# COULOMB SİNGÜLERLİĞİNE SAHİP STURM-LİOUVİLLE OPERATÖRÜNÜN İZ'İNİN (TRACE) HESAPLANMASI

Yaşar ÇAKMAK

YÜKSEK LİSANS TEZİ

MATEMATİK ANABİLİM DALI

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FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI

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## FEN BİLİMLERİ ENSTİTÜSÜ MÜDÜRLÜĞÜ' NE

Bu çalışma, jürimiz tarafından, Matematik Ana Bilim Dalı' nda Yüksek Lisans Tezi olarak kabul edilmiştir.

Başkan : Prof. Dr. Bünyamin Özbay

Üye : R. ... Prof. Dr. Rauf Emirov

Üye : Yrd. Doç. Dr. Yılmaz Çeven

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ONAY

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FEN BİLİMLERİ ENSTİTÜSÜ MÜDÜRÜ

Prof. Dr. Necati ÇELİK

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# İÇİNDEKİLER

# ÖZET

Yüksek Lisans Tezi

## COULOMB SİNGÜLERLİĞİNE SAHİP STURM-LİOUVİLLE OPERATÖRÜNÜN İZİ'NİN (TRACE) HESAPLANMASI




Yaşar ÇAKMAK

Cumhuriyet Üniversitesi

Fen Bilimleri Enstitüsü

Matematik Anabilim Dalı

Danışman: Prof.Dr.Rauf EMİROV



Bu çalışma II.mertebeden diferensiyel opetatörlerin spektral teorisine aittir. Sunduğumuz çalışmada Coulomb Singülerliğine sahip Sturm-Liouville operatörünün regülerize izi hesaplanmıştır.

Reğülerize izlerin hesaplanması, Sturm-Liouville operatörleri için inverse (ters) problemlerin çözümünde önem taşımaktadır. Çalışmanın en önemli sonuçlarından birisi de budur.



ANAHTAR KELİMELER: Operatör, Spektrum, Inverse Problem, Sturm-Liouville Operatör, İz (Trace)

SUMMARY

MsC Thesis

CALCULATION OF THE TRACE OF STURM-LİOUVİLLE OPERATOR WİTH COULOMB SİNGULARİTİES



Yaşar ÇAKMAK

Cumhuriyet University
Graduate School of Natural
and Applied Sciences
Department of Mathematics

Supervisor: Prof.Dr.Rauf AMIROV



This work belongs to the scope of spectral theories of second order differential operators. It present a calculation of regularized trace of Sturm-Liouville operators which possess Coulomb singularity.

The task of computing regularized traces have importance for solving inverse problems for Sturn-Liouville operators. This is indeed the main stream of this present work.



**Key words:** Operator, Spectrum, Trace, Sturm-Liouville operator, Inverse problem

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# GİRİŞ

Klasik analizin ve matematiksel fiziğin bir çok problemi, diferansiyel operatörler için ters (Inverse) problemlere indirgenebilmektedir.İki spektra göre ters (Inverse) problemlerin çözümünde ise diferansiyel operatörlerin iz (trace) inden faydalanılır. Bu alanda ilk adım 1953 'de I.M. Gelfand ve B.M. Levitan tarafından atılmış ve aşağıda vereceğimiz teorem kanıtlanmıştır.

$L_2[0,b]$ uzayında $Ly = -\dfrac{d^2y}{dx^2} + q(x)y$ regüler diferansiyel denklemin Dirichlet sınır koşulları altında (Yani, $y(0) = 0$, $y(b) = 0$ ) öz değerlerinin asimptotik ifadesini;

$$\mu_n = \frac{\pi^2 n^2}{b^2} + \frac{1}{b}\int_0^b q(t)\,dt + O(n^{-2})$$ şeklinde bulmuş ve

$\int_0^b q(t)\,dt = 0$ koşulu altında izini (trace) de;

$$\sum_n \left( \mu_n - \frac{\pi^2 n^2}{b^2} \right) = \frac{q(0) + q(b)}{4}$$ şeklinde almıştır.

Yaptığımız bu çalışma iki kısımdan oluşmaktadır. Giriş kısmında Sturm-Liouville Operatörü için İz (trace) formülünün gelişme tarihinden bahsedilmektedir.

Çalışmanın ikinci kısmı üç bölümden oluşmaktadır.Birinci bölümde Coulomb singülerliğine sahip Sturm-Liouville operatörünün fiziksel anlamı, Sturm-Liouville operatörü ve İz (Trace) kavramı ile ilgili ön bilgiler verilmiştir.İkinci bölümde ise Coulomb singülerliğine sahip Sturm-Liouville operatörünün İzi (trace) hesaplanmıştır.Üçüncü bölümde ise vektör değerli fonksiyon durumu araştırılmıştır.

# 1. BÖLÜM

## 1.1 Coulomb Singülerliğine sahip Sturm-Liouville Operatörünün Fiziksel Anlamı:

Kuantum teorisinde en çok önem taşıyan problemlerden birisi, Coulomb potansiyelli alanda elektronların hareketinin öğrenilmesidir. Bu tip problemlerin çözümü; bir tek Hidrojen atomu için değil,bir valentli atoma sahip Sodyum ve benzeri atomların spekturumunun ve enerji seviyelerinin bulunmasını sağlar.

Hidrojen atomu için, U potansiyel enerjisi (Coulomb potansiyeli) $U = -\frac{e^2}{r}$ şeklindedir. Burada r elektronun çekirdekten uzaklığı, -e elektronun yükü, +e protonun yüküdür. Buna göre;

$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x,y,z,t)\Psi$ , $\int_{R^3}|\Psi|^2 dxdydz = 1$ zamana bağlı Schrödinger denklemidir. Burada $\Psi$ dalga fonksiyonu, $\hbar$ plank sabiti, m ise elektronun kütlesidir. Bu denklemde $\widetilde{\Psi} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda t}\Psi dt$ Fourier dönüşümü yapılırsa;

$\frac{\hbar^2}{2m}\nabla^2\widetilde{\Psi} + \widetilde{U}\widetilde{\Psi} = E\widetilde{\Psi}$ konuma bağlı enerji denklemine dönüşecektir. Dolayısıyla Hidrojen atomunun Coulomb potansiyelli alanda enerji denklemi;

$-\frac{\hbar^2}{2m}\nabla^2\widetilde{\Psi} + \left(E + \frac{e^2}{r}\right)\widetilde{\Psi} = 0$ olur.

Bu hidrojen atomu bir başka potansiyel alana yerleştirilirse o zaman enerji denklemi;

$-\frac{\hbar^2}{2m}\nabla^2\widetilde{\Psi} + \left(E + \frac{e^2}{r} + q(x,y,z)\right)\widetilde{\Psi} = 0$ olur. Burada da gerekli dönüşümler yapıldığında;

$-y'' + \left(\frac{A}{x} + q(x)\right)y = \lambda y$ Coulomb potansiyelli Sturm-Liouville denklemi alınır. Buradaki $\lambda$ parametresi enerjiye karşılık gelen parametredir.

$\ell(y) = -y'' + q(x)y$ diferansiyel ifade olsun. Eğer, a ve b sonlu olmak üzere $x \in [a,b]$ ve $q(x)$ fonksiyonu $[a,b]$ aralığında integrallenebilirse $\ell(y)$ ifadesini **Regüler diferansiyel ifade** denir. Eğer, a ve b sayılarından herhangi biri sonsuz yada her ikiside sonsuza eşit veya $q(x)$ fonksiyonu $[a,b]$ aralığında integrallenemezse ve yada her iki durum birlikte söz konusu ise $\ell(y)$ ifadesini **Singüler diferansiyel ifade** denir.

Bu çalışmada incelenen diferansiyel ifade $\ell(y) = -y'' + Q(x)y,\ x \in [0,\pi]$ ve $Q(x) = \frac{A}{x} + q(x)$ şeklinde olduğu için $Q(x)$ fonksiyonu $[0,\pi]$ aralığında sonlu integrale sahip değildir. Dolayısıyla $\ell(y) = -y'' + Q(x)y$ diferansiyel ifadesi Singüler diferansiyel ifadedir.

Bu tip singüler diferansiyel ifadelerin tanımlı olduğu fonksiyonlar sınıfından olan fonksiyonlar için $y'(0)$ sonlu değere sahip olmadığından $y(0) = 0,\ y'(\pi) - Hy(\pi) = 0$ ayrık sınır değer koşullarını sağlayan fonksiyonlar sınıfı söz konusu diferansiyel ifadelerin tanım kümesi olarak alınır.

Sonuç olarak;

$$L = \begin{cases} -y'' + \left(\frac{A}{x} + q(x)\right)y = \lambda y \\ y(0) = 0 \\ y'(\pi) - Hy(\pi) = 0 \end{cases}$$

Coulomb singülerliğine sahip Sturm-Liouville denklemini ve $D(L) = \left\{ \begin{array}{l} y \,|\, Ly \in L_2[0,\pi], q(x) \in L_2[0,\pi],\ A,\ H \text{ reel sabitler,} \\ y(0) = 0, y'(\pi) - Hy(\pi) = 0 \end{array} \right\}$ L operatörünün tanım kümesidir.

## 1.2 Strum-Liouville Operatörü ve İz (Trace) ile ilgili ön bilgiler.

**Tanım 1.2.1:** L bir lineer operatör olmak üzere $Ly = \lambda y$ eşitliğini sağlayan $y \neq 0$ fonksiyonuna L operatörünün **öz fonksiyonu** denir. $\lambda$'ya ise L operatörünün $y(x)$'e karşılık gelen **öz değeri** denir.

**Teorem 1.2.2:** $\lambda_1, \lambda_2$, L operatörünün farklı iki öz değeri olsun. Bu öz değerlere karşılık gelen $y(x,\lambda_1)$, $y(x,\lambda_2)$ öz fonksiyonları otogonaldir. Yani,

$$\int_0^\pi y(x,\lambda_1)y(x,\lambda_2)dx = 0 \qquad \text{olur.}$$

Vede;Gram Schmidt teoremiyle öz fonksiyonları ortonormal bir sistem oluşturur.

**Teorem 1.2.3:** L operatörünün öz değerleri reeldir.

**Tanım 1.2.4:** $(L - \lambda I)^{-1}$ olmadığı $\lambda$ noktaları kümesini L operatörünün **spektrumu** denir. $\sigma(L)$ ile gösterilir. $\sigma(L) = \{\lambda | Ly = \lambda y,\ y \in D(L)\}$

L operatörü self-adjoint operatör olduğu için genel operatörler teorisinden L operatörünün spektrumu öz değerlerinden oluşmuştur.

Bunlara göre;

$$L = \begin{cases} -y'' + \left(\dfrac{A}{x} + q(x)\right)y = \lambda y & (1) \\ y(0) = 0 & \\ y'(\pi) - Hy(\pi) = 0 & (2) \end{cases}$$

$q(x) \in L_2[0,\pi]$, A, H reel sabitler olmak üzere,

$$y(x,\lambda) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x + \int_0^{\pi} \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t} + q(t)\right)y(t,\lambda)dt$$

L operatörğünün çözümüdür.

$\varphi(x,\lambda)$ ile $\varphi'(0,\lambda) = 1$, $\varphi(0,\lambda) = 0$ başlangıç koşullarını sağlayan (1) denkleminin çözümü olsun.

Gösterilirki, L operatörünün özdeğerleri; $\varphi'(\pi,\lambda) - H\varphi(\pi,\lambda) = 0$ denkleminin sıfirlarıdır.

$Ly = \lambda y$ *ise* $(L - \lambda I)y = 0$ dır. Teorem gereği, Ly = 0 denkleminin bir tek $y = 0$ çözümü varsa her $f(x) \in L_2[0,\pi]$ fonksiyonu için $Ly = \lambda y + f(x)$ denkleminin $y = (L - \lambda I)^{-1} f(x)$ gibi bir çözümü vardır.

Ayrıca; $Ly = \lambda y + f(x)$ denklemi ve $y(0) = 0, y'(\pi) - Hy(\pi) = 0$ problemi için sınır değer probleminden;

$$\left.\begin{array}{l} \varphi(x,\lambda) \ ; \ \varphi(o,\lambda) = 0 \ , \ \varphi'(o,\lambda) = 1 \\ \psi(o,\lambda) \ ; \ \psi(o,\lambda) = 1 \ , \ \psi'(o,\lambda) = H \end{array}\right\} \quad (*) \ ,$$

fonksiyonları L operatörünün lineer bağımsız çözümleri olsunlar. O halde $Ly = \lambda y + f(x)$ denkleminin (*) koşullarını sağlayan genel çözümü;

$y(x,\lambda) = \int_0^{\pi} G(x,\varsigma,\lambda) f(\varsigma) d\varsigma$ şeklindedir.

Burada $G(x,\varsigma,\lambda)$ operatörünün Green fonksiyonudur ve

$$G(x,\varsigma,\lambda) = \begin{cases} \dfrac{\varphi(x,\lambda)\ \psi(\varsigma,\lambda)}{W[\varphi,\psi]}, & o \le x < \varsigma \\ \dfrac{\varphi(\varsigma,\lambda)\ \psi(x,\lambda)}{W[\varphi,\psi]}, & o \le x < \pi \end{cases}$$

Şeklinde tanımlanır. Yukarıda da görüldügü gibi $W[\varphi,\psi]$ Wronskiyanı x' e bağlı değildir ve yalnızca $\lambda$ ya bağlı bir fonksiyondur yani, $W[\varphi,\psi] = \omega(\lambda)$ dır. Buradan, L operatörünün öz değerleri $G(x,\varsigma,\lambda)$ fonksiyonunun $(\lambda)$ düzlemindeki tekil noktalarıdır, yani $\omega(\lambda) = 0$ denkleminin kökleridir.

Şimdi gösterilir ki,
0=W($\lambda$)=$\varphi'(\pi,\lambda)$ + H$\varphi(\pi,\lambda)$ dır.

$$W_\pi[\varphi,\Psi]=\begin{vmatrix}\varphi(\pi,\lambda) & \psi(\pi,\lambda)\\ \varphi'(\pi,\lambda) & \psi'(\pi,\lambda)\end{vmatrix}=\varphi(\pi,\lambda)\psi'(\pi,\lambda)-\varphi'(\pi,\lambda)\psi(\pi,\lambda)=$$

$$\psi(\pi,\lambda)\left[\varphi(\pi,\lambda)\frac{\psi'(\pi,\lambda)}{\psi(\pi,\lambda)}-\varphi'(\pi,\lambda)\right]=\psi(\pi,\lambda)[H\varphi(\pi,\lambda)-\varphi'(\pi,\lambda)]=0 \quad \text{olur.}$$

$\psi(\pi,\lambda)$ sıfırdan farklı olduğundan;
$W(\lambda)=\varphi'(\pi,\lambda) - H\varphi(\pi,\lambda) = 0$ olur
$\varphi'(0,\lambda)=1$ ve $\varphi(0,\lambda)=0$ koşullarını sağlayan çözümünü $\varphi(x,\lambda)$ ile belirtilmişti.

Buradan kolayca gösterilebilirki, $\varphi(x,\lambda)$ fonksiyonu ;

$$\varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)\varphi(t,\lambda)dt \quad \text{şeklinde olur.}$$

Ardışık yaklaşmalar yapılarak $\varphi(x,\lambda)$'nın asimptotik ifadesi ;

$$\varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt$$
$$+\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)\left[\int_0^t\frac{\sin\sqrt{\lambda}(t-\tau)}{\sqrt{\lambda}}\left(\frac{A}{\tau}+q(\tau)\right)\frac{\sin\sqrt{\lambda}\tau}{\sqrt{\lambda}}d\tau\right]dt+$$
$$+\cdots$$

$$\varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x\frac{\sin\sqrt{\lambda}(x-t)}{\lambda}\left(\frac{A}{t}+q(t)\right)\sin\sqrt{\lambda}t\,dt+O\left(\frac{1}{\lambda^{3/2}}\right) \quad \text{alınır.}$$

Eğer $\varphi'(\pi,\lambda)-H\varphi(\pi,\lambda)=0$ denkleminden yararlanılırsa ve $q(x)\in L_2[0,\pi]$ olduğu göz önüne tutulduğunda L operatörünün öz degerlerini alınır.

### Operatörler için iz (Trace) kavramı:

Sonlu boyutlu durum için; A nxn tipinde bir matris olsun. Teorem gereği eğer $|A|\neq 0$ ve de A matrisinin lineer bağımsız n tane öz fonksiyonu varsa A matrisi köşegenleştirilebilir.

$$A=\begin{bmatrix}\lambda_1 & 0 & 0 \ldots & 0\\ 0 & \lambda_2 & 0 \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 \ldots & \lambda_n\end{bmatrix}$$ köşegen matrisi olmak üzere; Bu bir dönüşümdür ve her matris dönüşümüne bir operatör gibi bakılabilir.Burada $\sum_{k=1}^{n}\lambda_k$ toplamına A matrisinin izi (trace) denir.

Sonsuz boyutlu durum için;

$A: H \rightarrow H$ bir lineer operatör olsun. H bir Hilbert uzayıdır ve Hilbert uzayı Ayrılabilir uzay olduğundan ortonormal sistemler alınabilir.

$\{e_n\} \in H$ ortonormal sistemdir ve $\|e_n\|_H = 1$ dir.

$\{e_n\}$ sistemini A'nın özfonksiyonları şeklinde alınırsa $Ae_n = \lambda_n e_n$ olur. Buna göre;

$$\sum_{n=1}^{\infty} < Ae_n, e_n >= \sum_{n=1}^{\infty} < \lambda_n e_n, e_n >= \sum_{n=1}^{\infty} \lambda_n < e_n, e_n >= \sum_{n=1}^{\infty} \lambda_n$$

$\sum_{n=1}^{\infty} \lambda_n < +\infty$ ise bu seriye A operatörünün izi (trace) denir ve $tr(A) = \sum_{n=1}^{\infty} \lambda_n$ olur.

$L: L_2[0,\pi] \rightarrow L_2[0,\pi]$ , $L_2[0,\pi]$ Hilbert uzayı olduğu için $\{y_n\}$ ortonormal sistem alınabilir.

$\{y_n\}$ sistemini L'nın özfonksiyonları şeklinde alınırsa $Ly_n = \lambda_n y_n$ olur. Buna göre;

$$\sum_{n=1}^{\infty} < Ly_n, y_n >= \sum_{n=1}^{\infty} < \lambda_n y_n, y_n >= \sum_{n=1}^{\infty} \lambda_n < y_n, y_n >= \sum_{n=1}^{\infty} \lambda_n \qquad \text{şeklindedir.}$$

Ancak bu seri yakınsak değildir. Dolayısıyla Regülerize edilmiş ize bakılacaktır. $\mu_n$ ile $q(x) = 0$ olduğu duruma karşılık gelen (1)-(2) probleminin özdeğerlerini göstersin. $\sum_{n=1}^{\infty} [\lambda_n - \mu_n]$ ifadesine (1)-(2) probleminin Regülerize edilmiş izi denir. Bu iz (trace)'ler iki spektra göre invers problemlerin çözümünde anlamlıdır.

# 2. BÖLÜM

## 2.1 Coulomb singülerliğine sahip Sturm-Liouville operatörünün özdeğerinin ve İz'inin hesaplanması:

$$L=\begin{cases} -\dfrac{d^2y}{dx^2}+\left(\dfrac{A}{x}+q(x)\right)y=\lambda y \qquad 0\le x\le \pi,\ q(x)\in C^2[0,\pi] \\ y(0)=0 \\ y'(\pi)-Hy(\pi)=0 \end{cases}$$

operatörü alınsın. L operatörünün özdeğerleri bulunacak olursa;.
Homojen kısmın çözümü için;

$$\frac{d^2y}{dx^2}+\lambda y=0 \Rightarrow \qquad K^2+\lambda=0 \Rightarrow K=\mp\sqrt{\lambda}i$$

O halde $y_h=c_1\operatorname{Cos}\sqrt{\lambda}x+c_2\operatorname{Sin}\sqrt{\lambda}x$ olur.
Homojen olmayan kısmın çözümü için;

$$y_p=c_1(x)\operatorname{Cos}\sqrt{\lambda}x+c_2(x)\operatorname{Sin}\sqrt{\lambda}x \text{ olsun.}$$

$$y_p'=-\sqrt{\lambda}c_1(x)\operatorname{Sin}\sqrt{\lambda}x+\sqrt{\lambda}c_2(x)\operatorname{Cos}\sqrt{\lambda}x+\underbrace{c_1'(x)\operatorname{Cos}\sqrt{\lambda}x+c_2'(x)\operatorname{Sin}\sqrt{\lambda}x}_{0}$$

I.koşul

$$y_p'=-\sqrt{\lambda}c_1(x)\operatorname{Sin}\sqrt{\lambda}x+\sqrt{\lambda}c_2(x)\operatorname{Cos}\sqrt{\lambda}x$$

$$y_p''=-\lambda c_1(x)\operatorname{Cos}\sqrt{\lambda}x-\lambda c_2(x)\operatorname{Sin}\sqrt{\lambda}x-\sqrt{\lambda}c_1'(x)\operatorname{Sin}\sqrt{\lambda}x+\sqrt{\lambda}c_2'(x)\operatorname{Cos}\sqrt{\lambda}x$$

yerlerine yazılırsa;

$$-\lambda c_1(x)\operatorname{Cos}\sqrt{\lambda}x-\lambda c_2(x)\operatorname{Sin}\sqrt{\lambda}x-\sqrt{\lambda}c_1'(x)\operatorname{Sin}\sqrt{\lambda}x+\sqrt{\lambda}c_2'(x)\operatorname{Cos}\sqrt{\lambda}x+$$
$$+\lambda c_1(x)\operatorname{Cos}\sqrt{\lambda}x+\lambda c_2(x)\operatorname{Sin}\sqrt{\lambda}x=\left(\frac{A}{x}+q(x)\right)y$$

$$-\sqrt{\lambda}c_1'(x)\operatorname{Sin}\sqrt{\lambda}x+\sqrt{\lambda}c_2'(x)\operatorname{Cos}\sqrt{\lambda}x=\left(\frac{A}{x}+q(x)\right)y \qquad \text{II. koşul}$$

$$c_1'(x)\operatorname{Cos}\sqrt{\lambda}x+c_2'(x)\operatorname{Sin}\sqrt{\lambda}x=0$$
$$-\sqrt{\lambda}c_1'(x)\operatorname{Sin}\sqrt{\lambda}x+\sqrt{\lambda}c_2'(x)\operatorname{Cos}\sqrt{\lambda}x=\left(\frac{A}{x}+q(x)\right)y$$

$$c_1'(x)=\frac{\begin{vmatrix} 0 & \operatorname{Sin}\sqrt{\lambda}x \\ \left(\frac{A}{x}+q(x)\right)y & \sqrt{\lambda}\operatorname{Cos}\sqrt{\lambda}x \end{vmatrix}}{\begin{vmatrix} \operatorname{Cos}\sqrt{\lambda}x & \operatorname{Sin}\sqrt{\lambda}x \\ -\sqrt{\lambda}\operatorname{Sin}\sqrt{\lambda}x & \sqrt{\lambda}\operatorname{Cos}\sqrt{\lambda}x \end{vmatrix}}=-\frac{\operatorname{Sin}\sqrt{\lambda}x}{\sqrt{\lambda}}\left(\frac{A}{x}+q(x)\right)y$$

$$c_1(x) = -\int_0^x \frac{\mathrm{Sin}\sqrt{\lambda}t}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)y(t)dt \text{ olur.}$$

$$c_2'(x) = \frac{\begin{vmatrix} \mathrm{Cos}\sqrt{\lambda}x & 0 \\ -\sqrt{\lambda}\mathrm{Sin}\sqrt{\lambda}x & \left(\frac{A}{x}+q(x)\right)y \end{vmatrix}}{\begin{vmatrix} \mathrm{Cos}\sqrt{\lambda}x & \mathrm{Sin}\sqrt{\lambda}x \\ -\sqrt{\lambda}\mathrm{Sin}\sqrt{\lambda}x & \sqrt{\lambda}\mathrm{Cos}\sqrt{\lambda}x \end{vmatrix}} = \frac{\mathrm{Cos}\sqrt{\lambda}x}{\sqrt{\lambda}}\left(\frac{A}{x}+q(x)\right)y$$

$$c_2(x) = \int_0^x \frac{\mathrm{Cos}\sqrt{\lambda}t}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)y(t)dt \text{ olur.}$$

Buna göre; $y_p = c_1(x)\ \ \mathrm{Cos}\sqrt{\lambda}x + c_2(x)\mathrm{Sin}\sqrt{\lambda}x$ olduğundan;

$$y_p = Cos\sqrt{\lambda}x\int_0^x \frac{-Sin\sqrt{\lambda}t}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)y(t)dt + Sin\sqrt{\lambda}x\int_0^x \frac{Cos\sqrt{\lambda}t}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)y(t)dt$$

$$y_p = \int_0^x \frac{\mathrm{Sin}\sqrt{\lambda}x\mathrm{Cos}\sqrt{\lambda}t - \mathrm{Sin}\sqrt{\lambda}t\mathrm{Cos}\sqrt{\lambda}x}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)y(t)dt$$

$$y_p = \int_0^x \frac{\mathrm{Sin}\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)y(t)dt$$

O halde genel çözüm;

$$y = c_1\mathrm{Cos}\sqrt{\lambda}x + c_2\mathrm{Sin}\sqrt{\lambda}x + \int_0^x \frac{\mathrm{Sin}\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)y(t)dt \text{ olur.}$$

$\varphi(x,\lambda)$; ile $\varphi(0,\lambda)=0$, $\varphi'(0,\lambda)=$ koşulunu sağlayan çözümü olsun.

Buna göre,

$$\varphi(x,\lambda) = c_1\mathrm{Cos}\sqrt{\lambda}x + c_2\mathrm{Sin}\sqrt{\lambda}x + \int_0^x \frac{\mathrm{Sin}\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)\varphi(t,\lambda)dt \text{ olur.}$$

$\varphi(0,\lambda)=0$ ise $c_1=0$

$\varphi'(0,\lambda)=1$ ise $\sqrt{\lambda}c_2=1$ ise $c_2=\frac{1}{\sqrt{\lambda}}$ dir. Buna göre,

$$\varphi(x,\lambda) = \frac{\mathrm{Sin}\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \frac{\mathrm{Sin}\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)\varphi(t,\lambda)dt \text{ bulunur.}$$

Ardışık yaklaşmalar yapılırsa;

$$\varphi(x,\lambda)=\frac{Sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x\frac{Sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)\frac{Sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt+$$
$$+\int_0^x\frac{Sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)\left[\int_0^t\frac{Sin\sqrt{\lambda}(t-\tau)}{\sqrt{\lambda}}\left(\frac{A}{\tau}+q(\tau)\right)\frac{Sin\sqrt{\lambda}\tau}{\sqrt{\lambda}}\,d\tau\right]dt+\ldots$$

$$\varphi(x,\lambda)=\frac{Sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x\frac{Sin\sqrt{\lambda}(x-t)}{\lambda}\left(\frac{A}{t}+q(t)\right)Sin\sqrt{\lambda}t\,dt+O\left(\frac{1}{\lambda^{3/2}}\right)\text{ olur.}$$

Şimdi $\varphi'(\pi,\lambda)-H\varphi(\pi,\lambda)=0$ denkleminden özdeğerleri bulunabilir.

$$\varphi(\pi,\lambda)=\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\int_0^\pi\frac{Sin\sqrt{\lambda}(\pi-t)}{\lambda}\left(\frac{A}{t}+q(t)\right)Sin\sqrt{\lambda}t\,dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$=\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\int_0^\pi\frac{Sin\sqrt{\lambda}\pi Cos\sqrt{\lambda}t-Sin\sqrt{\lambda}tCos\sqrt{\lambda}\pi}{\lambda}Sin\sqrt{\lambda}t\left(\frac{A}{t}+q(t)\right)dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$=\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\frac{Sin\sqrt{\lambda}\pi}{2\lambda}\int_0^\pi Sin2\sqrt{\lambda}t\left(\frac{A}{t}+q(t)\right)dt-\frac{Cos\sqrt{\lambda}\pi}{\lambda}\int_0^\pi Sin^2\sqrt{\lambda}t\left(\frac{A}{t}+q(t)\right)dt+$$
$$+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$=\frac{Sh\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\frac{ASin\sqrt{\lambda}\pi}{2\lambda}\underbrace{\int_0^\pi\frac{Sin2\sqrt{\lambda}t}{t}dt}_{I}+\frac{Sin\sqrt{\lambda}\pi}{2\lambda}\underbrace{\int_0^\pi Sin2\sqrt{\lambda}t\,q(t)dt}_{II}-$$
$$-\frac{ACos\sqrt{\lambda}\pi}{\lambda}\underbrace{\int_0^\pi\frac{Sin^2\sqrt{\lambda}t}{t}dt}_{III}-\frac{Cos\sqrt{\lambda}\pi}{\lambda}\underbrace{\int_0^\pi Sin^2\sqrt{\lambda}t\,q(t)dt}_{IV}+O\left(\frac{1}{\lambda^{3/2}}\right)$$

I.İntegral için;

$$\left.\begin{array}{l}2\sqrt{\lambda}t=u\text{ ise }2\sqrt{\lambda}dt=du\\ t=0\text{ ise }u=0\\ t=\pi\text{ ise }u=2\sqrt{\lambda}\pi\end{array}\right\}\text{ ise }\int_0^\pi\frac{Sin2\sqrt{\lambda}t}{t}dt=\int_0^{2\sqrt{\lambda}\pi}\frac{Sinu}{u}du=\int_0^1\frac{Sinu}{u}du+\int_1^{2\sqrt{\lambda}\pi}\frac{Sinu}{u}du$$

$$\underbrace{\int_0^1\frac{Sinu}{u}du}_{N_1}\text{ yakınsaktır.}$$

$\frac{Sinu}{u}$ fonksiyonu u=0 noktasında süreklidir ve $\lim\limits_{u\to0^+}\frac{Sinu}{u}=1$ dir.

$$\int_1^{2\sqrt{\lambda}\pi}\frac{\text{Sinu}}{u}du=-\frac{\text{Cosu}}{u}\Bigg|_1^{2\sqrt{\lambda}\pi}-\underbrace{\int_1^{2\sqrt{\lambda}\pi}\frac{\text{Cosu}}{u^2}du}_{N_2}=-\frac{\text{Cos}2\sqrt{\lambda}\pi}{2\sqrt{\lambda}\pi}+\text{Cos}1+N_2$$

$$\int_0^{\pi}\frac{Sin2\sqrt{\lambda}t}{t}dt=-\frac{Cos2\sqrt{\lambda}\pi}{2\sqrt{\lambda}\pi}+Cos1+N_1-N_2$$ bulunur.

II.İntegral için;

$$\int_0^{\pi}\text{Sin}2\sqrt{\lambda}t\,q(t)dt=-\frac{1}{2\sqrt{\lambda}}\text{Cos}2\sqrt{\lambda}tq(t)\Bigg|_0^{\pi}+\frac{1}{2\sqrt{\lambda}}\int_0^{\pi}\text{Cos}2\sqrt{\lambda}t\,q'(t)dt$$

$$=-\frac{1}{2\sqrt{\lambda}}\text{Cos}2\sqrt{\lambda}\pi q(\pi)+\frac{1}{2\sqrt{\lambda}}q(0)+$$

$$+\frac{1}{2\sqrt{\lambda}}\left[\frac{1}{2\sqrt{\lambda}}\text{Sin}2\sqrt{\lambda}\pi q'(\pi)\Bigg|_0^{\pi}-\frac{1}{2\sqrt{\lambda}}\int_0^{\pi}\text{Sin}2\sqrt{\lambda}t\,q''(t)dt\right]$$

III.İntegral için;

$$\left.\begin{array}{l}\sqrt{\lambda}t=u \text{ ise } \sqrt{\lambda}dt=du\\ t=0 \text{ ise } u=0\\ t=x \text{ ise } u=\sqrt{\lambda}\pi\end{array}\right\}\text{ise}\int_0^{\pi}\frac{Sin^2\sqrt{\lambda}t}{t}dt=\int_0^{\sqrt{\lambda}\pi}\frac{Sin^2u}{u}du=\int_0^1\frac{Sin^2u}{u}du+\int_1^{\sqrt{\lambda}\pi}\frac{Sin^2u}{u}du$$

$\underbrace{\int_0^1\frac{\text{Sin}^2u}{u}du}_{N_3}$ yakınsaktır. Çünkü $\frac{\text{Sin}^2u}{u}\in C(0,1]$ ve $\lim\limits_{u\to 0^+}\frac{\text{Sin}^2u}{u}=0$ dır.

$$\int_1^{\sqrt{\lambda}\pi}\frac{\text{Sin}^2u}{u}du=\int_1^{\sqrt{\lambda}\pi}\frac{1-\text{Cos}2u}{2u}du=\frac{1}{2}\int_1^{\sqrt{\lambda}\pi}\frac{du}{u}-\int_1^{\sqrt{\lambda}\pi}\frac{\text{Ch}2u}{2u}du$$

$$=\frac{1}{2}\ln u\Bigg|_1^{\sqrt{\lambda}\pi}-\frac{1}{2}\left[\frac{\text{Sin}\theta}{\theta}\Bigg|_2^{2\sqrt{\lambda}\pi}+\underbrace{\int_2^{2\sqrt{\lambda}\pi}\frac{\text{Sin}\theta}{\theta^2}d\theta}_{N_4}\right]$$

$$=\frac{1}{2}\ln\sqrt{\lambda}\pi-\frac{\text{Sin}2\sqrt{\lambda}\pi}{4\sqrt{\lambda}\pi}+\frac{\text{Sin}2}{4}-\frac{N_4}{2}$$

$$\int_0^{\pi}\frac{\text{Sin}^2\sqrt{\lambda}t}{t}dt=\frac{\ln\sqrt{\lambda}\pi}{2}-\frac{\text{Sin}2\sqrt{\lambda}\pi}{4\sqrt{\lambda}\pi}+\frac{\text{Sin}2}{4}+N_3-\frac{N_4}{2}$$ bulunur.

IV.İntegral için;

$$\int_0^\pi Sin^2\sqrt{\lambda}t\, q(t)dt=\int_0^\pi \frac{1-Cos2\sqrt{\lambda}t}{2}\, q(t)dt=\frac{1}{2}\int_0^\pi q(t)dt-\frac{1}{2}\int_0^\pi Cos2\sqrt{\lambda}t\, q(t)dt$$

$$=\frac{1}{2}\int_0^\pi q(t)dt-\frac{1}{2}\left[\frac{1}{2\sqrt{\lambda}}Sin2\sqrt{\lambda}t\, q(t)\,\Big|_0^\pi-\frac{1}{2\sqrt{\lambda}}\int_0^\pi Sin2\sqrt{\lambda}t\, q'(t)dt\right]$$

$$=\frac{1}{2}\int_0^\pi q(t)dt-\frac{1}{4\sqrt{\lambda}}Sin2\sqrt{\lambda}\pi\, q(\pi)+$$

$$+\frac{1}{4\sqrt{\lambda}}\left[-\frac{1}{2\sqrt{\lambda}}Cos2\sqrt{\lambda}t\, q(t)\,\Big|_0^\pi+\frac{1}{2\sqrt{\lambda}}\int_0^\pi Cos2\sqrt{\lambda}t\, q''(t)dt\right]$$

bulunur. Bu integralleri yerlerine yazılırsa;

$$\varphi(\pi,\lambda)=\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\frac{ASin\sqrt{\lambda}\pi}{2\lambda}\left[-\frac{Cos2\sqrt{\lambda}\pi}{2\sqrt{\lambda}\pi}+Cos1+N_1-N_2\right]+$$

$$+\frac{Sin\sqrt{\lambda}\pi}{2\lambda}\left[-\frac{1}{2\sqrt{\lambda}}Cos2\sqrt{\lambda}t\, q(t)\,\Big|_0^\pi+\frac{1}{2\sqrt{\lambda}}\int_0^\pi Cos2\sqrt{\lambda}t\, q'(t)dt\right]-$$

$$-\frac{ACos\sqrt{\lambda}\pi}{\lambda}\left[-\frac{Sin2\sqrt{\lambda}\pi}{4\sqrt{\lambda}\pi}+\frac{\ln\sqrt{\lambda}\pi}{2}+\frac{Sin2}{4}+N_3-\frac{N_4}{2}\right]-$$

$$-\frac{Cos\sqrt{\lambda}\pi}{\lambda}\left[\frac{1}{2}\int_0^\pi q(t)dt-\frac{1}{4\sqrt{\lambda}}Sin2\sqrt{\lambda}t\, q(t)\,\Big|_0^\pi+\frac{1}{4\sqrt{\lambda}}\int_0^\pi Sin2\sqrt{\lambda}t\, q'(t)dt\right]+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\varphi(\pi,\lambda)=\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\frac{ASin\sqrt{\lambda}\pi}{2\lambda}\underbrace{\left(Cos1+N_1-N_2\right)}_{\alpha_1}-\frac{ACos\sqrt{\lambda}\pi}{\lambda}\underbrace{\left(\frac{Sin2}{4}+N_3-\frac{N_4}{2}\right)}_{\alpha_2}-$$

$$-\frac{A}{2}\frac{Cos\sqrt{\lambda}\pi}{\lambda}\ln\sqrt{\lambda}\pi-\frac{Cos\sqrt{\lambda}\pi}{2\lambda}\int_0^\pi q(t)dt\;+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\varphi(\pi,\lambda)=\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\frac{ASin\sqrt{\lambda}\pi}{2\lambda}\alpha_1-\frac{ACos\sqrt{\lambda}\pi}{\lambda}\alpha_2-\frac{A}{2}\frac{Cos\sqrt{\lambda}\pi}{\lambda}\ln\sqrt{\lambda}\pi-$$

$$-\frac{Cos\sqrt{\lambda}\pi}{2\lambda}\int_0^\pi q(t)dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

bulunur. Şimdi $\varphi'(\pi,\lambda)$ 'nün asimptotik ifadesine bakılırsa;

$$\varphi'(x,\lambda)=Cos\sqrt{\lambda}x+\int_0^x\frac{Cos\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}Sin\sqrt{\lambda}t\left(\frac{A}{t}+q(t)\right)dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\varphi'(\pi,\lambda)=Cos\sqrt{\lambda}\pi+\int_0^\pi\frac{Cos\sqrt{\lambda}(\pi-t)}{\sqrt{\lambda}}Sin\sqrt{\lambda}t\left(\frac{A}{t}+q(t)\right)dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$=Cos\sqrt{\lambda}\pi+\int_0^\pi\frac{Cos\sqrt{\lambda}\pi Cos\sqrt{\lambda}t+Sin\sqrt{\lambda}\pi Sin\sqrt{\lambda}t}{\sqrt{\lambda}}Sin\sqrt{\lambda}t\left(\frac{A}{t}+q(t)\right)dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$=Cos\sqrt{\lambda}\pi+\frac{Cos\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\int_0^\pi Sin2\sqrt{\lambda}t\left(\frac{A}{t}+q(t)\right)dt+\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\int_0^\pi Sin^2\sqrt{\lambda}t\left(\frac{A}{t}+q(t)\right)dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$=Cos\sqrt{\lambda}\pi+\frac{ACos\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\underbrace{\int_0^\pi\frac{Sin2\sqrt{\lambda}t}{t}dt}_{I}+\frac{Cos\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\underbrace{\int_0^\pi Sin2\sqrt{\lambda}t\,q(t)dt}_{II}+$$

$$+\frac{ASin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\underbrace{\int_0^\pi\frac{Sin^2\sqrt{\lambda}t}{t}dt}_{III}+\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\underbrace{\int_0^\pi Sin^2\sqrt{\lambda}t\,q(t)dt}_{IV}+O\left(\frac{1}{\lambda^{3/2}}\right)$$

Bu integrallerin değerleri daha önce bulunduğundan, Bu değerler yerlerine yazılırsa;

$$\varphi'(\pi,\lambda)=Cos\sqrt{\lambda}\pi+\frac{ACos\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\left[-\frac{Cos2\sqrt{\lambda}\pi}{2\sqrt{\lambda}\pi}+Cos1+N_1-N_2\right]+$$

$$+\frac{Cos\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\left[\frac{q(0)}{2\sqrt{\lambda}}-\frac{q(\pi)Cos2\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\right]+$$

$$+\frac{ASin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\left[\frac{\ln\sqrt{\lambda}\pi}{2}-\frac{Sin2\sqrt{\lambda}\pi}{4\sqrt{\lambda}\pi}+\frac{Sin2}{4}+N_3-\frac{N_4}{2}\right]+$$

$$+\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\left[\frac{1}{2}\int_0^\pi q(t)dt-\frac{1}{4\sqrt{\lambda}}Sin2\sqrt{\lambda}\pi\,q(\pi)\right]+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\varphi'(\pi,\lambda)=Cos\sqrt{\lambda}\pi-\frac{A}{4\lambda\pi}Cos\sqrt{\lambda}\pi Cos2\sqrt{\lambda}\pi+\frac{ACos\sqrt{\lambda}}{2\sqrt{\lambda}}\left(Cos1+N_1-N_2\right)+$$

$$+\frac{Cos\sqrt{\lambda}\pi}{\lambda}\frac{q(0)}{4}-\frac{Cos\sqrt{\lambda}\pi}{\lambda}Cos2\sqrt{\lambda}\pi\frac{q(\pi)}{4}+$$

$$+\frac{A}{2}\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\ln\sqrt{\lambda}\pi-\frac{A}{4\lambda\pi}Sin\sqrt{\lambda}\pi Sin2\sqrt{\lambda}\pi+\frac{ASin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\left(\frac{Sin2}{4}+N_3-\frac{N_4}{2}\right)+$$

$$+\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\frac{1}{2}\int_0^\pi q(t)dt-\frac{1}{4\lambda}Sin\sqrt{\lambda}\pi Sin2\sqrt{\lambda}\pi\, q(\pi)+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\varphi'(\pi,\lambda)=Cos\sqrt{\lambda}\pi-\frac{A}{4\pi}\left(\frac{Cos\sqrt{\lambda}\pi Cos2\sqrt{\lambda}\pi+Sin\sqrt{\lambda}\pi Sin2\sqrt{\lambda}\pi}{\lambda}\right)+\frac{ACos\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\alpha_1-$$

$$+\frac{Cos\sqrt{\lambda}\pi}{\lambda}\frac{q(0)}{4}-\frac{q(\pi)}{4}\left(\frac{Cos\sqrt{\lambda}\pi Cos2\sqrt{\lambda}\pi+Sin\sqrt{\lambda}\pi Sin2\sqrt{\lambda}\pi}{\lambda}\right)+$$

$$+\frac{A}{2}\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\ln\sqrt{\lambda}\pi+\frac{ASin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\alpha_2+\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\frac{1}{2}\int_0^\pi q(t)dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

$$\varphi'(\pi,\lambda)=Cos\sqrt{\lambda}\pi-\frac{Cos\sqrt{\lambda}\pi}{\lambda}\left(\frac{A}{4\pi}+\frac{q(\pi)-q(0)}{4}\right)+\frac{ACos\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\alpha_1+\frac{ASin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\alpha_2+$$
$$+\frac{A}{2}\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\ln\sqrt{\lambda}\pi+\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\frac{1}{2}\int_0^\pi q(t)dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

O halde $\varphi'(\pi,\lambda)-H\varphi(\pi,\lambda)=0$ denkleminden;

$$Cos\sqrt{\lambda}\pi-\frac{Cos\sqrt{\lambda}\pi}{\lambda}\left(\frac{q(\pi)-q(0)}{4}+\frac{A}{4\pi}\right)+\frac{ACos\sqrt{\lambda}\pi}{2\sqrt{\lambda}}\alpha_1+\frac{ASin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\alpha_2+$$
$$+\frac{A}{2}\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\ln\sqrt{\lambda}\pi+\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\frac{1}{2}\int_0^\pi q(t)dt-\frac{HSin\sqrt{\lambda}\pi}{\sqrt{\lambda}}-\frac{AH}{2}\frac{Sin\sqrt{\lambda}\pi}{\lambda}\alpha_1+$$
$$+AH\frac{Cos\sqrt{\lambda}\pi}{\lambda}\alpha_2+\frac{AH}{2}\frac{Cos\sqrt{\lambda}\pi}{\lambda}\ln\sqrt{\lambda}\pi+H\frac{Cos\sqrt{\lambda}\pi}{2\lambda}\int_0^\pi q(t)dt+O\left(\frac{1}{\lambda^{3/2}}\right)=0$$

$$Cos\sqrt{\lambda}\pi-\frac{Cos\sqrt{\lambda}\pi}{\lambda}\left[\frac{q(\pi)-q(0)}{4}+\frac{A}{4\pi}+AH\alpha_2+\frac{H}{2}\int_0^\pi q(t)dt\right]+$$
$$+\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\left[\frac{1}{2}\int_0^\pi q(t)dt+A\alpha_2-H\right]+\frac{A}{2}\frac{Cos\sqrt{\lambda}\pi}{\sqrt{\lambda}}\alpha_1+\frac{A}{2}\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\ln\sqrt{\lambda}\pi+$$
$$+\frac{AH}{2}\frac{Cos\sqrt{\lambda}\pi}{\lambda}\ln\sqrt{\lambda}\pi-\frac{AH}{2}\frac{Sin\sqrt{\lambda}\pi}{\lambda}\alpha_1+O\left(\frac{1}{\lambda^{3/2}}\right)=0$$

$$Cos\sqrt{\lambda}\pi \underbrace{- \frac{Cos\sqrt{\lambda}\pi}{\lambda}\beta_1 + \frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\beta_2 + \frac{A}{2}\frac{Cos\sqrt{\lambda}\pi}{\sqrt{\lambda}}\alpha_1 + \frac{A}{2}\frac{Sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\ln\sqrt{\lambda}\pi +}_{\Omega(\lambda)}$$

$$\underbrace{+\frac{AH}{2}\frac{Cos\sqrt{\lambda}\pi}{\lambda}\ln\sqrt{\lambda}\pi - \frac{AH}{2}\frac{Sin\sqrt{\lambda}\pi}{\lambda}\alpha_1 + O\left(\frac{1}{\lambda^{3/2}}\right)} = 0 \quad \text{olur.}$$

Burada

$$\beta_1 = \frac{q(\pi)-q(0)}{4} + \frac{A}{4\pi} + AH\alpha_2 + \frac{H}{2}\int_0^{\pi} q(t)\,dt \qquad \alpha_1 = Cos1 + N_1 - N_2$$

$$\beta_2 = \frac{1}{2}\int_0^{\pi} q(t)\,dt + A\alpha_2 - H \qquad \alpha_2 = \frac{Sin2}{4} + N_3 - \frac{N_4}{2}$$

dir.

Rouche Teoreminden yararlanılırsa; $Cos\sqrt{\lambda}\pi$ ve $\Omega(\lambda)$ fonksiyonlarının sıfırları sayısı aynı olur. Bu yüzden de;

$$Cos\sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\lambda}\pi = (2n+1)\frac{\pi}{2} \Rightarrow \sqrt{\lambda_n} = (n+1/2) \Rightarrow \lambda_n = (n+1/2)^2 \text{ olur.}$$

$\sqrt{\lambda_n} = (n+1/2) + \delta_n$ alırsak, $q(x) \in C^1[0,\pi]$ $\delta_n = O\left(\frac{1}{(n+1/2)^2}\right) = O\left(\frac{1}{n^2}\right)$ olmak üzere;

$$Cos\left(n+\frac{1}{2}+\delta_n\right)\pi + \frac{A}{2}\frac{Cos\left(n+\frac{1}{2}+\delta_n\right)\pi}{\left(n+\frac{1}{2}\right)+\delta_n}\alpha_1 + \frac{Sin\left(n+\frac{1}{2}+\delta_n\right)\pi}{\left(n+\frac{1}{2}\right)+\delta_n}\beta_2 +$$

$$+\frac{A}{2}\frac{Sin\left(n+\frac{1}{2}+\delta_n\right)\pi}{\left(n+\frac{1}{2}\right)+\delta_n}\ln\left(n+\frac{1}{2}+\delta_n\right)\pi + O\left(\frac{1}{n^2}\right) = 0$$

$$-(-1)^n\delta_n\pi \underbrace{- \frac{A\alpha_1}{2}\frac{1}{(n+1/2)}(-1)^n\delta_n\pi}_{O\left(\frac{1}{n^2}\right)} + \frac{(-1)^n\beta_2}{(n+1/2)} + \frac{A}{2}\frac{(-1)^n}{(n+1/2)}\ln\pi +$$

$$+\frac{A}{2}\frac{(-1)^n}{(n+1/2)}\ln(n+1/2) + O\left(\frac{1}{n^2}\right) = 0$$

$$-\delta_n\pi+\frac{\beta_2+\frac{A}{2}\ln\pi}{(n+1/2)}+\frac{A}{2}\frac{\ln(n+1/2)}{(n+1/2)}+O\left(\frac{1}{n^2}\right)=0$$

$$\delta_n\pi=\frac{\frac{1}{\pi}\left(\beta_2+\frac{A}{2}\ln\pi\right)}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+O\left(\frac{1}{n^2}\right)$$

$$\delta_n\pi=\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+O\left(\frac{1}{n^2}\right) \qquad c_1=\frac{1}{\pi}\left(\beta_2+\frac{A}{2}\ln\pi\right)$$ bulunur.

O zaman;

$$\sqrt{\lambda_n}=(n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}$$ olur.

$q(x)\in C^2[0,\pi]$ $\quad \delta_n=O\left(\frac{1}{(n+1/2)^3}\right)=O\left(\frac{1}{n^3}\right)$ olmak üzere;

$$\sqrt{\lambda_n}=(n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n$$ alınırsa,

$$\mathrm{Cos}\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi-$$

$$-\frac{\mathrm{Cos}\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi}{\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)^2}\beta_1+$$

$$+\frac{\mathrm{Sin}\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi}{\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)}\beta_2+$$

$$+\frac{A}{2}\frac{\mathrm{Cos}\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi}{\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)}\alpha_1+$$

$$+\frac{A}{2}\frac{Sin\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi}{\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)}\left[\ln\left((n+1/2)+\frac{c_1}{(n+1/2)}+\right.\right.$$

$$\left.\left.+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi\right]+$$

$$+\frac{AH}{2}\frac{Cos\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi}{\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)^2}\left[\ln\left((n+1/2)+\frac{c_1}{(n+1/2)}+\right.\right.$$

$$\left.\left.+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi\right]-$$

$$-\frac{AH}{2}\frac{\mathrm{Sin}\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)\pi}{\left((n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\right)^2}\alpha_1+O\left(\frac{1}{n^3}\right)=0$$

olur. Buradan;

$$-(-1)^n\left(\frac{c_1\pi}{(n+1/2)}+\frac{A}{2}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\pi+O\left(\frac{\ln^2 n}{n^4}\right)\right)+$$

$$(-1)^n\left(\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)\right)\left(\frac{c_1\pi}{n+1/2}+\frac{A}{2}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\pi+O\left(\frac{\ln^2 n}{n^4}\right)\right)\beta_1+$$

$$\frac{(-1)^n}{(n+1/2)}\left(1-\frac{c_1^2\pi^2}{(n+1/2)^2}-\frac{A^2}{4}\frac{\ln^2(n+1/2)}{(n+1/2)^2}-Ac_1\pi\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)\right)\beta_2+$$

$$-\frac{A(-1)^n}{2}\frac{1}{(n+1/2)}\left(\frac{c_1\pi}{(n+1/2)}+\frac{A}{2}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\pi+O\left(\frac{\ln^2 n}{n^4}\right)\right)\alpha_1+$$

$$+\frac{A(-1)^n}{2}\frac{1}{(n+1/2)}\left(1-\frac{c_1^{\,2}\pi^2}{(n+1/2)^2}-\frac{A^2}{4}\frac{\ln^2(n+1/2)}{(n+1/2)^2}-Ac_1\pi\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)\right)$$

$$\ln\pi+\ln(n+1/2)+\frac{c_1}{(n+1/2)^2}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)\Bigg)-$$

$$-\frac{(-1)^n AH}{2}\frac{1}{(n+1/2)^2}\left(\frac{c_1\pi}{n+1/2}+\frac{A}{2}\frac{\ln(n+1/2)}{(n+1/2)}+\delta_n\pi+O\left(\frac{\ln^2 n}{n^4}\right)\right)$$

$$\ln\pi+\ln(n+1/2)+\frac{c_1}{(n+1/2)^2}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)\Bigg)$$

$$-\frac{AH(-1)^n}{2}\frac{1}{(n+1/2)^2}\left(1-\frac{c_1^{\,2}\pi^2}{(n+1/2)^2}-\frac{A^2}{4}\frac{\ln^2(n+1/2)}{(n+1/2)^2}-Ac_1\pi\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)\right)\alpha_1$$

$$+O\left(\frac{\ln^2 n}{n^4}\right)=0$$

$$-\delta_n\pi-\frac{c_1\pi}{(n+1/2)}-\frac{A}{2}\frac{\ln(n+1/2)}{(n+1/2)}+\frac{\beta_2}{(n+1/2)}-\frac{A}{2}\frac{c_1\pi\alpha_1}{(n+1/2)^2}-\frac{A^2\alpha_1}{4}\frac{\ln(n+1/2)}{(n+1/2)^2}+$$

$$+\frac{A}{2}\frac{\ln\pi}{(n+1/2)}+\frac{A}{2}\frac{\ln(n+1/2)}{(n+1/2)}-\frac{AH}{2}\frac{\alpha_1}{(n+1/2)^2}=0$$

$$\delta_n\pi=\frac{\frac{A}{2}\ln\pi+\beta_2-c_1\pi}{(n+1/2)}+\frac{-\frac{Ac_1\pi\alpha_1}{2}-\frac{AH\alpha_1}{2}}{(n+1/2)^2}-\frac{A^2\alpha_1}{4}\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)$$

buradan;

$$\delta_n=\frac{c_2}{(n+1/2)}+\frac{c_3}{(n+1/2)^2}+c_4\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)\ \text{alınır.}$$

O halde

$$\sqrt{\lambda_n}=(n+1/2)+\frac{c_1}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\frac{c_2}{(n+1/2)}+\frac{c_3}{(n+1/2)^2}+$$

$$+\ c_4\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)$$

$$\sqrt{\lambda_n}=(n+1/2)+\frac{c_0}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)}+\frac{c_3}{(n+1/2)^2}+c_4\frac{\ln(n+1/2)}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^4}\right)$$

Buradan;

$$\lambda_n=(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)+2c_0+2c_4\frac{\ln(n+1/2)}{(n+1/2)}+\frac{2c_3}{(n+1/2)}+\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}+$$
$$+\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}+\gamma_3\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^3}\right)$$

özdeğerlerinin asimptotik ifadesi bulunur.Burada,

$$c_4=-\frac{A^2\alpha_1}{4\pi},\qquad c_3=-\frac{A\alpha_1}{2}\left(c_1+\frac{H}{\pi}\right),\quad \gamma_1=\frac{A^2}{4\pi^2},\ \ \gamma_2=\frac{A^2\ln\pi+2A\beta_2}{\pi^2}$$

$$\gamma_3=c_0{}^2=\left(\frac{A\ln\pi+2\beta_2}{\pi}\right)^2,\ \ \alpha_1=\mathrm{Cos}1+N_1-N_2,\quad \alpha_2=\frac{\mathrm{Sin}2}{4}+N_3-\frac{N_4}{2}$$

$$\beta_2=\frac{1}{2}\int_0^\pi q(t)dt+A\alpha_2-H=\frac{1}{2}\int_0^\pi q(t)dt+A\left(\frac{\mathrm{Sin}2}{4}+N_3-\frac{N_4}{2}\right)-H$$

$$\beta_1=\frac{q(\pi)-q(0)}{4}+\frac{A}{4\pi}+AH\alpha_2+\frac{H}{2}\int_0^\pi q(t)dt$$
$$=\frac{q(\pi)-q(0)}{4}+\frac{A}{4\pi}+AH\left(\frac{\mathrm{Sin}2}{4}+N_3-\frac{N_4}{2}\right)+\frac{H}{2}\int_0^\pi q(t)dt \quad \text{şeklindedir.}$$

$\varphi(x,\lambda)$ fonksiyonunun asimptotik ifadesinde $\lambda$ nın yerine $\lambda_n$ yazılırsa L operatörünün $\varphi(x,\lambda_n)$ öz fonksiyonlarının asimptotik ifadeleri bulunur.

$$\varphi(x,\lambda_n)=\frac{\sin(n+1/2)x}{(n+1/2)}+\frac{A}{2\pi}\frac{\ln(n+1/2)}{(n+1/2)^2}\mathrm{Cos}(n+1/2)x+O\left(\frac{1}{n^2}\right)\ \text{ alınır.}$$

Şimdi L operatörünün izi bulunacak olursa;

$$L:\begin{cases}-y''+\left(\frac{A}{x}+q(x)\right)y=\lambda y & 0\le x\le\pi\\ y(0)=0 & q(x)\in C^2[0,\pi]\\ y'(\pi)-Hy(\pi)=0\end{cases}$$

$\varphi(x,\lambda)$ ile $\varphi'(0,\lambda)=1,\varphi(0,\lambda)=0$ koşulunu sağlayan çözümü olsun.

O halde

$$\varphi(x,\lambda)=\frac{Sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_0^x\frac{Sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\left(\frac{A}{t}+q(t)\right)\frac{Sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt+O\left(\frac{1}{\lambda^{3/2}}\right)$$

şeklinde bulunmuştu. L operatörünün özdeğerleri için ise;

$$\lambda_n=(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)+2c_0+2c_1\frac{\ln(n+1/2)}{(n+1/2)}+\frac{2c_2}{(n+1/2)}+\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}+$$
$$+\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}+\gamma_3\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^3}\right)$$

asimptotik ifadesi bulunmuştu.

Şimdi bu L operatörünün izi (trace) bulunursa;

$$\sum_{n=0}^{\infty}\Bigg[\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-$$
$$-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}\Bigg]$$

toplamının değeri L operatörünün regülerize izini (trace) verir.

Bu toplam yakınsak ve $\varphi(x,\lambda)$, $\lambda$ parametresine göre $\frac{1}{2}$ göstergeli tam fonksiyon olduğundan Weierstrass teoreminden;

$\varphi'(\pi,\lambda)-H\varphi(\pi,\lambda)=A\Phi(\lambda)$ şeklinde yazılabilir.

$\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n,\ldots$ler $\Phi(\lambda)$'nın sıfırları olmak üzere ve $\Phi(\lambda)$ tam fonksiyon olduğundan $\Phi(\lambda)=\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right)$ olur.

$\lambda=-\mu$ ve $\mu>0$ alınırsa $\varphi'(\pi,-\mu)-H\varphi(\pi,-\mu)=A\Phi(-\mu)$ eşitliği alınır.

İlk önce eşitliğin sağ tarafındaki $\Phi(-\mu)$'nün asimptotik ifadesi bulunursa;

$$\Phi(\lambda)=\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right) \text{ ise } \Phi(-\mu)=\prod_{n=0}^{\infty}\left(1+\frac{\mu}{\lambda_n}\right)=\frac{\prod_{n=0}^{\infty}\left(1+\frac{\mu}{\lambda_n}\right)}{\prod_{n=0}^{\infty}\left(1+\frac{\mu}{(n+1/2)}\right)}Ch\sqrt{\mu}\pi$$

$$=\prod_{n=0}^{\infty}\left(\frac{(n+1/2)^2}{\lambda_n}\right)\prod_{n=0}^{\infty}\left[\frac{\lambda_n+\mu}{\mu+(n+1/2)^2}\right]\mathrm{Ch}\sqrt{\mu}\pi$$

$\Phi(-\mu)=c\Psi(\mu)\mathrm{Ch}\sqrt{\mu}\pi$ olur. Burada $c=\prod_{n=0}^{\infty}\left(\frac{(n+1/2)^2}{\lambda_n}\right)$ dir.

$\Psi(\mu)=\prod_{n=0}^{\infty}\left[\frac{\lambda_n+\mu}{\mu+(n+1/2)^2}\right]=\prod_{n=0}^{\infty}\left[1-\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right]$ ve $\Psi(\mu)$ yakınsaktır.
Dolayısıyla,

$\ln\Psi(\mu)=\sum_{n=0}^{\infty}\ln\left[1-\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right]$ olur. Buradan;

$\ln\Psi(\mu)=\ln\left(1-\frac{1/4-\lambda_0}{\mu+1/4}\right)+\sum_{n=1}^{\infty}\ln\left[1-\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right]$ dir.

$$\ln\left(1-\frac{1/4-\lambda_0}{\mu+1/4}\right)=-\frac{1/4-\lambda_0}{\mu+1/4}-\frac{1}{2}\left(\frac{1/4-\lambda_0}{\mu+1/4}\right)^2-\ \ldots=-\frac{4-\lambda_0}{\mu+1/4}+O\left(\frac{1}{\mu^2}\right)=$$

$$=\frac{1/4-\lambda_0}{\mu}\left(\frac{1}{1+1/4\mu}\right)+O\left(\frac{1}{\mu^2}\right)=$$

$$=\frac{\lambda_0-1/4}{\mu}\left[1-\frac{1}{4\mu}+\left(\frac{1}{4\mu}\right)^2+\ \ldots\ \right]+O\left(\frac{1}{\mu^2}\right)$$

$$=\frac{\lambda_0-1/4}{\mu}+O\left(\frac{1}{\mu^2}\right)$$

O halde

$$\ln\Psi(\mu)=\frac{\lambda_0-1/4}{\mu}+\sum_{n=1}^{\infty}\ln\left[1-\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right]+O\left(\frac{1}{\mu^2}\right)\ \text{olur.}$$

$\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ serisi yakınsak olduğundan yukarıdaki seri de yakınsaktır.

P $=\sum Pn$ yakınsak olması için gerek ve yeter şart $\sum(1-Pn)$ yakınsak
Buradan; $\ln P=\ln\prod Pn=\sum\ln Pn=\sum\ln(1-(1-Pn))$ dir.
Dolayısıyla;

$$\ln\Psi(\mu)=\frac{\lambda_0-1/4}{\mu}+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}-\frac{1}{k}\left(\frac{-\lambda_n+(n+1/2)^2}{(n+1/2)^2+\mu}\right)^k+O\left(\frac{1}{\mu^2}\right)\quad \text{olur.}$$

$$\ln\Psi(\mu)=\frac{\lambda_0-1/4}{\mu}-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right)^k+O\left(\frac{1}{\mu^2}\right)\quad \text{dir}$$

$\left(\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right)^k$ ifadesinden $\frac{A}{\pi}\ln(n+\frac{1}{2})$ eklenip çıkarılırsa;

$$\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n-\frac{A}{\pi}\ln(n+1/2)}{\mu+(n+1/2)^2}\right]^k=$$

$$=\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right]^k$$

$$=\sum_{j=0}^{k}\binom{k}{j}\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-j}\left(\frac{A}{\pi}\right)^j\left(\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)^j\quad \text{olur. Yerine}$$

yazılırsa ;

$$\ln\Psi(\mu)=-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-j}\left(\frac{A}{\pi}\right)^j\left(\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)^j$$

$$+\frac{\lambda_0-1/4}{\mu}+O\left(\frac{1}{\mu^2}\right)$$

$$=\frac{\lambda_0-1/4}{\mu}-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^k-$$

$$-\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k}k\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-1}\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}-$$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=2}^{\infty}\binom{k}{j}\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-j}\left(\frac{A}{\pi}\right)^j\left(\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)^j+O(\mu^{-2})$$

Şimdi $\frac{A}{2\pi}\sum_{n=1}^{\infty}\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-1}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}$ ifadesi alınır.

Özdeğer ifadesinden $\left|\lambda_n-\left(n+\frac{1}{2}\right)^2-\frac{A}{\pi}\ln\left(n+\frac{1}{2}\right)\right|\leq a$ dir. ( a-sabit)

$$\frac{A}{\pi}\sum_{n=1}^{\infty}\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-1}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\leq$$

$$\leq\frac{A}{\pi}\sum\frac{a^{k-1}}{\left[\mu+(n+1/2)^2\right]^{k-1}}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}$$

$$=\frac{A}{\pi}a^{k-1}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{\left[\mu+(n+1/2)^2\right]^k}$$

$$=\frac{A}{\pi}\frac{a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{\ln(x+1/2)}{\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]^k}\quad \text{olur.}$$

Buna göre,

$\ln\Psi(\mu)=$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right)^k-\sum_{k=1}^{\infty}\frac{A}{\pi}\frac{a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{\left[1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2\right]^k}$$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=2}^{k}\binom{k}{j}\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(\mathrm{n}+1/2)-\lambda_{\mathrm{n}}}{\mu+(n+1/2)^2}\right]^{k-j}\left(\frac{A}{\pi}\right)^{j}\left(\frac{\ln(\mathrm{n}+1/2)}{\mu+(n+1/2)^2}\right)^{j}+$$

$$+\frac{\lambda_0-1/4}{\mu}+O\left(\frac{1}{\mu^2}\right)$$

$$\ln\Psi(\mu)=-\sum_{n=1}^{\infty}\left(\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right)-\frac{A}{\pi}\frac{1}{\mu}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2}-$$

$$-\sum_{k=2}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right)^{k}-\sum_{k=2}^{\infty}\frac{A}{\pi}\frac{a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2}$$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=2}^{k}\binom{k}{j}\left[\frac{(n+1/2)^2+\frac{A}{\pi}\ln(\mathrm{n}+1/2)-\lambda_{\mathrm{n}}}{\mu+(n+1/2)^2}\right]^{k-j}\left(\frac{A}{\pi}\right)^{j}\left(\frac{\ln(\mathrm{n}+1/2)}{\mu+(n+1/2)^2}\right)^{j}+$$

$$+\frac{\lambda_0-1/4}{\mu}+O\left(\frac{1}{\mu^2}\right)$$

Burada;

$$\frac{1}{\mu}\sum_{n=1}^{\infty}\frac{\ln(\mathrm{n}+1/2)}{1+\left(\frac{\mathrm{n}+1/2}{\sqrt{\mu}}\right)^2}\le\frac{1}{\mu}\int_1^{\infty}\frac{\ln(\mathrm{x}+1/2)}{1+\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\right)^2}d\mathrm{x}=\frac{1}{\mu^{1/2}}\int_1^{\infty}\frac{\ln\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\sqrt{\mu}\right)}{1+\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\right)^2}d\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\right)$$

$$=\frac{1}{\mu^{1/2}}\int_1^{\infty}\frac{\ln\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\right)}{1+\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\right)^2}d\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\right)+\frac{\ln\sqrt{\mu}}{\mu^{1/2}}\int_1^{\infty}\frac{d\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\right)}{1+\left(\frac{\mathrm{x}+1/2}{\sqrt{\mu}}\right)^2}$$

$$= \frac{1}{\mu^{1/2}} \int_{3/2\sqrt{\mu}}^{\infty} \frac{\ln t}{1+t^2} dt + \frac{\ln\sqrt{\mu}}{\mu^{1/2}} \int_{3/2\sqrt{\mu}}^{\infty} \frac{dt}{1+t^2}$$

$$= \frac{1}{\mu^{1/2}} \left[ \left( \ln t \operatorname{arctg} t + t - \frac{t^3}{3^2} + \frac{t^5}{5^2} - \ldots \right) \Big|_{3/2\sqrt{\mu}}^{1} + \right.$$

$$\left. + \left( \ln t \operatorname{arctg} t + \frac{\pi}{2} \ln t + \frac{1}{t} - \frac{1}{3^2 t^3} + \ldots \right) \Big|_{1}^{\infty} \right] + \frac{\ln\sqrt{\mu}}{\mu^{1/2}} \operatorname{arctg} t \Big|_{3/2\sqrt{\mu}}^{\infty}$$

$$= \frac{1}{\mu^{1/2}} \left[ \left( 1 - \frac{1}{3^2} + \frac{1}{5^2} - \ldots \right) - \ln\left(3/2\sqrt{\mu}\right) \operatorname{arctg}\left(3/2\sqrt{\mu}\right) - \left(3/2\sqrt{\mu}\right) + \left(3/2\sqrt{\mu}\right)^3 \frac{1}{3^2} - \right.$$

$$\left. - \ldots - \left( 1 - \frac{1}{3^2} + \frac{1}{5^2} - \ldots \right) + \frac{\pi}{2} \frac{\ln\sqrt{\mu}}{\mu^{1/2}} - \frac{\ln\sqrt{\mu}}{\mu^{1/2}} \operatorname{arctg}\left(3/2\sqrt{\mu}\right) \right]$$

$$= -\frac{\ln\left(3/2\sqrt{\mu}\right)}{\mu^{1/2}} \left( \left(3/2\sqrt{\mu}\right) + \left(3/2\sqrt{\mu}\right)^3 \frac{1}{3} + \ldots \right) - 3/2\sqrt{\mu} + \frac{3^3}{2^3 3^2 \mu^2} - \ldots + \frac{\pi}{2} \frac{\ln\sqrt{\mu}}{\mu^{1/2}} -$$

$$- \frac{\ln\sqrt{\mu}}{\mu^{1/2}} \left( 3/2\sqrt{\mu} - \left(3/2\sqrt{\mu}\right)^3 \frac{1}{3} + \ldots \right)$$

$$= -\frac{3}{2\mu} \ln\left(3/2\sqrt{\mu}\right) - \frac{3}{2\mu} + \frac{\pi}{2} \frac{\ln\sqrt{\mu}}{\sqrt{\mu}} - \frac{3}{2\mu} \ln\sqrt{\mu} + O\left( \frac{\ln\sqrt{\mu}}{\mu^2} \right)$$

$$= \frac{\pi}{2} \frac{\ln\sqrt{\mu}}{\sqrt{\mu}} - \frac{3}{2\mu} \left(1 + \ln(3/2)\right) + O\left( \frac{\ln\sqrt{\mu}}{\mu^2} \right)$$

O halde;

$$\ln \Psi(\mu) = -\sum_{n=1}^{\infty} \left( \frac{(n+1/2)^2 + \frac{A}{\pi} \ln(n+1/2) - \lambda_n}{\mu + (n+1/2)^2} \right) + \frac{A}{2} \frac{\ln\sqrt{\mu}}{\sqrt{\mu}} + \frac{3A}{2\mu\pi} \left(1 + \ln(3/2)\right) +$$

$$-\sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left( \frac{(n+1/2)^2 + \frac{A}{\pi} \ln(n+1/2) - \lambda_n}{\mu + (n+1/2)^2} \right)^k - \sum_{k=2}^{\infty} \frac{A}{\pi} \frac{a^{k-1}}{\mu^k} \sum_{n=1}^{\infty} \frac{\ln(n+1/2)}{1 + \left( \frac{n+1/2}{\sqrt{\mu}} \right)^2} -$$

$$-\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=2}^{k} \binom{k}{j} \left[ \frac{(n+1/2)^2 + \frac{A}{\pi} \ln(n+1/2) - \lambda_n}{\mu + (n+1/2)^2} \right]^{k-j} \left( \frac{A}{\pi} \right)^j \left( \frac{\ln(n+1/2)}{\mu + (n+1/2)^2} \right)^j +$$

$$+\frac{\lambda_0-1/4}{\mu}+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)$$

Buradan,

$$\ln\Psi(\mu)=\sum_{n=1}^{\infty}\left(\frac{(n+1/2)^2+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right)+\frac{\lambda_0-1/4}{\mu}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+$$

$$+\frac{3A}{2\mu\pi}(1+\ln(3/2))+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)$$

alınır.Bu ifadeye uygun eklemeler ve çıkarmalar yapılırsa;

$$\ln\Psi(\mu)=$$

$$=\sum_{n=1}^{\infty}\left[\frac{\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}}{\mu+(n+1/2)^2}\right.$$

$$\left.-\frac{\gamma_2\frac{\ln\,(n+1/2)}{(n+1/2)^2}}{\mu+(n+1/2)^2}\right]+$$

$$+2c_0\sum_{n=1}^{\infty}\frac{1}{\mu+(n+1/2)^2}+2c_1\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+$$

$$+2c_2\sum_{n=1}^{\infty}\frac{1}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+\gamma_1\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+$$

$$+\gamma_2\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+\frac{\lambda_0-1/4}{\mu}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+$$

$$+\frac{3A}{2\pi\mu}(1+\ln(3/2))+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)$$

$$\frac{1}{\mu}\sum_{n=1}^{\infty}\left[\frac{\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}}{\mu+(n+1/2)^2}\right.$$

$$\left.-\frac{\gamma_2\frac{\ln\,(n+1/2)}{(n+1/2)^2}}{\mu+(n+1/2)^2}\right]\left[\mu+(n+1/2)^2-(n+1/2)^2\right]$$

$$+2c_0\sum_{n=1}^{\infty}\frac{1}{\mu+(n+1/2)^2}+2c_1\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+$$

$$+2c_2\sum_{n=1}^{\infty}\frac{1}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+\gamma_1\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+$$

$$+\gamma_2\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+\frac{\lambda_0-1/4}{\mu}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}$$

$$+\frac{3A}{2\pi\mu}\left(1+\ln(3/2)\right)+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)$$

$$\ln\Psi(\mu)=\frac{1}{\mu}\sum_{n=0}^{\infty}\left(\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\right.$$

$$\left.-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln\,(n+1/2)}{(n+1/2)^2}\right)$$

$$-\frac{1}{\mu}\sum_{n=1}^{\infty}\left(\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\right.$$

$$\left.-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln\,(n+1/2)}{(n+1/2)^2}\right)\frac{(n+1/2)^2}{\mu+(n+1/2)^2}$$

$$+2c_0\sum_{n=1}^{\infty}\frac{1}{\mu+(n+1/2)^2}+2c_1\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+$$

$$+2c_2\sum_{n=1}^{\infty}\frac{1}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+\gamma_1\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+$$

$$+\gamma_2\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+\frac{\lambda_0-1/4}{\mu}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+$$

$$+\frac{3A}{2\pi\mu}\left(1+\ln(3/2)\right)+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)$$

$$\ln\Psi(\mu)=\frac{1}{\mu}\sum_{n=0}^{\infty}\overbrace{\left(\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}\right)}^{s_\lambda}$$

$$-\frac{1}{\mu}\left(\lambda_0-\frac{1}{4}+\frac{A}{2\pi}\ln 2-2c_0+4c_1\ln 2-4c_2-2\gamma_1\ln^2 2+4\gamma_2\ln 2\right)-$$

$$-\frac{1}{\mu}\sum_{n=1}^{\infty}\left(\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}\right)\frac{(n+1/2)^2}{\mu+(n+1/2)^2}$$

$$+2c_0\sum_{n=1}^{\infty}\frac{1}{\mu+(n+1/2)^2}+2c_1\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+$$

$$+2c_2\sum_{n=1}^{\infty}\frac{1}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+\gamma_1\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+$$

$$+\gamma_2\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+\frac{\lambda_0-1/4}{\mu}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+$$

$$+\frac{3A}{2\pi\mu}\left(1+\ln(3/2)\right)+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)$$

Şimdi, bu toplamların değerleri hesaplanırsa;

$\sum_{n=1}^{\infty}\frac{1}{\mu+(n+1/2)^2}$ toplamının $\mu$ 'ye göre asimptotik ifadesi için;

Bunun için ;

$$\sum_{n=1}^{\infty}\frac{1}{\mu+(n+1/2)^2}=\frac{\pi tgh\sqrt{\mu}\pi}{2\sqrt{\mu}}-\frac{1}{2\mu} \text{ eşitliğinden,}$$

$$tgh\sqrt{\mu}\pi=\frac{e^{\sqrt{\mu}\pi}-e^{-\sqrt{\mu}\pi}}{e^{\sqrt{\mu}\pi}+e^{-\sqrt{\mu}\pi}}=\frac{1-e^{-2\sqrt{\mu}\pi}}{1+e^{-2\sqrt{\mu}\pi}}=1+O\left(e^{-2\sqrt{\mu}\pi}\right) \quad (\mu \text{ nün büyük değerleri için})$$

O halde;

$$\sum_{n=1}^{\infty}\frac{1}{\mu+(n+1/2)^2}=\frac{\pi}{2\sqrt{\mu}}-\frac{1}{2\mu}+O\left(e^{-2\sqrt{\mu}\pi}\right) \text{ olur.}$$

$\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu+(n+1/2)^2\right)}$ toplamı için; Bu ifade her $\mu$ için n'e göre düzgün yakınsaktır. Dolayısıyla,

$$\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu+(n+1/2)^2\right)}\le\int_1^{\infty}\frac{\ln(x+1/2)}{(x+1/2)\left(\mu+(x+1/2)^2\right)}dx$$

$$=\frac{1}{\mu}\int_1^{\infty}\frac{\ln(x+1/2)}{(x+1/2)\left(1+\left(\frac{(x+1/2)}{\sqrt{\mu}}\right)^2\right)}d(x+1/2)$$

$$=\frac{1}{\mu}\int_1^{\infty}\frac{\ln\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}d\left(\frac{x+1/2}{\sqrt{\mu}}\right)+\frac{1}{\mu}\int_1^{\infty}\frac{\ln\sqrt{\mu}\ \ d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}$$

$\frac{x+1/2}{\sqrt{\mu}}=t$ denirse $\qquad x=1$ iç in $t=3/2\sqrt{\mu}$

$\qquad x=\infty$ iç in $t=\infty$

$$=\frac{1}{\mu}\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln t}{t(1+t^2)}dt+\frac{\ln\sqrt{\mu}}{\mu}\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t(1+t^2)} \quad \text{olur.}$$

I. İntegral için;

$$\int \frac{\ln t}{t(1+t^2)}dt = \frac{1}{2}\int \frac{2t\,\ln t}{t^2(1+t^2)}dt = \frac{1}{4}\int \frac{2t\,2\ln t}{t^2(1+t^2)}dt = \frac{1}{4}\int \frac{\ln t^2}{t^2(1+t^2)}d(t^2) = \frac{1}{4}\int \frac{\ln u}{u(1+u)}du$$

$$= \frac{1}{4}\left(\frac{\ln^2 u}{2} - \int \frac{\ln u}{u+1}du\right) = \frac{1}{4}\left(\frac{\ln^2 u}{2} - \ln u \ln(1+u) + \int \frac{\ln(1+u)}{u}du\right)$$

Burada $\int \frac{\ln(1+u)}{u}du = \begin{cases} u - \frac{u^2}{2^2} + \frac{u^3}{3^2} - \frac{u^4}{4^2} + \ \dots & u^2 \langle\ 1 \\ \frac{\ln^2 u}{2} - \frac{1}{u} + \frac{1}{u^2 2^2} - \frac{1}{u^3 3^2} + \ \dots & u^2 \rangle\ 1 \end{cases}$ olduğundan

$$\frac{1}{4}\int \frac{\ln u}{u(1+u)}du = \begin{cases} \frac{1}{4}\left(\frac{\ln^2 u}{2} - \ln u \ln(1+u) + u - \frac{u^2}{2^2} + \frac{u^3}{3^2} - \frac{u^4}{4^2} + \ \dots\right) & u^2 \langle 1 \\ \frac{1}{4}\left(\frac{\ln^2 u}{2} - \ln u \ln(1+u) + \frac{\ln^2 u}{2} - \frac{1}{u} + \frac{1}{u^2 2^2} - \frac{1}{u^3 3^2} + \ \dots\right) & u^2 \rangle 1 \end{cases}$$

Dolayısıyla;

$$\frac{1}{\mu}\int_{3/2\sqrt{\mu}}^{\infty} \frac{\ln t}{t(1+t^2)}dt = \frac{1}{\mu}\int_{3/2\sqrt{\mu}}^{1} \frac{\ln t}{t(1+t^2)}dt + \frac{1}{\mu}\int_{1}^{\infty} \frac{\ln t}{t(1+t^2)}dt$$

$$= \frac{1}{4\mu}\left(\left(\frac{\ln^2(t^2)}{2} - \ln(t^2)\ln(1+t^2) + t^2 - \frac{t^4}{2^2} + \frac{t^6}{3^2} - \ \dots\right)\Bigg|_{3/2\sqrt{\mu}}^{1}\right) +$$

$$+ \frac{1}{4\mu}\left(\left(\frac{\ln^2(t^2)}{2} - \ln(t^2)\ln(1+t^2) + \frac{\ln^2(t^2)}{2} - \frac{1}{t^2} + \frac{1}{t^4 2^2} - \frac{1}{t^6 3^2} + \ \dots\right)\Bigg|_{1}^{\infty}\right)$$

$$= \frac{1}{2\mu}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \ \dots\right) - \frac{1}{4\mu}\ln(9/4\mu)\ln\left(\frac{9/4\mu}{1+9/4\mu}\right) + O\left(\frac{1}{\mu^2}\right)$$

$$= \frac{\pi^2}{24\mu} - \frac{1}{4\mu}\left(2\ln(3/2) - 2\ln\sqrt{\mu}\right)\left(2\ln(3/2) - 2\ln\sqrt{\mu} - \ln(1+9/4\mu)\right) + O\left(\frac{1}{\mu^2}\right)$$

$$=\frac{\pi^2}{24\mu}-\frac{1}{2\mu}\left(\ln(3/2)-\ln\sqrt{\mu}\right)\left[2\ln(3/2)-2\ln\sqrt{\mu}-\left(\frac{9}{4\mu}-\left(\frac{9}{4\mu}\right)^2\frac{1}{2}+\ \dots\ \right)\right]+$$

$$+O\left(\frac{1}{\mu^2}\right)$$

$$=\frac{\pi^2}{24\mu}-\frac{1}{4\mu}\left(\ln(3/2)-\ln\sqrt{\mu}\right)\left(2\ln(3/2)-2\ln\sqrt{\mu}\right)+O\left(\frac{1}{\mu^2}\right)$$

$$=\frac{\pi^2}{24\mu}-\frac{1}{\mu}\left(\ln^2(3/2)-2\ln(3/2)\ln\sqrt{\mu}+\ln^2\sqrt{\mu}\right)+O\left(\frac{1}{\mu^2}\right)$$

$$=\frac{\ln^2(3/2)-2\ln(3/2)\ln\sqrt{\mu}+\ln^2\sqrt{\mu}+\pi^2/24}{\mu}+O\left(\frac{1}{\mu^2}\right)\quad \text{olur.}$$

II. İntegral için;

$$\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t(1+t^2)}=\frac{1}{2}\int_{3/2\sqrt{\mu}}^{\infty}\frac{2tdt}{t^2(1+t^2)}=\frac{1}{2}\int_{3/2\sqrt{\mu}}^{\infty}\frac{d(t^2)}{t^2(1+t^2)}=\frac{1}{2}\int\frac{du}{u(1+u)}=\frac{1}{2}\ln\left(\frac{u}{1+u}\right)$$

$$=\frac{1}{2}\ln\left(\frac{t^2}{1+t^2}\right)\Bigg|_{3/2\sqrt{\mu}}^{\infty}=-\frac{1}{2}\ln\left(\frac{9/4\mu}{1+9/4\mu}\right)\quad \text{olur. Buradan;}$$

$$\frac{\ln\sqrt{\mu}}{\mu}\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t(1+t^2)}=\frac{\ln\sqrt{\mu}}{\mu}\left(-\frac{1}{2}\ln(9/4\mu)+\frac{1}{2}\ln(1+9/4\mu)\right)$$

$$=-\frac{\ln\sqrt{\mu}}{2\mu}\left(\ln(9/4\mu)-\ln(1+9/4\mu)\right)$$

$$=-\frac{\ln\sqrt{\mu}}{2\mu}\left[2\ln(3/2)-2\ln\sqrt{\mu}-\left(\frac{9}{4\mu}-\left(\frac{9}{4\mu}\right)^2\frac{1}{2!}+\left(\frac{9}{4\mu}\right)^3\frac{1}{3!}+\dots\right)\right]$$

$$=-\frac{\ln\sqrt{\mu}}{2\mu}\left[2\ln(3/2)-2\ln\sqrt{\mu}-\frac{9}{4\mu}+O\left(\frac{1}{\mu^2}\right)\right]$$

$$=\frac{\ln^2\sqrt{\mu}-\ln\sqrt{\mu}\ln(3/2)}{\mu}+\frac{9\ln\sqrt{\mu}}{8\mu^2}+O\left(\frac{\ln\sqrt{\mu}}{\mu^3}\right)$$

$$=\frac{\ln^2\sqrt{\mu}-\ln\sqrt{\mu}\ln(3/2)}{\mu}+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)\quad \text{olur.}$$

O halde;

$$\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu+(n+1/2)^2\right)}=\frac{\ln^2(3/2)-3\ln(3/2)\ln\sqrt{\mu}+2\ln^2\sqrt{\mu}+\pi^2/24}{\mu}+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)$$

bulunur.

$$\sum_{n=1}^{\infty}\frac{1}{(n+1/2)\left(\mu+(n+1/2)^2\right)}$$ toplamının $\mu$'ye göre asimptotik ifadesi için;

Seri her $\mu$ için n'e göre düzgün yakınsaktır. Dolayısıyla;

$$\sum_{n=1}^{\infty}\frac{1}{(n+1/2)\left(\mu+(n+1/2)^2\right)} \leq \int_1^{\infty}\frac{dx}{(x+1/2)\left(\mu+(x+1/2)^2\right)} = \frac{1}{\mu}\int_1^{\infty}\frac{d(x+1/2)}{(x+1/2)\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}$$

$$= \frac{1}{\mu}\int_1^{\infty}\frac{d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}$$

$$\frac{x+1/2}{\sqrt{\mu}} = t \quad \text{denirse} \quad \begin{array}{l} x=1 \ \text{iç in} \ \ t=3/2\sqrt{\mu} \\ x=\infty \ \text{iç in} \ \ t=\infty \end{array}$$

$$\frac{1}{\mu}\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t(1+t^2)} = \frac{1}{2\mu}\int_{3/2\sqrt{\mu}}^{\infty}\frac{2tdt}{t^2(1+t^2)} = \frac{1}{2\mu}\int_{3/2\sqrt{\mu}}^{\infty}\frac{d(t^2)}{t^2(1+t^2)} =$$

$$= \frac{1}{2\mu}\int\frac{du}{u(1+u)} = \frac{1}{2\mu}\ln\left(\frac{u}{1+u}\right) = \frac{1}{2\mu}\ln\left(\frac{t^2}{1+t^2}\right)\Bigg|_{3/2\sqrt{\mu}}^{\infty}$$

$$= -\frac{1}{2\mu}\ln\left(\frac{9/4\mu}{1+9/4\mu}\right)$$

$$= -\frac{1}{2\mu}\ln(9/4\mu) + \frac{1}{2\mu}\ln(1+9/4\mu)$$

$$= -\frac{1}{2\mu}\left[\ln(9/4\mu) - \ln(1+9/4\mu)\right]$$

$$= -\frac{1}{2\mu}\left[2\ln(3/2) - 2\ln\sqrt{\mu} - \frac{9}{4\mu} + O\left(\frac{1}{\mu^2}\right)\right]$$

$$= \frac{\ln\sqrt{\mu} - \ln(3/2)}{\mu} + \frac{9}{8\mu^2} + O\left(\frac{1}{\mu^2}\right)$$ bulunur.

$\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}$ toplamının $\mu$'ye göre asimptotik ifadesi için;

$$\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}\le\int_1^{\infty}\frac{\ln^2(x+1/2)}{(x+1/2)^2\left(\mu+(x+1/2)^2\right)}dx$$

$$=\frac{1}{\mu}\int_1^{\infty}\frac{\ln^2(x+1/2)d(x+1/2)}{(x+1/2)^2\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}$$

$$=\frac{1}{\mu^{3/2}}\int_1^{\infty}\frac{\ln^2\left(\frac{x+1/2}{\sqrt{\mu}}\right)d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}-\frac{1}{\mu^{3/2}}\int_1^{\infty}\frac{\ln^2\sqrt{\mu}\,d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}+$$

$$+\frac{2\ln\sqrt{\mu}}{\mu^{3/2}}\int_1^{\infty}\frac{\ln^2(x+1/2)\,d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}$$

Buradan;

$$=\frac{1}{\mu^{3/2}}\int_1^{\infty}\frac{\ln^2\left(\frac{x+1/2}{\sqrt{\mu}}\right)d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}-\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\int_1^{\infty}\frac{d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}+$$

$$+\frac{2\ln\sqrt{\mu}}{\mu^{3/2}}\int_1^{\infty}\frac{\ln^2\left(\frac{x+1/2}{\sqrt{\mu}}\right)d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}+\frac{2\ln^2\sqrt{\mu}}{\mu^{3/2}}\int_1^{\infty}\frac{d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left[1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right]}$$

$\frac{x+1/2}{\sqrt{\mu}}=t$ denirse $\qquad x=1$ iç in $t=3/2\sqrt{\mu}$

$x=\infty$ iç in $t=\infty$

$$=\underbrace{\frac{1}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln^2 t\,dt}{t^2(1+t^2)}}_{I}+\underbrace{\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t^2(1+t^2)}}_{II}+\underbrace{\frac{2\ln\sqrt{\mu}}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln t\,dt}{t^2(1+t^2)}}_{III}\quad \text{olur.}$$

I. İntegral için;

$$\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln^2 t\,dt}{t^2(1+t^2)}=\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln^2 t\,dt}{t^2}-\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln^2 t\,dt}{1+t^2}\qquad \begin{cases} t=\dfrac{1}{x} & t\to\infty\ ,\ x\to 0\\[2mm] dt=-\dfrac{1}{x^2}dx & t\to\dfrac{3}{2\sqrt{\mu}},\ x\to\dfrac{2\sqrt{\mu}}{3}\end{cases}$$

$$=\int_0^{2\sqrt{\mu}/3}\ln^2 x\,dx-\int_0^{2\sqrt{\mu}/3}\frac{\ln^2 x}{1+x^2}dx\qquad \text{olur.}$$

$$\int_0^{2\sqrt{\mu}/3}\ln^2 x\,dx=x\ln^2 x-2x\ln x+2x\,\Big|_0^{2\sqrt{\mu}/3}=\frac{2\sqrt{\mu}}{3}\ln^2\left(\frac{2\sqrt{\mu}}{3}\right)-\frac{4\sqrt{\mu}}{3}\ln\left(\frac{2\sqrt{\mu}}{3}\right)+\frac{4\sqrt{\mu}}{3}$$

$$\int_0^{2\sqrt{\mu}/3}\frac{\ln^2 x}{1+x^2}dx\qquad \text{için;}\qquad \begin{cases} u=\dfrac{1}{1+x^2} & dv=\ln^2 x\,dx\\[2mm] du=-\dfrac{2x}{(1+x^2)^2}dx & v=x\ln^2 x-2x+2x\end{cases}$$

$$\int_0^{2\sqrt{\mu}/3}\frac{\ln^2 x}{1+x^2}dx=\frac{x\ln^2 x-2x\ln x+2x}{1+x^2}\,\Big|_0^{2\sqrt{\mu}/3}+\underbrace{\int_0^{2\sqrt{\mu}/3}\frac{2x^2\ln^2 x-4x^2\ln x+4x^2}{(1+x^2)^2}dx}$$

$$\int_0^{2\sqrt{\mu}/3}\frac{2x^2\ln^2 x-4x^2\ln x+4x^2}{(1+x^2)^2}dx=$$

$$=2\int_0^{2\sqrt{\mu}/3}\frac{x^2\ln^2 x}{(1+x^2)^2}dx-4\int_0^{2\sqrt{\mu}/3}\frac{x^2\ln x}{(1+x^2)^2}dx+4\int_0^{2\sqrt{\mu}/3}\frac{x^2dx}{(1+x^2)^2}$$

$$= \int_0^{2\sqrt{\mu}/3} \frac{2\ln^2 x dx}{1+x^2} - \int_0^{2\sqrt{\mu}/3} \frac{2\ln^2 x dx}{\left(1+x^2\right)^2} - \int_0^{2\sqrt{\mu}/3} \frac{4\ln x dx}{1+x^2} +$$

$$+ \int_0^{2\sqrt{\mu}/3} \frac{4\ln x dx}{\left(1+x^2\right)^2} + \int_0^{2\sqrt{\mu}/3} \frac{4dx}{1+x^2} - \int_0^{2\sqrt{\mu}/3} \frac{4dx}{\left(1+x^2\right)^2}$$

$$\int_0^{2\sqrt{\mu}/3} \frac{\ln^2 x dx}{\left(1+x^2\right)^2} \quad \text{için;} \quad \begin{cases} u = \dfrac{1}{\left(1+x^2\right)^2} & dv = \ln^2 x dx \\ du = -\dfrac{4x}{\left(1+x^2\right)^2} dx & v = x\ln^2 x - 2x + 2x \end{cases}$$

$$\int_0^{2\sqrt{\mu}/3} \frac{\ln^2 x dx}{\left(1+x^2\right)^2} = \underbrace{\frac{x\ln^2 x - 2x\ln x + 2x}{\left(1+x^2\right)^2}\Bigg|_0^{2\sqrt{\mu}/3}}_{O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)} + \underbrace{\int_0^{2\sqrt{\mu}/3} \frac{2x^2\ln^2 x - 4x^2 + 4x^2 dx}{\left(1+x^2\right)^3}}_{O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)}$$

Dolayısıyla ,

$$\int_0^{2\sqrt{\mu}/3} \frac{\ln x dx}{\left(1+x^2\right)^2} = O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right) \quad \text{ve} \quad \int_0^{2\sqrt{\mu}/3} \frac{dx}{\left(1+x^2\right)^2} = O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right) \quad \text{olur.}$$

O halde;

$$\int_0^{2\sqrt{\mu}/3} \frac{2x^2\ln^2 x - 4x^2\ln x + 4x^2 dx}{\left(1+x^2\right)^2} = 2\int_0^{2\sqrt{\mu}/3} \frac{\ln^2 x dx}{1+x^2} - 4\int_0^{2\sqrt{\mu}/3} \frac{\ln x dx}{1+x^2} + 4\int_0^{2\sqrt{\mu}/3} \frac{dx}{1+x^2} + O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$$= 2\int_0^{2\sqrt{\mu}/3} \frac{\ln^2 x dx}{1+x^2} - 4\int_0^1 \frac{\ln x dx}{1+x^2} - 4\int_1^{2\sqrt{\mu}/3} \frac{\ln x dx}{1+x^2} + 4\operatorname{arctg} x \Big|_0^{2\sqrt{\mu}/3} + O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$$= 2\int_0^{2\sqrt{\mu}/3} \frac{\ln^2 x dx}{1+x^2} - 4\left[\ln x \operatorname{arctg} x \Big|_0^1 - \int_0^1 \frac{1}{x}\operatorname{arctg} x dx\right] - 4\left[\ln x \operatorname{arctg} x \Big|_1^{2\sqrt{\mu}/3} - \int_1^{2\sqrt{\mu}/3} \frac{\operatorname{arctg} x}{x} dx\right]$$

$$+4\operatorname{arctg} x \Big|_0^{2\sqrt{\mu}/3} + O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$$= 2\int_0^{2\sqrt{\mu}/3} \frac{\ln^2 x dx}{1+x^2} + 4\left(x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \ \dots \ \Big|_0^1\right) - 4\ln\left(2\sqrt{\mu}/3\right)\operatorname{arctg}\left(2\sqrt{\mu}/3\right) +$$

$$+4\left(\frac{\pi}{2}\ln x+\frac{1}{x}-\frac{1}{3^2x^3}+\frac{1}{5^2x^5}-\ \dots\ \Bigg|_1^{2\sqrt{\mu}/3}\right)+4\operatorname{arctg}\left(2\sqrt{\mu}/3\right)+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$$=2\int_0^{2\sqrt{\mu}/3}\frac{\ln^2 x dx}{1+x^2}+4\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\ \dots\right)-4\ln\left(2\sqrt{\mu}/3\right)\operatorname{arctg}\left(2\sqrt{\mu}/3\right)\ +2\pi n\left(2\sqrt{\mu}/3\right)$$

$$-4\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\ \dots\right)+4\left(\frac{3}{2\sqrt{\mu}}-\frac{3^3}{3^2 2^3\mu^{3/2}}+\ \dots\right)+4\operatorname{arctg}\left(2\sqrt{\mu}/3\right)+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$$=2\int_0^{2\sqrt{\mu}/3}\frac{\ln^2 xdx}{1+x^2}-4\ln\left(2\sqrt{\mu}/3\right)\operatorname{arctg}\left(2\sqrt{\mu}/3\right)\ +2\pi n\left(2\sqrt{\mu}/3\right)+\frac{6}{\sqrt{\mu}}+4\operatorname{arctg}\left(2\sqrt{\mu}/3\right)+$$

$$+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

O halde,

$$\int_0^{2\sqrt{\mu}/3}\frac{\ln^2 xdx}{1+x^2}=\frac{6\sqrt{\mu}\ln^2\left(2\sqrt{\mu}/3\right)-12\sqrt{\mu}\ln\left(2\sqrt{\mu}/3\right)+12\sqrt{\mu}}{9+4\mu}+2\int_0^{2\sqrt{\mu}/3}\frac{\ln^2 xdx}{1+x^2}-$$

$$-4\ln\left(2\sqrt{\mu}/3\right)\operatorname{arctg}\left(2\sqrt{\mu}/3\right)+2\pi\ln\left(2\sqrt{\mu}/3\right)+\frac{6}{\sqrt{\mu}}+4\operatorname{arctg}\left(2\sqrt{\mu}/3\right)+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$$\int_0^{2\sqrt{\mu}/3}\frac{\ln^2 xdx}{1+x^2}=-\frac{6\sqrt{\mu}\ln^2\left(2\sqrt{\mu}/3\right)-12\sqrt{\mu}\ln\left(2\sqrt{\mu}/3\right)+12\sqrt{\mu}}{9+4\mu}+4\ln\left(2\sqrt{\mu}/3\right)\operatorname{arctg}\left(2\sqrt{\mu}/3\right)$$

$$-2\pi\ln\left(2\sqrt{\mu}/3\right)-\frac{6}{\sqrt{\mu}}-4\operatorname{arctg}\left(2\sqrt{\mu}/3\right)+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

Dolayısıyla

$$\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln^2 t\, dt}{t^2\left(1+t^2\right)}=\frac{2\sqrt{\mu}}{3}\ln^2\left(2\sqrt{\mu}/3\right)-\frac{4\sqrt{\mu}}{3}\ln\left(2\sqrt{\mu}/3\right)+4\ln\left(2\sqrt{\mu}/3\right)\operatorname{arctg}\left(2\sqrt{\mu}/3\right)$$

$$-\frac{6\sqrt{\mu}\ln^2\left(2\sqrt{\mu}/3\right)-12\sqrt{\mu}\ln\left(2\sqrt{\mu}/3\right)+12\sqrt{\mu}}{9+4\mu}-$$

$$-2\pi\ln\left(2\sqrt{\mu}/3\right)-\frac{6}{\sqrt{\mu}}+\frac{4\sqrt{\mu}}{3}-4\operatorname{arctg}\left(2\sqrt{\mu}/3\right)+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$ bulunur. Buna

göre,

$$\frac{1}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln^2 t\,dt}{t^2(1+t^2)}=\frac{2\ln^2(2\sqrt{\mu}/3)-4\ln(2\sqrt{\mu}/3)+4}{3\mu}+\frac{4\ln(2\sqrt{\mu}/3)\mathrm{arctg}(2\sqrt{\mu}/3)}{\mu^{3/2}}-$$

$$-\frac{6\sqrt{\mu}\ln^2(2\sqrt{\mu}/3)-12\sqrt{\mu}\ln(2\sqrt{\mu}/3)+12\sqrt{\mu}}{(9+4\mu)\mu^{3/2}}-\frac{2\pi\ln(2\sqrt{\mu}/3)+4\mathrm{arctg}(2\sqrt{\mu}/3)}{\mu^{3/2}}$$

$$-\frac{6}{\mu^2}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)$$ bulunur.

II.İntegral için;

$$\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t^2(1+t^2)}=-\frac{1}{t}-\mathrm{arctgt}\Big|_{3/2\sqrt{\mu}}^{\infty}=-\frac{\pi}{2}+\frac{2\sqrt{\mu}}{3}\mathrm{arctg}(3/2\sqrt{\mu})$$

$\mu$'nün büyük değerleri için $\left(\frac{3}{2\sqrt{\mu}}\right)^2 < 1$ olduğundan

$$\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t^2(1+t^2)}=-\frac{\pi}{2}+\frac{2\sqrt{\mu}}{3}\left(\frac{3}{2\sqrt{\mu}}-\left(\frac{3}{2\sqrt{\mu}}\right)^3\frac{1}{3}+\ \dots\right)$$ dir.

$$=-\frac{\pi}{2}+\frac{2\sqrt{\mu}}{3}+\frac{3}{2\sqrt{\mu}}+O\left(\frac{1}{\mu^{3/2}}\right)$$ bulunur. Buradan ise,

$$\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t^2(1+t^2)}=-\frac{\pi}{2}\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}+\frac{2\ln^2\sqrt{\mu}}{3\mu}+\frac{3\ln^2\sqrt{\mu}}{2\mu^2}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)$$ olur.

III.İntegral için;

$$\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln t\,dt}{t^2(1+t^2)}=\int_{3/2\sqrt{\mu}}^{1}\frac{\ln t\,dt}{t^2(1+t^2)}+\int_{1}^{\infty}\frac{\ln t\,dt}{t^2(1+t^2)}$$

$$=\int_{3/2\sqrt{\mu}}^{1}\frac{\ln t\,dt}{t^2}-\int_{3/2\sqrt{\mu}}^{1}\frac{\ln t\,dt}{1+t^2}+\int_{1}^{\infty}\frac{\ln t\,dt}{t^2}-\int_{1}^{\infty}\frac{\ln t\,dt}{1+t^2}$$

$$=-\frac{\ln t}{t}-\frac{1}{t}\Big|_{3/2\sqrt{\mu}}^{1}-\frac{\ln t}{t}-\frac{1}{t}\Big|_{1}^{\infty}-\int_{3/2\sqrt{\mu}}^{1}\frac{\ln t\,dt}{1+t^2}-\int_{1}^{\infty}\frac{\ln t\,dt}{1+t^2}$$

$$=-1-\frac{\ln\left(3/2\sqrt{\mu}\right)}{3/2\sqrt{\mu}}+\frac{2\sqrt{\mu}}{3}+1-\int_{3/2\sqrt{\mu}}^{1}\frac{\ln t\,dt}{1+t^2}-\int_{1}^{\infty}\frac{\ln t\,dt}{1+t^2}$$

$$\int\frac{\ln t\,dt}{1+t^2}=-\ln t\,\mathrm{arctg}t+\begin{cases}t-\dfrac{t^3}{3^2}+\dfrac{t^5}{5^2}-\ \dots & t^2\langle 1\\[2ex] \dfrac{\pi}{2}\ln|t|+\dfrac{1}{t}-\dfrac{1}{3^2t^3}+\dfrac{1}{5^2t^5}-\ \dots & t^2\rangle 1\end{cases}$$

olduğundan,

$$\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln t\,dt}{t^2\left(1+t^2\right)}=\frac{2\sqrt{\mu}}{3}\left(\ln(3/2)-\ln\sqrt{\mu}+1\right)-$$

$$-\left(\ln t\,\mathrm{arctg}t\,\Big|_{3/2\sqrt{\mu}}^{1}+\left(-t+\frac{t^3}{3^2}-\frac{t^5}{5^2}+\ \dots\right)\Big|_{3/2\sqrt{\mu}}^{1}\right)$$

$$-\left(\ln t\,\mathrm{arctg}t\,\Big|_{1}^{\infty}-\frac{\pi}{2}\ln t-\frac{1}{t}+\frac{1}{3^2t^3}-\frac{1}{5^2t^5}+\ \dots\ \Big|_{1}^{\infty}\right)$$

$$=\frac{2\sqrt{\mu}}{3}\left(\ln(3/2)-\ln\sqrt{\mu}+1\right)+\ln\left(3/2\sqrt{\mu}\right)\mathrm{arctg}\left(3/2\sqrt{\mu}\right)-\left(-1+\frac{1}{3^2}-\frac{1}{5^2}+\ \dots\right)+$$

$$+\left(\frac{3}{2\sqrt{\mu}}-\left(\frac{3}{2\sqrt{\mu}}\right)^3\frac{1}{3^2}+\ \dots\right)-\left(\ln t\,\mathrm{arctg}t-\frac{\pi}{2}\ln t\,\Big|_{1}^{\infty}\right)-\left(1-\frac{1}{3^2}+\frac{1}{5^2}+\ \dots\right)$$

$$=\frac{2\sqrt{\mu}}{3}\left(\ln(3/2)-\ln\sqrt{\mu}+1\right)+\left(\ln(3/2)-\ln\sqrt{\mu}\right)\mathrm{arctg}\left(3/2\sqrt{\mu}\right)+\frac{3}{2\sqrt{\mu}}$$

$$-\underbrace{\left[\ln t\left(\mathrm{arctg}t-\frac{\pi}{2}\right)\Big|_{1}^{\infty}\right]}_{0}+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$=\frac{2\sqrt{\mu}}{3}\left(\ln(3/2)-\ln\sqrt{\mu}+1\right)+\left(\ln(3/2)-\ln\sqrt{\mu}\right)\mathrm{arctg}\left(3/2\sqrt{\mu}\right)+\frac{3}{2\sqrt{\mu}}+O\left(\frac{1}{\mu^{3/2}}\right)$$

olur.

O halde;

$$\frac{2\ln\sqrt{\mu}}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln t\,dt}{t^2\left(1+t^2\right)}=\frac{4}{3}\left(\frac{\ln(3/2)\ln\sqrt{\mu}-\ln^2\sqrt{\mu}+\ln\sqrt{\mu}}{\mu}\right)+\frac{3\ln\sqrt{\mu}}{\mu^2}+$$

$$+\frac{2\ln\sqrt{\mu}\ln(3/2)\mathrm{arctg}\left(3/2\sqrt{\mu}\right)-2\ln^2\sqrt{\mu}\,\mathrm{arctg}\left(3/2\sqrt{\mu}\right)}{\mu^{3/2}}+O\left(\frac{\ln\sqrt{\mu}}{\mu^3}\right)$$

$$\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}$$ serisinin $\mu$ ' ye göre asimptotik ifadesi için;

$$\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}\leq\int_1^{\infty}\frac{\ln(x+1/2)}{(x+1/2)^2\left(\mu+(x+1/2)^2\right)}dx$$

$$=\frac{1}{\mu}\int_1^{\infty}\frac{\ln(x+1/2)\,d(x+1/2)}{(x+1/2)^2\left(1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right)}=\frac{1}{\mu^{3/2}}\int_1^{\infty}\frac{\ln(x+1/2)\,d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left(1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right)}$$

$$=\frac{1}{\mu^{3/2}}\int_1^{\infty}\frac{\ln\left(\frac{x+1/2}{\sqrt{\mu}}\right)d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left(1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right)}+\frac{\ln\sqrt{\mu}}{\mu^{3/2}}\int_1^{\infty}\frac{d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\left(1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2\right)}$$

$\frac{x+1/2}{\sqrt{\mu}}=t$ denirse $\qquad$ $x=1$ iç in $t=3/2\sqrt{\mu}$

$x=\infty$ iç in $t=\infty$

$$=\frac{1}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln t\,dt}{t^2(1+t^2)}+\frac{\ln\sqrt{\mu}}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t^2(1+t^2)}$$ alınır.

Bu integrallerin değerleri daha önce bulunmuştu, yerlerine yazılırsa;

$$\frac{1}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{\ln t\,dt}{t^2(1+t^2)}=\frac{2}{3}\left(\frac{\ln(3/2)-\ln\sqrt{\mu}+1}{\mu}\right)+\frac{\operatorname{arctg}\left(3/2\sqrt{\mu}\right)\left(\ln(3/2)-\ln\sqrt{\mu}\right)}{\mu^{3/2}}+$$

$$+\frac{3}{2\mu^2}+O\left(\frac{1}{\mu^3}\right)$$

$$\frac{\ln\sqrt{\mu}}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{t^2(1+t^2)}=\frac{\ln\sqrt{\mu}}{\mu^{3/2}}\left[-\frac{\pi}{2}+\frac{2\sqrt{\mu}}{3}+\frac{3}{2\sqrt{\mu}}+O\left(\frac{1}{\mu^{3/2}}\right)\right]$$

$$=-\frac{\pi}{2}\frac{\ln\sqrt{\mu}}{\mu^{3/2}}+\frac{2\ln\sqrt{\mu}}{3\mu}+\frac{3}{2\mu^2}+O\left(\frac{\ln\sqrt{\mu}}{\mu^3}\right)$$ olur.

$$-\frac{1}{\mu}\sum_{n=1}^{\infty}\left[\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\right.$$

$$\left.-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}\right]\frac{(n+1/2)^2}{\mu+(n+1/2)^2}$$ toplamında

$$\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-$$

$$-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}=\frac{\gamma_3}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^3}\right)$$ olduğundan,

$$-\frac{1}{\mu}\sum_{n=1}^{\infty}\left[\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\right.$$

$$\left.-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}\right]\frac{(n+1/2)^2}{\mu+(n+1/2)^2}=$$

$$=-\frac{1}{\mu}\sum_{n=1}^{\infty}\left[\frac{\gamma_3}{\mu+(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^3}\right)\right]$$

alınır. Buradan ise,

$$-\frac{1}{\mu}\sum_{n=1}^{\infty}\frac{\gamma_3}{\mu+(n+1/2)^2}\le-\frac{\gamma_3}{\mu}\int_1^{\infty}\frac{dx}{\mu+(x+1/2)^2}=-\frac{\gamma_3}{\mu^{3/2}}\int_1^{\infty}\frac{d\left(\frac{x+1/2}{\sqrt{\mu}}\right)}{1+\left(\frac{x+1/2}{\sqrt{\mu}}\right)^2}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)$$

$\frac{x+1/2}{\sqrt{\mu}}=t$ denirse $\quad x=1$ için $t=3/2\sqrt{\mu}$; $\quad x=\infty$ için $t=\infty$

$$=-\frac{\gamma_3}{\mu^{3/2}}\int_{3/2\sqrt{\mu}}^{\infty}\frac{dt}{1+t^2}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)=-\frac{\gamma_3}{\mu^{3/2}}\operatorname{arctgt}\Big|_{3/2\sqrt{\mu}}^{\infty}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)$$

$$=-\frac{\pi}{2}\frac{\gamma_3}{\mu^{3/2}}+\frac{\gamma_3}{\mu^{3/2}}\operatorname{arctg}\left(3/2\sqrt{\mu}\right)+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)$$ olur.

O halde;

$$-\frac{1}{\mu}\sum_{n=1}^{\infty}\left[\lambda_n-(n+1/2)^2-\frac{A}{\pi}\ln(n+1/2)-2c_0-2c_1\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_2}{(n+1/2)}-\right.$$

$$\left.-\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_2\frac{\ln\ (n+1/2)}{(n+1/2)^2}\right]\frac{(n+1/2)^2}{\mu+(n+1/2)^2}=$$

$$=-\frac{\pi}{2}\frac{\gamma_3}{\mu^{3/2}}+\frac{\gamma_3}{\mu^{3/2}}\operatorname{arctg}\left(3/2\sqrt{\mu}\right)+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)$$

$$=-\frac{\pi}{2}\frac{\gamma_3}{\mu^{3/2}}+\frac{\gamma_3}{\mu^{3/2}}\left[\frac{3}{2\sqrt{\mu}}-\left(\frac{3}{2\sqrt{\mu}}\right)^3\frac{1}{3}+\ \dots\ \right]+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)$$

$$=-\frac{\pi}{2}\frac{\gamma_3}{\mu^{3/2}}+\frac{3\gamma_3}{2\mu^2}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)$$ olduğu görülür.

Bu bulunan toplamlar yerlerine yazıldığında $\ln\Psi(\mu)$' nin ifadesi;

$$\ln\Psi(\mu)=\frac{1}{\mu}S_\lambda-\frac{\pi\gamma_3}{2\mu^{3/2}}+\frac{3\gamma_3}{2\mu^2}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)+2c_0\left[\frac{\pi}{2\sqrt{\mu}}-\frac{1}{2\mu}+O\left(e^{-2\sqrt{\mu\pi}}\right)\right]+$$

$$+2c_1\left[\frac{\ln^2(3/2)-3\ln(3/2)\ln\sqrt{\mu}+2\ln^2\sqrt{\mu}+\pi^2/24}{\mu}+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)\right]+$$

$$+2c_2\left[\frac{\ln\sqrt{\mu}-\ln(3/2)}{\mu}+\frac{9}{8\mu^2}+O\left(\frac{1}{\mu^2}\right)\right]+$$

$$+\gamma_1\left[\left(-\frac{\pi}{2}\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}+\frac{2\ln^2\sqrt{\mu}}{3\mu}+\frac{3\ln^2\sqrt{\mu}}{2\mu^2}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)\right)\right]+$$

$$+\frac{4}{3}\left(\frac{\ln(3/2)\ln\sqrt{\mu}-\ln^2\sqrt{\mu}+\ln\sqrt{\mu}}{\mu}\right)+\frac{3\ln\sqrt{\mu}}{\mu^2}+$$

$$+\frac{2\ln\sqrt{\mu}\ln(3/2)\operatorname{arctg}\left(3/2\sqrt{\mu}\right)-2\ln^2\sqrt{\mu}\operatorname{arctg}\left(3/2\sqrt{\mu}\right)}{\mu^{3/2}}+O\left(\frac{\ln\sqrt{\mu}}{\mu^3}\right)$$

$$+\frac{2\ln^2\left(2\sqrt{\mu}/3\right)-4\ln\left(2\sqrt{\mu}/3\right)+4}{3\mu}+\frac{4\ln\left(2\sqrt{\mu}/3\right)\operatorname{arctg}\left(2\sqrt{\mu}/3\right)}{\mu^{3/2}}-$$

$$-\frac{6\sqrt{\mu}\ln^2\left(2\sqrt{\mu}/3\right)-12\sqrt{\mu}\ln\left(2\sqrt{\mu}/3\right)+12\sqrt{\mu}}{(9+4\mu)\mu^{3/2}}-\frac{2\pi\ln\left(2\sqrt{\mu}/3\right)+4\operatorname{arctg}\left(2\sqrt{\mu}/3\right)}{\mu^{3/2}}$$

$$-\frac{6}{\mu^2}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^3}\right)\Bigg]+$$

$$+\gamma_2\left[\frac{2}{3}\left(\frac{\ln(3/2)-\ln\sqrt{\mu}+1}{\mu}\right)+\frac{\operatorname{arctg}(3/2\sqrt{\mu})(\ln(3/2)-\ln\sqrt{\mu})}{\mu^{3/2}}+\frac{3}{2\mu^2}+O\left(\frac{1}{\mu^3}\right)\right.$$

$$\left.-\frac{\pi}{2}\frac{\ln\sqrt{\mu}}{\mu^{3/2}}+\frac{2\ln\sqrt{\mu}}{3\mu}+\frac{3}{2\mu^2}+O\left(\frac{\ln\sqrt{\mu}}{\mu^3}\right)\right]$$

$$-\frac{1}{\mu}\left(\lambda_0-\frac{1}{4}+\frac{A}{2\pi}\ln 2-2c_0+4c_1\ln 2-4c_2-2\gamma_1\ln^2 2+4\gamma_2\ln 2\right)-$$

$$+\frac{\lambda_0-1/4}{\mu}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\frac{3A}{2\pi\mu}(1+\ln(3/2))+O\left(\frac{\ln\sqrt{\mu}}{\mu^2}\right)$$

$$\ln\Psi(\mu)=\frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1-\frac{2\gamma_1}{3}\right)+\frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2)+2c_2+\frac{4\gamma_1}{3}(1+\ln(3/2))\right)-$$

$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}c_0\pi+$$

$$+\frac{1}{\mu}\left[S_\lambda-c_0+2c_1\ln^2(3/2)+c_1\frac{\pi^2}{12}-2c_2\ln(3/2)+\frac{4\gamma_1}{3}+\frac{2\gamma_2}{3}(1+\ln(3/2))\right.$$

$$\left.-\frac{A}{2\pi}\ln 2+2c_0-4c_1\ln 2+4c_2+2\gamma_1\ln^2 2-4\gamma_2\ln 2+\frac{3A}{2\pi}(1+\ln(3/2))\right]$$

$$+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$$\Psi(\mu)=\exp\left\{\frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1-\frac{2\gamma_1}{3}\right)+\frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2)+2c_2+\frac{4\gamma_1}{3}(1+\ln(3/2))\right)-\right.$$

$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}c_0\pi+$$

$$+\frac{1}{\mu}\left[S_\lambda-c_0+2c_1\ln^2(3/2)+c_1\frac{\pi^2}{12}-2c_2\ln(3/2)+\frac{4\gamma_1}{3}+\frac{2\gamma_2}{3}(1+\ln(3/2))\right.$$

$$\left.-\frac{A}{\pi}\ln 2+2c_0-4c_1\ln 2+4c_2+2\gamma_1\ln^2 2-4\gamma_2\ln 2+\frac{3A}{2\pi}(1+\ln(3/2))\right]$$

$$\left.+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)\right\}$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ \ldots$ olduğundan, $\Psi(\mu)$ fonksiyonunun $\mu$'nün büyük değerleri için

$$\Psi(\mu) = 1 + \Bigg\{ \frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1 - \frac{2\gamma_1}{3}\right) + \frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2) + 2c_2 + \frac{4\gamma_1}{3}(1+\ln(3/2))\right) -$$
$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}} + \frac{1}{\sqrt{\mu}}c_0\pi +$$
$$+\frac{1}{\mu}\Bigg[S_\lambda - c_0 + 2c_1\ln^2(3/2) + c_1\frac{\pi^2}{12} - 2c_2\ln(3/2) + \frac{4\gamma_1}{3} + \frac{2\gamma_2}{3}(1+\ln(3/2))$$
$$-\frac{A}{\pi}\ln 2 + 2c_0 - 4c_1\ln 2 + 4c_2 + 2\gamma_1\ln^2 2 - 4\gamma_2\ln 2 + \frac{3A}{\pi}(1+\ln(3/2))\ \Bigg]$$
$$+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)\Bigg\}\frac{1}{1!} +$$

$$+\Bigg\{ \frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1 - \frac{2\gamma_1}{3}\right) + \frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2) + 2c_2 + \frac{4\gamma_1}{3}(1+\ln(3/2))\right) -$$
$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}} + \frac{1}{\sqrt{\mu}}c_0\pi +$$
$$+\frac{1}{\mu}\Bigg[S_\lambda - c_0 + 2c_1\ln^2(3/2) + c_1\frac{\pi^2}{12} - 2c_2\ln(3/2) + \frac{4\gamma_1}{3} + \frac{2\gamma_2}{3}(1+\ln(3/2))$$
$$-\frac{A}{\pi}\ln 2 + 2c_0 - 4c_1\ln 2 + 4c_2 + 2\gamma_1\ln^2 2 - 4\gamma_2\ln 2 + \frac{3A}{\pi}(1+\ln(3/2))\ \Bigg]$$
$$+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)\Bigg\}^2\frac{1}{2!} + \ \ldots$$

$$\Psi(\mu) = 1 + \frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1 - \frac{2\gamma_1}{3}\right) + \frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2) + 2c_2 + \frac{4\gamma_1}{3}(1+\ln(3/2))\right) -$$
$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}} + \frac{1}{\sqrt{\mu}}c_0\pi +$$
$$+\frac{1}{\mu}\Bigg[S_\lambda - c_0 + 2c_1\ln^2(3/2) + c_1\frac{\pi^2}{12} - 2c_2\ln(3/2) + \frac{4\gamma_1}{3} + \frac{2\gamma_2}{3}(1+\ln(3/2))$$
$$-\frac{A}{\pi}\ln 2 + 2c_0 - 4c_1\ln 2 + 4c_2 + 2\gamma_1\ln^2 2 - 4\gamma_2\ln 2 + \frac{3A}{\pi}(1+\ln(3/2))\ \Bigg]$$

$$+\frac{A^2}{4}\frac{\ln^2\sqrt{\mu}}{\mu}+\frac{c_0{}^2\pi^2}{\mu}-Ac_0\pi\frac{\ln\sqrt{\mu}}{\mu}+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$$=1+\frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1-\frac{2\gamma_1}{3}+\frac{A^2}{4}\right)+\frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2)+2c_2+\frac{4\gamma_1}{3}(1+\ln(3/2))-Ac_0\pi\right)-$$

$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}c_0\pi+$$

$$+\frac{1}{\mu}\left[S_\lambda+c_0+2c_1\ln^2(3/2)+c_1\frac{\pi^2}{12}-2c_2\ln(3/2)+\frac{4\gamma_1}{3}+\frac{2\gamma_2}{3}(1+\ln(3/2))-\frac{A}{\pi}\ln 2-\right.$$

$$\left.-4c_1\ln 2+4c_2+2\gamma_1\ln^2 2-4\gamma_2\ln 2+\frac{3A}{2\pi}\left(1+\ln(3/2)+c_0{}^2\pi^2\right)\right]+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)$$

$\Phi(-\mu)=c\Psi(\mu)\text{Ch}\sqrt{\mu}\pi$ ve $c=\prod_{n=0}^{\infty}\frac{(n+1/2)^2}{\lambda_n}$ olduğundan

$$\Phi(-\mu)=c\left\{1+\frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1-\frac{2\gamma_1}{3}+\frac{A^2}{4}\right)+\right.$$

$$+\frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2)+2c_2+\frac{4\gamma_1}{3}(1+\ln(3/2))-Ac_0\pi\right)-$$

$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}c_0\pi+$$

$$+\frac{1}{\mu}\left[S_\lambda-c_0+2c_1\ln^2(3/2)+c_1\frac{\pi^2}{12}-2c_2\ln(3/2)+\frac{4\gamma_1}{3}+\frac{2\gamma_2}{3}(1+\ln(3/2))-\right.$$

$$\left.-\frac{A}{\pi}\ln 2+2c_0-4c_1\ln 2+4c_2+2\gamma_1\ln^2 2-4\gamma_2\ln 2+\frac{3A}{2\pi}(1+\ln(3/2))+c_0{}^2\pi^2\right]$$

$$\left.+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)\right\}\text{Ch}\sqrt{\mu}\pi$$

$$=c\left\{1+\frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1-\frac{2\gamma_1}{3}+\frac{A^2}{4}\right)+\right.$$

$$+\frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2)+2c_2+\frac{4\gamma_1}{3}(1+\ln(3/2))-Ac_0\pi\right)-$$

$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}c_0\pi+$$

$$+\frac{1}{\mu}\left[S_\lambda-c_0+2c_1\ln^2(3/2)+c_1\frac{\pi^2}{12}-2c_2\ln(3/2)+\frac{4\gamma_1}{3}+\frac{2\gamma_2}{3}(1+\ln(3/2))-\right.$$

$$-\frac{A}{\pi}\ln 2+2c_0-4c_1\ln 2+4c_2+2\gamma_1\ln^2 2-4\gamma_2\ln 2+\frac{3A}{2\pi}\left(1+\ln(3/2)\right)+c_0{}^2\pi^2\Bigg]$$

$$+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)\Bigg\}\left(\frac{e^{\sqrt{\mu}\pi}}{2}+O\left(e^{-2\sqrt{\mu}\pi}\right)\right)$$

$$=\frac{e^{\sqrt{\mu}\pi}}{2}c\Bigg\{\ 1+\frac{\ln^2\sqrt{\mu}}{\mu}\left(4c_1-\frac{2\gamma_1}{3}+\frac{A^2}{4}\right)+$$

$$+\frac{\ln\sqrt{\mu}}{\mu}\left(-6c_1\ln(3/2)+2c_2+\frac{4\gamma_1}{3}\left(1+\ln(3/2)\right)-Ac_0\pi\right)-$$

$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}c_0\pi+$$

$$+\frac{1}{\mu}\Bigg[S_\lambda+c_0+2c_1\ln^2(3/2)+c_1\frac{\pi^2}{12}-2c_2\ln(3/2)+\frac{4\gamma_1}{3}+\frac{2\gamma_2}{3}\left(1+\ln(3/2)\right)-$$

$$-\frac{A}{\pi}\ln 2-4c_1\ln 2+4c_2+2\gamma_1\ln^2 2-4\gamma_2\ln 2+\frac{3A}{2\pi}\left(1+\ln(3/2)\right)+c_0{}^2\pi^2\Bigg]$$

$$+O\left(\frac{\ln^2\sqrt{\mu}}{\mu^{3/2}}\right)\Bigg\}$$ bulunur.

Diğer taraftan $\underbrace{\varphi'(\pi,-\mu)-H\varphi(\pi,-\mu)}$ fonksiyonunun asimptotik ifadesi

hesaplanacak olursa;.
Bunun için;

$$\begin{cases} -\dfrac{d^2y}{dx^2}+\left(\dfrac{A}{x}+q(x)\right)y=-\mu y \\ y(0)=0 \\ y'(\pi)-Hy(\pi)=0 \end{cases}$$

problemi ele alınır. Homojen kısmın çözümü için;

$$\frac{d^2y}{dx^2}-\mu y=0\Rightarrow \qquad K^2-\mu=0\Rightarrow K=\mp\sqrt{\mu}$$

O halde $y_h=c_1e^{\sqrt{\mu}x}+c_2e^{-\sqrt{\mu}x}$ olur.
Homojen olmayan kısmın çözümü için;

$y_p=c_1(x)e^{\sqrt{\mu}x}+c_2(x)e^{-\sqrt{\mu}x}$ olsun.

$$y_p'=\sqrt{\mu}c_1(x)e^{\sqrt{\mu}x}-\sqrt{\mu}c_2(x)e^{-\sqrt{\mu}x}+\underbrace{c_1'(x)e^{\sqrt{\mu}x}+c_2'(x)e^{-\sqrt{\mu}x}}_{0}$$

I.koşul

$$y_p' = \sqrt{\mu}c_1(x)e^{\sqrt{\mu}x} - \sqrt{\mu}c_2(x)e^{-\sqrt{\mu}x}$$

$$y_p'' = \mu c_1(x)e^{\sqrt{\mu}x} + \mu c_2(x)e^{-\sqrt{\mu}x} + \sqrt{\mu}c_1'(x)e^{\sqrt{\mu}x} - \sqrt{\mu}c_2'(x)e^{-\sqrt{\mu}x}$$

yerlerine yazılırsa;

$$\mu c_1(x)e^{\sqrt{\mu}x} + \mu c_2(x)e^{-\sqrt{\mu}x} + \sqrt{\mu}c_1'(x)e^{\sqrt{\mu}x} - \sqrt{\mu}c_2'(x)e^{-\sqrt{\mu}x} - \mu c_1(x)e^{\sqrt{\mu}x} - \mu c_2(x)e^{-\sqrt{\mu}x} = \\ = \left(\frac{A}{x} + q(x)\right)y$$

$$\sqrt{\mu}c_1'(x)e^{\sqrt{\mu}x} - \sqrt{\mu}c_2'(x)e^{-\sqrt{\mu}x} = \left(\frac{A}{x} + q(x)\right)y \qquad \text{II. koşul}$$

$$c_1'(x)e^{\sqrt{\mu}x} + c_2'(x)e^{-\sqrt{\mu}x} = 0$$

$$\sqrt{\mu}c_1'(x)e^{\sqrt{\mu}x} - \sqrt{\mu}c_2'(x)e^{-\sqrt{\mu}x} = \left(\frac{A}{x} + q(x)\right)y$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & e^{-\sqrt{\mu}x} \\ \left(\frac{A}{x} + q(x)\right)y & -\sqrt{\mu}e^{-\sqrt{\mu}x} \end{vmatrix}}{\begin{vmatrix} e^{\sqrt{\mu}x} & e^{-\sqrt{\mu}x} \\ \sqrt{\mu}e^{\sqrt{\mu}x} & -\sqrt{\mu}\, e^{-\sqrt{\mu}x} \end{vmatrix}} = \frac{e^{-\sqrt{\mu}x}}{2\sqrt{\mu}}\left(\frac{A}{x} + q(x)\right)y$$

$$c_1(x) = \int_0^x \frac{e^{-\sqrt{\mu}t}}{2\sqrt{\mu}}\left(\frac{A}{t} + q(t)\right)y(t)dt \quad \text{olur.}$$

$$c_2'(x) = \frac{\begin{vmatrix} e^{\sqrt{\mu}x} & 0 \\ \sqrt{\mu}e^{\sqrt{\mu}x} & \left(\frac{A}{x} + q(x)\right)y \end{vmatrix}}{\begin{vmatrix} e^{\sqrt{\mu}x} & e^{-\sqrt{\mu}x} \\ \sqrt{\mu}e^{\sqrt{\mu}x} & -\sqrt{\mu}e^{-\sqrt{\mu}x} \end{vmatrix}} = -\frac{e^{\sqrt{\mu}x}}{2\sqrt{\mu}}\left(\frac{A}{x} + q(x)\right)y$$

$$c_2(x) = -\int_0^x \frac{e^{\sqrt{\mu}t}}{2\sqrt{\mu}}\left(\frac{A}{t} + q(t)\right)y(t)dt \quad \text{olur.}$$

Buna göre $y_p = c_1(x)e^{\sqrt{\mu}x} + c_2(x)e^{-\sqrt{\mu}x}$ olduğundan;

$$y_p = e^{\sqrt{\mu}x}\int_0^x \frac{e^{-\sqrt{\mu}t}}{2\sqrt{\mu}}\left(\frac{A}{t} + q(t)\right)y(t)dt - e^{-\sqrt{\mu}x}\int_0^x \frac{e^{\sqrt{\mu}t}}{2\sqrt{\mu}}\left(\frac{A}{t} + q(t)\right)y(t)dt$$

$$y_p = \int_0^x \frac{e^{\sqrt{\mu}(x-t)} - e^{-\sqrt{\mu}(x-t)}}{2\sqrt{\mu}}\left(\frac{A}{t} + q(t)\right)y(t)dt$$

$$y_p=\int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\sqrt{\mu}}\left(\frac{A}{t}+q(t)\right)y(t)dt$$

O halde genel çözüm;

$$y=c_1e^{\sqrt{\mu}x}+c_2e^{-\sqrt{\mu}x}+\int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\sqrt{\mu}}\left(\frac{A}{t}+q(t)\right)y(t)dt \text{ olur.}$$

$\varphi(x,-\mu)$; ile $\varphi(0,-\mu)=0$, $\varphi'(0,-\mu)=$ koşulunu sağlayan çözümü olsun. Dolayısıyla;

$$\varphi(x,-\mu)=c_1e^{\sqrt{\mu}x}+c_2e^{-\sqrt{\mu}x}+\int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\sqrt{\mu}}\left(\frac{A}{t}+q(t)\right)\varphi(t,-\mu)dt \text{ olur.}$$

$$\varphi(0,-\mu)=0 \Rightarrow c_1+c_2=0$$

$$\varphi'(0,-\mu)=1 \Rightarrow \sqrt{\mu}c_1-\sqrt{\mu}c_2=1 \Rightarrow c_1-c_2=\frac{1}{\sqrt{\mu}}$$

$$c_1+c_2=0$$

$$c_1-c_2=\frac{1}{\sqrt{\mu}} \qquad c_1=\frac{1}{2\sqrt{\mu}},\ c_2=-\frac{1}{2\sqrt{\mu}}$$

Dolayısıyla;

$$\varphi(x,-\mu)=\frac{1}{2\sqrt{\mu}}e^{\sqrt{\mu}x}-\frac{1}{2\sqrt{\mu}}e^{-\sqrt{\mu}x}+\int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\sqrt{\mu}}\left(\frac{A}{t}+q(t)\right)y(t)dt$$

$$\varphi(x,-\mu)=\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}+\int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\sqrt{\mu}}\left(\frac{A}{t}+q(t)\right)\varphi(t,-\mu)dt \text{ bulunur.}$$

Ardışık yaklaşmalar yapılırsa;

$$\varphi(x,-\mu)=\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}+\int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\sqrt{\mu}}\left(\frac{A}{t}+q(t)\right)\frac{Sh\sqrt{\mu}t}{\sqrt{\mu}}dt+$$

$$+\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}+\int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\sqrt{\mu}}\left(\frac{A}{t}+q(t)\right)\left[\int_0^t \frac{Sh\sqrt{\mu}(t-\tau)}{\sqrt{\mu}}\left(\frac{A}{\tau}+q(\tau)\right)\frac{Sh\sqrt{\mu}\tau}{\sqrt{\mu}}d\tau\right]dt$$

$$+\ldots$$

$$\varphi(x,-\mu)=\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}+\int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\mu}\left(\frac{A}{t}+q(t)\right)Sh\sqrt{\mu}t\,dt+O\left(\frac{1}{\mu^{3/2}}\right) \text{ bulunur.}$$

Şimdi $\varphi'(\pi,-\mu)-H\varphi(\pi,-\mu)$ ifadesinin asimptotik ifadesine bakılırsa;

$$\varphi(x,-\mu) = \frac{Sh\sqrt{\mu x}}{\sqrt{\mu}} + \int_0^x \frac{Sh\sqrt{\mu}(x-t)}{\mu}\left(\frac{A}{t} + q(t)\right) Sh\sqrt{\mu t}\,dt \; + O\left(\frac{1}{\mu^{3/2}}\right)$$

$$= \frac{Sh\sqrt{\mu x}}{\sqrt{\mu}} + \int_0^x \frac{Sh\sqrt{\mu x}Ch\sqrt{\mu t} - Sh\sqrt{\mu t}Ch\sqrt{\mu x}}{\mu} Sh\sqrt{\mu t}\left(\frac{A}{t} + q(t)\right)dt \; + O\left(\frac{1}{\mu^{3/2}}\right)$$

$$= \frac{Sh\sqrt{\mu x}}{\sqrt{\mu}} + \frac{Sh\sqrt{\mu x}}{2\mu}\int_0^x Sh2\sqrt{\mu t}\left(\frac{A}{t} + q(t)\right)dt \; - \frac{Ch\sqrt{\mu x}}{\mu}\int_0^x Sh^2\sqrt{\mu t}\left(\frac{A}{t} + q(t)\right)dt \; O\left(\frac{1}{\mu^{3/2}}\right)$$

$$= \frac{Sh\sqrt{\mu x}}{\sqrt{\mu}} + \frac{ASh\sqrt{\mu x}}{2\mu}\underbrace{\int_0^x \frac{Sh2\sqrt{\mu t}}{t}dt}_{I} + \frac{Sh\sqrt{\mu x}}{2\mu}\underbrace{\int_0^x Sh2\sqrt{\mu t}\,q(t)dt}_{II} \; -$$

$$- \frac{ACh\sqrt{\mu x}}{\mu}\underbrace{\int_0^x \frac{Sh^2\sqrt{\mu t}}{t}dt}_{III} \; - \frac{Ch\sqrt{\mu x}}{\mu}\underbrace{\int_0^x Sh^2\sqrt{\mu t}\,q(t)dt}_{IV} + O\left(\frac{1}{\mu^{3/2}}\right)$$

I.İntegral için;

$$\left.\begin{array}{l} 2\sqrt{\mu t} = u \Rightarrow 2\sqrt{\mu}dt = du \\ t = 0 \Rightarrow u = 0 \\ t = x \Rightarrow u = 2\sqrt{\mu x} \end{array}\right\} \text{ise} \int_0^x \frac{Sh2\sqrt{\mu t}}{t}dt = \int_0^{2\sqrt{\mu x}} \frac{Shu}{u}du = \int_0^1 \frac{Shu}{u}du + \int_1^{2\sqrt{\mu x}} \frac{Shu}{u}du$$

$\underbrace{\int_0^1 \frac{Shu}{u}du}_{M_1}$ yakınsaktır. Çünkü $0 < u \le 1$ de Shu sürekli, $\frac{1}{u}$ sürekli,

Dolayısıyla $\frac{Shu}{u}$ süreklidir ve $\lim\limits_{u \to 0^+} \frac{Shu}{u} = 1$ dir.

$$\int_1^{2\sqrt{\mu x}} \frac{Shu}{u}du = \left.\frac{Chu}{u}\right|_1^{2\sqrt{\mu x}} + \underbrace{\int_1^{2\sqrt{\mu x}} \frac{Chu}{u^2}du}_{M_2} = \frac{Ch2\sqrt{\mu x}}{2\sqrt{\mu x}} - Ch1 + M_2$$

$$\int_0^x \frac{Sh2\sqrt{\mu t}}{t}dt = \frac{Ch2\sqrt{\mu x}}{2\sqrt{\mu x}} - Ch1 + M_1 + M_2 \qquad \text{bulunur.}$$

II.İntegral için;

$$\int_0^x Sh2\sqrt{\mu t}\,q(t)dt = \frac{1}{2\sqrt{\mu}}Ch2\sqrt{\mu t}q(t)\Big|_0^x - \frac{1}{2\sqrt{\mu}}\int_0^x Ch2\sqrt{\mu t}\,q'(t)dt \qquad \text{bulunur.}$$

III.İntegral için;

$$\left.\begin{aligned}&\sqrt{\mu}t=u\Rightarrow\sqrt{\mu}dt=du\\&t=0\Rightarrow u=0\\&t=x\Rightarrow u=\sqrt{\mu x}\end{aligned}\right\}\Rightarrow\int_0^x\frac{Sh^2\sqrt{\mu}t}{t}dt=\int_0^{\sqrt{\mu x}}\frac{Sh^2u}{u}du=\int_0^1\frac{Sh^2u}{u}du+\int_1^{\sqrt{\mu x}}\frac{Sh^2u}{u}du$$

$\underbrace{\int_0^1\frac{Sh^2u}{u}du}_{M_3}$ yakınsaktır. Çünkü $\frac{Sh^2u}{u}\in C(0,1]$ ve $\lim\limits_{u\to 0^+}\frac{Sh^2u}{u}=0$ dır.

$$\int_1^{\sqrt{\mu x}}\frac{Sh^2u}{u}du=\int_1^{\sqrt{\mu x}}\frac{Ch2u-1}{2u}du=-\frac{1}{2}\int_1^{\sqrt{\mu x}}\frac{du}{u}+\int_1^{\sqrt{\mu x}}\frac{Ch2u}{2u}du$$

$$=-\frac{1}{2}\ln u\,\Big|_1^{\sqrt{\mu x}}+\frac{1}{2}\left[\frac{Sh\theta}{\theta}\Big|_2^{2\sqrt{\mu x}}+\underbrace{\int_2^{2\sqrt{\mu x}}\frac{Sh\theta}{\theta^2}d\theta}_{M_4}\right]$$

$$=-\frac{1}{2}\ln\sqrt{\mu x}+\frac{Sh2\sqrt{\mu x}}{4\sqrt{\mu x}}-\frac{Sh2}{4}+M_4$$

$$\int_0^x\frac{Sh^2\sqrt{\mu}t}{t}dt=\frac{Sh2\sqrt{\mu x}}{4\sqrt{\mu x}}-\frac{\ln\sqrt{\mu x}}{2}-\frac{Sh2}{4}+M_3+M_4 \text{ bulunur.}$$

IV.İntegral için;

$$\int_0^x Sh^2\sqrt{\mu}t\,q(t)dt=\int_0^x\frac{Ch2\sqrt{\mu}t-1}{2}\,q(t)dt=-\frac{1}{2}\int_0^x q(t)dt+\frac{1}{2}\int_0^x Ch2\sqrt{\mu}t\,q(t)dt$$

$$=-\frac{1}{2}\int_0^x q(t)dt+\frac{1}{2}\left[\frac{1}{2\sqrt{\mu}}Sh2\sqrt{\mu}t\,q(t)\,\Big|_0^x-\frac{1}{2\sqrt{\mu}}\int_0^x Sh2\sqrt{\mu}t\,q'(t)dt\right]$$

bulunur. Bu integraller yerlerine yazılırsa;

$$\varphi(x,-\mu)=\frac{Sh\sqrt{\mu x}}{\sqrt{\mu}}+\frac{ASh\sqrt{\mu x}}{2\mu}\left[\frac{Ch2\sqrt{\mu x}}{2\sqrt{\mu x}}-Ch1+M_1+M_2\right]+$$

$$+\frac{Sh\sqrt{\mu x}}{2\mu}\left[\frac{1}{2\sqrt{\mu}}Ch2\sqrt{\mu}tq(t)\,\Big|_0^x-\frac{1}{2\sqrt{\mu}}\int_0^x Ch2\sqrt{\mu}t\,q'(t)dt\right]-$$

$$-\frac{ACh\sqrt{\mu x}}{\mu}\left[\frac{Sh2\sqrt{\mu x}}{4\sqrt{\mu x}}-\frac{\ln\sqrt{\mu x}}{2}-\frac{Sh2}{4}+M_3+M_4\right]-$$

$$-\frac{Ch\sqrt{\mu}x}{\mu}\left[-\frac{1}{2}\int_0^x q(t)\,dt+\frac{1}{2}\left[\frac{1}{2\sqrt{\mu}}Sh2\sqrt{\mu}t\,q(t)\Big|_0^x-\frac{1}{2\sqrt{\mu}}\int_0^x Sh2\sqrt{\mu}t\,q'(t)\,dt\right]\right]$$
$$+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$\varphi(x,-\mu)=\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}+\frac{ASh\sqrt{\mu}x}{2\mu}\underbrace{(-Ch1+M_1+M_2)}_{\alpha_1}-\frac{ACh\sqrt{\mu}x}{\mu}\underbrace{\left(-\frac{Sh2}{4}+M_3+\frac{M_4}{2}\right)}_{\alpha_2}$$
$$+\frac{A}{2}\frac{Ch\sqrt{\mu}x}{\mu}\ln\sqrt{\mu}x+\frac{Ch\sqrt{\mu}x}{2\mu}\int_0^x q(t)dt+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$\varphi(x,-\mu)=\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}+\frac{ASh\sqrt{\mu}x}{2\mu}\alpha_1-\frac{ACh\sqrt{\mu}x}{\mu}\alpha_2+\frac{A}{2}\frac{Ch\sqrt{\mu}x}{\mu}\ln\sqrt{\mu}x+$$
$$+\frac{Ch\sqrt{\mu}x}{2\mu}\int_0^x q(t)dt+O\left(\frac{1}{\mu^{3/2}}\right)\quad \text{bulunur.}$$

Şimdi $\varphi'(x,-\mu)$ 'nün asimptotik ifadesine bakılırsa;

$$\varphi'(x,-\mu)=Ch\sqrt{\mu}x+\int_0^x\frac{Ch\sqrt{\mu}(x-t)}{\sqrt{\mu}}Sh\sqrt{\mu}t\left(\frac{A}{t}+q(t)\right)dt+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$=Ch\sqrt{\mu}x+\int_0^x\frac{Ch\sqrt{\mu}xCh\sqrt{\mu}t-Sh\sqrt{\mu}xSh\sqrt{\mu}t}{\sqrt{\mu}}Sh\sqrt{\mu}t\left(\frac{A}{t}+q(t)\right)dt+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$=Ch\sqrt{\mu}x+\frac{Ch\sqrt{\mu}x}{2\sqrt{\mu}}\int_0^x Sh2\sqrt{\mu}t\left(\frac{A}{t}+q(t)\right)dt-\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}\int_0^x Sh^2\sqrt{\mu}t\left(\frac{A}{t}+q(t)\right)dt+$$
$$+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$=Ch\sqrt{\mu}x+\frac{ACh\sqrt{\mu}x}{2\sqrt{\mu}}\underbrace{\int_0^x\frac{Sh2\sqrt{\mu}t}{t}dt}_{I}+\frac{Ch\sqrt{\mu}x}{2\sqrt{\mu}}\underbrace{\int_0^x Sh2\sqrt{\mu}t\,q(t)dt}_{II}-$$
$$-\frac{ASh\sqrt{\mu}x}{\sqrt{\mu}}\underbrace{\int_0^x\frac{Sh^2\sqrt{\mu}t}{t}dt}_{III}-\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}\underbrace{\int_0^x Sh^2\sqrt{\mu}t\,q(t)dt}_{IV}+O\left(\frac{1}{\mu^{3/2}}\right)$$

Bu integrallerin değerleri daha önce bulunmuştu. Bu değerler yerlerine yazılırsa;

$$\varphi'(x,-\mu)=Ch\sqrt{\mu}x+\frac{ACh\sqrt{\mu}x}{2\sqrt{\mu}}\left[\frac{Ch2\sqrt{\mu}x}{2\sqrt{\mu}x}-Ch1+M_1+M_2\right]+$$

$$+\frac{Ch\sqrt{\mu}x}{2\sqrt{\mu}}\left[\frac{1}{2\sqrt{\mu}}Ch\sqrt{\mu}t\,q(t)\,\Big|_0^x-\frac{1}{2\sqrt{\mu}}\int_0^x Ch\sqrt{\mu}t\,q'(t)\,dt\right]-$$

$$-\frac{ASh\sqrt{\mu}x}{\sqrt{\mu}}\left[\frac{Sh2\sqrt{\mu}x}{4\sqrt{\mu}x}-\frac{\ln\sqrt{\mu}x}{2}-\frac{Sh2}{4}+M_3+M_4\right]-$$

$$-\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}\left[-\frac{1}{2}\int_0^x q(t)dt+\frac{1}{4\sqrt{\mu}}Sh2\sqrt{\mu}t\,q(t)\,\Big|_0^x-\frac{1}{4\sqrt{\mu}}\int_0^x Sh2\sqrt{\mu}t\,q'(t)dt\right]+$$

$$+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$\varphi'(x,-\mu)=Ch\sqrt{\mu}x+\frac{ACh\sqrt{\mu}x}{2\sqrt{\mu}}\frac{Ch2\sqrt{\mu}x}{2\sqrt{\mu}x}+\frac{ACh\sqrt{\mu}x}{2\sqrt{\mu}}\alpha_1+$$

$$+\frac{Ch\sqrt{\mu}x}{2\sqrt{\mu}}\left[\frac{Ch2\sqrt{\mu}x\,q(x)}{2\sqrt{\mu}}-\frac{q(0)}{2\sqrt{\mu}}\right]-$$

$$-\frac{Ch\sqrt{\mu}x}{4\mu}\int_0^x Ch2\sqrt{\mu}t\,q'(t)dt-\frac{ASh\sqrt{\mu}x}{2\sqrt{\mu}}\frac{Sh2\sqrt{\mu}x}{2\sqrt{\mu}x}+$$

$$+\frac{ASh\sqrt{\mu}x}{2\sqrt{\mu}}\ln\sqrt{\mu}x-\frac{ASh\sqrt{\mu}x}{\sqrt{\mu}}\alpha_2+$$

$$+\frac{Sh\sqrt{\mu}x}{2\sqrt{\mu}}\int_0^x q(t)dt-\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}\left[\frac{Sh2\sqrt{\mu}x\,q(x)}{4\sqrt{\mu}}\right]+$$

$$+\frac{Sh\sqrt{\mu}x}{4\sqrt{\mu}}\int_0^x Sh2\sqrt{\mu}t\,q'(t)dt+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$\varphi'(x,-\mu)=Ch\sqrt{\mu}x+\frac{A}{4x}\left[\frac{Ch\sqrt{\mu}xCh2\sqrt{\mu}x-Sh\sqrt{\mu}xSh2\sqrt{\mu}x}{\mu}\right]+\frac{ACh\sqrt{\mu}x}{2\sqrt{\mu}}\alpha_1-\frac{ASh\sqrt{\mu}x}{\sqrt{\mu}}\alpha_2$$

$$+\frac{q(x)}{4}\left[\frac{Ch\sqrt{\mu}xCh2\sqrt{\mu}x-Sh\sqrt{\mu}xSh2\sqrt{\mu}x}{\mu}\right]-\frac{Ch\sqrt{\mu}x}{\mu}\frac{q(0)}{4}+\frac{ASh\sqrt{\mu}x}{2\sqrt{\mu}}\ln\sqrt{\mu}x+$$

$$+\frac{Sh\sqrt{\mu}x}{2\sqrt{\mu}}\int_0^x q(t)dt+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$\varphi'(x,-\mu) = Ch\sqrt{\mu}x + \frac{Ch\sqrt{\mu}x}{\mu}\frac{A}{4x} + \frac{A}{2}\frac{Ch\sqrt{\mu}x}{\sqrt{\mu}}\alpha_1 - \frac{ASh\sqrt{\mu}x}{\sqrt{\mu}}\alpha_2 +$$
$$+\frac{Ch\sqrt{\mu}x}{\mu}\frac{q(x)}{4} - \frac{Ch\sqrt{\mu}x}{\mu}\frac{q(0)}{4}$$
$$+\frac{ASh\sqrt{\mu}x}{2\sqrt{\mu}}\ln\sqrt{\mu}x + \frac{Sh\sqrt{\mu}x}{2\sqrt{\mu}}\int_0^x q(t)dt + O\left(\frac{1}{\mu^{3/2}}\right)$$

$$\varphi'(x,-\mu) = Ch\sqrt{\mu}x + \frac{Ch\sqrt{\mu}x}{\mu}\left[\frac{A}{4x} + \frac{q(x)-q(0)}{4}\right] + \frac{A}{2}\frac{Ch\sqrt{\mu}x}{\sqrt{\mu}}\alpha_1 - \frac{ASh\sqrt{\mu}x}{\sqrt{\mu}}\alpha_2 +$$
$$+\frac{ASh\sqrt{\mu}x}{2\sqrt{\mu}}\ln\sqrt{\mu}x + \frac{Sh\sqrt{\mu}x}{2\sqrt{\mu}}\int_0^x q(t)dt + O\left(\frac{1}{\mu^{3/2}}\right)$$

Dolayısıyla;

$$\varphi'(x,-\mu) = Ch\sqrt{\mu x} + \frac{Ch\sqrt{\mu x}}{\mu}\beta_1 + \frac{A}{2}\frac{Ch\sqrt{\mu x}}{\sqrt{\mu}}\alpha_1 - \frac{ASh\sqrt{\mu x}}{\sqrt{\mu}}\beta_2 + \frac{A}{2}\frac{Sh\sqrt{\mu x}}{\sqrt{\mu}}\ln\sqrt{\mu x} +$$
$$+ O\left(\frac{1}{\mu^{3/2}}\right)$$

bulunur. Burada

$$\beta_1 = \left[\frac{A}{4x} + \frac{q(x)-q(0)}{4}\right] \quad , \quad \alpha_1 = M_1 + M_2 - Ch1$$
$$\beta_2 = \frac{1}{2}\int_0^x q(t)dt - A\alpha_2 \quad , \quad \alpha_2 = M_3 + M_4 - \frac{Sh2}{4}$$

dir.

O halde $\varphi'(x,-\mu) - H\varphi(x,-\mu)$ ifadesinin asimptotik ifadesi;

$$\varphi'(x,-\mu) - H\varphi(x,-\mu) = Ch\sqrt{\mu}x + \frac{Ch\sqrt{\mu}x}{\mu}\beta_1 + \frac{A}{2}\frac{Ch\sqrt{\mu}x}{\sqrt{\mu}}\alpha_1 - \frac{ASh\sqrt{\mu}x}{\sqrt{\mu}}\beta_2 +$$
$$+\frac{A}{2}\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}\ln\sqrt{\mu}x - \frac{HSh\sqrt{\mu}x}{\sqrt{\mu}} - \frac{AH}{2}\frac{Sh\sqrt{\mu}x}{\mu}\alpha_1 -$$
$$-\frac{AH}{2}\frac{Ch\sqrt{\mu}x}{\mu}\ln\sqrt{\mu}x + AH\frac{Ch\sqrt{\mu}x}{\mu}\alpha_2 - H\frac{Ch\sqrt{\mu}x}{\mu}\frac{1}{2}\int_0^x q(t)dt +$$
$$+O\left(\frac{1}{\mu^{3/2}}\right)$$

$$= Ch\sqrt{\mu}x + \frac{Ch\sqrt{\mu}x}{\mu}\left(\beta_1 + AH\alpha_2 - \frac{H}{2}\int_0^x q(t)\,dt\right) + \frac{A}{2}\frac{Ch\sqrt{\mu}x}{\sqrt{\mu}}\alpha_1 - \frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}(\beta_2 - H) -$$

$$-\frac{Sh\sqrt{\mu}x}{\mu}\frac{AH\alpha_1}{2} + \frac{A}{2}\frac{Sh\sqrt{\mu}x}{\sqrt{\mu}}\ln\sqrt{\mu}x - \frac{AH}{2}\frac{Ch\sqrt{\mu}x}{\mu}\ln\sqrt{\mu}x + O\left(\frac{1}{\mu^{3/2}}\right)$$

Bu ifade $\mu$'nün derecelerine göre düzenlenirse;

$$= Ch\sqrt{\mu}x + \frac{1}{\sqrt{\mu}}\left[\frac{A}{2}Ch\sqrt{\mu}x\alpha_1 + Sh\sqrt{\mu}x(\beta_2 - H) + \frac{A}{2}Sh\sqrt{\mu}x\ln\sqrt{\mu}x\right] +$$

$$+\frac{1}{\mu}\left[\left(\beta_1 + AH\alpha_2 - \frac{H}{2}\int_0^x q(t)\,dt\right)Ch\sqrt{\mu}x - \frac{AH\alpha_1}{2}Sh\sqrt{\mu}x - \frac{AH}{2}Ch\sqrt{\mu}x\ln\sqrt{\mu}x\right] + O\left(\frac{1}{\mu^{3/2}}\right)$$

olur. Bu son ifadede $x = \pi$ alınırsa;

$$\varphi'(\pi,-\mu) - H\varphi(\pi,-\mu) =$$

$$= Ch\sqrt{\mu}\pi + \frac{1}{\sqrt{\mu}}\left[\frac{A}{2}Ch\sqrt{\mu}\pi\alpha_1 + Sh\sqrt{\mu}\pi(\beta_2 - H) + \frac{A}{2}Sh\sqrt{\mu}\pi\ln\sqrt{\mu}\pi\right] +$$

$$+\frac{1}{\mu}\left[\left(\beta_1 + AH\alpha_2 - \frac{H}{2}\int_0^\pi q(t)\,dt\right)Ch\sqrt{\mu}\pi - \frac{AH\alpha_1}{2}Sh\sqrt{\mu}\pi - \frac{AH}{2}Ch\sqrt{\mu}\pi\ln\sqrt{\mu}\pi\right] + O\left(\frac{1}{\mu^{3/2}}\right)$$

$$Ch\sqrt{\mu}\pi = \frac{e^{\sqrt{\mu}\pi} + e^{-\sqrt{\mu}\pi}}{2} = \frac{e^{\sqrt{\mu}\pi}}{2}\left(1 + e^{-2\sqrt{\mu}\pi}\right) = \frac{e^{\sqrt{\mu}\pi}}{2} + O\left(e^{-2\sqrt{\mu}\pi}\right)$$

$$Sh\sqrt{\mu}\pi = \frac{e^{\sqrt{\mu}\pi} - e^{-\sqrt{\mu}\pi}}{2} = \frac{e^{\sqrt{\mu}\pi}}{2}\left(1 - e^{-2\sqrt{\mu}\pi}\right) = \frac{e^{\sqrt{\mu}\pi}}{2} + O\left(e^{-2\sqrt{\mu}\pi}\right)$$

$$\varphi'(\pi,-\mu) - H\varphi(\pi,-\mu) = \frac{e^{\sqrt{\mu}\pi}}{2} + \frac{1}{\sqrt{\mu}}\left[\frac{A}{2}\frac{e^{\sqrt{\mu}\pi}}{2}\alpha_1 + \frac{e^{\sqrt{\mu}\pi}}{2}(\beta_2 - H) + \frac{A}{2}\frac{e^{\sqrt{\mu}\pi}}{2}\ln\sqrt{\mu}\pi\right] +$$

$$+\frac{1}{\mu}\left[\left(\beta_1 + AH\alpha_2 - \frac{H}{2}\int_0^\pi q(t)\,dt\right)\frac{e^{\sqrt{\mu}\pi}}{2} - \frac{AH\alpha_1}{2}\frac{e^{\sqrt{\mu}\pi}}{2} - \frac{AH}{2}\frac{e^{\sqrt{\mu}\pi}}{2}\ln\sqrt{\mu}\pi\right] + O\left(\frac{1}{\mu^{3/2}}\right)$$

$$= \frac{e^{\sqrt{\mu}\pi}}{2}\left[1 + \frac{1}{\sqrt{\mu}}\left(\frac{A\alpha_1}{2} + (\beta_2 - H) + \frac{A}{2}\ln\sqrt{\mu}\pi\right) + \right.$$

$$\left. + \frac{1}{\mu}\left(\beta_1 + AH\alpha_2 - \frac{H}{2}\int_0^\pi q(t)\,dt - \frac{AH\alpha_1}{2} - \frac{AH}{2}\ln\sqrt{\mu}\pi\right)\right] + O\left(\frac{1}{\mu^{3/2}}\right)$$

$$=\frac{e^{\sqrt{\mu}\pi}}{2}\left[1+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}\left(\frac{A\alpha_1}{2}+(\beta_2-H)+\frac{A\ln\pi}{2}\right)-\frac{AH}{2}\frac{\ln\sqrt{\mu}}{\mu}\right.$$
$$\left.+\frac{1}{\mu}\left(\beta_1+AH\alpha_2-\frac{H}{2}\int_0^\pi q(t)\,dt-\frac{AH\alpha_1}{2}-\frac{AH\ln\pi}{2}\right)\right]+O\left(\frac{1}{\mu^{3/2}}\right)\quad \text{bulunur.}$$

Sonuç olarak; $\varphi(\pi,-\mu)-H\varphi'(\pi,-\mu)=A\Phi(-\mu)$ eşitliğinden;

$$A=c=\prod_{n=0}^{\infty}\frac{(n+1/2)^2}{\lambda_n}$$

$$-\frac{AH}{2}=-6c_1\ln(3/2)+2c_2+\frac{4\gamma_1}{3}(1+\ln(3/2))-\frac{Ac_0\pi}{2}$$

$$\frac{A\alpha_1}{2}+(\beta_2-H)+\frac{A\ln\pi}{2}=c_0\pi$$

$$4c_1-\frac{2\gamma_1}{3}+\frac{A^2}{16}=0$$

$$\frac{A}{2}=\frac{A}{2}$$

$$S_\lambda+c_0+2c_1\ln^2(3/2)+c_1\frac{\pi^2}{12}-2c_2\ln(3/2)+\frac{4\gamma_1}{3}+\frac{2\gamma_2}{3}(1+\ln(3/2))-\frac{A}{\pi}\ln 2-4c_1\ln 2$$
$$+4c_2+2\gamma_1\ln^2 2-4\gamma_2\ln 2+\frac{3A}{2\pi}(1+\ln(3/2))+c_0^2\pi^2=$$
$$=\beta_1+AH\alpha_2-\frac{H}{2}\int_0^\pi q(t)dt-\frac{AH\alpha_1}{2}-\frac{AH\ln\pi}{2}$$

olur. Buradan,

$$S_\lambda=\beta_1+AH\alpha_2-\frac{H}{2}\int_0^\pi q(t)dt-\frac{AH\alpha_1}{2}-\frac{AH\ln\pi}{2}-c_0-2c_1\ln^2(3/2)-c_1\frac{\pi^2}{12}+$$

$$+2c_2\ln(3/2)-\frac{4\gamma_1}{3}-\frac{2\gamma_2}{3}(1+\ln(3/2))+\frac{A}{2\pi}\ln 2+4c_1\ln 2-4c_2-$$

$$-2\gamma_1\ln^2 2+4\gamma_2\ln 2-\frac{3A}{4\pi}(1+\ln(3/2))-c_0^2\pi^2$$

$$S_\lambda = \frac{q(\pi)-q(0)}{4} + \frac{A}{4\pi} + AH\left(M_3 + M_4 - \frac{Sh2}{4}\right) - \frac{H}{2}\int_0^\pi q(t)dt - \frac{AH(M_1 + M_2 - Ch1)}{2} -$$

$$-\frac{AH\ln\pi}{2} - \frac{A}{2\pi}(M_1 + M_2 - Ch1) - \frac{1}{\pi}\left[\frac{1}{2}\int q(t)dt - A\left(M_3 + M_4 - \frac{Sh2}{4}\right) - H\right] +$$

$$-\left[\frac{A}{2}(M_1 + M_2 - Ch1) + \left(\frac{1}{2}\int_0^\pi q(t)dt - A\left(M_3 + M_4 - \frac{Sh2}{4}\right) - H\right) + \frac{A\ln\pi}{2}\right]^2 -$$

$$-\frac{A\ln\pi}{\pi} - 2c_1\ln^2(3/2) - c_1\ell_1 + 2c_2\ln(3/2) - \frac{4\gamma_1}{3} - \frac{2\gamma_2}{3}(1+\ln(3/2)) + \frac{A\ln 2}{\pi} +$$

$$+4c_1\ln 2 - 4c_2 - 2\gamma_1\ln^2 2 + 4\gamma_2\ln 2 - \frac{3A}{2\pi}(1+\ln(3/2))$$

Bu bulunan $S_\lambda$ değeri L operatörünün izidir.

# 3. BÖLÜM

## 3.1. Coulomb singülerliğine sahip Sturm-Liouville operatörünün Vektör değerli fonksiyon durumunda İz'inin hesaplanması:

$L_r^2(0,\pi)=\{u=(u_1,u_2,\ldots,u_r):\ u_j\in L^2(0,\pi),\ 1\le j\le r\}$ tanımlayalım.

$u=(u_1,u_2,\ldots,u_r)$ ve $v=(v_1,v_2,\ldots,v_r)$

$\langle u,v\rangle=\int_0^\pi uv\,dx=\int_0^\pi (u_1v_1+u_2v_2+\ldots+u_rv_r)dx$ Hilbert uzayında bir iç çarpımdır.

$u\in L^2(0,\pi)$ için;

$$L_0u=-\frac{d^2u}{dx^2}+Q_0u+\frac{A_{ij}}{x}u \quad \text{ve} \quad Lu=L_0u+Q(x)u \qquad (1)$$

Burada $Q_0$ $r\ x\ r$ tipinde köşegen matrisdir. Yani;

$Q_0=\operatorname{diag}(a_1,a_2,\ldots,a_r)$ $a_1\le a_2\le\ldots\le a_r$ $Q(x)=\left[\frac{A_{ij}}{x}+q_{ij}(x)\right]$, $1\le i,j\le r$ reel ve simetriktir. Öyleki $q_{ij}(x)\in C^2[0,\pi]\subset L^2(0,\pi)$'dir. $A_{ij}$, $q_{ij}(x)$ köşegen matris olsun. ( Köşegen olmasa bile köşegenleştirilebilir.)

$$u(0)=0$$
$$u'(\pi)-Hu(\pi)=0 \qquad (2)$$

sınır koşullarıdır.

$L_0$ ve $L$ 'nin öz değerleri bulunacak olursa;

$$L_0=\begin{cases}-\dfrac{d^2u}{dx^2}+\left(Q_0u+\dfrac{A_{ij}}{x}\right)u=\mu u & 0\le x\le\pi,\ q(x)\in C^2[0,\pi]\\ u(0)=0 & 1\le i,j\le r\\ u'(\pi)-Hu(\pi)=0 & \end{cases}$$

için;

$$\left.\begin{array}{l}-\dfrac{d^2u_1}{dx^2}+a_1u_1+\dfrac{A_{11}u_1}{x}=\mu u_1\\ -\dfrac{d^2u_2}{dx^2}+a_2u_2+\dfrac{A_{22}u_2}{x}=\mu u_2\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ -\dfrac{d^2u_r}{dx^2}+a_ru_r+\dfrac{A_{rr}u_r}{x}=\mu u_r\end{array}\right\}$$

$1 \le k \le r$ için;

$$\begin{cases} -\dfrac{d^2u_k}{dx^2} + a_k u_k + \dfrac{A_{kk}u_k}{x} = \mu u_k \\ u_k(0) = 0 \\ u_k'(\pi) - Hu_k(\pi) = 0 \end{cases}$$ problemi alınırsa,.

Homojen kısmın çözümü için;

$$\frac{d^2u_k}{dx^2} + (\mu - a_k)u_k = 0 \text{ ise } \quad K^2 + (\mu - a_k) = 0 \text{ ise } K = \mp\sqrt{\mu - a_k}\, i$$

O halde $u_{k_h} = c_1 \mathrm{Cos}\sqrt{\mu - a_k}\,x + c_2 \mathrm{Sin}\sqrt{\mu - a_k}\,x$ olur.

Homojen olmayan kısmın çözümü için;

$$u_{k_p} = c_1(x)\mathrm{Cos}\sqrt{\mu - a_k}\,x + c_2(x)\mathrm{Sin}\sqrt{\mu - a_k}\,x \text{ olsun}$$

$$c_1'(x)\mathrm{Cos}\sqrt{\mu - a_k}\,x + c_2'(x)\mathrm{Sin}\sqrt{\mu - a_k}\,x = 0$$

$$-\sqrt{\mu - a_k}\,c_1'(x)\mathrm{Sin}\sqrt{\mu - a_k}\,x + \sqrt{\mu - a_k}\,c_2'(x)\mathrm{Cos}\sqrt{\mu - a_k}\,x = \frac{A_{kk}}{x}u_k$$

$$c_1(x) = -\int_0^x \frac{\mathrm{Sin}\sqrt{\mu - a_k}\,t}{\sqrt{\mu - a_k}} \frac{A_{kk}}{t} u_k(t)dt, \quad c_2(x) = \int_0^x \frac{\mathrm{Cos}\sqrt{\mu - a_k}\,t}{\sqrt{\mu - a_k}} \frac{A_{kk}}{t} u_k(t)dt$$

bulunur.

$$u_{k_p} = \int_0^x \frac{\mathrm{Sin}\sqrt{\mu - a_k}(x-t)}{\sqrt{\mu - a_k}} \frac{A_{kk}}{t} u_k(t)dt \text{ olur.}$$

O halde genel çözüm;

$$u_k(x,\mu) = c_1 \mathrm{Cos}\sqrt{\mu - a_k}\,x + c_2 \mathrm{Sin}\sqrt{\mu - a_k}\,x + \int_0^x \frac{\mathrm{Sin}\sqrt{\mu - a_k}(x-t)}{\sqrt{\mu - a_k}} \frac{A_{kk}}{t} u_k(t,\mu)dt$$

şeklindedir.

$\varphi_k(x,\mu)$; ile $\varphi_k(0,\mu) = 0$, $\varphi_k'(0,\mu) = 1$ koşulunu sağlayan çözümü olsun

Dolayısıyla;

$$\varphi_k(x,\mu) = c_1 \mathrm{Cos}\sqrt{\mu - a_k}\,x + c_2 \mathrm{Sin}\sqrt{\mu - a_k}\,x + \int_0^x \frac{\mathrm{Sin}\sqrt{\mu - a_k}(x-t)}{\sqrt{\mu - a_k}} \frac{A_{kk}}{t} \varphi_k(t,\mu)dt$$

olur.

$$\varphi_k(0,\mu) = 0 \text{ ise } c_1 = 0$$

$$\varphi_k'(0,\mu) = 1 \text{ ise } \sqrt{\mu - a_k}\,c_2 = 1 \text{ ise } c_2 = \frac{1}{\sqrt{\mu - a_k}}$$

Dolayısıyla;

$$\varphi_k(x,\mu)=\frac{\operatorname{Sin}\sqrt{\mu-a_k}\,x}{\sqrt{\mu-a_k}}+\int_0^x\frac{\operatorname{Sin}\sqrt{\mu-a_k}(x-t)}{\sqrt{\mu-a_k}}\frac{A_{kk}}{t}\varphi_k(t,\mu)dt$$ bulunur.

Ardışık yaklaşmalar yapılırsa $\varphi_k(x,\mu)$ 'nin asimptotik ifadesi

$$\varphi_k(x,\mu)=\frac{\operatorname{Sin}\sqrt{\mu-a_k}\,x}{\sqrt{\mu-a_k}}+\int_0^x\frac{\operatorname{Sin}\sqrt{\mu-a_k}(x-t)}{\mu-a_k}\frac{A_{kk}}{t}\operatorname{Sin}\sqrt{\mu-a_k}\,tdt+O\left(\frac{1}{(\mu-a_k)^{3/2}}\right)$$

alınır. Şimdi $\varphi_k'(\pi,\mu)-H\varphi_k(\pi,\mu)=0$ denkleminden;

$$\mu_{n_k}=a_k+(n+1/2)^2+\frac{A_{kk}}{\pi}\ln(n+1/2)+2c_0+2c_4\frac{\ln(n+1/2)}{(n+1/2)}+\frac{2c_3}{(n+1/2)}+$$

$$+\gamma_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}+\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^2}+\gamma_3\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^3}\right)$$

özdeğerlerinin asimptotik ifadesi bulunur. Burada,

$$\alpha_1=\operatorname{Cos}1+N_1-N_2\ ,\quad \alpha_2=\frac{\operatorname{Sin}2}{4}+N_3-\frac{N_4}{2}$$

$$c_0=c_1+c_2=\frac{\beta_2}{\pi}+\frac{A_{kk}}{2\pi}\ln\pi,\quad c_4=-\frac{A_{kk}{}^2\alpha_1}{4\pi},\quad c_3=-\frac{A_{kk}}{2}(c_1\alpha_1\pi-H)$$

$$\gamma_1=\frac{A_{kk}{}^2}{4\pi^2},\quad \gamma_2=\frac{A_{kk}{}^2\ln\pi+2A_{kk}\beta_2}{\pi^2},\quad \gamma_3=c_0{}^2=\left(\frac{A_{kk}\ln\pi+2\beta_2}{\pi}\right)^2$$

$$\beta_1=\frac{A_{kk}}{4\pi}+A_{kk}H\alpha_1\ ,\quad \beta_2=\frac{A_{kk}}{2}\alpha_1-H$$

şeklindedir. Şimdi de $Lu=L_0u+Q(x)u$, $Q(x)=\left[\frac{A_{ij}}{x}+q_{ij}(x)\right]$, $1\le i,j\le r$ için;

$$L=\begin{cases}-\dfrac{d^2u}{dx^2}+Q_0u+\left(\dfrac{A_{ij}}{x}+q_{ij}(x)\right)u=\lambda u & 0\le x\le\pi,\ q_{ij}(x)\in C^2[0,\pi]\\ u(0)=0 & 1\le i,j\le r\\ u'(\pi)-Hu(\pi)=0 & \end{cases}$$

problemi için;

$$\left.\begin{array}{l} -\dfrac{d^2u_1}{dx^2}+a_1u_1+\left(\dfrac{A_{11}}{x}+q_{11}(x)\right)u_1=\lambda u_1 \\ -\dfrac{d^2u_2}{dx^2}+a_2u_2+\left(\dfrac{A_{22}}{x}+q_{22}(x)\right)u_2=\lambda u_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -\dfrac{d^2u_r}{dx^2}+a_ru_r+\left(\dfrac{A_{rr}}{x}+q_{rr}(x)\right)u_r=\lambda u_r \end{array}\right\}$$

$1 \le k \le r$ için;

$$\begin{cases} -\dfrac{d^2u_k}{dx^2}+a_ku_k+\left(\dfrac{A_{kk}}{x}+q_{kk}(x)\right)u_k=\lambda u_k \\ u_k(0)=0 \\ u_k'(\pi)-Hu_k(\pi)=0 \end{cases}$$

problemi alınırsa; Homojen kısmın çözümü için;

$$\frac{d^2u_k}{dx^2}+(\lambda-a_k)u_k=0 \Rightarrow \qquad K^2+(\lambda-a_k)=0 \Rightarrow K=\mp\sqrt{\lambda-a_k}\,i$$

O halde $u_{k_h}=c_1\mathrm{Cos}\sqrt{\lambda-a_k}\,x+c_2\mathrm{Sin}\sqrt{\lambda-a_k}\,x$ olur.
Homojen olmayan kısmın çözümü için;

$$u_{k_p}=c_1(x)\mathrm{Cos}\sqrt{\lambda-a_k}\,x+c_2(x)\mathrm{Sin}\sqrt{\lambda-a_k}\,x \quad \text{olsun}$$

$$c_1'(x)\mathrm{Cos}\sqrt{\lambda-a_k}\,x+c_2'(x)\mathrm{Sin}\sqrt{\lambda-a_k}\,x=0$$

$$-\sqrt{\lambda-a_k}\,c_1'(x)\mathrm{Sin}\sqrt{\lambda-a_k}\,x+\sqrt{\lambda-a_k}\,c_2'(x)\mathrm{Cos}\sqrt{\lambda-a_k}\,x=\frac{A_{kk}}{x}u_k$$

$$c_1(x)=-\int_0^x\frac{\mathrm{Sin}\sqrt{\lambda-a_k}\,t}{\sqrt{\lambda-a_k}}\left(\frac{A_{kk}}{t}+q_{kk}(t)\right)u_k(t)dt$$

$$c_2(x)=\int_0^x\frac{\mathrm{Cos}\sqrt{\lambda-a_k}\,t}{\sqrt{\lambda-a_k}}\left(\frac{A_{kk}}{t}+q_{kk}(t)\right)u_k(t)dt$$

bulunur. Buradan;

$$u_{k_p}=\int_0^x\frac{\mathrm{Sin}\sqrt{\lambda-a_k}\,(x-t)}{\sqrt{\lambda-a_k}}\left(\frac{A_{kk}}{t}+q_{kk}(t)\right)u_k(t)dt \text{ olur.}$$

O halde genel çözüm;

$$u_k(x,\lambda)=c_1\mathrm{Cos}\sqrt{\lambda-a_k}\,x+c_2\mathrm{Sin}\sqrt{\lambda-a_k}\,x+$$
$$+\int_0^x\frac{\mathrm{Sin}\sqrt{\lambda-a_k}(x-t)}{\sqrt{\lambda-a_k}}\left(\frac{A_{kk}}{t}+q_{kk}(t)\right)u_k(t,\lambda)dt$$

olur. $\psi_k(x,\mu)$; ile $\psi_k(0,\mu)=0,\quad \psi_k'(0,\mu)=1$ koşulunu sağlayan çözümü olsun. Dolayısıyla;

$$\psi_k(x,\mu)=c_1\mathrm{Cos}\sqrt{\lambda-a_k}\,x+c_2\mathrm{Sin}\sqrt{\lambda-a_k}\,x+$$
$$+\int_0^x\frac{\mathrm{Sin}\sqrt{\lambda-a_k}(x-t)}{\sqrt{\lambda-a_k}}\left(\frac{A_{kk}}{t}+q_{kk}(t)\right)\psi_k(t,\lambda)dt$$

olur.

$$\psi_k(0,\mu)=0 \text{ ise } c_1=0$$
$$\psi_k'(0,\lambda)=1 \text{ ise } \sqrt{\lambda-a_k}\,c_2=1 \text{ ise } c_2=\frac{1}{\sqrt{\lambda-a_k}}$$

Dolayısıyla;

$$\psi_k(x,\mu)=\frac{\mathrm{Sin}\sqrt{\lambda-a_k}\,x}{\sqrt{\lambda-a_k}}+\int_0^x\frac{\mathrm{Sin}\sqrt{\lambda-a_k}(x-t)}{\sqrt{\mu-a_k}}\left(\frac{A_{kk}}{t}+q_{kk}(t)\right)\psi_k(t,\lambda)dt \quad \text{bulunur.}$$

Ardışık yaklaşmalar yapılırsa $\psi_k(x,\lambda)$'nin asimptotik ifadesi

$$\psi_k(x,\lambda)=\frac{\mathrm{Sin}\sqrt{\lambda-a_k}\,x}{\sqrt{\lambda-a_k}}+\int_0^x\frac{\mathrm{Sin}\sqrt{\lambda-a_k}(x-t)}{\lambda-a_k}\left(\frac{A_{kk}}{t}+q_{kk}(t)\right)\mathrm{Sin}\sqrt{\lambda-a_k}\,tdt+$$
$$+O\left(\frac{1}{(\lambda-a_k)^{3/2}}\right)$$

şeklinde alınır. Şimdi $\psi_k'(\pi,\lambda)-H\psi_k(\pi,\lambda)=0$ denkleminden;

$$\lambda_{n_k}=a_k+(n+1/2)^2+\frac{A_{kk}}{\pi}\ln(n+1/2)+2\tilde{c}_0+2\tilde{c}_4\frac{\ln(n+1/2)}{(n+1/2)}+\frac{2\tilde{c}_3}{(n+1/2)}+$$
$$+\tilde{\gamma}_1\frac{\ln^2(n+1/2)}{(n+1/2)^2}+\tilde{\gamma}_2\frac{\ln(n+1/2)}{(n+1/2)^2}+\tilde{\gamma}_3\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^3}\right)$$

özdeğerlerinin asimptotik ifadesi bulunur. Burada,

$$\tilde{\alpha}_1=\mathrm{Cos}1+N_1-N_2 \qquad \tilde{\alpha}_2=\frac{\mathrm{Sin}2}{4}+N_3-\frac{N_4}{2}$$
$$\tilde{c}_0=\tilde{c}_1+\tilde{c}_2=\frac{\tilde{\beta}_2}{\pi}+\frac{A_{kk}}{2\pi}\ln\pi,\ \tilde{c}_4=-\frac{A_{kk}{}^2\tilde{\alpha}_1}{4\pi},\ \tilde{c}_3=-\frac{A_{kk}}{2\pi}(\tilde{c}_1\tilde{\alpha}_1\pi-H)$$

$$\widetilde{\gamma}_1=\frac{A_{kk}^{\,2}}{2\pi^2},\qquad \widetilde{\gamma}_2=\frac{2A_{kk}^{\,2}\ln\pi+2A_{kk}\widetilde{\beta}_2}{\pi^2},\qquad \widetilde{\gamma}_3=\widetilde{c}_0^{\,2}=\left(\frac{A_{kk}\ln\pi+2\widetilde{\beta}_2}{\pi}\right)^2$$

$$\widetilde{\beta}_1=\frac{q_{kk}(\pi)-q_{kk}(0)}{4}+\frac{A_{kk}}{4\pi}+A_{kk}H\widetilde{\alpha}_2+\frac{H}{2}\int_0^{\pi}q_{kk}(t)dt$$

$$\widetilde{\beta}_2=\frac{A_{kk}}{2}\widetilde{\alpha}_2-H+\frac{1}{2}\int_0^{\pi}q_{kk}(t)dt$$ şeklindedir.

O halde vektör değerli fonksiyon durumunda İz (Trace);

$$\lim_{t\to 0}\frac{1}{t}\sum_n\left(e^{-\lambda_n t}-e^{-\mu_n t}\right)=\sum_n\left(\lambda_{n_k}-\mu_{n_k}\right)=\frac{q_{kk}(\pi)-q_{kk}(0)}{4}+\frac{A_{kk}}{4\pi}+A_{kk}HM-\frac{H}{2}\int_0^{\pi}q_{kk}(t)dt-$$
$$-\frac{A_{kk}HN}{2}-\frac{A_{kk}H\ln\pi}{2}-\frac{A_{kk}N}{2\pi}-\frac{1}{\pi}\left[\frac{1}{2}\int_0^{\pi}q_{kk}(t)dt-A_{kk}M-H\right]-\frac{A_{kk}\ln\pi}{2\pi}-S-$$
$$-\left[\frac{A_{kk}N}{2}+\left(\frac{1}{2}\int_0^{\pi}q_{kk}(t)dt-A_{kk}M-H\right)+\frac{A_{kk}\ln\pi}{2}\right]^2$$

$$tr\left(L-L_0\right)=\sum_n\left(\lambda_{n_k}-\mu_{n_k}\right)=\frac{trQ(\pi)-trQ(0)}{4}+\frac{trA}{4\pi}+(trA)HM-\frac{H}{2}\int_0^{\pi}q_{kk}(t)dt-$$
$$-\frac{(trA)HN}{2}-\frac{(trA)H\ln\pi}{2}-\frac{(trA)N}{2\pi}-$$
$$-\frac{1}{\pi}\left[\frac{1}{2}\int_0^{\pi}q_{kk}(t)dt-(trA)M-H\right]-\frac{(trA)\ln\pi}{2\pi}-S-$$
$$-\left[\frac{A_{kk}N}{2}+\left(\frac{1}{2}\int_0^{\pi}q_{kk}(t)dt-A_{kk}M-H\right)+\frac{A_{kk}\ln\pi}{2}\right]^2$$

alınır. Burada M, N, S bilinen sabitlerdir.

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## *ÖZGEÇMİŞ*

Yaşar ÇAKMAK 1970 yılında Sivas'da doğdu. İlk, orta ve liseyi Sivas' da bitirdi. 1988 yılında Cumhuriyet Üniversitesi, Fen-Edebiyat Fakültesi, Matematik bölümünü kazandı ve 1992 yılında mezun oldu. 1993 yılında Cumhuriyet Üniversitesi, Fen-Edebiyat Fakültesi, Matematik bölümününde Araştırma Görevlisi olarak işe başladı. 1995 yılında Cumhuriyet Üniversitesi, Fen Bilimleri Enstitüsü Matematik Anabilim dalında yüksek lisans öğrenimine başladı. Halen burada çalışmaktadır.