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ON THE ROOT FUNCTIONS OF ORDINARY DIFFERENTIAL OPERATORS

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CONTENTS

ÖZET

ADİ DİFERANSİYEL OPERATÖRLERİN KÖK FONKSİYONLARI

Bu tezin ana amacı, $L_2[0,1]$ uzayı içinde

$$l(y) = -y'' + q(x)\, y$$

diferansiyel ifadesi ve genel regüler olan fakat güçlü regüler olmayan sınır koşulları ile üretilen, öz eşlenik olmayan ikinci mertebeden Sturm-Liouville operatörünün kök fonksiyonları sistemini ve Riesz tabanı özelliğini incelemektir; burada q, [0,1] kümesi üzerinde kompleks değerli toplanabilir bir fonksiyondur.

Bu amaçla ilk önce, bu operatörlerin özdeğerleri ve özfonksiyonları için, $q \in L_1[0,1]$ durumunda ve q potansiyelinin mutlak sürekli fonksiyon olduğu durumda, ince asimptotik formüller inşa edilmiştir. Daha sonra, bu formüller kullanılarak, q potansiyeli üzerinde, genel regüler sınır koşullarına sahip Sturm-Liouville operatörünün kök fonksiyonları sisteminin Riesz tabanı oluşturmamasını sağlayan açık koşullar bulunmuştur.

Ayrıca, genel regüler sınır koşullarına sahip, öz eşlenik olmayan ikinci mertebeden Sturm-Liouville operatörünün küçük öz değerlerine nümerik yöntemler ile yaklaşımda bulunulmuştur. Son olarak da, hata analizi verilip, bazı nümerik örnekler sunulmuştur.

Anahtar sözcükler: Asimptotik formüller, Regüler sınır koşulları, Riesz tabanı, Küçük özdeğerlerin nümerik yaklaşımı.

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SUMMARY

ON THE ROOT FUNCTIONS OF ORDINARY DIFFERENTIAL OPERATORS

The main objective of this thesis is to investigate the system of the root functions and the Riesz basis property of the second order non-self-adjoint Sturm-Liouville operator generated in $L_2[0,1]$ by the differential expression

$$l(y) = -y'' + q(x)\,y$$

where q is a complex-valued summable function on [0,1], and general regular boundary conditions that are not strongly regular.

To this end, first we construct subtle asymptotic formulas for the eigenvalues and eigenfunctions of these operators for both cases $q \in L_1[0,1]$ and q is an absolutely continuous function. Then using these formulas we find explicit conditions on the potential q such that the system of the root functions of the Sturm-Liouville operator with general regular boundary conditions does not form a Riesz basis.

Also, we estimate the small eigenvalues of the second order non-self-adjoint Sturm-Liouville operators with general regular boundary conditions by the numerical methods. Finally, we give the error estimations and present some numerical examples.

Key Words: Asymptotic formulas, Regular boundary conditions, Riesz basis, Numerical estimations of the eigenvalues.

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1 INTRODUCTION AND PRELIMINARY FACTS

1.1 Introduction

Non-self-adjoint differential operators arise in the theory of open resonators, in problems of inelastic scattering, in problems of mathematical physics, when the Fourier method is used. The first works concerned with these operators were by G. Birkhoff [7-11], Ya.D. Tamarkin [67-69] in the beginning of the 20th century.

In this thesis we consider the operators generated in $L_2[0,1]$ by the differential expression

$$l\left(y\right)=-y''+q(x)y \tag{1.1}$$

and regular boundary conditions that are not strongly regular. Note that, if the boundary conditions are strongly regular, then the root functions (eigenfunctions and associated functions) form a Riesz basis (this result was proved independently in [24], [36] and [47]). In the case when an operator is associated with the regular but not strongly regular boundary conditions, the root functions generally do not form even a usual basis. However, Shkalikov [64, 65] proved that they can be combined in pairs, so that the corresponding 2-dimensional subspaces form a Riesz basis of subspaces. Note that the boundary conditions are strongly regular if and only if all large eigenvalues are far from each other. [65] This easify to investigate the perturbation theory and Riesz basis property. If the boundary conditions are not strongly regular then the eigenvalues are pairwise very close to each other. This situation complicates the investigation of the Riesz basis property. Therefore the regular cases which are not strongly regular are still investigated. Only the special cases, the periodic and antiperiodic problems, were investigated in detail. There are some interesting results [33-35] about the basis properties of the higher order differential operators with some regular boundary conditions. Besides, there are some important investigations about the Sturm-Liouville operators with singular potentials in [49, 50, 57-63]. Our aim is to consider the Riesz basis property for the regular boundary conditions in general form. We discuss only the second order differential operators for the case when the potential is from $L_1[0,1]$.

To describe the results of this thesis and preliminary results let us classify all regular boundary conditions that are not strongly regular for the second order differential operators. One can readily see from the pages 62-63 of [48] that all regular boundary conditions that are not strongly regular can be written in the form

$$a_1y_0' + b_1y_1' + a_0y_0 + b_0y_1 = 0,$$
$$c_0y_0 + d_0y_1 = 0, \tag{1.2}$$

if

$$b_1c_0 + a_1d_0 \neq 0 \tag{1.3}$$

and $\theta_0^2 - 4\theta_1\theta_{-1} = 0$, where , $a_i, b_i, c_0, d_0,\ i = 0, 1$, are complex numbers and θ_0, θ_1 and θ_{-1} are defined by

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = w_1(b_1c_0 + a_1d_0)\left(s + \frac{1}{s}\right) + 2(a_1c_0 + b_1d_0)w_1,\ \forall s \in \mathbb{C}\backslash\{0\}. \tag{1.4}$$

From (1.4) we obtain, $\theta_{-1} = \theta_1 = w_1(b_1c_0 + a_1d_0)$, $\theta_0 = 2(a_1c_0 + b_1d_0)w_1$, and hence the equality $\theta_0^2 - 4\theta_1\theta_{-1} = 0$ implies that

$$4\omega_1^2\left[(a_1c_0 + b_1d_0)^2 - (b_1c_0 + a_1d_0)^2\right] = 0,$$

that is, $(a_1^2 - b_1^2)(c_0^2 - d_0^2) = 0$ which means that at least one of the following conditions holds:

$$a_1 = \pm b_1,\ c_0 = \pm d_0.$$

First suppose that $a_1 = (-1)^\sigma b_1$, where $\sigma = 0, 1$. This with (1.3) implies that both a_1 and b_1 are not zero and at least one of c_0 and d_0 is not zero. If $c_0 \neq 0$, then (1.2) can be written in the form

$$y_0' + (-1)^\sigma y_1' + \alpha_1 y_1 = 0,$$
$$y_0 + \alpha_2 y_1 = 0, \tag{1.5}$$

where $\alpha_1 = \dfrac{b_0}{a_1} - \dfrac{a_0 d_0}{a_1 c_0}$, $\alpha_2 = \dfrac{d_0}{c_0}$, $a_1, c_0 \neq 0$ and $\alpha_2 \neq -(-1)^\sigma$ due to (1.3).

Similarly, if $d_0 \neq 0$, then (1.2) can be transformed to

$$y_0' + (-1)^\sigma y_1' + \alpha_3 y_0 = 0,$$
$$\alpha_4 y_0 + y_1 = 0, \tag{1.6}$$

where $\alpha_3 = \dfrac{a_0}{a_1} - \dfrac{b_0 c_0}{a_1 d_0}$, $\alpha_4 = \dfrac{c_0}{d_0}$, $a_1, d_0 \neq 0$ and by (1.3) $\alpha_4 \neq -(-1)^\sigma$.

Now suppose that $d_0 = (-1)^\sigma c_0$. Arguing as in the reductions of (1.5) and (1.6) we arrive at the boundary conditions

$$y_0' + \beta_1 y_1' + \beta_2 y_1 = 0,$$
$$y_0 + (-1)^\sigma y_1 = 0, \tag{1.7}$$

where $\beta_1 = \dfrac{b_1}{a_1}$, $\beta_2 = \left(\dfrac{b_0}{a_1} \mp \dfrac{a_0}{a_1}\right)$, $a_1, c_0 \neq 0$ and

$$\beta_1 \neq -(-1)^\sigma \tag{1.8}$$

and the boundary conditions

$$\beta_3 y_0' + y_1' + \beta_4 y_1 = 0,$$
$$y_0 + (-1)^\sigma y_1 = 0, \tag{1.9}$$

where $\beta_3 = \dfrac{a_1}{b_1}$, $\beta_4 = \dfrac{b_0}{b_1} \mp \dfrac{a_0}{b_1}$, $b_1, c_0 \neq 0$ and

$$\beta_3 \neq -(-1)^\sigma \tag{1.10}$$

for $\sigma = 0, 1$.

One can verify in the standard way that, the boundary conditions (1.5) and (1.6), are the adjoint boundary conditions to (1.9) and (1.7), respectively, where $\alpha_3 = -(-1)^\sigma \overline{\beta_2}$, $\alpha_4 = \overline{\beta_1}$ and $\alpha_1 = (-1)^\sigma \overline{\beta_4}$, $\alpha_2 = \overline{\beta_3}$.

Thus to consider all regular boundary conditions that are not strongly regular it is enough to investigate the boundary conditions (1.7) and (1.9). Note that these boundary conditions depend on two parameters. Let us describe the special cases that were investigated.

Case (a) The cases $\beta_2, \beta_4 = 0$, $\beta_1, \beta_3 = (-1)^\sigma$ in (1.7), (1.9) for $\sigma = 1$ and $\sigma = 0$ coincide with the periodic and antiperiodic boundary conditions respectively. These boundary conditions are the ones more commonly studied. We will briefly describe some historical developments related to the Riesz basis property of the root functions of the boundary value problems for such boundary conditions. Since the results for the periodic and antiperiodic problems can be found in a similar way, we will focus only on the periodic problem. The antiperiodic problem is similar to the periodic one. One of the important results was obtained by Kerimov and Mamedov [32]. They proved that if $q \in C^4[0,1]$ and $q(1) \neq q(0)$, then the root functions of the operator $L(q)$ generated by (1.1) and the periodic boundary conditions form a Riesz basis in $L_2[0,1]$. This result remains valid for the case when $q(x)$ is a smooth potential, satisfying

$$q^{(k)}(0) = q^{(k)}(1), \quad \forall\, k = 0, 1, ..., s-1$$

and $q^{(s)}(0) \neq q^{(s)}(1)$ for arbitrary $s \geq 1$.(see Corollary 2 of [66]).

The first result in terms of the Fourier coefficients of the potential q was obtained by Dernek and Veliev [18]. Makin [39] extended this result for the larger class of functions. Shkalikov and Veliev obtained in [66] more general results which cover all results about periodic and antiperiodic boundary conditions discussed above.

The other interesting results about the periodic and antiperiodic boundary conditions were obtained in [20-23, 26, 31, 37, 43, 44, 46, 70-72].

Case (b) The cases $\beta_2, \beta_4 \neq 0$ and $\beta_1, \beta_3 = (-1)^\sigma$ were investigated in [40, 41] and it was proved that the system of the root functions of the Sturm-Liouville operator corresponding to this case is a Riesz basis in $L_2(0,1)$ (see Theorem 1 of [40, 41]).

Case (c) The cases $\beta_2, \beta_4 = 0$ and $\beta_1, \beta_3 \neq (-1)^\sigma$ were investigated in [40, 41] and in Chapter 2 of this thesis. The results of Chapter 2 have been published in Boundary Value Problems (see [51]). To explain the difference between the two results, first let us give the following definition.

We call the boundary conditions (1.7) and (1.9) for $\beta_2, \beta_4 \neq 0$ and $\beta_1, \beta_3 \neq (-1)^\sigma$ which are different from the special cases (a) , (b) and (c) as the *general regular boundary conditions* that are not strongly regular. Note that in any case $\beta_1, \beta_3 \neq -(-1)^\sigma$ by (1.8) and (1.10). For the case (c) and general boundary conditions Makin [40, 41] proved that the systems of the root functions of the Sturm-Liouville operators corresponding to these cases are Riesz bases in $L_2(0,1)$ if and only if all large eigenvalues are multiple. Note that this result is not effective, since the conditions are given in implicit form and can not be verified for concrete potentials. In this thesis we find explicit conditions on the potential such that the system of the root functions of the Sturm-Liouville operator corresponding to each of the cases (c) and general boundary conditions does not form a Riesz basis.

Since we are interested also in the numerical estimations, let us mention the literature about the investigations of the small eigenvalues. There are a lot of papers about the estimations of the small eigenvalues for the strongly regular boundary conditions (see for example [13, 16, 25, 53-56]). In the numerical results about the regular but not strongly regular boundary conditions, the estimations of the small eigenvalues for the periodic and antiperiodic boundary conditions are the most widely-studied ones as (see for example [2, 6, 12, 17, 19, 29, 30, 42, 74]). There are also many papers concerning with the estimations of the small eigenvalues for the boundary conditions

$$a_{1,1}y(0) + a_{1,2}y'(0) = 0,$$

$$a_{2,1}y(1) + a_{2,2}y'(1) = 0,$$

where

$$a_{1,1}^2 + a_{1,2}^2 \neq 0, \ a_{2,1}^2 + a_{2,2}^2 \neq 0,$$

which contain some strongly regular boundary conditions including the Dirichlet and Neumann boundary conditions as special cases (see for example [1, 3, 5, 13-16, 25, 53-56]).

We are interested in the numerical estimations of the small eigenvalues for the regular boundary conditions that are not strongly regular in the case (c) There are only two papers [4, 28] containing the estimations of the small eigenvalues for such boundary conditions. In [4], C. J. Goh, K. L. Teo and R. P. Agarwal gave the estimations of the small eigenvalues in the case when the potential is continuous and there is no any example for the boundary conditions we are interested in. In [28], M. H. Annaby and R. M. Asharabi, estimated the small eigenvalues for the general boundary conditions but their numerical example concerning with the case (c) is for very simple potential. In this thesis we use a method different from the methods of the papers [4] and [28], to get subtle estimations for the small eigenvalues when the potential is in the form $q(x) = \sum_{k=1}^{\infty} q_k \cos 2\pi kx$. Note that, for this potential, it is impossible to compute the exact values of the eigenvalues. It consists of the transformation of the original problem that researching the eigenvalues to a new problem concerned with finding the root of some functions. The method used is inspired from [18].

The thesis is divided into four chapters. The first chapter presents preliminary definitions and formulations of some results to be used in Chapter 2 and Chapter 3.

In Chapter 2 of this thesis we find explicit conditions on the potential such that the system of the root functions of the Sturm-Liouville operator corresponding to the case (c) does not form a Riesz basis. Namely we prove that if

$$\lim_{n\to\infty} \frac{\ln|n|}{ns_{2n}} = 0, \tag{1.11}$$

where $q \in L_1(0,1)$, $s_n = (q, \sin 2\pi nt)$ and $(.,.)$ is the inner product in $L_2[0,1]$, then the large eigenvalues of each of the operators corresponding to these cases are simple for $\sigma = 1$. Moreover, if there exists a sequence $\{n_k\}$ such that (1.11) holds when n is replaced by n_k, then the root functions of these operators do not form a Riesz basis. Similarly, if the condition

$$\lim_{n\to\infty} \frac{\ln|n|}{ns_{2n+1}} = 0 \tag{1.11a}$$

holds instead of (1.11), then the same statements continue to hold for $\sigma = 0$.

In Chapter 3 of this thesis we find explicit conditions on potential q such that the system of the root functions of the Sturm-Liouville operator generated by (1.1) and the general regular boundary conditions does not form a Riesz basis. The main results of Chapter 3 can be described as follows:

Let $T_1^\sigma(q)$ and $T_2^\sigma(q)$ be the Sturm-Liouville operators associated by the boundary conditions (1.7) and (1.9), respectively. Without loss of generality we assume that

$$\int_0^1 q(t)\,dt = 0.$$

First we prove that if $q \in L_1[0,1]$ and

$$\int_0^1 \sin(2\pi nt)\, q(t)\,dt = o\left(\frac{1}{n}\right) \tag{1.12}$$

then the large eigenvalues of $T_1^\sigma(q)$ and $T_2^\sigma(q)$ for $\sigma = 1$, are simple. Moreover if there exists a subsequence $\{n_k\}$ such that (1.12) holds whenever n is replaced by n_k, then the system of the root functions of each operators $T_1^\sigma(q)$ and $T_2^\sigma(q)$ for $\sigma = 1$, does not form a Riesz basis. The same results continue to hold for $T_1^{\sigma}(q)$ and $T_2^{\sigma}(q)$ for $\sigma = 0$, if instead of (1.12) the condition

$$\int_0^1 \sin((2n+1)\pi t)\, q\,(t)\, dt = o\left(\frac{1}{n}\right) \tag{1.12a}$$

holds.

Now, if the potential q is an absolutely continuous function and

$$q\,(0) + (-1)^{\sigma}\, q\,(1) \neq \frac{2\beta_2^2}{1-\beta_1^2} \tag{1.13}$$

then the large eigenvalues of $T_1^{\sigma}(q)$ for $\sigma = 0, 1$ are simple and the system of the root functions of $T_1^{\sigma}(q)$ does not form a Riesz basis. Similarly, if the condition

$$q\,(0) + (-1)^{\sigma}\, q\,(1) \neq \frac{2\beta_4^2}{\beta_3^2 - 1} \tag{1.14}$$

holds instead of (1.13), then the same results remain valid for $T_2^{\sigma}(q)$ for $\sigma = 0, 1$. Moreover we obtain subtle asymptotic formulas for the eigenvalues and eigenfunctions for the operators $T_1^{\sigma}(q)$ and $T_2^{\sigma}(q)$ for both cases $q \in L_1[0,1]$ and q is an absolutely continuous function.

Note that the general cases we investigate in Chapter 3 are essentially different from the case (c) as the method of investigations and obtained results. The results of Chapter 3 have been submitted for publication. (see [52])

In Chapter 4 of this thesis we estimate the small eigenvalues of the operators defined in Chapter 2 by the numerical methods. Finally we give the error estimations and some numerical examples.

1.2 Preliminary Facts

Let us begin by introducing some basic definitions and formulations of some results.

1.2.1 Main Definitions and Formulations of Some Results

In this section, our aim is to present basic definitions and results which will be used in the subsequent chapters of this thesis.

A linear differential expression means an expression of the form

$$\widetilde{l}(y) = p_0(x)\, y^{(n)} + p_1(x)\, y^{(n-1)} + \ldots + p_n(x)\, y.$$

The functions $p_0(x), p_1(x), p_2(x), \ldots, p_n(x)$ are called the *coefficients* and the number n is called the *order* of the differential expression. [48]

We consider only the case $n = 2$ and the differential expression (1.1). Therefore we give the definitions and results only for the case $n = 2$.

We denote the values of the function y and its first derivatives at the boundary points 0 and 1 of the interval $[0, 1]$ by

$$y_0, y_0',; y_1, y_1'. \tag{1.15}$$

Let $U(y)$ be a linear form in the variables (1.15):

$$U(y) = \alpha_0 y_0 + \alpha_1 y_0' + \beta_0 y_1 + \beta_1 y_1'.$$

If two such forms $U_\nu(y)$ have been specified, $\nu = 1, 2$, and if the conditions

$$U_\nu(y) = 0,\ \ \nu = 1, 2 \tag{1.16}$$

are imposed on the functions y we call these the *boundary conditions* which y must satisfy.

Let $\mathcal{D}$ be the set of the functions $y \in L_2[0, 1]$ satisfying (1.16) such that $y' \in AC[0, 1]$ and $l(y) \in L_2[0, 1]$,where $AC[0, 1]$ is the set of all absolutely continuos functions on $[0, 1]$.

Definition 1.1 *The operator T is called the linear differential operator generated by the differential expression $l(y)$ and the boundary conditions (1.16) if $T(y) = l(y)$ for all $y \in \mathcal{D}$.*

Definition 1.2 *The problem of determining a function $y \in \mathcal{D}(T)$ which satisfies the conditions*

$$l(y) = 0$$

and (1.16) is called the homogeneous boundary-value problem.

We note that if T is the operator which is generated by the differential expression $l(y)$ and the boundary conditions (1.16), then the homogeneous boundary-value problem amounts to finding, in the domain of definition $\mathcal{D}$ of the operator T, a function y for which T vanishes. [48]

Definition 1.3 *The operator T^* is called the adjoint operator to T if the equation*

$$(Ty, z) = (y, T^*z)$$

holds for all y in the domain of definition of T and all z in the domain of definition of T^*, where $(.,.)$ denotes the inner product in $L_2[0,1]$.

An operator T is self-adjoint if $T = T^*$.

Definition 1.4 *A number λ is called an eigenvalue of an operator T if there exists in the domain of definition of the operator T a function $y \neq 0$ such that*

$$Ty = \lambda y. \tag{1.17}$$

The function y is called the eigenfunction of the operator T corresponding to the eigenvalue λ.

The operator T may be generated by the differential expression $l(y)$ and the boundary conditions (1.16). Since an eigenfunction y must belong to the domain of definition of the operator T, it must satisfy the conditions (1.16). Moreover, $Ty = l(y)$, and therefore (1.17) is equivalent to

$$l(y) = \lambda y. \tag{1.18}$$

Hence, the *eigenvalues* of an operator T are those values of the parameter λ for which the homogeneous boundary-value problem

$$l(y) = \lambda y,\ U_\nu(y) = 0,\ \nu = 1, 2 \tag{1.19}$$

has non-trivial solutions; each of these non-trivial solutions is an *eigenfunction* belonging to λ.

Consider the differential equation (1.18). It can be easily shown that there exists a set of linearly independent solutions which are entire in the parameter λ. Let this set be $\{y_1(x,\lambda), y_2(x,\lambda)\}$. The general solution of (1.18) and also the solution of the homogeneous boundary-value problem can be expressed in the form

$$y = c_1 y_1(x,\lambda) + c_2 y_2(x,\lambda)$$

where c_1, c_2 are certain constants. [48]

On imposing the two linearly independent boundary conditions (1.16), one gets a system of two linear, homogeneous equations in the two unknowns c_1, c_2

$$c_1 U_\nu(y_1) + c_2 U_\nu(y_2) = 0,\ \nu = 1, 2 \tag{1.20}$$

for the determination of the constants c_1, c_2.

Hence we have the following results (see [45]):

(a) The homogeneous boundary-value problem (1.20) has a non-trivial solution if and only if the determinant of the coefficient matrix vanishes.

(b) Since the functions $\{y_1(x,\lambda), y_2(x,\lambda)\}$ are entire functions of the parameter λ, this determinant, being a linear combination of entire functions in λ is itself entire.

The eigenvalues (or characteristic values) of the boundary value problem (1.19) are determined by the zeros of the *characteristic determinant* $\Delta(\lambda)$, which has the form

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}.$$

Definition 1.5 *An eigenvalue λ of the boundary-value problem (1.19) is said to have multiplicity p if λ is a root of multiplicity p of the function $\Delta(\lambda)$. An eigenvalue λ of (1.19) is called simple if λ is a simple zero of $\Delta(\lambda)$.*

Theorem 1.1 *If λ is an eigenvalue of multiplicity p of the operator T, then $\overline{\lambda}$ is an eigenvalue of the adjoint operator T^* and has the same multiplicity. [48]*

By the preceding discussions, $\Delta(\lambda)$ is an integral, analytic function of λ and the following theorems hold [48]:

Theorem 1.2 *The eigenvalues of the operator T are the zeros of the function $\Delta(\lambda)$. If $\Delta(\lambda)$ vanishes identically, then any number λ is an eigenvalue of the operator T.*

If, however, $\Delta(\lambda)$ is not identically zero, the operator T has at most denumerably many eigenvalues, and these eigenvalues can have no finite limit point.

If, in particular the function $\Delta(\lambda)$ has no zeros at all, then the operator T has no eigenvalues.

Definition 1.6 *Denote by $\psi_{n,0}(x) \equiv \psi_n(x)$ the eigenfunction of the operator T corresponding to the eigenvalue λ_n. The function $\psi_{n,p}(x)$ for $p = 1, 2, \ldots, m_p$ is said to be an associated function of order p corresponding to the same eigenvalue λ_n and the* *eigenfunction* $\psi_{n,0}(x)$ *if all the functions* $\psi_{n,p}(x)$ *satisfy the following equations*

$$(T-\lambda_n)\,\psi_{n,0}(x)=0,$$

$$(T-\lambda_n)\,\psi_{n,p}(x)=\psi_{n,p-1}(x)\,,\ \ p=1,2,\ldots,m_p,$$

where m_p *is called the length of the system of associated functions.*

An eigenfunction $\psi_n(x)$ is said to have *multiplicity* m if there is a system of functions associated with $\psi_n(x)$ of length $(m-1)$ but no system of length m. [48]

1.2.2 Some Auxiliary Statements

In this section, we present some auxiliary statements which will be used in the subsequent chapters of this thesis.

We denote by ω_1,ω_2 the different two roots of -1 arranged in an order in each case to suit later requirements.

By the transformation $\lambda=-\rho^2$, we divide the complex ρ-plane into 4 sectors S_k, $k=0,1,2,3$ defined by

$$\frac{k\pi}{2}\le\arg\rho\le\frac{(k+1)\,\pi}{2}. \tag{1.21}$$

For each of the sectors S_k the numbers ω_1,ω_2 can be ordered in such a way that, for all $\rho\in S_k$, the inequality

$$\mathcal{R}\,(\rho\omega_1)\le\mathcal{R}\,(\rho\omega_2) \tag{1.22}$$

holds, where $\mathcal{R}\,(z)$ means the real part of z. [48]

It can be obtained more general domains from the sectors S_k by a translation $\rho\to\rho-c$, where c is a fixed complex number. These new sectors with their vertices at the point $\rho=-c$ will correspondingly be denoted by T_k, $k=0,1,2,3$. Taking into account of the way in which the T_k are produced from the S_k by translation, we see that, for $\rho \in T_k$, the inequality

$$\mathcal{R}\left(\left(\rho+c\right)\omega_1\right) \leq \mathcal{R}\left(\left(\rho+c\right)\omega_2\right) \tag{1.23}$$

holds for a suitable ordering of the numbers ω_1, ω_2. In the sequel we shall let ρ vary in a fixed domain T_k and so we shall write simply S and T instead of S_k and T_k. The order of the numbers ω_1, ω_2 will be such that for $\rho \in T$ the inequality (1.23) is valid. [48]

The homogeneous, linear differential equation $y'' + \rho^2 y = 0$ has, for $\rho \neq 0$, the fundamental system

$$e^{i\rho x}, e^{-i\rho x}.$$

Now the following theorem gives us the asymptotic estimates for the fundamental set of solutions $y_1\left(x,\rho\right), y_2\left(x,\rho\right)$ and their first order derivatives of the inhomogeneous equation

$$y'' + \rho^2 y = q\left(x\right) y \tag{1.24}$$

as $|\rho| \to \infty$ in the sectors (1.21).

Theorem 1.3 *If the function $q\left(x\right)$ is an arbitrary summable function in the interval $[0,1]$, then the equation (1.24) has, for each region T of the complex plane, two linearly independent solutions y_1, y_2 which are regular for $\rho \in T$ and for sufficiently large $|\rho|$, and which, with their derivatives, can be expressed in the form*

$$\begin{aligned} y_k &= e^{\rho\omega_k x}\left[1 + O\left(\frac{1}{\rho}\right)\right], \\ \frac{dy_k}{dx} &= \rho e^{\rho\omega_k x}\left[\omega_k + O\left(\frac{1}{\rho}\right)\right], \end{aligned} \tag{1.25}$$

for $k = 1, 2$. [48]

For real and positive ρ it is often convenient to replace these solutions by the linear combinations of y_1, y_2:

$$\frac{y_1 + y_2}{2} = \cos \rho x + O\left(\frac{1}{\rho}\right),$$
$$\frac{y_1 - y_2}{2i} = \sin \rho x + O\left(\frac{1}{\rho}\right).$$

Consider the different systems $U_\nu(y) = 0$, $\nu = 1, 2$, of linear forms which define a given differential operator. If $y^{(k)}(0)$ or $y^{(k)}(1)$ appear explicitly in the form $U(y)$ but $y^{(\nu)}(0)$ and $y^{(\nu)}(1)$ do not, for any $\nu > k$, then we say that the form $U(y)$ has order k. From the way in which they are constructed the boundary conditions must have the form

$$U_\nu(y) = \alpha_\nu y^{(k_\nu)}(0) + \alpha_{\nu,0} y(0) + \beta_\nu y^{(k_\nu)}(1) + \beta_{\nu,0} y(1) = 0, \ \ \nu = 1, 2, \tag{1.26}$$

where $1 \geq k_1 \geq k_2 \geq 0$,and for each value of the suffix ν at least one of the numbers α_ν, β_ν is non-zero. [48]

Consider a fixed domain S_k; as before, we number ω_1, ω_2 so that, for $\rho \in S_k$, (1.22) holds.

Definition 1.7 *The boundary conditions (1.26) are said to be regular if the numbers θ_{-1} and θ_1 defined by the identity*

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s = \begin{vmatrix} (\alpha_1 + s\beta_1)\omega_1^{k_1} & (\alpha_1 + \frac{1}{s}\beta_1)\omega_2^{k_1} \\ (\alpha_2 + s\beta_2)\omega_1^{k_2} & (\alpha_2 + \frac{1}{s}\beta_2)\omega_2^{k_2} \end{vmatrix}$$

are different from zero.

This definition of regularity is independent of the choice of the region S for which the numbers ω_1, ω_2 were arranged in order.

Note that, since $\theta_0, \theta_1, \theta_{-1}$ depend only on the complex coefficients α_ν and β_ν $(\nu = 1, 2)$ of the highest order derivatives in (1.26), regularity also depends on α_ν and β_ν.

Definition 1.8 *The regular boundary conditions (1.26) are said to be strongly regular if* $\theta_0^2 - 4\theta_1\theta_{-1} \neq 0$.

The most general boundary conditions for $n = 2$ have the form

$$\begin{aligned} a_1 y_0' + b_1 y_1' + a_0 y_0 + b_0 y_1 &= 0, \\ c_1 y_0' + d_1 y_1' + c_0 y_0 + d_0 y_1 &= 0. \end{aligned} \tag{1.27}$$

The conditions (1.27) are regular in just these cases:

1. $a_1 d_1 - b_1 c_1 \neq 0$;

2. $a_1 = b_1 = c_1 = d_1 = 0$, $a_0 d_0 - b_0 c_0 \neq 0$;

3. $a_1 d_1 - b_1 c_1 = 0$, $|a_1| + |b_1| > 0$, $b_1 c_0 + a_1 d_0 \neq 0$.

In the first two cases $\theta_0 = 0$, $\theta_1 = -1$, $\theta_{-1} = 1$ and $\theta_0^2 - 4\theta_1\theta_{-1} = 4 \neq 0$, i.e., the boundary conditions are strongly regular. In the third case we can transform the conditions (1.27) so:

$$\begin{aligned} a_1 y_0' + b_1 y_1' + a_0 y_0 + b_0 y_1 &= 0, \\ c_0 y_0 + d_0 y_1 &= 0, \end{aligned}$$

$$\begin{aligned} \frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s &= \begin{vmatrix} (a_1 + s b_1)\,\omega_1 & -\left(a_1 + \frac{1}{s} b_1\right)\omega_1 \\ c_0 + s d_0 & c_0 + \frac{1}{s} d_0 \end{vmatrix} \\ &= \omega_1 \left(b_1 c_0 + a_1 d_0\right)\left(s + \frac{1}{s}\right) + 2\omega_1 \left(a_1 c_0 + b_1 d_0\right), \end{aligned}$$

$$\theta_1 = \theta_{-1} = \omega_1 \left(b_1 c_0 + a_1 d_0\right), \; \theta_0 = 2\omega_1 \left(a_1 c_0 + b_1 d_0\right);$$

the conditions are therefore regular if $b_1 c_0 + a_1 d_0 \neq 0$. [48]

For $\theta_0^2 - 4\theta_1\theta_{-1} = 0$, the following sequences are obtained:

$$\lambda_{k,j} = -(2k\pi)^2 \left\{ 1 + \frac{\mu \ln_0 \xi}{k\pi i} + O\left(\frac{1}{k^{3/2}}\right) \right\}, \ j = 1, 2. \tag{1.28}$$

where ξ is the double root, occurring in this case, of the equation

$$\theta_1 \xi^2 + \theta_0 \xi + \theta_{-1} = 0$$

(relative to the θ for the domain S_0). The upper or lower sign is to be taken according as $n = 4\nu$ or $n = 4\nu + 2$, respectively (Here $\ln_0 \xi$ is any fixed branch of the natural logarithm). [48]

Let y_1, y_2 be linearly independent solutions of the equation $l(y) + \rho^2 y = 0$ which satisfy the relations (1.25) in a certain domain T. An eigenfunction which belongs to a prescribed eigenvalue $\lambda = -\rho^2$ with $\rho \in T$ must be expressible as a linear combination of the functions y_1, y_2:

$$y = c_1 y_1 + c_2 y_2,$$

where the coefficients c_1, c_2 are non-trivial solutions of the system of homogeneous equations (1.21). For simplicity, we consider only a simple eigenvalue λ, for which the rank of the determinant $\Delta = \det [U_\nu (y_k)]$, $\nu, k = 1, 2$, is equal to 1. Then

$$y = \begin{vmatrix} y_1 & y_2 \\ U_2(y_1) & U_2(y_2) \end{vmatrix}$$

is an eigenfunction belonging to the eigenvalue λ. [48]

1.2.3 On the Riesz Basis

Let $\{\phi_j\}$ be an arbitrary orthonormal basis of the space $\mathfrak{D}$, and A some bounded and boundedly invertible linear operator. Then for any vector $f \in \mathfrak{D}$ one has

$$A^{-1} f = \sum_{j=1}^{\infty} \left(A^{-1} f, \phi_j\right) \phi_j = \sum_{j=1}^{\infty} \left(f, A^{*-1} \phi_j\right) \phi_j,$$

and consequently

$$f = \sum_{j=1}^{\infty} (f, \chi_j)\, \psi_j,$$

where

$$\psi_j = A\phi_j, \; \chi_j = A^{*-1}\phi_j \;\; (j = 1, 2, \ldots).$$

Obviously

$$\left(\psi_j, \chi_k\right) = \delta_{jk} \;\; (j = 1, 2, \ldots).$$

Therefore if

$$f = \sum_{j=1}^{\infty} c_j \psi_j, \tag{1.29}$$

then $c_j = \left(f, \chi_j\right)$ $(j = 1, 2, \ldots)$, i.e. the expansion (1.29) is unique.

Thus every bounded and boundedly invertible linear operator transforms any orthonormal basis into some other basis of the space $\mathfrak{D}$.

Definition 1.9 *A basis $\left\{\psi_j\right\}_1^{\infty}$ of the space $\mathfrak{D}$ which is obtained from an orthonormal basis by means of such a transformation is called a basis equivalent to an orthonormal basis (or a Riesz basis). In other words, $\left\{\psi_j\right\}_1^{\infty}$ is a Riesz basis if there exist a bounded and boundedly invertible linear operator A such that $\psi_j = A\phi_j$ for some orthonormal basis $\left\{\phi_j\right\}_1^{\infty}$. [27]*

We formulate a number of characteristic properties of Riesz bases. [27]

Theorem 1.4 *(N. K. Bari) The following assertions are equivalent.*

(i) The sequence $\left\{\psi_j\right\}_1^{\infty}$ forms a basis of the space $\mathfrak{D}$, equivalent to an orthonormal basis (i.e. $\left\{\psi_j\right\}_1^{\infty}$ is a Riesz basis.).

(ii) The sequence $\left\{\psi_j\right\}_1^{\infty}$ becomes an orthonormal basis of the space $\mathfrak{D}$ following the appropriate replacement of the scalar product $\langle f, g\rangle$ by some new one $\langle f, g\rangle_1$, topologically equivalent to the original one (i.e. if there exist positive constants c_1, c_2 such that $c_1 \langle f, f\rangle \leq \langle f, f\rangle_1 \leq c_2 \langle f, f\rangle$ $f \in \mathfrak{D}$.).

(iii) The sequence $\{\psi_j\}_1^\infty$ *is complete in* $\mathfrak{D}$*, and there exist positive constants* a_1, a_2 *such that for any positive integer* n *and any complex numbers* $\gamma_1, \gamma_2, \ldots, \gamma_n$ *one has*

$$a_2 \sum_{j=1}^{n} |\gamma_j|^2 \leq \left| \sum_{j=1}^{n} \gamma_j \psi_j \right|^2 \leq a_1 \sum_{j=1}^{n} |\gamma_j|^2 .$$

(iv) The sequence $\{\psi_j\}_1^\infty$ *is complete in* $\mathfrak{D}$*, and its Gram matrix* $\|(\psi_j, \psi_k)\|_1^\infty$ *generates a bounded invertible operator in the space* l_2*.*

(v) The sequence $\{\psi_j\}_1^\infty$ *is complete in* $\mathfrak{D}$*, there corresponds to it a complete biorthogonal sequence* $\{\chi_j\}_1^\infty$*, and for any* $f \in \mathfrak{D}$ *one has*

$$\sum_{j=1}^{\infty} |(f, \psi_j)|^2 < \infty, \quad \sum_{j=1}^{\infty} |(f, \chi_j)|^2 < \infty.$$

Definition 1.10 *A sequence* $\{\mathfrak{N}_k\}_1^\infty$ *of nonzero subspaces* $\mathfrak{N}_k \subset \mathfrak{D}$ *is said to be a basis (of subspaces) of the space* $\mathfrak{D}$*, if any vector* $x \in \mathfrak{D}$ *can be expanded in a unique way in a series of the form*

$$x = \sum_{k=1}^{\infty} x_k$$

where $x_k \in \mathfrak{N}_k \ (k = 1, 2, \ldots)$*.*

Theorem 1.5 *If the sequence of subspaces* $\{\mathfrak{N}_k\}_1^\infty$ *is a basis of the space* $\mathfrak{D}$ *equivalent to an orthogonal one, then any sequence* $\{\phi_k\}_1^\infty$*, obtained as the union of orthonormal bases of all the subspaces* $\mathfrak{N}_k \ (k = 1, 2, \ldots)$*, is a basis of the space* $\mathfrak{D}$ *equivalent to orthonormal one. [27]*

If the subspaces $\mathfrak{N}_k \ (k = 1, 2, \ldots)$ are one-dimensional, then they form a basis of the space $\mathfrak{D}$ if and only if unit vectors $\phi_k \in \mathfrak{N}_k \ (k = 1, 2, \ldots)$ form a vector basis of $\mathfrak{D}$. [27]

2 STURM-LIOUVILLE OPERATORS WITH SOME REGULAR BOUNDARY CONDITIONS

Let $T_1(q), T_2(q), T_3(q)$ and $T_4(q)$ be the operators generated in $L_2[0,1]$ by the differential expression (1.1) and the following boundary conditions:

$$y_0' + \beta y_1' = 0, \ \ y_0 - y_1 = 0, \tag{2.1}$$

$$y_0' + \beta y_1' = 0, \ \ y_0 + y_1 = 0, \tag{2.2}$$

$$y_0' - y_1' = 0, \ \ y_0 + \alpha y_1 = 0, \tag{2.3}$$

and

$$y_0' + y_1' = 0, \ \ y_0 + \alpha y_1 = 0 \tag{2.4}$$

respectively, where q is a complex-valued summable function on $[0,1]$, $\beta \neq \pm 1$ and $\alpha \neq \pm 1$.

In conditions (2.1), (2.2), (2.3) and (2.4) if $\beta = 1$, $\beta = -1$, $\alpha = 1$ and $\alpha = -1$ respectively, then any $\lambda \in \mathbb{C}$ is an eigenvalue of infinite multiplicity. In (2.1) and (2.3) if $\beta = -1$ and $\alpha = -1$ then they are periodic boundary conditions; In (2.2) and (2.4) if $\beta = 1$ and $\alpha = 1$ then they are antiperiodic boundary conditions.

We will focus only on the operator $T_1(q)$. The investigations of the operators $T_2(q), T_3(q)$ and $T_4(q)$ are similar. It is well-known that (see (47a) and (47b)) on page 65 of [48]) the eigenvalues of the operators $T_1(q)$ consist of the sequences $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}$ satisfying

$$\lambda_{n,j} = (2n\pi)^2 + O(n^{1/2}) \tag{2.5}$$

for $j = 1, 2$. From this formula one can easily obtain the following inequality

$$\left|\lambda_{n,j} - (2\pi k)^2\right| = |2(n-k)\pi| \, |2(n+k)\pi| + O(n^{\frac{1}{2}}) > n \tag{2.6}$$

for $j = 1, 2$; $k \neq n$; $k = 0, 1, ...$, and $n \geq N$, where N denotes a sufficiently large positive integer, that is, $N \gg 1$.

The eigenvalues of the operator $T_1(0)$ are $\lambda_n = (2\pi n)^2$ for $n = 0, 1, \ldots$ The eigenvalue 0 is simple and the corresponding eigenfunction is 1. The eigenvalues $\lambda_n = (2\pi n)^2$ for $n = 1, 2, \ldots$ are double and the corresponding eigenfunctions and associated functions are

$$y_n(x) = \cos 2\pi nx \quad \& \quad \phi_n(x) = \left(\frac{\beta}{1+\beta} - x\right)\frac{\sin 2\pi nx}{4\pi n}, \tag{2.7}$$

respectively. Note that for any constant c, $\phi_n(x) + cy_n(x)$ is also an associated function corresponding to λ_n, since one can easily verify that it satisfies the equation and boundary conditions for the associated functions. It can be shown that the adjoint operator $T_1^*(0)$ is associated with the boundary conditions:

$$y_1 + \overline{\beta} y_0 = 0, \;\; y_1' - y_0' = 0.$$

It is easy to see that, 0 is a simple eigenvalue of $T_1^*(0)$ and the corresponding eigenfunction is $y_0^*(x) = x - \dfrac{1}{1+\overline{\beta}}$. The other eigenvalues $\lambda_n^* = (2\pi n)^2$ for $n = 1, 2, \ldots$, are double and the corresponding eigenfunctions and associated functions are

$$y_n^*(x) = \sin 2\pi nx \quad \& \quad \phi_n^*(x) = \left(x - \frac{1}{1+\overline{\beta}}\right)\frac{\cos 2\pi nx}{4\pi n} \tag{2.8}$$

respectively.

Let

$$\varphi_n(x) := \frac{16\pi n(\beta+1)}{\beta-1}\phi_n(x) = \frac{4(\beta+1)}{\beta-1}\left(\frac{\beta}{1+\beta} - x\right)\sin 2\pi nx \tag{2.9}$$

and

$$\varphi_n^*(x) := \frac{16\pi n\left(\overline{\beta}+1\right)}{\overline{\beta}-1}\phi_n^*(x) = \frac{4\left(\overline{\beta}+1\right)}{\overline{\beta}-1}\left(x - \frac{1}{1+\overline{\beta}}\right)\cos 2\pi nx \tag{2.10}$$

(see (2.7) and (2.8)). The system of the root functions of $T_1^*(0)$ can be written as $\{f_n : n \in \mathbb{Z}\}$, where

$$f_{-n} = \sin 2\pi nx,\ \forall n > 0\ \ \&\ \ f_n = \varphi_n^*(x),\ \forall n \geq 0. \tag{2.11}$$

One can easily verify that it forms a basis in $L_2[0,1]$ and the biorthogonal system $\{g_n : n \in \mathbb{Z}\}$ is the system of the root functions of $T_1(0)$, where

$$g_{-n} = \varphi_n(x), \forall n > 0\ \ \&\ \ g_n = \cos 2\pi nx, \forall n \geq 0, \tag{2.12}$$

since $(f_n, g_m) = \delta_{n,m}$.

2.1 The Asymptotic Formulas for the Eigenvalues and Eigenfunctions of $T_1(q)$

To obtain the asymptotic formulas for the eigenvalues $\lambda_{n,j}$ and the corresponding normalized eigenfunctions $\Psi_{n,j}(x)$ of $T_1(q)$ we use (2.6) and the well-known relations

$$(\lambda_{N,j} - (2\pi n)^2)(\Psi_{N,j}, \sin 2\pi nx) = (q\Psi_{N,j}, \sin 2\pi nx) \tag{2.13}$$

and

$$\left(\lambda_{N,j} - (2\pi n)^2\right)(\Psi_{N,j}, \varphi_n^*) - \gamma_1 n (\Psi_{N,j}, \sin 2\pi nx) = (q\Psi_{N,j}, \varphi_n^*), \tag{2.14}$$

where

$$\gamma_1 = \frac{16\pi(\beta+1)}{\beta-1},$$

which can be obtained by multiplying both sides of the equality

$$-(\Psi_{N,j})'' + q(x)\Psi_{N,j} = \lambda_{N,j}\Psi_{N,j}$$

by $\sin 2\pi nx$ and φ_n^* respectively. It follows from (2.13) and (2.14) that

$$(\Psi_{N,j}, \sin 2\pi nx) = \frac{(q\Psi_{N,j}, \sin 2\pi nx)}{\lambda_{N,j} - (2\pi n)^2};\ \ N \neq n, \tag{2.15}$$

$$\left(\Psi_{N,j},\varphi_n^*\right)=\frac{\gamma_1 n\left(q\Psi_{N,j},\sin 2\pi nx\right)}{\left(\lambda_{N,j}-(2\pi n)^2\right)^2}+\frac{\left(q\Psi_{N,j},\varphi_n^*\right)}{\lambda_{N,j}-(2\pi n)^2};\ N\neq n. \tag{2.16}$$

Moreover, we use the following relations

$$\left(\Psi_{N,j},\overline{q}\sin 2\pi nx\right)=\sum_{n_1=0}^{\infty}[\left(q\varphi_{n_1},\sin 2\pi nx\right)\left(\Psi_{N,j},\sin 2\pi n_1x\right)+ \tag{2.17}$$

$$+\left(q\cos 2\pi n_1x,\sin 2\pi nx\right)\left(\Psi_{N,j},\varphi_{n_1}^*\right)],$$

$$\left(\Psi_{N,j},\overline{q}\varphi_n^*\right)=\sum_{n_1=0}^{\infty}\left[\left(q\varphi_{n_1},\varphi_n^*\right)\left(\Psi_{N,j},\sin 2\pi n_1x\right)+\left(q\cos 2\pi n_1x,\varphi_n^*\right)\left(\Psi_{N,j},\varphi_{n_1}^*\right)\right], \tag{2.18}$$

$$\left|\left(q\Psi_{N,j},\sin 2\pi nx\right)\right|<4M, \tag{2.19}$$

$$\left|\left(q\Psi_{N,j},\varphi_n^*\right)\right|<4M, \tag{2.20}$$

for $N\gg 1$,where $M=\sup\left|q_n\right|$. These relations are obvious for $q\in L_2(0,1)$, since to obtain (2.17) and (2.18) we can use the decomposition of $\overline{q}\sin 2\pi nx$ and $\overline{q}\varphi_n^*$ by the basis (2.11). For $q\in L_1(0,1)$ see Lemma 1 of [70].

To obtain the asymptotic formulas for the eigenvalues and eigenfunctions we iterate (2.13) and (2.14) by using (2.17) and (2.18). First let us prove the following obvious asymptotic formulas, namely (2.24), for the eigenfunctions $\Psi_{n,j}$. The expansion of $\Psi_{n,j}$ by the basis (2.12) can be written in the form

$$\Psi_{n,j}=u_{n,j}\varphi_n\left(x\right)+v_{n,j}\cos 2\pi nx+h_{n,j}\left(x\right), \tag{2.21}$$

where

$$u_{n,j}=\left(\Psi_{n,j},\sin 2\pi nx\right),\ v_{n,j}=\left(\Psi_{n,j},\varphi_n^*\right), \tag{2.22}$$

$$h_{n,j}\left(x\right)=\sum_{\substack{k=0\\k\neq n}}^{\infty}\left[\left(\Psi_{n,j},\sin 2\pi kx\right)\varphi_k\left(x\right)+\left(\Psi_{n,j},\varphi_k^*\right)\cos 2\pi kx\right],$$

and $\varphi_n\left(x\right)$, $\varphi_n^*\left(x\right)$ are defined in (2.9) and (2.10), respectively. Using (2.15), (2.16), (2.19) and (2.20) one can readily see that, there exists a constant C such that

$$\sup |h_{n,j}(x)| \leq C \left(\sum_{k \neq n} \left(\frac{1}{|\lambda_{n,j} - (2\pi k)^2|} + \frac{n}{\left| \left(\lambda_{n,j} - (2\pi k)^2\right)^2 \right|} \right) \right) = O\left(\frac{\ln n}{n}\right). \tag{2.23}$$

Hence by (2.21) and (2.23) we obtain

$$\Psi_{n,j} = u_{n,j}\varphi_n(x) + v_{n,j}\cos 2\pi nx + O\left(\frac{\ln n}{n}\right). \tag{2.24}$$

Since $\Psi_{n,j}$ is normalized, we have

$$1 = \|\Psi_{n,j}\|^2 = (\Psi_{n,j}, \Psi_{n,j}) = |u_{n,j}|^2 \|\varphi_n(x)\|^2 + |v_{n,j}|^2 \|\cos 2\pi nx\|^2 +$$

$$+u_{n,j}\overline{v_{n,j}}(\varphi_n(x), \cos 2\pi nx) + v_{n,j}\overline{u_{n,j}}(\cos 2\pi nx, \varphi_n(x)) + O\left(\frac{\ln n}{n}\right) =$$

$$= \left(\frac{8}{3}\frac{|\beta|^2 - \mathrm{Re}\beta + 1}{|\beta - 1|^2}\right)|u_{n,j}|^2 + \frac{1}{2}|v_{n,j}|^2 + O\left(\frac{\ln n}{n}\right),$$

that is,

$$a|u_{n,j}|^2 + \frac{1}{2}|v_{n,j}|^2 = 1 + O\left(\frac{\ln n}{n}\right), \tag{2.25}$$

where

$$a = \frac{8}{3}\frac{|\beta|^2 - \mathrm{Re}\beta + 1}{|\beta - 1|^2}.$$

Note that $a \neq 0$, since $|\beta|^2 + 1 > |\beta|$ and by (2.25) we see that at least one of $u_{n,j}$ and $v_{n,j}$ is different from zero.

Now let us iterate (2.13). Using (2.17) in (2.13) we get

$$\left(\lambda_{n,j} - (2\pi n)^2\right)(\Psi_{n,j}, \sin 2\pi nx) =$$

$$= \sum_{n_1=0}^{\infty} \left[(q\varphi_{n_1}, \sin 2\pi nx)(\Psi_{n,j}, \sin 2\pi n_1 x) + (q\cos 2\pi n_1 x, \sin 2\pi nx)\left(\Psi_{n,j}, \varphi_{n_1}^*\right) \right].$$

Isolating the terms in the right-hand side of this equality containing the multiplicands $(\Psi_{n,j}, \sin 2\pi nx)$ and $(\Psi_{n,j}, \varphi_n^*)$ (i.e., the case $n_1 = n$), using (2.15) and (2.16) for the terms $(\Psi_{n,j}, \sin 2\pi n_1 x)$ and $\left(\Psi_{n,j}, \varphi_{n_1}^*\right)$, respectively (in the case $n_1 \neq n$), we obtain

$$\left[\lambda_{n,j} - (2\pi n)^2 - (q\varphi_n, \sin 2\pi nx)\right](\Psi_{n,j}, \sin 2\pi nx) - (q\cos 2\pi nx, \sin 2\pi nx)(\Psi_{n,j}, \varphi_n^*) =$$

$$= \sum_{\substack{n_1=0 \\ n_1 \neq n}}^{\infty} \left[\left(q\varphi_{n_1}, \sin 2\pi nx\right)(\Psi_{n,j}, \sin 2\pi n_1 x) + (q\cos 2\pi n_1 x, \sin 2\pi nx)\left(\Psi_{n,j}, \varphi_{n_1}^*\right)\right]$$

$$= \sum_{n_1} \left[a_1(\lambda_{n,j})(q\Psi_{n,j}, \sin 2\pi n_1 x) + b_1(\lambda_{n,j})\left(q\Psi_{n,j}, \varphi_{n_1}^*\right)\right],$$

where

$$a_1(\lambda_{n,j}) = \frac{\left(q\varphi_{n_1}, \sin 2\pi nx\right)}{\lambda_{n,j} - (2\pi n_1)^2} + \frac{\gamma_1 n_1 (q\cos 2\pi n_1 x, \sin 2\pi nx)}{\left(\lambda_{n,j} - (2\pi n_1)^2\right)^2},$$

$$b_1(\lambda_{n,j}) = \frac{(q\cos 2\pi n_1 x, \sin 2\pi nx)}{\lambda_{n,j} - (2\pi n_1)^2}.$$

Using (2.17) and (2.18) for the terms $(q\Psi_{n,j}, \sin 2\pi n_1 x)$ and $\left(q\Psi_{n,j}, \varphi_{n_1}^*\right)$ of the last summation we obtain

$$\left[\lambda_{n,j} - (2\pi n)^2 - (q\varphi_n, \sin 2\pi nx)\right](\Psi_{n,j}, \sin 2\pi nx) - (q\cos 2\pi nx, \sin 2\pi nx)(\Psi_{n,j}, \varphi_n^*) =$$

$$= \sum_{n_1} \left[a_1(\lambda_{n,j})(q\Psi_{n,j}, \sin 2\pi n_1 x) + b_1(\lambda_{n,j})\left(q\Psi_{n,j}, \varphi_{n_1}^*\right)\right]$$

$$= \sum_{n_1} a_1 \left(\sum_{n_2=0}^{\infty} \left[\left(q\varphi_{n_2}, \sin 2\pi n_1 x\right)(\Psi_{n,j}, \sin 2\pi n_2 x) + (q\cos 2\pi n_2 x, \sin 2\pi n_1 x)\left(\Psi_{n,j}, \varphi_{n_2}^*\right)\right]\right) +$$

$$+ \sum_{n_1} b_1 \left(\sum_{n_2=0}^{\infty} \left[\left(q\varphi_{n_2}, \varphi_{n_1}^*\right)(\Psi_{n,j}, \sin 2\pi n_2 x) + \left(q\cos 2\pi n_2 x, \varphi_{n_1}^*\right)\left(\Psi_{n,j}, \varphi_{n_2}^*\right)\right]\right).$$

Now isolating the terms for $n_2 = n$ we get

$$\left[\lambda_{n,j} - (2\pi n)^2 - (q\varphi_n, \sin 2\pi nx)\right](\Psi_{n,j}, \sin 2\pi nx) - (q\cos 2\pi nx, \sin 2\pi nx)(\Psi_{n,j}, \varphi_n^*) =$$

$$= \sum_{n_1} \left[a_1(q\varphi_n, \sin 2\pi n_1 x) + b_1\left(q\varphi_n, \varphi_{n_1}^*\right)\right](\Psi_{n,j}, \sin 2\pi nx) +$$

$$+ \sum_{n_1} \left[a_1(q\cos 2\pi nx, \sin 2\pi n_1 x) + b_1\left(q\cos 2\pi nx, \varphi_{n_1}^*\right)\right](\Psi_{n,j}, \varphi_n^*) +$$

$$+ \sum_{n_1, n_2} \left\{\left[a_1\left(q\varphi_{n_2}, \sin 2\pi n_1 x\right) + b_1\left(q\varphi_{n_2}, \varphi_{n_1}^*\right)\right](\Psi_{n,j}, \sin 2\pi n_2 x)\right\} +$$

$$+ \sum_{n_1, n_2} \left\{\left[a_1(q\cos 2\pi n_2 x, \sin 2\pi n_1 x) + b_1\left(q\cos 2\pi n_2 x, \varphi_{n_1}^*\right)\right]\left(\Psi_{n,j}, \varphi_{n_2}^*\right)\right\}.$$

Here and below the summations are taken under the conditions $n_i \neq n$ and $n_i = 0, 1, ...$ for $i = 1, 2, ...$ Introduce the notations

$$C_1 =: a_1, \ M_1 =: b_1,$$

$$C_2 =: a_1 a_2 + b_1 A_2 = C_1 a_2 + M_1 A_2, \ M_2 =: a_1 b_2 + b_1 B_2 = C_1 b_2 + M_1 B_2,$$

$$C_{k+1} =: C_k a_{k+1} + M_k A_{k+1}, \ M_{k+1} =: C_k b_{k+1} + M_k B_{k+1}; \ k = 1, 2, \ldots,$$

where

$$a_{k+1} = a_{k+1}(\lambda_{n,j}) = \frac{\left(q\varphi_{n_{k+1}}, \sin 2\pi n_k x\right)}{\lambda_{n,j} - (2\pi n_{k+1})^2} + \frac{\gamma_1 n_{k+1} (q \cos 2\pi n_{k+1} x, \sin 2\pi n_k x)}{\left(\lambda_{n,j} - (2\pi n_{k+1})^2\right)^2},$$

$$b_{k+1} = b_{k+1}(\lambda_{n,j}) = \frac{(q \cos 2\pi n_{k+1} x, \sin 2\pi n_k x)}{\lambda_{n,j} - (2\pi n_{k+1})^2},$$

$$A_{k+1} = A_{k+1}(\lambda_{n,j}) = \frac{\left(q\varphi_{n_{k+1}}, \varphi^*_{n_k}\right)}{\lambda_{n,j} - (2\pi n_{k+1})^2} + \frac{\gamma_1 n_{k+1} \left(q \cos 2\pi n_{k+1} x, \varphi^*_{n_k}\right)}{\left(\lambda_{n,j} - (2\pi n_{k+1})^2\right)^2},$$

$$B_{k+1} = B_{k+1}(\lambda_{n,j}) = \frac{\left(q \cos 2\pi n_{k+1} x, \varphi^*_{n_k}\right)}{\lambda_{n,j} - (2\pi n_{k+1})^2}.$$

Using these notations and repeating this iteration k times we get

$$\left[\lambda_{n,j} - (2\pi n)^2 - (q\varphi_n, \sin 2\pi n x) - \widetilde{A}_k(\lambda_{n,j})\right] (\Psi_{n,j}, \sin 2\pi n x) =$$
$$= \left[(q \cos 2\pi n x, \sin 2\pi n x) + \widetilde{B}_k(\lambda_{n,j})\right] (\Psi_{n,j}, \varphi^*_n(x)) + R_k, \tag{2.26}$$

where

$$\widetilde{A}_k(\lambda_{n,j}) = \sum_{m=1}^{k} \alpha_m(\lambda_{n,j}), \ \widetilde{B}_k(\lambda_{n,j}) = \sum_{m=1}^{k} \beta_m(\lambda_{n,j}),$$

$$\alpha_k(\lambda_{n,j}) = \sum_{n_1, \ldots, n_k} \left[C_k (q\varphi_n, \sin 2\pi n_k x) + M_k \left(q\varphi_n, \varphi^*_{n_k}\right)\right],$$

$$\beta_k(\lambda_{n,j}) = \sum_{n_1, \ldots, n_k} \left[C_k (q \cos 2\pi n x, \sin 2\pi n_k x) + M_k \left(q \cos 2\pi n x, \varphi^*_{n_k}\right)\right],$$

$$R_k = \sum_{n_1, \ldots, n_{k+1}} \left\{C_{k+1} (q\Psi_{n,j}, \sin 2\pi n_{k+1} x) + M_{k+1} \left(q\Psi_{n,j}, \varphi^*_{n_{k+1}}\right)\right\}.$$

It follows from (2.6), (2.19) and (2.20) that

$$\alpha_k(\lambda_{n,j}) = O\left(\left(\frac{\ln|n|}{n}\right)^k\right),\ \beta_k(\lambda_{n,j}) = O\left(\left(\frac{\ln|n|}{n}\right)^k\right),\ R_k = O\left(\left(\frac{\ln|n|}{n}\right)^{k+1}\right). \tag{2.27}$$

Therefore letting k tend to infinity, we obtain

$$\left[\lambda_{n,j} - (2\pi n)^2 - Q_n - A(\lambda_{n,j})\right] u_{n,j} = \left[P_n + B(\lambda_{n,j})\right] v_{n,j},$$

where

$$P_n = (q\cos 2\pi nx, \sin 2\pi nx),\ Q_n = (q\varphi_n, \sin 2\pi nx), \tag{2.28}$$

$$A(\lambda_{n,j}) = \sum_{m=1}^{\infty} \alpha_m(\lambda_{n,j}),\ B(\lambda_{n,j}) = \sum_{m=1}^{\infty} \beta_m(\lambda_{n,j})$$

and by (2.27) we have

$$A(\lambda_{n,j}) = O\left(\frac{\ln|n|}{n}\right),\ B(\lambda_{n,j}) = O\left(\frac{\ln|n|}{n}\right). \tag{2.29}$$

Thus, iterating (2.13) we obtain (2.26). Now iterating (2.14) instead of (2.13), using (2.18) and (2.17) and arguing as in the previous iteration, we get

$$\left[\lambda_{n,j} - (2\pi n)^2 - P_n^* - A_k'(\lambda_{n,j})\right] v_{n,j} = \left[\gamma_1 n + Q_n^* + B_k'(\lambda_{n,j})\right] u_{n,j} + R_k', \tag{2.30}$$

where

$$P_n^* = (q\cos 2\pi nx, \varphi_n^*),\ Q_n^* = (q\varphi_n, \varphi_n^*), \tag{2.31}$$

$$A_k'(\lambda_{n,j}) = \sum_{m=1}^{k} \alpha_m'(\lambda_{n,j}),\ B_k'(\lambda_{n,j}) = \sum_{m=1}^{k} \beta_m'(\lambda_{n,j}),$$

$$\alpha_k'(\lambda_{n,j}) = \sum_{n_1,\ldots,n_k} \left[\widetilde{C}_k(q\cos 2\pi nx, \sin 2\pi n_k x) + \widetilde{M}_k\left(q\cos 2\pi nx, \varphi_{n_k}^*\right)\right],$$

$$\beta_k'(\lambda_{n,j}) = \sum_{n_1,\ldots,n_k} \left[\widetilde{C}_k(q\varphi_n, \sin 2\pi n_k x) + \widetilde{M}_k\left(q\varphi_n, \varphi_{n_k}^*\right)\right],$$

$$R_k' = \sum_{n_1,\ldots,n_{k+1}} \left\{\widetilde{C}_{k+1}(q\Psi_{n,j}, \sin 2\pi n_{k+1}x) + \widetilde{M}_{k+1}\left(q\Psi_{n,j}, \varphi_{n_{k+1}}^*\right)\right\},$$

$$\widetilde{C}_{k+1} = \widetilde{C}_k a_{k+1} + \widetilde{M}_k A_{k+1},\ \widetilde{M}_{k+1} = \widetilde{C}_k b_{k+1} + \widetilde{M}_k B_{k+1};\ k = 1, 2, \ldots,$$

$$\widetilde{C}_1 = A_1(\lambda_{n,j}) = \frac{(q\varphi_{n_1}, \varphi_n^*)}{\lambda_{n,j} - (2\pi n_1)^2} + \frac{\gamma_1 n_1 (q\cos 2\pi n_1 x, \varphi_n^*)}{\left(\lambda_{n,j} - (2\pi n_1)^2\right)^2},$$

$$\widetilde{M}_1 = B_1(\lambda_{n,j}) = \frac{(q\cos 2\pi n_1 x, \varphi_n^*)}{\lambda_{n,j} - (2\pi n_1)^2}.$$

Similar to (2.27) one can verify that

$$\alpha_k'(\lambda_{n,j}) = O\left(\left(\frac{\ln|n|}{n}\right)^k\right), \; \beta_k'(\lambda_{n,j}) = O\left(\left(\frac{\ln|n|}{n}\right)^k\right), \; R_k' = O\left(\left(\frac{\ln|n|}{n}\right)^{k+1}\right). \tag{2.32}$$

Now letting k tend to infinity in (2.30), we obtain

$$\left[\lambda_{n,j} - (2\pi n)^2 - P_n^* - A'(\lambda_{n,j})\right] v_{n,j} = \left[\gamma_1 n + Q_n^* + B'(\lambda_{n,j})\right] u_{n,j},$$

where

$$A'(\lambda_{n,j}) = \sum_{m=1}^{\infty} \alpha_m'(\lambda_{n,j}), \; B'(\lambda_{n,j}) = \sum_{m=1}^{\infty} \beta_m'(\lambda_{n,j})$$

and by (2.32) we have

$$A'(\lambda_{n,j}) = O\left(\frac{\ln|n|}{n}\right), \; B'(\lambda_{n,j}) = O\left(\frac{\ln|n|}{n}\right). \tag{2.33}$$

To get some main results of this chapter we use the following system of equations, obtained above, with respect to $u_{n,j}$ and $v_{n,j}$

$$\left[\lambda_{n,j} - (2\pi n)^2 - Q_n - A(\lambda_{n,j})\right] u_{n,j} = \left[P_n + B(\lambda_{n,j})\right] v_{n,j}, \tag{2.34}$$

$$\left[\lambda_{n,j} - (2\pi n)^2 - P_n^* - A'(\lambda_{n,j})\right] v_{n,j} = \left[\gamma_1 n + Q_n^* + B'(\lambda_{n,j})\right] u_{n,j}, \tag{2.35}$$

where

$$Q_n = -\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + \frac{2(\beta+1)}{\beta-1}(xq, \cos 4\pi nx) - \frac{2\beta}{\beta-1}(q, \cos 4\pi nx) \tag{2.36}$$

$$= -\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + o(1), \tag{2.37}$$

$$P_n^* = \frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + \frac{2(\beta+1)}{\beta-1}(xq, \cos 4\pi nx) - \frac{2}{\beta-1}(q, \cos 4\pi nx) \tag{2.38}$$

$$= \frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + o(1), \tag{2.39}$$

$$P_n = \frac{1}{2}(q, \sin 4\pi nx) = o(1), \tag{2.40}$$

$$Q_n^* = 8\left(\frac{\beta_1+1}{\beta_1-1}\right)^2 \int_0^1 q(x)\left(\frac{\beta_1}{1+\beta_1} - x\right)\left(x - \frac{1}{1+\beta_1}\right)\sin 4\pi nx dx = o(1) \tag{2.41}$$

(see (2.28) and (2.31)). Note that (2.34), (2.35) with (2.29), (2.33) give

$$\left[\lambda_{n,j} - (2\pi n)^2 - Q_n + O\left(\frac{\ln|n|}{n}\right)\right] u_{n,j} = \left[P_n + O\left(\frac{\ln|n|}{n}\right)\right] v_{n,j}, \tag{2.42}$$

$$\left[\lambda_{n,j} - (2\pi n)^2 - P_n^* + O\left(\frac{\ln|n|}{n}\right)\right] v_{n,j} = \left[\gamma_1 n + Q_n^* + O\left(\frac{\ln|n|}{n}\right)\right] u_{n,j}. \tag{2.43}$$

Introduce the notations

$$\begin{aligned} c_n &= (q, \cos 2\pi nx),\ s_n = (q, \sin 2\pi nx), \\ c_{n,1} &= (xq, \cos 2\pi nx),\ s_{n,1} = (xq, \sin 2\pi nx), \\ c_{n,2} &= \left(x^2 q, \cos 2\pi nx\right),\ s_{n,2} = \left(x^2 q, \sin 2\pi nx\right). \end{aligned} \tag{2.44}$$

Then, by (2.36)-(2.41) and (2.44) we have

$$Q_n = -\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + \frac{2(\beta+1)}{\beta-1}c_{2n,1} - \frac{2\beta}{\beta-1}c_{2n}, \tag{2.45}$$

$$P_n^* = \frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + \frac{2(\beta+1)}{\beta-1}c_{2n,1} - \frac{2}{\beta-1}c_{2n}, \tag{2.46}$$

$$P_n = \frac{1}{2}s_{2n}, \tag{2.47}$$

$$Q_n^* = -8\left(\frac{\beta+1}{\beta-1}\right)^2 s_{2n,2} + 8\left(\frac{\beta+1}{\beta-1}\right)^2 s_{2n,1} - \frac{8\beta}{(\beta-1)^2}s_{2n}. \tag{2.48}$$

Theorem 2.1 *The following statements hold:*

(a) Any eigenfunction $\Psi_{n,j}$ of $T_1(q)$ corresponding to the eigenvalue $\lambda_{n,j}$ defined in (2.5) satisfies

$$\Psi_{n,j} = \sqrt{2}\cos 2\pi nx + O\left(n^{-1/2}\right). \tag{2.49}$$

Moreover there exists N such that for all $n > N$ the geometric multiplicity of the eigenvalue $\lambda_{n,j}$ is 1.

(b) A complex number $\lambda \in U(n) =: \{\lambda : \left|\lambda - (2\pi n)^2\right| \leq n\}$ *is an eigenvalue of* $T_1(q)$ *if and only if it is a root of the equation*

$$\left[\lambda - (2\pi n)^2 - Q_n - A(\lambda)\right]\left[\lambda - (2\pi n)^2 - P_n^* - A'(\lambda)\right] -$$
$$- \left[P_n + B(\lambda)\right]\left[\gamma_1 n + Q_n^* + B'(\lambda)\right] = 0. \tag{2.50}$$

Moreover $\lambda \in U(n)$ *is a double eigenvalue of* $T_1(q)$ *if and only if it is a double root of (2.50) .*

Proof. (a) By (2.5) the left-hand side of (2.43) is $O(n^{1/2})$, which implies that $u_{n,j} = O(n^{-1/2})$. Therefore from (2.24) we obtain (2.49). Now suppose that there are two linearly independent eigenfunctions corresponding to $\lambda_{n,j}$. Then there exists an eigenfunction satisfying

$$\Psi_{n,j} = \sqrt{2}\sin 2\pi n x + o(1)$$

which contradicts (2.49).

(b) First we prove that the large eigenvalues $\lambda_{n,j}$ are the roots of the equation (2.50). It follows from (2.49), (2.22) and (2.10) that $v_{n,j} \neq 0$. If $u_{n,j} \neq 0$ then multiplying the equations (2.34) and (2.35) side by side and then canceling $v_{n,j}u_{n,j}$ we obtain (2.50) . If $u_{n,j} = 0$ then by (2.34) and (2.35) we have $P_n + B(\lambda_{n,j}) = 0$ and $\lambda_{n,j} - (2\pi n)^2 - P_n^* - A'(\lambda_{n,j}) = 0$ which mean that (2.50) holds. Thus in any case $\lambda_{n,j}$ is a root of (2.50).

Now we prove that the roots of (2.50) lying in $U(n)$ are the eigenvalues of $T_1(q)$. Let $F(\lambda)$ be the left-hand side of (2.50) which can be written as

$$F(\lambda) = (\lambda - (2\pi n)^2)^2 - (Q_n + A(\lambda) + P_n^* + A'(\lambda))\left(\lambda - (2\pi n)^2\right) + \tag{2.51}$$
$$+ (Q_n + A(\lambda))(P_n^* + A'(\lambda)) - (P_n + B(\lambda))(\gamma_1 n + Q_n^* + B'(\lambda))$$

and

$$G(\lambda) = (\lambda - (2\pi n)^2)^2.$$

One can easily verify that the inequality

$$| F(\lambda) - G(\lambda) |<| G(\lambda) |$$

holds for all λ from the boundary of $U(n)$. Since the function $G(\lambda)$ has two roots in the set $U(n)$, by the Rouche's theorem we find that $F(\lambda)$ has two roots in the same set. Thus T_1 has two eigenvalues (counting with multiplicities) lying in $U(n)$ that are the roots of (2.50). On the other hand, (2.50) has preciously two roots (counting with multiplicities) in $U(n)$. Therefore $\lambda \in U(n)$ is an eigenvalue of $T_1(q)$ if and only if (2.50) holds.

If $\lambda \in U(n)$ is a double eigenvalue of $T_1(q)$ then it has no other eigenvalues in $U(n)$ and hence (2.50) has no other roots. This implies that λ is a double root of (2.50). By the same way one can prove that if λ is a double root of (2.50) then it is a double eigenvalue of $T_1(q)$. ■

Let us consider (2.50) in detail. By (2.51) we have

$$F(\lambda) = 0. \tag{2.52}$$

If we substitute $t =: \lambda - (2\pi n)^2$ in (2.52), then it becomes

$$t^2 - (Q_n + A(\lambda) + P_n^* + A'(\lambda))\, t + \tag{2.53}$$
$$+ (Q_n + A(\lambda))(P_n^* + A'(\lambda)) - (P_n + B(\lambda))(\gamma_1 n + Q_n^* + B'(\lambda)) = 0.$$

The solutions of (2.53) are

$$t_{1,2} = \frac{(Q_n + P_n^* + A + A') \pm \sqrt{\Delta(\lambda)}}{2},$$

where

$$\Delta(\lambda) = (Q_n + P_n^* + A + A')^2 - 4(Q_n + A)(P_n^* + A') + 4(P_n + B)(\gamma_1 n + Q_n^* + B')$$

which can be written in the form

$$\Delta(\lambda) = (Q_n - P_n^* + A - A')^2 + 4(P_n + B)(\gamma_1 n + Q_n^* + B') \tag{2.54}$$

and, as we shall see below, $\sqrt{\Delta(\lambda)}$ can be defined as an analytic function on $U(n)$. Clearly the eigenvalue $\lambda_{n,j}$ is a root either of the equation

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left[(Q_n + P_n^* + A + A') - \sqrt{\Delta(\lambda)}\right] \tag{2.55}$$

or of the equation

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left[(Q_n + P_n^* + A + A') + \sqrt{\Delta(\lambda)}\right]. \tag{2.56}$$

Now let us examine $\Delta(\lambda_{n,j})$ in detail. If (1.11) holds then one can readily see from (2.29), (2.33), (2.45)-(2.48) and (2.54) that

$$\Delta(\lambda_{n,j}) = 2\gamma_1 n s_{2n}(1 + o(1)). \tag{2.57}$$

for $\lambda \in U(n)$. By (62) there exists appropriate choice of branch of $\sqrt{\Delta(\lambda)}$ (depending on n) which is analytic on $U(n)$. Taking into account (2.57), (2.29), (2.33), (2.45) and (2.46), we see that (2.55) and (2.56) have the form

$$\lambda = (2\pi n)^2 - \frac{\sqrt{2\gamma_1}}{2}\sqrt{n s_{2n}}(1 + o(1)), \tag{2.58}$$

$$\lambda = (2\pi n)^2 + \frac{\sqrt{2\gamma_1}}{2}\sqrt{n s_{2n}}(1 + o(1)). \tag{2.59}$$

Theorem 2.2 *If (1.11) holds, then the large eigenvalues $\lambda_{n,j}$ are simple and satisfy the following asymptotic formulas*

$$\lambda_{n,j} = (2\pi n)^2 + (-1)^j \frac{\sqrt{2\gamma_1}}{2}\sqrt{n s_{2n}}(1 + o(1)). \tag{2.60}$$

for $j = 1, 2$. Moreover, if there exists a sequence $\{n_k\}$ such that (1.11) holds when n is replaced by n_k, then the root functions of $T_1(q)$ do not form a Riesz basis.

Proof. To prove that the large eigenvalues $\lambda_{n,j}$ are simple let us show that one of the eigenvalues, say $\lambda_{n,1}$ satisfies (2.60) for $j=1$ and the other $\lambda_{n,2}$ satisfies (2.60) for $j=2$. Let us prove that each of the equations (2.55) and (2.56) has a unique root in $U(n)$ by proving that

$$(2\pi n)^2+\frac{1}{2}\left[(Q_n+P_n^*+A+A')\pm\sqrt{\Delta(\lambda)}\right]$$

is a contraction mapping. For this we show that there exist positive real numbers K_1,K_2,K_3 such that

$$\mid A(\lambda)-A(\mu)\mid<K_1\mid\lambda-\mu\mid,\quad \mid A'(\lambda)-A'(\mu)\mid<K_2\mid\lambda-\mu\mid, \tag{2.61}$$

$$\left|\sqrt{\Delta(\lambda)}-\sqrt{\Delta(\mu)}\right|<K_3\mid\lambda-\mu\mid \tag{2.62}$$

for $\lambda,\mu\in U(n)$, where $K_1+K_2+K_3<1$. The proof of (2.61) is similar to the proof of (56) of the paper [73].

Now let us prove (2.62). By (2.57) and (1.11) we have

$$\left(\sqrt{\Delta(\lambda)}\right)^{-1}=o(1).$$

On the other hand arguing as in the proof of (56) of the paper [73] we get

$$\frac{d}{d\lambda}\Delta(\lambda)=O(1).$$

Hence for the large values of n we have

$$\frac{d}{d\lambda}\sqrt{\Delta(\lambda)}=\frac{\frac{d}{d\lambda}\Delta(\lambda)}{2\sqrt{\Delta(\lambda)}}=o(1).$$

Thus by the fixed point theorem, each of the equations (2.55) and (2.56) has a unique root λ_1 and λ_2 respectively. Clearly by (2.58) and (2.59), we have $\lambda_1\neq\lambda_2$ which implies that the equation (2.50) has two simple roots in $U(n)$. Therefore by Theorem 2.1(b), λ_1 and λ_2 are the eigenvalues of $T_1(q)$ lying in $U(n)$, that is, they are $\lambda_{n,1}$ and $\lambda_{n,2}$, which proves the simplicity of the large eigenvalues and the validity of (2.60).

If there exists a sequence $\{n_k\}$ such that (1.11) holds when n is replaced by n_k, then by Theorem 2.1(a)

$$(\Psi_{n_k,1}, \Psi_{n_k,2}) = 1 + O\left(n_k^{-1/2}\right).$$

Now it follows from the theorems of [22, 23] (see also Lemma 3 of [71]) that the root functions of $T_1(q)$ do not form a Riesz basis. ■

2.2 The Asymptotic Formulas for the Eigenvalues and Eigenfunctions of $T_2(q)$, $T_3(q)$ and $T_4(q)$

Now let us consider the operators $T_2(q)$, $T_3(q)$ and $T_4(q)$. First we consider the operator $T_3(q)$.

It is well known that (see (47a) and (47b)) on page 65 of [48]) the eigenvalues of the operators $T_3(q)$ consist of the sequences $\{\lambda_{n,1,3}\}$, $\{\lambda_{n,2,3}\}$ satisfying (2.5) when $\lambda_{n,j}$ is replaced by $\lambda_{n,j,3}$. The eigenvalues, eigenfunctions and associated functions of $T_3(0)$ are

$$\lambda_{n,3} = (2\pi n)^2;\ n = 0, 1, 2, \ldots$$
$$y_{0,3}(x) = x - \frac{\alpha}{1+\alpha},\ y_{n,3}(x) = \sin 2\pi nx;\ n = 1, 2, \ldots$$
$$\phi_{n,3}(x) = \left(x - \frac{\alpha}{1+\alpha}\right)\frac{\cos 2\pi nx}{4\pi n};\ n = 1, 2, \ldots.$$

respectively. The biorthogonal systems analogous to (2.11) and (2.12) are

$$\left\{\cos 2\pi nx, \frac{4(1+\overline{\alpha})}{1-\overline{\alpha}}\left(\frac{1}{1+\overline{\alpha}} - x\right)\sin 2\pi nx\right\}_{n=0}^{\infty} \tag{2.63}$$

$$\left\{\sin 2\pi nx, \frac{4(1+\alpha)}{1-\alpha}\left(x - \frac{\alpha}{1+\alpha}\right)\cos 2\pi nx\right\}_{n=0}^{\infty} \tag{2.64}$$

respectively.

Analogous formulas to (2.13) and (2.14) are

$$\left(\lambda_{N,j,3}-(2\pi n)^{2}\right)(\Psi_{N,j,3},\cos 2\pi nx)=(q\Psi_{N,j,3},\cos 2\pi nx) \tag{2.65}$$

$$\left(\lambda_{N,j,3}-(2\pi n)^{2}\right)\left(\Psi_{N,j,3},\varphi_{n,3}^{*}\right)-\gamma_{3}n\,(\Psi_{N,j,3},\cos 2\pi nx)=\left(q\Psi_{N,j,3},\varphi_{n,3}^{*}\right) \tag{2.66}$$

respectively, where

$$\gamma_{3}=\frac{16\pi\,(1+\alpha)}{1-\alpha}.$$

Instead of (2.11)-(2.14) using (2.63)-(2.66) and arguing as in the proofs of Theorem 2.1 and Theorem 2.2 we obtain the following results for $T_{3}\,(q)$.

Theorem 2.3 *If (1.11) holds, then the large eigenvalues $\lambda_{n,j,3}$ are simple and satisfy the following asymptotic formulas*

$$\lambda_{n,j,3}=(2\pi n)^{2}+(-1)^{j}\,\frac{\sqrt{2\gamma_{3}}}{2}\sqrt{ns_{2n}}(1+o(1)).$$

for $j=1,2$. The eigenfunctions $\Psi_{n,j,3}$ corresponding to $\lambda_{n,j,3}$ obey

$$\Psi_{n,j,3}=\sqrt{2}\sin 2\pi nx+O\left(n^{-1/2}\right).$$

Moreover, if there exists a sequence $\{n_{k}\}$ such that (1.11) holds when n is replaced by n_{k}, then the root functions of $T_{3}\,(q)$ do not form a Riesz basis.

Now let us consider the operator $T_{2}\,(q)$. It is well-known that (see (47a) and (47b)) on page 65 of [48]) the eigenvalues of the operators $T_{2}(q)$ consist of the sequences $\{\lambda_{n,1,2}\},\{\lambda_{n,2,2}\}$ satisfying

$$\lambda_{n,j,2}=(2n\pi+\pi)^{2}+O(n^{1/2}), \tag{2.67}$$

for $j=1,2$. The eigenvalues, eigenfunctions and associated functions of $T_{2}\,(0)$ are

$$(\pi+2\pi n)^{2},\ y_{n,2}\,(x)=\cos\left(2n+1\right)\pi x,$$
$$\phi_{n,2}\,(x)=\left(\frac{\beta}{\beta-1}-x\right)\frac{\sin\left(2n+1\right)\pi x}{2\left(2n+1\right)\pi}$$

for $n = 0, 1, 2, \ldots$respectively. The biorthogonal systems analogous to (2.11) and (2.12) are

$$\left\{ \sin(2n+1)\pi x, \frac{4(\overline{\beta}-1)}{\overline{\beta}+1}\left(x+\frac{1}{\overline{\beta}-1}\right)\cos(2n+1)\pi x \right\}_{n=0}^{\infty} \tag{2.68}$$

$$\left\{ \cos(2n+1)\pi x, \frac{4(\beta-1)}{\beta+1}\left(\frac{\beta}{\beta-1}-x\right)\sin(2n+1)\pi x \right\}_{n=0}^{\infty} \tag{2.69}$$

respectively.

Analogous formulas to (2.13) and (2.14) are

$$\left(\lambda_{N,j,2} - ((2n+1)\pi)^2\right)(\Psi_{N,j,2}, \sin(2n+1)\pi x) = (q\Psi_{N,j,2}, \sin(2n+1)\pi x) \tag{2.70}$$

$$\left(\lambda_{N,j,2} - ((2n+1)\pi)^2\right)\left(\Psi_{N,j,2}, \varphi_{n,2}^*\right) - (2n+1)\gamma_2(\Psi_{N,j,2}, \sin(2n+1)\pi x) = \left(q\Psi_{N,j,2}, \varphi_{n,2}^*\right) \tag{2.71}$$

respectively, where

$$\gamma_2 = \frac{8\pi(\beta-1)}{\beta+1}.$$

Instead of (2.11)-(2.14) using (2.68)-(2.71) and arguing as in the proofs of Theorem 2.1 and Theorem 2.2 we obtain the following results for $T_2(q)$.

Theorem 2.4 *If (1.11a) holds, then the large eigenvalues $\lambda_{n,j,2}$ are simple and satisfy the following asymptotic formulas*

$$\lambda_{n,j,2} = ((2n+1)\pi)^2 + (-1)^j \frac{\sqrt{2\gamma_2}}{2}\sqrt{(2n+1)s_{2n+1}}(1+o(1)).$$

for $j = 1, 2$. The eigenfunctions $\Psi_{n,j,2}$ corresponding to $\lambda_{n,j,2}$ obey

$$\Psi_{n,j,2} = \sqrt{2}\cos(2n+1)\pi x + O\left(n^{-1/2}\right).$$

Moreover, if there exists a sequence $\{n_k\}$ such that (1.11a) holds when n is replaced by n_k, then the root functions of $T_2(q)$ do not form a Riesz basis.

Lastly we consider the operator $T_4(q)$. It is well known that (see (47a) and (47b)) on page 65 of [48]) the eigenvalues of the operators $T_4(q)$ consist of the sequences $\{\lambda_{n,1,4}\}, \{\lambda_{n,2,4}\}$ satisfying (2.67) when $\lambda_{n,j,2}$ is replaced by $\lambda_{n,j,4}$. The eigenvalues, eigenfunctions and associated functions of $T_4(0)$ are

$$\lambda_{n,4} = (\pi + 2\pi n)^2, \ y_{n,4}(x) = \sin(2n+1)\pi x,$$
$$\phi_{n,4}(x) = \left(\frac{\alpha}{1-\alpha} + x\right)\frac{\cos(2n+1)\pi x}{2(2n+1)\pi}$$

for $n = 0, 1, 2, \ldots$respectively. The biorthogonal systems analogous to (2.11) and (2.12) are

$$\left\{\cos(2n+1)\pi x, \frac{4(1-\overline{\alpha})}{1+\overline{\alpha}}\left(\frac{1}{1-\overline{\alpha}} - x\right)\sin(2n+1)\pi x\right\}_{n=0}^{\infty} \tag{2.72}$$
$$\left\{\sin(2n+1)\pi x, \frac{4(1-\alpha)}{1+\alpha}\left(\frac{\alpha}{1-\alpha} + x\right)\cos(2n+1)\pi x\right\}_{n=0}^{\infty} \tag{2.73}$$

respectively.

Analogous formulas to (2.13) and (2.14) are

$$\left(\lambda_{N,j,4} - (\pi + 2\pi n)^2\right)\left(\Psi_{N,j,4}, \cos(2n+1)\pi x\right) = \left(q\Psi_{N,j,4}, \cos(2n+1)\pi x\right), \tag{2.74}$$

$$\left(\lambda_{N,j,4} - ((2n+1)\pi)^2\right)\left(\Psi_{N,j,4}, \varphi_{n,4}^*\right) - (2n+1)\gamma_4\left(\Psi_{N,j,4}, \cos(2n+1)\pi x\right) = \left(q\Psi_{N,j,4}, \varphi_{n,4}^*\right) \tag{2.75}$$

respectively, where

$$\gamma_4 = \frac{8\pi(1-\alpha)}{1+\alpha}.$$

Instead of (2.11)-(2.14) using (2.72)-(2.75) and arguing as in the proofs of Theorem 2.1 and Theorem 2.2 we obtain the following results for $T_4(q)$.

Theorem 2.5 *If (1.11a) holds, then the large eigenvalues $\lambda_{n,j,4}$ are simple and satisfy the following asymptotic formulas*

$$\lambda_{n,j,4} = ((2n+1)\pi)^2 + (-1)^j \frac{\sqrt{2\gamma_4}}{2}\sqrt{(2n+1)s_{2n+1}}(1 + o(1)).$$

for $j = 1, 2$. *The eigenfunctions* $\Psi_{n,j,4}$ *corresponding to* $\lambda_{n,j,4}$ *obey*

$$\Psi_{n,j,4} = \sqrt{2} \sin (2n + 1) \pi x + O \left(n^{-1/2} \right).$$

Moreover, if there exists a sequence $\{n_k\}$ *such that (1.11a) holds when* n *is replaced by* n_k, *then the root functions of* $T_4(q)$ *do not form a Riesz basis.*

Now suppose that

$$\int_0^1 xq(x)\, dx \neq 0. \tag{2.76}$$

If

$$\frac{1}{2} s_{2n} + B = o\left(\frac{1}{n}\right), \tag{2.77}$$

where B is defined by (2.29), then one can readily see from (2.54), (2.29), (2.33) and (2.45)-(2.48) that there exists a positive constant K such that

$$|\Delta(\lambda)| > K$$

for $\lambda \in U(n)$ and for the large values of n. Therefore arguing as in the proof of Theorem 2.2, we obtain the following.

Theorem 2.6 *Suppose that (2.76) holds. If (2.77) holds, then the large eigenvalues of the operator* $T_1(q)$ *are simple. Moreover if there exists a sequence* $\{n_k\}$ *such that (2.77) holds when* n *is replaced by* n_k, *then the root functions of* $T_1(q)$ *do not form a Riesz basis. Similar results continue to hold for the operators* $T_2(q), T_3(q)$ *and* $T_4(q)$.

Remark 2.1 *Since the eigenvalues* $\lambda_{n,1}$ *and* $\lambda_{n,2}$ *are the fixed points of the equations (2.55) and (2.56) respectively, using the fixed point iteration one can determine these eigenvalues with arbitrary precision. Moreover, using these better approximations of the eigenvalues, one can also determine the better approximations for the eigenfunctions of the operator* $T_1(q)$. *Similar results can be obtained for the operators* $T_2(q), T_3(q)$ *and* $T_4(q)$.

3 STURM-LIOUVILLE OPERATORS WITH GENERAL REGULAR BOUNDARY CONDITIONS

In the present chapter we consider the non-self-adjoint linear differential operators $T_1^\sigma(q)$ and $T_2^\sigma(q)$ for $\sigma = 0, 1$, which are introduced in Section 1.1. We will focus only on the operator $T_1^1(q)$. The investigations of the operators $T_1^0(q), T_2^0(q)$ and $T_2^1(q)$ are similar. First let us prove the following simple proposition about $T_1^1(0)$. Note that the simplest case $q(x) \equiv 0$ was completely solved in [38]. Here we write the asymptotic formulas for the eigenvalues of $T_1^1(0)$ in the form we need.

3.1 The Asymptotic Formulas for the Eigenvalues and Eigenfunctions of $T_1^1(q)$

Proposition 3.1 *The square roots (with nonnegative real part) of the eigenvalues of the operator $T_1^1(0)$ consist of the sequences $\{\mu_{n,1}(0)\}$ and $\{\mu_{n,2}(0)\}$ satisfying*

$$\mu_{n,1}(0) = 2\pi n, \tag{3.1}$$

$$\mu_{n.2}(0) = 2\pi n + \frac{\beta_2}{\beta_1 - 1}\frac{1}{\pi n} + O\left(\frac{1}{n^2}\right). \tag{3.2}$$

Proof. Using the fundamental solutions $e^{i\mu x}$ and $e^{-i\mu x}$ of $-y'' = \lambda y$ where $\mu = \sqrt{\lambda}$, one can readily see that the characteristic determinant $\Delta_0(\mu)$ of $T_1^1(0)$ has the form

$$\Delta_0(\mu) = \left(1 - e^{i\mu}\right)\left(i\mu + \beta_1 i\mu e^{-i\mu} - \beta_2 e^{-i\mu}\right) + \left(i\mu + \beta_1 i\mu e^{i\mu} + \beta_2 e^{i\mu}\right)\left(1 - e^{-i\mu}\right) = 0.$$

After simplifying this equation, we have

$$\Delta_0(\mu) = \left(1 - e^{-i\mu}\right)\left[i\mu(\beta_1 - 1)\left(e^{i\mu} - 1\right) + \beta_2\left(e^{i\mu} + 1\right)\right] = 0 \tag{3.3}$$

which is equivalent to

$$1 - e^{-i\mu} = 0 \text{ or } f(\mu) = 0 \tag{3.4}$$

where

$$f(\mu) = e^{i\mu} - 1 - \frac{i\beta_2}{\beta_1 - 1}\frac{e^{i\mu}+1}{\mu} = e^{i\mu} - 1 + O\left(\frac{1}{\mu}\right) \tag{3.5}$$

The solution of the first equation in (3.4) is $\mu_{n,1}(0) = 2\pi n$ for $n \in \mathbb{Z}$, that is, (3.1) is proved.

To prove (3.2), we estimate the roots of (3.5). Using Rouche's theorem on the circle $\left\{\mu : \; |\mu - 2\pi n| = \frac{c}{n}\right\}$ for some constant c, one can easily see that, the roots of (3.5) has the form

$$\mu_{2,n}^{0} = 2\pi n + \xi \;\&\; \xi = O\left(\frac{1}{n}\right). \tag{3.6}$$

Now we prove that

$$\xi = \frac{\beta_2}{\beta_1 - 1}\frac{1}{\pi n} + O\left(\frac{1}{n^2}\right). \tag{3.7}$$

For this, let us consider the roots of (3.5) in detail. By (3.6) and (3.5) we have

$$e^{i(2\pi n+\xi)} - 1 = \frac{i\beta_2}{\beta_1 - 1}\frac{2 + O\left(\frac{1}{n}\right)}{2\pi n + O\left(\frac{1}{n}\right)} = \frac{2i\beta_2}{\beta_1 - 1}\frac{1}{2\pi n} + O\left(\frac{1}{n^2}\right). \tag{3.8}$$

On the other hand, using Maclaurin expansion of $e^{i\xi}$ and taking into account the second equality of (3.6) we see that

$$e^{i(2\pi n+\xi)} - 1 = i\xi + O\left(\frac{1}{n^2}\right)$$

This with (3.8) gives us (3.7). Now (3.2) follows from (3.6) and (3.7). Lemma is proved.

■

For $q \neq 0$ it is known that (see (21) of [41]) the characteristic polynomial of $T_1^1(q)$ has the form

$$\Delta(\mu) = \Delta_0(\mu) - \frac{\beta_1 + 1}{2}\left\{e^{i\mu}(c_\mu - is_\mu) - e^{-i\mu}(c_\mu + is_\mu)\right\} + o\left(\frac{1}{\mu}\right), \tag{3.9}$$

where $\Delta_0(\mu)$ is defined in (3.3) and

$$c_\mu = \int_0^1 \cos(2\mu t)\, q(t)\, dt, \;\; s_\mu = \int_0^1 \sin(2\mu t)\, q(t)\, dt. \tag{3.10}$$

After some arrangements (3.9) can be written in the form

$$\Delta(\mu) = \Delta_0(\mu) - \frac{\beta_1+1}{2} e^{-i\mu} \left\{ c_\mu \left(e^{2i\mu} - 1\right) - is_\mu \left(e^{2i\mu}+1\right) \right\} + o\left(\frac{1}{\mu}\right). \qquad (3.11)$$

Using (3.3) in this formula we obtain

$$\begin{aligned}\Delta(\mu) &= \left(1-e^{-i\mu}\right)\left[i\mu(\beta_1-1)\left(e^{i\mu}-1\right)+\beta_2\left(e^{i\mu}+1\right)\right] - \\ &\quad -\frac{\beta_1+1}{2}e^{-i\mu}\left\{c_\mu\left(e^{2i\mu}-1\right)-is_\mu\left(e^{2i\mu}+1\right)\right\}+o\left(\frac{1}{\mu}\right) \\ &= \left(1-e^{-i\mu}\right)\left[i\mu(\beta_1-1)\left(e^{i\mu}-1\right)+\beta_2\left(e^{i\mu}+1\right)-\frac{\beta_1+1}{2}c_\mu\left(e^{i\mu}+1\right)\right]+ \\ &\quad +i(\beta_1+1)s_\mu\cos\mu+o\left(\frac{1}{\mu}\right).\end{aligned}$$

Therefore the characteristic determinant $\Delta(\mu)$, can be written as

$$\Delta(\mu) = \Delta_1(\mu) + i(\beta_1+1)s_\mu\cos\mu + o\left(\frac{1}{\mu}\right). \qquad (3.12)$$

where

$$\Delta_1(\mu) = \left(1-e^{-i\mu}\right)\left[i\mu(\beta_1-1)\left(e^{i\mu}-1\right)+\left(\beta_2-\frac{\beta_1+1}{2}c_\mu\right)\left(e^{i\mu}+1\right)\right]. \qquad (3.13)$$

To obtain the asymptotic formulas for the eigenvalues of $T_1^1(q)$ first let us consider the roots of $\Delta_1(\mu)$.

Lemma 3.1 *The roots of the function* $\Delta_1(\mu)$ *consist of the sequences* $\left\{\mu_{n,1}^1\right\}$ *and* $\left\{\mu_{n,2}^1\right\}$ *such that*

$$\mu_{n,1}^1 = 2\pi n, \ \ n\in\mathbb{Z}, \qquad (3.14)$$

$$\mu_{n.2}^1 = 2\pi n + \frac{\beta_2}{\beta_1-1}\frac{1}{\pi n} + o\left(\frac{1}{n}\right). \qquad (3.15)$$

Proof. The zeros of $\Delta_1(\mu)$ are the zeros of the equations

$$1-e^{-i\mu}=0,$$

and

$$g(\mu) =: e^{i\mu} - 1 + \frac{1}{\beta_1 - 1}\left(\beta_2 - \frac{\beta_1 + 1}{2} c_\mu\right) \frac{e^{i\mu} + 1}{i\mu} = 0.$$

The roots of the first equation are $2\pi n$ for $n \in \mathbb{Z}$, that is (3.14) holds. By definition of $f(\mu)$ (see (3.5)) we have

$$g(\mu) = f(\mu) - \frac{\frac{\beta_1+1}{2} c_\mu}{\beta_1 - 1} \frac{e^{i\mu} + 1}{i\mu}.$$

Since $c_\mu = o(1)$, there exists a sequence δ_n such that $\delta_n = o(1)$ and

$$|g(\mu) - f(\mu)| < \frac{\delta_n}{n} \tag{3.16}$$

for $\mu \in U(2\pi n)$, where $U(2\pi n)$ is $O\left(\frac{1}{n}\right)$-neighborhood of $2\pi n$.

Now to estimate the zeros of $g(\mu)$, we use Rouche's theorem for the functions $f(\mu)$ and $g(\mu)$ on the circle

$$\gamma_n = \left\{\mu : \mid \mu - \mu_{n,2}(0) \mid = \frac{\varepsilon_n}{n}\right\}, \tag{3.17}$$

where $\mu_{n,2}(0)$ is defined in (3.2) and ε_n is chosen so that

$$\varepsilon_n = o(1) \;\&\; \delta_n = o(\varepsilon_n). \tag{3.18}$$

For this let us estimate $|f(\mu)|$ on γ_n by using the Taylor series of $f(\mu)$ about $\mu_{n,2}(0)$:

$$f(\mu) = f'\left(\mu_{n,2}\right)\left(\mu - \mu_{n,2}\right) + \frac{f''\left(\mu_{n,2}\right)}{2!}\left(\mu - \mu_{n,2}\right)^2 + \cdots$$

Since

$$f'(\mu) = ie^{i\mu} - \frac{i\beta_2}{\beta_1 - 1}\frac{ie^{i\mu}}{i\mu} + O\left(\frac{1}{n^2}\right) \sim 1,\; f''(\mu) \sim 1, \ldots,$$

there exist a constant $c > 0$ such that $|f'(\mu)| > c$ and

$$|f(\mu)| > c\frac{\varepsilon_n}{2n} \tag{3.19}$$

for $\mu \in \gamma_n$. Thus by (3.16)-(3.19) and Rouche's theorem, there exists a root $\mu_{n,2}^1$ of $g(\mu)$ inside the circle (3.17). Therefore (3.15), follows from (3.2). ■

Now using (3.12), (3.13) and Lemma 3.1, we get one of the main results of this thesis.

Theorem 3.1 (a) *If (1.12) holds, then the large eigenvalues of* $T_1^1(q)$ *are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences* $\{\mu_{n,1}(q)\}$ *and* $\{\mu_{n,2}(q)\}$ *satisfying the asymptotic formulas*

$$\mu_{n,1}(q) = 2\pi n + o\left(\frac{1}{n}\right), \tag{3.20}$$

$$\mu_{n,2}(q) = 2\pi n + \frac{\beta_2}{\beta_1 - 1}\frac{1}{\pi n} + o\left(\frac{1}{n}\right). \tag{3.21}$$

Moreover the normalized eigenfunctions $\varphi_{n,1}(x)$ *and* $\varphi_{n,2}(x)$ *corresponding to the eigenvalues* $(\mu_{n,1}(q))^2$ *and* $(\mu_{n,2}(q))^2$ *satisfy the same asymptotic formula*

$$\varphi_{n,j}(x) = \sqrt{2}\cos 2\pi n x + O\left(\frac{1}{n}\right) \tag{3.22}$$

for $j = 1, 2$

(b) *If there exists a subsequence* $\{n_k\}$ *such that (1.12) holds whenever* n *is replaced by* n_k*, then the system of the root functions of* $T_1^1(q)$ *does not form a Riesz basis.*

Proof. (a) To prove (3.20) and (3.21), we show that the large roots of $\Delta(\mu)$ lies in $o\left(\frac{1}{n}\right)$-neighborhood of the roots of $\Delta_1(\mu)$ by using Rouche's theorem for $\Delta(\mu)$ and $\Delta_1(\mu)$ on $\Gamma_1(r_n)$, $\Gamma_2(r_n)$, where

$$\Gamma_j(r_n) = \left\{\mu : \ \left|\mu - \mu_{n,j}^1\right| = r_n\right\}, r_n = o\left(\frac{1}{n}\right) \tag{3.23}$$

and $\mu_{n,j}^1$ for $j = 1, 2$ are the roots of $\Delta_1(\mu)$. If $\mu \in \Gamma_j(r_n)$ for $j = 1, 2$ then by (1.12) $s_\mu = o\left(\frac{1}{n}\right)$ and by (3.12)

$$a(\mu) =: |\Delta(\mu) - \Delta_1(\mu)| < b_n, \ b_n = o\left(\frac{1}{n}\right). \tag{3.24}$$

We can choose r_n so that

$$b_n = o(r_n). \tag{3.25}$$

Now let us estimate $\Delta_1(\mu)$ on the circles $\Gamma_1(r_n)$, $\Gamma_2(r_n)$. By (3.13)

$$\Delta_1(\mu) = \left(1 - e^{-i\mu}\right) i\mu h(\mu) \tag{3.26}$$

where

$$h(\mu) = (\beta_1 - 1)\left(e^{i\mu} - 1\right) + \left(\beta_2 - \frac{\beta_1 + 1}{2} c_\mu\right) \frac{e^{i\mu} + 1}{i\mu}. \tag{3.27}$$

It follows from (3.14), (3.15) and (3.23) that if $\mu \in \Gamma_1(r_n)$ and $\mu \in \Gamma_2(r_n)$ then $\mu = 2\pi n + r_n e^{i\theta}$ and $\mu = 2\pi n + \frac{\beta_2}{\beta_1 - 1}\frac{1}{\pi n} + r_n e^{i\theta} + o\left(\frac{1}{n}\right)$ respectively, where $\theta \in (0, 2\pi)$. Therefore

$$\left(1 - e^{-i\mu}\right) \sim r_n, \tag{3.28}$$

and

$$\left(1 - e^{-i\mu}\right) \sim \frac{1}{n}, \tag{3.29}$$

on $\Gamma_1(r_n)$ and $\Gamma_2(r_n)$ respectively, where $a_n \sim b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$.

Now let us consider $h(\mu)$ on $\Gamma_j(r_n)$, $j = 1, 2$. Since $\mu^1_{n,2}$ is the root of $h(\mu)$ the Taylor expansion of $h(\mu)$ about $\mu^1_{n,2}$ is

$$h(\mu) = h'\left(\mu^1_{n,2}\right)\left(\mu - \mu^1_{n,2}\right) + \frac{h''\left(\mu^1_{n,2}\right)}{2!}\left(\mu - \mu^1_{n,2}\right)^2 + \cdots. \tag{3.30}$$

By (3.27), we have

$$h'(\mu) = (\beta_1 - 1) i e^{i\mu} + \left(\beta_2 - \frac{\beta_1 + 1}{2} c_\mu\right) \frac{i e^{i\mu}}{i\mu} + O\left(\frac{1}{n^2}\right) \sim 1$$

for $\mu \in \Gamma_j(r_n)$, $j = 1, 2$. Clearly $h^{(k)}(\mu) \sim 1$ for $k > 1$ and $\mu \in \Gamma_j(r_n)$. On the other hand, $\left(\mu - \mu^1_{n,2}\right) \sim \frac{1}{n}$ for $\mu \in \Gamma_1(r_n)$ and $\left(\mu - \mu^1_{n,2}\right) \sim r_n$ for $\mu \in \Gamma_2(r_n)$. Therefore using (3.30) we obtain

$$h(\mu) \sim \frac{1}{n}, \ \forall \mu \in \Gamma_1(r_n),$$

$$h(\mu) \sim r_n, \ \forall \mu \in \Gamma_2(r_n).$$

These formulas with (3.26), (3.28) and (3.29) imply that

$$\Delta_1(\mu) \sim r_n, \ \forall \mu \in \ \Gamma_j(r_n) \tag{3.31}$$

for $j = 1, 2$. Thus by (3.24), (3.25), (3.31) and Rouche's theorem, each of the disks enclosed by the circles $\Gamma_1(r_n)$ and $\Gamma_2(r_n)$ contains an eigenvalue which proves (3.20) and (3.21).

Since the distance between the centres of the circles $\Gamma_1(r_n)$ and $\Gamma_2(r_n)$ is of order $\frac{1}{n}$, but $r_n = o\left(\frac{1}{n}\right)$, the eigenvalues inside the circles $\Gamma_1(r_n)$ and $\Gamma_2(r_n)$ are different, that is, they are simple.

Now let us prove (3.22). Since the equation

$$-y'' + q(x)y = \mu^2 y$$

has the fundamental solutions of the form

$$y_1(x,\mu) = e^{i\mu x} + O\left(\frac{1}{\mu}\right), \ y_2(x,\mu) = e^{-i\mu x} + O\left(\frac{1}{\mu}\right)$$

(see p. 52 of [48]) the eigenfunctions of $T_1^1(q)$ are

$$y_{n,j}(x) =$$

$$= \begin{vmatrix} e^{i\mu_{n,j}x} + O\left(\frac{1}{\mu_{n,j}}\right) & e^{-i\mu_{n,j}x} + O\left(\frac{1}{\mu_{n,j}}\right) \\ i\mu_{n,j}\left(1+\beta_1 e^{i\mu_{n,j}}\right) + \beta_2 e^{i\mu_{n,j}} + O\left(\frac{1}{\mu_{n,j}}\right) & -i\mu_{n,j}\left(1+\beta_1 e^{-i\mu_{n,j}}\right) + \beta_2 e^{-i\mu_{n,j}} + O\left(\frac{1}{\mu_{n,j}}\right) \end{vmatrix}$$

$$= \left[e^{i\mu_{n,j}x} + O\left(\frac{1}{\mu_{n,j}}\right)\right]\left[-i\mu_{n,j}\left(1+\beta_1 e^{-i\mu_{n,j}}\right) + \beta_2 e^{-i\mu_{n,1j}} + O\left(\frac{1}{\mu_{n,j}}\right)\right] -$$
$$- \left[e^{-i\mu_{n,j}x} + O\left(\frac{1}{\mu_{n,j}}\right)\right]\left[i\mu_{n,j}\left(1+\beta_1 e^{i\mu_{n,1}}\right) + \beta_2 e^{i\mu_{n,j}} + O\left(\frac{1}{\mu_{n,j}}\right)\right].$$

This with the formula

$$\mu_{n,j} = 2\pi n + O\left(\frac{1}{n}\right),$$

for $j = 1, 2$ (see (3.20) and (3.21)), implies (3.22).

(b) It is clear that if (1.12) holds for the subsequence $\{n_k\}$ then (3.22) holds for $\{n_k\}$ too. Therefore the angle between the eigenfunctions $\varphi_{n_k,1}(x)$ and $\varphi_{n_k,2}(x)$ corresponding to $\mu_{n_k,1}(q)$ and $\mu_{n_k,2}(q)$ tends to zero. Hence the system of the root functions of $T_1^1(q)$ does not form a Riesz basis (see [64]). Note that (b) follows also from (a) and Theorem 2 of [40, 41]. ■

Let q be an absolutely continuous function. Then using the integration by parts formula for s_μ and c_μ defined in (3.10) we obtain

$$s_\mu = \frac{1}{2\mu}\left[q(0) - q(1)\cos(2\mu)\right] + o(\frac{1}{\mu})$$

and

$$c_\mu = \frac{1}{2\mu} q(1)\sin(2\mu) + o\left(\frac{1}{\mu}\right).$$

If $\mu \in U(2\pi n)$, where $U(2\pi n)$ is defined in the proof of Lemma 3.1 , then

$$\cos\mu = 1 + O\left(\frac{1}{\mu}\right) \,\&\, \sin\mu = O\left(\frac{1}{\mu}\right)$$

Therefore we have

$$s_\mu = \frac{1}{2\mu}\left[q(0) - q(1)\right] + o(\frac{1}{\mu}), \;\; c_\mu = o\left(\frac{1}{\mu}\right)$$

and hence by (3.11)

$$\begin{aligned}\Delta(\mu) &= \Delta_0(\mu) + i(\beta_1 + 1)\, s_\mu \cos\mu + o\left(\frac{1}{\mu}\right) \\ &= \Delta_0(\mu) + \frac{a}{\mu} + o\left(\frac{1}{\mu}\right) \qquad (3.32)\end{aligned}$$

where

$$a = \frac{i(\beta_1 + 1)}{2}\left[q(0) - q(1)\right].$$

Now we are ready to state one of the main results of this thesis.

Theorem 3.2 *Let q be an absolutely continuous function and (1.13) for $\sigma = 1$ hold. Then* *(a) the large eigenvalues of $T_1^1(q)$ are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences $\{\mu_{n,1}(q)\}$ and $\{\mu_{n,2}(q)\}$ satisfying*

$$\mu_{n,1}(q) = 2\pi n + \frac{2\beta_2 - i\sqrt{D}}{4(\beta_1 - 1)\pi n} + o\left(\frac{1}{n}\right), \tag{3.33}$$

$$\mu_{n,2}(q) = 2\pi n + \frac{2\beta_2 + i\sqrt{D}}{4(\beta_1 - 1)\pi n} + o\left(\frac{1}{n}\right). \tag{3.34}$$

where $D = 2\left(1 - \beta_1^2\right)[q(0) - q(1)] - (2\beta_2)^2$

(b) the system of the root functions of $T_1^1(q)$ does not form a Riesz basis.

Proof. (*a*) By (3.32) $\mu_{n,j}(q)$ is a root of the equation

$$\mu\Delta_0(\mu) + a + o(1) = 0.$$

Using (3.3) in this equation we get

$$\mu\left(1 - e^{-i\mu}\right)\left[i\mu(\beta_1 - 1)\left(e^{i\mu} - 1\right) + \beta_2\left(e^{i\mu} + 1\right)\right] + a + o(1) = 0. \tag{3.35}$$

By the Taylor expansions of $e^{-i\mu}$ and $e^{i\mu}$ at $2\pi n$ we have

$$e^{-i\mu} = 1 - i(\mu - 2\pi n) + O\left(\frac{1}{n^2}\right),$$
$$e^{i\mu} = 1 + i(\mu - 2\pi n) + O\left(\frac{1}{n^2}\right)$$

for $\mu \in U(2\pi n)$. Therefore (3.35) can be written in the form

$$i\mu(\mu - 2\pi n)\left[-\mu(\beta_1 - 1)(\mu - 2\pi n) + 2\beta_2 + O\left(\frac{1}{\mu}\right)\right] + a + o(1) = 0. \tag{3.36}$$

To prove the formulas (3.33) and (3.34) we consider the equation (3.36). In (3.36) substituting $x = \mu(\mu - 2\pi n)$ and taking into account that $x = O(1)$ for $\mu \in U(2\pi n)$ we get

$$-i(\beta_1 - 1)x^2 + 2i\beta_2 x + a + o(1) = 0. \tag{3.37}$$

To solve (3.37) we compare the roots of the functions

$$f_1(\mu) = -i(\beta_1 - 1)x^2 + 2i\beta_2 x + a \tag{3.38}$$

and

$$f_2(\mu) = -i(\beta_1 - 1)x^2 + 2i\beta_2 x + a + \alpha_n \tag{3.39}$$

on the set $U(2\pi n)$, where $\alpha_n = o(1)$. The roots of $f_1(\mu)$ are

$$x_{1,2} = \frac{-2i\beta_2 \pm \sqrt{D}}{-2i(\beta_1 - 1)} \tag{3.40}$$

where

$$D = (2i\beta_2)^2 + 4i(\beta_1 - 1)a = (2i\beta_2)^2 - 2(\beta_1^2 - 1)[q(0) - q(1)] \neq 0. \tag{3.41}$$

by the assumption (1.13) for $\sigma = 1$. Therefore we have two different solutions x_1 and x_2.

On the other hand the solutions of the equations $\mu(\mu - 2\pi n) = x_1$ and $\mu(\mu - 2\pi n) = x_2$ with respect to μ are

$$\mu_{11} = O\left(\frac{1}{n}\right), \ \mu_{12} = 2\pi n + \frac{x_1}{2\pi n} + O\left(\frac{1}{n^2}\right)$$

and

$$\mu_{21} = O\left(\frac{1}{n}\right), \ \mu_{22} = 2\pi n + \frac{x_2}{2\pi n} + O\left(\frac{1}{n^2}\right)$$

respectively. Since $x_1 - x_2 \sim 1$ (see (3.40) and (3.41)), we have

$$\mu_{12} - \mu_{21} \sim n, \ \mu_{12} - \mu_{22} \sim \frac{1}{n}, \ \mu_{12} - \mu_{11} \sim n. \tag{3.42}$$

Now consider the roots of $f_2(\mu)$ by using Rouche's theorem on

$$\gamma_j(r_n) = \left\{\mu : \ \left|\mu - \mu_{j2}\right| = r_n\right\}, \tag{3.43}$$

for $j = 1, 2$, where r_n is chosen so that

$$r_n = o\left(\frac{1}{n}\right) \ \& \ \alpha_n = o(nr_n). \tag{3.44}$$

By (3.38), (3.39) and (3.44)

$$|f_1(\mu) - f_2(\mu)| = \alpha_n = o(1)$$

on $\gamma_1(r_n) \cap \gamma_2(r_n)$. Since the roots of $f_1(\mu)$ are μ_{ij} for $i, j = 1, 2$, we have

$$f_1(\mu) = A(\mu - \mu_{11})(\mu - \mu_{12})(\mu - \mu_{21})(\mu - \mu_{22}) \tag{3.45}$$

where A is a constant. One can easily verify by using (3.42) and (3.45) that

$$f'(\mu_{12}) = A(\mu_{12} - \mu_{11})(\mu_{12} - \mu_{21})(\mu_{12} - \mu_{22}) \sim n$$

Since $f(\mu)$ is a polynomial of order 4 we have

$$f''(\mu_{12}) = O(n^2),\ f'''(\mu_{12}) = O(n),\ f^{(4)}(\mu_{12}) = O(1),\ f^{(5)}(\mu_{12}) = 0.$$

Therefore using the Taylor series

$$f_1(\mu) = f_1'(\mu_{12})(\mu - \mu_{12}) + \cdots .$$

of $f_1(\mu)$ about μ_{12} for $\mu \in \gamma_1(r_n)$ and taking into account that $(\mu - \mu_{12}) \sim r_n$ we obtain

$$|f_1(\mu)| \sim nr_n.$$

On the other hand by (3.44) we have

$$|f_1(\mu) - f_2(\mu)| = \alpha_n = o(nr_n)$$

for $\mu \in \gamma_1(r_n)$. Therefore

$$|f_1(\mu) - f_2(\mu)| < |f_1(\mu)| \tag{3.46}$$

on $\gamma_1(r_n)$ In the same way we prove that (3.46) holds on $\gamma_2(r_n)$ too. Hence inside of each of the circles $\gamma_1(r_n)$ and $\gamma_2(r_n)$, there is one root of (3.35) denoted by $\mu_{n,1}(q)$ and $\mu_{n,2}(q)$ respectively. Since $r_n = o\left(\frac{1}{n}\right)$, $\mu_{n,1}(q)$ and $\mu_{n,2}(q)$ satisfy the formulas (3.33) and (3.34). To complete the proof of (a) it is enough to note that disks enclosed by the circles $\gamma_1(r_n)$ and $\gamma_2(r_n)$ have no common points and there are only two roots of (3.32) in the neighborhood of $2\pi n$. Thus (a) is proved.

(b) The proof of (b) is the same as the proof of Theorem 3.1(b). ■

3.2 The Asymptotic Formulas for the Eigenvalues and Eigenfunctions of $T_1^0(q)$, $T_2^0(q)$ and $T_2^1(q)$

Now consider the operators $T_1^0(q)$, $T_2^0(q)$ and $T_2^1(q)$. In this case the characteristic determinant of $T_1^0(0)$, $T_2^0(0)$ and $T_2^1(0)$ are

$$\Delta_0^0(\mu) = \left(1+e^{i\mu}\right)\left(i\mu+\beta_1 i\mu e^{-i\mu}-\beta_2 e^{-i\mu}\right)+\left(i\mu+\beta_1 i\mu e^{i\mu}+\beta_2 e^{i\mu}\right)\left(1+e^{-i\mu}\right)=0,$$

$$D_0^0(\mu) = \left(1+e^{i\mu}\right)\left(\beta_3 i\mu+i\mu e^{-i\mu}-\beta_4 e^{-i\mu}\right)+\left(\beta_3 i\mu+i\mu e^{i\mu}+\beta_4 e^{i\mu}\right)\left(1+e^{-i\mu}\right)=0$$

and

$$D_0^1(\mu) = \left(1-e^{i\mu}\right)\left(\beta_3 i\mu+i\mu e^{-i\mu}-\beta_4 e^{-i\mu}\right)+\left(\beta_3 i\mu+i\mu e^{i\mu}+\beta_4 e^{i\mu}\right)\left(1-e^{-i\mu}\right)=0$$

respectively. After simplifying these equations, we have

$$\Delta_0^0(\mu) = \left(1+e^{-i\mu}\right)\left[i\mu\left(\beta_1+1\right)\left(e^{i\mu}+1\right)+\beta_2\left(e^{i\mu}-1\right)\right]=0,$$

$$D_0^0(\mu) = \left(1+e^{-i\mu}\right)\left[i\mu\left(1+\beta_3\right)\left(e^{i\mu}+1\right)+\beta_4\left(e^{i\mu}-1\right)\right]=0$$

and

$$D_0^1(\mu) = \left(1-e^{-i\mu}\right)\left[i\mu\left(1-\beta_3\right)\left(e^{i\mu}-1\right)+\beta_4\left(e^{i\mu}+1\right)\right]=0.$$

The roots of these equations have the form

$$(2n+1)\pi,\ \ (2n+1)\pi+\frac{2\beta_2}{\beta_1+1}\frac{1}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right),$$

$$(2n+1)\pi,\ \ (2n+1)\pi+\frac{2\beta_4}{\beta_3+1}\frac{1}{(2n+1)\pi}+O\left(\frac{1}{n^2}\right)$$

and

$$2\pi n,\ 2\pi n+\frac{\beta_4}{1-\beta_3}\frac{1}{\pi n}+O\left(\frac{1}{n^2}\right)$$

respectively.

The characteristic determinants of $T_1^0(q)$, $T_2^0(q)$ and $T_2^1(q)$ can be written in the forms

$$\Delta^0(\mu) = \Delta_1^0(\mu) + i(\beta_1 - 1) s_\mu \cos\mu + o\left(\frac{1}{\mu}\right),$$
$$D^0(\mu) = D_1^0(\mu) + i(1 - \beta_3) s_\mu \cos\mu + o\left(\frac{1}{\mu}\right)$$

and

$$D^1(\mu) = D_1^1(\mu) + i(\beta_3 + 1) s_\mu \cos\mu + o\left(\frac{1}{\mu}\right),$$

where

$$\Delta_1^0(\mu) = \left(1 + e^{-i\mu}\right)\left[i\mu(\beta_1 + 1)\left(e^{i\mu} + 1\right) + \left(\beta_2 + \frac{1 - \beta_1}{2} c_\mu\right)\left(e^{i\mu} - 1\right)\right],$$
$$D_1^0(\mu) = \left(1 + e^{-i\mu}\right)\left[i\mu(1 + \beta_3)\left(e^{i\mu} + 1\right) + \left(\beta_4 + \frac{\beta_3 - 1}{2} c_\mu\right)\left(e^{i\mu} - 1\right)\right]$$

and

$$D_1^1(\mu) = \left(1 - e^{-i\mu}\right)\left[i\mu(1 - \beta_3)\left(e^{i\mu} - 1\right) + \left(\beta_4 - \frac{\beta_3 + 1}{2} c_\mu\right)\left(e^{i\mu} + 1\right)\right].$$

The investigation $T_2^1(q)$ is similar to the investigations of $T_1^1(q)$. The investigations of $T_1^0(q)$ and $T_2^0(q)$ are also similar to the investigations of $T_1^1(q)$. The difference is that, for the operators $T_1^0(q)$ and $T_2^0(q)$ we consider the functions and equations in $O\left(\frac{1}{n}\right)$-neighborhood of $(2n+1)\pi$ (we denote it by $U((2n+1)\pi)$) instead of $U(2\pi n)$, since the eigenvalues of $T_1^0(0)$ and $T_2^0(0)$ lie in $U((2n+1)\pi)$ while the eigenvalues of $T_1^1(0)$ and $T_2^1(0)$ lie in $U(2\pi n)$. Now instead of the triple $\{\Delta_0, \Delta_1, \Delta\}$ using the triples $\{\Delta_0^0, \Delta_1^0, \Delta^0\}$,$\{D_0^0, D_1^0, D^0\}$, $\{D_0^1, D_1^1, D^1\}$ and repeating the proof of Theorem 3.1 we obtain:

Theorem 3.3 *(a) If (1.12a) holds, then the large eigenvalues of $T_1^0(q)$ and $T_2^0(q)$ are simple and the square roots (with nonnegative real part) of the eigenvalues of these operators consist of the sequences $\{\mu_{n,1}^0\}$, $\{\mu_{n,2}^0\}$ and $\{\rho_{n,1}^0\}$, $\{\rho_{n,2}^0\}$ respectively, satisfying*

$$\mu_{n,1}^0 = (2n+1)\pi + o\left(\frac{1}{n}\right),$$

$$\mu_{n,2}^0 = (2n+1)\pi + \frac{2\beta_2}{\beta_1 + 1}\frac{1}{(2n+1)\pi} + o\left(\frac{1}{n}\right)$$

and

$$\rho_{n,1}^0 = (2n+1)\pi + o\left(\frac{1}{n}\right),$$

$$\rho_{n,2}^0 = (2n+1)\pi + \frac{2\beta_4}{\beta_3 + 1}\frac{1}{(2n+1)\pi} + o\left(\frac{1}{n}\right).$$

The normalized eigenfunctions corresponding to the eigenvalues $(\mu_{n,1}^0)^2$, $(\mu_{n,2}^0)^2$, $(\rho_{n,1}^0)^2$ *and* $(\rho_{n,2}^0)^2$ *have the same form*

$$\sqrt{2}\cos(2n+1)\pi x + O\left(\frac{1}{n}\right).$$

If there exists a subsequence $\{n_k\}$ *such that (1.12a) holds whenever* n *is replaced by* n_k, *then the systems of the root functions of* $T_1^0(q)$ *and* $T_2^0(q)$ *do not form Riesz bases.*

(b) If (1.12) holds, then the large eigenvalues of $T_2^1(q)$ *are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences* $\{\rho_{n,1}\}$ *and* $\{\rho_{n,2}\}$ *satisfying*

$$\rho_{n,1} = 2\pi n + o\left(\frac{1}{n}\right),$$

$$\rho_{n,2} = 2\pi n + \frac{\beta_4}{1-\beta_3}\frac{1}{\pi n} + o\left(\frac{1}{n}\right).$$

The normalized eigenfunctions $\phi_{n,1}(x)$ *and* $\phi_{n,2}(x)$ *corresponding to the eigenvalues* $(\rho_{n,1})^2$ *and* $(\rho_{n,2})^2$ *satisfy the same asymptotic formula*

$$\phi_{n,j}(x) = \sqrt{2}\cos 2\pi n x + O\left(\frac{1}{n}\right)$$

for $j = 1, 2$

If there exists a subsequence $\{n_k\}$ *such that (1.12) holds whenever* n *is replaced by* n_k, *then the system of the root functions of* $T_2^1(q)$ *does not form a Riesz basis.*

Now we investigate $T_1^0(q)$, $T_2^0(q)$ and $T_2^1(q)$ when q is an absolutely continuous function. The analogous formulas to (3.32) are

$$\Delta^0(\mu) = \Delta_0^0(\mu) + \frac{b}{\mu} + o\left(\frac{1}{\mu}\right) = 0, \tag{3.47}$$

$$D^0(\mu) = D_0^0(\mu) + \frac{d}{\mu} + o\left(\frac{1}{\mu}\right) = 0 \tag{3.48}$$

and

$$D^1(\mu) = D_0^1(\mu) + \frac{c}{\mu} + o\left(\frac{1}{\mu}\right) = 0, \tag{3.49}$$

where

$$b = \frac{i(1-\beta_1)}{2}\left[q(0) + q(1)\right],$$
$$d = \frac{i(\beta_3 - 1)}{2}\left[q(0) + q(1)\right]$$

and

$$c = \frac{i(\beta_3 + 1)}{2}\left[q(0) - q(1)\right].$$

Instead of (3.32) using (3.47), (3.48), (3.49) and repeating the proof of Theorem 3.2, we obtain:

Theorem 3.4 (*a*) *Let q be an absolutely continuous function. Suppose that for the operators $T_1^0(q)$ and $T_2^0(q)$ the conditions (1.13) and (1.14) for $\sigma = 0$ hold respectively. Then:*

The large eigenvalues of $T_1^0(q)$ and $T_2^0(q)$ are simple and the square roots (with nonnegative real part) of the eigenvalues of these operators consist of two sequences $\{\mu_{n,1}^0\}$, $\{\mu_{n,2}^0\}$ and $\{\rho_{n,1}^0\}$, $\{\rho_{n,2}^0\}$ respectively, satisfying

$$\mu_{n,1}^0 = (2n+1)\pi + \frac{2\beta_2 - i\sqrt{D_2}}{2(\beta_1+1)(2n+1)\pi} + o\left(\frac{1}{n}\right),$$

$$\mu_{n,2}^0 = (2n+1)\pi + \frac{2\beta_2 + i\sqrt{D_2}}{2(\beta_1+1)(2n+1)\pi} + o\left(\frac{1}{n}\right),$$

and

$$\rho_{n,1}^0 = (2n+1)\pi + \frac{2\beta_4 - i\sqrt{D_4}}{2(\beta_3+1)(2n+1)\pi} + o\left(\frac{1}{n}\right),$$

$$\rho_{n,2}^0 = (2n+1)\pi + \frac{2\beta_4 + i\sqrt{D_4}}{2(\beta_3+1)(2n+1)\pi} + o\left(\frac{1}{n}\right),$$

where $D_2 = 2\left(1-\beta_1^2\right)[q(0)+q(1)] - (2\beta_2)^2$ *and* $D_4 = 2\left(\beta_3^2-1\right)[q(0)+q(1)] - (2\beta_4)^2$.

The systems of the root functions of $T_1^0(q)$ *and* $T_2^0(q)$ *do not form a Riesz basis.*

(b) *Let* q *be an absolutely continuous function and (1.14) for* $\sigma = 1$ *hold.*

The large eigenvalues of $T_2^1(q)$ *are simple and the square roots (with nonnegative real part) of these eigenvalues consist of two sequences* $\{\rho_{n,1}\}$ *and* $\{\rho_{n,2}\}$ *satisfying*

$$\rho_{n,1} = 2\pi n + \frac{-2\beta_4 - i\sqrt{D_3}}{4(\beta_3-1)\pi n} + o\left(\frac{1}{n}\right),$$

$$\rho_{n,2} = 2\pi n + \frac{-2\beta_4 + i\sqrt{D_3}}{4(\beta_3-1)\pi n} + o\left(\frac{1}{n}\right),$$

where $D_3 = 2\left(\beta_3^2-1\right)[q(0)-q(1)] - (2\beta_4)^2$.

The system of the root functions of $T_2^1(q)$ *does not form a Riesz basis.*

4 NUMERICAL RESULTS

In the present chapter we estimate the small eigenvalues of the operators $T_1(q)$, $T_2(q)$, $T_3(q)$ and $T_4(q)$ defined in Chapter 2 by the numerical methods. We will focus only on the operator $T_1(q)$. The investigations of the operators $T_2(q), T_3(q)$ and $T_4(q)$ are similar. Our method is based on the equation (2.50) in Chapter 2 which gives the eigenvalues. To consider the small eigenvalues, first we prove (see Theorem 4.1) that the small eigenvalues also satisfy the equation (2.50) and using this equation we show that the eigenvalue $\lambda_{n,j}$ is either the root of (2.55) or the root of (2.56). To use the numerical methods, we take finite summations instead of the infinite series in the expressions (2.55) and (2.56) and show that the eigenvalues are close to the roots of the equations obtained by taking these finite summations. To find the roots of these equations, many numerical methods can be used such as the fixed point iteration and Newton method. Since it is not necessary to compute the derivatives of the functions $f_j(x)$, $j = 1, 2$, defined in (4.24), we choose the fixed point iteration method. Then using the Banach fixed point theorem, we prove that each of these equations containing the finite summations has a unique solution on the convenient set (see Theorem 4.2). Finally we give the error estimations and some examples.

For simplicity of calculations we assume that

$$q(x) = \sum_{k=1}^{\infty} q_k \cos 2\pi kx, \tag{4.1}$$

$$\sup |q(x)| := M < \infty, \tag{4.2}$$

$$\sum_{k=1}^{\infty} |q_k| := \frac{c}{2} < \infty, \tag{4.3}$$

and that

$$|\lambda_n(q) - \lambda_n(0)| \leq M, \quad \lambda_n(0) = (2\pi n)^2, \; n = 0, 1, 2, \ldots. \tag{4.4}$$

For $n_k \neq n$,we have

$$\left|\lambda_n - (2\pi n_k)^2\right| \geq \left|(2\pi n)^2 - (2\pi n_k)^2\right| - M \geq \left|4\pi^2 (n - n_k)(n + n_k)\right| - M \geq \delta(n), \quad (4.5)$$

where

$$\delta(n) = 4\pi^2 (2n - 1) - M.$$

To prove Theorem 4.1 we use the following lemmas.

Lemma 4.1 *If*

$$\delta(n) > \frac{4c}{3}, \quad (4.6)$$

then the following equalities hold:

$$\lim_{k\to\infty} R_k(\lambda_{n,j}) = 0, \quad (4.7)$$

and

$$\lim_{k\to\infty} R'_k(\lambda_{n,j}) = 0,$$

for $j = 1, 2$, *where* $R_k(\lambda_{n,j})$ *and* $R'_k(\lambda_{n,j})$ *are defined in (2.26) and (2.30), respectively.*

Proof. By the detailed estimations for C_{k+1} and M_{k+1} which were done in the Appendix, we have

$$|R_k(\lambda_{n,j})| = \left|\sum_{n_1,\ldots,n_{k+1}} \left\{C_{k+1}(q\Psi_{n,j}, \sin 2\pi n_{k+1}x) + M_{k+1}\left(q\Psi_{n,j}, \varphi^*_{n_{k+1}}\right)\right\}\right|$$

$$\leq \left|\sum_{n_1,\ldots,n_{k+1}} \frac{(q_{n-n_1} - q_{n+n_1})(q_{n_2-n_1} - q_{n_2+n_1})\ldots\left(q_{n_{k+1}-n_k} - q_{n_{k+1}+n_k}\right)(q\Psi_{n,j}, \sin 2\pi n_{k+1}x)}{2^{k+1}\left(\lambda_{n,j} - (2\pi n_1)^2\right)\left(\lambda_{n,j} - (2\pi n_2)^2\right)\ldots\left(\lambda_{n,j} - (2\pi n_{k+1})^2\right)}\right|.$$

One can easily see that there exists a nonnegative integer n_1^0 such that

$$|R_k(\lambda_{n,j})| \leq \left|\left(\sum_{n_1} \frac{(q_{n-n_1} - q_{n+n_1})}{\lambda_{n,j} - (2\pi n_1)^2}\right) S(n_2, n_3, \ldots, n_{k+1})\right|$$

where

$$S(n_2, n_3, \ldots, n_{k+1}) = \sum_{n_2,\ldots,n_{k+1}} \frac{\left(q_{n_2-n_1^0} - q_{n_2+n_1^0}\right)\ldots\left(q_{n_{k+1}-n_k} - q_{n_{k+1}+n_k}\right)(q\Psi_{n,j}, \sin 2\pi n_{k+1}x)}{2^{k+1}\left(\lambda_{n,j} - (2\pi n_1)^2\right)\left(\lambda_{n,j} - (2\pi n_2)^2\right)\ldots\left(\lambda_{n,j} - (2\pi n_{k+1})^2\right)}.$$

It follows from (4.3) and (4.5) that

$$\left|\sum_{n_1}\frac{(q_{n-n_1}-q_{n+n_1})}{\lambda_{n,j}-(2\pi n_1)^2}\right|\leq\frac{c}{\delta(n)}.$$

Repeating this process $k+1$ times and taking into account that $\|\Psi_{n,j}\|=1$ and that

$$|(q\Psi_{n,j},\sin 2\pi n_{k+1}x)|\leq\|q\Psi_{n,j}\|\,\|\sin 2\pi n_{k+1}x\|\leq\frac{M}{\sqrt{2}},$$

we obtain

$$|R_k(\lambda_{n,j})|\leq\frac{Mc^{k+1}}{\sqrt{2}2^{k+1}(\delta(n))^{k+1}}=\frac{M}{\sqrt{2}2^{k+1}}\left(\frac{c}{\delta(n)}\right)^{k+1}.$$

Thus this with (4.6) implies (4.7). In the same way we prove the same result for $R'_k(\lambda_{n,j})$. ∎

Lemma 4.2 *If (4.6) and the condition*

$$\delta(n)>C(\beta)M\left(\frac{1}{2}+\frac{121|\beta+1|^2}{4\pi^2|\beta-1|^2}+(A(\beta))^2\right)^{\frac{1}{2}},\tag{4.8}$$

hold, where

$$A(\beta)=\sup_{x\in[0,1]}\left|\frac{4(\beta+1)}{\beta-1}\left(x-\frac{1}{1+\beta}\right)\right|,\tag{4.9}$$

and $C(\beta)$ is defined in (4.14), then the inequality

$$|u_{n,j}|^2+|v_{n,j}|^2>0\tag{4.10}$$

is satisfied for $j=1,2$.

Proof. Suppose to the contrary that (4.10) does not hold. Then

$$u_{n,j}=0,\ v_{n,j}=0,$$

and since $\Psi_{n,j}$ is normalized we get by the formula (2.21) that

$$\|h_{n,j}\|=1,\tag{4.11}$$

where

$$h_{n,j}(x) = \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left[(\Psi_{n,j}, \sin 2\pi kx)\, \varphi_k(x) + (\Psi_{n,j}, \varphi_k^*) \cos 2\pi kx \right]. \tag{4.12}$$

To get a contradiction, it is enough to show that

$$\|h_{n,j}\| < 1 \tag{4.13}$$

for $j = 1, 2$. Since $\{g_i : i \in \mathbb{Z}\}$ (see (2.12)) is a Riesz basis, there exists a bounded and boundedly invertible operator A which takes the orthonormal basis $\{e_i : i \in \mathbb{Z}\}$ to this basis, say $Ae_{-k} = \varphi_k$ and $Ae_k = \cos 2\pi kx$ (see [27]). Thus there exists $C(\beta)$ such that

$$\|A\| \le C(\beta) \quad \& \quad A^{-1}h_{n,j}(x) = \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left[(\Psi_{n,j}, \sin 2\pi kx)\, e_{-k} + (\Psi_{n,j}, \varphi_k^*)\, e_k \right]. \tag{4.14}$$

Therefore by (4.14) and Parseval's equality we have

$$\|h_{n,j}\|^2 \le \left\|AA^{-1}h_{n,j}\right\|^2 \le C^2 \left\|A^{-1}h_{n,j}\right\|^2 = C^2 \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left[|(\Psi_{n,j}, \sin 2\pi kx)|^2 + |(\Psi_{n,j}, \varphi_k^*)|^2 \right]. \tag{4.15}$$

Now using (2.15), (4.5) and Bessel inequality we obtain that

$$\sum_{\substack{k=0 \\ k\neq n}}^{\infty} |(\Psi_{n,j}, \sin 2\pi kx)|^2 = \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left| \frac{(q\Psi_{n,j}, \sin 2\pi kx)}{\lambda_{n,j} - (2\pi k)^2} \right|^2$$

$$\le \frac{1}{(\delta(n))^2} \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left| \frac{1}{\sqrt{2}} \left(q\Psi_{n,j}, \sqrt{2}\sin 2\pi kx \right) \right|^2 \le \frac{\frac{1}{2}\|q\Psi_{n,j}\|^2}{(\delta(n))^2} \le \frac{\frac{1}{2}M^2}{(\delta(n))^2}. \tag{4.16}$$

By (2.16) we have

$$\sum_{\substack{k=0 \\ k\neq n}}^{\infty} |(\Psi_{n,j}, \varphi_k^*)|^2 = \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left| \frac{\gamma_1 k\, (q\Psi_{n,j}, \sin 2\pi kx)}{\left(\lambda_{n,j} - (2\pi k)^2\right)^2} + \frac{(q\Psi_{n,j}, \varphi_k^*)}{\lambda_{n,j} - (2\pi k)^2} \right|^2 \le 2(S_1 + S_2), \tag{4.17}$$

where

$$S_1 = \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left| \frac{\gamma_1 k\, (q\Psi_{n,j}, \sin 2\pi kx)}{\left(\lambda_{n,j} - (2\pi k)^2\right)^2} \right|^2, \quad S_2 = \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left| \frac{(q\Psi_{n,j}, \varphi_k^*)}{\lambda_{n,j} - (2\pi k)^2} \right|^2,$$

and γ_1 is defined in (2.14). Using (4.6) and taking into account that $c \geq 2M$, we obtain $\frac{12}{11}\pi^2(2n-1) > M$ and

$$\left|\lambda_{n,j} - (2\pi k)^2\right| > \left|(2\pi n)^2 - (2\pi k)^2\right| - M = 4\pi^2 \left|(n-k)(n+k)\right| - M$$
$$> \frac{32}{11}\pi^2 \left|(n-k)(n+k)\right| > \frac{32}{11}\pi^2 k.$$

Therefore, using the definition of γ_1 and arguing as in the proof of (4.16) we obtain

$$S_1 \leq \sum_{\substack{k=0\\k\neq n}}^{\infty} \left| \frac{11\,(\beta+1)}{2\pi\,(\beta-1)} \frac{(q\Psi_{n,j}, \sin 2\pi kx)}{(\lambda_{n,j} - (2\pi k)^2)} \right|^2 \leq \frac{121\,|\beta+1|^2 M^2}{8\,|\beta-1|^2 \pi^2 (\delta(n))^2}. \tag{4.18}$$

To estimate S_2, we use (2.10), (4.9) and the equality

$$(q\Psi_{n,j}, \varphi_k^*) = \left(\left[\frac{4\,(\beta+1)}{\beta-1} \left(x - \frac{1}{1+\beta} \right) \right] q\Psi_{n,j}, \cos 2\pi nx \right),$$

and then repeat the proof of (4.16) and get

$$S_2 = \sum_{\substack{k=0\\k\neq n}}^{\infty} \left| \frac{(q\Psi_{n,j}, \varphi_k^*)}{\lambda_{n,j} - (2\pi k)^2} \right|^2 \leq \frac{\frac{1}{2}(A(\beta)M)^2}{(\delta(n))^2}. \tag{4.19}$$

Thus using (4.15)-(4.19) we obtain

$$\|h_{n,j}\|^2 \leq C^2 \left(\frac{\frac{1}{2}M^2}{(\delta(n))^2} + \frac{121\,|\beta+1|^2 M^2}{4\,|\beta-1|^2 \pi^2 (\delta(n))^2} + \frac{(A(\beta)M)^2}{(\delta(n))^2} \right),$$

and hence

$$\|h_{n,j}\| \leq \frac{1}{\delta(n)} CM \left(\frac{1}{2} + \frac{121\,|\beta+1|^2}{4\,|\beta-1|^2 \pi^2} + (A(\beta))^2 \right)^{\frac{1}{2}},$$

which contradicts to (4.11) and completes the proof of the lemma. ■

Now we are ready to prove the following theorem.

Theorem 4.1 *If (4.6) and (4.8) hold then $\lambda_{n,j}$ is an eigenvalue of T_1 if and only if it is a root of the equation*

$$\left[\lambda - (2\pi n)^2 - Q_n - A(\lambda)\right] \left[\lambda - (2\pi n)^2 - P_n^* - A'(\lambda)\right] -$$
$$- \left[P_n + B(\lambda)\right] \left[\gamma_1 n + Q_n^* + B'(\lambda)\right] = 0. \tag{4.20}$$

Moreover $\lambda \in U(n) := \left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$ *is a double eigenvalue of* T_1 *if and only if it is a double root of (4.20).*

Proof. Using Lemma 4.1 and arguing as in the proof of Theorem 2.1 (b) in Chapter 2, we obtain

$$\left[\lambda_{n,j} - (2\pi n)^2 - Q_n - A(\lambda_{n,j})\right] u_{n,j} = \left[P_n + B(\lambda_{n,j})\right] v_{n,j}, \tag{4.21}$$

$$\left[\lambda_{n,j} - (2\pi n)^2 - P_n^* - A'(\lambda_{n,j})\right] v_{n,j} = \left[\gamma_1 n + Q_n^* + B'(\lambda_{n,j})\right] u_{n,j}, \tag{4.22}$$

We have the following cases:

Case 1. $u_{n,j} = 0$ then by Lemma 4.2 we have $v_{n,j} \neq 0$. Therefore from (4.21) and (4.22) we obtain $P_n + B(\lambda_{n,j}) = 0$ and $\lambda_{n,j} - (2\pi n)^2 - P_n^* - A'(\lambda_{n,j}) = 0$ which mean that (4.20) holds.

Case 2. $v_{n,j} = 0$ then again by Lemma 4.2 we have $u_{n,j} \neq 0$. Therefore from (4.21) and (4.22) we obtain $\lambda_{n,j} - (2\pi n)^2 - Q_n - A(\lambda_{n,j}) = 0$ and $\gamma_1 n + Q_n^* + B'(\lambda_{n,j}) = 0$ which mean that (4.20) again holds.

Case 3. Both $v_{n,j} \neq 0$ and $u_{n,j} \neq 0$. Multiplying the equations (4.21) and (4.22) side by side and then canceling $v_{n,j} u_{n,j}$ we obtain (4.20) . Thus in any case $\lambda_{n,j}$ is a root of (4.20).

The other parts of the proof are the same as in the proof of Theorem 2.1 (b) in Chapter 2. ■

By Theorem 4.1, the eigenvalue $\lambda_{n,j}$ is either the root of (2.55) or the root of (2.56). To use the numerical methods, we take finite summations instead of the infinite series in the expressions (2.55) and (2.56), and get

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left(Q_n + P_n^*\right) + f_j(\lambda), \tag{4.23}$$

for $j = 1$ and for $j = 2$, where

$$f_j(\lambda) = \frac{1}{2}\left(\widetilde{A}_{k,s}(\lambda) + A'_{k,s}(\lambda)\right) + (-1)^j \frac{1}{2}\sqrt{\Delta_{k,s}(\lambda)}, \tag{4.24}$$

and the functions $\widetilde{A}_{k,s}(\lambda)$, $A'_{k,s}(\lambda)$ and $\Delta_{k,s}(\lambda)$ are defined and investigated in the Appendix. (see (A.5), (A.6), and (A.7)) By (A.1) and (A.2) in the Appendix we have

$$(Q_n + P_n^*) = 0.$$

Therefore (4.23) becomes

$$\lambda = (2\pi n)^2 + f_j(\lambda). \tag{4.25}$$

Now we prove that the eigenvalues of T_1 are close to the roots of (4.25).

Theorem 4.2 *Let (4.6) and (4.8) hold. Then for all x and y from $\left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$ the inequality*

$$|f_j(x) - f_j(y)| < K_n |x - y| \tag{4.26}$$

holds for $j = 1, 2,$ where

$$K_n = \frac{c^2}{4(\delta(n))(\delta(n) - c)} < \frac{9}{16}, \tag{4.27}$$

and for each j, the equation (4.25) has a unique solution $r_{n,j}$ on $\left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$.

Moreover

$$|\lambda_{n,j} - r_{n,j}| \leq \frac{2c^{k+2}}{2^k (\delta(n))^k (2\delta(n) - c)(1 - K_n)}, \tag{4.28}$$

for $j = 1, 2$ and $s \geq k$.

Proof. First let us prove (4.26) by using the mean-value theorem. For this we estimate $\left|f'_j(\lambda)\right|$. By (4.24) we have

$$\begin{aligned} \left|f'_j(\lambda)\right| &= \left|\frac{1}{2}\left(\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda) + \frac{d}{d\lambda}A'_{k,s}(\lambda)\right) + (-1)^j \frac{1}{4}\frac{\frac{d}{d\lambda}\Delta_{k,s}(\lambda)}{\sqrt{\Delta_{k,s}(\lambda)}}\right| \\ &\leq \frac{1}{2}\left(\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda)\right| + \left|\frac{d}{d\lambda}A'_{k,s}(\lambda)\right| + \frac{1}{2}\left|\frac{\frac{d}{d\lambda}\Delta_{k,s}(\lambda)}{\sqrt{\Delta_{k,s}(\lambda)}}\right|\right). \end{aligned} \tag{4.29}$$

By the estimations (A.11), (A.12) and (A.13) in the Appendix we prove that

$$\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda)\right| \leq \frac{K_n}{2}, \tag{4.30}$$

$$\left|\frac{d}{d\lambda}A'_{k,s}(\lambda)\right| \leq \frac{K_n}{2}, \tag{4.31}$$

and

$$\frac{\left|\frac{d}{d\lambda}\Delta_{k,s}(\lambda)\right|}{\left|\sqrt{\Delta_{k,s}(\lambda)}\right|} \leq 2K_n, \tag{4.32}$$

respectively. Hence by (4.29)-(4.32) we obtain

$$\left|f'_j(\lambda)\right| \leq K_n$$

and since K_n can be written as

$$K_n = \frac{c^2}{4(\delta(n))^2}\sum_{j=0}^{\infty}\left(\frac{c}{\delta(n)}\right)^j,$$

we get by (4.6) and the geometric series formula that $K_n < \dfrac{9}{16}$.

Since the inequality

$$\left|f'_j(\lambda)\right| \leq K_n < 1 \tag{4.33}$$

holds for all x and y from $\left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$, by the mean value theorem (4.26) holds, and the equation (4.25) has a unique solution $r_{n,j}$ on $\left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$ for each j $(j = 1, 2)$, by the contraction mapping theorem.

Now let us prove (4.28). Let

$$H_j(x) = x - (2\pi n)^2 - f_j(x). \tag{4.34}$$

Using the definition of $\{r_{n,j}\}$, we obtain

$$H_j(r_{n,j}) = 0,$$

for $j=1,2$. Therefore by (2.55) and (2.56) we have

$$|H_j(\lambda_{n,j})-H_j(r_{n,j})|=|H_j(\lambda_{n,j})|$$
$$=\left|\lambda_{n,j}-(2\pi n)^2-\frac{1}{2}(Q_n+P_n^*)-\frac{1}{2}\left(\widetilde{A}_{k,s}(\lambda_{n,j})+A'_{k,s}(\lambda_{n,j})\right)+(-1)^j\frac{1}{2}\sqrt{\Delta_{k,s}(\lambda_{n,j})}\right|$$
$$=\left|\frac{1}{2}(A(\lambda_{n,j})+A'(\lambda_{n,j}))+(-1)^j\frac{1}{2}\sqrt{\Delta(\lambda_{n,j})}-\frac{1}{2}\left(\widetilde{A}_{k,s}+A'_{k,s}\right)+(-1)^j\frac{1}{2}\sqrt{\Delta_{k,s}}\right|$$
$$\leq\frac{1}{2}\left(\left|A'(\lambda_{n,j})-A'_{k,s}(\lambda_{n,j})\right|+\left|A(\lambda_{n,j})-\widetilde{A}_{k,s}(\lambda_{n,j})\right|+\left|\sqrt{\Delta(\lambda_{n,j})}-\sqrt{\Delta_{k,s}(\lambda_{n,j})}\right|\right). \tag{4.35}$$

First let us estimate the first term of the right-hand side of (4.35).From the formula (A.6) in the Appendix and by the definition of $A'(\lambda_{n,j})$ in Section 2.1 we get

$$\left|\left(A'(\lambda_{n,j})-A'_{k,s}(\lambda_{n,j})\right)\right|\leq$$
$$\leq 2\{\sum_{n_1,\ldots,n_{k+1}=1}^{\infty}\frac{\left|(q_{n-n_1}+q_{n+n_1})(q_{n_2-n_1}+q_{n_2+n_1})\ldots\left(q_{n_{k+1}-n_k}+q_{n_{k+1}+n_k}\right)\left(q_{n-n_{k+1}}+q_{n+n_{k+1}}\right)\right|}{2^{k+2}\left(\lambda_{n,j}-(2\pi n_1)^2\right)\left(\lambda_{n,j}-(2\pi n_2)^2\right)\ldots\left(\lambda_{n,j}-(2\pi n_k)^2\right)\left(\lambda_{n,j}-(2\pi n_{k+1})^2\right)}+$$
$$+\sum_{n_1,\ldots,n_{k+2}=1}^{\infty}\frac{\left|(q_{n-n_1}+q_{n+n_1})(q_{n_2-n_1}+q_{n_2+n_1})\ldots\left(q_{n_{k+2}-n_{k+1}}+q_{n_{k+2}+n_{k+1}}\right)\left(q_{n-n_{k+2}}+q_{n+n_{k+2}}\right)\right|}{2^{k+3}\left(\lambda_{n,j}-(2\pi n_1)^2\right)\left(\lambda_{n,j}-(2\pi n_2)^2\right)\ldots\left(\lambda_{n,j}-(2\pi n_{k+1})^2\right)\left(\lambda_{n,j}-(2\pi n_{k+2})^2\right)}+$$
$$+\cdots\},$$

for $s\geq k$ (see (A.15) in the Appendix). Arguing as in the proof of Lemma 4.1 and using the geometric series formula, we obtain

$$\left|\left(A'(\lambda_{n,j})-A'_{k,s}(\lambda_{n,j})\right)\right|\leq\frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)}, \tag{4.36}$$

Similarly, from the formula (A.5) in the Appendix and by the definition of $A(\lambda_{n,j})$ in Section 2.1, for the second term of the right-hand side of (4.35), we get

$$\left|A(\lambda_{n,j})-\widetilde{A}_{k,s}(\lambda_{n,j})\right|\leq\frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)} \tag{4.37}$$

for $s\geq k$ (see (A.14) in the Appendix). Using (A.7) and (A.8) in the Appendix, for the third term of the right-hand side of (4.35) we get

$$\begin{aligned}\left|\sqrt{\Delta\left(\lambda_{n,j}\right)}-\sqrt{\Delta_{k,s}\left(\lambda_{n,j}\right)}\right| &= \left|\left(A\left(\lambda_{n,j}\right)-A'\left(\lambda_{n,j}\right)-q_{2n}\right)-\left(\widetilde{A}_{k,s}\left(\lambda_{n,j}\right)-A'_{k,s}\left(\lambda_{n,j}\right)-q_{2n}\right)\right| \\ &\leq \left|A\left(\lambda_{n,j}\right)-\widetilde{A}_{k,s}\left(\lambda_{n,j}\right)\right|+\left|A'\left(\lambda_{n,j}\right)-A'_{k,s}\left(\lambda_{n,j}\right)\right| \\ &\leq \frac{2c^{k+2}}{2^k\left(\delta\left(n\right)\right)^k\left(2\delta\left(n\right)-c\right)} \end{aligned} \tag{4.38}$$

by (4.36) and (4.37).

Hence by (4.35)-(4.38) we obtain

$$\left|H_j\left(\lambda_{n,j}\right)-H_j\left(r_{n,j}\right)\right| \leq \frac{2c^{k+2}}{2^k\left(\delta\left(n\right)\right)^k\left(2\delta\left(n\right)-c\right)}, \tag{4.39}$$

for $j=1,2$.

To apply the mean value theorem we estimate $\left|H'_j\left(\lambda\right)\right|$:

$$\left|H'_j\left(\lambda\right)\right| = \left|1-f'_j\left(\lambda\right)\right| \geq \left|1-\left|f'_j\left(\lambda\right)\right|\right| \geq 1-K_n. \tag{4.40}$$

By the mean value formula, (4.39) and (4.40) we get

$$\left|H_j\left(\lambda_{n,j}\right)-H_j\left(r_{n,j}\right)\right| = \left|H'_j\left(\xi\right)\right|\left|\lambda_{n,j}-r_{n,j}\right|, \quad \xi\in\left[\left(2\pi n\right)^2-M,\left(2\pi n\right)^2+M\right],$$

$$\begin{aligned}\left|\lambda_{n,j}-r_{n,j}\right| &= \frac{\left|H_j\left(\lambda_{n,j}\right)-H_j\left(r_{n,j}\right)\right|}{\left|H'_j\left(\xi\right)\right|} \\ &\leq \frac{2c^{k+2}}{2^k\left(\delta\left(n\right)\right)^k\left(2\delta\left(n\right)-c\right)\left(1-K_n\right)},\end{aligned}$$

for $j=1,2$. ■

Now let us approximate $r_{n,j}$ by the fixed point iterations:

$$x_{n,i+1} = \left(2\pi n\right)^2+f_1\left(x_{n,i}\right), \tag{4.41}$$

and

$$y_{n,i+1} = \left(2\pi n\right)^2+f_2\left(y_{n,i}\right), \tag{4.42}$$

where $f_j\left(x\right)$ $\left(j=1,2\right)$ is defined in (4.24).

First, using (A.7), (A.9) and (A.10) in the Appendix, we get

$$
\begin{aligned}
|f_j(\lambda_{n,j})| &\leq \frac{1}{2}\left|\widetilde{A}_{k,s}(\lambda_{n,j}) + A'_{k,s}(\lambda_{n,j})\right| + \left|(-1)^j\frac{1}{2}\sqrt{\Delta_{k,s}(\lambda_{n,j})}\right| \\
&= \frac{1}{2}\left(\left|\widetilde{A}_{k,s}(\lambda_{n,j})\right| + \left|A'_{k,s}(\lambda_{n,j})\right| + \sqrt{\Delta_{k,s}(\lambda_{n,j})}\right) \\
&\leq \frac{|q_{2n}|}{2} + \frac{c^2}{2\delta(n) - c}. \qquad (4.43)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left|f_j\left((2\pi n)^2\right)\right| &\leq \frac{1}{2}\left|\left(\widetilde{A}_{k,s}\left((2\pi n)^2\right) + A'_{k,s}\left((2\pi n)^2\right)\right)\right| + \frac{1}{2}\sqrt{\Delta_{k,s}\left((2\pi n)^2\right)} \\
&\leq \frac{1}{2}\left(\left|\widetilde{A}_{k,s}\left((2\pi n)^2\right)\right| + \left|A'_{k,s}\left((2\pi n)^2\right)\right| + \sqrt{\Delta_{k,s}\left((2\pi n)^2\right)}\right) \\
&\leq \frac{1}{2}\left(\frac{c^2}{8\pi^2(2n-1)}\sum_{j=0}^{k}\left(\frac{c}{8\pi^2(2n-1)}\right)^j + |-q_{2n}| + \frac{c^2}{8\pi^2(2n-1)-c}\right) \\
&\leq \frac{1}{2}\left(\frac{c^2}{8\pi^2(2n-1)}\frac{1}{1-\frac{c}{8\pi^2(2n-1)}} + |q_{2n}| + \frac{c^2}{8\pi^2(2n-1)-c}\right) \\
&= \frac{1}{2}\left(\frac{c^2}{8\pi^2(2n-1)-c} + |q_{2n}| + \frac{c^2}{8\pi^2(2n-1)-c}\right) \\
&= \frac{|q_{2n}|}{2} + \frac{c^2}{8\pi^2(2n-1)-c}. \qquad (4.44)
\end{aligned}
$$

Theorem 4.3 *If (4.6) and (4.8) hold then for the sequence $\{x_{n,i}\}$ and $\{y_{n,i}\}$ defined by (4.41) and (4.42), the following estimations hold:*

$$
|x_{n,i} - r_{n,1}| \leq K_n^i\left(\frac{|q_{2n}|}{2(1-K_n)} + \frac{c^2}{(1-K_n)\left(8\pi^2(2n-1)-c\right)}\right), \qquad (4.45)
$$

$$
|y_{n,i} - r_{n,2}| \leq K_n^i\left(\frac{|q_{2n}|}{2(1-K_n)} + \frac{c^2}{(1-K_n)\left(8\pi^2(2n-1)-c\right)}\right), \qquad (4.46)
$$

for $i = 1, 2, 3, \ldots$, where K_n is defined in (4.27).

Proof. Without loss of generality we can take $x_{n,0} = (2\pi n)^2$. By (4.34) and (4.41) we have

$$
\begin{aligned}
|x_{n,i} - r_{n,1}| &= \left|(2\pi n)^2 + f_1(x_{n,i-1}) - \left((2\pi n)^2 + f_1(r_{n,1})\right)\right| = |f_1(x_{n,i-1}) - f_1(r_{n,1})| \\
&< K_n|x_{n,i-1} - r_{n,1}| < K_n^i|x_{n,0} - r_{n,1}|.
\end{aligned}
$$

Therefore it is enough to estimate $|x_{n,0} - r_{n,1}|$. By definitions of $r_{n,j}$ and $x_{n,0}$ we obtain

$$r_{n,1} - x_{n,0} = f_1(r_{n,1}) + (2\pi n)^2 - x_{n,0} = f_1(r_{n,1}) - f_1(x_{n,0}) + f_1\left((2\pi n)^2\right)$$

and by the mean value theorem there exists $x \in \left[(2\pi n)^2 - M, (2\pi n)^2 + M\right]$ such that

$$f_1(r_{n,1}) - f_1(x_{n,0}) = f_1'(x)(r_{n,1} - x_{n,0}).$$

These two equalities imply that

$$(r_{n,j} - x_{n,0})(1 - f_1'(x)) = f_1\left((2\pi n)^2\right).$$

Hence by (4.33) and (4.44) we get

$$(r_{n,1} - x_{n,0}) \le \frac{f_1\left((2\pi n)^2\right)}{1 - K_n} \le \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right)$$

and

$$|x_{n,i} - r_{n,1}| \le K_n^i \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right).$$

One can easily show in a similar way to (4.45) that

$$|y_{n,i} - r_{n,2}| \le K_n^i \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right)$$

for the iteration (4.42). ■

Thus by (4.28), (4.45) and (4.46) we have the approximations $x_{n,i}$ and $y_{n,i}$ for $\lambda_{n,1}$ and $\lambda_{n,2}$, respectively, with the errors

$$E_{n,i} =: |\lambda_{n,1} - x_{n,i}|$$
$$< \frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n) - c)(1 - K_n)} + K_n^i \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right)$$

$$E'_{n,i} =: |\lambda_{n,2} - y_{n,i}|$$
$$< \frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n) - c)(1 - K_n)} + K_n^i \left(\frac{|q_{2n}|}{2(1 - K_n)} + \frac{c^2}{(1 - K_n)(8\pi^2(2n-1) - c)}\right).$$

Remark 4.1 *If $q(x)=\sum_{k=1}^{p} q_k\cos(2k\pi x)$, where p is a finite positive integer, then it follows from the formulas (4.41) and (4.42) that for $n\geq s+p+1$*

$$x_{n,i}=y_{n,i}=(2\pi n)^2\,,$$

since the multiplicands $(q_{n-n_1}-q_{n+n_1})$ and $(q_{n-n_1}+q_{n+n_1})$ in $f_1(x_{n,i})$ and $f_2(y_{n,i})$ are zero.

4.1 Numerical Examples

First we give the algorithm for the estimations of the small eigenvalues.

Step1. Compute the followings for $n=1,2,\ldots 10$.

$$\begin{aligned}
c_n &= \int_0^1 q(x)\cos 2\pi nx dx,\ s_n=\int_0^1 q(x)\sin 2\pi nx dx,\\
c_{n,1} &= \int_0^1 xq(x)\cos 2\pi nx dx,\ s_{n,1}=\int_0^1 xq(x)\sin 2\pi nx dx,\\
c_{n,2} &= \int_0^1 x^2q(x)\cos 2\pi nx dx,\ s_{n,2}=\int_0^1 x^2q(x)\sin 2\pi nx dx.
\end{aligned}$$

$$\begin{aligned}
Q_n &= -\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx+\frac{2(\beta+1)}{\beta-1}c_{2n,1}-\frac{2\beta}{\beta-1}c_{2n},\\
P_n^* &= \frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx+\frac{2(\beta+1)}{\beta-1}c_{2n,1}-\frac{2}{\beta-1}c_{2n},\\
P_n &= \frac{1}{2}s_{2n},\\
Q_n^* &= -8\left(\frac{\beta+1}{\beta-1}\right)^2 s_{2n,2}+8\left(\frac{\beta+1}{\beta-1}\right)^2 s_{2n,1}-\frac{8\beta}{(\beta-1)^2}s_{2n},
\end{aligned}$$

$$\gamma_1=\frac{16\pi(\beta+1)}{\beta-1}.$$

Step2. Define the following functions:

$$\varphi_n(x):=\frac{4(\beta+1)}{\beta-1}\left(\frac{\beta}{1+\beta}-x\right)\sin 2\pi nx,$$

$$\varphi_n^*(x) := \frac{4\left(\overline{\beta}+1\right)}{\overline{\beta}-1}\left(x-\frac{1}{1+\overline{\beta}}\right)\cos 2\pi n x,$$

$$a_1(\lambda) = \frac{\left(q\varphi_{n_1}, \sin 2\pi n x\right)}{\lambda-(2\pi n_1)^2} + \frac{\gamma_1 n_1\left(q\cos 2\pi n_1 x, \sin 2\pi n x\right)}{\left(\lambda-(2\pi n_1)^2\right)^2},$$
$$b_1(\lambda) = \frac{\left(q\cos 2\pi n_1 x, \sin 2\pi n x\right)}{\lambda-(2\pi n_1)^2},$$

$$a_{k+1}(\lambda) = \frac{\left(q\varphi_{n_{k+1}}, \sin 2\pi n_k x\right)}{\lambda-(2\pi n_{k+1})^2} + \frac{\gamma_1 n_{k+1}\left(q\cos 2\pi n_{k+1} x, \sin 2\pi n_k x\right)}{\left(\lambda-(2\pi n_{k+1})^2\right)^2},$$
$$b_{k+1}(\lambda) = \frac{\left(q\cos 2\pi n_{k+1} x, \sin 2\pi n_k x\right)}{\lambda-(2\pi n_{k+1})^2},$$
$$A_{k+1}(\lambda) = \frac{\left(q\varphi_{n_{k+1}}, \varphi_{n_k}^*\right)}{\lambda-(2\pi n_{k+1})^2} + \frac{\gamma_1 n_{k+1}\left(q\cos 2\pi n_{k+1} x, \varphi_{n_k}^*\right)}{\left(\lambda-(2\pi n_{k+1})^2\right)^2},$$
$$B_{k+1}(\lambda) = \frac{\left(q\cos 2\pi n_{k+1} x, \varphi_{n_k}^*\right)}{\lambda-(2\pi n_{k+1})^2};\ k=1,2,\ldots.$$

$$C_1(\lambda) =: a_1(\lambda),\ M_1(\lambda) =: b_1(\lambda),$$
$$C_2(\lambda) =: a_1a_2 + b_1A_2 = C_1a_2 + M_1A_2,\ M_2(\lambda) =: a_1b_2 + b_1B_2 = C_1b_2 + M_1B_2,$$
$$C_{k+1}(\lambda) =: C_k a_{k+1} + M_k A_{k+1},\ M_{k+1}(\lambda) =: C_k b_{k+1} + M_k B_{k+1};\ k=1,2,\ldots,$$

$$\alpha_{m,s}(\lambda) = \sum_{n_1,\ldots,n_m=1}^{s}\left[C_m\left(q\varphi_n, \sin 2\pi n_m x\right) + M_m\left(q\varphi_n, \varphi_{n_m}^*\right)\right],$$
$$\beta_{m,s}(\lambda) = \sum_{n_1,\ldots,n_m=1}^{s}\left[C_m\left(q\cos 2\pi n x, \sin 2\pi n_m x\right) + M_m\left(q\cos 2\pi n x, \varphi_{n_m}^*\right)\right],$$
$$\widetilde{A}_{k,s}(\lambda) = \sum_{m=1}^{k}\alpha_{m,s}(\lambda),\ \widetilde{B}_{k,s}(\lambda) = \sum_{m=1}^{k}\beta_{m,s}(\lambda),$$

$$\widetilde{C}_1(\lambda) = A_1(\lambda) = \frac{\left(q\varphi_{n_1}, \varphi_n^*\right)}{\lambda - (2\pi n_1)^2} + \frac{\gamma_1 n_1 \left(q\cos 2\pi n_1 x, \varphi_n^*\right)}{\left(\lambda - (2\pi n_1)^2\right)^2},$$
$$\widetilde{M}_1(\lambda) = B_1(\lambda) = \frac{\left(q\cos 2\pi n_1 x, \varphi_n^*\right)}{\lambda - (2\pi n_1)^2}.$$

$$\widetilde{C}_{k+1}(\lambda) = \widetilde{C}_k a_{k+1} + \widetilde{M}_k A_{k+1},\ \widetilde{M}_{k+1}(\lambda) = \widetilde{C}_k b_{k+1} + \widetilde{M}_k B_{k+1};\ k = 1, 2, \ldots,$$

$$\alpha'_{m,s}(\lambda) = \sum_{n_1,\ldots,n_m=1}^{s} \left[\widetilde{C}_m\left(q\cos 2\pi n x, \sin 2\pi n_m x\right) + \widetilde{M}_m\left(q\cos 2\pi n x, \varphi_{n_m}^*\right)\right],$$
$$\beta'_{m,s}(\lambda) = \sum_{n_1,\ldots,n_m=1}^{s} \left[\widetilde{C}_m\left(q\varphi_n, \sin 2\pi n_m x\right) + \widetilde{M}_m\left(q\varphi_n, \varphi_{n_m}^*\right)\right],$$
$$A'_{k,s}(\lambda) = \sum_{m=1}^{k} \alpha'_{m,s}(\lambda),\ B'_{k,s}(\lambda) = \sum_{m=1}^{k} \beta'_{m,s}(\lambda).$$

Step3. Define the following function.

$$\Delta_{k,s}(\lambda) = \left(Q_n - P_n^* + \widetilde{A}_{k,s}(\lambda) - A'_{k,s}(\lambda)\right)^2 + 4\left(P_n + \widetilde{B}_{k,s}(\lambda)\right)\left(\gamma_1 n + Q_n^* + B'_{k,s}(\lambda)\right).$$

Step4. Compute $x_{n,i+1}$ by the iteration (4.41) corresponding to the eigenvalue $\lambda_{n,1}$, with the initial value $x_{n,0}$:

$$x_{n,1} = (2\pi n)^2 + \frac{1}{2}\left(Q_n + P_n^*\right) + \frac{1}{2}\left(\widetilde{A}_k(x_{n,0}) + A'_k(x_{n,0})\right) - \frac{1}{2}\sqrt{\Delta_k(x_{n,0})},$$
$$x_{n,2} = (2\pi n)^2 + \frac{1}{2}\left(Q_n + P_n^*\right) + \frac{1}{2}\left(\widetilde{A}_k(x_{n,1}) + A'_k(x_{n,1})\right) - \frac{1}{2}\sqrt{\Delta_k(x_{n,1})},$$
$$\cdots$$
$$x_{n,i+1} = (2\pi n)^2 + \frac{1}{2}\left(Q_n + P_n^*\right) + \frac{1}{2}\left(\widetilde{A}_k(x_{n,i}) + A'_k(x_{n,i})\right) - \frac{1}{2}\sqrt{\Delta_k(x_{n,i})}.$$

Step5. Compute $y_{n,i+1}$ by the iteration (4.42) corresponding to the eigenvalue $\lambda_{n,2}$, with the initial value $y_{n,0}$:

$$y_{n,1} = (2\pi n)^2 + \frac{1}{2}(Q_n + P_n^*) + \frac{1}{2}\left(\widetilde{A}_k(y_{n,0}) + A'_k(y_{n,0})\right) + \frac{1}{2}\sqrt{\Delta_k(y_{n,0})},$$
$$y_{n,2} = (2\pi n)^2 + \frac{1}{2}(Q_n + P_n^*) + \frac{1}{2}\left(\widetilde{A}_k(y_{n,1}) + A'_k(y_{n,1})\right) + \frac{1}{2}\sqrt{\Delta_k(y_{n,1})},$$
$$\ldots$$
$$y_{n,i+1} = (2\pi n)^2 + \frac{1}{2}(Q_n + P_n^*) + \frac{1}{2}\left(\widetilde{A}_k(y_{n,i}) + A'_k(y_{n,i})\right) + \frac{1}{2}\sqrt{\Delta_k(y_{n,i})}.$$

Since $q(x) = cos(2\pi x)$ is a famous Mathieu potential and $q(x) = cos(2\pi x) + cos(4\pi x)$ is the generalization of the Mathieu potential, we consider these potentials in our examples.

Example 4.1 *For $q(x) = cos(2\pi x)$, $\beta = 2$, $k = 3$ and $s = 5$ with the initial approximations $x_{n,0} = 0$ and $y_{n,0} = 0$, we have the following table for the estimations of the small eigenvalues of $T_1(q)$. According to Remark 4.1, it is enough to compute only the first 6 eigenvalues for this case.*

From the table we can see that we have the same values for both of the iterations (4.41) and (4.42) corresponding to $\lambda_{n,1}$ and $\lambda_{n,2}$, respectively. This shows that the eigenvalues $\lambda_{n,1}$ and $\lambda_{n,2}$ are very close to each other or equal and they are close to $(2\pi n)^2$. In this table $x_{n,i}$ and $y_{n,i}$ denote the estimations for $\lambda_{n,1}$ and $\lambda_{n,2}$, respectively, where i is the number of the iterations.

n	i	$x_{n,i}$	$\lvert x_{n,i+1} - x_{n,i}\rvert$	$y_{n,i}$	$\lvert y_{n,i+1} - y_{n,i}\rvert$
1	0	0		0	
	1	39.478417597303	39.4784175973	39.478417597303	39.4784175973
	2	39.478417590249	7.05405e-09	39.478417590249	7.05405e-09
	3	39.478417590249	0	39.478417590249	0
2	0	0		0	
	1	157.907337842668	157.9073378427	157.907337842668	157.9073378427
	2	157.915781384907	0.00844354	157.915781384907	0.00844354
	3	157.915781234413	1.50494e-07	157.915781234413	1.50494e-07
	4	157.915781234416	2.70006e-12	157.915781234416	2.70006e-12
	5	157.915781234416	0	157.915781234416	0
3	0	0		0	
	1	355.304175232077	355.3041752321	355.304175232077	355.3041752321
	2	355.307024967954	0.00284974	355.307024967954	0.00284974
	3	355.307024949669	1.82848e-08	355.307024949669	1.82848e-08
	4	355.307024949669	1.13687e-13	355.307024949669	1.13687e-13
	5	355.307024949669	0	355.307024949669	0
4	0	0		0	
	1	631.653978047253	631.6539780473	631.653978047253	631.6539780473
	2	631.655586327175	0.00160828	631.655586327175	0.00160828
	3	631.655586321910	5.26495e-09	631.655586321910	5.26495e-09
	4	631.655586321910	1.13687e-13	631.655586321910	1.13687e-13
	5	631.655586321910	0	631.655586321910	0
5	0	0		0	
	1	986.960044322621	986.9600443226	986.960044322621	986.9600443226
	2	986.961143729834	0.00109941	986.961143729834	0.00109941
	3	986.961143729834	2.17722e-09	986.961143729834	2.17722e-09
	4	986.961143729834	0	986.961143729834	0
6	0	0		0	
	1	1421.222780453807	1421.2227804538	1421.222780453807	1421.2227804538
	2	1421.223609446167	0.000828992	1421.223609446167	0.000828992
	3	1421.223609445068	1.09912e-09	1421.223609445068	1.09912e-09
	4	1421.223609445068	0	1421.223609445068	0

Example 4.2 *For $q(x) = cos(2\pi x) + cos(4\pi x)$, $\beta = 2$, $k = 3$ and $s = 5$ with the initial approximations $x_{n,0} = 0$ and $y_{n,0} = 0$ we have the following table for the estimations of the small eigenvalues of $T_1(q)$. $x_{n,i}$ is the estimation for $\lambda_{n,1}$ and $y_{n,i}$ is the estimation for $\lambda_{n,2}$. According to Remark 4.1, it is enough to compute only the first 7 eigenvalues for this case.*

From the following table we can see that the first eigenvalues $\lambda_{1,1}$ and $\lambda_{1,2}$ are far from each but the other eigenvalues $\lambda_{n,1}$ and $\lambda_{n,2}$ are close to each other or equal and they are close to $(2\pi n)^2$. In this table $x_{n,i}$ and $y_{n,i}$ denote the estimations for $\lambda_{n,1}$ and $\lambda_{n,2}$, respectively, where i is the number of the iterations.

n	i	$x_{n,i}$	$\|x_{n,i+1} - x_{n,i}\|$	$y_{n,i}$	$\|y_{n,i+1} - y_{n,i}\|$
1	0	0		0	
	1	38.97842204110939	38.9784	39.97842204110939	39.9784
	2	38.97842421532967	2.17422e-06	39.97842429233646	2.25123e-06
	3	38.97842421532983	1.63425e-13	39.97842429233664	1.77636e-13
	4	38.97842421532983	0	39.97842429233664	0
2	0	0		0	
	1	157.90727564526605	157.9073	157.90743559977980	157.9074
	2	157.91576453269735	0.00848889	157.91578226434297	0.00834666
	3	157.91576438313135	1.49566e-07	157.91578211477969	1.49563e-07
	4	157.91576438313396	2.6148e-12	157.91578211478236	2.67164e-12
	5	157.91576438313396	0	157.91578211478236	0
3	0	0		0	
	1	355.29780466405339	355.2978	355.29796405424344	355.2980
	2	355.30781619970361	0.0100115	355.30781871744506	0.00985466
	3	355.30781611026259	8.9441e-08	355.30781862924766	8.81974e-08
	4	355.30781611026339	7.95808e-13	355.30781862924846	7.95808e-13
	5	355.30781611026339	0	355.30781862924846	0
4	0	0		0	
	1	631.65239992994259	631.6524	631.65239992994259	631.6524
	2	631.65611341414319	0.00371348	631.65611341414319	0.00371348
	3	631.65611339781697	1.63262e-08	631.65611339781697	1.63262e-08
	4	631.65611339781708	1.13687e-13	631.65611339781708	1.13687e-13
	5	631.65611339781708	0	631.65611339781708	0
5	0	0		0	
	1	986.95934179624237	986.9593	986.95934179624237	986.9593
	2	986.96154063378719	0.00219884	986.96154063378719	0.00219884
	3	986.96154062804419	5.743e-09	986.96154062804419	5.743e-09
	4	986.96154062804419	0	986.96154062804419	0
6	0	0		0	
	1	1421.22238506661188	1421.2224	1421.22238506661188	1421.2224
	2	1421.22392680582675	0.00154174	1421.22392680582675	0.00154174
	3	1421.22392680316057	2.66618e-09	1421.22392680316057	2.66618e-09
	4	1421.22392680316057	0	1421.22392680316057	0
7	0	0		0	
	1	1934.44220931027280	1934.4422	1934.44220931027280	1934.4422
	2	1934.44272647093135	0.000517161	1934.44272647093135	0.000517161
	3	1934.44272647078719	0.000000000144	1934.44272647078719	0.000000000144
	4	1934.44272647078719	0	1934.44272647078719	0

Appendix

Since $q(x)$ has the form (4.1), by (2.44)-(2.48) we have the followings:

$$\int_0^1 xq(x)dx = 0,$$

$$\begin{aligned} c_n &= \int_0^1 q(x)\cos 2\pi nx dx = \frac{q_n}{2}, \\ s_n &= \int_0^1 q(x)\sin 2\pi nx dx = 0, \\ c_{n,1} &= \int_0^1 xq(x)\cos 2\pi nx dx = \frac{q_n}{4}, \\ s_{n,1} &= \int_0^1 xq(x)\sin 2\pi nx dx = s_{n,2} = \int_0^1 x^2 q(x)\sin 2\pi nx dx, \end{aligned}$$

$$Q_n = -\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + \frac{2(\beta+1)}{\beta-1}c_{2n,1} - \frac{2\beta}{\beta-1}c_{2n} = -\frac{q_{2n}}{2}, \quad \text{(A.1)}$$

$$P_n^* = \frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + \frac{2(\beta+1)}{\beta-1}c_{2n,1} - \frac{2}{\beta-1}c_{2n} = \frac{q_{2n}}{2}, \quad \text{(A.2)}$$

$$P_n = \frac{1}{2}s_{2n} = 0, \quad \text{(A.3)}$$

$$Q_n^* = -8\left(\frac{\beta+1}{\beta-1}\right)^2 s_{2n,2} + 8\left(\frac{\beta+1}{\beta-1}\right)^2 s_{2n,1} - \frac{8\beta}{(\beta-1)^2}s_{2n} = 0, \quad \text{(A.4)}$$

$$Q_n - P_n^* = -2c_{2n} = -q_{2n},$$

and

$$Q_n + P_n^* = 0.$$

We also evaluate the following functions defined in Section 2.1:

$$\begin{aligned} a_1(\lambda) &= \frac{(q\varphi_{n_1}, \sin 2\pi nx)}{\lambda - (2\pi n_1)^2} + \frac{\gamma_1 n_1 (q\cos 2\pi n_1 x, \sin 2\pi nx)}{\left(\lambda - (2\pi n_1)^2\right)^2} = \frac{q_{n-n_1} - q_{n+n_1}}{2\left(\lambda - (2\pi n_1)^2\right)}, \\ b_1(\lambda) &= \frac{(q\cos 2\pi n_1 x, \sin 2\pi nx)}{\lambda - (2\pi n_1)^2} = 0, \end{aligned}$$

$$a_{k+1}(\lambda)=\frac{\left(q\varphi_{n_{k+1}},\sin 2\pi n_k x\right)}{\lambda-(2\pi n_{k+1})^2}+\frac{\gamma_1 n_{k+1}\left(q\cos 2\pi n_{k+1}x,\sin 2\pi n_k x\right)}{\left(\lambda-(2\pi n_{k+1})^2\right)^2}=\frac{q_{n_{k+1}-n_k}-q_{n_{k+1}+n_k}}{2\left(\lambda-(2\pi n_{k+1})^2\right)},$$

$$b_{k+1}(\lambda)=\frac{\left(q\cos 2\pi n_{k+1}x,\sin 2\pi n_k x\right)}{\lambda-(2\pi n_{k+1})^2}=0,$$

$$A_{k+1}(\lambda)=\frac{\left(q\varphi_{n_{k+1}},\varphi^*_{n_k}\right)}{\lambda-(2\pi n_{k+1})^2}+\frac{\gamma_1 n_{k+1}\left(q\cos 2\pi n_{k+1}x,\varphi^*_{n_k}\right)}{\left(\lambda-(2\pi n_{k+1})^2\right)^2}=\frac{\gamma_1 n_{k+1}\left(q_{n_{k+1}-n_k}+q_{n_{k+1}+n_k}\right)}{2\left(\lambda-(2\pi n_{k+1})^2\right)^2},$$

$$B_{k+1}(\lambda)=\frac{\left(q\cos 2\pi n_{k+1}x,\varphi^*_{n_k}\right)}{\lambda-(2\pi n_{k+1})^2}=\frac{q_{n_{k+1}-n_k}+q_{n_{k+1}+n_k}}{2\left(\lambda-(2\pi n_{k+1})^2\right)};\ k=1,2,\ldots,$$

$$C_1(\lambda)=a_1(\lambda)=\frac{q_{n-n_1}-q_{n+n_1}}{2\left(\lambda-(2\pi n_1)^2\right)},\quad M_1(\lambda)=b_1(\lambda)=0,$$

$$C_2(\lambda)=C_1a_2+M_1A_2=a_1a_2,\quad M_2(\lambda)=C_1b_2+M_1B_2=0,$$

$$C_{k+1}(\lambda)=C_ka_{k+1}+M_kA_{k+1}=a_1a_2\ldots a_ka_{k+1},$$

$$M_{k+1}(\lambda)=C_kb_{k+1}+M_kB_{k+1}=0;\quad k=1,2,\ldots,$$

$$\widetilde{C}_1(\lambda)=A_1(\lambda)=\frac{\left(q\varphi_{n_1},\varphi^*_n\right)}{\lambda-(2\pi n_1)^2}+\frac{\gamma_1 n_1\left(q\cos 2\pi n_1x,\varphi^*_n\right)}{\left(\lambda-(2\pi n_1)^2\right)^2}$$
$$=\frac{\gamma_1 n_1\left(q\cos 2\pi n_1x,\varphi^*_n\right)}{\left(\lambda-(2\pi n_1)^2\right)^2}=\frac{\gamma_1 n_1\left(q_{n-n_1}+q_{n+n_1}\right)}{2\left(\lambda-(2\pi n_1)^2\right)^2},$$

$$\widetilde{M}_1(\lambda)=B_1(\lambda)=\frac{\left(q\cos 2\pi n_1x,\varphi^*_n\right)}{\lambda-(2\pi n_1)^2}=\frac{\left(q_{n-n_1}+q_{n+n_1}\right)}{2\left(\lambda-(2\pi n_1)^2\right)},$$

$$\widetilde{C}_2(\lambda)=\widetilde{C}_1a_2+\widetilde{M}_1A_2$$
$$=\frac{\gamma_1 n_1\left(q_{n-n_1}+q_{n+n_1}\right)}{2\left(\lambda-(2\pi n_1)^2\right)^2}\frac{\left(q_{n_2-n_1}-q_{n_2+n_1}\right)}{2\left(\lambda-(2\pi n_2)^2\right)}+\frac{\left(q_{n-n_1}+q_{n+n_1}\right)}{2\left(\lambda-(2\pi n_1)^2\right)}\frac{\gamma_1 n_2\left(q_{n_2-n_1}+q_{n_2+n_1}\right)}{2\left(\lambda-(2\pi n_2)^2\right)^2}$$
$$=\frac{\gamma_1 n_1\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)}{2^2\left(\lambda-(2\pi n_1)^2\right)^2\left(\lambda-(2\pi n_2)^2\right)}+\frac{\gamma_1 n_2\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)}{2^2\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)^2},$$

$$\widetilde{M}_2(\lambda)=\widetilde{C}_1b_2+\widetilde{M}_1B_2=\widetilde{M}_1B_2=\frac{\left(q_{n-n_1}+q_{n+n_1}\right)}{2\left(\lambda-(2\pi n_1)^2\right)}\frac{\left(q_{n_2-n_1}+q_{n_2+n_1}\right)}{2\left(\lambda-(2\pi n_2)^2\right)}$$
$$=\frac{\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)}{2^2\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)},$$

and

$$\widetilde{M}_{k+1}(\lambda) = \widetilde{C}_k b_{k+1} + \widetilde{M}_k B_{k+1} = \widetilde{M}_k B_{k+1} = B_1 B_2 \ldots B_{k+1}$$

$$= \frac{(q_{n-n_1} + q_{n+n_1})(q_{n_2-n_1} + q_{n_2+n_1}) \ldots (q_{n_{k+1}-n_k} + q_{n_{k+1}+n_k})}{2^{k+1}\left(\lambda - (2\pi n_1)^2\right)\left(\lambda - (2\pi n_2)^2\right) \ldots \left(\lambda - (2\pi n_{k+1})^2\right)}; \; k = 1, 2, \ldots.$$

Using these functions, we obtain the followings:

$$\alpha_{m,s}(\lambda) := \sum_{n_1,\ldots,n_m=1}^{s} \left[C_m(q\varphi_n, \sin 2\pi n_m x) + M_m\left(q\varphi_n, \varphi^*_{n_m}\right)\right]$$

$$= \sum_{n_1,\ldots,n_m=1}^{s} a_1 a_2 \ldots a_m (q\varphi_n, \sin 2\pi n_m x)$$

$$= \sum_{n_1,\ldots,n_m=1}^{s} \left\{ \frac{q_{n-n_1} - q_{n+n_1}}{2\left(\lambda - (2\pi n_1)^2\right)} \frac{q_{n_2-n_1} - q_{n_2+n_1}}{2\left(\lambda - (2\pi n_2)^2\right)} \ldots \frac{q_{n_m-n_{m-1}} - q_{n_m+n_{m-1}}}{2\left(\lambda - (2\pi n_m)^2\right)} \frac{q_{n-n_m} - q_{n+n_m}}{2} \right\}$$

$$= \sum_{n_1,\ldots,n_m=1}^{s} \left\{ \frac{(q_{n-n_1} - q_{n+n_1})(q_{n_2-n_1} - q_{n_2+n_1}) \ldots (q_{n_m-n_{m-1}} - q_{n_m+n_{m-1}})(q_{n-n_m} - q_{n+n_m})}{2^{m+1}\left(\lambda - (2\pi n_1)^2\right)\left(\lambda - (2\pi n_2)^2\right) \ldots \left(\lambda - (2\pi n_m)^2\right)} \right\},$$

$$\beta_{m,s}(\lambda) := \sum_{n_1,\ldots,n_m=1}^{s} \left[C_m(q\cos 2\pi n x, \sin 2\pi n_m x) + M_m\left(q\cos 2\pi n x, \varphi^*_{n_m}\right)\right] = 0,$$

$$\widetilde{A}_{k,s}(\lambda) := \sum_{m=1}^{k} \alpha_{m,s}(\lambda)$$

$$= \sum_{m=1}^{k} \sum_{n_1,\ldots,n_m=1}^{s} \frac{(q_{n-n_1} - q_{n+n_1})(q_{n_2-n_1} - q_{n_2+n_1}) \ldots (q_{n_m-n_{m-1}} - q_{n_m+n_{m-1}})(q_{n-n_m} - q_{n+n_m})}{2^{m+1}\left(\lambda - (2\pi n_1)^2\right)\left(\lambda - (2\pi n_2)^2\right) \ldots \left(\lambda - (2\pi n_m)^2\right)}$$

$$= \sum_{n_1=1}^{s} \frac{(q_{n-n_1} - q_{n+n_1})^2}{2^2\left(\lambda - (2\pi n_1)^2\right)} + \sum_{n_1,n_2=1}^{s} \frac{(q_{n-n_1} - q_{n+n_1})(q_{n_2-n_1} - q_{n_2+n_1})(q_{n-n_2} - q_{n+n_2})}{2^3\left(\lambda - (2\pi n_1)^2\right)\left(\lambda - (2\pi n_2)^2\right)} +$$

(A.5)

$$+ \ldots + \sum_{n_1,\ldots,n_k=1}^{s} \frac{(q_{n-n_1} - q_{n+n_1})(q_{n_2-n_1} - q_{n_2+n_1}) \ldots (q_{n_k-n_{k-1}} - q_{n_k+n_{k-1}})(q_{n-n_k} - q_{n+n_k})}{2^{k+1}\left(\lambda - (2\pi n_1)^2\right)\left(\lambda - (2\pi n_2)^2\right) \ldots \left(\lambda - (2\pi n_k)^2\right)},$$

$$\widetilde{B}_{k,s}(\lambda) := \sum_{m=1}^{k} \beta_{m,s}(\lambda) = 0,$$

$$\alpha'_{m,s}(\lambda) := \sum_{n_1,\ldots,n_m=1}^{s} \left[\widetilde{C}_m\left(q\cos 2\pi nx, \sin 2\pi n_m x\right) + \widetilde{M}_m\left(q\cos 2\pi nx, \varphi^*_{n_m}\right)\right]$$

$$= \sum_{n_1,\ldots,n_m=1}^{s} \widetilde{M}_m\left(q\cos 2\pi nx, \varphi^*_{n_m}\right)$$

$$= \sum_{n_1,\ldots,n_m=1}^{s} \frac{\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\ldots\left(q_{n_m-n_{m-1}}+q_{n_m+n_{m-1}}\right)}{2^{m+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\ldots\left(\lambda-(2\pi n_m)^2\right)}\left(q_{|n-n_m|}+q_{n+n_m}\right),$$

$$A'_{k,s}(\lambda) := \sum_{m=1}^{k} \alpha'_{m,s}(\lambda)$$

$$= \sum_{m=1}^{k}\sum_{n_1,\ldots,n_m=1}^{s} \frac{\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\ldots\left(q_{n_m-n_{m-1}}+q_{n_m+n_{m-1}}\right)\left(q_{n-n_m}+q_{n+n_m}\right)}{2^{m+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\ldots\left(\lambda-(2\pi n_m)^2\right)}$$

$$= \sum_{n_1=1}^{s} \frac{\left(q_{n-n_1}+q_{n+n_1}\right)^2}{2^2\left(\lambda-(2\pi n_1)^2\right)} + \sum_{n_1,n_2=1}^{s} \frac{\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\left(q_{n-n_2}+q_{n+n_2}\right)}{2^3\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)} +$$

$$+\ldots+\sum_{n_1,\ldots,n_k=1}^{s} \frac{\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\ldots\left(q_{n_k-n_{k-1}}+q_{n_k+n_{k-1}}\right)\left(q_{n-n_k}+q_{n+n_k}\right)}{2^{k+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\ldots\left(\lambda-(2\pi n_k)^2\right)}, \tag{A.6}$$

$$\Delta_{k,s}(\lambda) := \left(Q_n - P^*_n + \widetilde{A}_{k,s}(\lambda) - A'_{k,s}(\lambda)\right)^2 + 4\left(P_n + \widetilde{B}_{k,s}(\lambda)\right)\left(\gamma_1 n + Q^*_n + B'_{k,s}(\lambda)\right)$$

$$= \left(Q_n - P^*_n + \widetilde{A}_{k,s}(\lambda) - A'_{k,s}(\lambda)\right)^2$$

$$= \left(\widetilde{A}_{k,s}(\lambda) - A'_{k,s}(\lambda) - q_{2n}\right)^2, \tag{A.7}$$

and

$$\Delta(\lambda) = \left(A(\lambda) - A'(\lambda) - q_{2n}\right)^2. \tag{A.8}$$

Using (4.3) and (4.5), in (A.5) and (A.6), we obtain

$$\left|\widetilde{A}_{k,s}(\lambda)\right| = \left|\sum_{m=1}^{k}\alpha_{m,s}(\lambda)\right|$$

$$= \left|\sum_{m=1}^{k}\sum_{n_1,\dots,n_m=1}^{s}\frac{(q_{n-n_1}-q_{n+n_1})(q_{n_2-n_1}-q_{n_2+n_1})\dots\left(q_{n_m-n_{m-1}}-q_{n_m+n_{m-1}}\right)(q_{n-n_m}-q_{n+n_m})}{2^{m+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\dots\left(\lambda-(2\pi n_m)^2\right)}\right|$$

$$= \left|\begin{array}{c}\sum_{n_1=1}^{s}\frac{\left(q_{n-n_1}-q_{n+n_1}\right)^2}{2^2\left(\lambda-(2\pi n_1)^2\right)}+\sum_{n_1,n_2=1}^{s}\frac{\left(q_{n-n_1}-q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)\left(q_{n-n_2}-q_{n+n_2}\right)}{2^3\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)}+\\ +\dots+\sum_{n_1,\dots,n_k=1}^{s}\frac{\left(q_{n-n_1}-q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)\dots\left(q_{n_k-n_{k-1}}-q_{n_k+n_{k-1}}\right)\left(q_{n-n_k}-q_{n+n_k}\right)}{2^{k+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\dots\left(\lambda-(2\pi n_k)^2\right)}\end{array}\right|$$

$$\leq \frac{c^2}{2^2\delta(n)}+\frac{c^3}{2^3(\delta(n))^2}+\dots+\frac{c^{k+1}}{2^{k+1}(\delta(n))^k}$$

$$= \frac{c^2}{2^2\delta(n)}\sum_{j=0}^{k}\left(\frac{c}{2\delta(n)}\right)^j, \tag{A.9}$$

and

$$\left|A'_{k,s}(\lambda)\right| = \left|\sum_{m=1}^{k}\alpha'_{m,s}(\lambda)\right|$$

$$= \left|\sum_{m=1}^{k}\sum_{n_1,\dots,n_m=1}^{s}\frac{(q_{n-n_1}+q_{n+n_1})(q_{n_2-n_1}+q_{n_2+n_1})\dots\left(q_{n_m-n_{m-1}}+q_{n_m+n_{m-1}}\right)(q_{n-n_m}+q_{n+n_m})}{2^{m+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\dots\left(\lambda-(2\pi n_m)^2\right)}\right|$$

$$= \left|\begin{array}{c}\sum_{n_1=1}^{s}\frac{\left(q_{n-n_1}+q_{n+n_1}\right)^2}{2^2\left(\lambda-(2\pi n_1)^2\right)}+\sum_{n_1,n_2=1}^{s}\frac{\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\left(q_{n-n_2}+q_{n+n_2}\right)}{2^3\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)}+\\ +\dots+\sum_{n_1,\dots,n_k=1}^{s}\frac{\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\dots\left(q_{n_k-n_{k-1}}+q_{n_k+n_{k-1}}\right)\left(q_{n-n_k}+q_{n+n_k}\right)}{2^{k+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\dots\left(\lambda-(2\pi n_k)^2\right)}\end{array}\right|$$

$$\leq \frac{c^2}{2^2\delta(n)}+\frac{c^3}{2^3(\delta(n))^2}+\dots+\frac{c^{k+1}}{2^{k+1}(\delta(n))^k}$$

$$= \frac{c^2}{2^2\delta(n)}\sum_{j=0}^{k}\left(\frac{c}{2\delta(n)}\right)^j$$

$$\leq \frac{c^2}{2(2\delta(n)-c)}, \tag{A.10}$$

by the geometric series formula.

Moreover, we shall use the followings for the proof of Theorem 4.2:

$$\left|\frac{d}{d\lambda}C_1(\lambda)\right| = \left|\frac{d}{d\lambda}a_1(\lambda)\right| = \left|-\frac{q_{n-n_1}-q_{n+n_1}}{2\left(\lambda-(2\pi n_1)^2\right)^2}\right|,$$

$$\left|\frac{d}{d\lambda}a_{k+1}(\lambda)\right| = \left|-\frac{q_{n_{k+1}-n_k}-q_{n_{k+1}+n_k}}{2\left(\lambda-(2\pi n_{k+1})^2\right)^2}\right|,$$

$$\left|\frac{d}{d\lambda}C_2(\lambda)\right| = \left|\left(\frac{d}{d\lambda}C_1\right)a_2 + C_1\left(\frac{d}{d\lambda}a_2\right)\right|$$

$$= \left|\left(-\frac{q_{n-n_1}-q_{n+n_1}}{2\left(\lambda-(2\pi n_1)^2\right)^2}\right)\frac{q_{n_2-n_1}-q_{n_2+n_1}}{2\left(\lambda-(2\pi n_2)^2\right)} + \frac{q_{n-n_1}-q_{n+n_1}}{2\left(\lambda-(2\pi n_1)^2\right)}\left(-\frac{q_{n_2-n_1}-q_{n_2+n_1}}{2\left(\lambda-(2\pi n_2)^2\right)^2}\right)\right|$$

$$\leq \frac{2\left(q_{n-n_1}-q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)}{2^2\left(\delta(n)\right)^3},$$

$$\left|\frac{d}{d\lambda}C_3(\lambda)\right| = \left|\left(\frac{d}{d\lambda}C_2\right)a_3 + C_2\left(\frac{d}{d\lambda}a_3\right)\right|$$

$$= \left|\left[-\frac{q_{n-n_1}-q_{n+n_1}}{2\left(\lambda-(2\pi n_1)^2\right)^2}\frac{q_{n_2-n_1}-q_{n_2+n_1}}{2\left(\lambda-(2\pi n_2)^2\right)} + \frac{q_{n-n_1}-q_{n+n_1}}{2\left(\lambda-(2\pi n_1)^2\right)}\left(-\frac{q_{n_2-n_1}-q_{n_2+n_1}}{2\left(\lambda-(2\pi n_2)^2\right)^2}\right)\right] * \right.$$
$$* \frac{q_{n_3-n_2}-q_{n_3+n_2}}{2\left(\lambda-(2\pi n_2)^2\right)} +$$
$$\left. + \frac{q_{n-n_1}-q_{n+n_1}}{2\left(\lambda-(2\pi n_1)^2\right)}\frac{q_{n_2-n_1}-q_{n_2+n_1}}{2\left(\lambda-(2\pi n_2)^2\right)}\left(-\frac{q_{n_3-n_2}-q_{n_3+n_2}}{2\left(\lambda-(2\pi n_3)^2\right)^2}\right)\right|$$

$$\leq \frac{3\left(q_{n-n_1}-q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)\left(q_{n_3-n_2}-q_{n_3+n_2}\right)}{2^3\left(\delta(n)\right)^4},$$

$$\left|\frac{d}{d\lambda}C_{k+1}(\lambda)\right| = \left|\left(\frac{d}{d\lambda}C_k\right)a_{k+1} + C_k\left(\frac{d}{d\lambda}a_{k+1}\right)\right|,$$

$$\leq \frac{(k+1)\left(q_{n-n_1}-q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)\ldots\left(q_{n_{k+1}-n_k}-q_{n_{k+1}+n_k}\right)}{2^{k+1}\left(\delta(n)\right)^{k+2}};\ k=1,2,\ldots,$$

$$\left|\frac{d}{d\lambda}\alpha_{k,s}(\lambda)\right| = \left|\sum_{n_1,\ldots,n_k=1}^{s}\left(\frac{d}{d\lambda}C_k\right)\left(q\varphi_n, \sin 2\pi n_k x\right)\right|$$

$$\leq \sum_{n_1,\ldots,n_k=1}^{s}\frac{k\left|q_{n-n_1}-q_{n+n_1}\right|\left|q_{n_2-n_1}-q_{n_2+n_1}\right|\ldots\left|q_{n_k-n_{k-1}}-q_{n_k+n_{k-1}}\right|}{2^k\left(\delta(n)\right)^{k+1}}\frac{\left|q_{n-n_k}-q_{n+n_k}\right|}{2},$$

$$\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda)\right| = \left|\sum_{m=1}^{k}\frac{d}{d\lambda}\alpha_{m,s}(\lambda)\right|$$

$$\leq \sum_{m=1}^{k}\sum_{n_1,\ldots,n_m=1}^{s}\frac{m\left(q_{n-n_1}-q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)\ldots\left(q_{n_m-n_{m-1}}-q_{n_m+n_{m-1}}\right)\left(q_{n-n_m}-q_{n+n_m}\right)}{2^{m+1}\left(\delta(n)\right)^{m+1}}$$

$$= \sum_{n_1=1}^{s}\frac{\left(q_{n-n_1}-q_{n+n_1}\right)^2}{2^2\left(\delta(n)\right)^2} + \sum_{n_1,n_2=1}^{s}\frac{2\left(q_{n-n_1}-q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)\left(q_{n-n_2}-q_{n+n_2}\right)}{2^3\left(\delta(n)\right)^3} +$$

$$+ \ldots + \sum_{n_1,\ldots,n_k=1}^{s}\frac{k\left(q_{n-n_1}-q_{n+n_1}\right)\left(q_{n_2-n_1}-q_{n_2+n_1}\right)\ldots\left(q_{n_k-n_{k-1}}-q_{n_k+n_{k-1}}\right)\left(q_{n-n_k}-q_{n+n_k}\right)}{2^{k+1}\left(\delta(n)\right)^{k+1}}$$

$$
\begin{aligned}
&\leq \frac{c^2}{2^2\left(\delta\left(n\right)\right)^2}+\frac{2c^3}{2^3\left(\delta\left(n\right)\right)^3}+\ldots+\frac{kc^{k+1}}{2^{k+1}\left(\delta\left(n\right)\right)^{k+1}}\\
&=\frac{c}{2\left(\delta\left(n\right)\right)}\sum_{i=1}^{k} i\left(\frac{c}{2\delta\left(n\right)}\right)^{i}\\
&\leq \frac{c^2}{2^3\left(\delta\left(n\right)\right)^2}\sum_{i=0}^{k}\left(\frac{c}{\delta\left(n\right)}\right)^{i}\\
&\leq \frac{c^2}{2^3\left(\delta\left(n\right)\right)^2}\frac{1}{1-\dfrac{c}{\delta\left(n\right)}}\\
&\leq \frac{c^2}{2^3\delta\left(n\right)\left(\delta\left(n\right)-c\right)},
\end{aligned}
\tag{A.11}
$$

$$
\left|\frac{d}{d\lambda}\widetilde{M}_1\left(\lambda\right)\right|=\left|\frac{d}{d\lambda}B_1\left(\lambda\right)\right|=\left|-\frac{\left(q_{n-n_1}+q_{n+n_1}\right)}{2\left(\lambda-\left(2\pi n_1\right)^2\right)^2}\right|,
$$

$$
\left|\frac{d}{d\lambda}B_{k+1}\left(\lambda\right)\right|=\left|-\frac{\left(q\cos 2\pi n_{k+1}x,\varphi^*_{n_k}\right)}{\left(\lambda-\left(2\pi n_{k+1}\right)^2\right)^2}\right|=\frac{\left(q_{n_{k+1}-n_k}+q_{n_{k+1}+n_k}\right)}{2\left(\lambda-\left(2\pi n_{k+1}\right)^2\right)^2};\ k=1,2,\ldots,
$$

$$
\begin{aligned}
&\left|\frac{d}{d\lambda}\widetilde{M}_2\left(\lambda\right)\right|=\left|\frac{d}{d\lambda}\left(\widetilde{M}_1B_2\right)\right|=\left|\left(\frac{d}{d\lambda}\widetilde{M}_1\right)B_2+\widetilde{M}_1\left(\frac{d}{d\lambda}B_2\right)\right|\\
&=\left|\left(-\frac{\left(q_{n-n_1}+q_{n+n_1}\right)}{2\left(\lambda-\left(2\pi n_1\right)^2\right)^2}\right)\frac{\left(q_{n_2-n_1}+q_{n_2+n_1}\right)}{2\left(\lambda-\left(2\pi n_2\right)^2\right)}+\frac{\left(q_{n-n_1}+q_{n+n_1}\right)}{2\left(\lambda-\left(2\pi n_1\right)^2\right)}\left(-\frac{\left(q_{n_2-n_1}+q_{n_2+n_1}\right)}{2\left(\lambda-\left(2\pi n_2\right)^2\right)^2}\right)\right|\\
&\leq \frac{2\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)}{2^2\left(\delta\left(n\right)\right)^3},
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{d}{d\lambda}\widetilde{M}_{k+1}\left(\lambda\right)\right|=\left|\frac{d}{d\lambda}\left(\widetilde{M}_kB_{k+1}\right)\right|=\left|\left(\frac{d}{d\lambda}\widetilde{M}_k\right)B_{k+1}+\widetilde{M}_k\left(\frac{d}{d\lambda}B_{k+1}\right)\right|\\
&\leq \frac{(k+1)\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\ldots\left(q_{n_{k+1}-n_k}+q_{n_{k+1}+n_k}\right)}{2^{k+1}\left(\delta\left(n\right)\right)^{k+2}};\ k=1,2,\ldots,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{d}{d\lambda}\alpha'_{k,s}\left(\lambda\right)\right|=\left|\sum_{n_1,\ldots,n_k=1}^{s}\left(\frac{d}{d\lambda}\widetilde{M}_k\right)\left(q\cos 2\pi nx,\varphi^*_{n_k}\right)\right|\\
&\leq \sum_{n_1,\ldots,n_k=1}^{s}\frac{k\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\ldots\left(q_{n_k-n_{k-1}}+q_{n_k+n_{k-1}}\right)}{2^k\left(\delta\left(n\right)\right)^{k+1}}\left(q\cos 2\pi nx,\varphi^*_{n_k}\right);\\
&k=1,2,\ldots,
\end{aligned}
$$

$$\left|\frac{d}{d\lambda}A'_{k,s}(\lambda)\right| = \left|\sum_{m=1}^{k}\frac{d}{d\lambda}\alpha'_{m,s}(\lambda)\right|$$

$$\leq \sum_{m=1}^{k}\sum_{n_1,\ldots,n_m=1}^{s}\frac{m\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\ldots\left(q_{n_m-n_{m-1}}+q_{n_m+n_{m-1}}\right)\left(q_{n-n_m}+q_{n+n_m}\right)}{2^{m+1}\left(\delta(n)\right)^{m+1}}$$

$$= \sum_{n_1=1}^{s}\frac{\left(q_{n-n_1}+q_{n+n_1}\right)^2}{2^2\left(\delta(n)\right)^2}+\sum_{n_1,n_2=1}^{s}\frac{2\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\left(q_{n-n_2}+q_{n+n_2}\right)}{2^3\left(\delta(n)\right)^3}+$$

$$+\ldots+\sum_{n_1,\ldots,n_k=1}^{s}\frac{k\left(q_{n-n_1}+q_{n+n_1}\right)\left(q_{n_2-n_1}+q_{n_2+n_1}\right)\ldots\left(q_{n_k-n_{k-1}}+q_{n_k+n_{k-1}}\right)\left(q_{n-n_k}+q_{n+n_k}\right)}{2^{k+1}\left(\delta(n)\right)^{k+1}}$$

$$\begin{aligned}
&\leq \frac{c^2}{2^2\left(\delta(n)\right)^2}+\frac{2c^3}{2^3\left(\delta(n)\right)^3}+\ldots+\frac{kc^{k+1}}{2^{k+1}\left(\delta(n)\right)^{k+1}}\\
&= \frac{c}{2\delta(n)}\sum_{i=1}^{k} i\left(\frac{c}{2\delta(n)}\right)^i\\
&\leq \frac{c^2}{2^3\left(\delta(n)\right)^2}\sum_{i=0}^{k}\left(\frac{c}{\delta(n)}\right)^i\\
&\leq \frac{c^2}{2^3\left(\delta(n)\right)^2}\frac{1}{1-\dfrac{c}{\delta(n)}}\\
&\leq \frac{c^2}{2^3\delta(n)\left(\delta(n)-c\right)}, \qquad \text{(A.12)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\left|\frac{d}{d\lambda}\Delta_{k,s}(\lambda)\right|}{\left|\sqrt{\Delta_{k,s}(\lambda)}\right|} &= \frac{2\left|Q_n-P_n^*+\widetilde{A}_{k,s}(\lambda_n)-A'_{k,s}(\lambda_n)\right|\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda)-\frac{d}{d\lambda}A'_{k,s}(\lambda)\right|}{\left|Q_n-P_n^*+\widetilde{A}_{k,s}(\lambda_n)-A'_{k,s}(\lambda_n)\right|}\\
&\leq 2\left(\left|\frac{d}{d\lambda}\widetilde{A}_{k,s}(\lambda)\right|+\left|\frac{d}{d\lambda}A'_{k,s}(\lambda)\right|\right)\\
&\leq \frac{c^2}{2\delta(n)\left(\delta(n)-c\right)}. \qquad \text{(A.13)}
\end{aligned}$$

By (A.5) and (A.6), and using the definitions of $A(\lambda_{n,j})$, $\widetilde{A}_k(\lambda_{n,j})$, $A'(\lambda_{n,j})$ and

$A'_k(\lambda_{n,j})$ in Section 2.1, for $s \geq k$ we obtain

$$\left|A(\lambda_{n,j}) - \widetilde{A}_{k,s}(\lambda_{n,j})\right| \leq \left|A(\lambda_{n,j}) - \widetilde{A}_k(\lambda_{n,j})\right| +$$

$$+\left|\sum_{m=1}^{k}\sum_{n_1,\ldots,n_m=s+1}^{\infty}\frac{(q_{n-n_1}-q_{n+n_1})(q_{n_2-n_1}-q_{n_2+n_1})\ldots(q_{n_m-n_{m-1}}-q_{n_m+n_{m-1}})(q_{n-n_m}-q_{n+n_m})}{2^{m+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\ldots\left(\lambda-(2\pi n_m)^2\right)}\right|$$

$$\leq 2\{\sum_{n_1,\ldots,n_{k+1}=1}^{\infty}\frac{\left|(q_{n-n_1}-q_{n+n_1})(q_{n_2-n_1}-q_{n_2+n_1})\ldots(q_{n_{k+1}-n_k}-q_{n_{k+1}+n_k})(q_{n-n_{k+1}}-q_{n+n_{k+1}})\right|}{2^{k+2}\left(\lambda_{n,j}-(2\pi n_1)^2\right)\left(\lambda_{n,j}-(2\pi n_2)^2\right)\ldots\left(\lambda_{n,j}-(2\pi n_k)^2\right)\left(\lambda_{n,j}-(2\pi n_{k+1})^2\right)}+$$

$$+\sum_{n_1,\ldots,n_{k+2}=1}^{\infty}\frac{\left|(q_{n-n_1}-q_{n+n_1})(q_{n_2-n_1}-q_{n_2+n_1})\ldots(q_{n_{k+2}-n_{k+1}}-q_{n_{k+2}+n_{k+1}})(q_{n-n_{k+2}}-q_{n+n_{k+2}})\right|}{2^{k+3}\left(\lambda_{n,j}-(2\pi n_1)^2\right)\left(\lambda_{n,j}-(2\pi n_2)^2\right)\ldots\left(\lambda_{n,j}-(2\pi n_{k+1})^2\right)\left(\lambda_{n,j}-(2\pi n_{k+2})^2\right)}+$$

$$\leq \frac{c^{k+2}}{2^{k+1}(\delta(n))^{k+1}} + \frac{c^{k+3}}{2^{k+2}(\delta(n))^{k+2}} + \cdots$$

$$= \frac{c^{k+2}}{2^{k+1}(\delta(n))^{k+1}}\sum_{j=0}^{\infty}\left(\frac{c}{2\delta(n)}\right)^j$$

$$= \frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)}, \tag{A.14}$$

and

$$\left|\left(A'(\lambda_{n,j}) - A'_{k,s}(\lambda_{n,j})\right)\right| \leq \left|A'(\lambda_{n,j}) - A'_k(\lambda_{n,j})\right| +$$

$$+\left|\sum_{m=1}^{k}\sum_{n_1,\ldots,n_m=s+1}^{\infty}\frac{(q_{n-n_1}+q_{n+n_1})(q_{n_2-n_1}+q_{n_2+n_1})\ldots(q_{n_m-n_{m-1}}+q_{n_m+n_{m-1}})(q_{n-n_m}+q_{n+n_m})}{2^{m+1}\left(\lambda-(2\pi n_1)^2\right)\left(\lambda-(2\pi n_2)^2\right)\ldots\left(\lambda-(2\pi n_m)^2\right)}\right|$$

$$\leq 2\{\sum_{n_1,\ldots,n_{k+1}=1}^{\infty}\frac{\left|(q_{n-n_1}+q_{n+n_1})(q_{n_2-n_1}+q_{n_2+n_1})\ldots(q_{n_{k+1}-n_k}+q_{n_{k+1}+n_k})(q_{n-n_{k+1}}+q_{n+n_{k+1}})\right|}{2^{k+2}\left(\lambda_{n,j}-(2\pi n_1)^2\right)\left(\lambda_{n,j}-(2\pi n_2)^2\right)\ldots\left(\lambda_{n,j}-(2\pi n_k)^2\right)\left(\lambda_{n,j}-(2\pi n_{k+1})^2\right)}+$$

$$+\sum_{n_1,\ldots,n_{k+2}=1}^{\infty}\frac{\left|(q_{n-n_1}+q_{n+n_1})(q_{n_2-n_1}+q_{n_2+n_1})\ldots(q_{n_{k+2}-n_{k+1}}+q_{n_{k+2}+n_{k+1}})(q_{n-n_{k+2}}+q_{n+n_{k+2}})\right|}{2^{k+3}\left(\lambda_{n,j}-(2\pi n_1)^2\right)\left(\lambda_{n,j}-(2\pi n_2)^2\right)\ldots\left(\lambda_{n,j}-(2\pi n_{k+1})^2\right)\left(\lambda_{n,j}-(2\pi n_{k+2})^2\right)}+$$

$$\leq \frac{c^{k+2}}{2^{k+1}(\delta(n))^{k+1}} + \frac{c^{k+3}}{2^{k+2}(\delta(n))^{k+2}} + \cdots$$

$$= \frac{c^{k+2}}{2^{k+1}(\delta(n))^{k+1}}\sum_{j=0}^{\infty}\left(\frac{c}{2\delta(n)}\right)^j$$

$$= \frac{c^{k+2}}{2^k(\delta(n))^k(2\delta(n)-c)}. \tag{A.15}$$

5 CONCLUSIONS

In this work we constructed subtle asymptotic formulas for the eigenvalues and eigenfunctions of non-self-adjoint Sturm-Liouville operators with general regular boundary conditions for both cases $q \in L_1[0,1]$ and q is an absolutely continuous function. Using these formulas we found explicit conditions on potential q such that the system of the root functions of the Sturm-Liouville operator with general regular boundary conditions does not form a Riesz basis. Also we estimated the small eigenvalues of the operators defined in Chapter 2 by the numerical methods.

The results of this work for the differential operators may be extended for the nth order differential operators or when the potential function $q(x)$ is chosen from Sobolev spaces.

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RESUME

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