148	**-32,613**	-45,818	-123,966	-44,783	-39,260	-39,260
Switzerland	-41,081	**-33,025**	-40,154	-127,719	-48,045	-42,320	-46,711
UK	-44,254	**-31,332**	-46,221	-89,066	-51,636	-36,096	-
Ukraine	-25,354	**-20,550**	-26,742	-56,829	-27,859	-25,558	-26,899
US	-33,597	**-28,024**	-33,601	-60,594	-37,624	-35,427	-36,672

According to the likelihood results, RH model gives the maximum likelihood for all countries. CBD model gives the minimum likelihood for all countries. CBD3 model can not be fitted for the following countries; Canada, Denmark, Ireland, Italy, Netherlands, Norway and UK. It takes too long time, but there is no result. Therefore, the program is cancelled (Table 6.2).

Table 6.3 BIC results of the models

	BIC						
	LC	**RH**	**Currie**	**CBD**	**CBD1**	**CBD2**	**CBD3**
Australia	-40,625	**-32,664**	-38,944	-88,187	-50,634	-40,578	-47,613
Belgium	-50,787	**-36,634**	-54,256	-115,495	-47,118	-41,356	-46,881
Bulgaria	-43,702	**-30,442**	-45,861	-89,271	-49,808	-35,734	-47,861
Canada	-37,953	**-32,029**	-39,470	-94,214	-50,009	-41,142	-
Denmark	-43,836	**-38,086**	-48,939	-86,047	-46,409	-40,468	-
France	-50,787	**-36,634**	-54,256	-115,495	-47,118	-41,356	-46,881
Ireland	-36,256	**-29,469**	-35,843	-68,096	-51,394	-45,058	-
Italy	-51,290	**-36,834**	-71,125	-169,841	-59,234	-54,643	-
Japan	-42,424	**-24,188**	-38,770	-86,018	-35,983	-34,078	-33,293
Netherlands	-39,011	**-34,512**	-51,911	-111,905	-51,131	-38,908	-
Norway	-52,133	**-37,145**	-61,241	-142,831	-50,515	-45,962	-
Poland	-23,450	**-20,621**	-22,719	-41,607	-24,438	-22,712	-23,126
Portugal	-46,990	**-32,658**	-48,603	-154,715	-52,397	-46,824	-52,010
Russia	-26,342	**-22,522**	-29,602	-62,748	-30,974	-27,519	-28,784
Spain	-59,546	**-40,871**	-73,721	-175,696	-78,631	-73,816	-70,755
Sweden	-43,242	**-34,797**	-47,256	-124,671	-46,204	-41,029	-41,029
Switzerland	-42,174	**-35,208**	-41,592	-128,425	-49,465	-44,089	-48,140
UK	-45,348	**-33,515**	-47,659	-89,772	-53,056	-37,864	-
Ukraine	-26,270	**-22,378**	-27,863	-57,256	-28,843	-26,750	-27,891
US	-34,673	**-30,171**	-35,007	-61,270	-38,999	-37,135	-38,055

Table 6.3 gives the results of BIC. RH gives the maximum BIC for all countries. However, CBD gives the minimum BIC for all countries.

Table 6.4 Number of the parameters in the models

	Number of Parameter						
	LC	**RH**	**Currie**	**CBD**	**CBD1**	**CBD2**	**CBD3**
Australia	248	**495**	326	160	322	401	324
Belgium	248	**495**	326	160	322	401	324
Bulgaria	232	**463**	294	128	274	337	276
Canada	248	**495**	326	160	322	401	-
Denmark	248	**495**	326	160	322	401	-
France	248	**495**	326	160	322	401	324
Ireland	228	**455**	286	120	262	321	-
Italy	248	**495**	326	160	322	401	-
Japan	231	**461**	292	126	271	333	273
Netherlands	248	**495**	326	160	322	401	-
Norway	248	**495**	326	160	322	401	-
Poland	220	**439**	270	104	238	289	240
Portugal	238	**475**	306	140	292	361	294
Russia	219	**437**	268	102	235	285	237
Spain	248	**495**	326	160	322	401	324
Sweden	248	**495**	326	160	322	401	401
Switzerland	248	**495**	326	160	322	401	324
UK	248	**495**	326	160	322	401	-
Ukraine	219	**437**	268	102	235	285	237
US	245	**489**	320	154	313	389	315

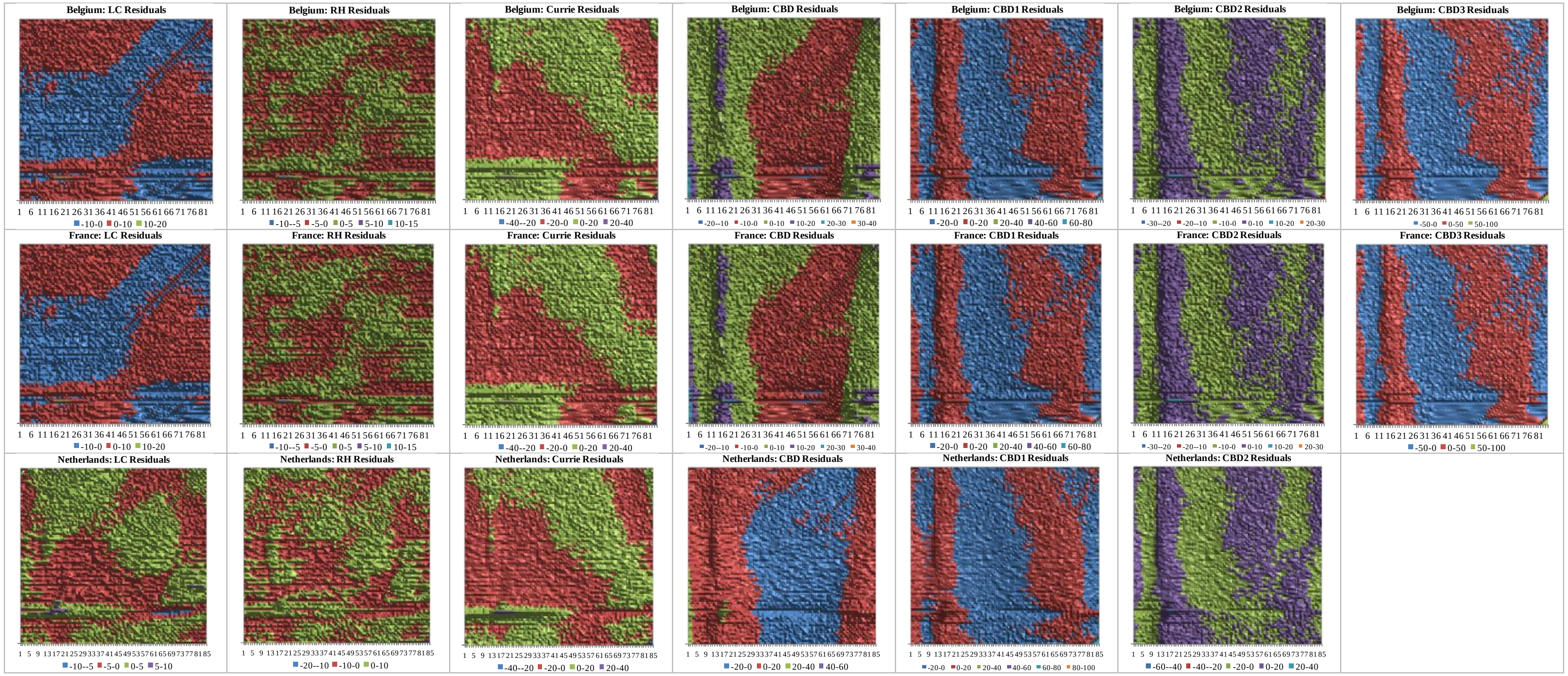
RH has the maximum number of parameters in the models and CBD model has the minimum number of parameters (Table 6.4).

Table 6.5 Variance of the model residuals

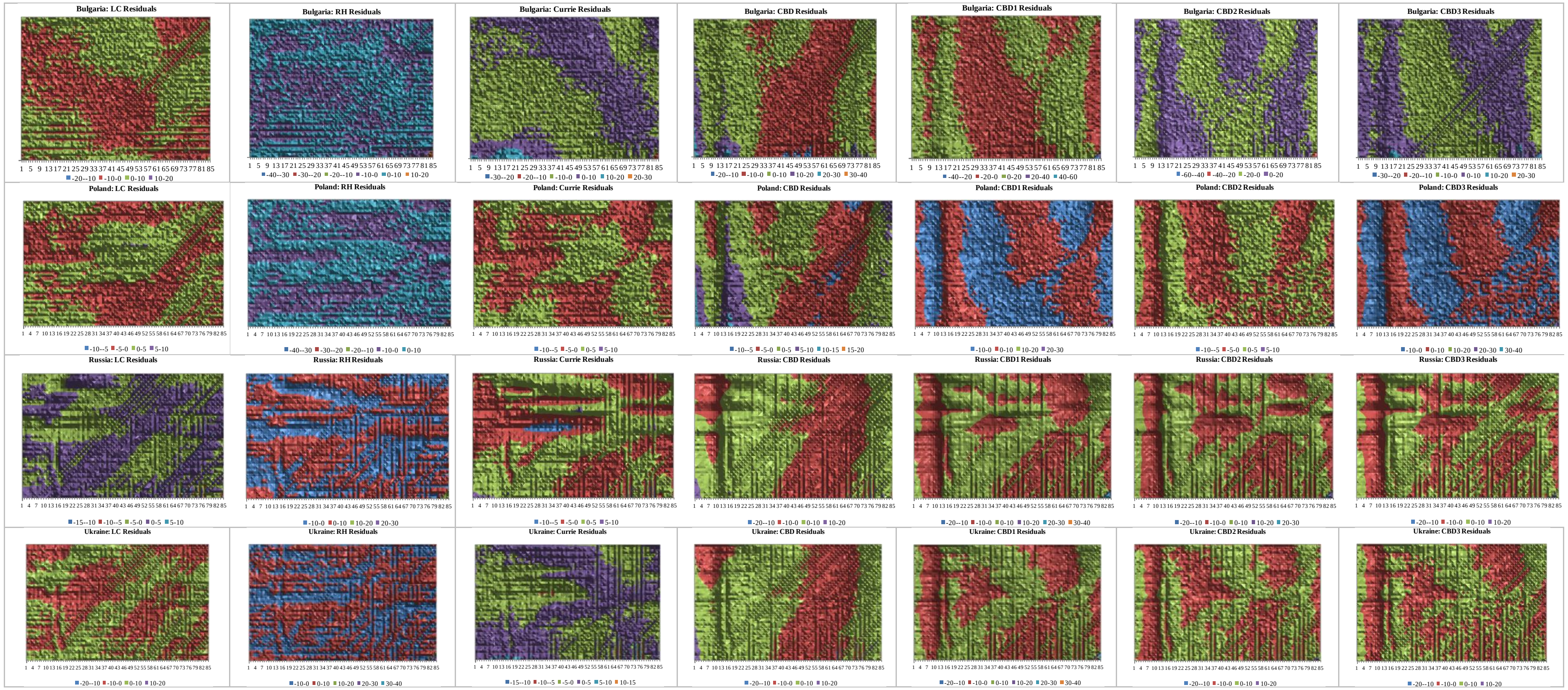
	Variance of Residuals						
	LC	**RH**	**Currie**	**CBD**	**CBD1**	**CBD2**	**CBD3**
Australia	3.9	**2.5**	3.5	22.4	22.4	6.6	14.1
Belgium	6.9	**2.3**	9.0	30.3	11.1	4.4	15.9
Bulgaria	7.8	**3.8**	9.9	29.8	29.8	7.7	9.9
Canada	3.1	**1.0**	4.1	24.1	11.9	6.1	-
Denmark	4.8	**2.8**	6.7	21.2	9.6	4.8	-
France	6.9	**2.3**	9.0	30.3	11.1	4.4	15.9
Ireland	6.4	**3.9**	6.1	21.9	13.6	11.7	-
Italy	7.5	**2.4**	15.6	50.6	20.6	8.7	-
Japan	8.1	**2.3**	7.5	29.4	9.5	4.7	5.0
Netherlands	3.5	**2.1**	9.2	33.5	14.6	6.3	-
Norway	7.7	**5.1**	12.1	41.8	24.5	6.0	-
Poland	2.4	3.0	**2.0**	11.7	4.1	2.0	3.9
Portugal	7.5	**2.9**	3.5	53.1	13.8	8.0	15.2
Russia	3.6	**2.2**	5.0	21.3	6.7	4.7	4.8
Spain	9.5	**3.6**	15.4	49.1	18.6	6.6	18.5
Sweden	4.8	**2.5**	6.5	35.1	19.4	4.7	4.7
Switzerland	4.3	**2.2**	4.4	37.1	13.1	5.5	15.3
UK	5.4	**3.1**	6.5	23.3	10.5	5.5	-
Ukraine	3.7	**3.1**	4.4	18.8	6.5	4.1	4.4
US	2.4	**0.9**	2.6	11.8	5.2	3.1	5.8

Table 6.5 shows the variance of residuals for the LC, RH, Currie, CBD, CBD1, CBD2 and CBD3. Except Poland, RH model gives the minimum variance and Currie model gives the minimum variance for Poland. High variance of CBD, CBD1 and CBD3 models are noteworthy.

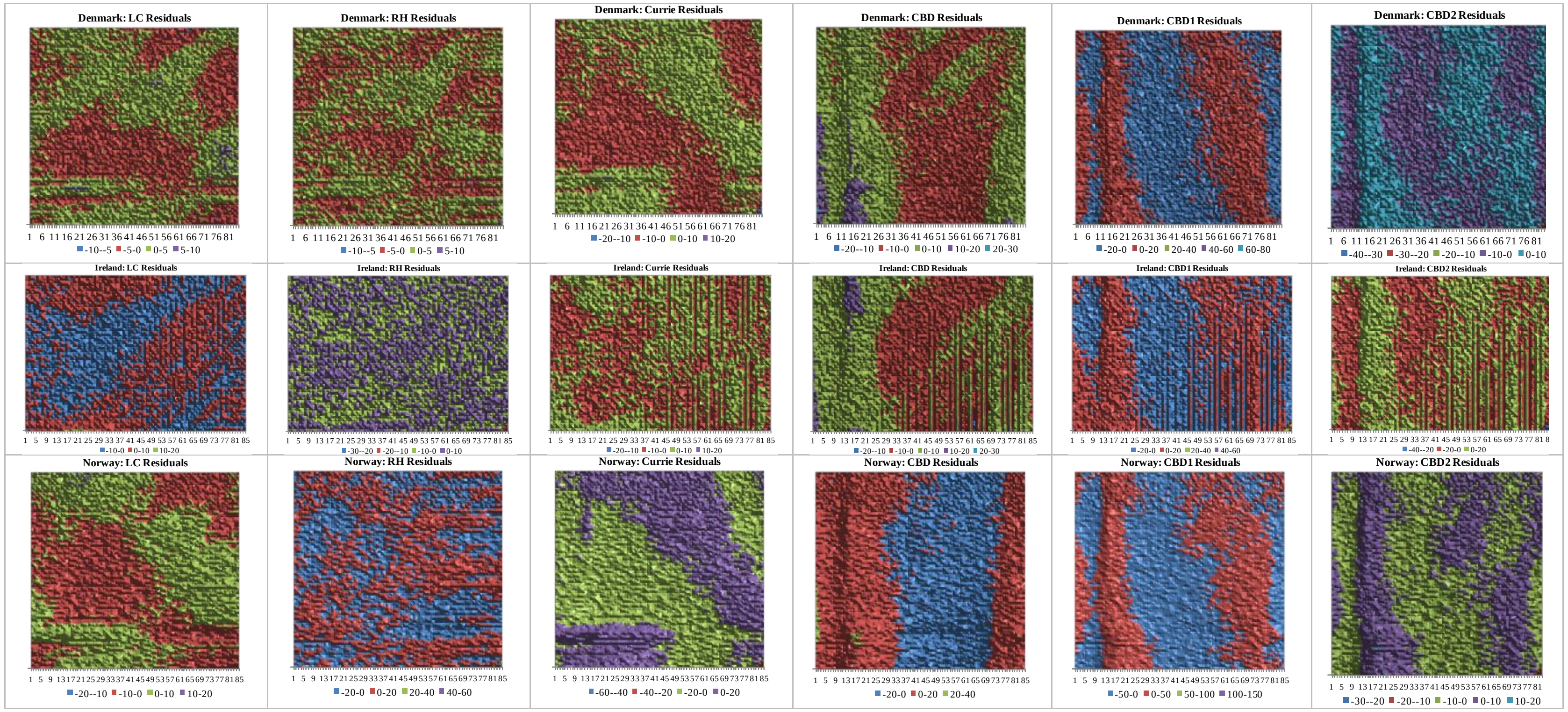
According to likelihood, BIC and variance of the residuals, RH is the most appropriate model in the models which are investigated.

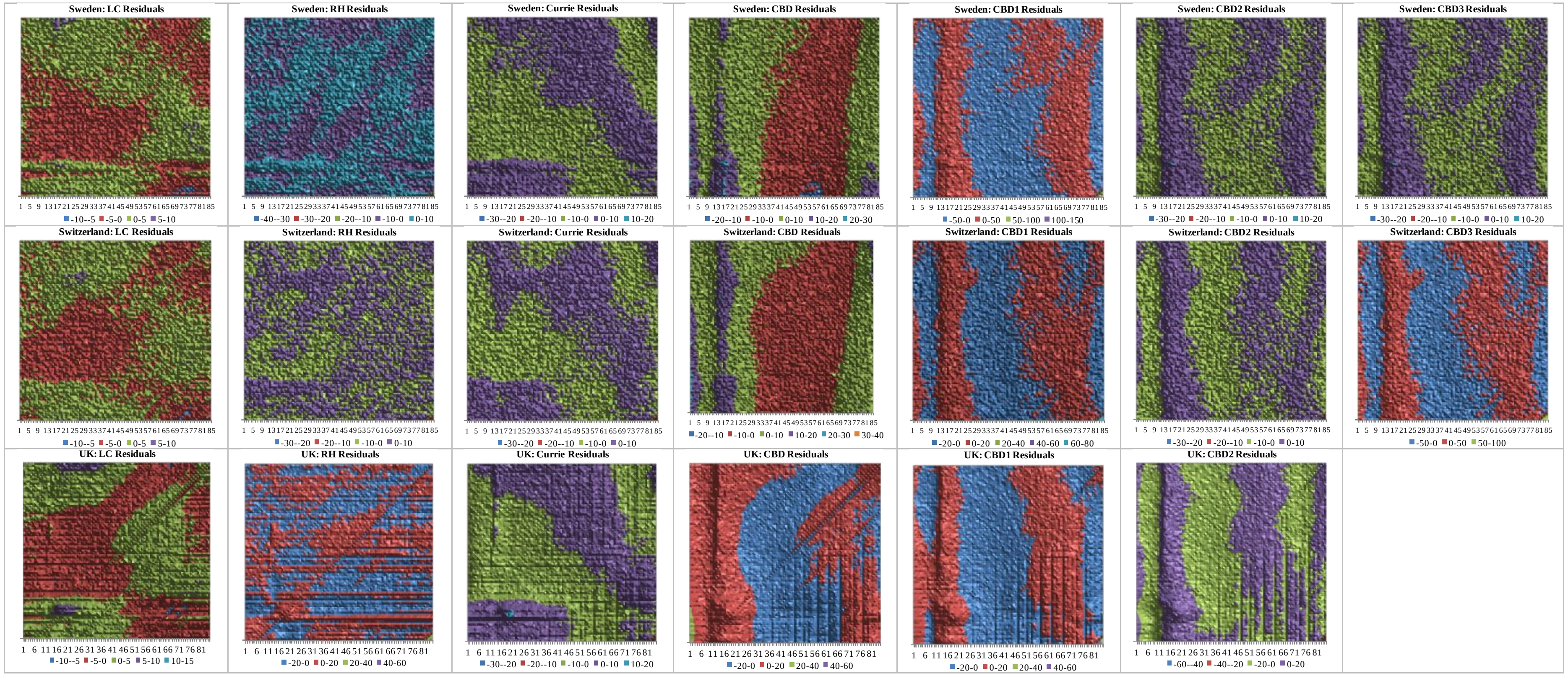


a) Western Europe Countries

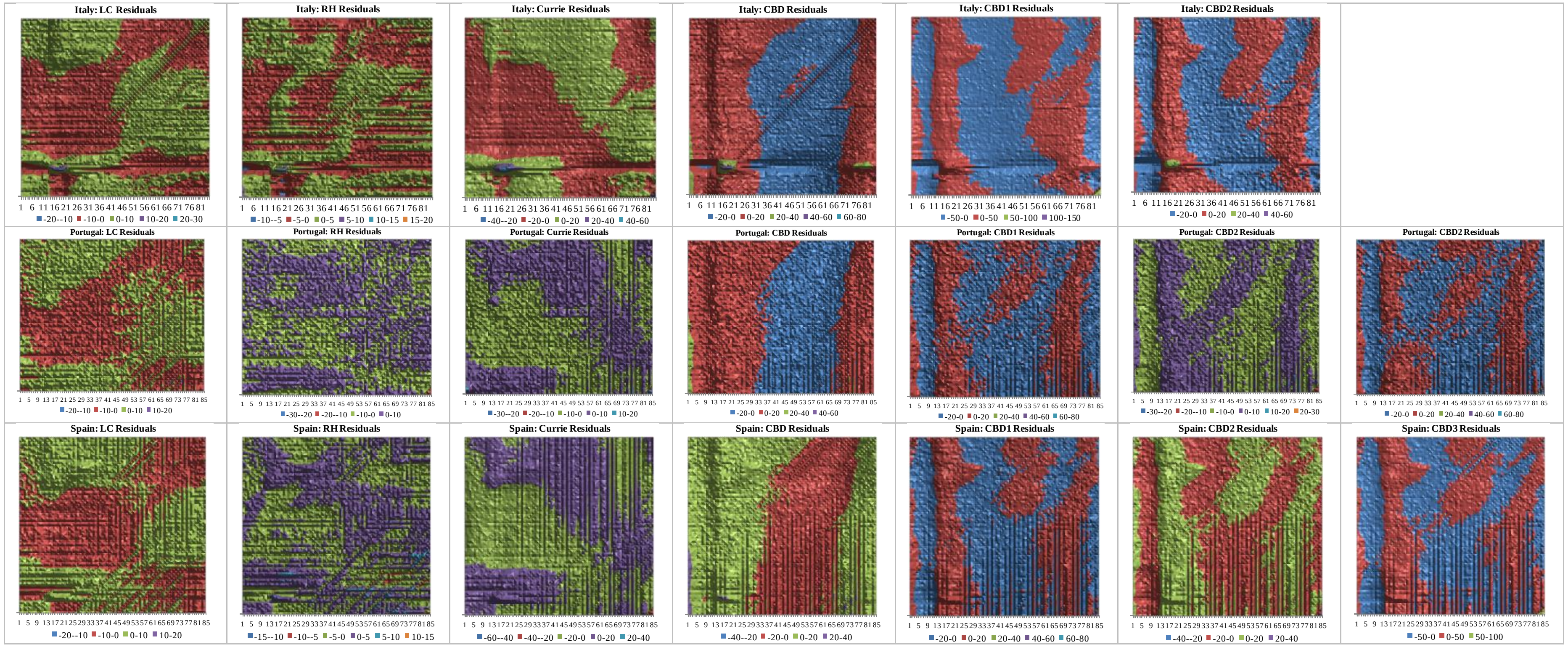


b) Eastern Europe Countries

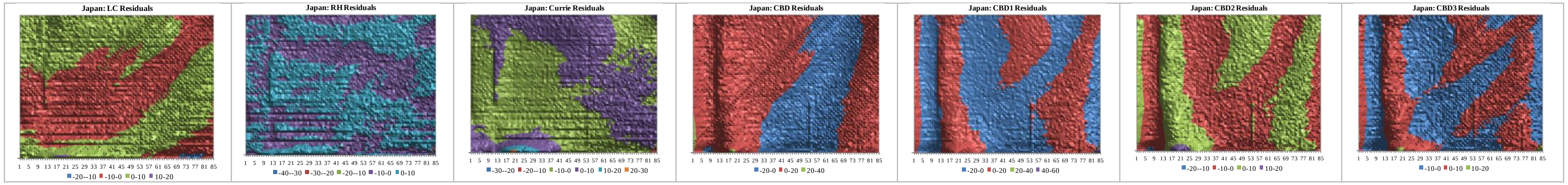

Denmark: LC Residuals
-10--5 -5-0 0-5 5-10
Denmark: RH Residuals
-10--5 -5-0 0-5 5-10
Denmark: Currie Residuals
-20--10 -10-0 0-10 10-20
Denmark: CBD Residuals
-20--10 -10-0 0-10 10-20 20-30
Denmark: CBD1 Residuals
-20-0 0-20 20-40 40-60 60-80
Denmark: CBD2 Residuals
-40--30 -30--20 -20--10 -10-0 0-10
Ireland: LC Residuals
-10-0 0-10 10-20
Ireland: RH Residuals
-30--20 -20--10 -10-0 0-10
Ireland: Currie Residuals
-20--10 -10-0 0-10 10-20
Ireland: CBD Residuals
-20--10 -10-0 0-10 10-20 20-30
Ireland: CBD1 Residuals
-20-0 0-20 20-40 40-60
Ireland: CBD2 Residuals
-40--20 -20-0 0-20
Norway: LC Residuals
-20--10 -10-0 0-10 10-20
Norway: RH Residuals
-20-0 0-20 20-40 40-60
Norway: Currie Residuals
-60--40 -40--20 -20-0 0-20
Norway: CBD Residuals
-20-0 0-20 20-40
Norway: CBD1 Residuals
-50-0 0-50 50-100 100-150
Norway: CBD2 Residuals
-30--20 -20--10 -10-0 0-10 10-20



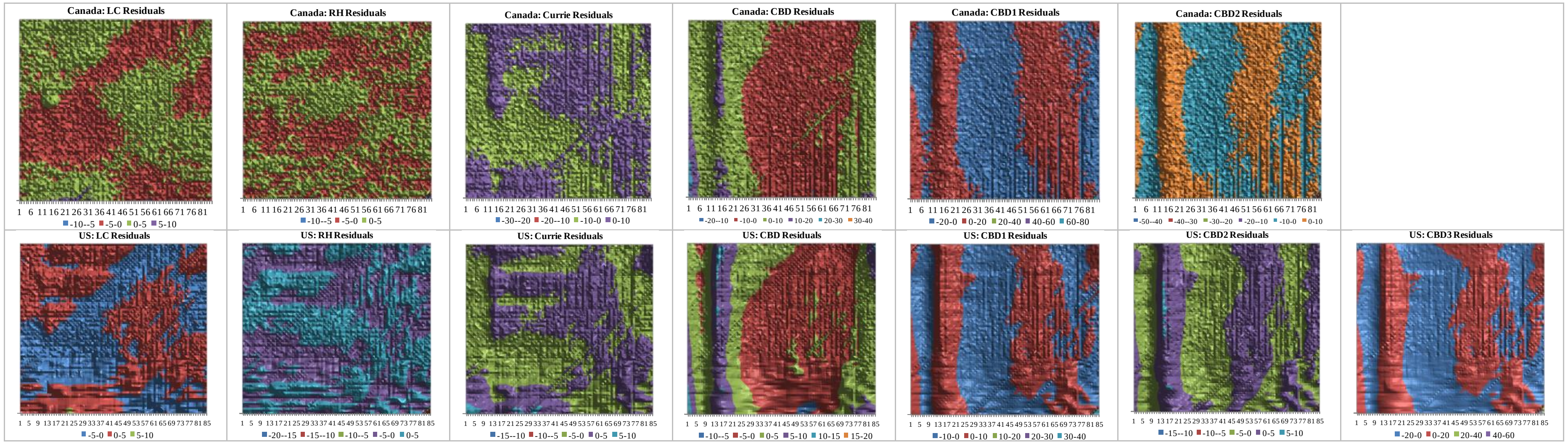
c) Northern Europe Countries



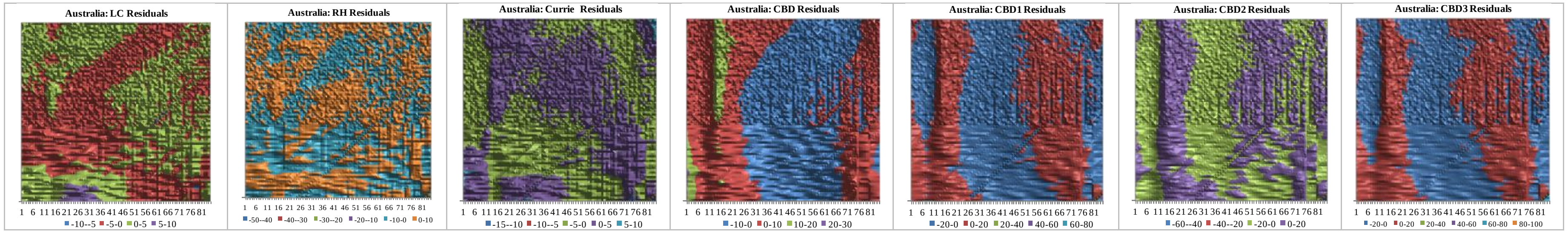
c) Southern Europe Countries



d) Asian Countries



e) American Countries

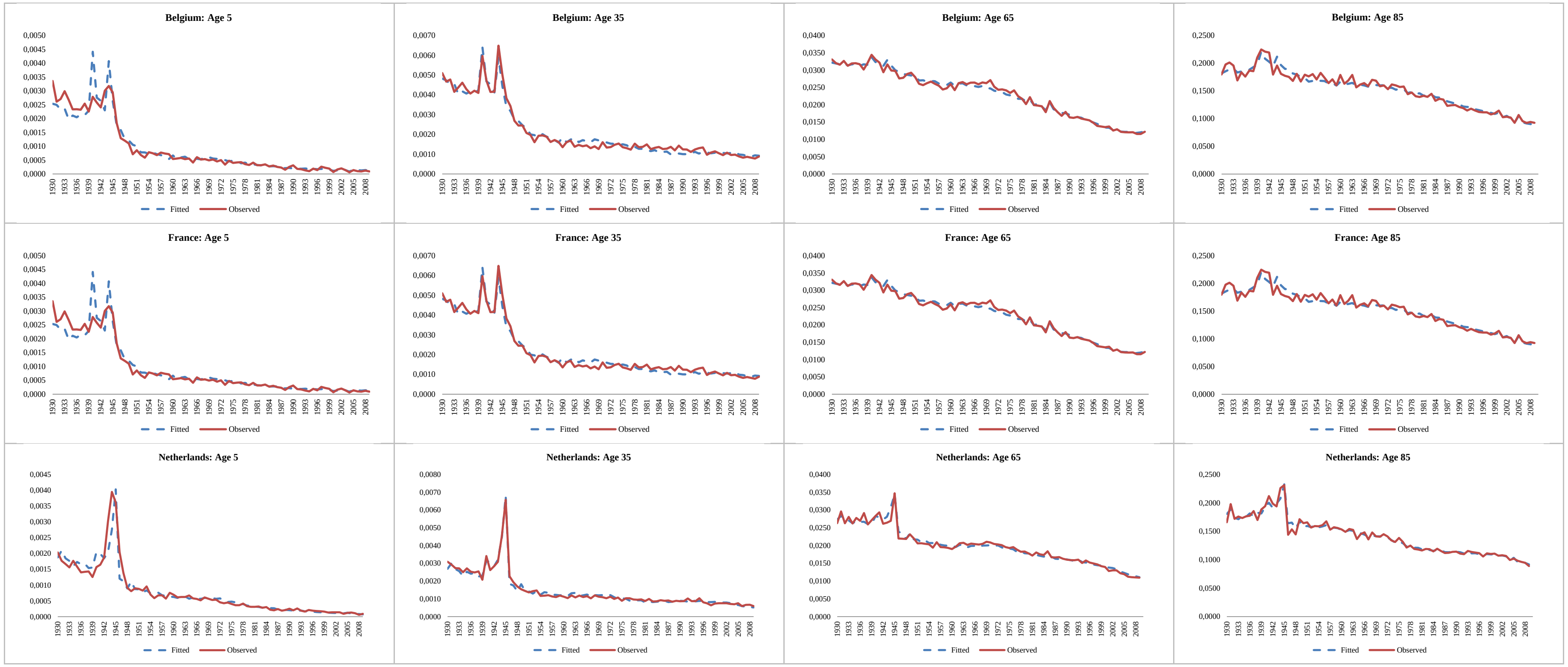


e) Australia

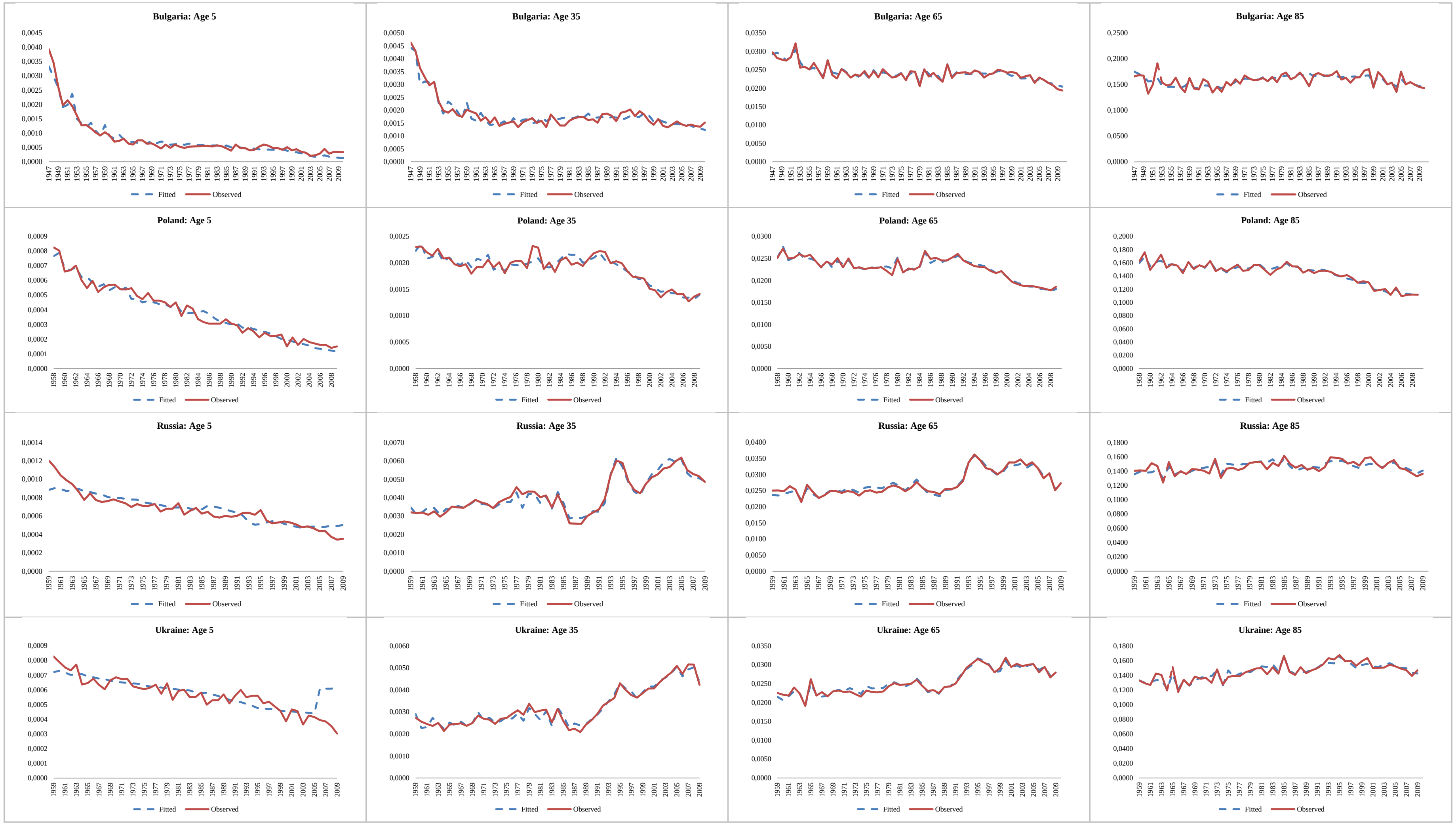
Figure 6.2. Residuals of the models

Figure 6.2 shows clear evidence of clustering that violate the assumption of residuals, especially along the diagonals and age ranges such as early age groups, middle age groups etc. CBD, CBD1, CBD2 and CBD3 models have the negative and positive clustering for all countries. There is no clear evidence of clustering for RH model. CBD models show clustering for the different age ranges. The age range “30-60” model gives the over fitted values than the observed. The approximately age range “60-80” is under fitted and “80+” is over fitted. Western Europe Countries, Northern Europe Countries, Southern Europe Countries, American Countries have similar residual paths. Eastern Europe Countries can be separated two parts Bulgaria-Poland and Russia-Ukraine. The paths of these countries have similar with each other.

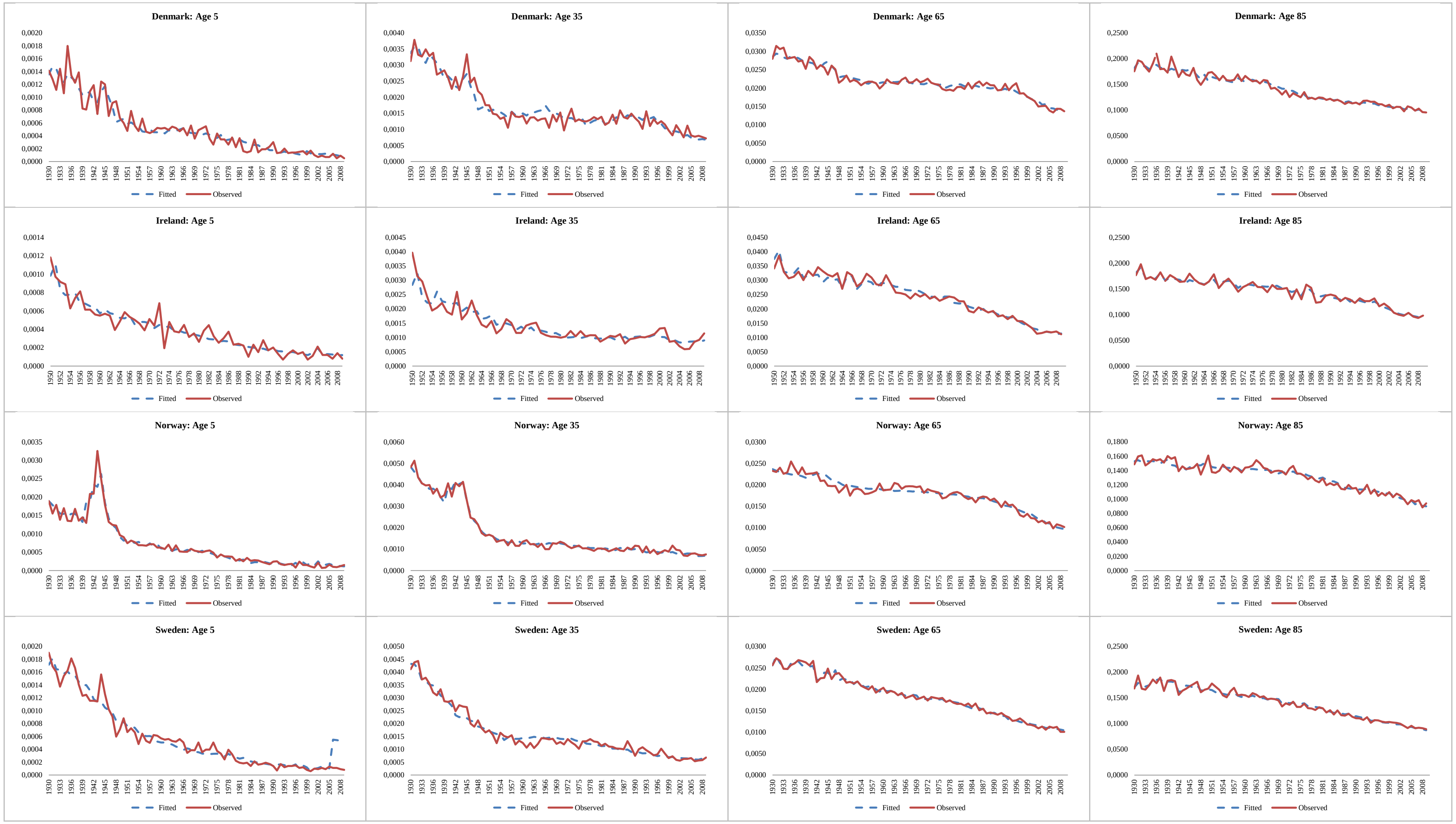
Figure 6.3 gives Renshaw and Haberman model fitted and observed values for the ages 5, 35, 65 and 85 because it is wanted to see the performance of the model for the young, middle and older ages. It is seen from the figures the results are satisfactory.

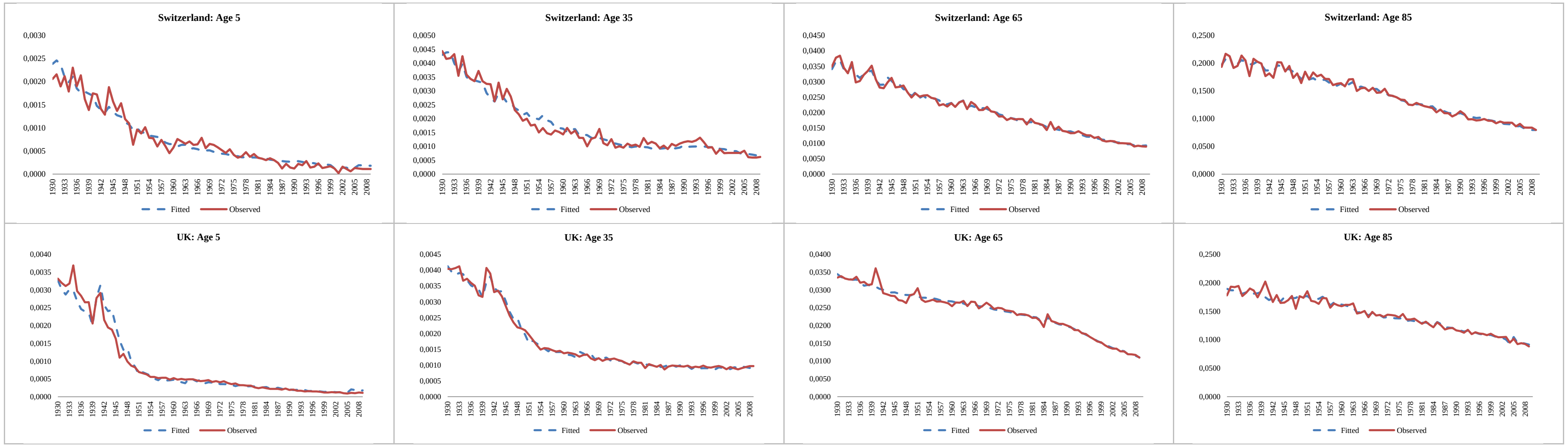


a) Western Europe Countries

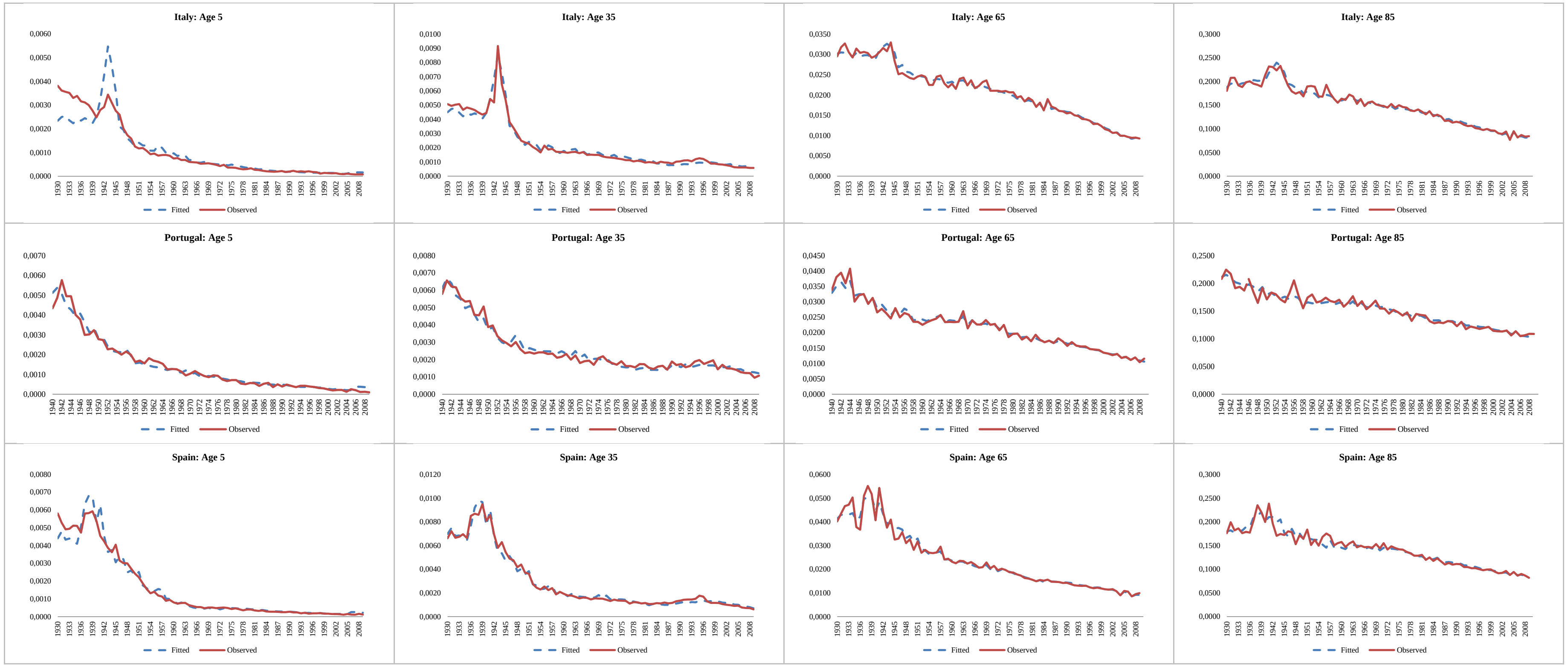


b) Eastern Europe Countries

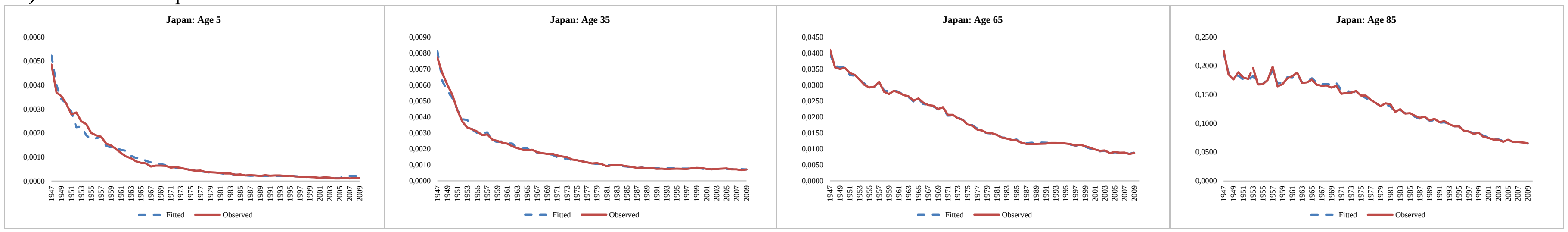

Denmark: Age 5
Denmark: Age 35
Denmark: Age 65
Denmark: Age 85
Ireland: Age 5
Ireland: Age 35
Ireland: Age 65
Ireland: Age 85
Norway: Age 5
Norway: Age 35
Norway: Age 65
Norway: Age 85
Sweden: Age 5
Sweden: Age 35
Sweden: Age 65
Sweden: Age 85
Fitted
Observed



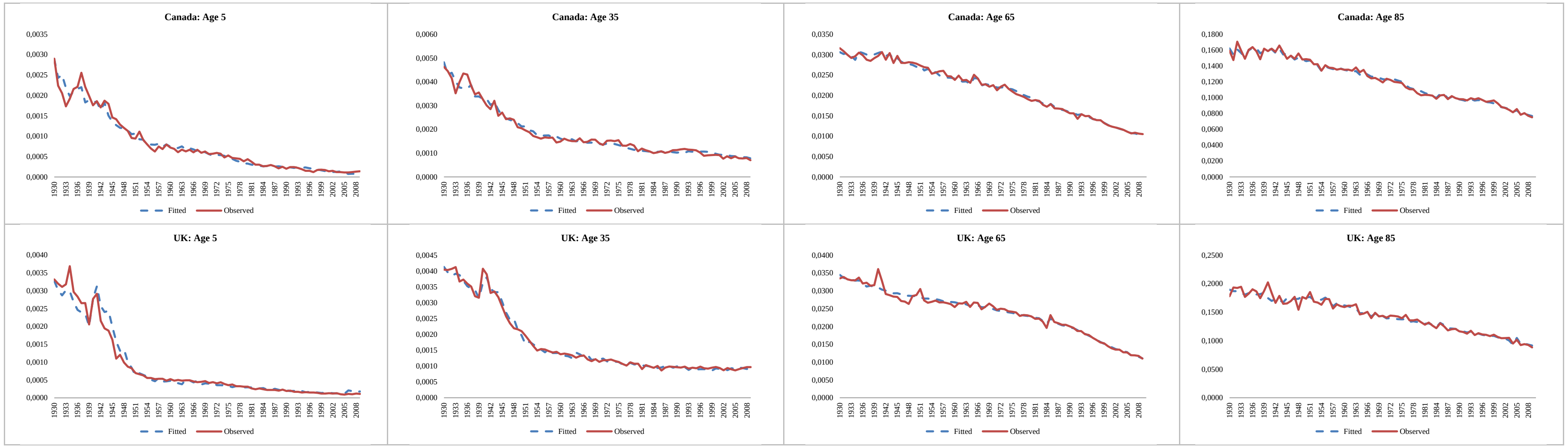
c) Northern Europe Countries



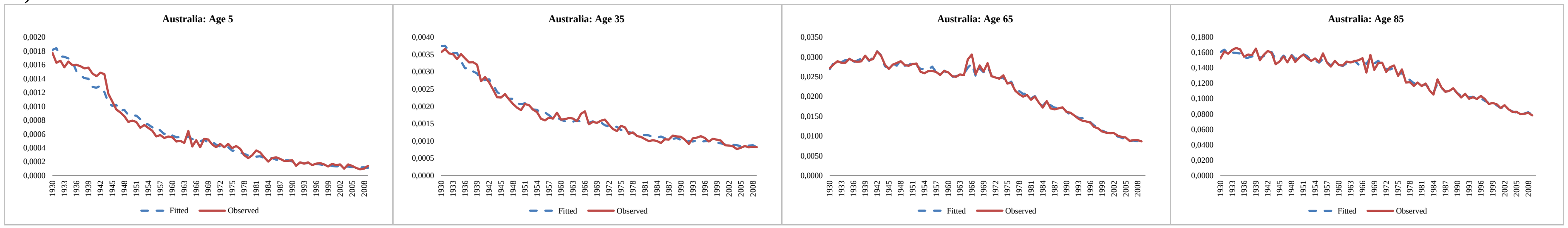
d) Southern Europe Countries



e) Asian Countries



f) American Countries



g) Australia

Figure 6.3 Renshaw and Haberman model fitted and observed values

6.3 Application of Stochastic Model

6.3.1 Data and Methodology

Luciano and Vigna (2005, 2008) states that the calibrated models can also be used for mortality extrapolation or future mortality prediction and there are two different ways to look at future mortality: the first one is within a certain generation; the second one is between different generations. At the first method the intensity process can be calibrated on the observed data and the evolution of the survival function can be forecasted in the future by considering its right tail after the last evolution. At the second method it can be considered how the different calibrated parameters of the intensity process change when changing generation. Both methods consider the members of generation are not all dead. The best calibration is collecting the data until the observation date for the related generation. For instance, if the calibration is done in 2005 for the generation was born in 1905. The data will be the observed mortality rates of this generation for 100 years. However, the data related with the generation is available only the members are all dead. It can be extrapolated the data by firstly collecting the mortality rates yearly from 1905 to 2005 and taking the diagonal starting from q_0 in 1905 to q_{100} in 2005. They stated that this procedure does not give exactly the mortality rates but it is a good approximation.

The analysis base is the run-off triangle so it is needed to produce the triangle in Figure 6.4. As Luciana and Vigna (2008) state the triangle is derived using their approximation. Figure 6.4 shows the run-off triangle of the death numbers. It is designed according to the birth year (cohort year) and age.

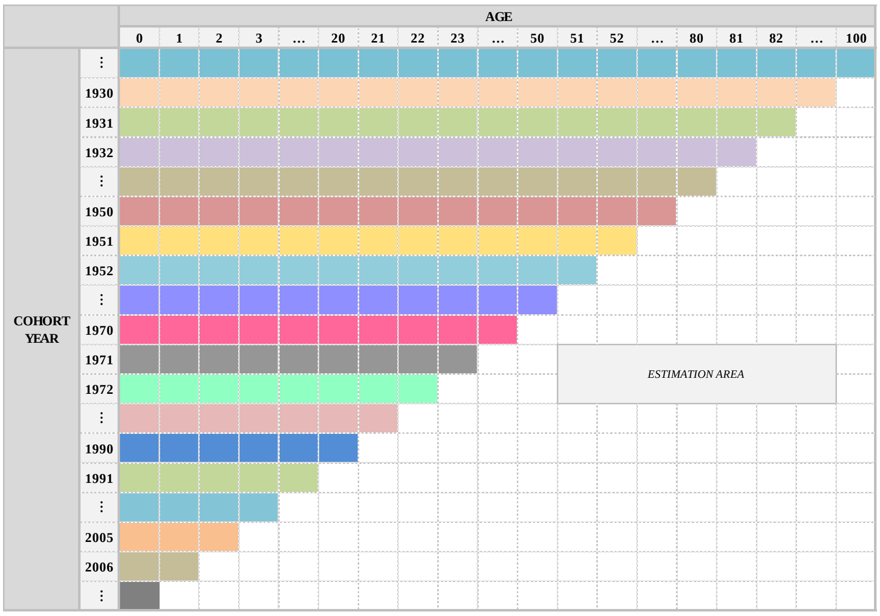


Figure 6.4 The run-off triangle

UK data is used to analyse the new method performance and to test other frequently used methods. The primary reason of data selection is that UK is already examined by several academically studies frequently known and cited and also known as a reliable data set. The inside of the triangle is filled using the following procedure;

1. Life tables are taken from the Human Mortality Database for the UK between the years 1922 and 2011,

2. LT_i shows the life table for the year i,

3. The first column of the triangle shows the number of deaths for the age 0. Therefore this is equal the following equation;

$$D_{i,j} = LT_i^{Age=j},\ j = 0. \tag{6.1}$$

For instance, LT_{1922}^{0} is the number of deaths for the age 0 at 1922. Actually this is the number for the new-born and dying in the first year at 1922. Consequently, the first column gives the development for the number of deaths at age 0 by the years.

4) The other columns are derived using the following approach;

$$D_{i,j} = LT_{i+j}^{Age=j},\ j > 0. \quad (6.2)$$

This equation is the simple approximately approach to the cohort year base number of death. As it is mentioned, this is an approach because there is no life table giving the cohort year statistics for the large range of years. However, it can be said that the approach mathematically, practically is consistent with its results.

Standard chain ladder approach, Mack method, Reversible Jump Markov Chain Monte Carlo, bootstrap method and the new proposed method are used for forecasting the number of deaths by cohort base. The out-sample forecast quality of the proposed model, Lee and Carter, Renshaw and Haberman model is also compared. Lee and Carter method also is used for the compare the results with the proposed method. The reason is in many studies the Lee Carter method is used for a benchmark method and it is used also as a benchmark methodology by the US Bureau of the Census. Renshaw and Haberman model is also selected to compare the results with the proposed model because as it is seen from the fitted results at Chapter 6. Renshaw and Haberman model gives the best results.

21 years are used for the out-sample forecasts because of this reason the life tables between 1991 and 2011 years are put aside. There are the tables between the years from 1922 to 1990 to estimate the GBM parameters. The run-off triangle is constituted by the life tables 1943 – 1990 to satisfy the triangle format. Firstly the parameters of the used models are estimated and according to these parameters the forecasting procedure is derived. Sum of Pearson residuals, sum square of Pearson residuals, standard deviation of Pearson residuals and mean square of Pearson residuals are used for the comparison of the model results.

Analyses and simulations are made using the following software; Chain Ladder package (Gesmann et al., 2013) which is prepared for R project is used for the estimation of Mack method, the Bootstrap method is programmed with R project according to the paper of Verrall and England, IBNRS software is used for the RJMCMC method, the R project package LifeMetrics is used for the parameter estimations of the stochastic mortality models. *ARIMA(1,1,0)* model is used for the parameter forecast of κ for the Lee and Carter method and $\gamma^{(3)}$ for Renshaw and Haberman model. After determining the parameters of the new proposed model, 5,000 simulations are done and generated 5,000 different loss development factors. According to these development factors, 5,000 different completed number of deaths triangles are estimated. The mean of the estimated numbers is taken for the comparison with other models and also 10,000 simulations are done for the RJMCMC and bootstrap methods. The mean of these methods is also taken to compare with the results of other methods.

6.3.2 Estimation of $\boldsymbol{\lambda(i,j)}$

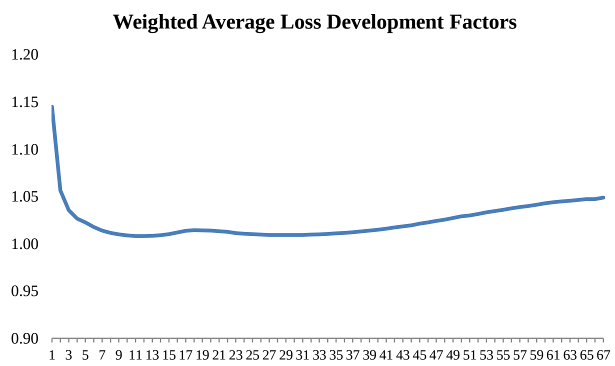


Figure 6.5 Weighted average loss development factors of the number of death by cohort base

Figure 6.5 gives the weighted average of loss development factors. The graphic of loss development factors of the number of death is similar to force of mortality. Early ages loss development factors are higher than the older ages. These trend mimics the number of deaths is relatively higher for one year transition for these ages. The older age factors are also higher than the middle ages and have an increasing trend. The number of deaths within a year increases for advancing ages.

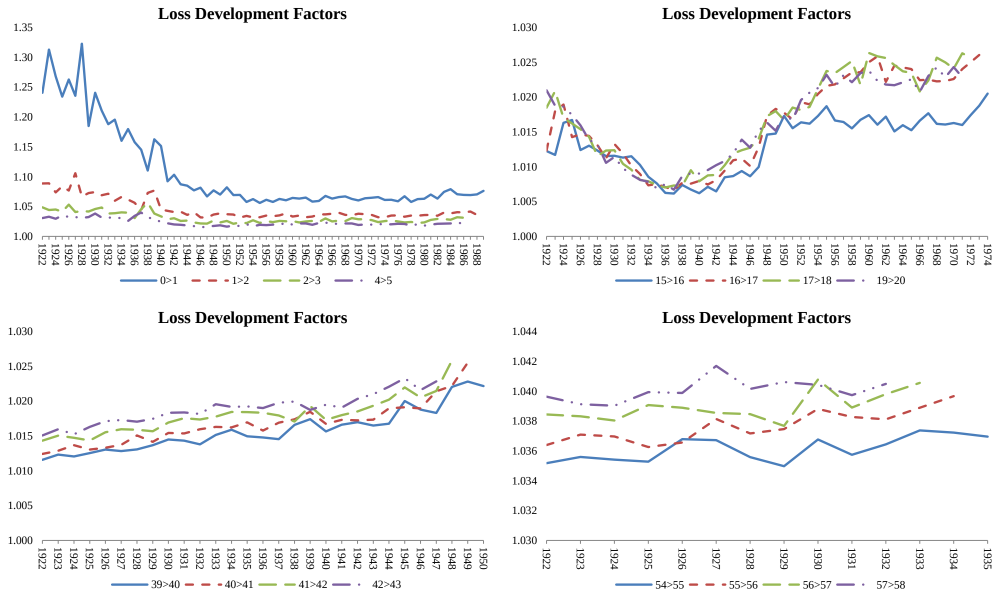


Figure 6.6 Loss development factors by cohort years for different age transitions

Figure 6.6 shows the loss development factors by the cohort years or in other words column base factors. It can be seen from the first graph that the earlier periods have a decreasing trend. Recent years have lower factors than the earlier periods. This shows that the infant mortality rate decreases.

Second graph shows the youth ages development. There are two trends in the graph firstly there is a decreasing trend from 1922 to 1934 and secondly there is an increasing trend. Number of deaths relatively increases for the youth ages. It is known that the probability of death decreases from past to now, therefore the number of death for early ages decreases and extends to older ages because of increasing life expectancy. For example the transition between from age 15 to age 16 the loss development factor 1.006 for the period 1940 and 1.021 for the period 1974. In other words at 1940 till to age 15 the number of death is 8,226 and till to age 16 the number of death is 8,277 so the number of death is 51 within a year. At 1974 till to age 15 the number of death is 2.146 and till to age 16 the number of death is 2,190 so the number of death is 44. When it is compared with the period 1940 it is seen that the number is proportionally higher than the 1940.

Third and fourth graphs have an increasing trend. Proportionally the number of deaths is higher for the recent years.

The trend and volatility of the factors change from past to now and it is seen that it may continue in the future. Therefore it is not correct to use the weighted average factors or most recent period factors in estimations. Loss development factors should be modelled to obtain more accurate estimations. For this aim it is used geometric Brownian motion approach to model the loss development factors as mentioned in Chapter 5. Appendix 5 gives the estimated number of deaths used in out-sample forecasts and an example for loss development factors.

Table 6.6 Estimates of μ and σ in GBM

Development Period	$\hat{\mu}$	$\hat{\sigma}$	**Development Period**	$\hat{\mu}$	$\hat{\sigma}$
1	-0.002123	0.023116	**26**	0.000225	0.000925
2	-0.000753	0.008532	**27**	0.000199	0.000975
3	-0.000306	0.004819	**28**	0.000211	0.000870
4	-0.000162	0.002755	**29**	0.000262	0.000895
5	-0.000133	0.002373	**30**	0.000255	0.000837
6	-0.000046	0.002080	**31**	0.000303	0.000798
7	-0.000079	0.001685	**32**	0.000311	0.000867
8	-0.000029	0.001520	**33**	0.000301	0.000690
9	-0.000004	0.001295	**34**	0.000301	0.000993
10	0.000007	0.001193	**35**	0.000351	0.001060
11	0.000026	0.001043	**36**	0.000396	0.000765
12	0.000001	0.000936	**37**	0.000419	0.000825
13	0.000039	0.001083	**38**	0.000394	0.001098
14	0.000067	0.001181	**39**	0.000414	0.000940
15	0.000069	0.001150	**40**	0.000372	0.001176
16	0.000156	0.001523	**41**	0.000477	0.001024
17	0.000264	0.001688	**42**	0.000434	0.001193
18	0.000142	0.001690	**43**	0.000306	0.000797
19	0.000037	0.001377	**44**	0.000277	0.000965
20	0.000114	0.001418	**45**	0.000399	0.000927
21	0.000058	0.001285	**46**	0.000336	0.000890
22	0.000029	0.001331	**47**	0.000292	0.000832
23	0.000112	0.001298	**48**	0.000201	0.000776
24	0.000131	0.001213	**49**	0.000125	0.000881
25	0.000165	0.000999			

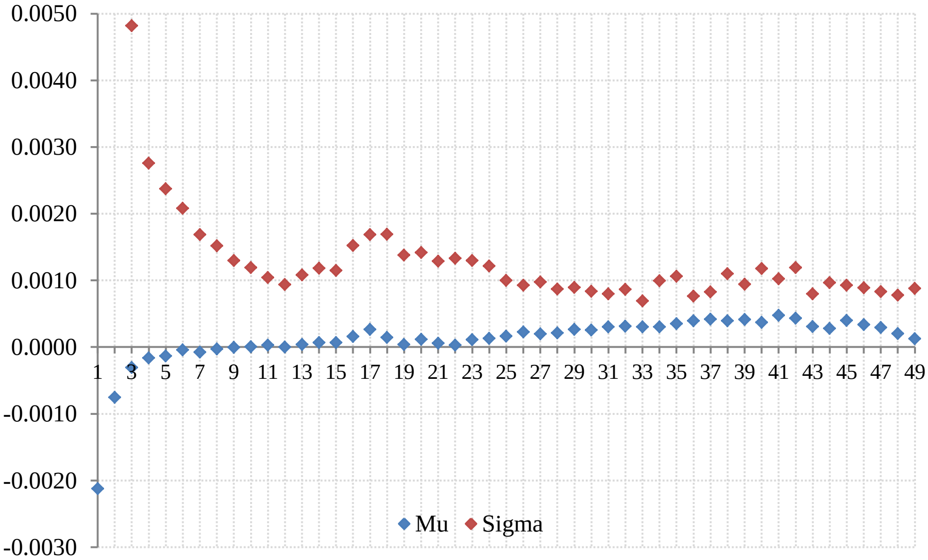


Figure 6.7 μ and σ parameters

The parameters are estimated from the column 1 to *(n-20)*. Because it is meaningless to estimate the μ with limited data for example for the period *(n-1)*, μ is estimated by two development factors. Table 6.6 gives the μ and σ estimations.

Figure 6.7 gives the μ and σ parameters of the loss development factors by columns. X axis is the development period another words columns at the Figure 6.4. σ has a decreasing trend for the first 11 periods and after this period its level is stable and 0.001. The decreasing trend shows the volatility is high for the childhood periods. μ has an increasing trend for the first 9 periods and it has negative values. It shows that the deaths in childhood period have a decreasing trend.

6.3.3 Lee and Carter vs The New Method

It is needed triangle format as in Figure 6.4 for Lee and Carter and Renshaw and Haberman models estimations to compare the methods. The number of deaths is derived from Lee and Carter and Renshaw and Haberman model using the following steps;

1. ARIMA(1,1,0) model is used for the parameter forecast of κ for LC and $\gamma^{(3)}$ for RH. These parameters are estimated for the periods 1991-2011,

2. Using the $\beta^{(1)}$ and $\beta^{(2)}$ parameters and κ, probability of death is derived for the ages and calendar years for LC and $\kappa^{(2)}$, $\beta^{(1)}$, $\beta^{(2)}$ and $\beta^{(3)}$ for RH model,

3. Radix is taken 100,000 to be compatible with the data taken from the Human Mortality Database and by multiplying the probabilities with the radix the number of deaths are derived,

4. The procedure mentioned above is used to complete the run-off triangle for the 1943-1990 periods using the forecasted life tables for 1991-2011. This triangle is used for the comparison of out-sample forecasts.

Loss development factors are estimated by using the new method. Savitzky and Golay filter is used to smooth the simulation patterns and to eliminate the noisy structure of the factors. The main aim is to keep the pattern of the number of death by ages with eliminating the noises by this way. Moving average, Savitzky and Golay, polynomial fit and local regression smoothing techniques are tried to find the best smoothing technique. However, Savitzky and Golay filter gives the best residual results. The results of other methods are given at the Appendix III.

Shortly, the method is proposed by Savitzky and Golay (Savitzky and Golay, 1964) based on local least-squares polynomial approximation. Savitzky and Golay show that fitting a polynomial to a set of input samples and then evaluating the resulting polynomial at a single point within the approximation interval is equivalent to discrete convolution with a fixed impulse response. Least squares smoothing reduces noise while maintaining the shape and height of waveform peaks are demonstrated by Savitzky and Golay (Schafer, 2011).

The main advantage of the Savitzky and Golay filter has over moving average and other finite impulse response (FIR) filters is its ability to preserve higher moments in the data and thus reduce smoothing on peak heights (Lynch et al. 2004).

The frame size and polynomial degree are determined by giving the minimum residuals for this aim from 5 to 45 for the frame size and from 3 to 7 for polynomial degree are estimated. The final result is the frame size for the filter and polynomial degree are selected 15 and 6.

Table 6.7 gives the results; Lee and Carter model and the new method's sum of Pearson residuals, sum square of Pearson residuals, mean square of Pearson residuals and standard deviation of Pearson residuals by cohort years.

Table 6.7 Lee and Carter model and the New Method residuals

Cohort Year	Number Of Estimated Cell	Standard Deviation of Pearson Residuals			Sum of Pearson Residuals			Sum Square of Pearson Residuals			Mean Square of Pearson Residuals		
		Lee Carter	New Method	New Method (SG Filter)	Lee Carter	New Method	New Method (SG Filter)	Lee Carter	New Method	New Method (SG Filter)	Lee Carter	New Method	New Method (SG Filter)
1943	1	-	-	-	0.38	1.00	0.94	0.15	1.01	0.88	0.15	1.01	0.88
1944	2	0.73	0.95	0.67	-1.38	0.25	0.32	1.48	0.94	0.51	0.74	0.47	0.25
1945	3	0.67	0.59	0.42	-1.12	2.14	1.91	1.32	2.21	1.56	0.44	0.74	0.52
1946	4	0.37	0.48	0.19	-6.03	-2.12	-1.58	9.50	1.81	0.74	2.37	0.45	0.19
1947	5	0.85	1.05	0.66	0.54	1.34	0.85	2.93	4.76	1.89	0.59	0.95	0.38
1948	6	0.85	0.70	0.45	-3.44	12.64	11.51	5.57	29.06	23.09	0.93	4.84	3.85
1949	7	0.68	1.13	0.91	-2.72	12.81	12.06	3.80	31.11	25.79	0.54	4.44	3.68
1950	8	0.83	1.82	1.43	0.16	16.93	17.23	4.86	58.90	51.44	0.61	7.36	6.43
1951	9	0.65	1.52	1.35	2.18	15.97	15.21	3.89	46.76	40.36	0.43	5.20	4.48
1952	10	0.57	1.38	1.16	2.66	16.57	15.69	3.65	44.70	36.73	0.36	4.47	3.67
1953	11	0.73	1.62	1.39	8.82	21.73	20.96	12.38	69.24	59.16	1.13	6.29	5.38
1954	12	0.83	1.80	1.48	10.17	22.23	21.77	16.15	76.71	63.47	1.35	6.39	5.29
1955	13	0.88	1.93	1.64	8.34	18.24	17.76	14.59	70.07	56.71	1.12	5.39	4.36
1956	14	0.78	1.82	1.58	9.06	16.24	15.84	13.79	61.69	50.49	0.99	4.41	3.61
1957	15	0.55	1.56	1.43	14.09	18.75	17.95	17.54	57.45	50.21	1.17	3.83	3.35
1958	16	0.70	1.57	1.35	13.21	14.52	13.24	18.34	50.35	38.35	1.15	3.15	2.40
1959	17	0.85	1.36	1.07	16.59	13.53	12.91	27.78	40.16	28.25	1.63	2.36	1.66
1960	18	0.90	1.44	1.13	18.40	10.82	10.34	32.64	41.69	27.46	1.81	2.32	1.53
1961	19	0.90	1.26	1.05	22.42	9.55	8.31	41.16	33.50	23.47	2.17	1.76	1.24
1962	20	1.01	1.15	0.87	25.34	1.69	0.85	51.48	25.33	14.50	2.57	1.27	0.73
1963	21	1.02	1.02	0.80	27.44	1.74	0.89	56.54	21.07	12.98	2.69	1.00	0.62
1964	21	1.09	0.93	0.77	31.13	-0.31	-0.89	69.85	17.19	11.85	3.33	0.82	0.56
1965	21	1.15	1.01	0.73	39.17	6.56	6.21	99.48	22.58	12.60	4.74	1.08	0.60
1966	21	1.13	1.16	0.81	40.30	2.92	2.32	102.79	27.41	13.53	4.89	1.31	0.64
1967	21	1.19	0.83	0.56	43.27	3.86	1.73	117.28	14.61	6.50	5.58	0.70	0.31
1968	21	1.18	0.81	0.62	49.11	1.52	1.26	142.46	13.30	7.84	6.78	0.63	0.37
1969	21	1.49	0.70	0.49	49.05	-1.39	-0.47	159.02	9.77	4.86	7.57	0.47	0.23
1970	21	1.56	0.68	0.59	46.82	-4.69	-4.65	152.90	10.36	8.00	7.28	0.49	0.38
1971	21	1.43	0.62	0.67	50.00	-3.67	-4.52	160.12	8.42	10.03	7.62	0.40	0.48
1972	21	1.49	0.73	0.62	50.65	4.72	3.78	166.64	11.76	8.28	7.94	0.56	0.39
1973	21	1.51	0.67	0.55	50.92	6.15	5.80	168.99	10.79	7.74	8.05	0.51	0.37
1974	21	1.82	0.75	0.60	48.06	3.75	4.43	176.42	11.85	8.16	8.40	0.56	0.39
1975	21	1.77	0.79	0.73	48.09	9.30	9.54	172.76	16.69	14.90	8.23	0.79	0.71
1976	21	1.71	0.85	0.78	41.18	16.48	15.60	139.26	27.41	23.83	6.63	1.31	1.13
1977	21	1.67	0.73	0.71	41.58	16.70	15.96	137.85	23.98	22.08	6.56	1.14	1.05
1978	21	1.55	0.76	0.67	33.25	12.49	12.68	100.98	18.86	16.72	4.81	0.90	0.80
1979	21	1.56	0.89	0.81	34.31	13.03	12.81	104.60	23.87	21.07	4.98	1.14	1.00
1980	21	1.45	0.86	0.78	34.47	21.00	20.83	98.83	35.83	32.87	4.71	1.71	1.57
1981	21	1.55	1.04	0.89	29.13	25.42	24.97	88.21	52.48	45.49	4.20	2.50	2.17
1982	21	1.16	0.79	0.76	27.92	24.82	24.66	63.97	41.67	40.56	3.05	1.98	1.93
1983	21	1.31	0.92	0.86	23.36	23.77	23.66	60.24	43.78	41.60	2.87	2.08	1.98
1984	21	1.19	0.94	0.90	17.14	20.51	19.82	42.13	37.53	34.89	2.01	1.79	1.66
1985	21	1.01	0.97	0.95	14.83	17.46	17.56	30.75	33.28	32.63	1.46	1.58	1.55
1986	21	1.13	1.01	0.98	13.38	13.08	12.64	33.86	28.40	26.92	1.61	1.35	1.28
1987	21	0.87	0.92	0.86	10.42	10.96	10.42	20.46	22.84	20.02	0.97	1.09	0.95
1988	21	0.82	0.95	0.94	9.73	9.51	9.77	18.00	22.25	22.24	0.86	1.06	1.06
1989	21	0.75	0.98	0.99	9.09	13.60	13.14	15.18	27.90	27.67	0.72	1.33	1.32
1990	21	0.67	0.93	0.90	4.46	10.05	9.76	9.86	22.01	20.67	0.47	1.05	0.98
TOTAL	**798**	**1.41**	**1.16**	**1.04**	**1,055.9**	**504.2**	**483.8**	**2,992.4**	**1,405.3**	**1,151.6**	**3.75**	**1.76**	**1.44**

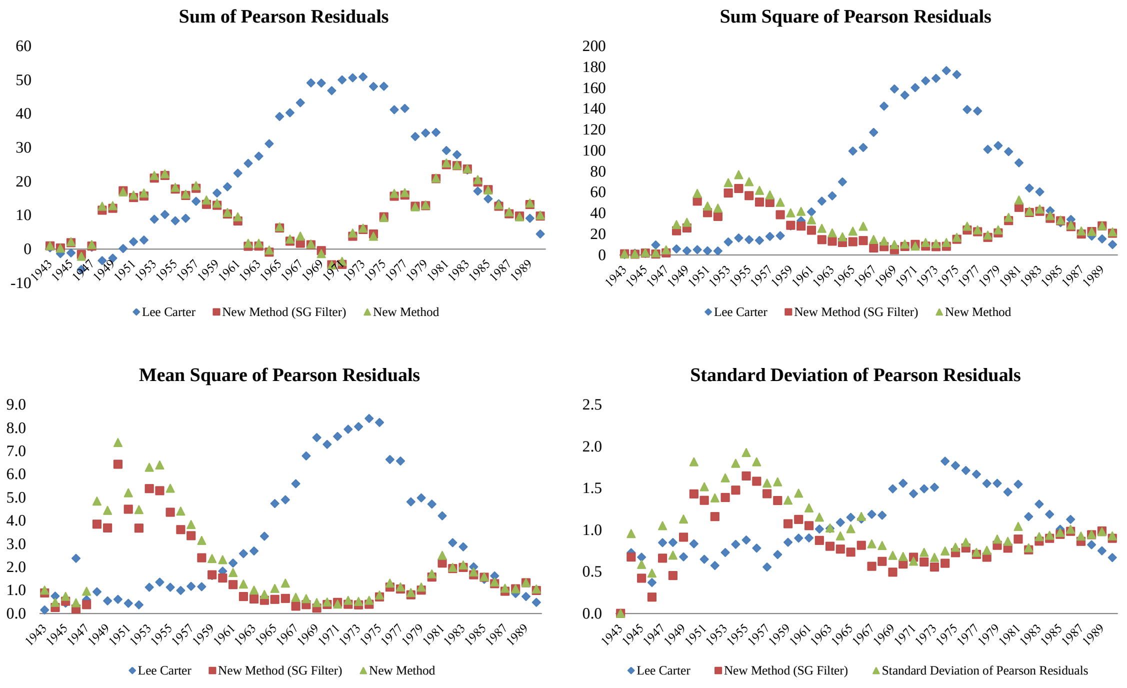


Figure 6.8 Descriptives of residuals for Lee and Carter and the New Method

The new method gives the best overall sum of residual, sum square of residual, mean square of residual and standard deviation of residual results than Lee and Carter method. The forecasts for the older ages of Lee and Carter method are better than the new method and the forecasts of early ages of the new method are better than Lee and Carter method. Nevertheless total results show that the results of the new method are better than the Lee and Carter results (Figure 6.8).

6.3.4 The Other Methods vs The New Method

In this section, non-life methods in Chapter 4 are used for the mortality rate estimation and their results are compared with the new method.

Figure 6.9 shows the RJMCMC results. First 1,000 simulations are burned for RJMCMC and the truncation index k is estimated to be 65. Value at risk (TVaR) is implemented to determine the estimated bound tail which is a measure often used in financial mathematics. It is the average of the results greater than the determined quantile. Deterministic result shows the TVaR (50%).

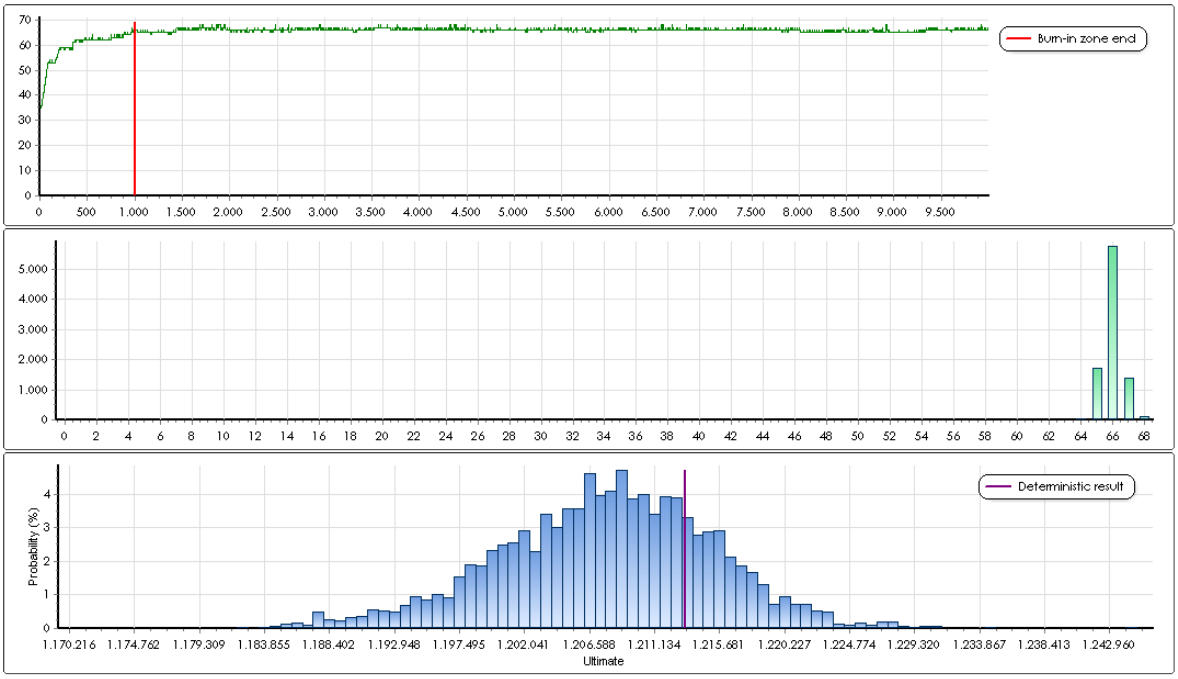


Figure 6.9 Burn in zone end, the distribution of truncation index k and histogram of ultimate estimations for RJMCMC

The development factors of Mack method are set by the result of an ordinary regression of $C_{i,k+1}$ against $C_{i,k}$ with intercept 0. Table 6.8 and Figure 6.10 give the Mack method fitting results. IBNR shows the expected number of death by the cohort periods. For example for the 1990 cohort period the observed number of death for the new-born is 801 and according to the Mack method, it is expected to die 3,303 people within 68 years.

Table 6.9 and Figure 6.11 give the comparison of the methods used in non-life analyses and the new method. According to sum of Pearson residuals, sum square of Pearson residuals, mean square of Pearson residuals and standard deviation of Pearson residuals by cohort years, the new method gives the best results.

Table 6.8 Mack method results

	Latest	Dev. To. Date	Ultimate	IBNR	Mack S.E.	CV(IBNR)
1922	43,419	100.0%	43,419	0	0.0	-
1923	40,402	95.4%	42,339	1,937	0.8	0.000
1924	38,885	91.0%	42,742	3,857	5.8	0.002
1925	36,260	86.9%	41,746	5,486	46.0	0.008
1926	33,804	82.9%	40,761	6,957	47.0	0.007
1927	31,307	79.2%	39,507	8,200	52.8	0.006
1928	29,261	75.8%	38,610	9,349	58.9	0.006
1929	27,366	72.5%	37,729	10,363	64.9	0.006
1930	24,787	69.5%	35,673	10,886	71.4	0.007
1931	23,608	66.6%	35,432	11,824	77.6	0.007
1932	21,659	64.0%	33,853	12,194	85.0	0.007
1933	19,983	61.5%	32,483	12,500	89.7	0.007
1934	18,422	59.2%	31,109	12,687	95.7	0.008
1935	17,479	57.1%	30,623	13,144	103.0	0.008
1936	16,528	55.1%	30,000	13,472	107.1	0.008
1937	15,494	53.2%	29,099	13,605	111.5	0.008
1938	14,042	51.5%	27,248	13,206	118.4	0.009
1939	13,363	50.0%	26,749	13,386	123.5	0.009
1940	13,235	48.5%	27,285	14,050	133.3	0.009
1941	12,464	47.1%	26,439	13,975	138.2	0.010
1942	11,250	45.9%	24,514	13,264	143.7	0.011
1943	10,508	44.8%	23,477	12,969	151.8	0.012
1944	9,721	43.7%	22,241	12,520	160.8	0.013
1945	8,967	42.8%	20,973	12,006	172.0	0.014
1946	8,415	41.9%	20,093	11,678	187.0	0.016
1947	8,367	41.1%	20,363	11,996	200.4	0.017
1948	6,706	40.4%	16,614	9,908	209.7	0.021
1949	6,386	39.7%	16,087	9,701	219.9	0.023
1950	5,992	39.1%	15,326	9,334	232.1	0.025
1951	5,812	38.5%	15,079	9,267	243.2	0.026
1952	5,490	38.0%	14,432	8,942	253.7	0.028
1953	5,235	37.6%	13,932	8,697	262.9	0.030
1954	4,975	37.1%	13,392	8,417	273.3	0.032
1955	4,759	36.8%	12,948	8,189	282.9	0.035
1956	4,567	36.4%	12,553	7,986	291.6	0.037
1957	4,380	36.0%	12,155	7,775	299.6	0.039
1958	4,215	35.7%	11,802	7,587	307.5	0.041
1959	4,064	35.4%	11,481	7,417	314.4	0.042
1960	3,926	35.1%	11,184	7,258	321.0	0.044
1961	3,801	34.8%	10,920	7,119	326.7	0.046
1962	3,761	34.5%	10,897	7,136	332.0	0.047
1963	3,584	34.2%	10,472	6,888	337.2	0.049
1964	3,429	33.9%	10,105	6,676	342.9	0.051
1965	3,222	33.6%	9,581	6,359	348.6	0.055
1966	3,101	33.3%	9,308	6,207	354.5	0.057
1967	2,957	33.0%	8,964	6,007	361.7	0.060
1968	2,894	32.6%	8,869	5,975	371.3	0.062
1969	2,768	32.2%	8,586	5,818	387.5	0.067
1970	2,673	31.8%	8,399	5,726	405.1	0.071
1971	2,595	31.4%	8,262	5,667	421.5	0.074
1972	2,394	31.0%	7,726	5,332	438.2	0.082
1973	2,284	30.6%	7,470	5,186	453.9	0.088
1974	2,190	30.2%	7,254	5,064	466.6	0.092
1975	2,054	29.8%	6,881	4,827	473.4	0.098
1976	1,841	29.5%	6,232	4,391	478.2	0.109
1977	1,786	29.3%	6,100	4,314	481.2	0.112
1978	1,696	29.0%	5,842	4,146	483.1	0.117
1979	1,657	28.8%	5,754	4,097	484.8	0.118
1980	1,528	28.6%	5,351	3,823	486.7	0.127
1981	1,383	28.3%	4,887	3,504	488.5	0.139
1982	1,335	28.0%	4,766	3,431	491.0	0.143
1983	1,252	27.7%	4,524	3,272	494.1	0.151
1984	1,175	27.3%	4,310	3,135	499.0	0.159
1985	1,133	26.8%	4,235	3,102	507.8	0.164
1986	1,112	26.1%	4,261	3,149	522.9	0.166
1987	1,056	25.4%	4,164	3,108	539.3	0.174
1988	1,007	24.4%	4,126	3,119	572.2	0.183
1989	903	22.9%	3,939	3,036	705.1	0.232
1990	801	19.5%	4,104	3,303	1,616.8	0.490

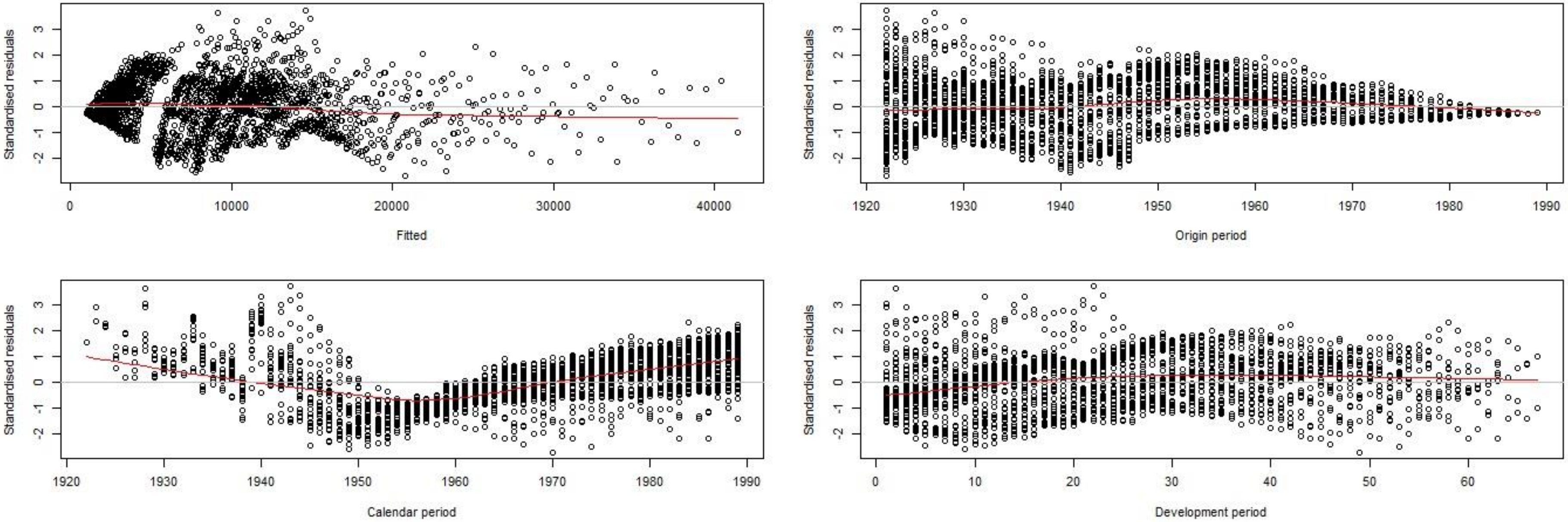


Figure 6.10 Standardised residuals with respect to Mack method

Table 6.9 Residual results of the New Method and non-life methods

Cohort Year	Number Of Estimated Cell	Standard Deviation of Pearson Residuals						Sum of Pearson Residuals						Sum Square of Pearson Residuals						Mean Square of Pearson Residuals					
		SCL	Mack	Bootstrap	RJMCMC	New Method (SG Filter)	New Method	SCL	Mack	Bootstrap	RJMCMC	New Method (SG Filter)	New Method	SCL	Mack	Bootstrap	RJMCMC	New Method (SG Filter)	New Method	SCL	Mack	Bootstrap	RJMCMC	New Method (SG Filter)	New Method
1943	1	-	-	-	-	-		2.6	2.7	2.5	3.1	0.9	1.0	6.6	7.3	6.2	9.9	0.9	1.0	6.58	7.28	6.25	9.90	0.88	1.01
1944	2	0.39	0.39	0.44	0.16	0.67	0.95	4.5	4.7	4.4	5.9	0.3	0.3	10.1	11.4	9.8	17.5	0.5	0.9	5.06	5.69	4.92	8.75	0.25	0.47
1945	3	0.60	0.63	0.67	0.42	0.42	0.59	9.5	10.1	9.5	10.3	1.9	2.1	30.9	34.9	31.1	35.9	1.6	2.2	10.29	11.63	10.37	11.96	0.52	0.74
1946	4	0.42	0.47	0.44	0.56	0.19	0.48	9.1	10.1	9.3	11.3	-1.6	-2.1	21.3	26.3	22.1	32.7	0.7	1.8	5.33	6.58	5.53	8.18	0.19	0.45
1947	5	0.44	0.38	0.46	0.47	0.66	1.05	16.1	17.4	16.1	14.1	0.9	1.3	52.8	61.3	52.9	40.5	1.9	4.8	10.55	12.26	10.57	8.11	0.38	0.95
1948	6	0.21	0.15	0.17	0.26	0.45	0.70	31.4	33.1	31.5	33.6	11.5	12.6	164.9	183.2	166.0	188.2	23.1	29.1	27.49	30.53	27.66	31.37	3.85	4.84
1949	7	0.57	0.52	0.56	0.69	0.91	1.13	38.9	41.0	38.8	45.4	12.1	12.8	218.4	241.5	217.5	297.3	25.8	31.1	31.21	34.50	31.07	42.47	3.68	4.44
1950	8	0.92	0.83	0.92	1.29	1.43	1.82	51.7	54.3	51.5	57.2	17.2	16.9	339.7	373.9	337.8	421.2	51.4	58.9	42.46	46.74	42.22	52.64	6.43	7.36
1951	9	0.78	0.71	0.81	0.97	1.35	1.52	58.2	61.3	57.8	61.1	15.2	16.0	380.6	421.7	376.7	422.1	40.4	46.8	42.29	46.86	41.86	46.90	4.48	5.20
1952	10	0.56	0.50	0.58	0.78	1.16	1.38	67.5	71.5	67.4	67.6	15.7	16.6	458.0	514.1	457.0	462.0	36.7	44.7	45.80	51.41	45.70	46.20	3.67	4.47
1953	11	0.80	0.67	0.80	0.96	1.39	1.62	82.8	87.8	82.9	79.9	21.0	21.7	628.9	705.7	630.9	590.0	59.2	69.2	57.17	64.15	57.35	53.64	5.38	6.29
1954	12	0.91	0.80	0.96	1.29	1.48	1.80	93.6	99.2	93.6	105.8	21.8	22.2	738.7	827.0	739.4	950.9	63.5	76.7	61.56	68.92	61.62	79.25	5.29	6.39
1955	13	1.28	1.14	1.27	1.29	1.64	1.93	99.3	105.6	98.9	95.8	17.8	18.2	777.9	873.9	771.3	726.7	56.7	70.1	59.84	67.23	59.33	55.90	4.36	5.39
1956	14	1.31	1.11	1.28	1.50	1.58	1.82	106.5	114.3	106.8	120.2	15.8	16.2	831.8	949.4	836.4	1,060.9	50.5	61.7	59.42	67.81	59.74	75.78	3.61	4.41
1957	15	1.19	1.00	1.20	1.14	1.43	1.56	119.6	128.3	119.2	123.1	18.0	18.7	973.2	1,111.2	967.9	1,029.1	50.2	57.5	64.88	74.08	64.53	68.60	3.35	3.83
1958	16	1.33	1.13	1.35	1.17	1.35	1.57	124.3	133.9	123.9	119.1	13.2	14.5	992.2	1,138.9	986.9	906.9	38.3	50.3	62.01	71.18	61.68	56.68	2.40	3.15
1959	17	1.16	0.94	1.13	1.18	1.07	1.36	134.3	145.5	134.4	134.1	12.9	13.5	1,082.8	1,260.0	1,083.0	1,080.0	28.3	40.2	63.69	74.12	63.71	63.53	1.66	2.36
1960	18	1.29	1.05	1.26	1.30	1.13	1.44	142.7	154.5	142.6	144.9	10.3	10.8	1,159.6	1,344.3	1,156.1	1,195.9	27.5	41.7	64.42	74.68	64.23	66.44	1.53	2.32
1961	19	1.23	1.08	1.25	1.42	1.05	1.26	151.8	165.1	151.4	154.4	8.3	9.5	1,239.3	1,455.5	1,235.1	1,290.5	23.5	33.5	65.22	76.60	65.00	67.92	1.24	1.76
1962	20	1.18	1.02	1.18	1.57	0.87	1.15	155.2	169.5	155.0	179.1	0.8	1.7	1,231.3	1,456.3	1,227.6	1,650.5	14.5	25.3	61.57	72.81	61.38	82.53	0.73	1.27
1963	21	1.26	1.11	1.24	1.38	0.80	1.02	165.9	181.3	165.9	177.0	0.9	1.7	1,343.1	1,588.9	1,341.3	1,529.1	13.0	21.1	63.95	75.66	63.87	72.81	0.62	1.00
1964	21	1.27	1.16	1.27	1.36	0.77	0.93	168.0	184.2	167.8	169.6	-0.9	-0.3	1,376.3	1,641.9	1,372.7	1,407.4	11.9	17.2	65.54	78.19	65.37	67.02	0.56	0.82
1965	21	1.69	1.59	1.67	1.82	0.73	1.01	179.9	196.7	179.8	189.4	6.2	6.6	1,598.0	1,893.8	1,594.9	1,774.2	12.6	22.6	76.10	90.18	75.95	84.49	0.60	1.08
1966	21	1.61	1.51	1.58	1.61	0.81	1.16	176.1	193.8	176.5	167.8	2.3	2.9	1,529.0	1,833.9	1,532.4	1,393.4	13.5	27.4	72.81	87.33	72.97	66.35	0.64	1.31
1967	21	1.77	1.79	1.79	1.83	0.56	0.83	176.5	194.7	177.0	171.3	1.7	3.9	1,546.8	1,869.1	1,556.2	1,464.1	6.5	14.6	73.66	89.01	74.10	69.72	0.31	0.70
1968	21	1.67	1.73	1.69	1.83	0.62	0.81	179.0	196.6	178.5	189.0	1.3	1.5	1,581.6	1,900.5	1,574.7	1,767.4	7.8	13.3	75.31	90.50	74.99	84.16	0.37	0.63
1969	21	2.15	2.23	2.15	2.22	0.49	0.70	175.2	192.5	174.7	171.9	-0.5	-1.4	1,555.1	1,864.5	1,545.9	1,506.2	4.9	9.8	74.05	88.79	73.61	71.73	0.23	0.47
1970	21	2.26	2.45	2.23	2.41	0.59	0.68	167.5	184.6	166.7	177.8	-4.7	-4.7	1,438.2	1,743.3	1,423.4	1,621.5	8.0	10.4	68.48	83.02	67.78	77.22	0.38	0.49
1971	21	2.05	2.29	2.08	2.35	0.67	0.62	164.3	182.0	164.7	193.2	-4.5	-3.7	1,369.2	1,682.2	1,377.4	1,888.5	10.0	8.4	65.20	80.11	65.59	89.93	0.48	0.40
1972	21	2.19	2.46	2.19	2.40	0.62	0.73	170.8	187.5	170.4	180.8	3.8	4.7	1,484.4	1,795.4	1,478.4	1,672.8	8.3	11.8	70.69	85.49	70.40	79.66	0.39	0.56
1973	21	2.17	2.48	2.23	2.51	0.55	0.67	168.6	185.5	167.8	178.8	5.8	6.2	1,448.4	1,761.8	1,440.8	1,648.5	7.7	10.8	68.97	83.90	68.61	78.50	0.37	0.51
1974	21	2.50	2.78	2.51	2.76	0.60	0.75	161.5	178.6	161.6	156.3	4.4	3.8	1,366.7	1,674.2	1,369.1	1,315.3	8.2	11.8	65.08	79.73	65.20	62.63	0.39	0.56
1975	21	2.53	2.78	2.51	2.77	0.73	0.79	162.5	178.8	162.1	163.2	9.5	9.3	1,384.9	1,677.1	1,377.3	1,420.6	14.9	16.7	65.95	79.86	65.58	67.65	0.71	0.79
1976	21	2.53	2.87	2.56	2.71	0.78	0.85	160.3	175.3	160.0	161.1	15.6	16.5	1,351.5	1,628.8	1,349.6	1,382.7	23.8	27.4	64.36	77.56	64.27	65.84	1.13	1.31
1977	21	2.72	3.07	2.73	2.57	0.71	0.73	153.5	168.4	153.6	135.5	16.0	16.7	1,270.5	1,539.1	1,272.5	1,006.2	22.1	24.0	60.50	73.29	60.60	47.92	1.05	1.14
1978	21	2.82	3.17	2.82	2.71	0.67	0.76	139.2	152.7	139.9	129.3	12.7	12.5	1,082.3	1,311.8	1,090.6	943.3	16.7	18.9	51.54	62.47	51.93	44.92	0.80	0.90
1979	21	3.04	3.41	3.04	2.85	0.81	0.89	132.7	145.3	133.9	118.6	12.8	13.0	1,024.0	1,237.1	1,038.5	831.4	21.1	23.9	48.76	58.91	49.45	39.59	1.00	1.14
1980	21	3.00	3.39	3.04	3.22	0.78	0.86	134.4	145.3	134.6	150.8	20.8	21.0	1,040.9	1,234.8	1,047.2	1,289.5	32.9	35.8	49.57	58.80	49.87	61.40	1.57	1.71
1981	21	3.37	3.69	3.37	3.16	0.89	1.04	129.7	139.0	130.1	112.3	25.0	25.4	1,027.3	1,192.8	1,033.6	800.3	45.5	52.5	48.92	56.80	49.22	38.11	2.17	2.50
1982	21	3.17	3.51	3.18	3.23	0.76	0.79	121.5	129.2	120.9	127.4	24.7	24.8	904.3	1,040.8	898.0	981.4	40.6	41.7	43.06	49.56	42.76	46.73	1.93	1.98
1983	21	3.40	3.69	3.39	3.37	0.86	0.92	110.4	117.1	110.4	108.4	23.7	23.8	811.8	925.3	810.5	786.5	41.6	43.8	38.66	44.06	38.60	37.45	1.98	2.08
1984	21	3.32	3.64	3.32	3.39	0.90	0.94	96.8	101.8	96.7	103.0	19.8	20.5	666.0	759.0	665.7	734.8	34.9	37.5	31.71	36.14	31.70	34.99	1.66	1.79
1985	21	3.22	3.52	3.20	3.53	0.95	0.97	84.9	89.1	85.1	106.4	17.6	17.5	549.8	626.3	549.8	787.6	32.6	33.3	26.18	29.83	26.18	37.50	1.55	1.58
1986	21	3.33	3.62	3.34	3.29	0.98	1.01	71.2	73.5	71.3	66.9	12.6	13.1	464.0	519.5	465.8	429.2	26.9	28.4	22.10	24.74	22.18	20.44	1.28	1.35
1987	21	3.08	3.36	3.10	3.26	0.86	0.92	58.7	60.4	59.8	72.5	10.4	11.0	353.0	399.9	362.0	463.0	20.0	22.8	16.81	19.04	17.24	22.05	0.95	1.09
1988	21	3.10	3.41	3.13	3.30	0.94	0.95	50.0	50.1	50.6	61.7	9.8	9.5	311.7	352.0	317.4	399.2	22.2	22.2	14.84	16.76	15.11	19.01	1.06	1.06
1989	21	3.20	3.53	3.21	3.34	0.99	0.98	44.3	42.2	44.3	54.3	13.1	13.6	298.4	333.4	299.7	362.9	27.7	27.9	14.21	15.88	14.27	17.28	1.32	1.33
1990	21	3.21	3.57	3.18	3.22	0.90	0.93	23.3	16.9	22.9	22.5	9.8	10.1	231.9	267.8	227.9	230.8	20.7	22.0	11.04	12.75	10.85	10.99	0.98	1.05
TOTAL	**798**	**3.07**	**3.41**	**3.07**	**3.16**	**1.04**	**1.16**	**5,226.4**	**5,653.2**	**5,225.1**	**5,382.7**	**483.8**	**504.2**	**41,748.3**	**49,293.1**	**41,724.9**	**44,246.8**	**1,151.6**	**1,405.3**	**52.32**	**61.77**	**52.29**	**55.45**	**1.44**	**1.76**

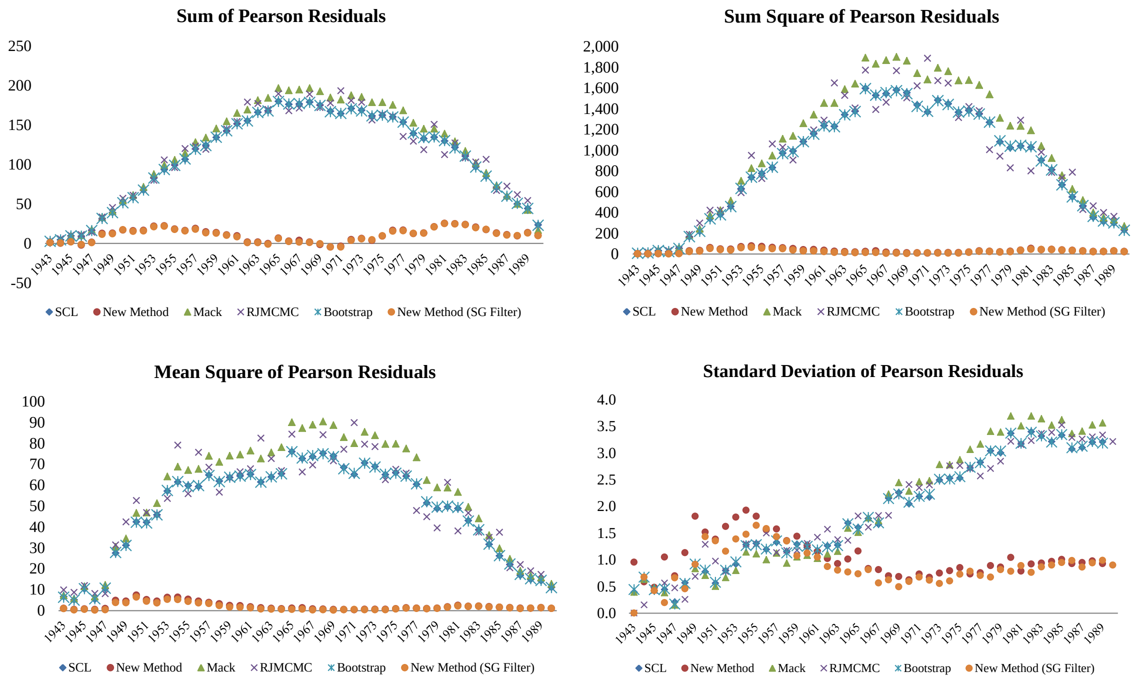


Figure 6.11 Residual results of the New Method and non-life methods

The new approach has the following desirable properties; ease of implementation, transparency, ability to generate sample paths, incorporation of cohort effects, non-trivial correlation structure, parsimony and appropriate for all ages.

The main disadvantage is that if all ages are wanted to include the model, it is needed to have large mortality table history. For example the most recent mortality table is 2010, if 0-100 age is wanted to model and 20 years are used to estimate the μ and σ then from 1890 to 2010 mortality tables should be reached. However, it is not easy to reach historic tables especially for many countries.

Table 6.10 and Figure 6.12 give the summary of the results.

Table 6.10 Summary of the overall residual results of the all methods

	Standard Deviation of Pearson Residuals	Sum of Pearson Residuals	Sum Square of Pearson Residuals	Mean Square of Pearson Residuals
LC	1.41	1,055.93	2,992.42	3.75
Renshaw and Haberman	2.42	1,263.95	12,375.49	15.51
The New Method	1.16	504.17	1,405.32	1.76
The New Method (SG)	1.04	483.75	1,151.57	1.44
SCL	3.07	5,226.40	41,748.29	52.32
Mack	3.41	5,653.24	49,293.05	61.77
Bootstrap	3.07	5,225.15	41,724.92	52.29
RJMCMC	3.16	5,382.69	44,246.77	55.45

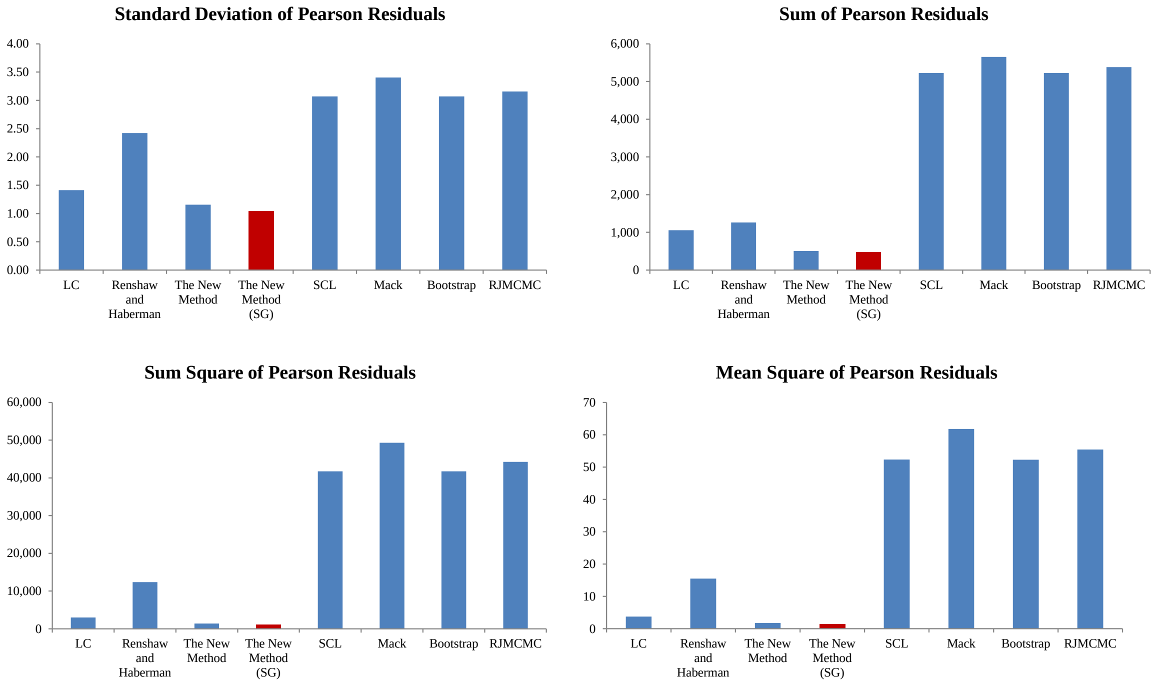


Figure 6.12 Summary of the overall residual results of the all methods

The residuals are based on the number of death. The characteristic of the observed and estimated data is discrete and positive. Root sum square error (RSSE) is calculated for the one year probability of death to counter check the results. RSSE and the residual (ε) are as follows;

$$\varepsilon_{x,t} = \log\left(q_{x,t}\right) - \log\left(\hat{q}_{x,t}\right) \qquad (6.3)$$

$$\text{RSSE} = \sqrt{\sum_{x,t} \varepsilon_{x,t}^{2}} \qquad (6.4)$$

Let unexplained variance be as follows;

$$\mathrm{UV_x} = \frac{Var(\varepsilon_{x,t})}{Var(\log[q_{x,t}])} \tag{6.5}$$

Table 6.11 RSSE, variance of residuals and unexplained variance of LC, RH and the New Method

	RSSE	Var(ε)	UV
Lee and Carter	2.72	0.005	0.042
Renshaw and Haberman	5.36	0.032	0.266
The New Method (SG)	1.60	0.002	0.020

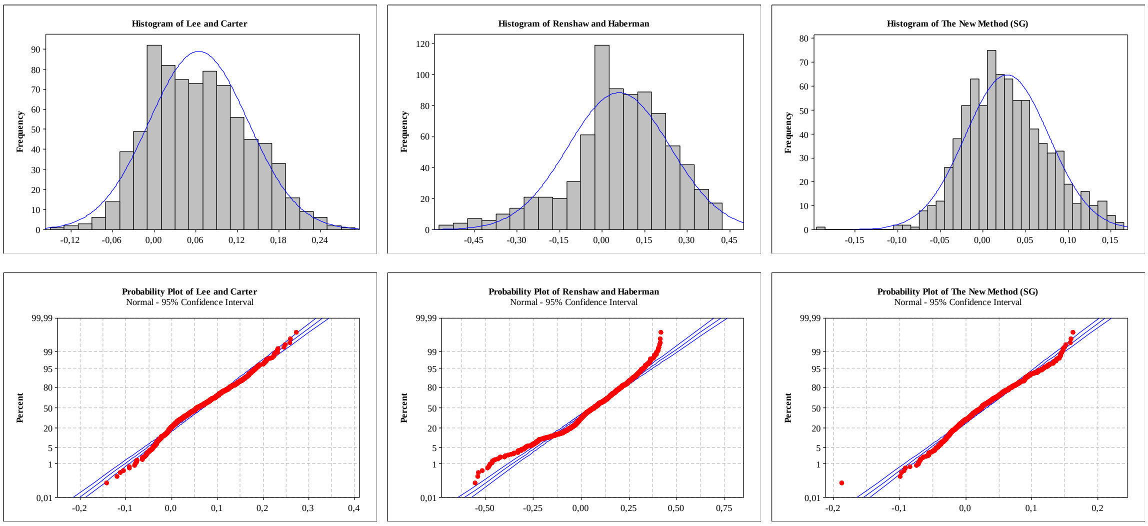


Figure 6.13 Histogram and probability plot of residuals

Table 6.11 gives the root sum square errors, variance of errors and unexplained variance of errors. All indicators demonstrate the new approach noticeably perform well according to the other frequently used and preferred models. Figure 6.13 gives the histogram and probability plot of residuals.

RSSE for Lee and Carter, Renshaw and Haberman and the new method (SG) are as follows; 2.72, 5.36 and 1.6. As mentioned before Renshaw and Haberman gives the best results for all countries nevertheless out-sample forecast is not as good as its fitting results. The new method (SG) prevails Lee and Carter and also Renshaw and Haberman methods.

CHAPTER 7: CONCLUSION

Longevity and mortality risks are handled for many years. On the other hand, the importance and focus has increased for the last few years in both academic and also practical studies with the global crises, narrowing financial area, increasing the awareness of the importance of risks and new estimation and measure methods of risks by this way managing the asset liability balance. Critical point is that the risk is the long term risk besides the effect is deteriorating. The one of the most important point of this risk is that it affects not only the insurance companies but also the other units such as government programs, banks, private companies etc. For instance asset liability balance of the insurance companies, government health and pension expenditures, price levels of the stocks in financial markets are the examples. It can be additionally added the following environmental factors such as food, water, environmental pollution to effected factors by the risk. To sum up the longevity and mortality risks have the critical roles in financial system.

In this thesis, it is demonstrated how the non-life insurance mathematics can be adapted to derive the mortality rates and how the IBNR methods are adjusted and improved for this purpose. It is provided a new estimation procedure on this perspective. First time loss development factors are modelled by geometric Brownian motion in the standard chain ladder approach used for estimation of IBNR in non-life mathematics and first time non-life methods are used to model the mortality rates used in life insurance mathematics. The new approach enables the simulation applications thereby it can easily be used for derivative pricing, longevity risk measurements and scenario analysis.

The life expectancies have an increasing trend for all investigated countries and mortality rates have volatile pattern are introduced in Chapter 2. In recent times, studies on population changes and life expectancies on financial and economic system have great attraction. The literature review is also provided in Chapter 2. Economic results show that the saving rations, returns and international cash flow are related with the demographic changes. The stock returns increases when the middle age population increases and decreases when the old age population increases. The volume of middle age cohort and stock price movements are correlated. There is a positive relationship between life expectancy, population and GDP. The aging can increase the investments and labour productivity. Demographic changes have a statistically significant effect on the excess of return. The problem of funding the elder people in the future will be more important. These changes indicate the need to further researches on longevity and mortality risks.

In Chapter 3, how a stochastic mortality model can be constructed and most commonly preferred following models are discussed, Lee and Carter model (1992), Currie model (2006), Renshaw and Haberman model (2003), the models of Cairns, Blake and Dowd (2009). Lee and Carter model has no cohort effect. However, it is easily implementable and good fitness results. Renshaw and Haberman model provides good fit to the historical data where countries have significant cohort effect. One drawback of this model is lack of robustness. Currie model is the generalized case of the Lee and Carter model. Cairns et al. (2006) propose the model and focus on higher ages (60 to 89). The model is robust and has simple age effect, but there is no cohort effect. It is good for a big picture but overall fit not as good as Lee and Carter. Cairns et al. (2007) generalise their model first model by adding cohort effect and they generalize their second generalisation with adding a quadratic term to the age effect. Third generalisation includes the adjusting the cohort effect. There are some critical properties such as parsimony, transparency, ability to generate sample paths, cohort effect etc. These points are explained in Chapter 3. There is a significant issue how the mortality models can be compared with other models. On this frame, Bayesian information criteria, standardized residuals are given. In Chapter 6, the best mortality model is determined for 20 countries using Lee and Carter model (1992), Currie model (2006), Renshaw and Haberman model (2003), the models of Cairns, Blake and Dowd (2009). Bayesian information criteria, likelihood estimations and variance of residuals demonstrate that the best model is Renshaw and Haberman model for all countries.

The backbone of the new approach is IBNR estimations in non-life insurance mathematics. In Chapter 4, the details of standard chain ladder, Mack method, bootstrap method and reversible jump markov chain monte carlo method which models can be used in the new approach are given in detail. Chain ladder methodology is one of the mostly preferred methodologies to estimate unpaid claims. The basic of this method is that the development of the future pattern of losses will be similar for the past losses. In the new approach it is used number of deaths instead of claim amounts in standard chain ladder method.

The theoretical background of the new approach is explained in Chapter 5. Loss development factors are modelled in the new approach by geometric Brownian motion since classical IBNR methods can not represent the historical trend of the development factors and also geometric Brownian motion enables the simulation studies.

The application of the all models and comparison are given, in Chapter 6. The approach is cohort base. Consider the illustrative case if the mortality tables from 1902 to 2010 are available, one can easily estimate the mortality rates for the cohort years from 1922 to 2010 and the ages from 0 to 68. 20 years are used to estimate the parameters of the geometric Brownian motion. Simulation spending time for 1,000 simulations is 410 seconds for the cohort years from 1922 to 2010 and the ages from 0 to 48. The results are taken from AMD Turion 64 X2 configuration computer. It is obvious that the new age generation computers perform much shorter spending time.

The age is not limited when applying the new approach to the UK mortality data. This is critical point because Cairns, Blake and Dowd models focus on older ages. This situation may restrict the flexibility of the model and also there is no agreed way to determine the age ranges. As it is stated before geometric Brownian motion is used to model loss development factors therefore all it is needed just two parameters for each columns. The approach is very parsimonious when compared with the other stochastic mortality models.

The out sample forecast performance of Renshaw and Haberman model does not well when compared with the new approach. The residual results of the new approach are better than the Renshaw and Haberman model. Lee and Carter (1992) is the other stochastic mortality model used to compare with the new approach. It is frequently used model to compare the new suggested stochastic mortality models in the literature by academicians and practitioners. The reasons why professionals prefer are that Lee and Carter model is a benchmark model in US Bureau of the Census, generally accepted model in the literature, flexible, easily implementable and successful at estimations of out sample forecasts.

However, the out sample forecast residual results show that the forecasted mortality rates of Lee and Carter model is not better than the new approach.

As mentioned before the originalities of this thesis are that it is introduced non-life techniques to estimate the number of deaths and so mortality rates and also loss development factors in IBNR estimations are modelled by geometric Brownian motion. The backbone of the approach is IBNR estimations. There are many techniques to estimate IBNR in non-life literature. Standard chain ladder, Mack, reversible jump markov chain monte carlo, bootstrap are also utilized methods to estimate number of deaths for the mortality rate estimations. The forecast performance of non-life methods is very disappointed since methods can not catch the historical trend of loss development factors. This is the reason why geometric Brownian motion is preferred.

Savitzky and Golay filter is used to eliminate the noise of the Monte Carlo simulation and keep the pattern of the number of death. The results show the filtered version of the method is better than the original one.

The new approach is simple to fit because all need is to use two factors in each column and also there is no age limitation such as new born, young or elder ages. As it is recognized generalisations of Cairn, Blake and Dowd models are designed for older ages. Forecasting performance is better than Lee and Carter and Renshaw and Haberman models. There is a cohort effect as it is known Lee and Carter model has not. In summary, the new approach has advantages of the stochastic mortality models in the literature such as lack of age range, cohort base, appropriate for simulations, parsimonious and the successful performance in out sample forecasts highlights the new approach.

In conclusion, the new model represents an alternative approach to estimate the stochastic mortality models and also can be used in the mortality scenario generation in order to improve models for the pricing of mortality-linked securities, to picture the impact of mortality risks, to estimate reserves, to draw solvency requirements. Main drawback is the lack of long period mortality tables.

Further research, the traditional non-life IBNR estimation methods can not catch the trend of the loss development factors so the estimation results fail. The proposed method eliminates this situation limitedly. It can be improved by adding the jump term. Markov Chain Monte Carlo base estimation and Brownian Bridge base modelling are also investigated in the future research. Main drawback of this approach is limited data. Therefore it should be investigated how this point can be overwhelmed. The method can be used for mortality derivative pricing. The performance also should be investigated. In this thesis the new approach is applied to UK data. It can be applied to other countries and check the results.

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Appendix I: **Survival Probability Modelling with Affine Term Structures**

The mortality intensity process is defined on the probability space $(\Omega, \mathcal{F}, P)$, and it is modelled under the objective measure.

S(t) is the survival function shows a new born will survive at least t years and defined as follows;

$$S(t) = P(T_0 > t) = 1 - F_{T_0}(t) \tag{A.1}$$

where T_0 is the duration life of a new born individual. F_{T_0} is the distribution of the duration life. Force of mortality is the derivative of the survival function and it shows the instantaneous death rate for the person:

$$\mu_x = -\frac{d}{dx} logS(x), \tag{A.2}$$

$$P(x < T_0 \leq x + \Delta x | T_0 > x) \cong \mu_x \Delta x. \tag{A.3}$$

Let τ be the remaining life time of an individual. It is a doubly stochastic stopping time with the intensity λ a stochastic process for the age *x*, then the survival probability of a person age *x* at time *t* live to τ:

$$P(\tau > s | \mathcal{F}_t) = E\left[e^{-\int_t^s \lambda_x(u)du} | \mathcal{F}_t\right]. \tag{A.4}$$

$\mathcal{F}_t$ is the information at time *t*. Actually this equation is the price of the zero coupon bond at time t.

The stochastic differential equation of λ is as follows (Dahl, 2004);

$$d\lambda_x(t) = f\big(t,\lambda_x(t)\big)dt + g\big(t,\lambda_x(t)\big)dw(t). \tag{A.5}$$

w is the Brownian motion, *f* is the drift term and g is the diffusion term. The probability of survive at age *x* at time *t* live to *T* is given as follows;

$$p(\lambda_x(t),t,T) = E\left[e^{-\int_t^T \lambda_x(u)du} | \mathcal{F}_t\right] = e^{A(T-t)+B(T-t)\lambda_x(t)}. \tag{A.6}$$

Equation A.6 is the survival probability from time *t* to *T* at a person at age *x*. More information about the affine models can be found at the studies, Dahl (2004), Luciano and Vigna (2005, 2008).

$A(T-t)$ or $A(t,T)$ and $B(T-t)$ or $B(t,T)$ are deterministic functions for the following models Vasicek, Cox Ingersoll Ross (CIR), Non Mean Reverting CIR (NMRCIR), NMRCIR with jump, Ornstein Uhlenbeck, Ornstein Uhlenbeck with jump are given at the following paragraphs as stated Luciano and Vigna (2005).

When the force of mortality is modelled by Vasicek (1977);

$$d\lambda_x(t) = (\beta - \alpha\lambda_x(t))dt + \sigma dw(t), \tag{A.7}$$

$$\begin{cases} B_t(t,T) - \alpha B(t,T) = -1 \\ B(T,T) = 0 \\ A_t(t,T) = \beta B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0 \end{cases} \tag{A.8}$$

when $B(t,T)$ is solved and $A_t(t,T)$ is integrated, $B(t,T)$ and $A(t,T)$ are obtained as follows;

$$A(t,T) = A(\tau) = \frac{(B(t,T)-T+t)\left(\alpha\beta - \frac{1}{2}\sigma^2\right)}{\alpha^2} - \frac{\sigma^2 B^2(t,T)}{4\alpha}, \tag{A.9}$$

$$B(t,T) = B(\tau) = \frac{1}{\alpha}[1 - e^{-\alpha(T-t)}]. \quad \text{(A.10)}$$

When the force of mortality is modelled by CIR (1985);

$$d\lambda_x(t) = \kappa(\theta - \lambda_x(t))dt + \sigma\sqrt{\lambda_x(t)}dw(t), \quad \text{(A.11)}$$

$$A(t,T) = A(\tau) = \frac{2\kappa\theta}{\sigma^2}\ln\left(\frac{2he^{\frac{(\kappa+h)(T-t)}{2}}}{2h+(\kappa+h)\left(e^{(T-t)h}-1\right)}\right), \quad \text{(A.12)}$$

$$B(t,T) = B(\tau) = \frac{2(e^{(T-t)h}-1)}{2h+(\kappa+h)(e^{(T-t)h}-1)}, \quad \text{(A.13)}$$

$$h = \sqrt{\kappa^2 + 2\sigma^2}. \quad \text{(A.14)}$$

When the force of mortality is modelled by Non Mean Reverting CIR (NMRCIR);

$$d\lambda_x(t) = \varphi\lambda_x(t)dt + \sigma\sqrt{\lambda_x(t)}dw(t), \quad \text{(A.15)}$$

$$A(t,T) = 0, \quad \text{(A.16)}$$

$$B(t,T) = \frac{1-e^{bt}}{c+de^{bt}}, \quad \text{(A.17)}$$

b, *c*, *d* are as follows;

$$b = -\sqrt{\varphi^2 + 2\sigma^2}, \quad \text{(A.18)}$$

$$c = \frac{b+\varphi}{2}, \quad \text{(A.19)}$$

$$d = \frac{b-\varphi}{2}. \quad \text{(A.20)}$$

When the force of mortality is modelled by Ornstein Uhlenbeck;

$$d\lambda_x(t) = \varphi\lambda_x(t)dt + \sigma dw(t) \qquad \text{(A.21)}$$

with $\varphi > 0$ and $\sigma \geq 0$. Solving the differential equations:

$$A(t,T) = \frac{\sigma^2}{2\varphi^2}t - \frac{\sigma^2}{\varphi^3}e^{at} + \frac{\sigma^2}{4\varphi^3}e^{2at} + \frac{3\sigma^2}{4\varphi^3}, \qquad \text{(A.22)}$$

$$B(t,T) = \frac{1}{\varphi}(1 - e^{\varphi t}). \qquad \text{(A.23)}$$

When the force of mortality is modelled by Ornstein Uhlenbeck with jump;

$$d\lambda_x(t) = \varphi\lambda_x(t)dt + \sigma dw(t) + dJ(t). \qquad \text{(A.24)}$$

Where J is a pure compound Poisson jump process, with arrival times of intensity $l > 0$ and exponentially distributed jump sizes with mean $\mu < 0$. It is assumed that the Brownian motion and jump process are independent. It is allowed only negative jumps which indicate the sudden improvements in the force of mortality.

Some steps such deriving the differential equations are left out. *A(t,T)* and *B(t, T)* are as follows;

$$A(t,T) = \left(\frac{\sigma^2}{2\varphi^2} + \frac{l\varphi}{\varphi-\mu}\right)t - \frac{\sigma^2}{\varphi^3}e^{at} + \frac{\sigma^2}{4\varphi^3}e^{2at} + \frac{3\sigma^2}{4\varphi^3} + \frac{l}{\varphi-\mu}\ln(1 - \frac{\mu}{\varphi} + \frac{\mu}{\varphi}e^{\varphi t}), \qquad \text{(A.25)}$$

$$B(t,T) = \frac{1}{\varphi}(1 - e^{\varphi t}). \qquad \text{(A.26)}$$

The solution is satisfied by the following condition;

$$1-\frac{\mu}{\varphi}+\frac{\mu}{\varphi}e^{\varphi t}>0, \forall t\geq 0. \quad \text{(A.27)}$$

When the force of mortality is modelled by Non Mean Reverting CIR with jump;

$$d\lambda_x(t)=\varphi\lambda_x(t)dt+\sigma\sqrt{\lambda_x(t)}dw(t)+dJ(t) \quad \text{(A.28)}$$

where *J* is the pure jump process. *A(t,T)* and *B(t;T)* are as follows;

$$A(t,T)=\frac{l\mu}{c-\mu}t-\frac{l\mu(c+d)}{b(d+\mu)(c-\mu)}[\ln(\mu-c-(d+\mu)e^{bt})-\ln(-c-d)], \quad \text{(A.29)}$$

$$B(t,T)=\frac{1-e^{bt}}{c+de^{bt}}, \quad \text{(A.30)}$$

$$b=-\sqrt{\varphi^2+2\sigma^2}, \quad \text{(A.31)}$$

$$c=\frac{b+\varphi}{2}, \quad \text{(A.32)}$$

$$d=\frac{b-\varphi}{2}. \quad \text{(A.33)}$$

b, c and d are the negative coefficients. The requirement of the introduction to of the jump component is $\mu-c-(d+\mu)e^{bt}>0$.

***Appendix I.I:* An Extension of Thiele and Makeham**

Schrager (2004) propose an affine form model which has a close form solution. The model is continuous time so it can be used for the derivative pricing.

The proposed model is as follows;

$$\mu_x(t)=y_1(t)\exp(-\tau_1 x)+y_2(t)\exp(-\tau_2(x-\eta)^2)+y_3(t)\exp(\tau_3 x), \quad \text{(A.34)}$$

$$dy(t) = A\big(\theta - y(t)\big)dt + \sum dw_t^P, \tag{A.35}$$

$$y(0) = \bar{y} \tag{A.36}$$

where *A=diag(a)* and $a = [a_1; a_2; a_3]'$ is a vector in R_+^3. P is the probability measure P. This is an extension of Thiele (1867). The model proposed by Thiele is as follows;

$$\mu_x = y_1 \exp(-\tau_1 x) + y_2 \exp(-\tau_2 (x-\eta)^2) + y_3 \exp(\tau_3 x) \tag{A.37}$$

where all parameters are positive.

The affine form of the model of Schrager (2004) is as follows;

$$_{T-t}p_{x+t}(t) = \exp(C(x,t,T) - D_1(x,t,T)y_1(t) - D_2(x,t,T)y_2(t) - D_3(x,t,T)y_3(t)). \tag{A.38}$$

Details can be found in the paper.

Makeham proposed the following functional for the mortality intensity;

$$\mu_x = y_1 + y_2 c^x \tag{A.39}$$

where y_1 and y_2 are positive and *c>1*.

The extension form of the model;

$$\mu_x(t) = y_1(t) + y_2(t)c^x. \tag{A.40}$$

y_t follows an Ornstein Uhlenbeck process;

$$dy_i(t) = a_i\big(\theta_i - y_i(t)\big)dt + \sigma_i dw_{it}^P,\ y_i(0) = \overline{y_i},\ \text{i=1, 2,} \qquad \text{(A.41)}$$

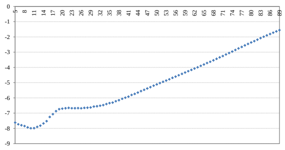
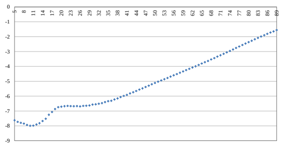
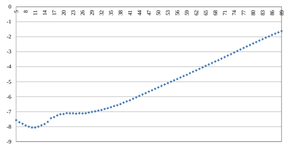
$$dw_{1t}^P dw_{2t}^P = \rho dt. \qquad \text{(A.42)}$$

The affine form is;

$${}_{T-t}p_{x+t}(t) = \exp(C(x,t,T) - D_1(x,t,T)y_1(t) - D_2(x,t,T)y_2(t)). \qquad \text{(A.43)}$$

The details of *C* and *D*s can be found at Schrager (2004).

Appendix II: Model Parameters for Each Countries

BELGIUM 	**FRANCE** 	**NETHERLANDS** 
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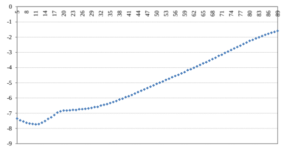
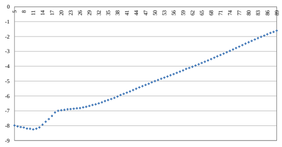
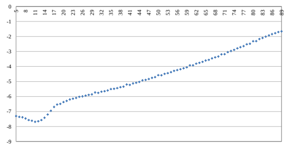
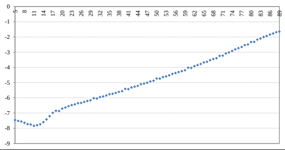
BULGARIA 	**POLAND** 	**RUSSIA** 
UKRAINE 		
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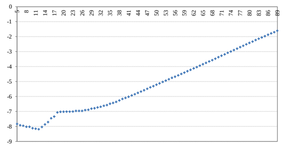
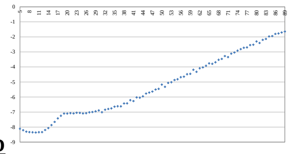
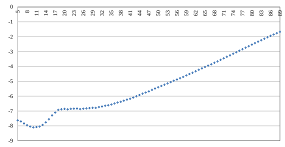
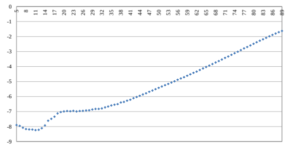
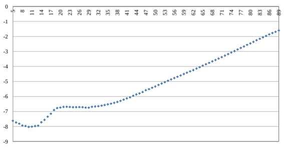
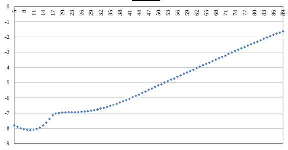
DENMARK 	 **IRELAND**	**NORWAY** 
SWEDEN 	**SWITZERLAND** 	**UK** 
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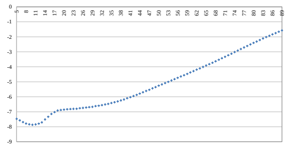
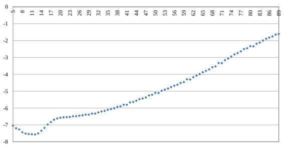
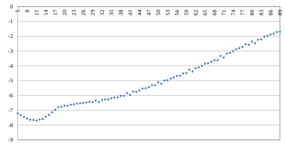
ITALY 	**PORTUGAL** 	**SPAIN** 
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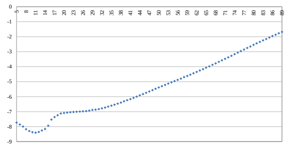
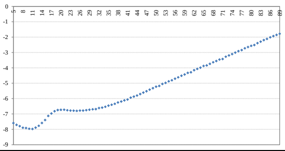
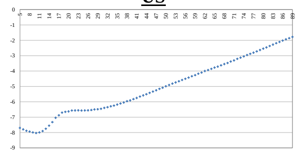
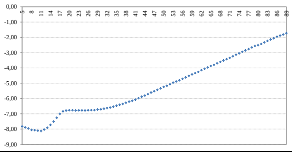
JAPAN 	**CANADA** 	**US** 
e) Asian Countries	f) American Countries	
AUSTRALIA 		
g) Australia		

Figure A.1 LC, $\beta^{(1)}$

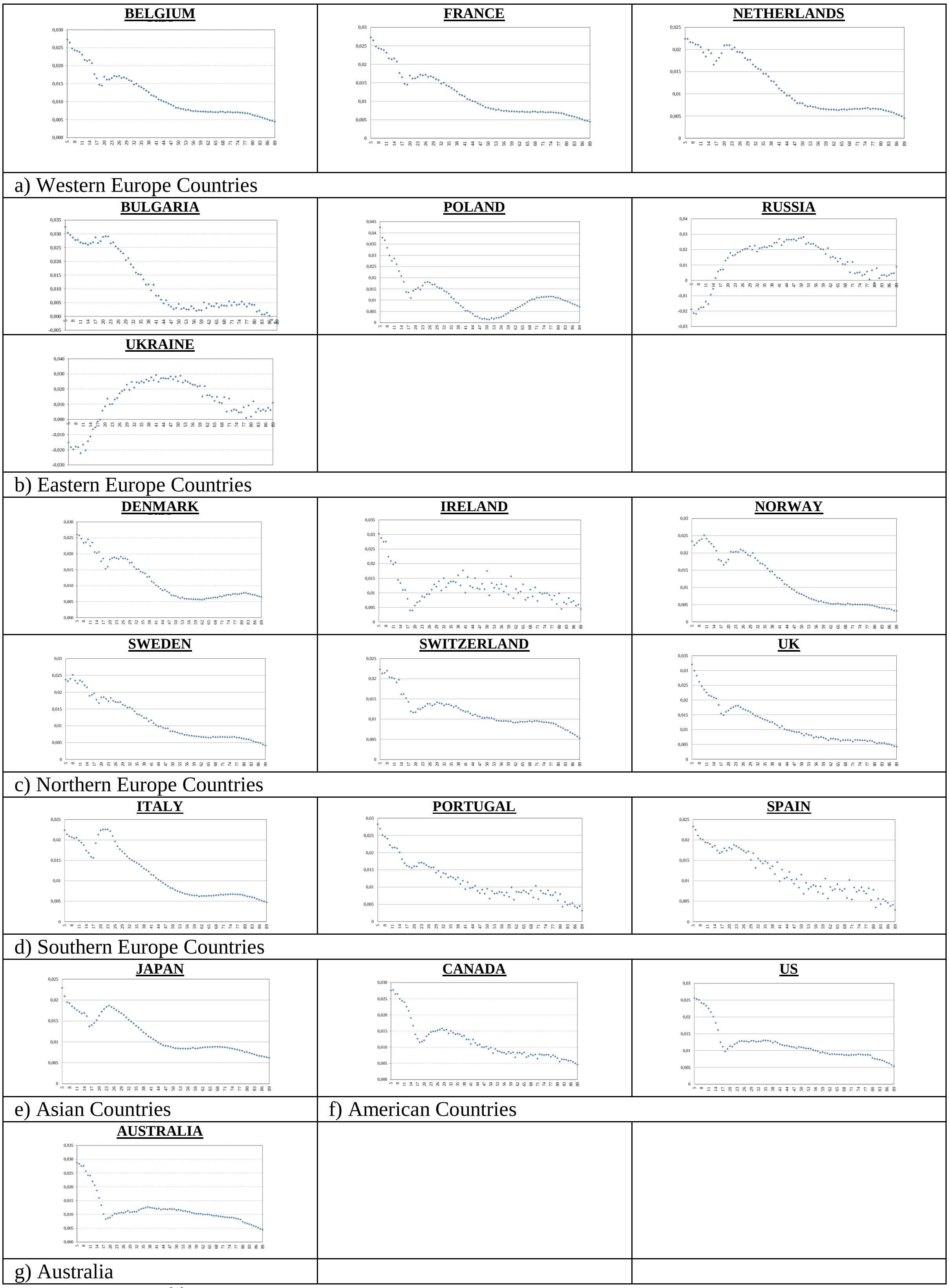


Figure A.2 LC, $\beta^{(2)}$

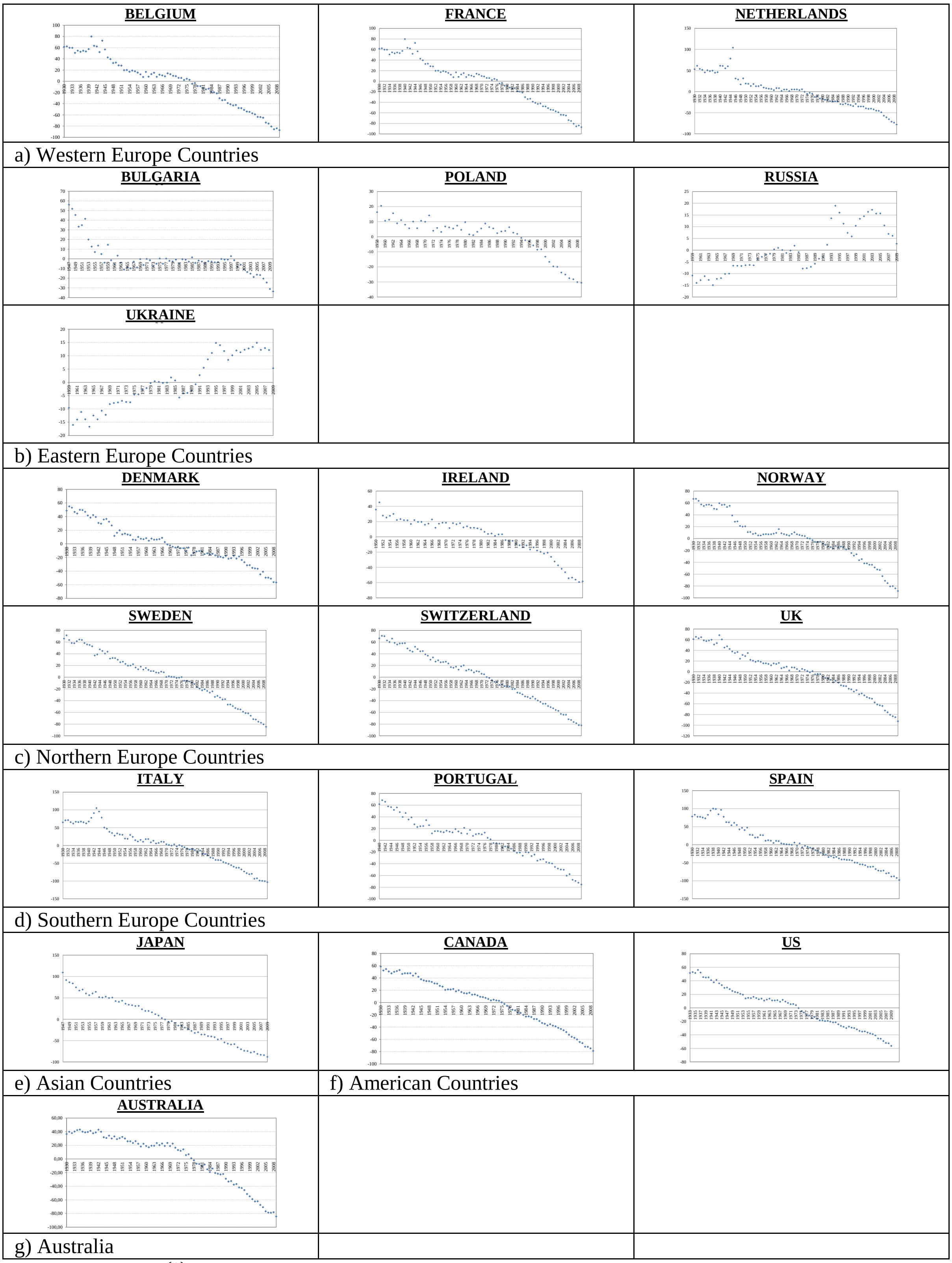


Figure A.3 LC, $\kappa^{(2)}$

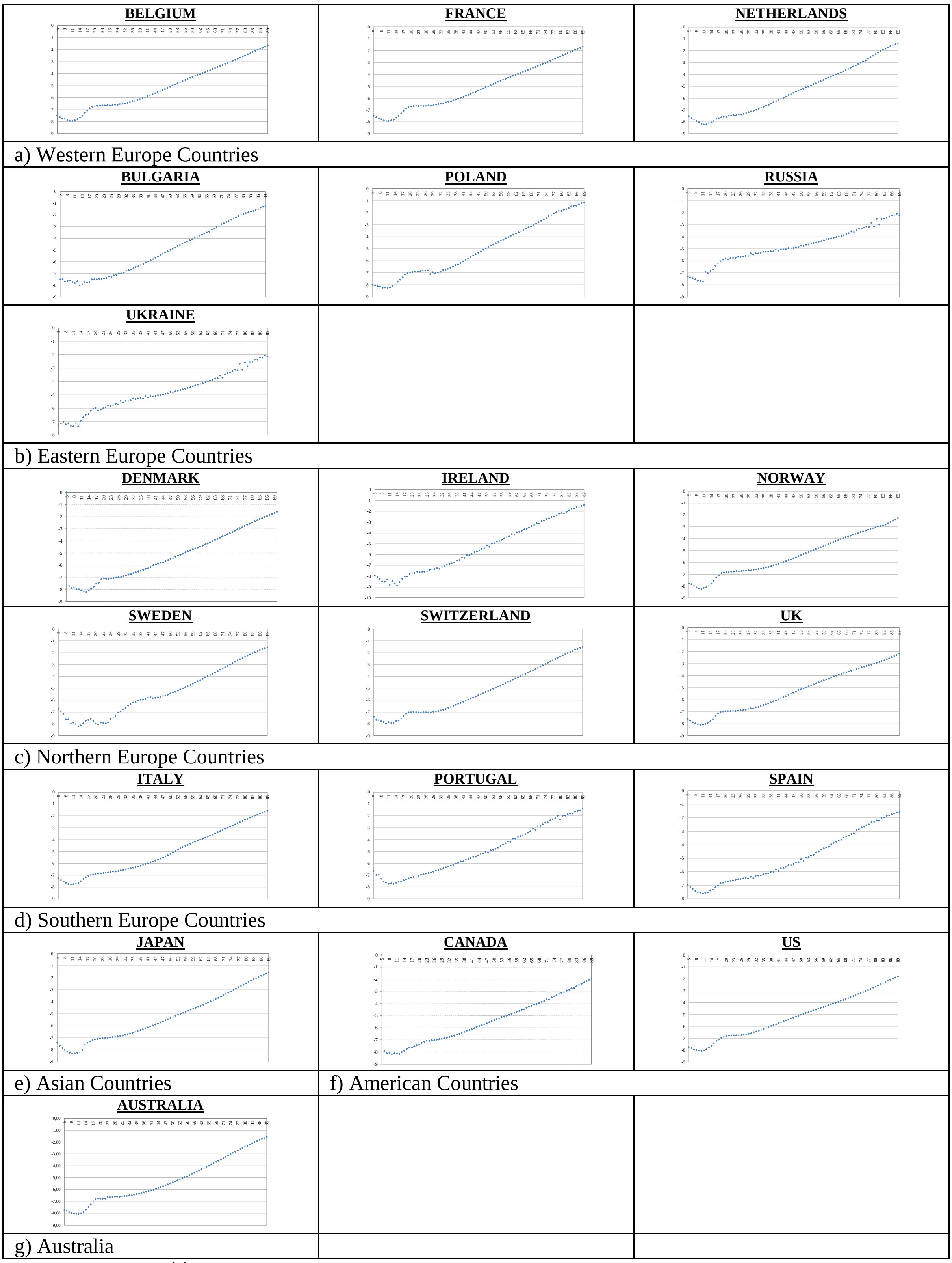


Figure A.4 RH, $\beta^{(1)}$

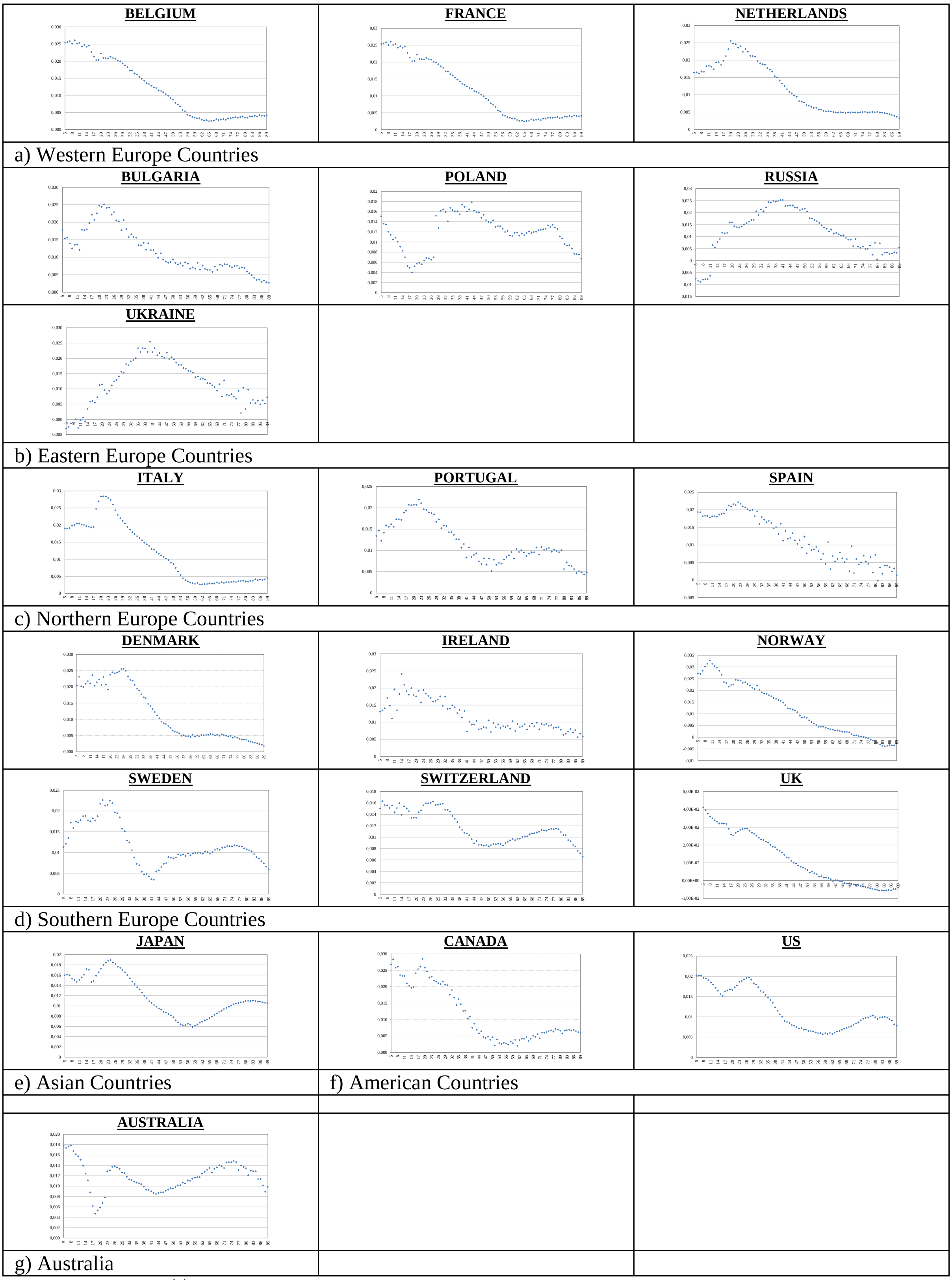


Figure A.5 RH, $\beta^{(2)}$

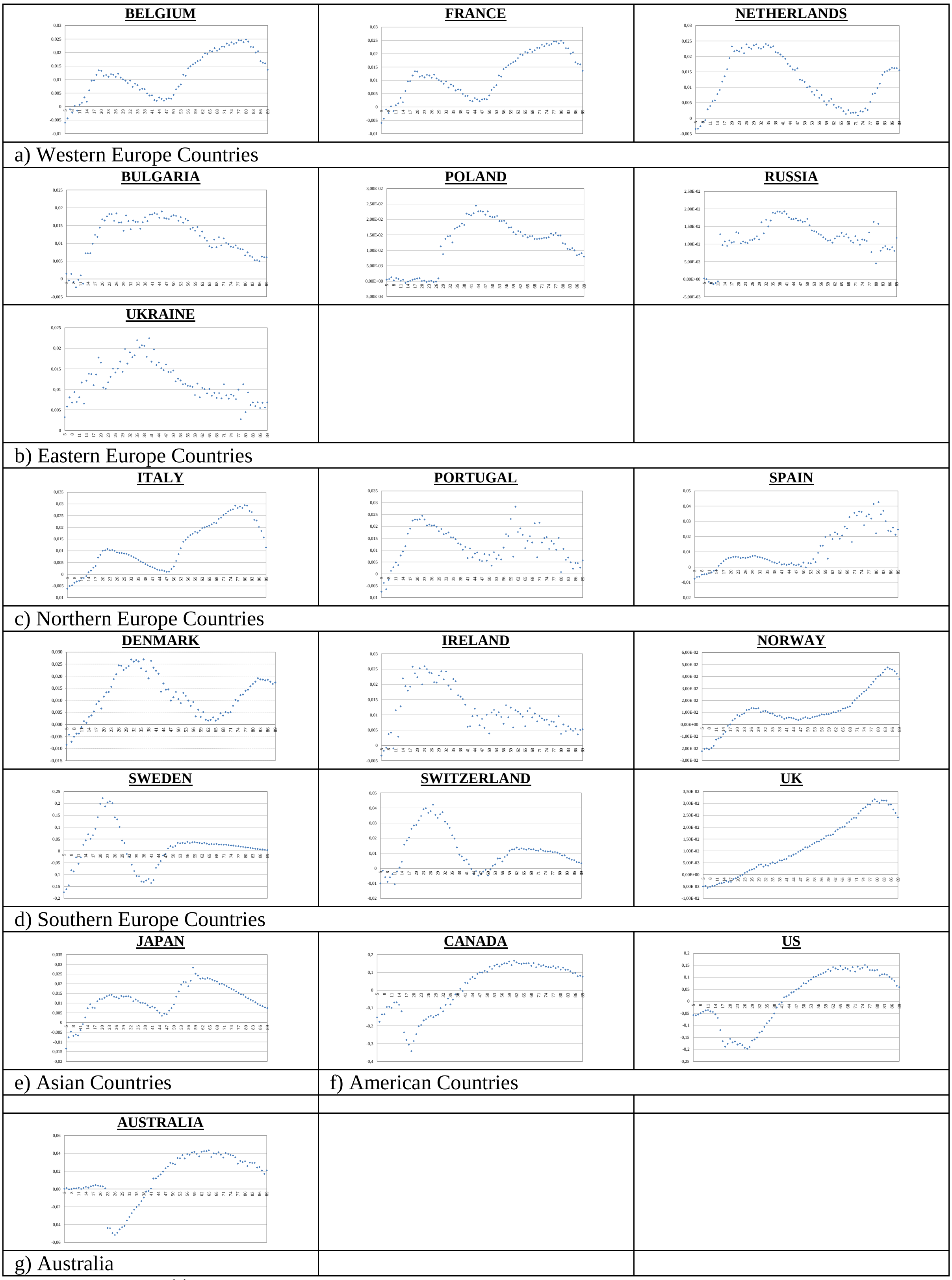


Figure A.6 RH, $\beta^{(3)}$

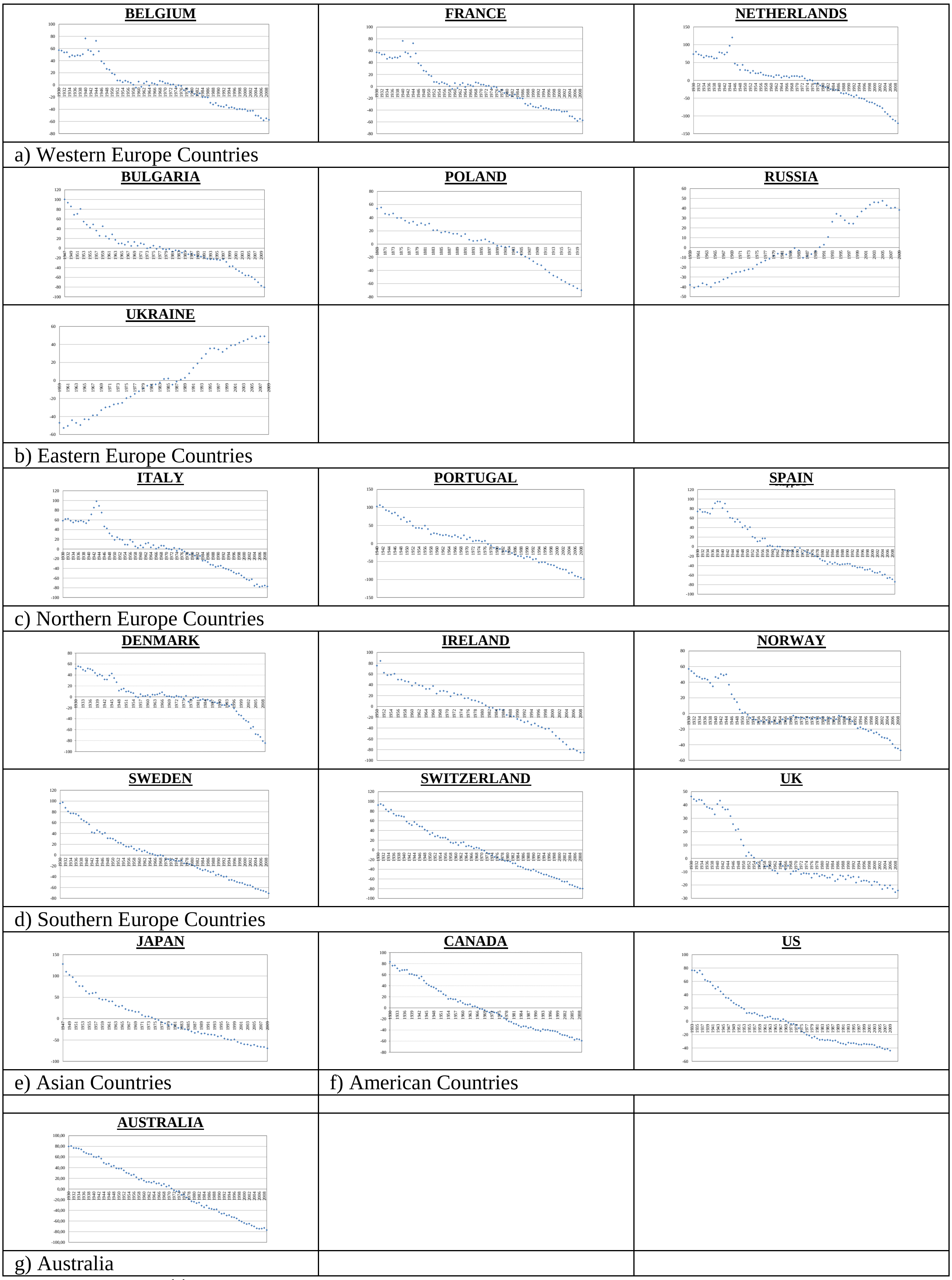


Figure A.7 RH, $\kappa^{(2)}$

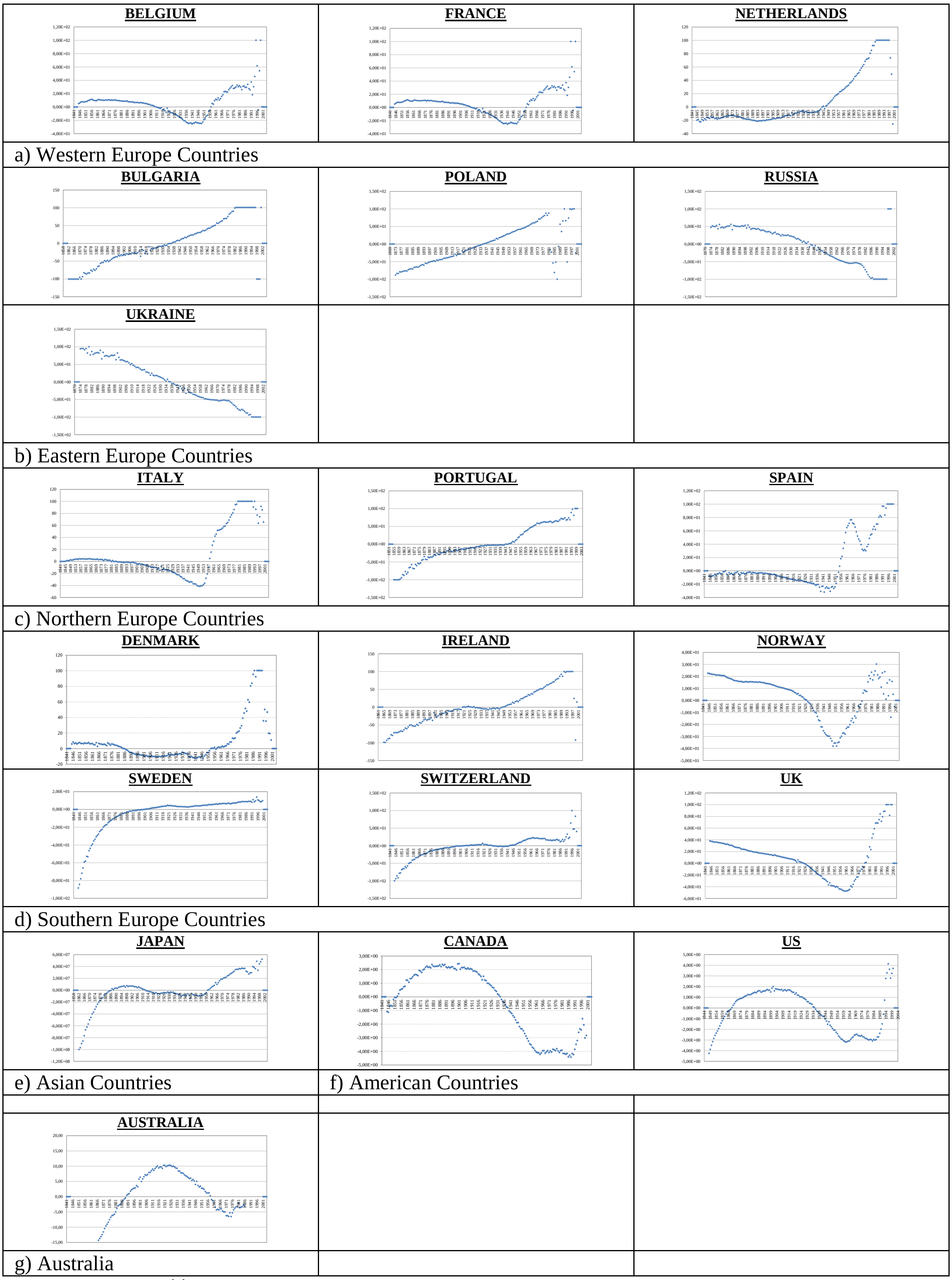


Figure A.8 RH, $\gamma^{(3)}$

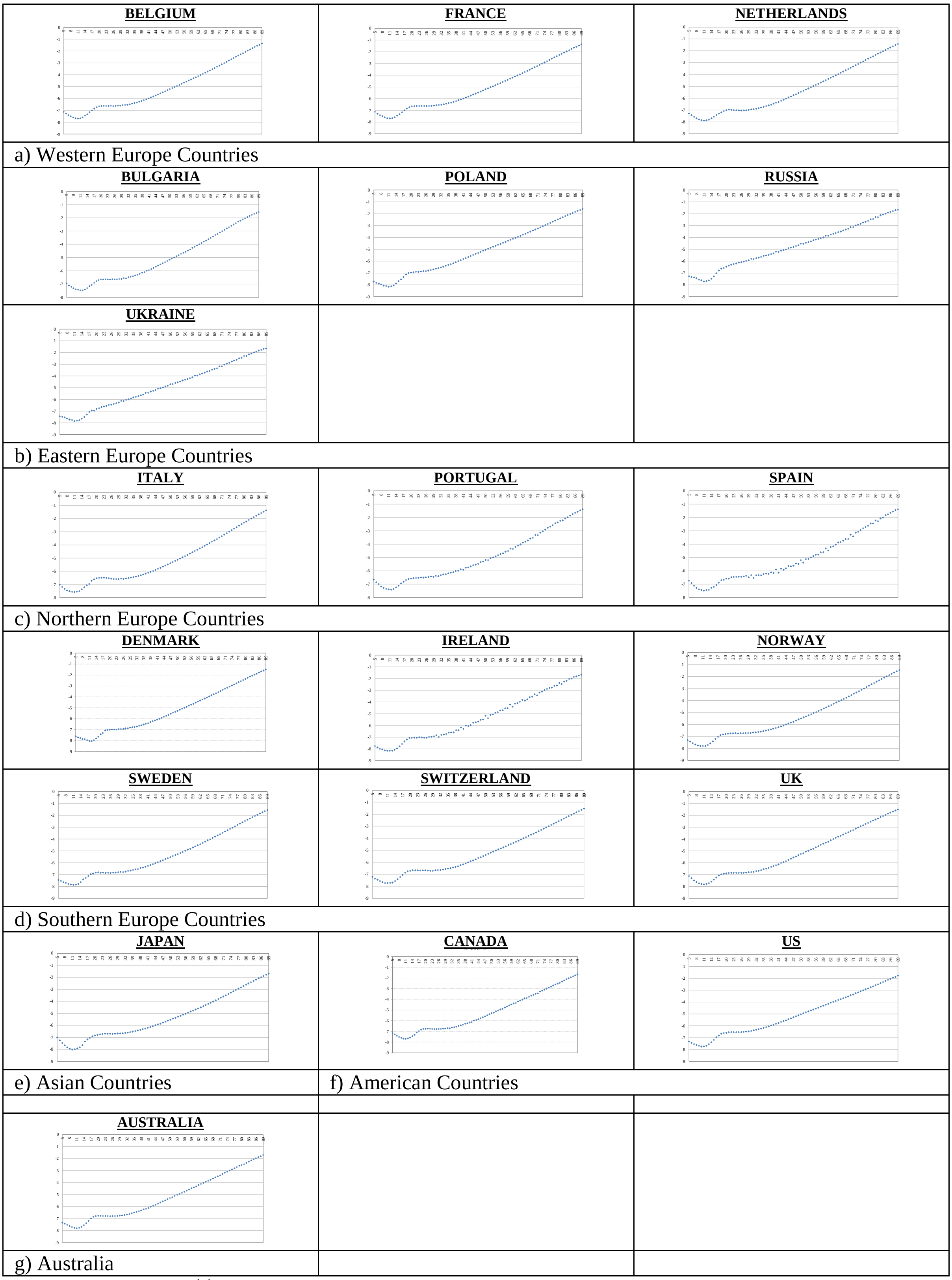


Figure A.9 Currie, $\beta^{(1)}$

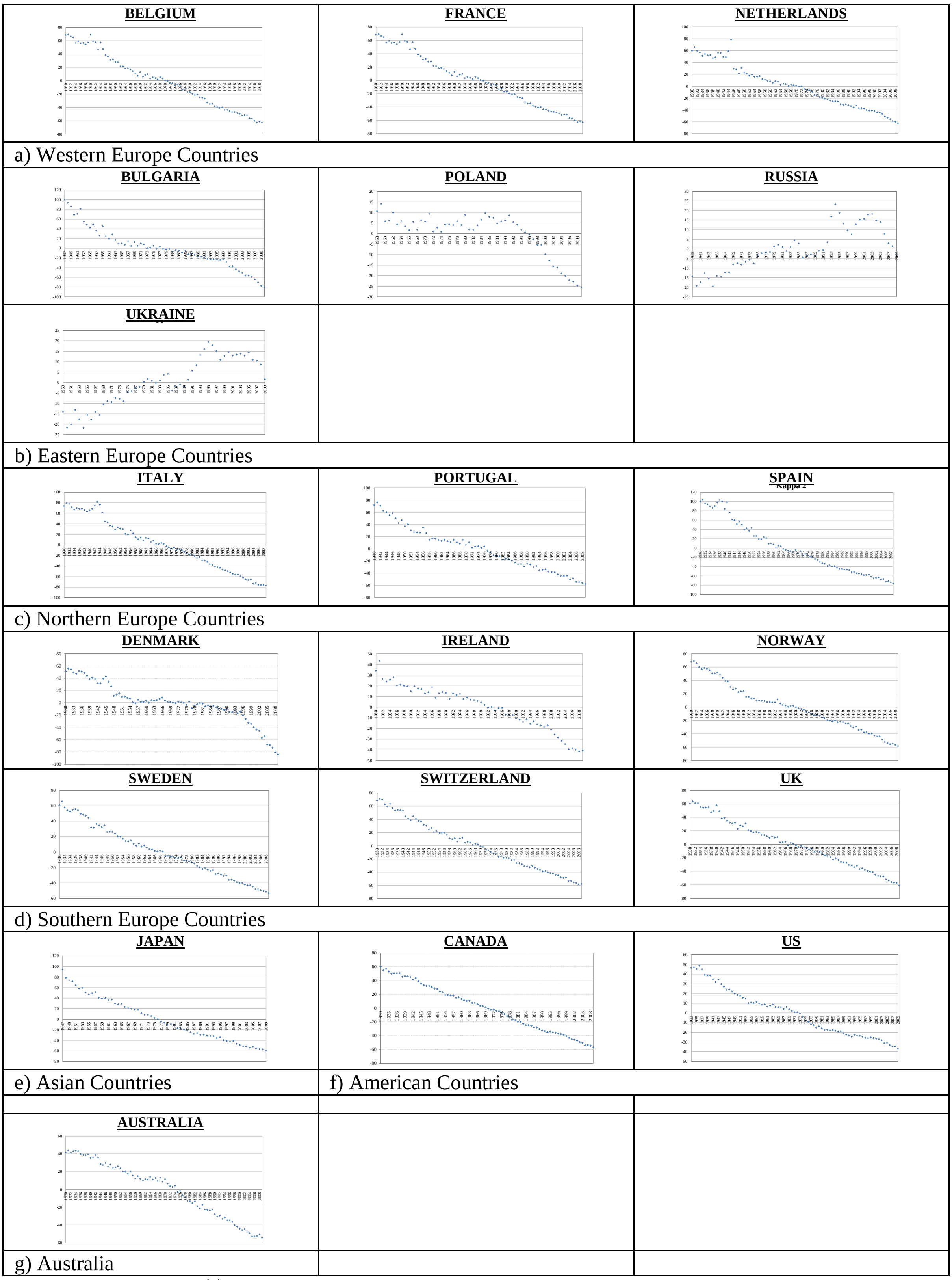


Figure A.10 Currie, $\kappa^{(2)}$

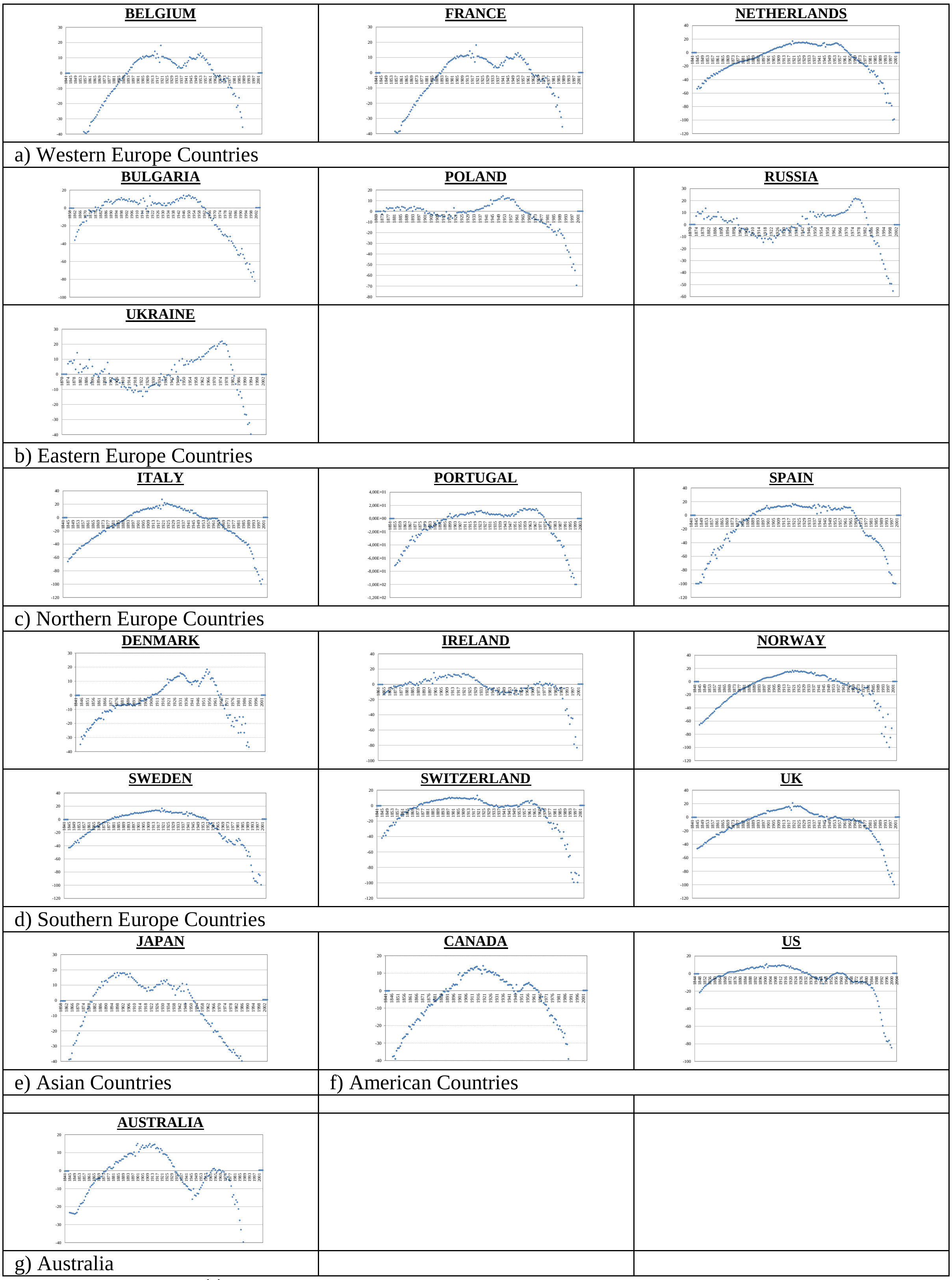


Figure A.11 Currie, $\gamma^{(3)}$

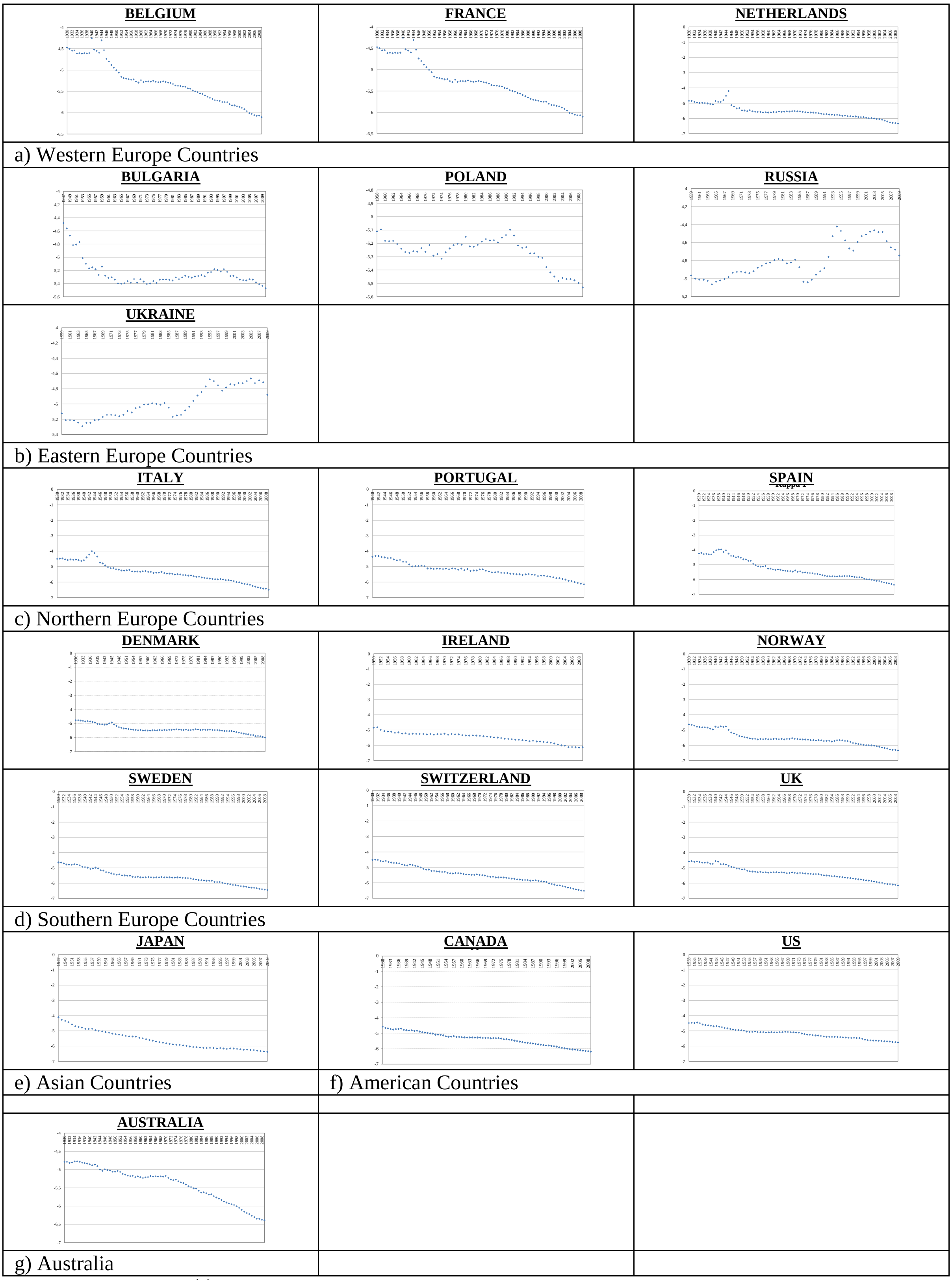


Figure A.12 CBD, $\kappa^{(1)}$

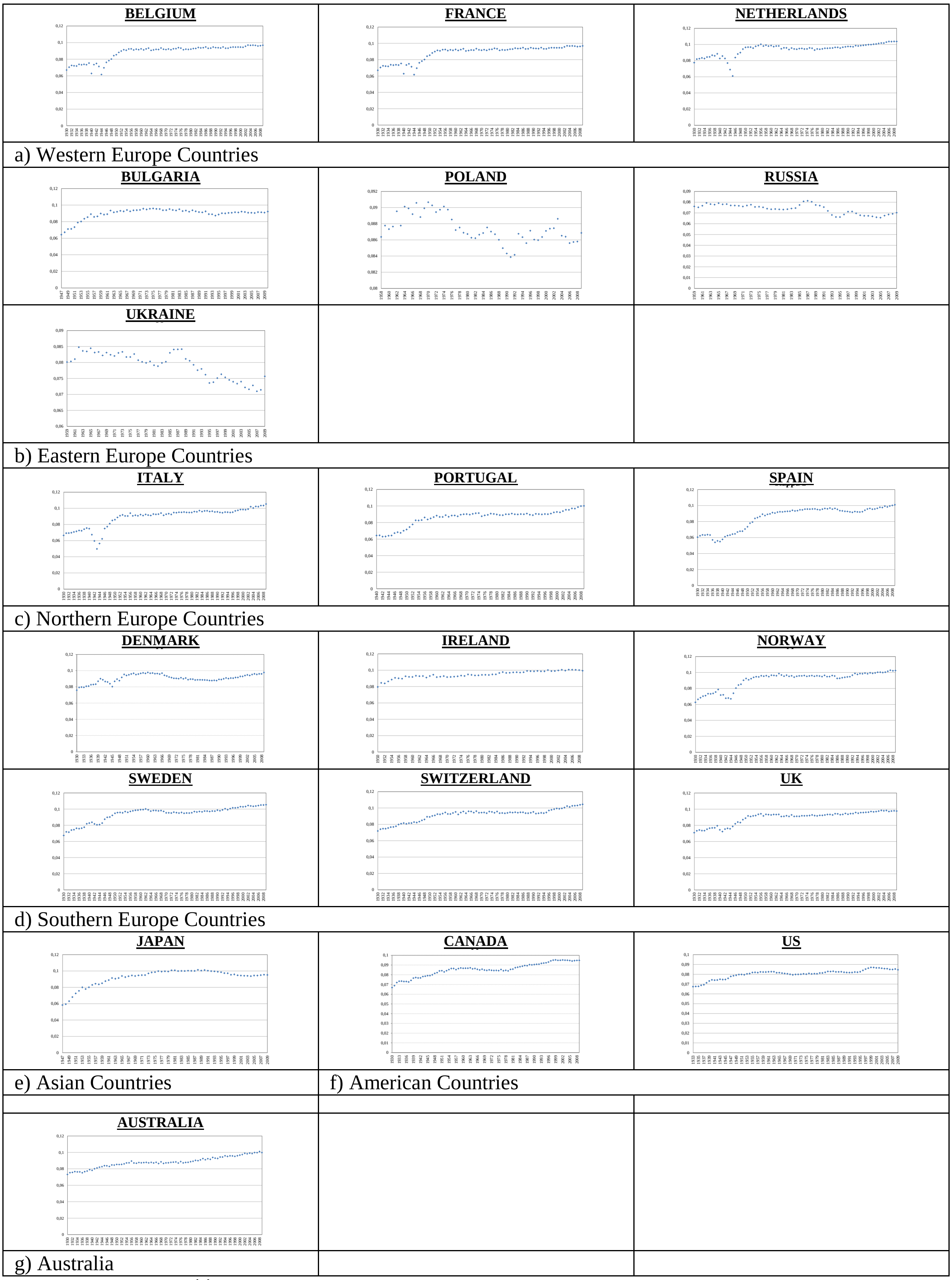


Figure A.13 CBD, $\kappa^{(2)}$

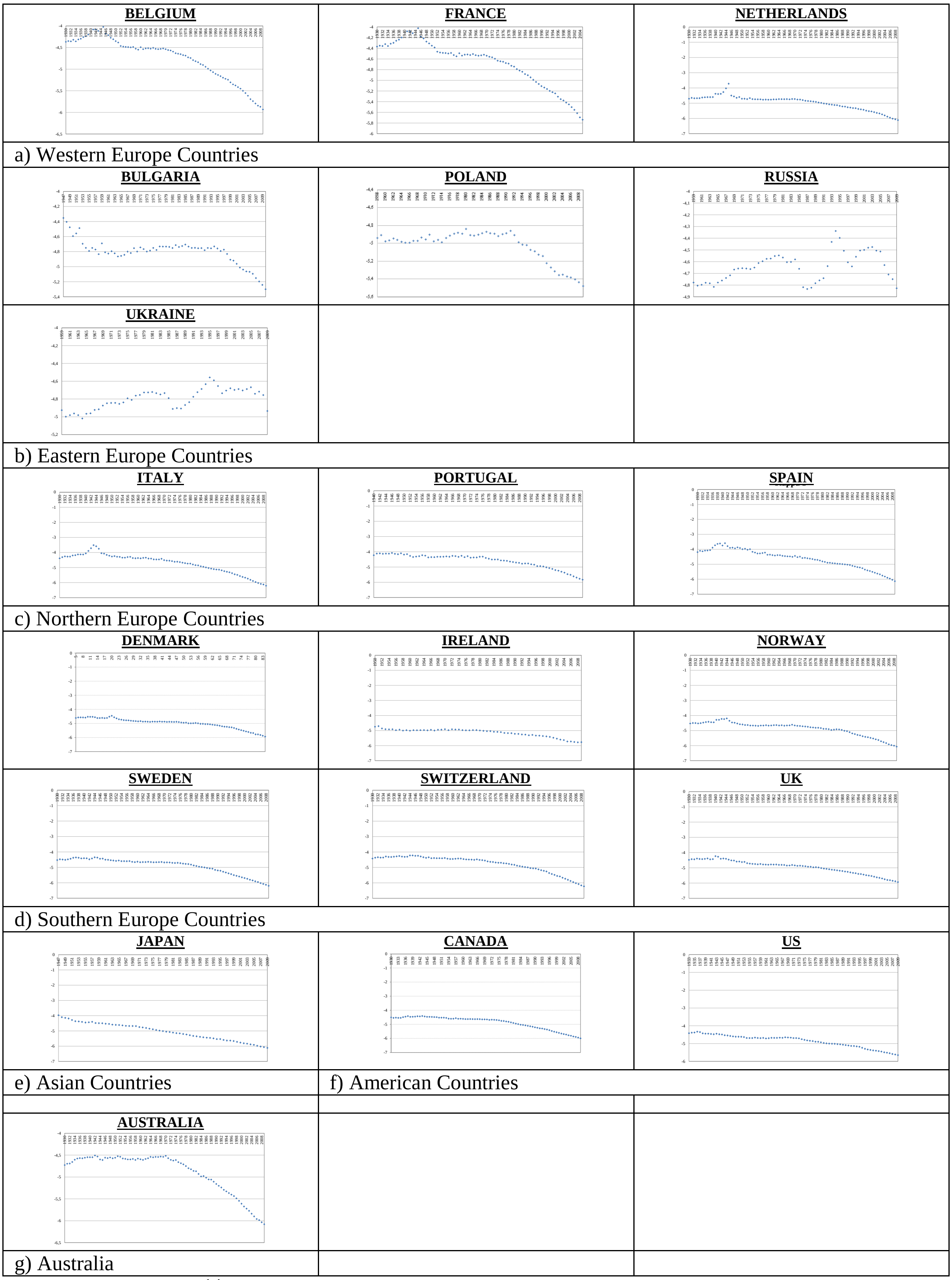


Figure A.14 CBD1, $\kappa^{(1)}$

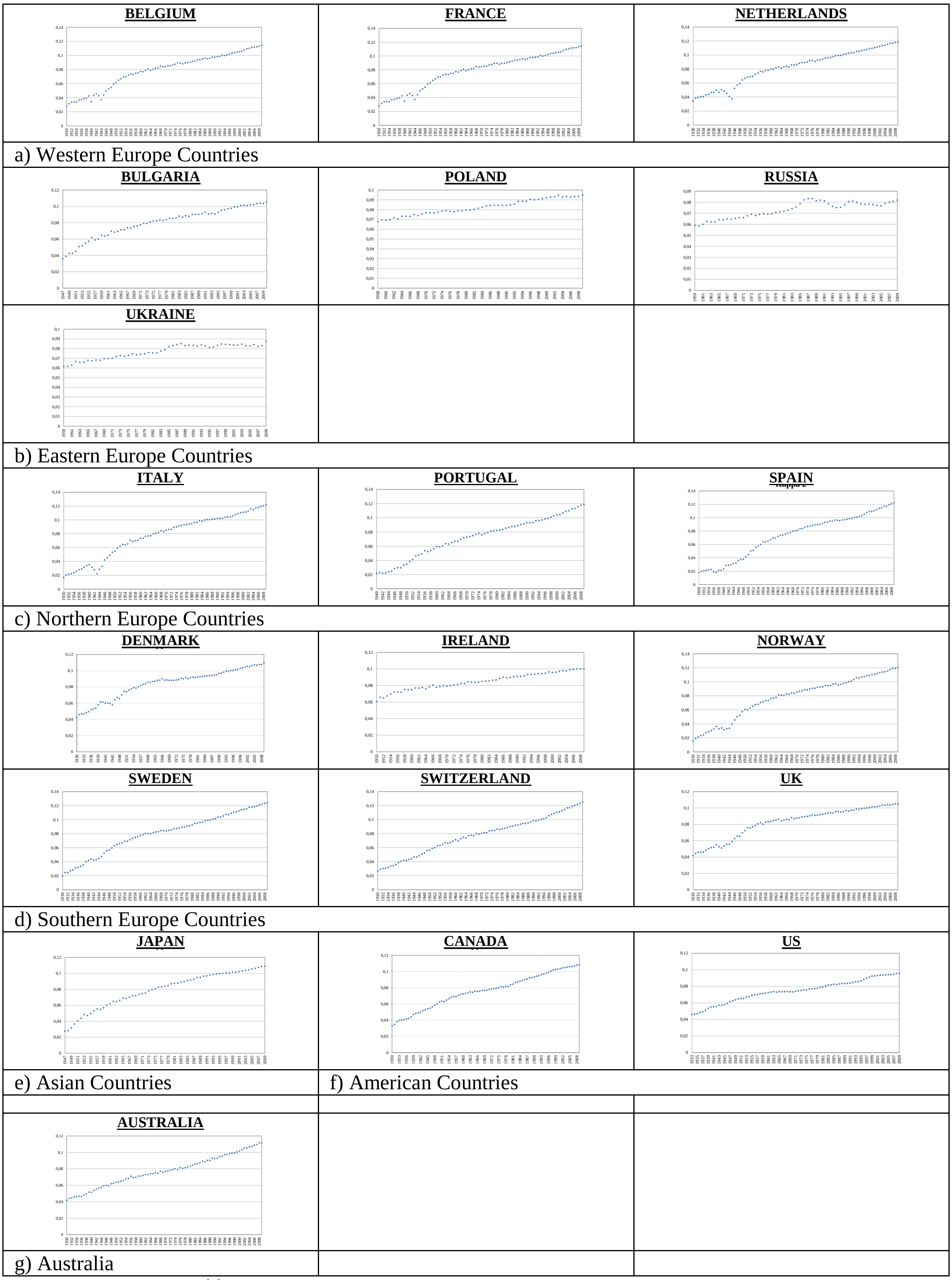


Figure A.15 CBD1, $\kappa^{(2)}$

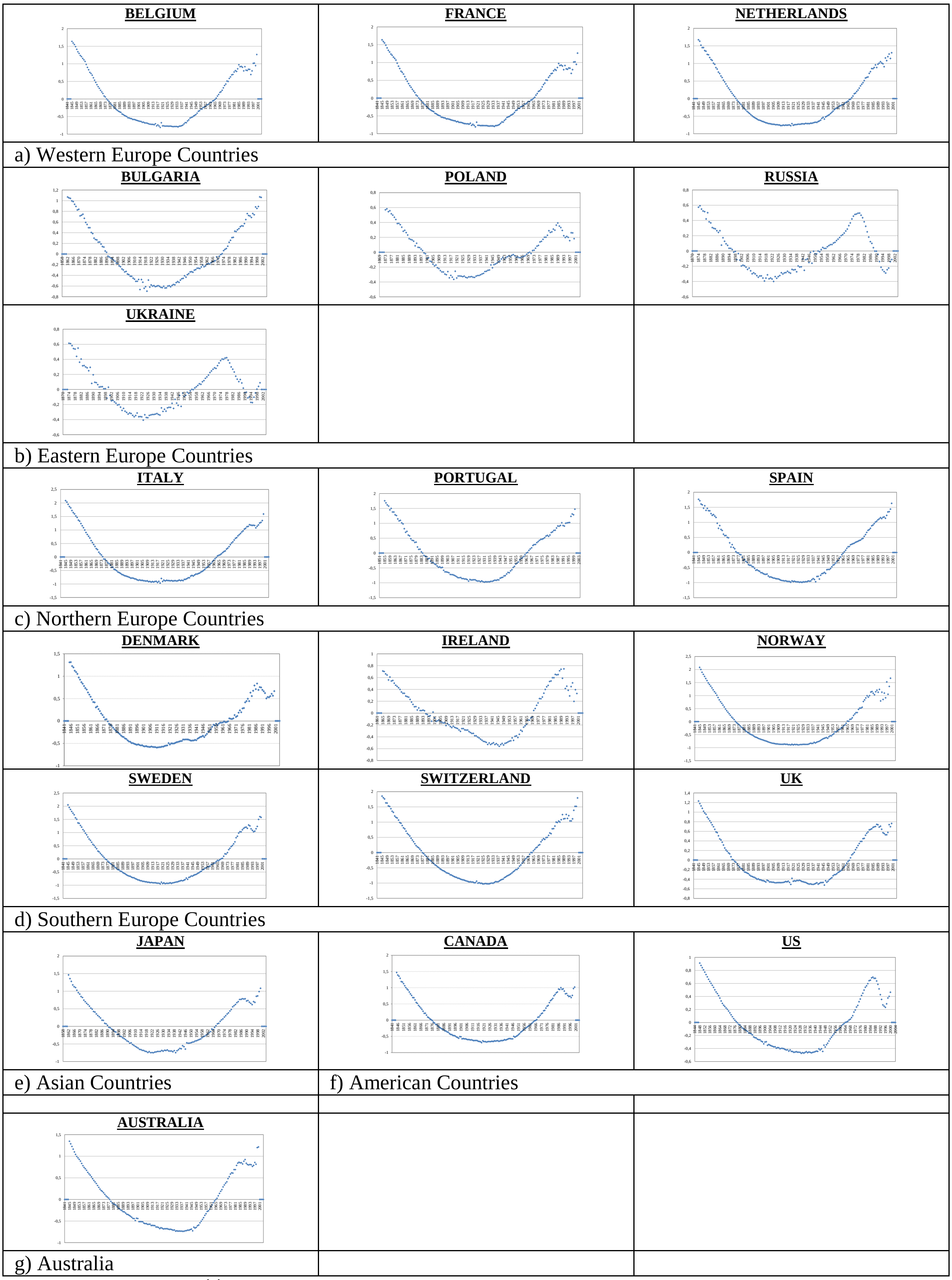


Figure A.16 CBD1, $\gamma^{(3)}$

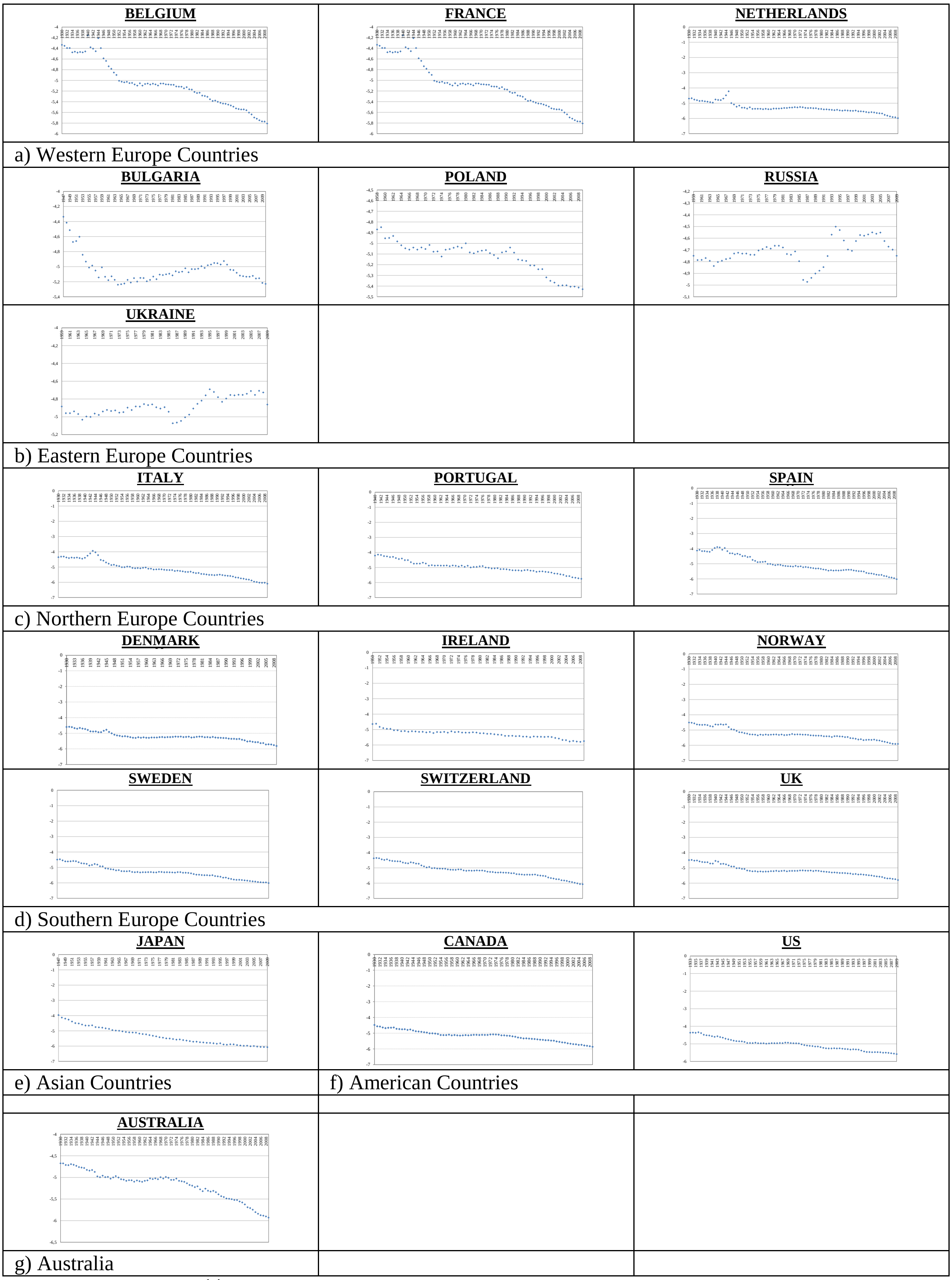


Figure A.17 CBD2, $\kappa^{(1)}$

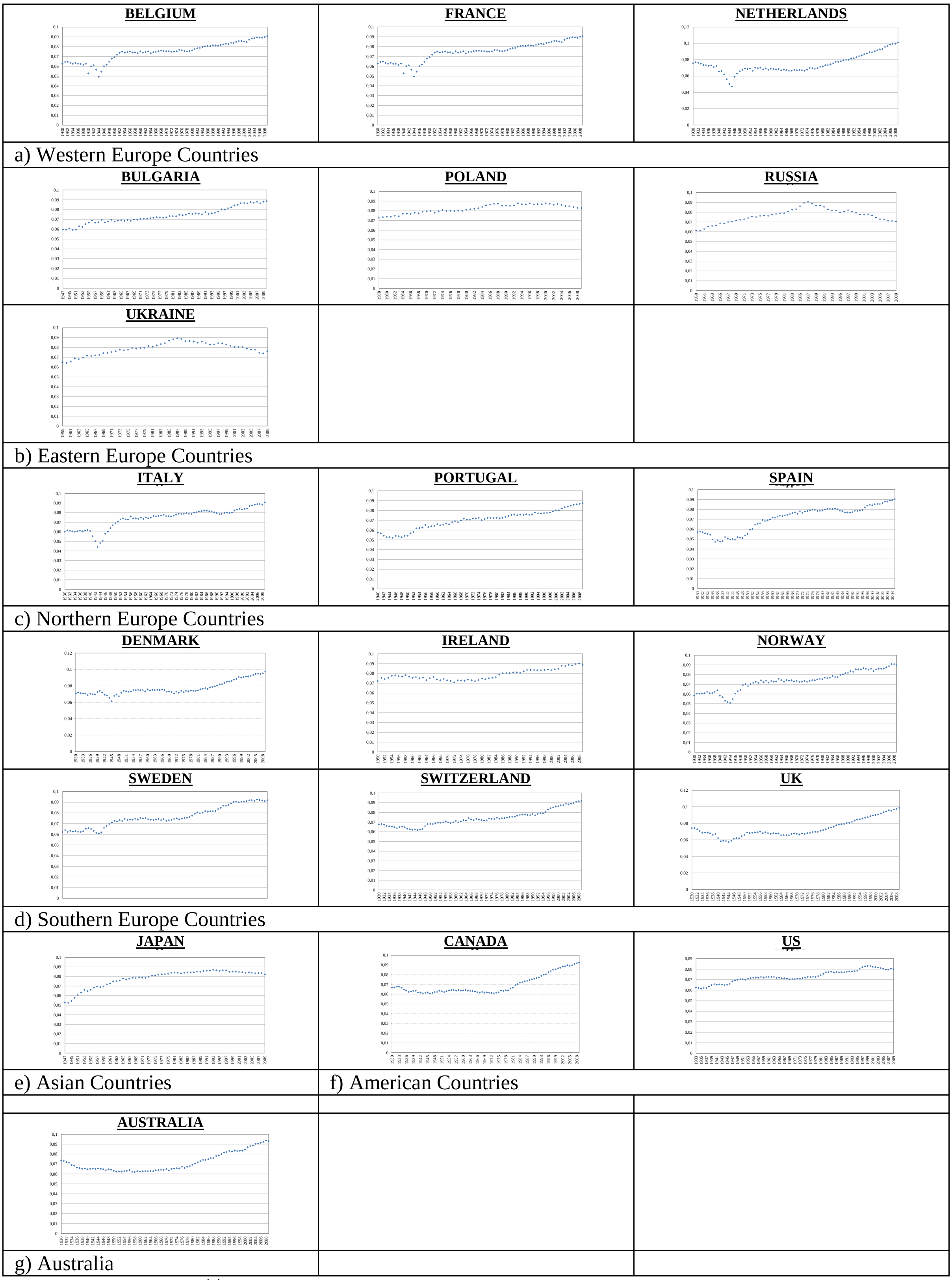


Figure A.18 CBD2, $\kappa^{(2)}$

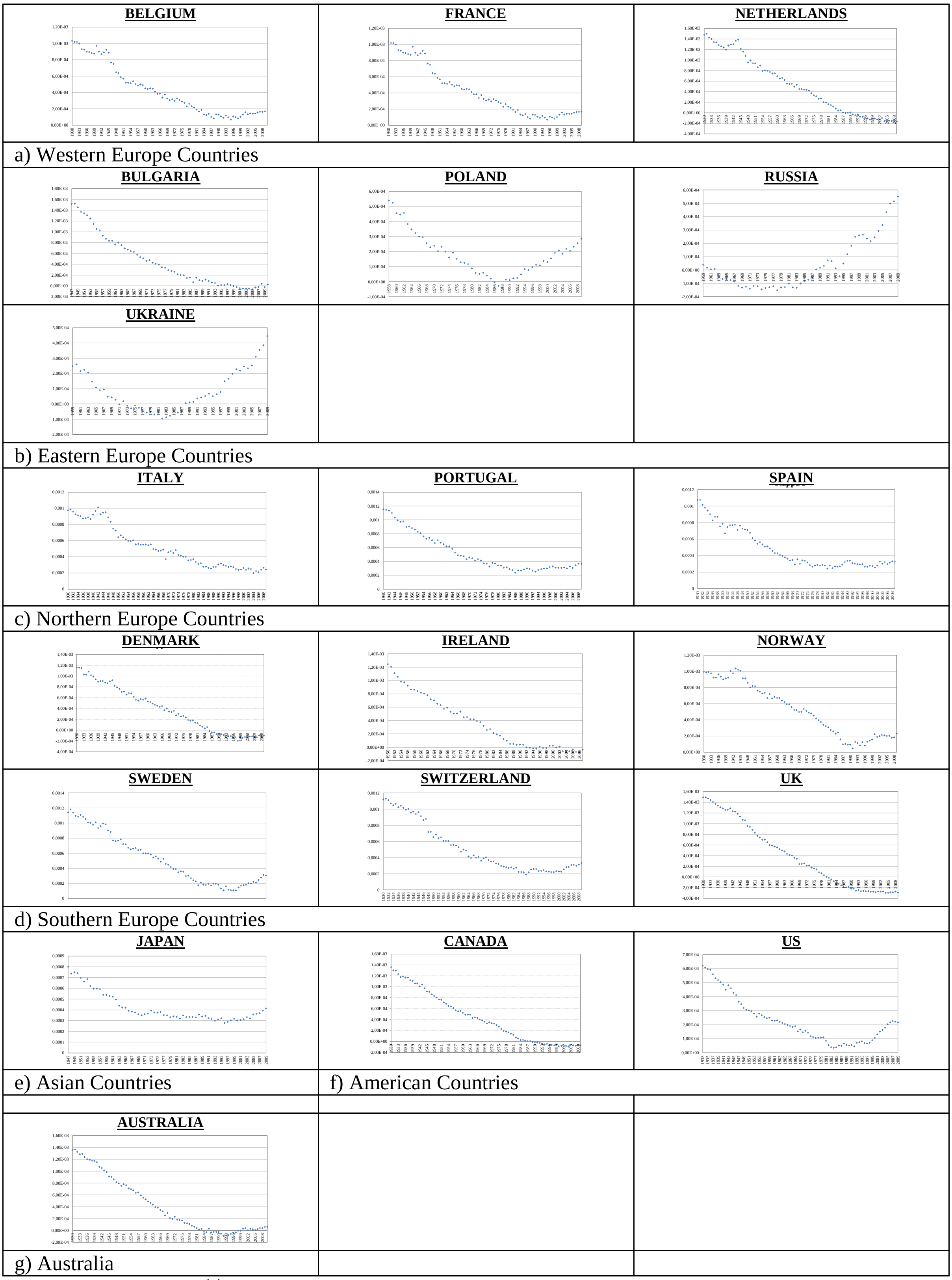


Figure A.19 CBD2, $\kappa^{(3)}$

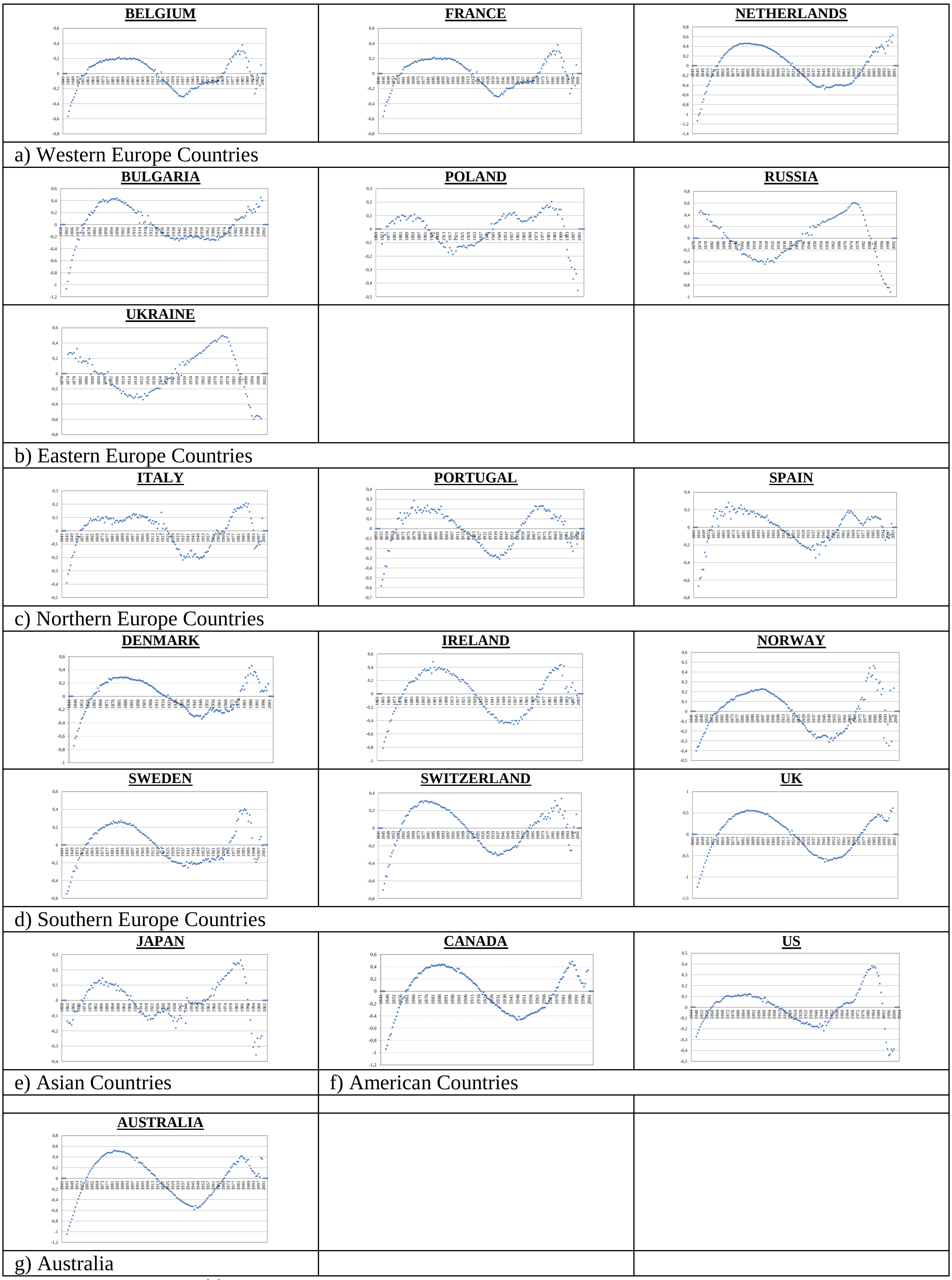
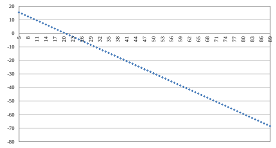
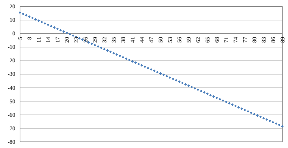


Figure A.20 CBD2, $\gamma^{(4)}$

BELGIUM 	**FRANCE** 	**NETHERLANDS**
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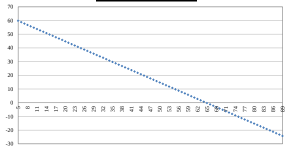
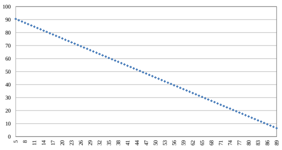
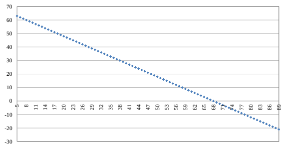
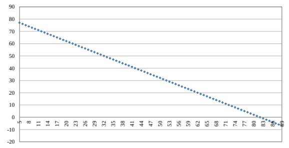
BULGARIA 	**POLAND** 	**RUSSIA** 
UKRAINE 		
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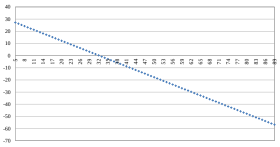
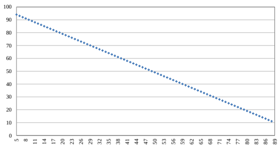
ITALY	**PORTUGAL** 	**SPAIN** 
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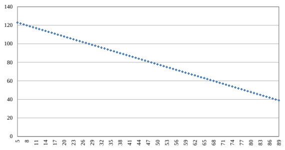
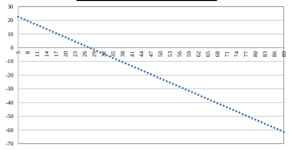
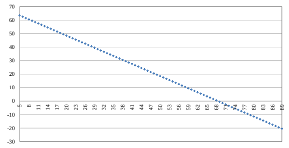
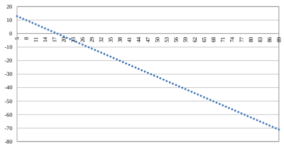
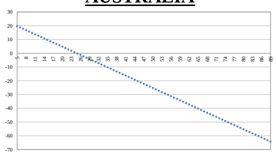
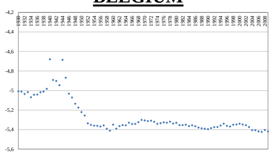
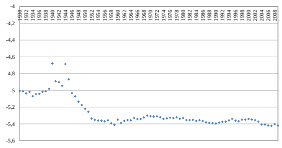
SWEDEN 	**SWITZERLAND** 	**UK**
d) Southern Europe Countries		
JAPAN 	**CANADA**	**US** 
e) Asian Countries	f) American Countries	
AUSTRALIA 		
g) Australia		

Figure A.21 CBD3, $\beta^{(3)}$

BELGIUM



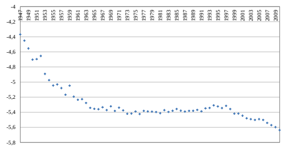
FRANCE



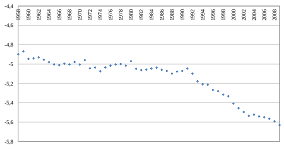
NETHERLANDS

a) Western Europe Countries

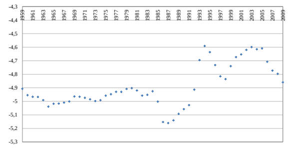
BULGARIA



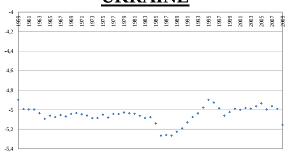
POLAND



RUSSIA



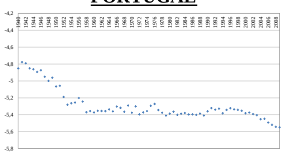
UKRAINE



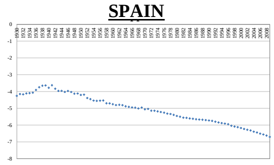
b) Eastern Europe Countries

ITALY

PORTUGAL



SPAIN



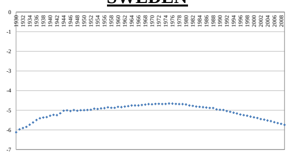
c) Northern Europe Countries

DENMARK

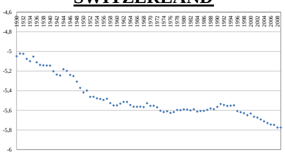
IRELAND

NORWAY

SWEDEN



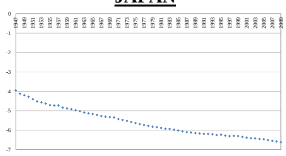
SWITZERLAND



UK

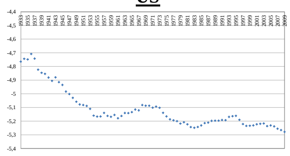
d) Southern Europe Countries

JAPAN



CANADA

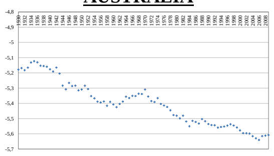
US



e) Asian Countries

f) American Countries

AUSTRALIA



g) Australia

Figure A.22 CBD3, $\kappa^{(1)}$

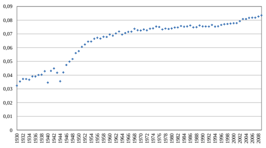
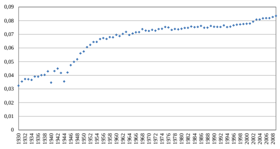
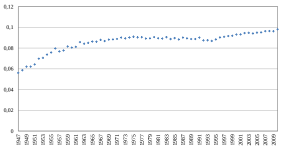
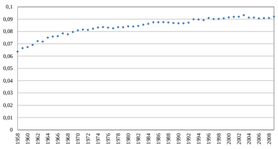
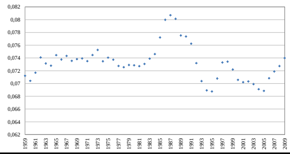
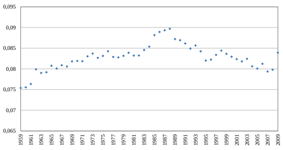
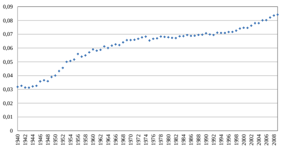
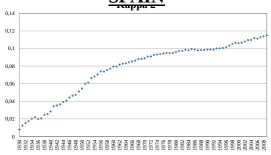
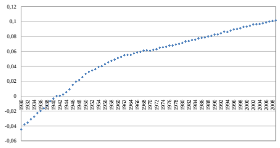
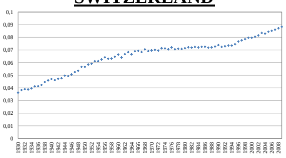
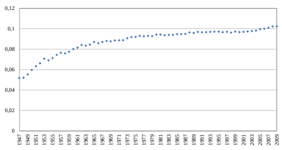
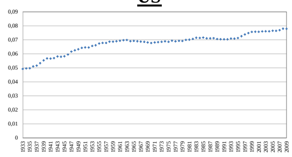
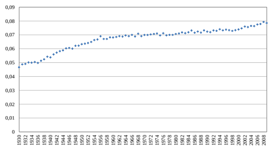
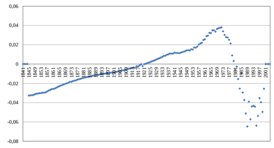
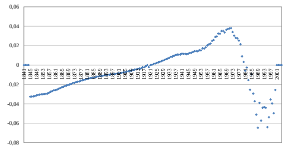
BELGIUM 	**FRANCE** 	**NETHERLANDS**
a) Western Europe Countries		
BULGARIA 	**POLAND** 	**RUSSIA** 
UKRAINE 		
b) Eastern Europe Countries		
ITALY	**PORTUGAL** 	**SPAIN** 
c) Northern Europe Countries		
DENMARK	**IRELAND**	**NORWAY**
SWEDEN 	**SWITZERLAND** 	**UK**
d) Southern Europe Countries		
JAPAN 	**CANADA**	**US** 
e) Asian Countries	f) American Countries	
AUSTRALIA 		
g) Australia		

Figure A.23 CBD3, $\kappa^{(2)}$

BELGIUM



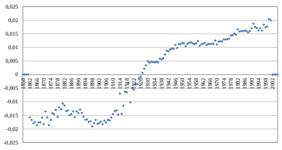
FRANCE



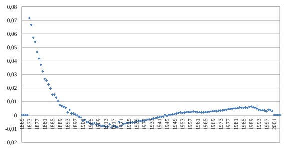
NETHERLANDS

a) Western Europe Countries

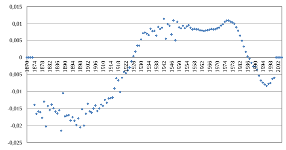
BULGARIA



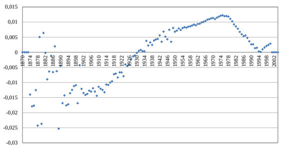
POLAND



RUSSIA



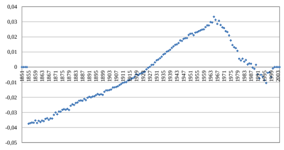
UKRAINE



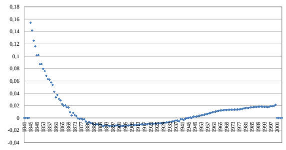
b) Eastern Europe Countries

ITALY

PORTUGAL



SPAIN



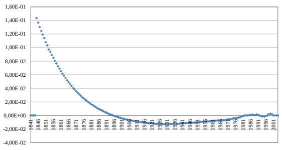
c) Northern Europe Countries

DENMARK

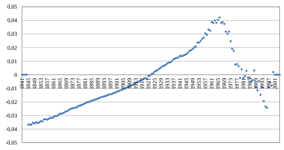
IRELAND

NORWAY

SWEDEN



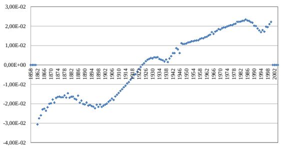
SWITZERLAND



UK

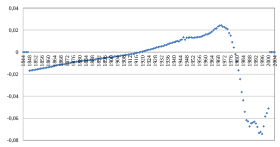
d) Southern Europe Countries

JAPAN



CANADA

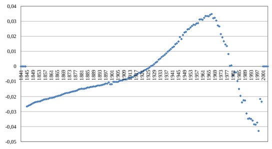
US



e) Asian Countries

f) American Countries

AUSTRALIA



g) Australia

Figure A.24 CBD3, $\gamma^{(3)}$

Appendix III: Fitting Results of the Smoothing Techniques

Table A.1 Fitting results of the smoothing techniques

	Standard Deviation of Pearson Residuals	Sum of Pearson Residuals	Sum Square of Pearson Residuals	Mean Square of Pearson Residuals
Savitzky Golay	1.038	483.75	1,151.57	1.44
Moving Average	1.368	515.67	1,824.30	2.29
Robust Loess (degree=5)	1.933	630.01	3,475.55	4.36
Robust Loess (degree=9)	1.505	191.70	1,850.89	2.32
Robust Loess (degree=15)	1.621	176.72	2,132.23	2.67
Loess (degree=5)	1.585	802.24	2,809.87	3.52
Loess (degree=9)	1.695	602.07	2,743.47	3.44
Loess (degree=15)	1.736	419.42	2,623.31	3.29
Polyfit (degree=3)	1.368	727.81	4,138.20	5.19
Robust Lowess (degree=5)	1.550	606.85	2,375.83	2.98
Robust Lowess (degree=9)	1.489	389.96	1,958.23	2.45
Robust Lowess (degree=15)	1.427	683.09	2,208.60	2.77
Lowess (degree=5)	1.270	517.60	1,621.68	2.03
Lowess (degree=9)	1.327	658.60	1,946.32	2.44
Lowess (degree=15)	1.566	662.48	2,503.60	3.14

Appendix IV: ARIMA(1,1,0) Results of RH Model $\gamma^{(3)}$ Parameter

Table A.2 ARIMA parameters for RH model

	COEF	SE COEF	T	P
AR(1)	-0.3055	0.0762	-4.010	0.000
Constant	0.6586	0.1421	4.640	0.000

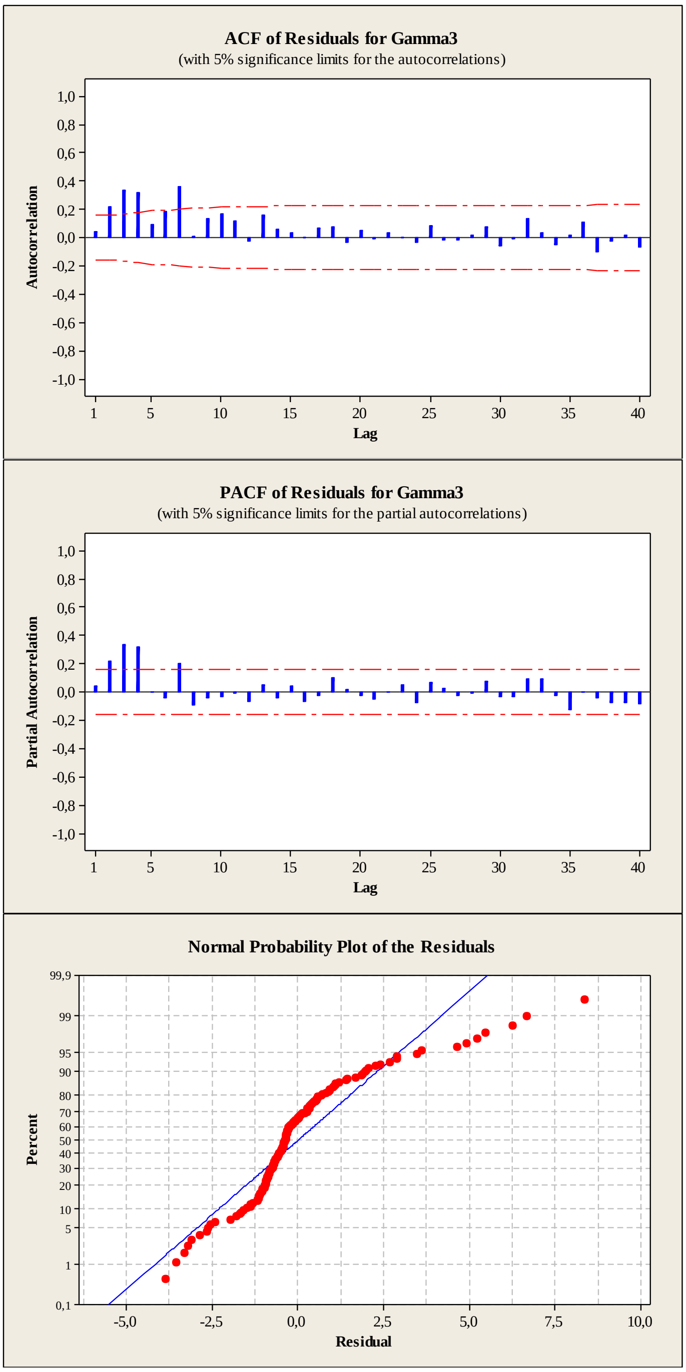


Figure A.25 ACF, PACF and probability plot of RH residuals

Appendix V: ARIMA(1,1,0) Results of LC Model $\kappa^{(2)}$ Parameter

Table A.3 ARIMA parameters for LC model

	COEF	SE COEF	T	P
AR(1)	-0.4357	0.1116	-3.90	0.000
Constant	-2.9945	0.5676	-5.28	0.000

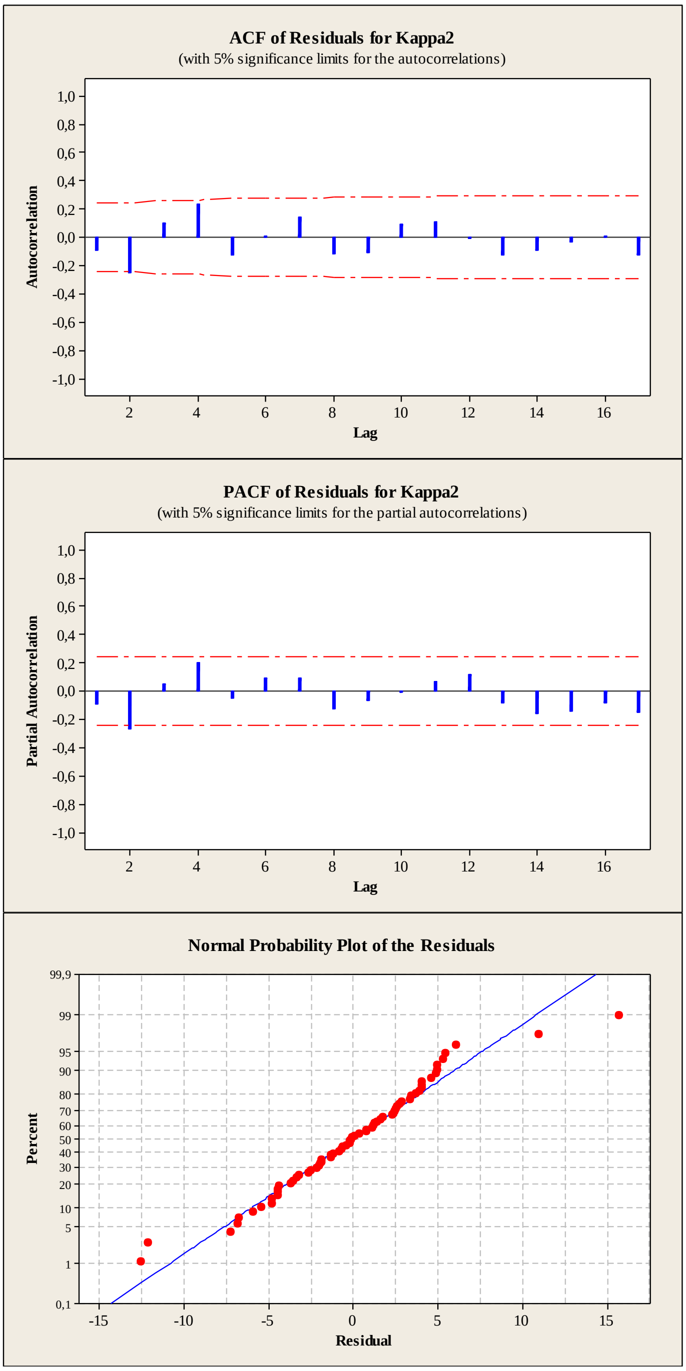


Figure A.26 ACF, PACF and probability plot of LC residuals

Appendix VI: Estimated Number of Death by the New Approach

COHORT YEAR	AGE																																																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
1922	7,695	1,853	848	508	336	288	233	200	166	136	125	125	126	123	129	156	160	163	247	286	258	296	307	251	208	189	164	159	151	138	117	116	130	122	127	140	146	150	173	187	202	219	256	273	291	325	362	420	466
1923	6,949	2,176	811	439	342	293	247	193	150	136	121	119	115	110	125	139	146	227	268	246	275	295	271	203	192	170	153	147	129	119	125	121	125	119	126	136	141	159	171	191	206	218	258	278	290	324	380	419	450
1924	7,614	2,043	713	468	322	332	244	186	150	136	133	117	108	116	124	136	211	249	228	254	266	262	219	197	168	156	150	128	117	119	118	108	115	115	131	138	139	157	177	191	206	236	258	271	299	340	366	424	452
1925	7,596	1,779	786	420	370	302	226	178	161	144	121	103	101	107	111	180	212	184	212	232	238	207	194	167	162	140	133	107	112	109	110	105	113	114	119	131	138	152	173	181	206	217	242	279	314	334	377	407	456
1926	7,139	1,877	693	518	333	289	216	185	176	142	112	98	105	100	152	191	153	183	195	205	184	187	163	156	139	126	109	106	108	108	104	103	111	111	123	133	143	155	171	184	206	213	252	282	298	331	380	412	449
1927	6,980	1,644	913	390	321	282	239	208	140	123	107	109	91	128	153	144	156	175	176	173	162	154	159	129	117	104	101	94	100	101	97	94	110	109	116	131	139	152	171	182	195	211	249	274	293	346	369	416	439
1928	6,788	2,191	600	407	310	290	258	177	145	119	109	93	120	127	119	143	148	159	142	165	149	146	123	110	108	94	90	89	94	95	93	96	97	110	120	128	134	153	167	181	196	229	245	267	300	351	368	395	419
1929	7,469	1,381	643	396	320	337	225	173	139	116	95	110	121	104	120	132	137	135	150	130	136	124	117	104	93	92	86	87	92	90	89	90	105	105	119	121	138	142	157	174	201	211	237	268	284	318	351	393	429
1930	6,363	1,531	587	390	339	273	219	166	142	104	124	126	97	103	114	117	125	145	137	128	118	109	104	99	88	96	84	86	90	91	88	94	101	109	113	126	133	133	161	172	196	212	236	260	292	311	345	383	418
1931	6,761	1,426	566	423	290	260	200	164	117	131	122	88	94	106	105	109	124	132	117	111	103	99	92	94	89	81	92	86	82	90	96	94	109	106	114	128	132	141	157	169	195	212	246	262	289	306	333	382	422
1932	6,666	1,254	566	326	277	244	201	135	142	132	91	92	94	96	92	110	121	108	103	96	95	90	83	81	80	88	79	81	81	84	92	91	98	100	117	116	122	134	149	163	179	210	232	247	274	296	330	366	387
1933	6,332	1,239	452	311	260	246	158	178	153	98	89	87	80	72	82	94	102	90	82	83	81	87	86	81	80	77	78	80	81	89	88	90	95	103	103	111	124	137	141	166	186	203	225	252	274	301	330	341	371
1934	6,166	987	473	307	243	192	211	179	110	100	99	77	74	73	73	85	81	70	76	76	77	84	79	78	71	77	80	75	79	88	88	85	91	94	106	117	125	141	151	163	186	193	223	236	267	302	315	349	356
1935	6,069	1,091	441	303	206	256	224	127	114	105	80	65	64	62	72	71	71	71	70	66	79	76	78	79	79	71	81	72	78	86	91	95	89	93	110	109	125	137	141	158	173	199	220	234	268	277	305	335	371
1936	6,260	988	408	235	271	268	164	127	107	84	69	62	59	58	52	59	58	66	66	71	65	81	79	79	75	76	73	80	81	85	82	88	93	91	100	114	124	131	143	157	169	183	216	228	257	281	298	326	353
1937	6,121	890	308	334	306	201	151	123	93	66	66	57	48	52	52	55	55	60	66	61	80	75	75	74	79	76	79	73	81	85	85	84	97	95	99	114	126	138	136	153	161	190	205	230	244	267	286	312	349
1938	5,585	616	453	376	223	182	147	104	75	69	57	51	45	49	50	48	60	63	60	72	73	82	77	77	75	73	73	76	78	69	76	79	92	95	101	116	121	130	135	154	170	181	181	215	237	268	283	305	331
1939	5,289	860	472	254	199	177	132	85	77	59	51	44	43	39	45	49	53	60	76	73	76	83	83	77	72	80	74	75	74	77	81	87	94	91	107	112	117	134	146	168	175	189	201	199	230	259	275	301	320
1940	5,894	892	306	238	178	154	101	87	60	51	44	41	42	45	41	52	51	66	66	76	80	75	81	74	71	77	77	72	72	77	82	82	84	92	103	109	124	126	149	154	162	176	185	212	227	260	287	300	338
1941	6,161	569	288	202	159	104	95	70	62	44	46	35	37	39	43	40	57	60	71	78	77	73	73	76	76	68	73	70	75	74	76	80	92	93	96	103	113	129	133	140	167	177	187	202	217	252	272	293	316
1942	5,517	570	249	193	131	114	85	65	58	44	37	36	34	39	38	47	47	59	65	76	77	79	70	71	72	73	66	70	71	70	79	74	83	92	94	98	109	123	133	129	157	162	177	198	218	239	265	294	303
1943	5,295	462	239	155	119	95	75	61	49	39	36	32	40	35	43	42	58	65	71	76	77	79	72	74	68	66	71	70	70	75	78	71	84	91	97	108	114	122	131	138	146	156	177	196	215	240	249	286	294
1944	4,979	423	195	148	105	84	66	54	40	38	36	32	32	40	35	45	55	70	77	78	79	74	76	68	63	68	69	68	66	77	82	84	85	85	98	104	108	111	119	137	140	161	175	195	213	232	252	271	283
1945	4,608	356	201	121	91	80	60	50	41	33	34	29	40	33	37	38	55	66	74	84	75	70	66	65	71	67	66	66	75	73	76	81	89	88	96	104	105	113	125	135	157	153	179	194	213	234	248	261	268
1946	4,630	378	159	112	79	67	52	45	31	30	30	28	30	35	34	40	50	59	75	76	74	69	62	61	58	62	65	70	61	66	72	68	80	89	85	88	94	98	112	129	143	147	162	174	186	217	239	255	263
1947	4,616	309	155	109	86	64	56	41	40	30	36	30	37	30	37	44	57	74	82	90	75	72	72	72	63	71	71	69	70	70	69	80	82	88	90	100	104	112	117	129	141	168	172	187	206	225	244	261	272
1948	3,393	262	134	100	68	61	44	43	34	35	29	29	32	32	38	39	64	77	78	75	67	68	68	66	63	69	65	64	68	71	72	70	77	81	86	96	90	105	119	125	138	142	169	161	173	187	203	217	225
1949	3,266	229	135	85	69	54	47	39	37	32	30	30	31	35	35	41	62	78	78	67	64	72	67	69	73	62	70	73	65	71	76	71	74	79	84	92	104	109	108	125	139	159	159	166	173	182	196	214	227
1950	3,012	248	121	88	57	54	43	39	32	28	32	31	31	33	37	43	68	71	68	72	82	73	76	71	67	67	67	67	73	78	68	75	82	79	87	98	96	109	114	125	130	148	153	160	168	179	193	208	217
1951	3,043	211	119	71	62	51	44	38	36	37	29	31	32	35	38	49	61	67	75	69	72	76	74	68	64	64	65	68	66	68	72	73	80	85	79	84	100	109	116	131	141	149	155	162	169	180	194	209	219
1952	2,877	200	97	74	57	52	44	41	40	35	30	26	33	36	38	45	61	73	70	77	70	78	63	65	67	62	68	69	64	69	74	70	75	78	90	93	99	111	119	130	140	147	152	158	166	177	191	205	213
1953	2,777	160	101	68	63	52	45	41	35	30	31	29	33	34	39	44	58	69	69	78	79	69	72	63	62	69	71	71	64	65	68	70	75	78	89	96	105	113	117	129	139	146	151	157	165	175	188	202	210
1954	2,620	164	89	78	50	47	49	40	32	32	29	34	32	36	40	38	59	71	75	77	78	77	71	71	69	63	65	67	63	65	62	71	78	87	96	95	105	109	115	128	137	144	149	155	162	172	185	199	207
1955	2,564	144	87	63	56	51	41	39	37	32	31	33	32	33	34	39	62	73	82	82	77	71	74	68	62	61	62	59	65	66	69	68	77	85	86	94	104	108	115	127	136	143	148	154	161	171	183	197	205
1956	2,487	153	92	70	53	47	40	40	34	33	28	30	33	35	29	39	54	72	79	74	77	68	71	62	64	58	65	63	68	60	71	75	79	83	81	93	103	109	116	127	135	143	148	154	161	170	182	195	203
1957	2,429	140	87	64	54	49	43	38	33	32	35	28	27	28	37	39	52	73	80	78	71	69	64	62	58	62	61	65	65	63	70	74	72	78	83	93	103	109	115	126	135	142	147	153	160	169	181	194	201
1958	2,353	148	88	67	52	47	37	38	35	32	28	27	31	33	34	43	48	74	81	73	75	68	62	62	60	60	60	55	64	61	66	73	80	79	85	92	102	108	115	126	135	142	146	152	159	168	180	193	199
1959	2,315	140	95	65	60	48	42	33	36	34	26	27	26	26	34	40	51	73	69	76	69	65	61	60	53	59	57	58	61	64	67	74	76	79	84	93	102	109	116	126	135	142	146	152	159	168	180	192	198
1960	2,291	148	81	65	51	48	38	34	32	26	31	25	25	26	30	34	52	76	82	76	69	57	59	49	55	54	59	58	62	67	66	73	76	79	84	93	102	109	116	127	135	142	146	152	159	168	180	192	198
1961	2,241	142	83	58	55	45	37	35	30	28	32	26	25	29	31	32	47	77	79	70	67	59	57	57	52	53	60	60	63	71	71	73	76	79	84	94	102	109	116	127	136	143	147	153	159	168	180	192	198
1962	2,274	148	78	63	56	43	39	34	27	32	26	29	25	27	30	35	51	67	79	69	68	59	58	54	54	56	56	57	67	69	73	76	77	81	87	96	105	112	120	131	140	147	151	157	164	173	184	197	202
1963	2,196	128	77	62	48	44	31	35	31	30	24	28	27	28	28	34	43	71	73	66	66	58	59	60	58	57	57	65	65	69	72	74	76	80	85	95	104	111	118	130	138	145	149	155	161	170	182	194	199
1964	2,099	125	82	56	53	46	40	34	33	26	26	23	25	27	30	31	44	68	68	65	63	60	62	57	57	61	68	66	67	68	70	73	76	79	84	94	103	110	117	129	137	145	148	154	160	169	180	192	197
1965	1,975	134	78	66	52	41	37	33	29	24	23	23	21	25	32	30	40	64	64	63	63	59	58	62	64	62	64	63	63	66	69	71	74	77	82	92	100	108	115	126	134	141	145	150	156	165	176	187	191
1966	1,957	124	79	54	47	43	40	31	27	26	24	22	24	24	29	35	43	59	56	57	60	62	60	58	60	61	63	63	63	65	68	71	74	77	82	92	100	108	115	126	134	141	144	150	156	164	175	186	190
1967	1,870	123	79	55	45	40	38	32	27	24	22	24	23	23	30	34	44	57	58	61	62	60	62	64	61	61	62	62	62	65	67	70	73	76	81	91	99	107	114	124	133	140	143	148	155	163	173	184	188
1968	1,867	125	72	53	46	43	33	30	25	22	26	23	22	26	29	33	40	56	66	64	67	62	64	62	62	61	62	62	63	66	68	71	74	77	83	92	101	109	116	127	135	142	146	151	157	166	176	187	191
1969	1,833	115	69	62	45	38	30	28	25	28	21	24	21	24	28	35	39	55	63	59	62	64	61	62	61	60	61	62	63	65	68	71	74	77	82	92	101	108	115	126	135	142	145	150	156	165	175	186	189
1970	1,828	110	74	58	40	35	31	25	23	25	22	21	21	23	27	31	39	55	60	62	63	63	61	62	60	60	61	62	63	65	68	71	74	77	83	92	101	109	116	127	136	143	146	151	157	166	176	186	190
1971	1,800	115	71	57	41	37	29	28	27	21	22	21	23	22	29	33	38	58	65	58	64	62	60	62	61	60	62	62	63	66	69	72	75	78	84	94	103	110	117	129	137	144	148	153	159	167	178	188	191
1972	1,696	110	65	51	38	32	29	25	26	20	22	21	23	25	24	31	39	57	60	60	60	58	58	59	58	57	59	60	60	63	66	69	72	75	80	90	99	106	113	124	132	139	142	147	153	161	171	181	183
1973	1,667	110	57	44	39	32	30	25	20	18	18	20	23	22	28	32	41	58	57	59	59	57	57	58	57	57	58	59	60	63	65	69	71	75	80	89	98	105	112	124	132	138	142	147	153	160	170	180	182
1974	1,637	100	56	46	37	31	27	23	23	21	21	23	19	26	25	31	44	53	57	59	57	56	56	57	56	56	58	59	59	62	65	68	71	74	80	89	98	105	112	123	132	138	142	147	152	160	170	179	181
1975	1,560	96	58	46	36	31	27	21	21	20	20	18	20	24	24	32	43	51	55	56	55	54	54	55	54	54	56	57	58	60	63	67	69	72	77	87	96	103	110	120	128	135	138	143	148	156	165	174	176
1976	1,424	84	53	39	34	26	23	22	19	21	17	17	18	19	25	32	40	47	50	51	51	49	49	51	50	50	52	52	53	56	58	62	64	67	72	81	89	95	102	112	119	125	128	133	138	145	153	162	163
1977	1,395	94	49	36	33	24	26	21	18	16	15	18	20	21	25	32	39	46	50	51	50	49	49	50	49	50	51	52	53	56	58	62	64	67	72	81	89	95	102	112	119	125	128	133	138	145	153	162	163
1978	1,351	78	50	36	27	26	22	21	17	17	17	17	17	19	24	31	38	45	48	49	48	47	47	48	48	48	50	50	51	54	57	60	62	65	70	79	87	93	99	109	117	123	125	130	135	141	150	158	159
1979	1,330	83	49	33	31	24	20	20	16	15	17	19	18	19	23	30	38	45	48	49	48	46	46	48	47	48	50	50	51	54	57	60	63	66	70	79	87	94	100	110	118	123	126	130	135	142	150	158	159
1980	1,236	78	47	32	25	22	19	15	21	17	16	17	16	17	22	29	36	42	45	46	45	43	43	45	44	45	47	48	48	51	54	57	59	62	67	75	83	89	95	104	111	117	120	124	128	135	142	150	151
1981	1,110	78	43	34	25	22	20	20	16	15	15	15	15	16	20	26	33	39	41	42	41	40	40	41	41	42	43	44	45	47	50	53	55	58	62	70	77	83	88	97	104	109	111	115	119	125	132	139	140
1982	1,088	69	40	35	26	22	19	20	16	16	15	15	15	16	20	26	33	38	41	41	40	39	39	41	40	41	43	43	44	47	49	52	55	57	61	69	76	82	88	97	103	108	111	114	119	124	132	138	139
1983	1,020	76	44	33	25	20	19	15	14	14	14	14	14	15	19	25	31	36	39	39	38	37	37	39	39	39	41	42	43	45	47	50	52	55	59	67	74	79	85	93	99	104	107	110	115	120	127	133	134
1984	962	76	40	30	24	23	20	17	14	13	13	14	14	14	18	24	30	35	37	37	37	35	36	37	37	38	39	40	41	43	46	49	51	53	57	64	71	77	82	90	96	101	103	107	111	116	123	129	129
1985	946	67	41	34	26	19	18	15	13	13	13	13	14	14	18	23	29	35	37	37	36	35	35	37	36	37	39	40	41	43	45	48	50	53	57	64	71	76	81	90	96	101	103	106	110	116	122	128	128
1986	950	66	41	33	22	20	17	15	14	13	13	13	14	14	18	24	30	35	37	37	36	35	35	37	37	37	39	40	41	43	46	49	51	54	58	65	72	77	83	91	98	102	105	108	112	117	124	130	130
1987	922	64	41	29	22	19	17	15	14	13	12	13	13	13	17	23	29	34	36	36	35	34	34	36	36	37	38	39	40	43	45	48	50	53	57	64	71	76	81	90	96	101	103	107	110	116	122	128	128
1988	908	64	35	25	21	19	17	15	13	12	12	13	13	13	17	23	28	33	35	35	34	33	33	35	35	36	38	39	40	42	45	48	50	52	56	63	70	76	81	89	96	100	103	106	110	115	121	127	127
1989	839	64	35	23	19	18	16	14	12	12	11	12	12	12	16	21	27	31	33	33	32	31	31	33	33	34	36	36	37	40	42	45	47	49	53	60	66	72	77	85	90	95	97	100	104	109	115	120	120
1990	801	58	32	21	18	17	15	13	11	11	11	11	11	12	15	20	25	30	32	32	31	29	30	31	31	32	34	35	36	38	41	43	45	48	51	58	64	69	74	82	87	92	94	97	100	105	111	116	115

Yellow shaded cells are forecasted number of deaths and used to compare out-sample forecast performance with other methods.

Appendix VII: Estimated LDFs by the New Approach

COHORT YEAR	AGE																								
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1922		1.2408	1.0888	1.0489	1.0308	1.0256	1.0202	1.0170	1.0139	1.0112	1.0102	1.0101	1.0101	1.0097	1.0101	1.0121	1.0123	1.0123	1.0185	1.0210	1.0186	1.0209	1.0212	1.0170	1.0139
1923		1.3131	1.0889	1.0442	1.0330	1.0273	1.0224	1.0171	1.0131	1.0117	1.0103	1.0100	1.0096	1.0091	1.0102	1.0113	1.0117	1.0180	1.0209	1.0188	1.0206	1.0216	1.0195	1.0143	1.0133
1924		1.2683	1.0738	1.0451	1.0297	1.0297	1.0212	1.0158	1.0126	1.0113	1.0109	1.0095	1.0087	1.0092	1.0098	1.0106	1.0163	1.0189	1.0170	1.0186	1.0192	1.0185	1.0152	1.0135	1.0113
1925		1.2342	1.0838	1.0413	1.0350	1.0276	1.0201	1.0155	1.0138	1.0122	1.0101	1.0085	1.0083	1.0087	1.0090	1.0144	1.0167	1.0143	1.0162	1.0175	1.0176	1.0150	1.0139	1.0118	1.0113
1926		1.2629	1.0769	1.0534	1.0326	1.0274	1.0199	1.0167	1.0156	1.0124	1.0097	1.0084	1.0089	1.0084	1.0127	1.0157	1.0124	1.0147	1.0154	1.0159	1.0141	1.0141	1.0121	1.0115	1.0101
1927		1.2355	1.1059	1.0409	1.0323	1.0275	1.0227	1.0193	1.0128	1.0111	1.0095	1.0096	1.0079	1.0111	1.0131	1.0122	1.0130	1.0144	1.0143	1.0139	1.0128	1.0120	1.0123	1.0098	1.0088
1928		1.3228	1.0668	1.0425	1.0310	1.0282	1.0244	1.0163	1.0132	1.0107	1.0097	1.0082	1.0104	1.0109	1.0101	1.0121	1.0123	1.0131	1.0115	1.0133	1.0118	1.0114	1.0095	1.0084	1.0082
1929		1.1849	1.0727	1.0417	1.0324	1.0330	1.0213	1.0161	1.0127	1.0105	1.0085	1.0097	1.0106	1.0090	1.0103	1.0112	1.0115	1.0112	1.0123	1.0106	1.0109	1.0099	1.0092	1.0081	1.0072
1930		1.2406	1.0744	1.0460	1.0382	1.0296	1.0231	1.0171	1.0144	1.0104	1.0123	1.0123	1.0094	1.0098	1.0108	1.0110	1.0116	1.0133	1.0124	1.0114	1.0104	1.0095	1.0090	1.0085	1.0075
1931		1.2109	1.0691	1.0483	1.0316	1.0275	1.0206	1.0165	1.0116	1.0128	1.0118	1.0084	1.0089	1.0100	1.0098	1.0100	1.0113	1.0119	1.0104	1.0098	1.0090	1.0086	1.0079	1.0080	1.0075
1932		1.1881	1.0715	1.0384	1.0314	1.0268	1.0215	1.0142	1.0147	1.0135	1.0092	1.0092	1.0093	1.0094	1.0089	1.0106	1.0115	1.0102	1.0096	1.0088	1.0087	1.0082	1.0075	1.0072	1.0071
1933		1.1957	1.0597	1.0388	1.0312	1.0286	1.0179	1.0198	1.0167	1.0105	1.0094	1.0091	1.0083	1.0074	1.0084	1.0096	1.0103	1.0090	1.0081	1.0081	1.0079	1.0084	1.0082	1.0077	1.0075
1934		1.1601	1.0661	1.0403	1.0306	1.0235	1.0252	1.0209	1.0126	1.0113	1.0110	1.0085	1.0081	1.0079	1.0079	1.0091	1.0086	1.0073	1.0079	1.0079	1.0079	1.0085	1.0080	1.0078	1.0071
1935		1.1798	1.0616	1.0399	1.0261	1.0316	1.0268	1.0148	1.0131	1.0119	1.0090	1.0072	1.0070	1.0068	1.0078	1.0077	1.0076	1.0075	1.0074	1.0069	1.0082	1.0078	1.0080	1.0080	1.0079
1936		1.1578	1.0563	1.0307	1.0343	1.0328	1.0195	1.0148	1.0123	1.0095	1.0077	1.0069	1.0065	1.0064	1.0057	1.0064	1.0063	1.0071	1.0070	1.0075	1.0068	1.0084	1.0082	1.0081	1.0076
1937		1.1454	1.0439	1.0456	1.0400	1.0253	1.0185	1.0148	1.0110	1.0077	1.0077	1.0066	1.0055	1.0059	1.0059	1.0062	1.0062	1.0067	1.0073	1.0067	1.0087	1.0081	1.0080	1.0079	1.0083
1938		1.1103	1.0731	1.0565	1.0317	1.0251	1.0198	1.0137	1.0098	1.0089	1.0073	1.0065	1.0057	1.0061	1.0062	1.0059	1.0074	1.0077	1.0073	1.0087	1.0087	1.0097	1.0090	1.0089	1.0086
1939		1.1626	1.0768	1.0384	1.0289	1.0250	1.0182	1.0115	1.0103	1.0078	1.0067	1.0057	1.0056	1.0050	1.0058	1.0063	1.0067	1.0076	1.0095	1.0091	1.0093	1.0101	1.0100	1.0092	1.0085
1940		1.1513	1.0451	1.0336	1.0243	1.0205	1.0132	1.0112	1.0076	1.0064	1.0055	1.0051	1.0052	1.0056	1.0050	1.0064	1.0062	1.0080	1.0079	1.0090	1.0094	1.0088	1.0094	1.0085	1.0081
1941		1.0924	1.0428	1.0288	1.0220	1.0141	1.0127	1.0092	1.0081	1.0057	1.0059	1.0045	1.0047	1.0050	1.0054	1.0050	1.0071	1.0075	1.0088	1.0095	1.0093	1.0088	1.0087	1.0090	1.0089
1942		1.1033	1.0409	1.0305	1.0201	1.0171	1.0125	1.0095	1.0084	1.0063	1.0053	1.0051	1.0048	1.0055	1.0053	1.0065	1.0065	1.0081	1.0088	1.0102	1.0103	1.0104	1.0091	1.0092	1.0092
1943		1.0873	1.0415	1.0259	1.0193	1.0152	1.0118	1.0095	1.0075	1.0060	1.0055	1.0048	1.0060	1.0052	1.0064	1.0062	1.0085	1.0095	1.0102	1.0108	1.0109	1.0110	1.0099	1.0101	1.0092
1944		1.0850	1.0361	1.0264	1.0183	1.0144	1.0111	1.0090	1.0066	1.0062	1.0059	1.0052	1.0052	1.0064	1.0056	1.0071	1.0087	1.0109	1.0119	1.0119	1.0119	1.0110	1.0112	1.0099	1.0091
1945		1.0773	1.0405	1.0234	1.0172	1.0149	1.0110	1.0091	1.0074	1.0059	1.0060	1.0051	1.0070	1.0057	1.0064	1.0065	1.0094	1.0112	1.0124	1.0139	1.0122	1.0113	1.0105	1.0102	1.0111
1946		1.0816	1.0317	1.0217	1.0150	1.0125	1.0096	1.0082	1.0056	1.0054	1.0054	1.0050	1.0053	1.0062	1.0060	1.0070	1.0087	1.0101	1.0127	1.0127	1.0123	1.0113	1.0100	1.0098	1.0092
1947		1.0669	1.0315	1.0215	1.0166	1.0121	1.0105	1.0076	1.0074	1.0055	1.0065	1.0054	1.0066	1.0053	1.0066	1.0078	1.0100	1.0128	1.0140	1.0152	1.0125	1.0118	1.0117	1.0115	1.0100
1948		1.0772	1.0367	1.0264	1.0175	1.0154	1.0110	1.0106	1.0083	1.0085	1.0069	1.0069	1.0076	1.0075	1.0088	1.0090	1.0146	1.0174	1.0173	1.0163	1.0144	1.0144	1.0142	1.0136	1.0128
1949		1.0701	1.0386	1.0234	1.0186	1.0143	1.0122	1.0100	1.0094	1.0081	1.0075	1.0075	1.0076	1.0086	1.0085	1.0099	1.0148	1.0183	1.0180	1.0152	1.0143	1.0158	1.0145	1.0147	1.0154
1950		1.0823	1.0371	1.0260	1.0164	1.0153	1.0120	1.0108	1.0087	1.0076	1.0086	1.0083	1.0082	1.0086	1.0096	1.0111	1.0173	1.0178	1.0167	1.0174	1.0195	1.0170	1.0174	1.0160	1.0149
1951		1.0693	1.0366	1.0210	1.0180	1.0145	1.0124	1.0106	1.0099	1.0101	1.0078	1.0083	1.0085	1.0092	1.0099	1.0126	1.0155	1.0168	1.0185	1.0167	1.0172	1.0178	1.0170	1.0154	1.0143
1952		1.0695	1.0315	1.0233	1.0175	1.0157	1.0131	1.0121	1.0116	1.0101	1.0085	1.0073	1.0092	1.0100	1.0104	1.0122	1.0164	1.0193	1.0181	1.0196	1.0175	1.0191	1.0152	1.0154	1.0156
1953		1.0576	1.0344	1.0224	1.0203	1.0164	1.0140	1.0126	1.0106	1.0090	1.0092	1.0085	1.0096	1.0098	1.0111	1.0124	1.0162	1.0190	1.0186	1.0206	1.0205	1.0175	1.0180	1.0155	1.0150
1954		1.0626	1.0320	1.0271	1.0169	1.0157	1.0161	1.0129	1.0102	1.0101	1.0091	1.0105	1.0098	1.0109	1.0120	1.0113	1.0173	1.0205	1.0212	1.0213	1.0211	1.0204	1.0185	1.0181	1.0173
1955		1.0562	1.0321	1.0225	1.0196	1.0175	1.0138	1.0130	1.0122	1.0104	1.0100	1.0105	1.0101	1.0103	1.0105	1.0119	1.0187	1.0216	1.0238	1.0232	1.0213	1.0192	1.0197	1.0177	1.0159
1956		1.0615	1.0348	1.0256	1.0189	1.0165	1.0138	1.0136	1.0114	1.0109	1.0092	1.0097	1.0106	1.0111	1.0091	1.0122	1.0167	1.0218	1.0234	1.0215	1.0219	1.0189	1.0194	1.0166	1.0168
1957		1.0576	1.0339	1.0241	1.0199	1.0177	1.0152	1.0133	1.0114	1.0109	1.0118	1.0093	1.0089	1.0092	1.0120	1.0125	1.0164	1.0227	1.0243	1.0232	1.0206	1.0196	1.0178	1.0170	1.0156
1958		1.0629	1.0352	1.0259	1.0196	1.0174	1.0134	1.0136	1.0124	1.0112	1.0097	1.0092	1.0105	1.0111	1.0113	1.0141	1.0155	1.0236	1.0252	1.0221	1.0223	1.0197	1.0177	1.0173	1.0165
1959		1.0605	1.0387	1.0255	1.0229	1.0179	1.0154	1.0119	1.0129	1.0120	1.0091	1.0093	1.0089	1.0088	1.0114	1.0133	1.0167	1.0236	1.0218	1.0235	1.0208	1.0192	1.0177	1.0171	1.0148
1960		1.0646	1.0332	1.0258	1.0197	1.0182	1.0142	1.0125	1.0116	1.0093	1.0110	1.0088	1.0087	1.0090	1.0103	1.0115	1.0174	1.0250	1.0263	1.0238	1.0211	1.0171	1.0174	1.0142	1.0157
1961		1.0634	1.0348	1.0235	1.0218	1.0174	1.0141	1.0132	1.0111	1.0103	1.0116	1.0093	1.0089	1.0102	1.0108	1.0110	1.0160	1.0259	1.0259	1.0223	1.0209	1.0180	1.0171	1.0168	1.0151
1962		1.0651	1.0322	1.0252	1.0218	1.0164	1.0147	1.0126	1.0099	1.0116	1.0093	1.0103	1.0088	1.0094	1.0103	1.0119	1.0172	1.0222	1.0256	1.0218	1.0210	1.0179	1.0173	1.0158	1.0156
1963		1.0583	1.0331	1.0258	1.0195	1.0175	1.0121	1.0135	1.0118	1.0113	1.0089	1.0103	1.0099	1.0101	1.0100	1.0121	1.0151	1.0245	1.0246	1.0217	1.0213	1.0183	1.0183	1.0183	1.0173
1964		1.0596	1.0369	1.0243	1.0224	1.0190	1.0163	1.0136	1.0130	1.0101	1.0100	1.0088	1.0095	1.0101	1.0111	1.0114	1.0160	1.0243	1.0237	1.0221	1.0210	1.0196	1.0198	1.0179	1.0176
1965		1.0678	1.0370	1.0302	1.0231	1.0178	1.0158	1.0138	1.0120	1.0098	1.0093	1.0092	1.0083	1.0099	1.0125	1.0116	1.0152	1.0240	1.0235	1.0226	1.0221	1.0202	1.0195	1.0204	1.0207
1966		1.0634	1.0380	1.0250	1.0212	1.0190	1.0174	1.0132	1.0114	1.0108	1.0099	1.0090	1.0097	1.0096	1.0115	1.0137	1.0166	1.0224	1.0208	1.0208	1.0214	1.0217	1.0205	1.0194	1.0197
1967		1.0658	1.0396	1.0265	1.0212	1.0184	1.0172	1.0142	1.0118	1.0104	1.0094	1.0102	1.0097	1.0096	1.0124	1.0138	1.0177	1.0225	1.0224	1.0230	1.0229	1.0217	1.0219	1.0221	1.0198
1968		1.0670	1.0361	1.0257	1.0217	1.0199	1.0150	1.0134	1.0110	1.0096	1.0112	1.0098	1.0093	1.0109	1.0120	1.0135	1.0162	1.0223	1.0257	1.0243	1.0248	1.0224	1.0226	1.0222	1.0200
1969		1.0627	1.0354	1.0307	1.0216	1.0179	1.0139	1.0128	1.0113	1.0125	1.0092	1.0105	1.0091	1.0103	1.0118	1.0146	1.0161	1.0223	1.0250	1.0228	1.0235	1.0237	1.0226	1.0224	1.0201
1970		1.0602	1.0382	1.0288	1.0193	1.0166	1.0145	1.0115	1.0104	1.0112	1.0098	1.0092	1.0092	1.0099	1.0116	1.0131	1.0163	1.0226	1.0241	1.0243	1.0241	1.0237	1.0226	1.0225	1.0202
1971		1.0639	1.0371	1.0287	1.0201	1.0178	1.0137	1.0130	1.0124	1.0095	1.0099	1.0093	1.0101	1.0096	1.0125	1.0141	1.0160	1.0240	1.0263	1.0229	1.0242	1.0237	1.0226	1.0226	1.0204
1972		1.0649	1.0360	1.0273	1.0198	1.0163	1.0146	1.0124	1.0127	1.0097	1.0105	1.0099	1.0108	1.0116	1.0110	1.0140	1.0174	1.0250	1.0257	1.0229	1.0244	1.0238	1.0227	1.0228	1.0205
1973		1.0660	1.0321	1.0240	1.0208	1.0167	1.0154	1.0126	1.0100	1.0089	1.0088	1.0097	1.0111	1.0105	1.0132	1.0149	1.0188	1.0261	1.0259	1.0229	1.0245	1.0238	1.0227	1.0228	1.0207
1974		1.0611	1.0322	1.0257	1.0201	1.0165	1.0142	1.0119	1.0118	1.0106	1.0105	1.0114	1.0093	1.0126	1.0120	1.0147	1.0205	1.0263	1.0260	1.0230	1.0245	1.0239	1.0228	1.0229	1.0207
1975		1.0615	1.0350	1.0268	1.0205	1.0173	1.0148	1.0113	1.0112	1.0105	1.0104	1.0093	1.0102	1.0122	1.0120	1.0158	1.0207	1.0266	1.0262	1.0231	1.0246	1.0239	1.0228	1.0231	1.0209
1976		1.0590	1.0351	1.0250	1.0213	1.0159	1.0139	1.0131	1.0111	1.0122	1.0097	1.0096	1.0101	1.0106	1.0138	1.0160	1.0208	1.0269	1.0263	1.0231	1.0247	1.0240	1.0230	1.0232	1.0210
1977		1.0674	1.0329	1.0234	1.0210	1.0149	1.0159	1.0127	1.0107	1.0094	1.0088	1.0104	1.0115	1.0119	1.0138	1.0160	1.0210	1.0272	1.0265	1.0232	1.0248	1.0240	1.0229	1.0233	1.0211
1978		1.0577	1.0350	1.0243	1.0178	1.0169	1.0140	1.0132	1.0106	1.0104	1.0103	1.0102	1.0101	1.0120	1.0139	1.0161	1.0211	1.0275	1.0266	1.0232	1.0249	1.0241	1.0229	1.0234	1.0212
1979		1.0624	1.0347	1.0226	1.0207	1.0157	1.0129	1.0127	1.0101	1.0093	1.0105	1.0116	1.0101	1.0120	1.0139	1.0162	1.0212	1.0278	1.0268	1.0233	1.0251	1.0242	1.0229	1.0234	1.0213
1980		1.0631	1.0358	1.0235	1.0179	1.0155	1.0132	1.0103	1.0142	1.0114	1.0106	1.0116	1.0101	1.0121	1.0139	1.0162	1.0214	1.0281	1.0269	1.0234	1.0252	1.0241	1.0230	1.0235	1.0215
1981		1.0703	1.0362	1.0276	1.0198	1.0171	1.0152	1.0150	1.0118	1.0110	1.0106	1.0116	1.0101	1.0121	1.0139	1.0163	1.0216	1.0283	1.0270	1.0234	1.0254	1.0242	1.0230	1.0236	1.0217
1982		1.0634	1.0346	1.0292	1.0211	1.0175	1.0148	1.0154	1.0121	1.0109	1.0107	1.0116	1.0102	1.0122	1.0139	1.0163	1.0217	1.0286	1.0272	1.0234	1.0255	1.0242	1.0231	1.0237	1.0219
1983		1.0745	1.0401	1.0289	1.0213	1.0167	1.0156	1.0121	1.0121	1.0110	1.0107	1.0116	1.0101	1.0122	1.0139	1.0164	1.0218	1.0290	1.0274	1.0235	1.0256	1.0243	1.0231	1.0238	1.0221
1984		1.0790	1.0385	1.0278	1.0217	1.0203	1.0173	1.0120	1.0120	1.0109	1.0106	1.0117	1.0101	1.0123	1.0140	1.0164	1.0220	1.0293	1.0275	1.0236	1.0257	1.0243	1.0232	1.0239	1.0222
1985		1.0708	1.0405	1.0323	1.0239	1.0171	1.0172	1.0119	1.0120	1.0109	1.0106	1.0117	1.0101	1.0123	1.0140	1.0164	1.0222	1.0295	1.0277	1.0236	1.0258	1.0244	1.0233	1.0241	1.0223
1986		1.0695	1.0404	1.0312	1.0202	1.0169	1.0173	1.0117	1.0119	1.0109	1.0106	1.0117	1.0101	1.0123	1.0141	1.0166	1.0223	1.0298	1.0278	1.0237	1.0259	1.0244	1.0233	1.0241	1.0225
1987		1.0694	1.0416	1.0282	1.0199	1.0168	1.0173	1.0115	1.0119	1.0109	1.0107	1.0118	1.0101	1.0123	1.0142	1.0166	1.0225	1.0301	1.0279	1.0237	1.0260	1.0245	1.0233	1.0243	1.0227
1988		1.0705	1.0360	1.0277	1.0198	1.0168	1.0172	1.0115	1.0119	1.0109	1.0107	1.0118	1.0101	1.0123	1.0143	1.0167	1.0227	1.0305	1.0281	1.0237	1.0261	1.0245	1.0234	1.0243	1.0228
1989		1.0763	1.0356	1.0272	1.0196	1.0166	1.0172	1.0115	1.0119	1.0109	1.0106	1.0118	1.0101	1.0123	1.0143	1.0167	1.0229	1.0307	1.0281	1.0238	1.0262	1.0245	1.0234	1.0243	1.0230
1990		1.0743	1.0346	1.0268	1.0195	1.0165	1.0173	1.0114	1.0118	1.0108	1.0106	1.0119	1.0101	1.0124	1.0144	1.0168	1.0230	1.0311	1.0282	1.0238	1.0263	1.0245	1.0235	1.0245	1.0231

COHORT YEAR	AGE																							
	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
1922	1.0124	1.0106	1.0102	1.0096	1.0087	1.0073	1.0072	1.0080	1.0074	1.0077	1.0084	1.0087	1.0089	1.0101	1.0108	1.0116	1.0124	1.0143	1.0151	1.0158	1.0174	1.0191	1.0217	1.0236
1923	1.0117	1.0104	1.0099	1.0086	1.0078	1.0082	1.0078	1.0080	1.0076	1.0080	1.0085	1.0088	1.0098	1.0105	1.0116	1.0123	1.0129	1.0150	1.0160	1.0164	1.0180	1.0208	1.0224	1.0236
1924	1.0104	1.0099	1.0084	1.0076	1.0077	1.0075	1.0068	1.0072	1.0072	1.0081	1.0085	1.0085	1.0095	1.0106	1.0113	1.0121	1.0137	1.0147	1.0153	1.0166	1.0186	1.0196	1.0223	1.0232
1925	1.0097	1.0091	1.0072	1.0075	1.0073	1.0073	1.0069	1.0074	1.0074	1.0077	1.0084	1.0087	1.0095	1.0108	1.0111	1.0125	1.0130	1.0144	1.0163	1.0181	1.0189	1.0209	1.0221	1.0242
1926	1.0091	1.0078	1.0075	1.0076	1.0075	1.0072	1.0071	1.0076	1.0075	1.0083	1.0089	1.0094	1.0101	1.0111	1.0118	1.0130	1.0133	1.0156	1.0171	1.0178	1.0194	1.0219	1.0232	1.0247
1927	1.0078	1.0075	1.0069	1.0073	1.0073	1.0070	1.0067	1.0078	1.0077	1.0081	1.0091	1.0096	1.0104	1.0115	1.0121	1.0129	1.0137	1.0160	1.0173	1.0182	1.0211	1.0220	1.0243	1.0250
1928	1.0071	1.0067	1.0066	1.0070	1.0070	1.0068	1.0070	1.0070	1.0079	1.0085	1.0090	1.0093	1.0106	1.0114	1.0122	1.0131	1.0151	1.0159	1.0171	1.0188	1.0216	1.0222	1.0233	1.0242
1929	1.0071	1.0066	1.0066	1.0069	1.0067	1.0066	1.0066	1.0077	1.0076	1.0086	1.0087	1.0098	1.0100	1.0109	1.0120	1.0137	1.0142	1.0157	1.0175	1.0182	1.0200	1.0216	1.0237	1.0253
1930	1.0081	1.0070	1.0071	1.0074	1.0075	1.0072	1.0076	1.0081	1.0087	1.0089	1.0098	1.0103	1.0102	1.0122	1.0129	1.0145	1.0155	1.0169	1.0183	1.0202	1.0211	1.0230	1.0249	1.0265
1931	1.0068	1.0077	1.0071	1.0067	1.0073	1.0078	1.0076	1.0087	1.0084	1.0089	1.0099	1.0102	1.0107	1.0118	1.0126	1.0143	1.0154	1.0176	1.0184	1.0199	1.0207	1.0220	1.0247	1.0267
1932	1.0077	1.0069	1.0070	1.0070	1.0072	1.0078	1.0077	1.0082	1.0083	1.0096	1.0094	1.0098	1.0107	1.0118	1.0127	1.0138	1.0160	1.0174	1.0182	1.0198	1.0210	1.0229	1.0248	1.0256
1933	1.0072	1.0072	1.0074	1.0074	1.0081	1.0079	1.0080	1.0084	1.0091	1.0090	1.0096	1.0106	1.0116	1.0118	1.0137	1.0152	1.0163	1.0178	1.0196	1.0209	1.0225	1.0241	1.0243	1.0258
1934	1.0076	1.0078	1.0073	1.0076	1.0084	1.0083	1.0080	1.0085	1.0087	1.0097	1.0106	1.0112	1.0125	1.0133	1.0141	1.0159	1.0162	1.0185	1.0192	1.0213	1.0236	1.0240	1.0260	1.0258
1935	1.0071	1.0080	1.0071	1.0076	1.0083	1.0087	1.0090	1.0084	1.0087	1.0102	1.0100	1.0114	1.0123	1.0125	1.0139	1.0150	1.0170	1.0184	1.0193	1.0216	1.0219	1.0236	1.0253	1.0273
1936	1.0077	1.0073	1.0080	1.0080	1.0083	1.0080	1.0085	1.0089	1.0086	1.0094	1.0106	1.0114	1.0119	1.0128	1.0139	1.0148	1.0158	1.0183	1.0190	1.0210	1.0225	1.0233	1.0250	1.0264
1937	1.0080	1.0082	1.0075	1.0083	1.0086	1.0085	1.0084	1.0096	1.0093	1.0096	1.0110	1.0120	1.0130	1.0126	1.0140	1.0145	1.0169	1.0180	1.0198	1.0206	1.0221	1.0231	1.0247	1.0269
1938	1.0083	1.0083	1.0085	1.0087	1.0076	1.0083	1.0086	1.0099	1.0101	1.0107	1.0121	1.0125	1.0132	1.0136	1.0153	1.0166	1.0174	1.0171	1.0200	1.0216	1.0239	1.0246	1.0259	1.0274
1939	1.0094	1.0086	1.0086	1.0085	1.0087	1.0091	1.0097	1.0104	1.0099	1.0116	1.0120	1.0123	1.0140	1.0150	1.0170	1.0174	1.0185	1.0193	1.0188	1.0213	1.0235	1.0243	1.0260	1.0269
1940	1.0087	1.0086	1.0080	1.0079	1.0084	1.0089	1.0088	1.0089	1.0097	1.0108	1.0113	1.0127	1.0127	1.0148	1.0151	1.0157	1.0167	1.0173	1.0195	1.0205	1.0230	1.0248	1.0253	1.0278
1941	1.0079	1.0084	1.0080	1.0085	1.0083	1.0084	1.0088	1.0101	1.0101	1.0103	1.0109	1.0119	1.0134	1.0136	1.0141	1.0166	1.0173	1.0180	1.0191	1.0201	1.0229	1.0242	1.0254	1.0267
1942	1.0093	1.0083	1.0087	1.0088	1.0086	1.0096	1.0089	1.0099	1.0109	1.0110	1.0113	1.0125	1.0139	1.0148	1.0142	1.0170	1.0172	1.0185	1.0203	1.0220	1.0235	1.0255	1.0276	1.0277
1943	1.0089	1.0094	1.0092	1.0091	1.0097	1.0100	1.0090	1.0106	1.0113	1.0119	1.0131	1.0137	1.0144	1.0153	1.0159	1.0165	1.0174	1.0194	1.0210	1.0226	1.0247	1.0250	1.0280	1.0279
1944	1.0097	1.0098	1.0095	1.0092	1.0106	1.0112	1.0113	1.0113	1.0112	1.0128	1.0134	1.0137	1.0139	1.0147	1.0167	1.0168	1.0190	1.0202	1.0221	1.0236	1.0251	1.0266	1.0283	1.0281
1945	1.0103	1.0101	1.0100	1.0112	1.0108	1.0111	1.0117	1.0127	1.0124	1.0134	1.0143	1.0143	1.0151	1.0165	1.0175	1.0200	1.0191	1.0220	1.0233	1.0250	1.0268	1.0270	1.0286	1.0283
1946	1.0097	1.0101	1.0108	1.0093	1.0100	1.0108	1.0101	1.0117	1.0129	1.0121	1.0124	1.0131	1.0135	1.0152	1.0173	1.0188	1.0190	1.0205	1.0216	1.0226	1.0272	1.0273	1.0290	1.0285
1947	1.0111	1.0110	1.0106	1.0106	1.0105	1.0103	1.0118	1.0119	1.0126	1.0128	1.0140	1.0144	1.0153	1.0157	1.0170	1.0183	1.0214	1.0215	1.0229	1.0230	1.0276	1.0277	1.0293	1.0288
1948	1.0138	1.0128	1.0125	1.0131	1.0135	1.0135	1.0129	1.0141	1.0146	1.0153	1.0168	1.0155	1.0178	1.0198	1.0204	1.0221	1.0222	1.0259	1.0232	1.0232	1.0280	1.0280	1.0296	1.0290
1949	1.0128	1.0143	1.0147	1.0129	1.0139	1.0147	1.0135	1.0139	1.0147	1.0154	1.0166	1.0184	1.0190	1.0184	1.0210	1.0228	1.0255	1.0263	1.0235	1.0235	1.0285	1.0283	1.0299	1.0292
1950	1.0146	1.0144	1.0142	1.0153	1.0161	1.0138	1.0150	1.0162	1.0153	1.0166	1.0184	1.0177	1.0198	1.0203	1.0218	1.0222	1.0261	1.0268	1.0238	1.0238	1.0289	1.0287	1.0302	1.0294
1951	1.0141	1.0141	1.0145	1.0139	1.0141	1.0147	1.0147	1.0159	1.0166	1.0152	1.0159	1.0187	1.0200	1.0208	1.0231	1.0225	1.0265	1.0272	1.0240	1.0242	1.0293	1.0290	1.0304	1.0296
1952	1.0143	1.0154	1.0154	1.0141	1.0150	1.0158	1.0147	1.0155	1.0159	1.0181	1.0184	1.0192	1.0211	1.0222	1.0235	1.0229	1.0271	1.0276	1.0243	1.0244	1.0297	1.0293	1.0307	1.0298
1953	1.0164	1.0166	1.0164	1.0145	1.0145	1.0150	1.0152	1.0160	1.0164	1.0184	1.0195	1.0209	1.0221	1.0226	1.0239	1.0233	1.0275	1.0281	1.0246	1.0247	1.0301	1.0297	1.0311	1.0300
1954	1.0155	1.0158	1.0160	1.0148	1.0151	1.0142	1.0160	1.0173	1.0189	1.0205	1.0199	1.0216	1.0224	1.0229	1.0243	1.0237	1.0280	1.0285	1.0250	1.0250	1.0305	1.0300	1.0314	1.0302
1955	1.0154	1.0154	1.0144	1.0157	1.0157	1.0161	1.0156	1.0174	1.0189	1.0188	1.0202	1.0220	1.0229	1.0234	1.0247	1.0241	1.0285	1.0290	1.0253	1.0253	1.0309	1.0303	1.0316	1.0303
1956	1.0150	1.0166	1.0158	1.0168	1.0146	1.0170	1.0177	1.0183	1.0189	1.0181	1.0205	1.0223	1.0233	1.0238	1.0251	1.0245	1.0289	1.0294	1.0256	1.0255	1.0313	1.0306	1.0319	1.0306
1957	1.0164	1.0159	1.0167	1.0164	1.0157	1.0171	1.0178	1.0170	1.0181	1.0183	1.0208	1.0228	1.0237	1.0242	1.0256	1.0249	1.0294	1.0298	1.0259	1.0258	1.0318	1.0310	1.0322	1.0308
1958	1.0162	1.0160	1.0144	1.0165	1.0155	1.0165	1.0180	1.0193	1.0185	1.0186	1.0212	1.0232	1.0241	1.0245	1.0260	1.0253	1.0298	1.0302	1.0262	1.0261	1.0322	1.0313	1.0325	1.0310
1959	1.0163	1.0155	1.0155	1.0161	1.0166	1.0171	1.0185	1.0196	1.0187	1.0188	1.0216	1.0236	1.0245	1.0250	1.0264	1.0256	1.0303	1.0307	1.0265	1.0264	1.0326	1.0317	1.0328	1.0312
1960	1.0152	1.0163	1.0158	1.0166	1.0177	1.0171	1.0188	1.0200	1.0191	1.0191	1.0219	1.0240	1.0250	1.0254	1.0268	1.0260	1.0308	1.0312	1.0269	1.0267	1.0330	1.0320	1.0331	1.0315
1961	1.0152	1.0169	1.0166	1.0172	1.0190	1.0174	1.0191	1.0203	1.0194	1.0194	1.0223	1.0244	1.0254	1.0259	1.0272	1.0263	1.0313	1.0316	1.0272	1.0269	1.0334	1.0323	1.0335	1.0317
1962	1.0159	1.0156	1.0157	1.0181	1.0193	1.0176	1.0194	1.0206	1.0197	1.0197	1.0226	1.0248	1.0258	1.0263	1.0276	1.0267	1.0318	1.0321	1.0275	1.0272	1.0338	1.0327	1.0337	1.0319
1963	1.0167	1.0165	1.0185	1.0183	1.0196	1.0179	1.0197	1.0209	1.0200	1.0199	1.0230	1.0252	1.0262	1.0268	1.0281	1.0271	1.0323	1.0326	1.0278	1.0275	1.0342	1.0331	1.0340	1.0321
1964	1.0185	1.0202	1.0187	1.0185	1.0199	1.0182	1.0200	1.0212	1.0203	1.0203	1.0233	1.0255	1.0267	1.0272	1.0285	1.0275	1.0328	1.0330	1.0281	1.0278	1.0347	1.0334	1.0343	1.0323
1965	1.0196	1.0205	1.0188	1.0187	1.0201	1.0185	1.0204	1.0215	1.0206	1.0206	1.0236	1.0260	1.0270	1.0275	1.0289	1.0279	1.0333	1.0334	1.0284	1.0281	1.0351	1.0337	1.0346	1.0325
1966	1.0198	1.0207	1.0191	1.0189	1.0205	1.0187	1.0207	1.0218	1.0209	1.0209	1.0240	1.0264	1.0275	1.0280	1.0294	1.0282	1.0338	1.0339	1.0287	1.0284	1.0355	1.0341	1.0349	1.0328
1967	1.0199	1.0209	1.0192	1.0191	1.0207	1.0190	1.0210	1.0222	1.0212	1.0213	1.0244	1.0268	1.0279	1.0284	1.0298	1.0286	1.0342	1.0343	1.0290	1.0287	1.0359	1.0345	1.0352	1.0330
1968	1.0201	1.0211	1.0194	1.0193	1.0210	1.0192	1.0213	1.0225	1.0216	1.0216	1.0248	1.0272	1.0283	1.0288	1.0302	1.0290	1.0347	1.0347	1.0293	1.0290	1.0363	1.0348	1.0355	1.0332
1969	1.0203	1.0214	1.0196	1.0196	1.0212	1.0195	1.0216	1.0228	1.0219	1.0219	1.0251	1.0276	1.0288	1.0291	1.0307	1.0294	1.0352	1.0351	1.0296	1.0293	1.0368	1.0352	1.0359	1.0333
1970	1.0204	1.0216	1.0198	1.0198	1.0214	1.0197	1.0219	1.0231	1.0222	1.0223	1.0254	1.0280	1.0292	1.0295	1.0311	1.0298	1.0357	1.0355	1.0299	1.0296	1.0372	1.0356	1.0362	1.0335
1971	1.0206	1.0218	1.0200	1.0200	1.0217	1.0200	1.0222	1.0234	1.0225	1.0226	1.0258	1.0283	1.0297	1.0299	1.0315	1.0303	1.0362	1.0359	1.0302	1.0298	1.0375	1.0359	1.0365	1.0337
1972	1.0208	1.0220	1.0202	1.0203	1.0220	1.0202	1.0225	1.0237	1.0228	1.0229	1.0262	1.0288	1.0301	1.0303	1.0319	1.0307	1.0367	1.0363	1.0305	1.0301	1.0379	1.0363	1.0368	1.0339
1973	1.0209	1.0223	1.0204	1.0205	1.0222	1.0205	1.0228	1.0240	1.0232	1.0232	1.0266	1.0292	1.0304	1.0307	1.0324	1.0311	1.0372	1.0368	1.0308	1.0304	1.0384	1.0366	1.0371	1.0341
1974	1.0211	1.0225	1.0207	1.0207	1.0225	1.0207	1.0232	1.0244	1.0235	1.0235	1.0270	1.0296	1.0309	1.0312	1.0328	1.0315	1.0377	1.0372	1.0311	1.0307	1.0387	1.0370	1.0374	1.0343
1975	1.0213	1.0227	1.0209	1.0210	1.0227	1.0210	1.0234	1.0247	1.0238	1.0238	1.0273	1.0300	1.0313	1.0315	1.0333	1.0318	1.0382	1.0377	1.0314	1.0310	1.0392	1.0373	1.0377	1.0346
1976	1.0215	1.0230	1.0211	1.0212	1.0230	1.0213	1.0237	1.0250	1.0241	1.0241	1.0277	1.0304	1.0317	1.0319	1.0337	1.0323	1.0387	1.0382	1.0317	1.0313	1.0395	1.0377	1.0380	1.0348
1977	1.0216	1.0233	1.0212	1.0214	1.0233	1.0215	1.0240	1.0253	1.0244	1.0243	1.0280	1.0308	1.0321	1.0323	1.0341	1.0327	1.0391	1.0387	1.0321	1.0316	1.0400	1.0380	1.0383	1.0350
1978	1.0218	1.0235	1.0214	1.0216	1.0236	1.0218	1.0243	1.0256	1.0247	1.0247	1.0284	1.0313	1.0325	1.0327	1.0345	1.0331	1.0396	1.0390	1.0324	1.0319	1.0404	1.0383	1.0386	1.0352
1979	1.0220	1.0237	1.0216	1.0218	1.0239	1.0221	1.0246	1.0259	1.0251	1.0250	1.0288	1.0317	1.0329	1.0331	1.0350	1.0334	1.0401	1.0394	1.0327	1.0321	1.0409	1.0387	1.0390	1.0354
1980	1.0222	1.0239	1.0218	1.0220	1.0241	1.0224	1.0249	1.0262	1.0254	1.0253	1.0291	1.0321	1.0334	1.0335	1.0353	1.0338	1.0406	1.0398	1.0331	1.0324	1.0413	1.0390	1.0393	1.0357
1981	1.0224	1.0242	1.0220	1.0222	1.0244	1.0227	1.0252	1.0266	1.0257	1.0256	1.0294	1.0324	1.0339	1.0339	1.0357	1.0341	1.0411	1.0403	1.0334	1.0327	1.0417	1.0393	1.0396	1.0359
1982	1.0225	1.0243	1.0222	1.0225	1.0247	1.0230	1.0254	1.0269	1.0260	1.0259	1.0298	1.0328	1.0343	1.0344	1.0362	1.0345	1.0416	1.0407	1.0337	1.0331	1.0421	1.0396	1.0399	1.0361
1983	1.0227	1.0246	1.0225	1.0227	1.0248	1.0233	1.0257	1.0272	1.0263	1.0262	1.0302	1.0332	1.0348	1.0348	1.0366	1.0349	1.0421	1.0412	1.0341	1.0333	1.0425	1.0400	1.0402	1.0363
1984	1.0229	1.0248	1.0227	1.0229	1.0251	1.0236	1.0261	1.0275	1.0266	1.0265	1.0306	1.0336	1.0353	1.0352	1.0370	1.0352	1.0426	1.0416	1.0344	1.0336	1.0430	1.0403	1.0405	1.0365
1985	1.0231	1.0250	1.0229	1.0231	1.0254	1.0238	1.0263	1.0278	1.0269	1.0268	1.0309	1.0340	1.0357	1.0356	1.0375	1.0356	1.0431	1.0420	1.0347	1.0339	1.0434	1.0406	1.0408	1.0367
1986	1.0233	1.0253	1.0231	1.0233	1.0256	1.0241	1.0267	1.0281	1.0273	1.0272	1.0313	1.0344	1.0362	1.0360	1.0380	1.0360	1.0437	1.0425	1.0350	1.0342	1.0438	1.0410	1.0411	1.0368
1987	1.0235	1.0255	1.0233	1.0235	1.0259	1.0243	1.0270	1.0284	1.0276	1.0275	1.0316	1.0348	1.0366	1.0365	1.0385	1.0363	1.0442	1.0428	1.0353	1.0345	1.0442	1.0414	1.0414	1.0370
1988	1.0236	1.0258	1.0236	1.0237	1.0262	1.0246	1.0273	1.0288	1.0279	1.0279	1.0320	1.0353	1.0371	1.0369	1.0389	1.0366	1.0447	1.0433	1.0356	1.0347	1.0446	1.0417	1.0417	1.0372
1989	1.0239	1.0260	1.0237	1.0239	1.0265	1.0248	1.0276	1.0290	1.0282	1.0283	1.0325	1.0357	1.0375	1.0374	1.0394	1.0370	1.0453	1.0438	1.0359	1.0350	1.0450	1.0421	1.0420	1.0374
1990	1.0241	1.0262	1.0239	1.0242	1.0267	1.0251	1.0280	1.0294	1.0285	1.0286	1.0328	1.0360	1.0379	1.0378	1.0399	1.0373	1.0457	1.0443	1.0363	1.0353	1.0454	1.0425	1.0423	1.0376

Yellow shaded cells are estimated loss development factors for a simulation.

CURRICULUM VITAE

Şirzat Çetinkaya was born in 1981. He graduated from Hacettepe University, Department of Statistics in 2004. He commenced his MSc studies at the same time both at the Middle East Technical University, Financial Mathematics Department and Hacettepe University, Actuarial Sciences Department. He completed his studies with thesis in 2007. He was a research assistant at Actuarial Sciences Department at Hacettepe University between 2004 and 2008. He has worked as an actuary at different private companies since 2008 in İstanbul.