ARALIĞIN İÇ NOKTASINDA SÜREKSİZLİĞE SAHİP İKİNCİ MERTEBEDEN SİNGÜLER DİFERANSİYEL OPERATÖRLER İÇİN DÜZ ve TERS PROBLEMLER

Baki KESKİN

DOKTORA TEZİ

MATEMATİK ANABİLİM DALI

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FEN BİLİMLERİ ENSTİTÜSÜ MÜDÜRLÜĞÜ'NE

Bu çalışma, jürimiz tarafından, Matematik Anabilim Dalı'nda Doktora Tezi olarak kabul edilmiştir.

Başkan : Prof.Dr. Abdullah ALTIN..............................

Üye : Prof.Dr. Rauf AMİROV..............................

Üye : Prof.Dr. Fahreddin ABDULLAH..............................

Üye : Yrd.Doç.Dr. Mustafa YILDIRIM..............................

Üye : Yrd.Doç.Dr. Anar ADİLOĞLU..............................

ONAY

Yukarıdaki imzaların, adı geçen öğretim üyelerine ait olduğunu onaylarım.

.... /.... / 2007

FEN BİLİMLERİ ENSTİTÜSÜ MÜDÜRÜ

Prof.Dr. Halil GÜRSOY

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İÇİNDEKİLER

ÖNSÖZ

Bu çalışmayı yöneten ve yardımlarını esirgemeyen danışman hocam Prof. Dr. Rauf AMİROV' a ve tüm emeği geçenlere içten teşekkürlerimi sunarım.

SUMMARY

Ph.D Thesis

DIRECT AND INVERSE PROBLEMS FOR SECOND ORDER SINGULAR DİFFERETIAL OPERATORS WITH DISCONTINUITY CONDITIONS INSIDE AN INTERVAL

Baki KESKİN
Graduate School of Natural and Applied
Science of Department of Mathematics
Supervisor: Prof. Dr. Rauf AMİROV

This study belongs to spectral theory of Sturm-Liouville operators. Integral representation of solution, properties and behaviours of spectral characteristics, properties of Weyl function and Weyl solution and finally uniqueness theorems are investigated for second order singular Sturm-Liouville operators with discontinuity conditions inside a finite interval.

Keywords: Operator, Spectrum, Inverse Problem, Dirac Operator, Weyl Function, Weyl Solution.

ÖZET

Doktora Tezi

ARALIĞIN İÇ NOKTASINDA SÜREKSİZLİĞE SAHİP İKİNCİ MERTEBEDEN SİNGÜLER DİFERANSİYEL OPERATÖRLER İÇİN DÜZ ve TERS PROBLEMLER

Baki KESKİN
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı
Danışman: Prof. Dr. Rauf AMİROV

Bu çalışma Sturm-Liouville operatörlerin spektral teorisine aittir. Sunulan bu çalışmada, sonlu aralığın iç noktasında süreksizlik koşullarına sahip İkinci mertebeden singüler Sturm-Liouville operatörü için çözümün bir integral gösterimi, spektral karakteristiklerin özellikleri, davranışları, Weyl fonksiyonu ve Weyl çözümünün özellikleri ile ters problem için teklik teoremleri incelenmiştir.

Anahtar Kelimeler: Operatör, Spektrum, Ters Problem, Dirac Operatör, Weyl Fonksiyon, Weyl Çözümü.

GIRIŞ

Operatörlerin spektral teorisi; matematik, fizik ve mekaniğin çeşitli alanlarında geniş bir şekilde kullanılmaktadır. Matemetiksel fiziğin bir sıra problemlerinde özellikle Kuantum mekaniğinde singuleriteye sahip diferansiyel operatörlerin öğrenilmesi önem kazanmaktadır. Kuantum mekaniğinin önemli problemlerinden birisi Coulomb potansiyelli alanda parçacıkların hareketini öğrenmektir. Bu tip problemlerin çözümü bir tek Hidrojen atomunun spektrumunu değil bir valentli atomların da öğrenilmesinde, örneğin Sodyum (Na) atomu spektrumunun öğrenilmesinde, önem taşımaktadır. Lineer operatörlerin spektral teorisinin esas kaynakları bir yandan lineer cebir olmak üzere diğer yandan titreşim teorisinin problemleridir (telin titreşimi vb.). Lineer cebir problemleri ve titreşim teorisi problemleri arasındaki benzerliklerin farkına varılması çok eskilere dayanır. Integral denklemler teorisinde yapılan çalışmalarda bu benzerliklerden sürekli faydalanan ilk olarak Hilbert olmuştur. Bunların sonucu olarak önce l_2 uzayı daha sonraları ise genel Hilbert uzayı meydana gelmiştir.

Matematikte l_2 ve H soyut Hilbert uzayı tanımlandıktan sonra H de lineer self-adjoint operatörler teorisi hızla gelişmeye başlamıştır. XIX. ve XX. yüzyıllarda birçok matematikçiler sayesinde bu teori mükemmel bir seviyeye ulaşmıştır. Özel olarak bu çalışmalarda özdeğerler, özfonksiyonlar, spektral fonksiyon, normalleştirici sayılar, vs. spektral veriler tanımlanmış ve farklı yöntemlerle bunlar için asimptotik formüller bulunmuştur.

Regüler ve singüler olmak üzere iki tür diferansiyel operatör tanımlanmış ve bunların spektral teorileri yapılandırılmıştır.

Tanım 0.1. Tanım bölgesi sonlu ve katsayıları toplanabilir fonksiyonlar olan diferansiyel operatöre regüler, tanım bölgesi sonsuz veya katsayıları(bazıları veya tamamı) toplanabilir olmayan diferansiyel operatörlere singülerdir denir.

Ikinci mertebeden regüler operatörler için spektral teori günümüzde Sturm-Liouville teorisi olarak bilinir. XIX. yüzyılın sonlarında ikinci mertebeden diferansiyel operatörler için sonlu aralıkta regüler sınır şartları sağlanacak şekilde adi diferansiyel operatörlerin özdeğerlerinin dağılımı Birkoff tarafından incelenmiştir. Diskret spektruma sahip ve uzayın tamamında tanımlı operatörlerin özdeğerlerinin dağılımı, özellikle Kuantum mekaniğinde çok önem taşımaktadır. Birinci mertebeden iki denklemin regüler sistemleri daha sonraki yıllarda ele alınmıştır. Singüler operatörler için spektral teori ilk olarak Weyl tarafından incelenmiştir. Daha sonra Rietsz, Neumann, Friedrichs ve diğer matematikçiler tarafından simetrik ve self-adjoint operatörlerin genel spektral teorisi oluşturulmuştur. Simetrik operatörlerin tüm self-adjoint genişlemelerinin bulunması problemi Neumann tarafından bir süre sonra yapılmıştır.

İkinci mertebeden singüler operatörlerin spektral teorisine yeni bir yaklaşımı 1946 yılında Titchmarsh vermiştir. Doğru ekseninde tanımlı azalan(artan) potansiyelli

$$L = -\frac{d^2}{dx^2} + q(x)$$

Sturm-Liouville operatörleri için özdeğerlerin dağılımı formülü Titchmarsh tarafından bulunmuştur. Son yıllarda bu operatöre bir boyutlu $q(x)$ potansiyelli Schrödinger denklemi de denir. Aynı zamanda bu çalışmada Schrödinger operatörü için özdeğerlerin dağılım formülü de verilmiştir.

Singüler diferansiyel operatörlerin incelenmesine ilişkin ve diferansiyel operatörlerin spektral teorisinde önemli bir yere sahip olan çalışmalar 1949 yılında Levitan tarafından yapılmıştır. Levitan bu çalışmalarında spektral teoriyi esaslandırmak için kendine has bir yöntem vermiştir. Farklı singüler durumlarda diferansiyel operatörlerin spektral teorisi, özellikle özdeğerlerin, özfonksiyonların asimptotiğine ve özfonksiyonların tamlığına ilşkin konular Courant, Carleman, Birman, Salamyak, Maslov, Keldish vs. matematikçiler tarafından geliştirilmiştir.

Tanım 0.2: L diferansiyel operatörü verildiğinde spektral karakteristiklerinin bulunması problemine düz problem, spektral karakteristikleri verildiğinde bu Sturm-Liouville tipinde hangi L diferansiyel operatörünün spektral karakteristikleri olduğu proble- mine ise ters problem denir.

Ters problemler teorisi, lineer diferansiyel operatörlerin spektral analizinde önemli bir yere sahiptir ve de fonksiyonel analizin bir sıra problemleri ile sıkı bağlantılıdır. Diferansiyel denklemler için ters problemler teorisinin başlangıcı sayılan ilk çalışma Ambartsumyan' a (1929) aittir. 1929 yılında Ambartsumyan Sturm-Liouville operatörleri için ters problemlerle ilgili aşağıdaki teoremi ispatlamıştır:

Teorem 0.3: $q(x)$, $[0, \pi]$ aralığında gerçel değerli sürekli fonksiyon olmak üzere

$\lambda_0, \lambda_1, \cdots, \lambda_n, \cdots$ ler

$$y'' + \{\lambda - q(x)\} y = 0, \quad (0 < x < \pi), \tag{0.1}$$

$$y'(0) = y'(\pi) = 0, \tag{0.2}$$

probleminin özdeğerleri olsun. Eğer $\lambda_n = n^2$ $(n = 0, 1, ...)$ ise $q(x) \equiv 0$ dır.

Ambartsumyan' ın bu çalışmasından sonra ters problemler teorisinde bu tip problemlerin çözümü için farklı yöntemler ve farklı problemler ortaya çıkmıştır. Bu problemlerle ilgili en önemli sonuçlardan birisi Borg' a aittir ve elde ettiği sonuç, aşağıdaki teoremle ifade edilebilir:

Teorem 0.4: $\lambda_0, \lambda_1, \cdots, \lambda_n, \cdots$ ler (0.1) diferansiyel denklemi ve

$$y'(0) - hy(0) = 0, \tag{0.3}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{0.4}$$

sınır koşulları ile verilen problemin, $\mu_0, \mu_1, \cdots, \mu_n, \cdots$ ler ise (0.1) denklemi ve

$$y'(0) - h_1 y(0) = 0, \qquad (h \neq h_1) \tag{0.5}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{0.6}$$

sınır koşullarıyla verilen problemin özdeğerleri olsun. O halde $\{\lambda_n\}_{n \geq 0}$ ve $\{\mu_n\}_{n \geq 0}$ dizileri $q(x)$ fonksiyonunu ve h, h_1 ve H sayılarını tek olarak belirtir. (h, h_1 ve H sonlu gerçel sayılardır.)

Borg' un 1945 yılındaki çalışmasında , $\{\lambda_n\}_{n \geq 0}$ ve $\{\mu_n\}_{n \geq 0}$ dizileri verilen operatörün farklı spektrumları olduğu farz edilir ve operatör bu dizilerin yardımıyla belirlenir. Yani, bu tip operatörün varlığı önceden kabul edilir. Borg aynı çalışmada, bu tip diferansiyel operatörün tek olarak belirtilmesi için bir tek $\{\lambda_n\}_{n \geq 0}$ spektrumunun yeterli olmadığını göstermiştir. O yüzden de, Ambartsumyan'ın sonucu istisna bir durum olarak düşünülmektedir.

Bu çalışmadan sonra potansiyelin $q(\pi - x) = q(x)$ simetriklik koşulunu sağlaması durumunda bir spektruma göre Sturm-Liouville operatörünün belirlenebileceğini Levinson (1949), [3] ve [4]' de ispatlamıştır. Bu problemin tam çözümü Guseinov ve Nabiyev(1995) tarafından verilmiştir. Ayrıca, Levinson negatif özdeğerlerin mevcut olmadığı durumda, saçılma fazının, potansiyeli birebir olarak tanımladığını göstermiştir.

Sturm-Liouville denkleminin inceleme sürecinde kullanılan yöntemlerden biri de ters problemin çözümlerinde önemli bir araç olan dönüşüm operatörü kavramı olmuştur. Bu kavram operatörlerin genelleştirilmiş ötelemesi teorisinde Delsarte, Lions (1938), (1956) ve Levitan, Gasymov (1964) tarafından verilmiştir. Keyfi Sturm-Liouville denklemleri için dönüşüm operatörünün yapısını ilk olarak Povzner (1948) kendi çalışmalarında incelemiştir.

II. mertebeden lineer diferansiyel operatörler için ters problemler teorisinde bir sonraki en önemli aşamalardan birisi Marchenko (1950) tarafından kaydedilmiştir. Marchenko bu çalışmasında ters problemlerin çözümünde Sturm-Liouville operatörünün spektral fonksiyonundan yararlanmıştır.

$\varphi(x,\lambda)$ fonksiyonu (0.1) diferansiyel denkleminin

$$\varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = h, \tag{0.7}$$

başlangıç koşullarını sağlayan çözümü, $\varphi(x,\lambda_n) = \varphi_n(x)$ fonksiyonları ise (0.1) diferansiyel denklemi ve ayrık sınır koşullarının ürettiği operatörün özfonksiyonları olsun. Bu durumda

$$\alpha_n = \int_0^{\pi} \varphi^2(x,\lambda_n)dx \tag{0.8}$$

sayıları verilen operatörün normalleştirici sayıları,

$$\rho(\lambda) = \sum_{\lambda_n<\lambda} \frac{1}{\alpha_n}$$

fonksiyonu ise bu operatörün spektral fonksiyonu olmak üzere Marchenko Borg' un ispatladığı teoremin benzerini $\rho(\lambda)$ spektral fonksiyonu yardımıyla vermiştir. Ayrıca bu çalışmada, $\rho(\lambda)$ fonksiyonun Sturm-Liouville tipinde bir diferansiyel operatörün spektral fonksiyonu olması için gerek ve yeter koşulu verilmiştir. Marchenko' nun çalışmaları ile hemen hemen aynı zamanda Krein (1951) ve (1954) çalışmalarında Sturm-Liouville tipindeki diferansiyel operatörü $\{\lambda_n\}_{n\geq 0}$ ve $\{\mu_n\}_{n\geq 0}$ dizilerine göre belirtmek için etkili yöntem vermiştir. Fakat, bu çalışmalarda verilen gerekli ve yeterli koşul, $\{\lambda_n\}_{n\geq 0}$ ve $\{\mu_n\}_{n\geq 0}$ dizileri yardımıyla değil, bu dizilerin yardımıyla kurulan yardımcı fonksiyon kullanılarak verilmiştir.

1949 yılında Marchenko' nun çalışması yayınlanmadan önce Tikhonov (1949) tarafından Marchenko' nun ispatladığı teklik teoremine denk olan bir teorem ispatlanmıştır. Tikhonov'un çalışmasında ispatlanan teoremin ifadesi aşağıdaki şekildedir:

Teorem 0.5: $\lambda < 0$ olduğunda

$$U'' + \lambda\rho^2(x)U = 0, \qquad x > 0, \qquad U(\infty) = 0$$

probleminin çözümü $U(x,\lambda)$ olsun. Burada $\rho(x)$ parçalı analitik fonksiyon ve $\rho(x) \geq \rho_0 > 0$ dır. $R(\lambda) = \dfrac{U'(0,\lambda)}{U(0,\lambda)}$ olsun. O halde $\lambda < 0$ olduğunda $R(\lambda)$ fonksiyonuna göre $\rho(x)$ fonksiyonu tek olarak belirtilir.

1951 yılında Gelfand ve Levitan çalışmalarında , $\rho(\lambda)$ monoton fonksiyonunun Sturm-Liouville operatörünün spektral fonksiyonu olması için gerekli ve yeterli şartları verdiler. Ayrıca, bu çalışmada Sturm-Liouville operatörünün belirtilmesi için etkili bir yöntem verilmiştir.

Klasik Sturm-Liouville operatörünün $\{\lambda_n\}_{n\geq 0}$ ve $\{\alpha_n\}_{n\geq 0}$ $(\alpha_n > 0)$ dizilerine göre belirlenmesi için, yani verilen dizilerin sırasıyla Sturm-Liouville probleminin spektrumu ve normalleştirici sayıları olması için gerekli ve yeterli koşul aşağıda verilen klasik asimptotik eşitliklerin sağlanmasıdır:

$$\begin{aligned}
\sqrt{\lambda_n} &= n + \frac{a_0}{n} + \cdots + \frac{a_{[\frac{m}{2}]}}{n^{2[\frac{m}{2}]+1}} + \frac{\gamma_n}{n^{2[\frac{m}{2}]+1}}, \\
\alpha_n &= \frac{\pi}{2} + \frac{b_0}{n^2} + \cdots + \frac{b_{[\frac{m}{2}]}}{n^{2[\frac{m+1}{2}]}} + \frac{\tau_n}{n^{2[\frac{m+1}{2}]}}
\end{aligned}$$

burada $a_0 = \dfrac{1}{\pi}\left[h + H + \dfrac{1}{2}\displaystyle\int_0^{\pi} q(t)dt\right]$ dir. Eğer m çift sayı ise $\sum \gamma_n^2 < \infty$ ve $\sum \left(\dfrac{\tau_n}{n}\right)^2 < \infty$, eğer m tek ise $\sum \left(\dfrac{\gamma_n}{n}\right)^2 < \infty$ ve $\sum \tau_n^2 < \infty$ dir.

Fakat, bu çalışmalarda ters problemin iki spektrumuna göre tam çözümü verilmemiştir. Regüler Sturm-Liouville operatörleri için bu problemin yani, iki spektruma göre regüler Sturm-Liouville operatörünün belirlenmesi problemi Levitan ve Gasymov'un (1964) çalışmasında verilmiştir. Bu çalışmada, verilen problemin $\{\alpha_n\}_{n\geq 0}$ normalleştirici sayılarının iki spektruma bağlı olduğunu gösteren

$$\alpha_n = \frac{h_1 - h}{\mu_n - \lambda_n} \prod_{k=0}^{\infty}{}' \frac{\lambda_k - \lambda_n}{\mu_k - \lambda_n} \tag{0.9}$$

şeklinde önemli bir formül elde edilmiştir. Burada $\prod'$ sembolü, sonsuz çarpımda $k = n$. çarpanın bulunmadığını gösterir. (0.9) formülü iki spektruma göre ters problemin çözümünü vermektedir. Gerçekten de, eğer $\{\lambda_n\}_{n\geq 0}$ ve $\{\mu_n\}_{n\geq 0}$ dizileri

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + O\left(\frac{1}{n^4}\right) \tag{0.9'}$$
$$\sqrt{\mu_n} = n + \frac{a_0'}{n} + \frac{a_1'}{n^3} + O\left(\frac{1}{n^4}\right)$$

şeklindeki klasik asimtotik formülleri sağlarsa, (0.9) formülünden yararlanarak $\{\alpha_n\}_{n\geq 0}$ sayılarının asimptotik ifadesi bulunur. Buradan, $q(x)$ sürekli fonksiyon olduğu durumda $\{\lambda_n\}_{n\geq 0}$ ve $\{\alpha_n\}_{n\geq 0}$ dizilerinin (0.1) formundaki denklemin iki spektrumu olması için gerek ve yeter koşullar alınır. Bu koşullar aşağıdaki şekilde sıralanabilir:

1) $\{\lambda_n\}_{n\geq 0}$ ve $\{\mu_n\}_{n\geq 0}$ dizileri sıralıdır, yani $\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots$ şaklindedir

2) λ_n ve μ_n'ler (0.9') asimptotik formüllerine sahiptir.

3) $a_0 \neq a_0'$

Şimdi ise Dirac operatörünün spektral teorisine ait bazı önemli sonuçları hatırlatalım. Dirac operatörünün spektral analizi ile ilgili ilk çalışmalar doğal olarak fizikçiler Prats ve Toll (1959), Moses (1957) ve diğerleri tarafından yapılmıştır. Dirac operatörü için $(0, \infty)$ yarı ekseninde spektral fonksiyona göre ters problem Gasimov ve Levitan (1966) tarafından çözülmüştür. Bu çalışmada $p(x)$ ve $q(x)$ $[0, \infty)$ yarı ekseninin her sonlu aralığında sürekli, gerçel değerli fonksiyonlar ve

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}$$

olmak üzere

$$B\frac{dy}{dx} + Q(x)y = \lambda y, \quad 0 < x < \pi \tag{0.10}$$

$$y_1(0) = 0, \quad y_2(\pi) + Hy_1(\pi) = 0 \tag{0.11}$$

$$(y_1(0) = 0, \quad y_2(\pi) + H_1 y_1(\pi) = 0 \quad H_1 \neq H) \tag{0.11'}$$

sınır problemi ele alınmıştır. Bu takdirde $\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$, (0.10) denkleminin

$$\varphi_1(0,\lambda) = 0, \quad \varphi_2(0,\lambda) = -1 \tag{0.12}$$

başlangıç şartlarını sağlayan çözümü, monoton artan $\rho(\lambda)$ $(-\infty < \lambda < \infty)$ fonksiyonu (0.10), (0.11) probleminin spektral fonksiyonu ve her $f(x) \in L_2(0, \infty)$ fonksiyonu için

$$F_n(\lambda) = \int_0^n f^T(x)\varphi(x,\lambda)dx, \quad \lim_{n \to \infty} \int_{-\infty}^{\infty} \{F(\lambda) - F_n(\lambda)\}^2 d\rho(\lambda) = 0$$

olmak üzere

$$\int_0^{\infty} f^T(x)f(x)dx = \int_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda) \tag{0.13}$$

Parseval eşitliğinin sağlandığı gösterilmiştir.

Ayrıca, bu çalışmada aşağıdaki önemli sonuçlar elde edilmiştir:

Teorem 0.6:

$$\sigma(\lambda) = \rho(\lambda) - \frac{\lambda}{\pi}$$

ve

$$F(x,y) = \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^{\infty} \begin{pmatrix} (1-\cos\lambda x)/\lambda \\ -\sin\lambda x/\lambda \end{pmatrix} \begin{pmatrix} \dfrac{1-\cos\lambda y}{\lambda} & -\dfrac{\sin\lambda y}{\lambda} \end{pmatrix} d\rho(\lambda)$$

olmak üzere $y \leq x$ için $K(x,y)$ matris fonksiyonu

$$F(x,y) + K(x,y) + \int_0^x K(x,s)F(s,y)ds = 0 \tag{0.14}$$

integral denklemini sağlar.

Teorem 0.7: $\rho(\lambda)$ fonksiyonu aşağıdaki şartları sağlasın:

1. $g(x) \in L_2(0, \infty)$ keyfi sonlu vektör fonksiyon ve

$$G(\lambda) = \int_0^{\infty} g^T(x)s(x,\lambda)dx, \qquad s(x,\lambda) = \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix}$$

olmak üzere

$$\int_{-\infty}^{\infty} G^2(\lambda)d\rho(\lambda) = 0$$

ise $g(x) \equiv 0$ dır.

$$\sigma(\lambda) = \rho(\lambda) - \frac{\lambda}{\pi}, \qquad c(x,\lambda) = \begin{pmatrix} (1-\cos\lambda x)/\lambda \\ -\sin\lambda x/\lambda \end{pmatrix}$$

olacak biçimde

$$f(x,y) = \int_{-\infty}^{\infty} c(x,\lambda)c^T(y,\lambda)d\sigma(\lambda)$$

matris fonksiyonu ikinci mertebeden sürekli $f''(x,y) \equiv F(x,y)$ türeve sahiptir.

Bu takdirde her sabitlenmiş $x \geq 0$ için (0.14) integral denklemi her iki değişkene göre sürekli olan tek $K(x,y)$ çözümüne sahiptir.

Teorem 0.8: $Q(x)$ sürekli matris fonksiyonu olmak üzere, monoton artan $\rho(\lambda)$ fonksiyonunun (0.10), (0.11) sınır probleminin spektral fonksiyonu olması için aşağıdaki şartların sağlanması gerek ve yeterdir:

1. Eğer $g(x) \in L_2(0,\infty)$ keyfi sonlu vektör fonksiyon ve

$$G(\lambda) = \int_0^{\infty} g^T(x)s(x,\lambda)dx$$

olmak üzere

$$\int_{-\infty}^{\infty} G^2(\lambda)d\rho(\lambda) = 0$$

ise $g(x) \equiv 0$ dır.

$$f(x,y) = \int_{-\infty}^{\infty} c(x,\lambda)c^T(y,\lambda)d\left\{\rho(\lambda) - \frac{\lambda}{\pi}\right\}$$

matris fonksiyonu $F_{11}(x,0) = F_{21}(x,0) = 0$ olmak üzere ikinci mertebeden sürekli $f''(x,y) \equiv F(x,y)$ türeve sahiptir.

Iki spektruma göre regüler Dirac operatörünün belirlenmesi problemi Gasimov ve Cebiyev (1975) tarafından yapılan çalışmada verilmiştir. Bu çalışmada aşağıdaki önemli teoremler ispatlanmıştır:

Teorem 0.9: $\{\lambda_n\}_{-\infty}^{\infty}$ ve $\{\mu_n\}_{-\infty}^{\infty}$ dizileri sırası ile (0.10), (0.11) ve (0.10) (0.11$'$) problemlerinin özdeğerleri ise

$$\alpha_n = \frac{H_1 - H}{\mu_n - \lambda_n} \prod_{k=-\infty}^{\infty}{}' \frac{\lambda_k - \lambda_n}{\mu_n - \mu_k}, \quad (n = 0, \pm 1, \pm 2, \ldots) \qquad (0.15)$$

dir. Burada, $\prod'$ sembolü, sonsuz çarpımda $k = n$. çarpanın bulunmadığını gösterir.

Teorem 0.10: $p(x)$ ve $q(x)$, $[0,\pi]$ aralığında tanımlı reel fonksiyonlar ve k. mertebeden türevleri $L_2(0,\pi)$ de olacak biçimde $\{\lambda_n\}_{-\infty}^{\infty}$ ve $\{\mu_n\}_{-\infty}^{\infty}$ dizileri sırası ile (0.10), (0.11) ve (0.10), (0.11$'$) problemlerinin spektrumları olması için

1. $\{\lambda_n\}$ ve $\{\mu_n\}$ sayılarının sıralı olması, yani

$$\ldots < \lambda_{-n} < \mu_{-n} < \lambda_{-n+1} < \ldots < \lambda_0 < \mu_0 < \lambda_1 < \ldots < \lambda_n < \mu_n < \lambda_{n+1} < \ldots$$

2. $\alpha \neq \beta$, $0 \leq \beta, \alpha \leq \pi$ **ve** $\sum\limits_{n=-\infty}^{\infty} |\alpha_{n,k}|^2$ **ve** $\sum\limits_{n=-\infty}^{\infty} \left|\beta_{n,k}\right|^2$ **serileri yakınsak olmak üzere**

$$\begin{aligned}
\lambda_n &= n - \frac{\alpha}{\pi} + \frac{\alpha_1}{n} + \ldots + \frac{\alpha_{k-1}}{n^{k-1}} + \frac{\alpha_{n,k}}{n^k} \\
\mu_n &= n - \frac{\beta}{\pi} + \frac{\beta_1}{n} + \ldots + \frac{\beta_{k-1}}{n^{k-1}} + \frac{\beta_{n,k}}{n^k}
\end{aligned}$$

asimptotik formüllerinin sağlanması gerek ve yeterdir.

Dirac operatörü için özvektör(özfonksiyon) fonksiyonlarının tamlığı, Cauchy probleminin çözümü, self-adjointlik durumunda spektrumun diskretliği ve sürekliliği, regülerize izin hesaplanması, periyodik ve antiperiyodik problemler, açılım teoremleri, özvektör fonksiyonlarının asimptotiği, 2n mertebeli Dirac denklemler sistemi için ters saçılma problemi, kısmen çakışmayan iki spektruma göre ters problem sırası ile Sargsjan (1966) ve Martynov (1965) çalışmalarında incelenmiştir.

Diğer taraftan $W_2^{-1}(0,1)$ uzayında singüler reel değerli potansiyellere sahip Sturm-Liouville operatörler sınıfı için ters spektral problem Hryniv ve Mkytyuk (2003) tarafından yapılan çalışmada incelenmiştir.

Bu çalışmada, $q \in W_2^{-1}(0,1)$ reel değerli dağılım(distrubition) fonksiyonu olmak üzere $H := L_2(0,1)$ Hilbert uzayında

$$l := -\frac{d^2}{dx^2} + q \tag{0.16}$$

diferansiyel ifadesine karşılık gelen T Sturm-Liouville operatörü tanımlanmış ve Savchuk ve Shkalikov'un (1999) çalışmasına göre, regülarizasyon yöntemi ile Dirichlet sınır koşullarından bahsedilmiştir.

Distrubition anlamında $\sigma' = q$ olacak şekilde reel değerli $\sigma \in H$ alınmış ve

$$D(T_\sigma) = \left\{ u \in W_1^1(0,1) \,\middle|\, u' - \sigma u \in W_1^1(0,1), l_\sigma(u) \in H, u(0) = u(1) = 0 \right\} \tag{0.17}$$

kümesinde tanımlı

$$Tu = T_\sigma u = l_\sigma(u) := -(u' - \sigma u)' - \sigma u' \tag{0.18}$$

operatörü yazılmıştır.

Burada, distrubition anlamında bütün $u \in D(T_\sigma)$ için $l_\sigma(u) = -u'' + qu$ ifadesi incelendiğinde özellikle T_σ operatörü, regüler potansiyeller için ilkel σ nın özel seçimine bağlı değildir ve (0.16)-ya karşılık gelen standart Dirichlet Sturm-Liouville operatörü ile

çakışır. Ayrıca T_σ, ilkel $\sigma \in H$'ye düzgün resolvent anlamında sürekli olarak bağlıdır ve böylece T_σ, herhangi bir $q = \sigma' \in W_2^{-1}(0,1)$ için (0.16)-ya ait standart Dirichlet Sturm-Liouville operatörüdür. Ele alınan potansiyeller sınıfı Dirac $\delta-$ tipli ve $\frac{1}{x}$-Coulomb tipli potansiyelleri içerir ve matematiksel fizik ve kuantum mekaniğinde geniş olarak kullanılır (Albeverio, Gesztesy ve Ark,1988) ve (Albeverio ve Kurasov, 2000).

Savchuk ve Shkalikov'un (1999) çalışmasından iyi bilinir ki, her reel değerli $\sigma \in H$ için yukarda tanımlanan T_σ operatörü, diskret basit $\left(\lambda_k^2\right), k \in N$ spektrumlu self adjoint operatördür ve λ_k, $\lambda_k = \pi k + \mu_k$ ($\mu_k \in \ell_2$ olan dizi) şeklinde asimptotiğe sahiptir(Savchuk ve Shkalikov, 1999, Savchuk,2001 ve Hryniv .2003). Regüler q potansiyelleri için yukardaki asimptotikler $\mu_k = O(\frac{1}{k})$ olacak şekilde yazılır.

Bu çalışmada, reel ikişerli farklı sayılardan oluşan ve yukarda ifade edilen asimtotiklere sahip hangi $\left(\lambda_k^2\right)$ dizileri, $W_2^{-1}(0,1)$ den olan singüler potansiyelli Sturm-Liouville operatörlerinin spektrumudur? sorusunun cevabı araştırılmıştır. Bu soru, ele alınan potansiyeller için ters spektral probleme götürür. Yani; bu durum, karşılık gelen spektral parametreye dayanan q potansiyelinin kurulmasıdır.

Regüler durumda, yukarıda bahsedilen problemin çözümü için sadece $\left(\lambda_k^2\right)$ spektrumunun yetersiz olduğu bilinmektedir. Aynı dirichlet spektrumlu Sturm-Liouville operatörlerinin ürettiği bir çok farklı q potansiyelleri(isospectral) vardır. Pöschel ve Trubowitz (1987) verilen $\left(\lambda_k^2\right)$ spektrumlu (reel, basit ve $\lambda_k = \pi k + O(\frac{1}{k})$ asimptotiğine ait) H Hilbert uzayındaki bütün potansiyellerin kümesinin, analitik olarak $w_n = n$ ağırlıkları ile $\ell_2(w_n)$ ağırlıklı uzaya difeomorfik olduğunu göstermişlerdir.

q potansiyelini yeniden tek olarak elde etmek için spektrumun yanında bazı ek bilgiler verilmelidir. Bu bilgiler, $(0,1)$ aralığının yarısı üzerindeki potansiyelin bilinmesi veya farklı sınır koşulları olan aynı diferansiyel ifade ile verilen Sturm-Liouville operatörünün spektrumu veya biri bütün aralık için ve diğerleri aralığın eşit iki yarısı için olan üç spektrum olabilir.

Çevirme operatörlerine dayanan regüler Sturm-Liouville operatörünün spektral verisinden, q potansiyelini yeniden elde etmenin algoritması Marchenko (1950) ve Gelfand (1951) tarafından geliştirilen Gelfand-Levitan-Marchenko denklemi olarak adlandırılır. Iki spektrum ile q potansiyelinin kurulumu için bir alternatif metod, Krein (1951) tarafından geliştirildi. Daha sonra H Hilbert uzayından potansiyellere sahip Sturm-

Liouville operatörler sınıfı için Trubowitz ve Pöschel (1987) tarafından farklı bir yaklaşım önerildi. Yazarlar, spektral veriyi ve H' deki potansiyeller arasındaki dönüşümü ayrıntılı olarak çalışmışlar ve ters spektral problemin çözülebilirliğini ispatlamışlardır. Özellikle spektral veriyi tam olarak karakterize etmişlerdir.

Hryniv and Mykytyuk'un, (2003) çalışmasında Gelfand, Levitan ve Marchenko'ya göre, klasik yaklaşım genelleştirilmiş ve $W_2^{-1}(0,1)$ den singüler potansiyellere sahip Sturm-Liouville operatörler sınıfı için ters spektral problem tam olarak çözülmüştür. Şöyle ki, spektral veriler kümesinin açık bir şekli verilmiş ve bu kümenin keyfi bir elemanından q'nun yeniden nasıl elde edildiği açıklanmıştır.

Diğer singülerite tiplerine(örneğin Sturm-Liouville operatörler sınıfı için a süreksizlik noktası, $1/x^{\gamma}$'ya benzer potansiyeller, vs.), Hald (1984), Andersson (1988), Carlson, (1994), Hald ve McLaughlin (1998), Yurko (2000), Yurko ve Freiling (2002), Amirov ve Yurko (2001) bakmışlardır.

Aralığın iç noktasında singüleriteye ve süreksizlik koşullarına sahip diferansiyel operatörler, Amirov ve Yurko (2001) tarafından çalışılmıştır. Bu çalışmada $x = 0$ noktasında singüleriteye sahip self-adjoint olmayan Bessel potansiyelli Sturm-Liouville operatörü için sonlu aralığın iç noktasında çözümün süreksizliğe sahip olduğu durumu incelenmiştir ve verilen operatörün spektral özellikleri ve bu spektral özelliklere göre ters problemin konumu ve çözümü için teklik teoremleri ispatlanmıştır.

Benzer şekilde Amirov (2002) çalışmasında, self-adjoint olmayan Bessel potansiyelli Sturm-Liouville operatörü için sonlu aralıkta sonlu sayıda süreksizlik noktalarına sahip olduğu durum incelenmiştir. Burada verilen diferansiyel operatörü üreten diferansiyel denklemin çözümlerinin davranışları, operatörün spektral özellikleri, spektrumu basit olduğu durumda yani yalnızca özdeğerlerden oluştuğu durumda, özdeğerlere karşılık gelen özfonksiyon ve koşulmuş fonksiyonlara göre operatörün ayrılışımı, spektral parametrelere göre ters problemin konumu ve bu ters problemlerin çözümü için teklik teoremleri ispatlanmıştır.

Amirov'un (2006) çalışmasında, sonlu aralığın iç noktasında süreksizliğe sahip Sturm-Liouville diferansiyel operatörler sınıfı için ve (2005) çalışmasında Dirac operatörü için çevirme operatörü, çekirdek fonksiyonunun bazı özellikleri, spektral karakteristiklerin özellikleri ve ters problem için teklik teoremleri öğrenilmiştir.

Aralığın iç noktasında süreksizliğe sahip Bessel potansiyelli Sturm- Liouville operatörü için düz ve ters problemlerin araştırıldığı bu tezde aşağıdaki yol izlenmiştir.

1. Bölümde tezde kullanılan temel tanım, teoremler ile bir boyutlu Dirac sistemi ve özellikleri verilmiştir.

2. Bölümde, sonlu aralıkta Sturm Liouville diferansiyel denklem birinci mertebeden denklem sistemine indirgenmiş ve bu sistemin çözümünün bir gösterilişi ve özellikleri incelenmiştir.

2.1 alt bölümünde; $q(x)\equiv 0$ ve $q(x)\not\equiv 0$ olduğu durumlarda

$$\begin{cases} y_3' + ky_1 = -ku_1(x)y_1 + \dfrac{1}{k}q(x)x^{-2\ell}y_1 \\ y_1' - ky_3 = ku_2(x)y_3 \end{cases} \tag{2.1.1}$$

$$y_1(0) = 0 \tag{2.1.2}$$

$$y_1(\pi) = 0 \tag{2.1.3}$$

olmak üzere (2.1.1), (2.1.2), (2.1.3) probleminin çözümü için integral gösterilim elde edilmiştir. $d\in(0,\pi)$ noktasında

$$\begin{aligned} y_1(d+0) &= \alpha y_1(d-0) \\ y_3(d+0) &= \alpha^{-1}y_3(d-0) \end{aligned} \tag{2.1.4}$$

süreksizlik koşuluna sahip (2.1.1)-(2.1.4) problemi L ile gösterilsin. Bu problemin $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ i \end{pmatrix}$ başlangıç koşulunu ve (2.1.4) süreksizlik koşulunu sağlayan $y(x,k) = \begin{pmatrix} y_1 \\ y_3 \end{pmatrix}(x,k)$ çözümünün;

$x<d$ ise,

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} e^{ikx} + a(x)e^{ikx} + \displaystyle\int_{-x}^{x} K_{11}(x,t)e^{ikt}dt + i\int_{-x}^{x} K_{12}(x,t)e^{ikt}dt \\ ie^{ikx} + ia(x)e^{ikx} + \displaystyle\int_{-x}^{x} K_{21}(x,t)e^{ikt}dt + i\int_{-x}^{x} K_{22}(x,t)e^{ikt}dt \end{pmatrix} \tag{2.1.7}$$

$x > d$ ise,

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_{10} + a(x)e^{ikx} + b(x)e^{ik(2d-x)} + \displaystyle\int_{-x}^{x} K_{11}(x,t)e^{ikt}dt \\ + i\displaystyle\int_{-x}^{x} K_{12}(x,t)e^{ikt}dt \\ y_{30} + ia(x)e^{ikx} - ib(x)e^{ik(2d-x)} + \displaystyle\int_{-x}^{x} K_{21}(x,t)e^{ikt}dt \\ + i\displaystyle\int_{-x}^{x} K_{22}(x,t)e^{ikt}dt \end{pmatrix} \qquad (2.1.8)$$

şeklinde bir gösterilime sahip olduğu gösterilmiştir. Ayrıca farklı bölgelerde $K_{ij}(x,t)$ $(i,j = 1,2)$ fonksiyonları için integral denklemleri sistemleri elde edilmiştir.

2.2 alt bölümünde; 2.1 alt bölümünde elde edilen integral denklemleri sisteminin uygun bölgede çözümünün varlığı ve tekliği gösterilmiştir

3. Bölümde, verilen L operatörünün spektrumunun özellikleri, Weyl çözümü ve Weyl fonksiyonunun özellikleri ile L probleminin belirlenmesi için Weyl fonksiyonuna ve diskret spektral verilere göre ters problemin çözümü incelenmiştir.

3.1 alt bölümünde, $q(x) = 0$ olduğu duruma karşılık gelen L_0 probleminin

$$\Delta_0(k) = (a_{10}(\pi) + \alpha^+)\sin k\pi + (b_{10}(\pi) + \alpha^-)\sin k(2d-\pi) + a_{20}(\pi)\cos k\pi$$
$$+ b_{20}(\pi)\cos k(2d-\pi) + \int_0^\pi \tilde{K}_{110}(\pi,t)\sin kt\,dt + \int_0^\pi \tilde{K}_{120}(\pi,t)\cos kt\,dt$$

karakteristik fonksiyonunun özellikleri incelenmiştir.

3.2 alt bölümünde, L probleminin spektral karakteristiklerinin n'nin yeterince büyük değerlerinde davranışları öğrenilmiştir.

3.3 alt bölümünde, L probleminin Weyl çözümü ve Weyl fonksiyonunun özellikleri araştırılmıştır.

3.4. alt bölümde, L probleminin belirlenmesi için Weyl fonksiyonuna ve diskret spektral verilere göre ters problemin çözümü için teklik teoremleri verilmiştir.

I.BÖLÜM

1.1 Temel Tanım ve Teoremler

Bu bölümde, diferansiyel operatörlerin spektral teorisinde sık sık kullanılan önemli kavramlar ve teoremler verilmiştir.

Tanım 1.1.1: $L_2[a,b]$ **uzayı** $a \le t \le b$ **olmak üzere ,**

$$L_2[a,b] = \left\{ x(t) : \int_a^b [x(t)]^2 dt < \infty \right\}$$

şeklinde tanımlanır. Bu uzayda iç çarpım ise

$$< f, g > = \int_a^b f(x)\overline{g(x)}dx$$

şeklinde tanımlanır

Tanım 1.1.2: ℓ_2 **uzayı,**

$$\ell_2 = \left(x = (x_1, x_2, \ldots, x_n, \ldots) \mid x_i \in K, \sum_{n=1}^{\infty} |x_n|^2 < \infty \right)$$

şeklinde tanımlanır. Burada K **kompleks veya reel sayılar cismidir.**

Tanım 1.1.3: L, $D(L)$ **tanım kümesinde sınırlı lineer bir operatör olmak üzere**

$$Ly \equiv -y'' + q(x)y = \lambda y$$

eşitliğini sağlayan $y(x) \neq 0$ **fonksiyonu mevcut ise** λ **sayısına** L **operatörünün özdeğeri,** $y(x,\lambda)$ **fonksiyonuna ise** λ **ya karşılık gelen özfonksiyonu denir.**

Tanım 1.1.4: $\{\lambda_n\}$ **dizisinin terimleri** L **operatörünün özdeğerleri ve** $y(x,\lambda_n)$ **ler bu özdeğerlere karşılık gelen öz fonksiyonlar olmak üzere**

$$\alpha_n = \int_a^b y(x,\lambda_n)\overline{y(x,\lambda_n)}dx$$

sayılarına L **operatörünün normalleştirici sayıları denir.**

Tanım 1.1.5: L, $D(L)$ **tanım kümesinde sınırlı lineer bir operatör ve**

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}$$

olmak üzere

$$Ly \equiv By' + Q(x)y = \lambda y$$

eşitliğini sağlayan $y(x) \neq 0$ vektör fonksiyonu mevcut ise λ sayısına L operatörünün özdeğeri, $y(x,\lambda)$ fonksiyonuna ise λ ya karşılık gelen özfonksiyonu denir.

Tanım 1.1.6: $\{\lambda_n\}$ dizisi L operatörünün özdeğerleri ve $y(x,\lambda_n)$ ler bu özdeğerlere karşılık gelen özfonksiyonlar olmak üzere

$$\alpha_n = \int_a^b \left(y_1^2(x,\lambda_n) + y_2^2(x,\lambda_n) \right) dx$$

sayılarına L operatörünün normalleştirici sayıları denir.

Tanım 1.1.7: $L - \lambda I$ operatörünün uzayın tümünde tanımlı, sınırlı $(L - \lambda I)^{-1}$ tersinin mevcut olmadığı λ'lar kümesine L operatörünün spektrumu denir.

Tanım(Adjoint Operatör) 1.1.8: H_1 ve H_2 iki Hilbert uzayları ve $L : H_1 \to H_2$ sınırlı lineer bir operatör olsun. Eğer L^* operatörü $< Lx, y > = < x, L^*y >$ şartını sağlıyorsa L^* operatörüne L nin adjointi denir. Eğer $L = L^*$ ise L operatörüne self adjoint operatör denir.

Tanım(Dönüşüm Operatörü) 1.1.9: E lineer topolojik uzay, A ve B de $A : E \to E$, $B : E \to E$ şeklinde tanımlı iki lineer operatör olsun. E_1 ile E_2 de E lineer uzayının kapalı alt uzayları olmak üzere E uzayının tamamında tanımlı, E_1 den E_2 ye dönüşüm yapan lineer terse sahip X operatörü

i) X ve X^{-1} operatörleri E uzayında süreklidir,

ii) $AX = XB$ operatör denklemi sağlanır

şartlarını sağlıyorsa, X e A ve B operatörler çifti için dönüşüm operatörü denir.

Tanım 1.1.10: $f(z)$ fonksiyonu kompleks düzlemin bir z_0 noktasının δ komşuluğunun tüm noktalarında türevlenebilirse, $f(z)$ fonksiyonuna z_0 noktasında analitiktir denir.

Tanım 1.1.11: $f(z)$ fonksiyonu kompleks düzlemin tüm noktalarında analitik ise $f(z)$ ye tam fonksiyon denir.

Teorem(Rouche Teoremi) 1.1.12: f ve g kompleks düzlemin bir B bölgesinde sonlu sayıda sıfır yeri olan ve sonlu sayıda kutup yerleri dışında analitik olan fonksiyonlar olsunlar. Eğer γ, f ve g nin hiçbir sıfır ve kutup yerinden geçmeyen, B içinde bulunan basit kapalı bir eğri ve de γ üzerinde $|g(z)| < |f(z)|$ olsun. Bu durumda $f(z)$ ve $f(z) + g(z)$ fonksiyonlarının γ içindeki sıfırlarının sayısı katlılığı ile birlikte aynıdır.

Teorem(Cauchy Integral Teoremi) 1.1.13: $f(z)$ basit bağlantılı G bölgesinde birebir analitik fonksiyon, γ ise G de bulunan keyfi düzlendirilebilir kapalı eğri olacak biçimde $f(z)$ nin γ eğrisi üzerinden integrali sıfıra eşittir:

$$\int_{\gamma} f(z)dz = 0$$

Teorem(Cauchy Integral Formülü) 1.1.14: B bir bölge ve γ bu bölge içinde bir kapalı eğri olsun. Eğer a, γ içinde bir nokta ve $f(z)$, B de analitik ise,

$$f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z - a}dz$$

dir.

Tanım 1.1.15: Analitik $f(z)$ fonksiyonunun ayrık tekil noktası z_0 olsun. Eğer,

$$\lim_{z \to z_0} f(z) = \infty$$

ise z_0 noktasına $f(z)$ nin kutup noktası denir.

Teorem(Rezidü Teoremi) 1.1.16: D bölgesinde ($f(z)$ nin sonlu sayıda ayrık tekil $z_1, z_2, \ldots, z_n$ noktaları hariç) ve D nin Γ sınırında analitik $f(z)$ fonksiyonu için

$$\int_{\Gamma} f(z)dz = 2\pi i \sum_{k=1} \operatorname*{Res}_{z=z_k} f(z)$$

eşitliği sağlanır. z_0 noktası $f(z)$ nin k katlı kutup noktası ise

$$\operatorname*{Res}_{z=z_k} f(z) = \frac{1}{(k-1)!}\lim_{z \to z_0}\frac{d^{k-1}}{dz^{k-1}}\left[f(z)(z-z_0)^k\right]$$

z_0 noktası $f(z)$ nin basit kutup noktası olduğunda ise

$$\operatorname*{Res}_{z=z_k} f(z) = \lim_{z \to z_0}[f(z)(z-z_0)]$$

dir. $f(z)$ analitik fonksiyon olmak üzere

$$R = (\overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|})^{-1}$$

formülü ile tanımlı R sayısı

$$f(z) = \sum_{n=0}^{\infty} a_k z_k$$

serisinin yakınsaklık yarıçapı ve $M(r) = \max\limits_{|z|=r} |f(z)|$ olsun.

Tanım 1.1.17: $r > R$ için

$$M(r) < \exp(r^{\mu})$$

olacak şekilde sonlu $\mu > 0$ sayısı varsa, $f(z)$ tam fonksiyonu sonlu mertebelidir ve yukarıda verilen eşitsizliği sağlayan μ sayılar kümesinin

$$\rho = \overline{\lim_{r \to \infty}} \frac{\ln \ln M(r)}{\ln r}$$

formülü ile tanımlı $\rho = \inf \{\mu\}$ sayısına $f(z)$ nin mertebesi denir.

Tanım 1.1.18: $f(z)$ tam fonksiyonunun mertebesi sonlu $\rho (0 < \rho < \infty)$ olmak üzere $r > R$ için

$$M(r) < \exp(a r^{\rho}) \qquad (1.1.1)$$

olacak şekilde $a > 0$ sayısı varsa $f(z)$ sonlu tipe sahiptir denir.

(1.1.1) eşitsizliğini sağlayan $\sigma = \inf \{a\}$ sayısına $f(z)$ fonksiyonunun tipi denir ve

$$\sigma = \overline{\lim_{r \to \infty}} \frac{\ln M(r)}{r^{\rho}}$$

formülüyle hesaplanır.dir.

Tanım 1.1.19: $\sigma = 0$, $0 < \sigma < \infty$, $\sigma = \infty$ olmak üzere $\rho (0 < \rho < \infty)$ mertebeli $f(z)$ tam fonksiyonu sırası ile minimal, normal, maksimal tipe sahiptir denir.

Tanım(Mittag-Leffler Açılımı) 1.1.18: Bir $f(z)$ fonksiyonunun sonlu düzlemdeki aykırılıkları mutlak değer büyüklüğüne göre sıralanmış, basit $a_1, a_2, a_3, \ldots$ kutup yerleri, ve bu noktalardaki rezidüleri sırasıyla $b_1, b_2, b_3, \ldots$ olsun. Eğer C_N hiçbir kutup yerinden geçmeyen, üzerinde $|f(z)| < M$ eşitsizliğinin gerçeklendiği R_N yarıçaplı çember ise ve $N \to \infty$ iken $R_N \to \infty$ oluyorsa

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right)$$

yazılır.

Tanım1.1.18: $W_2^{-1}[a,b]$ uzayı $a \leq t \leq b$ olmak üzere ,

$$W_2^{-1}[a,b] = \left\{ x(t) : \int x(t)dt \in L_2[a,b] \right\}$$

1.2. Bir Boyutlu Dirac Sistemi ve Özellikleri

$p_{ik}(x)$ ler, $(i,k=1,2)$ $[0,\pi]$ aralığında tanımlı ve sürekli reel değerli fonksiyonlar olmak üzere

$$L=\begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix},\quad p_{12}(x)\equiv p_{21}(x) \qquad (1.2.1)$$

bir matris operatörü olsun. $y(x)$ iki bileşenli bir vektör fonksiyonu

$$y(x)=\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\ B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

olmak üzere

$$\left(B\frac{d}{dx}+L(x)-\lambda I\right)y=0 \qquad (1.2.2)$$

denklemi

$$\begin{cases} y_2' + p_{11}(x)y_1 + p_{12}(x)y_2 = \lambda y_1 \\ -y_1' + p_{21}(x)y_1 + p_{22}(x)y_2 = \lambda y_2 \end{cases} \qquad (1.2.2')$$

şeklinde iki tane birinci mertebeden adi diferansiyel denklemler sistemine denktir.

$V(x)$ − potansiyel fonksiyon, m − zerreciğin kütlesi olmak üzere $p_{12}(x)=p_{21}(x)\equiv 0$, $p_{11}(x)=V(x)+m$, $p_{22}(x)=V(x)-m$ ise (1.2.2$'$) sistemi, relativistic kuantum teorisinde bir boyutlu stasyoner dirac sistemi olarak bilinmektedir.

Sabit, ortogonal ve normalleştirilmiş tabana göre; iki boyutlu uzayın herhangi düzgün ortogonal dönüşümü,

$$H(x)=\begin{pmatrix} \cos\varphi(x) & -\sin\varphi(x) \\ \sin\varphi(x) & \cos\varphi(x) \end{pmatrix}$$

şeklinde bir matris ile tanımlanır (Levitan and Sargsjan,1970)

$$BH=HB$$

olduğu kolayca görülür.

(1.2.2) 'de $y=H(x)z$ dönüşümü yapılır ve her tarafı H^{-1} ile soldan çarpılırsa;

$$H^{-1}B\frac{d}{dx}(Hz)+H^{-1}LHz=H^{-1}\lambda Hz$$

veya

$$B\frac{dz}{dx}+\left(H^{-1}B\frac{d}{dx}H+H^{-1}LH\right)z=\lambda z \qquad (1.2.3)$$

elde edilir.

$$Q = H^{-1}B\frac{d}{dx}H + H^{-1}LH$$

matrisi hesaplanacak olursa,

$$H^{-1}(x) = \begin{pmatrix} \cos\varphi(x) & \sin\varphi(x) \\ -\sin\varphi(x) & \cos\varphi(x) \end{pmatrix}, \quad \frac{d}{dx}H = \begin{pmatrix} \varphi'(x)\sin\varphi(x) & -\varphi'(x)\cos\varphi(x) \\ \varphi'(x)\cos\varphi(x) & -\varphi'(x)\sin\varphi(x) \end{pmatrix},$$

$$H^{-1}B\frac{d}{dx}H =$$

$$= \begin{pmatrix} \cos\varphi(x) & \sin\varphi(x) \\ -\sin\varphi(x) & \cos\varphi(x) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \varphi'(x)\sin\varphi(x) & -\varphi'(x)\cos\varphi(x) \\ \varphi'(x)\cos\varphi(x) & -\varphi'(x)\sin\varphi(x) \end{pmatrix} =$$

$$= \begin{pmatrix} \varphi'(x) & 0 \\ 0 & \varphi'(x) \end{pmatrix},$$

$$H^{-1}LH =$$

$$\begin{pmatrix} \cos\varphi(x) & \sin\varphi(x) \\ -\sin\varphi(x) & \cos\varphi(x) \end{pmatrix}\begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}\begin{pmatrix} \cos\varphi(x) & -\sin\varphi(x) \\ \sin\varphi(x) & \cos\varphi(x) \end{pmatrix} =$$

$$= \begin{pmatrix} p_{11}\cos^2\varphi + p_{12}\sin 2\varphi + p_{22}\sin^2\varphi & p_{12}\cos 2\varphi + \frac{1}{2}(p_{22} - p_{11})\sin 2\varphi \\ p_{12}\cos 2\varphi + \frac{1}{2}(p_{22} - p_{11})\sin 2\varphi & p_{11}\sin^2\varphi - p_{12}\sin 2\varphi + p_{22}\cos^2\varphi \end{pmatrix}$$

olduğundan

$$Q(x) = \begin{pmatrix} q_{11}(x) & q_{12}(x) \\ q_{21}(x) & q_{22}(x) \end{pmatrix} =$$

$$\begin{pmatrix} \varphi'(x) + p_{11}\cos^2\varphi + p_{12}\sin 2\varphi + p_{22}\sin^2\varphi & p_{12}\cos 2\varphi + \frac{1}{2}(p_{22} - p_{11})\sin 2\varphi \\ p_{12}\cos 2\varphi + \frac{1}{2}(p_{22} - p_{11})\sin 2\varphi & \varphi'(x) + p_{11}\sin^2\varphi - p_{12}\sin 2\varphi + p_{22}\cos^2\varphi \end{pmatrix}$$

ifadesi elde edilir. $q_{12}(x) \equiv 0$ olarak seçilirse,

$p_{12}(x)\cos 2\varphi(x) + \frac{1}{2}(p_{22}(x) - p_{11}(x))\sin 2\varphi(x) = 0$ olur.

Böylece eğer $p_{11}(x) \neq p_{22}(x)$ ise

$$\varphi(x) = \frac{1}{2}\arctan\frac{2p_{12}(x)}{p_{11}(x) - p_{22}(x)}$$

olarak elde edilir ve $Q(x)$ matrisi,

$$Q(x) = \begin{pmatrix} q_{11}(x) & 0 \\ 0 & q_{22}(x) \end{pmatrix} \equiv \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}$$

şeklinde olur. Buna göre (1.2.3) denklemi,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dz}{dx} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} z = \lambda z \tag{1.2.4}$$

şeklinde yazılabilir. Bu denkleme Dirac denkleminin I. kanonik formu denir.

Şimdi $izQ(x) = q_{11}(x) + q_{22}(x) = 0$ olmak üzere $\varphi(x)$ fonksiyonu seçilsin. Bu durumda $2\varphi'(x) + p_{11}(x) + p_{22}(x) = 0$ olacağından

$$\varphi(x) = -\frac{1}{2}\int_0^x \{p_{11}(s) + p_{22}(s)\}\, ds$$

elde edilir. Buna göre (1.2.3) denklemi

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dz}{dx} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} z = \lambda z \tag{1.2.5}$$

şeklinde yazılabilir. Bu denkleme Dirac denkleminin II. kanonik formu denir. (1.2.4) ve (1.2.5) denklemlerine (1.2.2) sisteminin kanonik formları da denir. (1.2.2) denklem sistemlerinin spektral teorisinin çeşitli problemlerini incelerken bu veya diğer kanonik formlardan faydalanmak kolaylık sağlar. Örneğin, özdeğerlerin ve özfonksiyonların asimptotik davranışı araştırılırken ve de keyfi vektör değerli fonksiyonların (0 ve π noktalarında homojen sınır koşulları altında) (1.2.2) denklem sisteminin özfonksiyonlarına göre açılımı incelenirken (1.2.4) kanonik denklemini kullanmak uygundur. Sonsuz aralıkta verilmiş (1.2.2) denklem sisteminin özdeğerlerinin asimptotik davranışı ve ters problem incelenirken de (1.2.5) kanonik denkleminden faydalanmak kolaylık sağlar.

(1.2.4) kanonik denklem sistemi için $p(x)$ ve $r(x)$, $[0, \pi]$ aralığında reel değerli ve sürekli fonksiyonlar olmak üzere

$$y_2' + \{p(x) - \lambda\} y_1 = 0, \quad y_1' - \{r(x) - \lambda\} y_2 = 0 \tag{1.2.6}$$

$$y_1(0) \sin\alpha + y_2(0)\cos\alpha = 0 \tag{1.2.7}$$

$$y_1(\pi) \sin\beta + y_2(\pi)\cos\beta = 0 \tag{1.2.8}$$

sınır problemini gözönüne alalım. Herhangi bir λ_0 için bu problemin sıfırdan farklı çözümü $y(x, \lambda_0) = \begin{pmatrix} y_1(x, \lambda_0) \\ y_2(x, \lambda_0) \end{pmatrix}$ olsun. Bu durumda λ_0'a özdeğer, buna karşılık gelen $y(x, \lambda_0)$'a özfonksiyon denir.

Lemma 1.2.1: $\lambda_1 \neq \lambda_2$ olmak üzere λ_1 ve λ_2 özdeğerlerine karşılık gelen $y(x, \lambda_1)$ ve $z(x, \lambda_2)$ özfonksiyonları ortogonaldir, yani,

$$\int_0^{\pi} \{y_1(x, \lambda_1) z_1(x, \lambda_2) + y_2(x, \lambda_1) z_2(x, \lambda_2)\}\, dx = 0$$

dır.

İspat: $y(x, \lambda_1)$ ve $z(x, \lambda_2)$ özfonksiyonları (1.2.6) sisteminin çözümleri olduğundan,

$$\begin{aligned} y_2'(x, \lambda_1) + \{p(x) - \lambda_1\}\, y_1(x, \lambda_1) &= 0 \\ y_1'(x, \lambda_1) - \{r(x) - \lambda_1\}\, y_2(x, \lambda_1) &= 0 \\ z_2'(x, \lambda_2) + \{p(x) - \lambda_2\}\, z_1(x, \lambda_2) &= 0 \\ z_1'(x, \lambda_2) - \{r(x) - \lambda_2\}\, z_2(x, \lambda_2) &= 0 \end{aligned}$$

dır. Bu denklemleri sırası ile $z_1(x, \lambda_2)$, $-z_2(x, \lambda_2)$, $-y_1(x, \lambda_1)$ ve $y_2(x, \lambda_1)$ ile çarpar ve sonuçları toplarsak,

$$\begin{aligned} &\frac{d}{dx} \{y_2(x, \lambda_1) z_1(x, \lambda_2) - y_1(x, \lambda_1) z_2(x, \lambda_2)\} = \\ &= (\lambda_1 - \lambda_2) \{y_1(x, \lambda_1) z_1(x, \lambda_2) + y_2(x, \lambda_1) z_2(x, \lambda_2)\} \end{aligned}$$

elde ederiz. Son eşitlik 0'dan π'ye x'e göre integrallenirse,

$$\begin{aligned} &(\lambda_1 - \lambda_2) \int_0^{\pi} \{y_1(x, \lambda_1) z_1(x, \lambda_2) + y_2(x, \lambda_1) z_2(x, \lambda_2)\}\, dx = \\ &= \{y_2(x, \lambda_1) z_1(x, \lambda_2) - y_1(x, \lambda_1) z_2(x, \lambda_2)\}|_0^{\pi} \end{aligned}$$

bulunur. Öte yandan,

$\{y_2(x, \lambda_1) z_1(x, \lambda_2) - y_1(x, \lambda_1) z_2(x, \lambda_2)\}|_0^{\pi} = 0$ olduğundan

$$(\lambda_1 - \lambda_2) \int_0^{\pi} \{y_1(x, \lambda_1) z_1(x, \lambda_2) + y_2(x, \lambda_1) z_2(x, \lambda_2)\}\, dx = 0$$

veya

$$(\lambda_1 - \lambda_2) \int_0^{\pi} y^T(x, \lambda_1) z(x, \lambda_2) dx = 0$$

olur. $\lambda_1 \neq \lambda_2$ olduğundan, $y(x, \lambda_1)$ ve $z(x, \lambda_2)$ özfonksiyonları ortogonaldirler.

Lemma 1.2.2: (1.2.6)-(1.2.8) sınır-değer probleminin özdeğerleri reeldir.

İspat: Kabul edelim ki, $\lambda_1 = u + iv$ kompleks özdeğer olsun. $p(x)$ ve $r(x)$ reel değerli ve $[0, \pi]$ de sürekli fonksiyonlar, α, β sayıları reel olduğundan, $\lambda_2 = \overline{\lambda_1} = u - iv$ sayısıda problemin $\overline{y}(x, \lambda_1)$ özfonksiyonuna karşılık gelen özdeğerdir.

Bu durumda, Lemma 1.2.1 'den,

$$\int_0^{\pi} \{ y_1(x,\lambda_1)\overline{y}_1(x,\lambda_1) + y_2(x,\lambda_1)\overline{y}_2(x,\lambda_1) \} \, dx = 0$$

ve

$$\int_0^{\pi} \left\{ |y_1(x,\lambda_1)|^2 + |y_2(x,\lambda_1)|^2 \right\} dx = 0$$

olur. Buradan $y_1(x,\lambda_1)$ ve $y_2(x,\lambda_1)$ sıfır olur ki, bu da özfonksiyonların sıfır olmaması ile çelişir. O halde özdeğerler reeldir.

II. BÖLÜM

ÇÖZÜMÜN INTEGRAL GÖSTERILIMI VE ÖZELLIKLERI

2.1. Integral Denklemin Oluşturulması

Bu tezde çalışılan diferansiyel ifade,

$$\ell(y) := \left(-y'' + \frac{\nu_0}{x^2} + q(x)\right) y,\ 0 < x < \pi,\ \nu_0 = \left(l + \tfrac{1}{2}\right)^2 - \frac{1}{4},\ |l| < \frac{1}{2},\ q(x) \in L_2(0,\pi)$$

şeklinde olup, burada $Q(x) = \frac{\nu_0}{x^2} + q(x)$ fonksiyonu $x = 0$ noktasında singüleriteye sahip olduğundan Tanım 0.1 gereği bu ifade singüler diferansiyel ifadedir. Dolayısıyla $D(L) = \{y(x) : y(x), y'(x) \in AC(0,d),\ y(x), y'(x) \in AC(d,\pi],\ \ell y \in L_2(0,\pi),\ y(d+0) = \alpha y(d-0),\ y'(d+0) = \alpha^{-1} y'(d-0),\ d \in (\frac{\pi}{2},\pi),\ \alpha > 0,\ \alpha \neq 1\}$ kümesinde bir singüler diferansiyel operatörü üretmektedir. O halde eğer $\left(-y'' + \frac{\nu_0}{x^2} + q(x)\right) y = \lambda y$ veya bir başka gösterimle $\ell(y) = \lambda y$ diferasiyel denklemi alınırsa bu denklem bir singüler diferansiyel denklemdir. Ayrıca $y(0)$ ve $y'(0)$ değerleri mevcut değildir. Dolayısıyla ilk olarak verilen diferansiyel operatörün bu ifadelere benzer değerleri de tanımlı olacak şekilde yeni bir operatör tanımlamak gerekir. Bu operatörse verilen operatörün self-adjoint genişlemesi olarak alınabilir.

Amirov ve Guseinov'un (2002) çalışmasında $\ell(y) := -y'' + \frac{c}{x^\alpha} + \frac{\nu_0}{x^2} y + q(x)y$ şeklinde verilen diferansiyel operatörlerin sınır değer koşulları dilinde self-adjoint genişlemeleri verilmiştir. Burada $c \in R$, $1 < \alpha < 2$, $\nu_0 = \left(l + \frac{1}{2}\right)^2 - \frac{1}{4}$, $|l| < \frac{1}{2}$, $q(x) \in L_2(0,\pi)$ şeklindedir. Bu çalışmada tezde de kullanılacak olan aşağıdaki lemmayı ispatlamışlardır.

Lemma 2.1.1 $y \in D(L^*)$ olmak üzere,

$(\Gamma_1 y)(x) = x^\ell y(x),\ (\Gamma_2 y)(x) = x^{-\ell-1}[xy' + \ell y] - \sigma_{\alpha,l}(x) x^\ell y(x)$

fonksiyonlarının $x \to 0^+$ iken limitleri vadır. Yani,

$$\lim_{x \to 0^+} (\Gamma_i y)(x) = (\Gamma_i y)(0), \quad i = 0, 1.$$

Burada L_0^* verilen L_0 diferansiyel opratörünün eşlenik operatörüdür. L_0 ise $D_0' = C_0^\infty(0,1)$ kümesinde tanımlı $L_0' := L_0' y = \ell(y)$ operatörünün kapanışıdır. Dolayısıyla L_0' operatörü L_0 operatörünün minimal operatörüdür. Belli ki L_0' operatörü $L_2(0,1)$ uzayında simetrik operatördür. Ayrıca burada $\sigma_{\alpha,l}(x)$ fonksiyonu

$$\sigma_{\alpha,l}(x) = \begin{cases} 0 & \alpha < 1 - 2l \\ c\ln x & \alpha = 1 - 2l \\ (2-\alpha)\sum\limits_{k=1}^{n-1} b_k x^{2k-1-\alpha k-2l} + c a_{n-1}\ln x, & \alpha = \dfrac{2n-1-2l}{n},\ n = 2,3,\dots \\ (2-\alpha)\sum\limits_{k=1} b_k x^{2k-1-\alpha k-2l} + c a_{n-1}\ln x, & \alpha \in \left(\dfrac{2n-1-2l}{n}, \dfrac{2n+1-2l}{n+1}\right), \\ & n = 1,2,\dots \end{cases}$$

$a_0 = 1,\ a_k = \frac{1}{k!}\frac{c^k}{k!(2-a)^k}\prod_{p=1}^{k}(2p-1-\alpha p-2l)^{-1},\ b_1 = a_1,\ b_k = ka_k - b_1a_{k-1} - \dots - b_{k-1}a_1,$ $k > 1$. şeklindedir.

Şimdi $-y'' + \ell(\ell+1)x^{-2}y + q(x)y = \lambda y$ diferansiyel denklemi için sınır değer problemi yazılsın. Belli ki bu diferansiyel denklemin $x \to 0^+$ iken $y_1(x) = x^{l+1}[1 + o(1)]$, $y_2(x) = x^l(l+1)[1 + o(1)]$ koşullarını sağlayan asimtotik çözümleri mevcuttur. Fakat $|2\ell| < 1$ durumunda $y(0)$ ve $y'(0)$ değerleri mevcut olmadığından sınır koşulları ancak $(\Gamma_1 y)(x)$ ve $(\Gamma_2 y)(x)$ fonksiyonları dilinde verilebilir. Bunun için alınan diferansiyel denklemin $(\Gamma_1 y)(x)$ ve $(\Gamma_2 y)(x)$ fonksiyonları yardımıyla sisteme indirgenerek incelenmesi gerekir.

Bu tezde yalnızca singüler sınır değer problemi değil, ayrıca $y(x)$ fonksiyonunun bir $d \in (\frac{\pi}{2}, \pi)$ noktasında süreksizliğe sahip olması durumunda elde edilen problem incelenecektir. Volk (1953) çalışmasında süreksizlik noktası olmadığı durumda bu tip problem için çevirme operatörünü incelemiş ve çözümün,

$y(x,\lambda) = j_\nu(x,\lambda) + \int_0^x K(x,t)j_\nu(x,\lambda)dt$ şeklinde bir gösterime sahip olduğunu ispatlamıştır. Burada $j_\nu(x,\lambda) = \sqrt{\lambda x}J_\nu(\lambda x)$ fonksiyonu birinci çeşit Bessel fonksiyonudur. Fakat süreksizlik olması durumunda bu problemler bu tip çevirme operatörüne sahip değillerdir. Bu yüzden tezde çevirme operatörü değil de çözüm için çevirme operatörü tipinde bir integral gösterilimin varlığı ispatlanmıştır.

Tezde,

$$-y'' + \ell(\ell+1)x^{-2}y + q(x)y = \lambda y,\ \lambda = k^2,\ x \in (0,\pi],\ |2\ell| < 1$$

$$\lim_{x\to 0^+} x^{2\ell}y(x) = 0,\ \ y(\pi) = 0$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}(d+0) = A\begin{pmatrix} y \\ y' \end{pmatrix}(d-0),\ A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

problemi incelenecektir. Burada $q(x)$ reel değerli fonksiyon, $\alpha > 0$, $\alpha \neq 1$, $d \in (\frac{\pi}{2}, \pi]$.

Şimdi alınan diferansiyel denklem $(\Gamma_1 y)(x)$ ve $(\Gamma_2 y)(x)$ fonksiyonları yardımıyla sisteme indirgensin;

$-y'' + \ell(\ell+1)x^{-2}y + q(x)y = -x^{\ell}[x^{-\ell}y' + \ell x^{-\ell-1}y]' + q(x)y = \lambda y$ eşitliğinde her iki taraf x^{ℓ} ile bölünürse,

$-[x^{-\ell}y' + \ell x^{-\ell-1}y]' + q(x)x^{-2\ell}x^{\ell}y = \lambda x^{-2\ell}x^{\ell}y$ elde edilir.

$(\Gamma_2 y)(x) = y_2 = x^{-\ell-1}[xy' + \ell y]$ ve $(\Gamma_1 y)(x) = y_1 = x^{\ell}y$ olarak alınırsa, diferansiyel denklem

$$\begin{cases} -y_2' + q(x)x^{-2\ell}y_1 = \lambda x^{-2\ell}y_1 \\ y_1' = x^{2\ell}y_2 \end{cases}$$

sistemine indirgenmiş olur. $y_2 = ky_3$, $x^{-2\ell} - 1 = u_1(x)$ ve $x^{2\ell} - 1 = u_2(x)$ olarak alınırsa,

$$\begin{cases} y_3' + ky_1 = -ku_1(x)y_1 + \frac{1}{k}q(x)x^{-2\ell}y_1 \\ y_1' - ky_3 = ku_2(x)y_3 \end{cases}$$

sistemi elde edilmiş olur.

$$\begin{cases} y_3' + ky_1 = -ku_1(x)y_1 + \frac{1}{k}q(x)x^{-2\ell}y_1 \\ y_1' - ky_3 = ku_2(x)y_3 \end{cases} \tag{2.1.1}$$

$$y_1(0) = 0 \tag{2.1.2}$$

$$y_1(\pi) = 0 \tag{2.1.3}$$

$$\begin{aligned} y_1(d+0) &= \alpha y_1(d-0) \\ y_3(d+0) &= \alpha^{-1}y_3(d-0) \end{aligned} \tag{2.1.4}$$

(2.1.1)-(2.1.4) problemi ele alınsın. Burada $q(x)$ reel değerli fonksiyon, $\alpha > 0$, $\alpha \neq 1$, $d \in \left(\frac{\pi}{2}, \pi\right]$ ..

$\begin{cases} y_3' + ky_1 = 0 \\ y_1' - ky_3 = 0 \end{cases}$ homojen denklem sisteminin iki lineer bağımsız çözümü:

$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix}$ ve $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} -\cos kx \\ \sin kx \end{pmatrix}$ şeklindedir.

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_1(x)\sin kx - c_2(x)\cos kx \\ c_1(x)\cos kx + c_2(x)\sin kx \end{pmatrix} \text{ ve}$$

$$\begin{pmatrix} y_1' \\ y_3' \end{pmatrix} = \begin{pmatrix} c_1'(x)\sin kx - c_2'(x)\cos kx + kc_1(x)\cos kx + kc_2(x)\sin kx \\ c_1'(x)\cos kx + c_2'(x)\sin kx - kc_1(x)\sin kx + kc_2(x)\cos kx \end{pmatrix}$$

alınır ve (2.1.1) denkleminde yerine yazılıp parametrelerin değişimi yöntemi uygulanırsa;

$$c_1(x) = -k\int_0^x u_1(t)y_1(t)\cos kt\,dt + \frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\cos kt\,dt + k\int_0^x u_2(t)y_3(t)\sin kt\,dt + c_0$$

$$c_2(x) = -k\int_0^x u_1(t)y_1(t)\sin kt\,dt + \frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\sin kt\,dt - k\int_0^x u_2(t)y_3(t)\cos kt\,dt + c_1$$

bulunur. $c_1(x)$ ve $c_2(x)$' in Ifadeleri denklemde yerine yazılırsa;

$$\begin{aligned}
y_1(x) ={}& c_0\sin kx - c_1\cos kx - k\int_0^x u_1(t)y_1(t)\cos kt\sin kx\,dt + \frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\cos kt\sin kx\,dt \\
&+ k\int_0^x u_2(t)y_3(t)\sin kt\sin kx\,dt + k\int_0^x u_1(t)y_1(t)\sin kt\cos kx\,dt \\
&- \frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\sin kt\cos kx\,dt + k\int_0^x u_2(t)y_3(t)\cos kt\cos kx\,dt
\end{aligned}$$

$$\begin{aligned}
y_3(x) ={}& c_0\cos kx + c_1\sin kx - k\int_0^x u_1(t)y_1(t)\cos kt\cos kx\,dt \\
&+ \frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\cos kt\cos kx\,dt + k\int_0^x u_2(t)y_3(t)\sin kt\cos kx\,dt - k\int_0^x u_1(t)y_1(t)\sin kt\sin kx\,dt \\
&+ \frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\sin kt\sin kx\,dt - k\int_0^x u_2(t)y_3(t)\cos kt\cos kx\,dt.
\end{aligned}$$

denklemleri elde edilir. Gerekli işlemler yapılırsa;

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_0 \sin kx - c_1 \cos kx - k\int_0^x u_1(t)y_1(t)\sin k(x-t)dt \\ +\frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\sin k(x-t)dt + k\int_0^x u_2(t)y_3(t)\cos k(x-t)dt \\ c_0 \cos kx + c_1 \sin kx - k\int_0^x u_1(t)y_1(t)\cos k(x-t)dt \\ +\frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\cos k(x-t)dt - k\int_0^x u_2(t)y_3(t)\sin k(x-t)dt \end{pmatrix}$$

bulunur.

Dolayısıyla (2.1.1) sisteminin $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ i \end{pmatrix}$ başlangıç koşulunu sağlayan çözümü;

$\begin{pmatrix} y_{01} \\ y_{03} \end{pmatrix} = \begin{pmatrix} e^{ikx} \\ ie^{ikx} \end{pmatrix}$ şeklinde olup

$x < d$ iken

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} e^{ikx} - k\int_0^x u_1(t)y_1(t)\sin k(x-t)dt + \frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\sin k(x-t)dt \\ +k\int_0^x u_2(t)y_3(t)\cos k(x-t)dt \\ ie^{ikx} - k\int_0^x u_1(t)y_1(t)\cos k(x-t)dt + \frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\cos k(x-t)dt \\ +k\int_0^x u_2(t)y_3(t)\sin k(x-t)dt \end{pmatrix}$$

olarak elde edilir. $x > d$ iken çözüm

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} A(k)e^{ikx} + B(k)e^{-ikx} - k\int\limits_0^x u_1(t)y_1(t)\sin k(x-t)dt \\ +\frac{1}{k}\int\limits_0^x t^{-2\ell}q(t)y_1(t)\sin k(x-t)dt + k\int\limits_0^x u_2(t)y_3(t)\cos k(x-t)dt \\ iA(k)e^{ikx} - iB(k)e^{-ikx} - k\int\limits_0^x u_1(t)y_1(t)\cos k(x-t)dt \\ +\frac{1}{k}\int\limits_0^x t^{-2\ell}q(t)y_1(t)\cos k(x-t)dt + k\int\limits_0^x u_2(t)y_3(t)\sin k(x-t)dt \end{pmatrix}$$

şeklinde aranısın. (2.1.4) süreksizlik koşulu kullanılarak $A(k)$ ve $B(k)$ fonksiyonları,

$$\begin{aligned} A(k) = {} & \alpha^+ + \frac{ke^{-ikd}}{2}\int\limits_0^d \left(u_1(t)y_1(t)\sin k(d-t) - u_2(t)y_3(t)\cos k(d-t)\right)dt \\ & - \frac{e^{-ikd}}{2k}\int\limits_0^d t^{-2\ell}q(t)y_1(t)\sin k(d-t)dt - \frac{i\alpha^{-1}e^{-ikd}}{2k}\int\limits_0^d t^{-2\ell}q(t)y_1(t)\cos k(d-t)dt \\ & - \frac{ike^{-ikd}}{2}\int\limits_0^d \left(u_1(t)y_1(t)\cos k(d-t) + u_2(t)y_3(t)\sin k(d-t)\right)dt \\ & + \frac{ie^{-ikd}}{2k}\int\limits_0^d t^{-2\ell}q(t)y_1(t)\cos k(d-t)dt + \frac{\alpha e^{-ikd}}{2k}\int\limits_0^d t^{-2\ell}q(t)y_1(t)\sin k(d-t)dt \\ & - \frac{\alpha ke^{-ikd}}{2}\int\limits_0^d \left(u_1(t)y_1(t)\sin k(d-t) - u_2(t)y_3(t)\cos k(d-t)\right)dt \\ & + \frac{i\alpha^{-1}ke^{-ikd}}{2}\int\limits_0^d \left(u_1(t)y_1(t)\cos k(d-t) + u_2(t)y_3(t)\sin k(d-t)\right)u_2(t)y_3(t))dt \end{aligned}$$

$$\begin{aligned} B(k) = {} & \alpha^- e^{2ikd} + \frac{ke^{ikd}}{2}\int\limits_0^d \left(u_1(t)y_1(t)\sin k(d-t) - u_2(t)y_3(t)\cos k(d-t)\right)dt \\ & - \frac{e^{ikd}}{2k}\int\limits_0^d t^{-2\ell}q(t)y_1(t)\sin k(d-t)dt - \frac{ie^{ikd}}{2k}\int\limits_0^d t^{-2\ell}q(t)y_1(t)\cos k(d-t)dt \\ & + \frac{ike^{ikd}}{2}\int\limits_0^d \left(u_1(t)y_1(t)\cos k(d-t) + u_2(t)y_3(t)\sin k(d-t)\right)dt \end{aligned}$$

$$-\frac{\alpha k e^{ikd}}{2}\int_0^d \left(u_1(t)y_1(t)\sin k(d-t) - u_2(t)y_3(t)\cos k(d-t)\right)dt$$

$$+\frac{\alpha e^{ikd}}{2k}\int_0^d t^{-2\ell}q(t)y_1(t)\sin k(d-t)dt + \frac{i\alpha^{-1}e^{ikd}}{2k}\int_0^d t^{-2\ell}q(t)y_1(t)\cos k(d-t)dt$$

$$-\frac{i\alpha^{-1}ke^{ikd}}{2}\int_0^d \left(u_1(t)y_1(t)\cos k(d-t) + u_2(t)y_3(t)\sin k(d-t)\right) u_2(t)y_3(t))dt$$

olarak bulunur. Bu sabitler yerine yazılıp gerekli düzenlemeler yapılırsa, $\alpha^+ = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)$, $\alpha^- = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)$ olmak üzere $y_1(x,k)$ ve $y_2(x,k)$ fonksiyonları için $x > d$ iken

$$\begin{aligned}
y_1(x,k) = {} & \alpha^+ e^{ikx} + \alpha^- e^{ik(2d-x)} \\
& - k\int_0^d \left(\alpha^+ \sin k(x-t) - \alpha^- \sin k(x+t-2d)\right) u_1(t)y_1(t)dt \\
& + k\int_0^d \left(\alpha^+ \cos k(x-t) + \alpha^- \cos k(x+t-2d)\right) u_2(t)y_3(t)dt \\
& + \frac{1}{k}\int_0^d \left(\alpha^+ \sin k(x-t) - \alpha^- \sin k(x+t-2d)\right) t^{-2\ell}q(t)y_1(t)dt \\
& - k\int_d^x \left(\sin k(x-t)u_1(t)y_1(t) - \cos k(x-t)u_2(t)y_3(t)\right)dt \\
& + \frac{1}{k}\int_d^x \sin k(x-t)t^{-2\ell}q(t)y_1(t)dt
\end{aligned} \qquad (2.1.5)$$

$$\begin{aligned}
y_3(x,k) = {} & i\alpha^+ e^{ikx} - i\alpha^- e^{ik(2d-x)} \\
& - k\int_0^d \left(\alpha^+ \cos k(x-t) - \alpha^- \cos k(x+t-2d)\right) u_1(t)y_1(t)dt \\
& - k\int_0^d \left(\alpha^+ \sin k(x-t) + \alpha^- \sin k(x+t-2d)\right) u_2(t)y_3(t)dt \\
& + \frac{1}{k}\int_0^d \left(\alpha^+ \cos k(x-t) - \alpha^- \cos k(x+t-2d)\right) t^{-2\ell}q(t)y_1(t)dt
\end{aligned}$$

$$-k\int_d^x \left(\cos k(x-t)u_1(t)y_1(t) + \sin k(x-t)u_2(t)y_3(t)\right) dt + \frac{1}{k}\int_d^x \cos k(x-t)\, t^{-2\ell} q(t) y_1(t)\, dt \qquad (2.1.6)$$

ifadeleri bulunur.

$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ i \end{pmatrix}$ başlangıç koşulunu sağlayan çözüm;

$x < d$ ise,

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} e^{ikx} + a(x)e^{ikx} + \int_{-x}^{x} K_{11}(x,t)e^{ikt}dt + i\int_{-x}^{x} K_{12}(x,t)e^{ikt}dt \\ ie^{ikx} + ia(x)e^{ikx} + \int_{-x}^{x} K_{21}(x,t)e^{ikt}dt + i\int_{-x}^{x} K_{22}(x,t)e^{ikt}dt \end{pmatrix} \qquad (2.1.7)$$

$x > d$ ise,

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_{10} + a(x)e^{ikx} + b(x)e^{ik(2d-x)} + \int_{-x}^{x} K_{11}(x,t)e^{ikt}dt \\ + i\int_{-x}^{x} K_{12}(x,t)e^{ikt}dt \\ y_{30} + ia(x)e^{ikx} - ib(x)e^{ik(2d-x)} + \int_{-x}^{x} K_{21}(x,t)e^{ikt}dt \\ + i\int_{-x}^{x} K_{22}(x,t)e^{ikt}dt \end{pmatrix} \qquad (2.1.8)$$

şeklinde gösterimin varlığı gösterilsin. Burada $\begin{pmatrix} y_{10} \\ y_{30} \end{pmatrix} = \begin{pmatrix} \alpha^{+} e^{ikx} + \alpha^{-} e^{ik(2d-x)} \\ i\alpha^{+} e^{ikx} - i\alpha^{-} e^{ik(2d-x)} \end{pmatrix}$ ve $K_{11}(x,t)$,$K_{12}(x,t)$, $K_{21}(x,t), K_{22}(x,t)$ fonksiyonları reel değerli , $a(x)$ ve $b(x)$ fonksiyonları, $a(x) = a_1(x) + ia_2(x), b(x) = b_1(x) + ib_2(x)$ olacak şekilde komplex değerli fonksiyonlardır.. (2.1.7) ve (2.1.8) ifadeleri (2.1.5) ve (2.1.6) çözümünde yerine yazılırsa,

$$
\begin{aligned}
& a(x)e^{ikx} + b(x)e^{ik(2d-x)} + \int_{-x}^{x} K_{11}(x,t)e^{ikt}dt + i\int_{-x}^{x} K_{11}(x,t)e^{ikt}dt = \\
& -k\int_{0}^{d} \left(\alpha^{+}\sin k(x-t) - \alpha^{-}\sin k(x+t-2d)\right) u_1(t) \left\{ e^{ikt} + a(t)e^{ikt} + \int_{-t}^{t} K_{11}(t,s)e^{iks}ds \right. \\
& \left. + i\int_{-t}^{t} K_{12}(t,s)e^{iks}ds \right\} dt \\
& + k\int_{0}^{d} \left(\alpha^{+}\cos k(x-t) + \alpha^{-}\cos k(x+t-2d)\right) u_2(t) \left\{ ie^{ikt} + ia(t)e^{ikt} + \int_{-t}^{t} K_{21}(t,s)e^{iks}ds \right. \\
& \left. + i\int_{-t}^{t} K_{22}(t,s)e^{iks}ds \right\} dt \\
& + \frac{1}{k}\int_{0}^{d} \left(\alpha^{+}\sin k(x-t) - \alpha^{-}\sin k(x+t-2d)\right) t^{-2\ell} q(t) \left\{ e^{ikt} + a(t)e^{ikt} + \int_{-t}^{t} K_{11}(t,s)e^{iks}ds \right. \\
& \left. + i\int_{-t}^{t} K_{12}(t,s)e^{iks}ds \right\} dt \\
& - k\int_{d}^{x} \sin k(x-t) u_1(t) \left\{ \alpha^{+}e^{ikt} + \alpha^{-}e^{ik(2d-t)} + a(t)e^{ikt} + b(t)e^{ik(2d-t)} + \int_{-t}^{t} K_{11}(t,s)e^{iks}ds \right. \\
& \left. + i\int_{-t}^{t} K_{12}(t,s)e^{iks}ds \right\} dt \\
& + k\int_{d}^{x} \cos k(x-t) u_2(t) \left\{ i\alpha^{+}e^{ikt} - i\alpha^{-}e^{ik(2d-t)} + ia(t)e^{ikt} - ib(t)e^{ik(2d-t)} + \int_{-t}^{t} K_{21}(t,s)e^{iks}ds \right. \\
& \left. + i\int_{-t}^{t} K_{22}(t,s)e^{iks}ds \right\} dt \\
& + \frac{1}{k}\int_{d}^{x} \sin k(x-t) t^{-2\ell} q(t) \left\{ \alpha^{+}e^{ikt} + \alpha^{-}e^{ik(2d-t)} + a(t)e^{ikt} + b(t)e^{ik(2d-t)} + \int_{-t}^{t} K_{11}(t,s)e^{iks}ds \right. \\
& \left. + i\int_{-t}^{t} K_{12}(t,s)e^{iks}ds \right\} dt = I_1 + I_2 + I_3 + \ldots + I_{42}
\end{aligned}
$$

elde edlir. Burada;

$$I_1 = -\alpha^+ k \int_0^d \sin k(x-t) u_1(t) e^{ikt} dt = \frac{i\alpha^+ k}{2} e^{ikx} \int_0^d u_1(t) dt - \frac{i\alpha^+ k}{4} \int_{-x}^{2d-x} u_1(\tfrac{x+\zeta}{2}) e^{ik\zeta} d\zeta$$

$$I_2 = -\alpha^+ k \int_0^d \sin k(x-t) u_1(t) a(t) e^{ikt} dt = \frac{i\alpha^+ k}{2} e^{ikx} \int_0^d u_1(t) a(t) dt$$

$$- \frac{i\alpha^+ k}{4} \int_{-x}^{2d-x} u_1(\tfrac{x+\zeta}{2}) a(\tfrac{x+\zeta}{2}) e^{ik\zeta} d\zeta$$

$$I_3 = \alpha^- k \int_0^d \sin k(x+t-2d) u_1(t) e^{ikt} dt = \frac{i\alpha^- k}{2} e^{ik(2d-x)} \int_0^d u_1(t) dt$$

$$- \frac{i\alpha^- k}{4} \int_{x-2d}^{x} u_1(d - \tfrac{x-\zeta}{2}) e^{ik\zeta} d\zeta$$

$$I_4 = \alpha^- k \int_0^d \sin k(x+t-2d) u_1(t) a(t) e^{ikt} dt = \frac{i\alpha^- k}{2} e^{ik(2d-x)} \int_0^d u_1(t) a(t) dt$$

$$- \frac{i\alpha^- k}{4} \int_{x-2d}^{x} u_1(d - \tfrac{x-\zeta}{2}) a(d - \tfrac{x-\zeta}{2}) e^{ik\zeta} d\zeta$$

$$I_5 = -\alpha^+ k \int_0^d \sin k(x-t) u_1(t) \left\{ \int_{-t}^{t} K_{11}(t,s) e^{iks} ds \right\} dt =$$

$$\frac{i\alpha^+ k}{2} \int_{x-2d}^{x} \left\{ \int_{\frac{x-\zeta}{2}}^{d} K_{11}(t, \zeta + t - x) u_1(t) dt \right\} e^{ik\zeta} d\zeta$$

$$- \frac{i\alpha^+ k}{2} \int_{-x}^{2d-x} \left\{ \int_{\frac{x+\zeta}{2}}^{d} K_{11}(t, \zeta + x - t) u_1(t) dt \right\} e^{ik\zeta} d\zeta$$

$$I_6 = \alpha^- k \int_0^d \sin k(x+t-2d) u_1(t) \left\{ \int_{-t}^{t} K_{11}(t,s) e^{iks} ds \right\} dt =$$

$$- \frac{i\alpha^- k}{2} \int_{x-2d}^{x} \left\{ \int_{d-\frac{x-\zeta}{2}}^{d} K_{11}(t, \zeta + 2d - x - t) u_1(t) dt \right\} e^{ik\zeta} d\zeta$$

$$+\frac{i\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{11}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$I_7 = -i\alpha^+ k\int_0^d \sin k(x-t)u_1(t)\left\{\int_{-t}^{t} K_{12}(t,s)e^{iks}ds\right\}dt =$$

$$-\frac{\alpha^+ k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+ k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$I_8 = i\alpha^- k\int_0^d \sin k(x+t-2d)u_1(t)\left\{\int_{-t}^{t} K_{12}(t,s)e^{iks}ds\right\}dt =$$

$$+\frac{\alpha^- k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+2d-x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$I_9 = i\alpha^+ k\int_0^d \cos k(x-t)u_2(t)e^{ikt}dt = \frac{i\alpha^+ k}{2}e^{ikx}\int_0^d u_2(t)dt + \frac{i\alpha^+ k}{4}\int_{-x}^{2d-x} u_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{10} = i\alpha^- k\int_0^d \cos k(x+t-2d)u_2(t)e^{ikt}dt = \frac{i\alpha^- k}{2}e^{ik(2d-x)}\int_0^d u_2(t)dt$$

$$+\frac{i\alpha^- k}{4}\int_{x-2d}^{x} u_2(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{11} = i\alpha^+ k\int_0^d \cos k(x-t)u_2(t)a(t)e^{ikt}dt = \frac{i\alpha^+ k}{2}e^{ikx}\int_0^d u_2(t)a(t)dt$$

$$+\frac{i\alpha^+ k}{4}\int_{-x}^{2d-x} u_2(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{12} = i\alpha^- k \int_0^d \cos k(x + t - 2d) u_2(t) a(t) e^{ikt} dt = \frac{i\alpha^- k}{2} e^{ik(2d - x)} \int_0^d u_2(t) a(t) dt$$

$$+ \frac{i\alpha^- k}{4} \int_{x-2d}^{x} u_2(d - \tfrac{x-\zeta}{2}) a(d - \tfrac{x-\zeta}{2}) e^{ik\zeta} d\zeta$$

$$I_{13} = \alpha^+ k \int_0^d \cos k(x - t) u_2(t) \left\{ \int_{-t}^{t} K_{21}(t,s) e^{iks} ds \right\} dt =$$

$$\frac{\alpha^+ k}{2} \int_{x-2d}^{x} \left\{ \int_{\frac{x-\zeta}{2}}^{d} K_{21}(t, \zeta + t - x) u_2(t) dt \right\} e^{ik\zeta} d\zeta$$

$$+ \frac{\alpha^+ k}{2} \int_{-x}^{2d-x} \left\{ \int_{\frac{x+\zeta}{2}}^{d} K_{21}(t, \zeta + x - t) u_2(t) dt \right\} e^{ik\zeta} d\zeta$$

$$I_{14} = \alpha^- k \int_0^d \cos k(x + t - 2d) u_2(t) \left\{ \int_{-t}^{t} K_{21}(t,s) e^{iks} ds \right\} dt =$$

$$\frac{\alpha^- k}{2} e^{ikx} \int_{x-2d}^{x} \left\{ \int_{d-\frac{x-\zeta}{2}}^{d} K_{21}(t, \zeta + 2d - x - t) u_2(t) dt \right\} e^{ik\zeta} d\zeta$$

$$+ \frac{\alpha^- k}{2} \int_{-x}^{2d-x} \left\{ \int_{d-\frac{x+\zeta}{2}}^{d} K_{21}(t, \zeta + x + t - 2d) u_2(t) dt \right\} e^{ik\zeta} d\zeta$$

$$I_{15} = i\alpha^+ k \int_0^d \cos k(x - t) u_2(t) \left\{ \int_{-t}^{t} K_{22}(t,s) e^{iks} ds \right\} dt =$$

$$\frac{i\alpha^+ k}{2} \int_{x-2d}^{x} \left\{ \int_{\frac{x-\zeta}{2}}^{d} K_{22}(t, \zeta + t - x) u_2(t) dt \right\} e^{ik\zeta} d\zeta$$

$$+ \frac{i\alpha^+ k}{2} \int_{-x}^{2d-x} \left\{ \int_{\frac{x+\zeta}{2}}^{d} K_{22}(t, \zeta + x - t) u_2(t) dt \right\} e^{ik\zeta} d\zeta$$

$$I_{16} = i\alpha^{-} k \int_0^d \cos k(x+t-2d)u_2(t)\left\{\int_{-t}^{t} K_{22}(t,s)e^{iks}ds\right\}dt =$$

$$\frac{i\alpha^{-} k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{22}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^{-} k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{22}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$I_{17} = \frac{\alpha^{+}}{k}\int_0^d \sin k(x-t)t^{-2\ell}q(t)e^{ikt}dt = -\frac{i\alpha^{+}}{2k}e^{ikx}\int_0^d t^{-2\ell}q(t)dt$$

$$+\frac{i\alpha^{+}}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$I_{18} = \frac{\alpha^{+}}{k}\int_0^d \sin k(x-t)t^{-2\ell}q(t)a(t)e^{ikt}dt = -\frac{i\alpha^{+}}{2k}e^{ikx}\int_0^d t^{-2\ell}q(t)a(t)dt$$

$$+\frac{i\alpha^{+}}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)a\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$I_{19} = -\frac{\alpha^{-}}{k}\int_0^d \sin k(x+t-2d)t^{-2\ell}q(t)e^{ikt}dt = -\frac{i\alpha^{-}}{2k}e^{i(2d-x)}\int_0^d t^{-2\ell}q(t)dt$$

$$+\frac{i\alpha^{-}}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$I_{20} = -\frac{\alpha^{-}}{k}\int_0^d \sin k(x+t-2d)t^{-2\ell}q(t)a(t)e^{ikt}dt = -\frac{i\alpha^{-}}{2k}e^{i(2d-x)}\int_0^d t^{-2\ell}q(t)a(t)dt$$

$$+\frac{i\alpha^{-}}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)a\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$I_{21} = \frac{\alpha^{+}}{k}\int_0^d \sin k(x-t)t^{-2\ell}q(t)\left\{\int_{-t}^{t} K_{11}(t,s)e^{iks}ds\right\}dt =$$

$$\frac{\alpha^{+}}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$I_{22}=\frac{-\alpha^-}{k}\int_0^d \sin k(x+t-2d)t^{-2\ell}q(t)\left\{\int_{-t}^{t}K_{11}(t,s)e^{iks}ds\right\}dt=$$

$$-\frac{\alpha^-}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d}K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$I_{23}=\frac{i\alpha^+}{k}\int_0^d \sin k(x-t)t^{-2\ell}q(t)\left\{\int_{-t}^{t}K_{12}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{i\alpha^+}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s}K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$I_{24}=\frac{-i\alpha^-}{k}\int_0^d \sin k(x+t-2d)t^{-2\ell}q(t)\left\{\int_{-t}^{t}K_{12}(t,s)e^{iks}ds\right\}dt=$$

$$-\frac{i\alpha^-}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d}K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$I_{25}=-\alpha^+k\int_d^x \sin k(x-t)u_1(t)e^{ikt}dt=\frac{i\alpha^+k}{2}e^{ikx}\int_d^x u_1(t)dt-\frac{i\alpha^+k}{4}\int_{2d-x}^{x}u_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$I_{26}=-\alpha^-k\int_d^x \sin k(x-t)u_1(t)e^{ik(2d-t)}dt=-\frac{i\alpha^-k}{2}e^{ik(2d-x)}\int_d^x u_1(t)dt$$

$$-\frac{i\alpha^-k}{4}\int_x^{2d-x}u_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$I_{27}=-k\int_d^x \sin k(x-t)u_1(t)a(t)e^{ikt}dt=\frac{ik}{2}e^{ikx}\int_d^x u_1(t)a(t)dt-\frac{ik}{4}\int_{2d-x}^{x}u_1\left(\tfrac{x+\zeta}{2}\right)a\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$I_{28}=-k\int_d^x \sin k(x-t)u_1(t)b(t)e^{ik(2d-t)}dt=-\frac{ik}{2}e^{ik(2d-x)}\int_d^x u_1(t)b(t)dt$$

$$-\frac{ik}{4}\int_x^{2d-x}u_1\left(d-\tfrac{x-\zeta}{2}\right)b\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$I_{29}=-k\int_d^x \sin k(x-t)u_1(t)\left\{\int_{-t}^{t}K_{11}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{ik}{2}\int\limits_{-x}^{x-2d}\left\{\int\limits_{\frac{x-\zeta}{2}}^{x} K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta+\frac{ik}{2}\int\limits_{x-2d}^{x}\left\{\int\limits_{d}^{x} K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{ik}{2}\int\limits_{-x}^{2d-x}\left\{\int\limits_{d}^{x} K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{ik}{2}\int\limits_{2d-x}^{x}\left\{\int\limits_{\frac{x+\zeta}{2}}^{x} K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$I_{30}=-ik\int\limits_{d}^{x}\sin k(x-t)u_1(t)\left\{\int\limits_{-t}^{t} K_{12}(t,s)e^{iks}ds\right\}dt=$$

$$-\frac{k}{2}\int\limits_{-x}^{x-2d}\left\{\int\limits_{\frac{x-\zeta}{2}}^{x} K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int\limits_{x-2d}^{x}\left\{\int\limits_{d}^{x} K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int\limits_{-x}^{2d-x}\left\{\int\limits_{d}^{x} K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int\limits_{2d-x}^{x}\left\{\int\limits_{\frac{x+\zeta}{2}}^{x} K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$I_{31}=i\alpha^{+}k\int\limits_{d}^{x}\cos k(x-t)u_2(t)e^{ikt}dt=\frac{i\alpha^{+}k}{2}e^{ikx}\int\limits_{d}^{x}u_2(t)dt-\frac{i\alpha^{+}k}{4}\int\limits_{2d-x}^{x}u_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{32}=-i\alpha^{-}k\int\limits_{d}^{x}\cos k(x-t)u_2(t)e^{ik(2d-t)}dt=-\frac{i\alpha^{-}k}{2}e^{ik(2d-x)}\int\limits_{d}^{x}u_2(t)dt$$

$$+\frac{i\alpha^{-}k}{4}\int\limits_{x}^{2d-x}u_2(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{33}=ik\int\limits_{d}^{x}\cos k(x-t)u_2(t)a(t)e^{ikt}dt=\frac{ik}{2}e^{ikx}\int\limits_{d}^{x}u_2(t)a(t)dt+\frac{ik}{4}\int\limits_{2d-x}^{x}u_2(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{34}=-ik\int\limits_{d}^{x}\cos k(x-t)u_2(t)b(t)e^{ik(2d-t)}dt=-\frac{ik}{2}e^{ik(2d-x)}\int\limits_{d}^{x}u_2(t)b(t)dt$$

$$+\frac{ik}{4}\int\limits_{x}^{2d-x}u_2(d+\tfrac{x-\zeta}{2})b(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{35}=k\int\limits_{d}^{x}\cos k(x-t)u_2(t)\left\{\int\limits_{-t}^{t} K_{21}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$I_{36}=ik\int_{d}^{x}\cos k(x-t)u_2(t)\left\{\int_{-t}^{t} K_{22}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{ik}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{ik}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{ik}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{ik}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$I_{37}=\frac{\alpha^+}{k}\int_{d}^{x}\sin k(x-t)t^{-2\ell}q(t)e^{ikt}dt=-\frac{i\alpha^+}{2k}e^{ikx}\int_{d}^{x}t^{-2\ell}q(t)dt+\frac{i\alpha^+}{4k}\int_{2d-x}^{x}(\tfrac{x+\zeta}{2})^{-2\ell}q(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{38}=\frac{\alpha^-}{k}\int_{d}^{x}\sin k(x-t)t^{-2\ell}q(t)e^{ik(2d-t)}dt=\frac{i\alpha^-}{2k}e^{ik(2d-x)}\int_{d}^{x}t^{-2\ell}q(t)dt$$

$$+\frac{i\alpha^-}{4k}\int_{x}^{2d-x}(d+\tfrac{x-\zeta}{2})^{-2\ell}q(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{39}=\frac{1}{k}\int_{d}^{x}\sin k(x-t)t^{-2\ell}q(t)a(t)e^{ikt}dt=-\frac{i}{2k}e^{ikx}\int_{d}^{x}t^{-2\ell}q(t)a(t)dt$$

$$+\frac{i}{4k}\int_{2d-x}^{x}(\tfrac{x+\zeta}{2})^{-2\ell}q(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{40}=\frac{1}{k}\int_{d}^{x}\sin k(x-t)t^{-2\ell}q(t)b(t)e^{ik(2d-t)}dt=\frac{i}{2k}e^{ik(2d-x)}\int_{d}^{x}t^{-2\ell}q(t)b(t)dt$$

$$+\frac{i}{4k}\int_{x1}^{2d-x}(d+\tfrac{x-\zeta}{2})^{-2\ell}q(d+\tfrac{x-\zeta}{2})b(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$I_{41}=\frac{1}{k}\int_d^x \sin k(x-t)\,t^{-2\ell}q(t)\left\{\int_{-t}^{t} K_{11}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{1}{2}\int_{-x}^{x}\left\{\int_d^x q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$I_{42}=\frac{i}{k}\int_d^x \sin k(x-t)\,t^{-2\ell}q(t)\left\{\int_{-t}^{t} K_{12}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{i}{2}\int_{-x}^{x}\left\{\int_d^x q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

şeklindedir. Bu hesaplamalar yerine yazılırsa,

$$a(x)e^{ikx}+b(x)e^{ik(2d-x)}+\int_{-x}^{x}K_{11}(x,t)e^{ikt}dt+i\int_{-x}^{x}K_{12}(x,t)e^{ikt}dt=$$

$$\frac{i\alpha^+k}{2}e^{ikx}\int_0^d u_1(t)dt-\frac{i\alpha^+k}{4}\int_{-x}^{2d-x}u_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{i\alpha^+k}{2}e^{ikx}\int_0^d u_1(t)a(t)dt$$

$$-\frac{i\alpha^+k}{4}\int_{-x}^{2d-x}u_1\left(\tfrac{x+\zeta}{2}\right)a\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{i\alpha^-k}{2}e^{ik(2d-x)}\int_0^d u_1(t)dt-\frac{i\alpha^-k}{4}\int_{x-2d}^{x}u_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^-k}{2}e^{ik(2d-x)}\int_0^d u_1(t)a(t)dt-\frac{i\alpha^-k}{4}\int_{x-2d}^{x}u_1\left(d-\tfrac{x-\zeta}{2}\right)a\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^+k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^+k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^-k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+2d-x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^-k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+ k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+ k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+2d-x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^+ k}{2}e^{ikx}\int_{0}^{d}u_2(t)dt+\frac{i\alpha^+ k}{4}\int_{-x}^{2d-x}u_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta+\frac{i\alpha^- k}{2}e^{ik(2d-x)}\int_{0}^{d}u_2(t)dt$$

$$+\frac{i\alpha^- k}{4}\int_{x-2d}^{x}u_2(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{i\alpha^+ k}{2}e^{ikx}\int_{0}^{d}u_2(t)a(t)dt+\frac{i\alpha^+ k}{4}\int_{-x}^{2d-x}u_2(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^- k}{2}e^{ik(2d-x)}\int_{0}^{d}u_2(t)a(t)dt+\frac{i\alpha^- k}{4}\int_{x-2d}^{x}u_2(d-\tfrac{x-\zeta}{2})a(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+ k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+ k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{2}e^{ikx}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{21}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{21}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^+ k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^+ k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^- k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{22}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{22}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^+}{2k}e^{ikx}\int_0^d t^{-2\ell}q(t)dt+\frac{i\alpha^+}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{i\alpha^+}{2k}e^{ikx}\int_0^d t^{-2\ell}q(t)a(t)dt$$

$$+\frac{i\alpha^+}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)a\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{i\alpha^-}{2k}e^{i(2d-x)}\int_0^d t^{-2\ell}q(t)dt$$

$$+\frac{i\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{i\alpha^-}{2k}e^{i(2d-x)}\int_0^d t^{-2\ell}q(t)a(t)dt$$

$$+\frac{i\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)a\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s}K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$-\frac{\alpha^-}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d}K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$+\frac{i\alpha^+}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s}K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$-\frac{i\alpha^-}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d}K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$+\frac{i\alpha^+k}{2}e^{ikx}\int_d^x u_1(t)dt-\frac{i\alpha^+k}{4}\int_{2d-x}^x u_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta-\frac{i\alpha^-k}{2}e^{ik(2d-x)}\int_d^x u_1(t)dt$$

$$-\frac{i\alpha^-k}{4}\int_x^{2d-x}u_1(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{ik}{2}e^{ikx}\int_d^x u_1(t)a(t)dt-\frac{ik}{4}\int_{2d-x}^x u_1(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{ik}{2}e^{ik(2d-x)}\int_d^x u_1(t)b(t)dt-\frac{ik}{4}\int_x^{2d-x}u_1(d-\tfrac{x-\zeta}{2})b(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{ik}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta+\frac{ik}{2}\int_{x-2d}^x\left\{\int_d^x K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{ik}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{ik}{2}\int_{2d-x}^x\left\{\int_{\frac{x+\zeta}{2}}^x K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{x-2d}^x\left\{\int_d^x K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{2d-x}^x\left\{\int_{\frac{x+\zeta}{2}}^x K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^+k}{2}e^{ikx}\int_d^x u_2(t)dt-\frac{i\alpha^+k}{4}\int_{2d-x}^x u_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta-\frac{i\alpha^-k}{2}e^{ik(2d-x)}\int_d^x u_2(t)dt$$

$$+\frac{i\alpha^-k}{4}\int_x^{2d-x}u_2(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{ik}{2}e^{ikx}\int_d^x u_2(t)a(t)dt+\frac{ik}{4}\int_{2d-x}^x u_2(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{ik}{2}e^{ik(2d-x)}\int_d^x u_2(t)b(t)dt+\frac{ik}{4}\int_x^{2d-x}u_2(d+\tfrac{x-\zeta}{2})b(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{x-2d}^x\left\{\int_d^x K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^x K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{ik}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{ik}{2}\int_{x-2d}^{x}\left\{\int_d^x K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{ik}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{ik}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^x K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^+}{2k}e^{ikx}\int_d^x t^{-2\ell}q(t)dt+\frac{i\alpha^+}{4k}\int_{2d-x}^x \left(\tfrac{x+\zeta}{2}\right)^{-2\ell}q\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{i\alpha^-}{2k}e^{ik(2d-x)}\int_d^x t^{-2\ell}q(t)dt$$

$$+\frac{i\alpha^-}{4k}\int_x^{2d-x}\left(d+\tfrac{x-\zeta}{2}\right)^{-2\ell}q\left(d+\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{i}{2k}e^{ikx}\int_d^x t^{-2\ell}q(t)a(t)dt$$

$$+\frac{i}{4k}\int_{2d-x}^x\left(\tfrac{x+\zeta}{2}\right)^{-2\ell}q\left(\tfrac{x+\zeta}{2}\right)a\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{i}{2k}e^{ik(2d-x)}\int_d^x t^{-2\ell}q(t)b(t)dt$$

$$+\frac{i}{4k}\int_x^{2d-x}\left(d+\tfrac{x-\zeta}{2}\right)^{-2\ell}q\left(d+\tfrac{x-\zeta}{2}\right)b\left(d+\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{1}{2}\int_{-x}^x\left\{\int_d^x q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s}K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt+\frac{i}{2}\int_{-x}^x\left\{\int_d^x q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s}K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

Aynı şekilde,

$$ia(x)e^{ikx}-ib(x)e^{ik(2d-x)}+\int_{-x}^x K_{21}(x,t)e^{ikt}dt+i\int_{-x}^x K_{22}(x,t)e^{ikt}dt=$$

$$-k\int_0^d(\alpha^+\cos k(x-t)-\alpha^-\cos k(x+t-2d))\,u_1(t)\left\{e^{ikt}+a(t)e^{ikt}+\int_{-t}^t K_{11}(t,s)e^{iks}ds\right.$$

$$\left.+i\int_{-t}^t K_{12}(t,s)e^{iks}ds\right\}dt$$

$$-k\int_0^d(\alpha^+\sin k(x-t)+\alpha^-\sin k(x+t-2d))\,u_2(t)\left\{ie^{ikt}+ia(t)e^{ikt}+\int_{-t}^t K_{21}(t,s)e^{iks}ds\right.$$

$$+i\int_{-t}^{t} K_{22}(t,s)e^{iks}ds \Bigg\} dt$$

$$+\frac{1}{k}\int_{0}^{d} (\alpha^{+}\cos k(x-t) - \alpha^{-}\cos k(x+t-2d))\, t^{-2\ell}q(t) \Bigg\{ e^{ikt} + a(t)e^{ikt} + \int_{-t}^{t} K_{11}(t,s)e^{iks}ds$$

$$+i\int_{-t}^{t} K_{12}(t,s)e^{iks}ds \Bigg\} dt$$

$$-k\int_{d}^{x} \cos k(x-t)u_1(t) \Bigg\{ \alpha^{+}e^{ikt} + \alpha^{-}e^{ik(2d-t)} + a(t)e^{ikt} + b(t)e^{ik(2d-t)} + \int_{-t}^{t} K_{11}(t,s)e^{iks}ds$$

$$+i\int_{-t}^{t} K_{12}(t,s)e^{iks}ds \Bigg\} dt$$

$$-k\int_{d}^{x} \sin k(x-t)u_2(t) \Bigg\{ i\alpha^{+}e^{ikt} - i\alpha^{-}e^{ik(2d-t)} + ia(t)e^{ikt} - ib(t)e^{ik(2d-t)} + \int_{-t}^{t} K_{21}(t,s)e^{iks}ds$$

$$+i\int_{-t}^{t} K_{22}(t,s)e^{iks}ds \Bigg\} dt$$

$$+\frac{1}{k}\int_{d}^{x} \cos k(x-t)t^{-2\ell}q(t) \Bigg\{ \alpha^{+}e^{ikt} + \alpha^{-}e^{ik(2d-t)} + a(t)e^{ikt} + b(t)e^{ik(2d-t)} + \int_{-t}^{t} K_{11}(t,s)e^{iks}ds$$

$$+i\int_{-t}^{t} K_{12}(t,s)e^{iks}ds \Bigg\} dt = T_1 + T_2 + ... + T_{42}$$

ile gösterilsin. Burada;

$$T_1 = -\alpha^{+}k\int_{0}^{d} \cos k(x-t)u_1(t)e^{ikt}dt = -\frac{\alpha^{+}k}{2}e^{ikx}\int_{0}^{d} u_1(t)dt - \frac{\alpha^{+}k}{4}\int_{-x}^{2d-x} u_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_2 = -\alpha^{+}k\int_{0}^{d} \cos k(x-t)u_1(t)a(t)e^{ikt}dt = -\frac{\alpha^{+}k}{2}e^{ikx}\int_{0}^{d} u_1(t)a(t)dt$$

$$-\frac{\alpha^{+}k}{4}\int_{-x}^{2d-x} u_1(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_3 = \alpha^{-}k\int_{0}^{d} \cos k(x+t-2d)u_1(t)e^{ikt}dt = \frac{\alpha^{-}k}{2}e^{ik(2d-x)}\int_{0}^{d} u_1(t)dt$$

$$+\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$T_4=\alpha^- k\int_0^d \cos k(x+t-2d)u_1(t)a(t)e^{ikt}dt=\frac{\alpha^- k}{2}e^{ik(2d-x)}\int_0^d u_1(t)a(t)dt$$

$$+\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_1\left(d-\tfrac{x-\zeta}{2}\right)a\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$T_5=-\alpha^+ k\int_0^d \cos k(x-t)u_1(t)\left\{\int_{-t}^{t}K_{11}(t,s)e^{iks}ds\right\}dt=$$

$$-\frac{\alpha^+ k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+ k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_6=\alpha^- k\int_0^d \cos k(x+t-2d)u_1(t)\left\{\int_{-t}^{t}K_{11}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{\alpha^- k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+2d-x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_7=-i\alpha^+ k\int_0^d \cos k(x-t)u_1(t)\left\{\int_{-t}^{t}K_{12}(t,s)e^{iks}ds\right\}dt=$$

$$-\frac{i\alpha^+ k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^+ k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x-t)u_1(t)dt\right\} e^{ik\zeta}d\zeta$$

$$T_8 = i\alpha^- k\int_0^d \cos k(x+t-2d)u_1(t)\left\{\int_{-t}^{t} K_{12}(t,s)e^{iks}ds\right\}dt =$$

$$\frac{i\alpha^- k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+2d-x-t)u_1(t)dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x+t-2d)u_1(t)dt\right\} e^{ik\zeta}d\zeta$$

$$T_9 = -i\alpha^+ k\int_0^d \sin k(x-t)u_2(t)e^{ikt}dt = -\frac{\alpha^+ k}{2}e^{ikx}\int_0^d u_2(t)dt + \frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{10} = -i\alpha^- k\int_0^d \sin k(x+t-2d)u_2(t)e^{ikt}dt = \frac{\alpha^- k}{2}e^{ik(2d-x)}\int_0^d u_2(t)dt$$

$$-\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_2(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{11} = -i\alpha^+ k\int_0^d \sin k(x-t)u_2(t)a(t)e^{ikt}dt = -\frac{\alpha^+ k}{2}e^{ikx}\int_0^d u_2(t)a(t)dt$$

$$+\frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_2(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{12} = -i\alpha^- k\int_0^d \sin k(x+t-2d)u_2(t)a(t)e^{ikt}dt = \frac{\alpha^- k}{2}e^{ik(2d-x)}\int_0^d u_2(t)a(t)dt$$

$$-\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_2(d-\tfrac{x-\zeta}{2})a(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{13} = -\alpha^+ k\int_0^d \sin k(x-t)u_2(t)\left\{\int_{-t}^{t} K_{21}(t,s)e^{iks}ds\right\}dt =$$

$$\frac{i\alpha^{+}k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^{+}k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{14}=-\alpha^{-}k\int_{0}^{d}\sin k(x+t-2d)u_2(t)\left\{\int_{-t}^{t}K_{21}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{i\alpha^{-}k}{2}e^{ikx}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{21}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^{-}k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{21}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{15}=-i\alpha^{+}k\int_{0}^{d}\sin k(x-t)u_2(t)\left\{\int_{-t}^{t}K_{22}(t,s)e^{iks}ds\right\}dt=$$

$$-\frac{\alpha^{+}k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{+}k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{16}=-i\alpha^{-}k\int_{0}^{d}\sin k(x+t-2d)u_2(t)\left\{\int_{-t}^{t}K_{22}(t,s)e^{iks}ds\right\}dt=$$

$$-\frac{\alpha^{-}k}{2}e^{ikx}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{22}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{22}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{17} = \frac{\alpha^+}{k}\int_0^d \cos k(x-t)t^{-2\ell}q(t)e^{ikt}dt = \frac{\alpha^+}{2k}e^{ikx}\int_0^d t^{-2\ell}q(t)dt + \frac{\alpha^+}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$T_{18} = \frac{\alpha^+}{k}\int_0^d \cos k(x-t)t^{-2\ell}q(t)a(t)e^{ikt}dt = \frac{\alpha^+}{2k}e^{ikx}\int_0^d t^{-2\ell}q(t)a(t)dt$$

$$+\frac{\alpha^+}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)a\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$T_{19} = -\frac{\alpha^-}{k}\int_0^d \cos k(x+t-2d)t^{-2\ell}q(t)e^{ikt}dt = -\frac{\alpha^-}{2k}e^{ik(2d-x)}\int_0^d t^{-2\ell}q(t)dt$$

$$-\frac{\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$T_{20} = -\frac{\alpha^-}{k}\int_0^d \cos k(x+t-2d)t^{-2\ell}q(t)a(t)e^{ikt}dt = -\frac{\alpha^-}{2k}e^{ik(2d-x)}\int_0^d t^{-2\ell}q(t)a(t)dt$$

$$-\frac{\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)a\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$T_{21} = \frac{\alpha^+}{k}\int_0^d \cos k(x-t)t^{-2\ell}q(t)\left\{\int_{-t}^{t}K_{11}(t,s)e^{iks}ds\right\}dt =$$

$$\frac{\alpha^+}{2k}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+t-x)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{2k}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x-t)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$T_{22} = -\frac{\alpha^-}{k}\int_0^d \cos k(x+t-2d)t^{-2\ell}q(t)\left\{\int_{-t}^{t}K_{11}(t,s)e^{iks}ds\right\}dt =$$

$$-\frac{\alpha^-}{2k}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{11}(t,\zeta+2d-x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-}{2k}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{11}(t,\zeta+x+t-2d)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{23}=\frac{i\alpha^+}{k}\int_0^d \cos k(x-t)t^{-2\ell}q(t)\left\{\int_{-t}^{t} K_{12}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{i\alpha^+}{2k}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+t-x)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^+}{2k}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x-t)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$T_{24}=-\frac{i\alpha^-}{k}\int_0^d \cos k(x+t-2d)t^{-2\ell}q(t)\left\{\int_{-t}^{t} K_{12}(t,s)e^{iks}ds\right\}dt=$$

$$-\frac{i\alpha^-}{2k}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+2d-x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^-}{2k}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x+t-2d)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{25}=-\alpha^+k\int_d^x \cos k(x-t)u_1(t)e^{ikt}dt=-\frac{\alpha^+k}{2}e^{ikx}\int_d^x u_1(t)dt-\frac{\alpha^+k}{4}\int_{2d-x}^{x} u_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{26}=-\alpha^-k\int_d^x \cos k(x-t)u_1(t)e^{ik(2d-t)}dt=-\frac{\alpha^-k}{2}e^{ik(2d-x)}\int_d^x u_1(t)dt$$

$$+\frac{\alpha^-k}{4}\int_{x}^{2d-x} u_1(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{27}=-k\int_d^x \cos k(x-t)u_1(t)a(t)e^{ikt}dt=-\frac{k}{2}e^{ikx}\int_d^x u_1(t)a(t)dt-\frac{k}{4}\int_{2d-x}^{x} u_1(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{28} = -k \int_d^x \cos k(x-t) u_1(t) b(t) e^{ik(2d-t)} dt = -\frac{k}{2} e^{ik(2d-x)} \int_d^x u_1(t) b(t) dt$$

$$+\frac{k}{4} \int_x^{2d-x} u_1(d+\tfrac{x-\zeta}{2}) b(d+\tfrac{x-\zeta}{2}) e^{ik\zeta} d\zeta$$

$$T_{29} = -k \int_d^x \cos k(x-t) u_1(t) \left\{ \int_{-t}^t K_{11}(t,s) e^{iks} ds \right\} dt =$$

$$-\frac{k}{2} \int_{-x}^{x-2d} \left\{ \int_{\frac{x-\zeta}{2}}^x K_{11}(t,\zeta+t-x) u_1(t) dt \right\} e^{ik\zeta} d\zeta - \frac{k}{2} \int_{x-2d}^x \left\{ \int_d^x K_{11}(t,\zeta+t-x) u_1(t) dt \right\} e^{ik\zeta} d\zeta$$

$$-\frac{k}{2} \int_{-x}^{2d-x} \left\{ \int_d^x K_{11}(t,\zeta+x-t) u_1(t) dt \right\} e^{ik\zeta} d\zeta - \frac{k}{2} \int_{2d-x}^x \left\{ \int_{\frac{x+\zeta}{2}}^x K_{11}(t,\zeta+x-t) u_1(t) dt \right\} e^{ik\zeta} d\zeta$$

$$T_{30} = -ik \int_d^x \cos k(x-t) u_1(t) \left\{ \int_{-t}^t K_{12}(t,s) e^{iks} ds \right\} dt =$$

$$-\frac{ik}{2} \int_{-x}^{x-2d} \left\{ \int_{\frac{x-\zeta}{2}}^x K_{12}(t,\zeta+t-x) u_1(t) dt \right\} e^{ik\zeta} d\zeta - \frac{ik}{2} \int_{x-2d}^x \left\{ \int_d^x K_{12}(t,\zeta+t-x) u_1(t) dt \right\} e^{ik\zeta} d\zeta$$

$$-\frac{ik}{2} \int_{-x}^{2d-x} \left\{ \int_d^x K_{12}(t,\zeta+x-t) u_1(t) dt \right\} e^{ik\zeta} d\zeta - \frac{ik}{2} \int_{2d-x}^x \left\{ \int_{\frac{x+\zeta}{2}}^x K_{12}(t,\zeta+x-t) u_1(t) dt \right\} e^{ik\zeta} d\zeta$$

$$T_{31} = -i\alpha^+ k \int_d^x \sin k(x-t) u_2(t) e^{ikt} dt = -\frac{\alpha^+ k}{2} e^{ikx} \int_d^x u_2(t) dt + \frac{\alpha^+ k}{4} \int_{2d-x}^x u_2(\tfrac{x+\zeta}{2}) e^{ik\zeta} d\zeta$$

$$T_{32} = i\alpha^- k \int_d^x \sin k(x-t) u_2(t) e^{ik(2d-x)} dt = -\frac{\alpha^- k}{2} e^{ik(2d-x)} \int_d^x u_2(t) dt$$

$$-\frac{\alpha^- k}{4} \int_x^{2d-x} u_2(d+\tfrac{x-\zeta}{2}) e^{ik\zeta} d\zeta$$

$$T_{33} = -ik \int_d^x \sin k(x-t) u_2(t) a(t) e^{ikt} dt = -\frac{k}{2} e^{ikx} \int_d^x u_2(t) a(t) dt + \frac{k}{4} \int_{2d-x}^x u_2(\tfrac{x+\zeta}{2}) a(\tfrac{x+\zeta}{2}) e^{ik\zeta} d\zeta$$

$$T_{34} = ik\int_d^x \sin k(x-t)u_2(t)b(t)e^{ik(2d-t)}dt = -\frac{k}{2}e^{ik(2d-x)}\int_d^x u_2(t)b(t)dt$$

$$-\frac{k}{4}\int_x^{2d-x} u_2(d+\tfrac{x-\zeta}{2})b(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{35} = -k\int_d^x \sin k(x-t)u_2(t)\left\{\int_{-t}^t K_{21}(t,s)e^{iks}ds\right\}dt =$$

$$\frac{ik}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{ik}{2}\int_{x-2d}^{x}\left\{\int_d^x K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{ik}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta-\frac{ik}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^x K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{36} = -ik\int_d^x \sin k(x-t)u_2(t)\left\{\int_{-t}^t K_{22}(t,s)e^{iks}ds\right\}dt =$$

$$-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{x-2d}^{x}\left\{\int_d^x K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^x K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{37} = \frac{\alpha^+}{k}\int_d^x \cos k(x-t)t^{-2\ell}q(t)e^{ikt}dt = \frac{\alpha^+}{2k}e^{ikx}\int_d^x t^{-2\ell}q(t)dt+\frac{\alpha^+}{4k}\int_{2d-x}^x (\tfrac{x+\zeta}{2})^{-2\ell}q(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{38} = \frac{\alpha^-}{k}\int_d^x \cos k(x-t)t^{-2\ell}q(t)e^{ik(2d-t)}dt = \frac{\alpha^-}{2k}e^{ik(2d-x)}\int_d^x t^{-2\ell}q(t)dt$$

$$-\frac{\alpha^-}{4k}\int_x^{2d-x} (d+\tfrac{x-\zeta}{2})^{-2\ell}q(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{39} = \frac{1}{k}\int_d^x \cos k(x-t)t^{-2\ell}q(t)a(t)e^{ikt}dt = \frac{1}{2k}e^{ikx}\int_d^x t^{-2\ell}q(t)a(t)dt$$

$$+\frac{1}{4k}\int_{2d-x}^x (\tfrac{x+\zeta}{2})^{-2\ell}q(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$T_{40}=\frac{1}{k}\int_{d}^{x}\cos k(x-t)\,t^{-2\ell}q(t)b(t)e^{ik(2d-t)}dt=\frac{1}{2k}e^{ik(2d-x)}\int_{d}^{x}t^{-2\ell}q(t)b(t)dt$$

$$-\frac{1}{4k}\int_{x}^{2d-x}\left(d+\tfrac{x-\zeta}{2}\right)^{-2\ell}q\left(d+\tfrac{x-\zeta}{2}\right)b\left(d+\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$T_{41}=\frac{1}{k}\int_{d}^{x}\cos k(x-t)\,t^{-2\ell}q(t)\left\{\int_{-t}^{t}K_{11}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{1}{2k}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x}K_{11}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{x-2d}^{x}\left\{\int_{d}^{x}K_{11}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{-x}^{2d-x}\left\{\int_{d}^{x}K_{11}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x}K_{11}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$T_{42}=\frac{i}{k}\int_{d}^{x}\cos k(x-t)\,t^{-2\ell}q(t)\left\{\int_{-t}^{t}K_{12}(t,s)e^{iks}ds\right\}dt=$$

$$\frac{i}{2k}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x}K_{12}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i}{2k}\int_{x-2d}^{x}\left\{\int_{d}^{x}K_{12}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i}{2k}\int_{-x}^{2d-x}\left\{\int_{d}^{x}K_{12}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i}{2k}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x}K_{12}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

şeklindedir. Bu hesaplamalar yerine yazılırsa,

$$ia(x)e^{ikx}-ib(x)e^{ik(2d-x)}+\int_{-x}^{x}K_{21}(x,t)e^{ikt}dt+i\int_{-x}^{x}K_{22}(x,t)e^{ikt}dt=$$

$$- \frac{\alpha^+ k}{2} e^{ikx}\int_0^d u_1(t)dt - \frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_1(\tfrac{x+\zeta}{2}) e^{ik\zeta} d\zeta - \frac{\alpha^+ k}{2} e^{ikx}\int_0^d u_1(t)a(t)dt$$

$$- \frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_1(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2}) e^{ik\zeta} d\zeta + \frac{\alpha^- k}{2} e^{ik(2d-x)}\int_0^d u_1(t)dt$$

$$+ \frac{\alpha^- k}{4}\int_{x-2d}^{x} u_1(d-\tfrac{x-\zeta}{2}) e^{ik\zeta} d\zeta + \frac{\alpha^- k}{2} e^{ik(2d-x)}\int_0^d u_1(t)a(t)dt$$

$$+ \frac{\alpha^- k}{4}\int_{x-2d}^{x} u_1(d-\tfrac{x-\zeta}{2})a(d-\tfrac{x-\zeta}{2}) e^{ik\zeta} d\zeta$$

$$- \frac{\alpha^+ k}{2}\int_{x-2d}^{x} \left\{ \int_{\frac{x-\zeta}{2}}^{d} K_{11}(t,\zeta+t-x)u_1(t)dt \right\} e^{ik\zeta} d\zeta$$

$$- \frac{\alpha^+ k}{2}\int_{-x}^{2d-x} \left\{ \int_{\frac{x+\zeta}{2}}^{d} K_{11}(t,\zeta+x-t)u_1(t)dt \right\} e^{ik\zeta} d\zeta$$

$$+ \frac{\alpha^- k}{2}\int_{x-2d}^{x} \left\{ \int_{d-\frac{x-\zeta}{2}}^{d} K_{11}(t,\zeta+2d-x-t)u_1(t)dt \right\} e^{ik\zeta} d\zeta$$

$$+ \frac{\alpha^- k}{2}\int_{-x}^{2d-x} \left\{ \int_{d-\frac{x+\zeta}{2}}^{d} K_{11}(t,\zeta+x+t-2d)u_1(t)dt \right\} e^{ik\zeta} d\zeta$$

$$- \frac{i\alpha^+ k}{2}\int_{x-2d}^{x} \left\{ \int_{\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+t-x)u_1(t)dt \right\} e^{ik\zeta} d\zeta$$

$$- \frac{i\alpha^+ k}{2}\int_{-x}^{2d-x} \left\{ \int_{\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x-t)u_1(t)dt \right\} e^{ik\zeta} d\zeta$$

$$+ \frac{i\alpha^- k}{2}\int_{x-2d}^{x} \left\{ \int_{d-\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+2d-x-t)u_1(t)dt \right\} e^{ik\zeta} d\zeta$$

$$+\frac{i\alpha-k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{12}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha+k}{2}e^{ikx}\int_{0}^{d}u_2(t)dt+\frac{\alpha+k}{4}\int_{-x}^{2d-x}u_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{\alpha-k}{2}e^{ik(2d-x)}\int_{0}^{d}u_2(t)dt$$

$$-\frac{\alpha-k}{4}\int_{x-2d}^{x}u_2\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{\alpha+k}{2}e^{ikx}\int_{0}^{d}u_2(t)a(t)dt+\frac{\alpha+k}{4}\int_{-x}^{2d-x}u_2\left(\tfrac{x+\zeta}{2}\right)a\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{\alpha-k}{2}e^{ik(2d-x)}\int_{0}^{d}u_2(t)a(t)dt-\frac{\alpha-k}{4}\int_{x-2d}^{x}u_2\left(d-\tfrac{x-\zeta}{2}\right)a\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha+k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha+k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha-k}{2}e^{ikx}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{21}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha-k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{21}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha+k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha+k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^- k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{22}(t,\zeta+2d-x-t)u_2(t)dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{22}(t,\zeta+x+t-2d)u_2(t)dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{2k}e^{ikx}\int_0^d t^{-2\ell}q(t)dt+\frac{\alpha^+}{4k}\int_{-x}^{2d-x}\left(\frac{x+\zeta}{2}\right)^{-2l}q\left(\frac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{\alpha^+}{2k}e^{ikx}\int_0^d t^{-2\ell}q(t)a(t)dt$$

$$+\frac{\alpha^+}{4k}\int_{-x}^{2d-x}\left(\frac{x+\zeta}{2}\right)^{-2l}q\left(\frac{x+\zeta}{2}\right)a\left(\frac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{\alpha^-}{2k}e^{i(2d-x)}\int_0^d t^{-2\ell}q(t)dt$$

$$-\frac{\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\frac{x-\zeta}{2}\right)^{-2l}q\left(d-\frac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{\alpha^-}{2k}e^{i(2d-x)}\int_0^d t^{-2\ell}q(t)a(t)dt$$

$$-\frac{\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\frac{x-\zeta}{2}\right)^{-2l}q\left(d-\frac{x-\zeta}{2}\right)a\left(d-\frac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{2k}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{11}(t,\zeta+t-x)q(t)t^{-2\ell}dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{2k}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{11}(t,\zeta+x-t)q(t)t^{-2\ell}dt\right\} e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-}{2k}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{11}(t,\zeta+2d-x-t)t^{-2\ell}q(t)dt\right\} e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-}{2k}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{11}(t,\zeta+x+t-2d)t^{-2\ell}q(t)dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^+}{2k}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+t-x)q(t)t^{-2\ell}dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{i\alpha^+}{2k}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x-t)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^-}{2k}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+2d-x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{i\alpha^-}{2k}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x+t-2d)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+k}{2}e^{ikx}\int_d^x u_1(t)dt-\frac{\alpha^+k}{4}\int_{2d-x}^{x}u_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta-\frac{\alpha^-k}{2}e^{ik(2d-x)}\int_d^x u_1(t)dt$$

$$+\frac{\alpha^-k}{4}\int_x^{2d-x}u_1(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta-\frac{k}{2}e^{ikx}\int_d^x u_1(t)a(t)dt-\frac{k}{4}\int_{2d-x}^{x}u_1(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}e^{ik(2d-x)}\int_d^x u_1(t)b(t)dt+\frac{k}{4}\int_x^{2d-x}u_1(d+\tfrac{x-\zeta}{2})b(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x}K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{x-2d}^{x}\left\{\int_d^x K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x}K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{ik}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x}K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{ik}{2}\int_{x-2d}^{x}\left\{\int_d^x K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{ik}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{ik}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x}K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+k}{2}e^{ikx}\int_d^x u_2(t)dt+\frac{\alpha^+k}{4}\int_{2d-x}^{x}u_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta-\frac{\alpha^-k}{2}e^{ik(2d-x)}\int_d^x u_2(t)dt$$

$$-\frac{\alpha^- k}{4}\int_x^{2d-x} u_2(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta-\frac{k}{2}e^{ikx}\int_d^x u_2(t)a(t)dt+\frac{k}{4}\int_{2d-x}^x u_2(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}e^{ik(2d-x)}\int_d^x u_2(t)b(t)dt-\frac{k}{4}\int_x^{2d-x} u_2(d+\tfrac{x-\zeta}{2})b(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{ik}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{ik}{2}\int_{x-2d}^x\left\{\int_d^x K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{ik}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta-\frac{ik}{2}\int_{2d-x}^x\left\{\int_{\frac{x+\zeta}{2}}^x K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{x-2d}^x\left\{\int_d^x K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_d^x K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{2d-x}^x\left\{\int_{\frac{x+\zeta}{2}}^x K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{2k}e^{ikx}\int_d^x t^{-2\ell}q(t)dt+\frac{\alpha^+}{4k}\int_{2d-x}^x(\tfrac{x+\zeta}{2})^{-2\ell}q(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta+\frac{\alpha^-}{2k}e^{ik(2d-x)}\int_d^x t^{-2\ell}q(t)dt$$

$$-\frac{\alpha^-}{4k}\int_x^{2d-x}(d+\tfrac{x-\zeta}{2})^{-2\ell}q(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{1}{2k}e^{ikx}\int_d^x t^{-2\ell}q(t)a(t)dt$$

$$+\frac{1}{4k}\int_{2d-x}^x(\tfrac{x+\zeta}{2})^{-2\ell}q(\tfrac{x+\zeta}{2})a(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta+\frac{1}{2k}e^{ik(2d-x)}\int_d^x t^{-2\ell}q(t)b(t)dt$$

$$-\frac{1}{4k}\int_x^{2d-x}(d+\tfrac{x-\zeta}{2})^{-2\ell}q(d+\tfrac{x-\zeta}{2})b(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^x K_{11}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{x-2d}^x\left\{\int_d^x K_{11}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{-x}^{2d-x}\left\{\int_{d}^{x}K_{11}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x}K_{11}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i}{2k}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x}K_{12}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i}{2k}\int_{x-2d}^{x}\left\{\int_{d}^{x}K_{12}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i}{2k}\int_{-x}^{2d-x}\left\{\int_{d}^{x}K_{12}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{i}{2k}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x}K_{12}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

bulunur. Bu ifadeler kullanılırsa;

$$\int_{-x}^{x}K_{11}(x,t)e^{ikt}dt=\frac{\alpha^{+}k}{4}\int_{-x}^{2d-x}u_1\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{\alpha^{-}k}{4}\int_{x-2d}^{x}u_1\left(d-\tfrac{x-\zeta}{2}\right)a_2\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^{+}}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{\alpha^{-}}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)a_2\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^{+}k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{+}k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{-}k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{12}(t,\zeta+2d-x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_2(\tfrac{x+\zeta}{2})a_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta - \frac{\alpha^- k}{4}\int_{x-2d}^{x} u_2(d-\tfrac{x-\zeta}{2})a_2(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+ k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+ k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{2}e^{ikx}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{21}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{21}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$\frac{\alpha^+}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s}K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$-\frac{\alpha^-}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d}K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$+\frac{k}{4}\int_{2d-x}^{x}u_1(\tfrac{x+\zeta}{2})a_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta+\frac{k}{4}\int_{x}^{2d-x}u_1(d-\tfrac{x-\zeta}{2})b_2(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x}K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x}K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x}K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x}K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{4}\int_{2d-x}^{x} u_2\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta - \frac{k}{4}\int_{x}^{2d-x} u_2\left(d+\tfrac{x-\zeta}{2}\right)b_2\left(d+\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta + \frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta + \frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{1}{4k}\int_{2d-x}^{x}\left(\tfrac{x+\zeta}{2}\right)^{-2\ell}q\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta + \frac{1}{2}\int_{-x}^{x}\left\{\int_{d}^{x} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{11}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$\int_{-x}^{x} K_{12}(x,t)e^{ikt}dt = -\frac{\alpha^+k}{4}\int_{-x}^{2d-x} u_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta - \frac{\alpha^-k}{4}\int_{x-2d}^{x} u_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+k}{4}\int_{-x}^{2d-x} u_1\left(\tfrac{x+\zeta}{2}\right)a_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta - \frac{\alpha^-k}{4}\int_{x-2d}^{x} u_1\left(d-\tfrac{x-\zeta}{2}\right)a_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)a_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta + \frac{\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)a_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{11}(t,\zeta+2d-x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^-k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{11}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$
+\frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_2\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_2\left(\tfrac{x+\zeta}{2}\right)a_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta
$$

$$
+\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_2\left(d-\tfrac{x-\zeta}{2}\right)a_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{\alpha^+ k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta
$$

$$
+\frac{\alpha^+ k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta
$$

$$
+\frac{\alpha^- k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d} K_{22}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta
$$

$$
+\frac{\alpha^- k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d} K_{22}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta
$$

$$
+\frac{\alpha^+}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta
$$

$$
+\frac{\alpha^+}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s}K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt
$$

$$
-\frac{\alpha^-}{2}\int_{-x}^{x}\left\{\int_0^d q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d}K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt
$$

$$
-\frac{\alpha^+ k}{4}\int_{2d-x}^{x} u_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{\alpha^- k}{4}\int_{x}^{2d-x} u_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{k}{4}\int_{2d-x}^{x} u_1\left(\tfrac{x+\zeta}{2}\right)a_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta
$$

$$
-\frac{k}{4}\int_{x}^{2d-x} u_1\left(d-\tfrac{x-\zeta}{2}\right)b_1\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta
$$

$$
+\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta
$$

$$-\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta+\frac{\alpha^+ k}{4}\int_{2d-x}^{x} u_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^- k}{4}\int_{x}^{2d-x} u_2(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{k}{4}\int_{2d-x}^{x} u_2(\tfrac{x+\zeta}{2})a_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{k}{4}\int_{x}^{2d-x} u_2(d+\tfrac{x-\zeta}{2})b_1(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{\alpha^+}{4k}\int_{2d-x}^{x}(\tfrac{x+\zeta}{2})^{-2\ell}q(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^-}{4k}\int_{x}^{2d-x}(d+\tfrac{x-\zeta}{2})^{-2\ell}q(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{1}{4k}\int_{2d-x}^{x}(\tfrac{x+\zeta}{2})^{-2\ell}q(\tfrac{x+\zeta}{2})a_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{1}{4k}\int_{x}^{2d-x}(d+\tfrac{x-\zeta}{2})^{-2\ell}q(d+\tfrac{x-\zeta}{2})b_1(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{1}{2}\int_{-x}^{x}\left\{\int_{d}^{x} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{12}(s,\zeta)d\zeta ds\right\}e^{ikt}dt$$

$$\int_{-x}^{x} K_{21}(x,t)e^{ikt}dt=-\frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta+\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_1(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_1(\tfrac{x+\zeta}{2})a_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta+\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_1(d-\tfrac{x-\zeta}{2})a_1(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{4k}\int_{-x}^{2d-x}(\tfrac{x+\zeta}{2})^{-2l}q(\tfrac{x+\zeta}{2})a_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta-\frac{\alpha^-}{4k}\int_{x-2d}^{x}(d-\tfrac{x-\zeta}{2})^{-2l}q(d-\tfrac{x-\zeta}{2})a_1(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta$$

$$
\begin{aligned}
&-\frac{\alpha^{+}k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+t-x)u_{1}(t)dt\right\}e^{ik\zeta}d\zeta\\
&-\frac{\alpha^{+}k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x-t)u_{1}(t)dt\right\}e^{ik\zeta}d\zeta\\
&+\frac{\alpha^{-}k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+2d-x-t)u_{1}(t)dt\right\}e^{ik\zeta}d\zeta\\
&+\frac{\alpha^{-}k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x+t-2d)u_{1}(t)dt\right\}e^{ik\zeta}d\zeta\\
&+\frac{\alpha^{+}k}{4}\int_{-x}^{2d-x}u_{2}(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta-\frac{\alpha^{-}k}{4}\int_{x-2d}^{x}u_{2}(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{\alpha^{+}k}{4}\int_{-x}^{2d-x}u_{2}(\tfrac{x+\zeta}{2})a_{1}(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta\\
&-\frac{\alpha^{-}k}{4}\int_{x-2d}^{x}u_{2}(d-\tfrac{x-\zeta}{2})a_{1}(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta-\frac{\alpha^{+}k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{22}(t,\zeta+t-x)u_{2}(t)dt\right\}e^{ik\zeta}d\zeta\\
&+\frac{\alpha^{+}k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{22}(t,\zeta+x-t)u_{2}(t)dt\right\}e^{ik\zeta}d\zeta\\
&-\frac{\alpha^{-}k}{2}e^{ikx}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{22}(t,\zeta+2d-x-t)u_{2}(t)dt\right\}e^{ik\zeta}d\zeta\\
&+\frac{\alpha^{-}k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{22}(t,\zeta+x+t-2d)u_{2}(t)dt\right\}e^{ik\zeta}d\zeta\\
&+\frac{\alpha^{+}}{4k}\int_{-x}^{2d-x}(\tfrac{x+\zeta}{2})^{-2l}q(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta-\frac{\alpha^{-}}{4k}\int_{x-2d}^{x}(d-\tfrac{x-\zeta}{2})^{-2l}q(d-\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta
\end{aligned}
$$

$$+\frac{\alpha^+}{2k}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+t-x)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+}{2k}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x-t)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-}{2k}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{11}(t,\zeta+2d-x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-}{2k}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{11}(t,\zeta+x+t-2d)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+k}{4}\int_{2d-x}^{x}u_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta+\frac{\alpha^-k}{4}\int_{x}^{2d-x}u_1(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta-\frac{k}{4}\int_{2d-x}^{x}u_1(\tfrac{x+\zeta}{2})a_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$+\frac{k}{4}\int_{x}^{2d-x}u_1(d+\tfrac{x-\zeta}{2})b_1(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x}K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x}K_{11}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x}K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x}K_{11}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta+\frac{\alpha^+k}{4}\int_{2d-x}^{x}u_2(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-k}{4}\int_{x}^{2d-x}u_2(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta+\frac{k}{4}\int_{2d-x}^{x}u_2(\tfrac{x+\zeta}{2})a_1(\tfrac{x+\zeta}{2})e^{ik\zeta}d\zeta$$

$$-\frac{k}{4}\int_{x}^{2d-x}u_2(d+\tfrac{x-\zeta}{2})b_1(d+\tfrac{x-\zeta}{2})e^{ik\zeta}d\zeta-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x}K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x}K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x}K_{22}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{22}(t,\zeta+x-t)u_2(t)dt\right\} e^{ik\zeta}d\zeta + \frac{\alpha^+}{4k}\int_{2d-x}^{x}\left(\tfrac{x+\zeta}{2}\right)^{-2\ell}q\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-}{4k}\int_{x}^{2d-x}\left(d+\tfrac{x-\zeta}{2}\right)^{-2\ell}q\left(d+\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta + \frac{1}{4k}\int_{2d-x}^{x}\left(\tfrac{x+\zeta}{2}\right)^{-2\ell}q\left(\tfrac{x+\zeta}{2}\right)a_1\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$-\frac{1}{4k}\int_{x}^{2d-x}\left(d+\tfrac{x-\zeta}{2}\right)^{-2\ell}q\left(d+\tfrac{x-\zeta}{2}\right)b_1\left(d+\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{11}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{11}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{11}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{11}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\} e^{ik\zeta}d\zeta$$

$$\int_{-x}^{x} K_{22}(x,t)e^{ikt}dt = -\frac{\alpha^+k}{4}\int_{-x}^{2d-x} u_1\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^-k}{4}\int_{x-2d}^{x} u_1\left(d-\tfrac{x-\zeta}{2}\right)a_2\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta + \frac{\alpha^+}{4k}\int_{-x}^{2d-x}\left(\tfrac{x+\zeta}{2}\right)^{-2l}q\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^-}{4k}\int_{x-2d}^{x}\left(d-\tfrac{x-\zeta}{2}\right)^{-2l}q\left(d-\tfrac{x-\zeta}{2}\right)a_2\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d} K_{12}(t,\zeta+t-x)u_1(t)dt\right\} e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^+k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d} K_{12}(t,\zeta+x-t)u_1(t)dt\right\} e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{-}k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{12}(t,\zeta+2d-x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{-}k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{12}(t,\zeta+x+t-2d)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{+}k}{2}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^{+}k}{2}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{-}k}{2}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{21}(t,\zeta+2d-x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^{-}k}{2}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{21}(t,\zeta+x+t-2d)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{+}}{2k}\int_{x-2d}^{x}\left\{\int_{\frac{x-\zeta}{2}}^{d}K_{12}(t,\zeta+t-x)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^{+}}{2k}\int_{-x}^{2d-x}\left\{\int_{\frac{x+\zeta}{2}}^{d}K_{12}(t,\zeta+x-t)q(t)t^{-2\ell}dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^{-}}{2k}\int_{x-2d}^{x}\left\{\int_{d-\frac{x-\zeta}{2}}^{d}K_{12}(t,\zeta+2d-x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{\alpha^{-}}{2k}\int_{-x}^{2d-x}\left\{\int_{d-\frac{x+\zeta}{2}}^{d}K_{12}(t,\zeta+x+t-2d)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{4}\int_{2d-x}^{x} u_1\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta+\frac{k}{4}\int_{x}^{2d-x} u_1\left(d+\tfrac{x-\zeta}{2}\right)b_2\left(d+\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{12}(t,\zeta+t-x)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{12}(t,\zeta+x-t)u_1(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{\alpha^+ k}{4}\int_{-x}^{2d-x} u_2\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{\alpha^- k}{4}\int_{x-2d}^{x} u_2\left(d-\tfrac{x-\zeta}{2}\right)a_2\left(d-\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta+\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{21}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{21}(t,\zeta+x-t)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$-\frac{k}{2}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta-\frac{k}{2}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{22}(t,\zeta+t-x)u_2(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{4k}\int_{2d-x}^{x}\left(\tfrac{x+\zeta}{2}\right)^{-2\ell}q\left(\tfrac{x+\zeta}{2}\right)a_2\left(\tfrac{x+\zeta}{2}\right)e^{ik\zeta}d\zeta-\frac{1}{4k}\int_{x}^{2d-x}\left(d+\tfrac{x-\zeta}{2}\right)^{-2\ell}q\left(d+\tfrac{x-\zeta}{2}\right)b_2\left(d+\tfrac{x-\zeta}{2}\right)e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{-x}^{x-2d}\left\{\int_{\frac{x-\zeta}{2}}^{x} K_{12}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{x-2d}^{x}\left\{\int_{d}^{x} K_{12}(t,\zeta+t-x)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{-x}^{2d-x}\left\{\int_{d}^{x} K_{12}(t,\zeta+x-t)t^{-2\ell}q(t)dt\right\}e^{ik\zeta}d\zeta$$

$$+\frac{1}{2k}\int_{2d-x}^{x}\left\{\int_{\frac{x+\zeta}{2}}^{x} K_{12}(t,\zeta+x-t)\,t^{-2\ell}q(t)\,dt\right\}e^{ik\zeta}d\zeta$$

$$a_1(x)=-\frac{\alpha^{+}k}{2}\int_0^d u_2(t)a_2(t)dt-\frac{k}{2}\int_d^x u_1(t)a_2(t)dt-\frac{k}{2}\int_d^x u_2(t)a_2(t)dt$$
$$+\frac{1}{2k}\int_d^x t^{-2\ell}q(t)a_2(t)dt-\frac{\alpha^{+}k}{2}\int_0^d u_1(t)a_2(t)dt+\frac{\alpha^{+}}{2k}\int_0^d t^{-2\ell}q(t)a_2(t)dt$$

$$a_2(x)=\frac{\alpha^{+}k}{2}\int_0^d u_1(t)dt+\frac{\alpha^{+}k}{2}\int_0^d u_2(t)dt+\frac{\alpha^{+}k}{2}\int_0^d u_2(t)a_1(t)dt-\frac{\alpha^{+}}{2k}\int_0^d t^{-2\ell}q(t)dt$$
$$+\frac{\alpha^{+}k}{2}\int_d^x u_1(t)dt+\frac{k}{2}\int_d^x u_1(t)a_1(t)dt+\frac{\alpha^{+}k}{2}\int_d^x u_2(t)dt+\frac{k}{2}\int_d^x u_2(t)a_1(t)dt$$
$$-\frac{\alpha^{+}}{2k}\int_d^x t^{-2\ell}q(t)dt-\frac{1}{2k}\int_d^x t^{-2\ell}q(t)a_1(t)dt+\frac{\alpha^{+}k}{2}\int_0^d u_1(t)a_1(t)dt-\frac{\alpha^{+}}{2k}\int_0^d t^{-2\ell}q(t)a_1(t)dt$$

$$b_1(x)=-\frac{\alpha^{-}k}{2}\int_0^d u_2(t)a_2(t)dt+\frac{k}{2}\int_d^x u_1(t)b_2(t)dt+\frac{k}{2}\int_d^x u_2(t)b_2(t)dt$$
$$-\frac{1}{2k}\int_d^x t^{-2\ell}q(t)b_2(t)dt-\frac{\alpha^{-}k}{2}\int_0^d u_1(t)a_2(t)dt+\frac{\alpha^{-}}{2k}\int_0^d t^{-2\ell}q(t)a_2(t)dt$$

$$b_2(x)=\frac{\alpha^{-}k}{2}\int_0^d u_1(t)dt+\frac{\alpha^{-}k}{2}\int_0^d u_2(t)dt+\frac{\alpha^{-}k}{2}\int_0^d u_2(t)a_1(t)dt-\frac{\alpha^{-}}{2k}\int_0^d t^{-2\ell}q(t)dt$$
$$-\frac{\alpha^{-}k}{2}\int_d^x u_1(t)dt-\frac{k}{2}\int_d^x u_1(t)b_1(t)dt-\frac{\alpha^{-}k}{2}\int_d^x u_2(t)dt-\frac{k}{2}\int_d^x u_2(t)b_1(t)dt+\frac{\alpha^{-}}{2k}\int_d^x t^{-2\ell}q(t)dt$$
$$+\frac{1}{2k}\int_d^x t^{-2\ell}q(t)b_1(t)dt+\frac{\alpha^{-}k}{2}\int_0^d u_1(t)a_1(t)dt-\frac{\alpha^{-}}{2k}\int_0^d t^{-2\ell}q(t)a_1(t)dt$$

Şimdi ,

1 -) $d<x<2d,\ -x<t<x-2d<2d-x$

2 -) $2d<x,\ -x<t<2d-x$

3 -) $d<x<2d,\ x-2d<t<2d-x$

$4-)\ 2d < x,\ -x < t < x - 2d$

$5-)\ 2d < x,\ 2d - x < t < x$

$6-)\ d < x < 2d,\ x - 2d < t < x$

bölgeleri için $K_{11}(x,t)$, $K_{12}(x,t)$, $K_{21}(x,t)$ ve $K_{22}(x,t)$ fonksiyonlarının ifadeleri yazılsın;

$1-)\ d < x < 2d,\ -x < t < x - 2d < 2d - x$ **aralığı için:**

$$
\begin{aligned}
K_{11}(x,t) &= \frac{\alpha^{+}k}{4}u_1\left(\tfrac{x+t}{2}\right)a_2\left(\tfrac{x+t}{2}\right) + \frac{\alpha^{-}k}{4}u_1\left(d-\tfrac{x-t}{2}\right)a_2\left(d-\tfrac{x-t}{2}\right)\\
&- \frac{\alpha^{+}}{4k}\left(\tfrac{x+t}{2}\right)^{-2l}q\left(\tfrac{x+t}{2}\right)a_2\left(\tfrac{x+t}{2}\right) - \frac{\alpha^{-}}{4k}\left(d-\tfrac{x-t}{2}\right)^{-2l}q\left(d-\tfrac{x-t}{2}\right)a_2\left(d-\tfrac{x-t}{2}\right)\\
&- \frac{\alpha^{+}k}{4}u_2\left(\tfrac{x+t}{2}\right)a_2\left(\tfrac{x+t}{2}\right) - \frac{\alpha^{-}k}{4}u_2\left(d-\tfrac{x-t}{2}\right)a_2\left(d-\tfrac{x-t}{2}\right)\\
&- \frac{\alpha^{+}k}{2}\int_{\frac{x-t}{2}}^{d}K_{12}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta + \frac{\alpha^{+}k}{2}\int_{\frac{x+t}{2}}^{d}K_{12}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta\\
&+ \frac{\alpha^{-}k}{2}\int_{d-\frac{x-t}{2}}^{d}K_{12}(\zeta,t+2d-x-\zeta)u_1(\zeta)d\zeta - \frac{\alpha^{-}k}{2}\int_{d-\frac{x+t}{2}}^{d}K_{12}(\zeta,t+x+\zeta-2d)u_1(\zeta)d\zeta\\
&+ \frac{\alpha^{+}k}{2}\int_{\frac{x-t}{2}}^{d}K_{21}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta + \frac{\alpha^{+}k}{2}\int_{\frac{x+t}{2}}^{d}K_{21}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta\\
&+ \frac{\alpha^{-}k}{2}\int_{d-\frac{x-t}{2}}^{d}K_{21}(\zeta,t+2d-x-\zeta)u_2(\zeta)d\zeta + \frac{\alpha^{-}k}{2}\int_{d-\frac{x+t}{2}}^{d}K_{21}(\zeta,t+x+\zeta-2d)u_2(\zeta)d\zeta\\
&+ \frac{\alpha^{+}}{2}\int_{0}^{d}q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s}K_{11}(s,\zeta)d\zeta ds - \frac{\alpha^{-}}{2}\int_{0}^{d}q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d}K_{11}(s,\zeta)d\zeta ds\\
&- \frac{k}{2}\int_{\frac{x-t}{2}}^{x}K_{12}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta - \frac{k}{2}\int_{d}^{x}K_{12}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta\\
&+ \frac{k}{2}\int_{d}^{x}K_{12}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta + \frac{k}{2}\int_{\frac{x-t}{2}}^{x}K_{21}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta\\
&+ \frac{k}{2}\int_{d}^{x}K_{21}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta + \frac{k}{2}\int_{d}^{x}K_{21}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta
\end{aligned}
$$

$$+\frac{1}{2}\int_{d}^{x} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{11}(s,\zeta)d\zeta ds$$

$$K_{12}(x,t)= -\frac{\alpha^{+}k}{4}u_1(\tfrac{x+t}{2})a_1(\tfrac{x+t}{2}) - \frac{\alpha^{-}k}{4}u_1(d-\tfrac{x-t}{2})a_1(d-\tfrac{x-t}{2})$$

$$+\frac{\alpha^{+}}{4k}(\tfrac{x+t}{2})^{-2l}q(\tfrac{x+t}{2})a_1(\tfrac{x+t}{2}) + \frac{\alpha^{-}}{4k}(d-\tfrac{x-t}{2})^{-2l}q(d-\tfrac{x-t}{2})a_1(d-\tfrac{x-t}{2})$$

$$-\frac{\alpha^{+}k}{4}u_1(\tfrac{x+t}{2}) - \frac{\alpha^{-}k}{4}u_1(d-\tfrac{x-t}{2}) + \frac{\alpha^{+}k}{4}u_2(\tfrac{x+t}{2}) + \frac{\alpha^{-}k}{4}u_2(d-\tfrac{x-t}{2}) + \frac{\alpha^{+}k}{4}u_2(\tfrac{x+t}{2})a_1(\tfrac{x+t}{2})$$

$$+\frac{\alpha^{-}k}{4}u_2(d-\tfrac{x-t}{2})a_1(d-\tfrac{x-t}{2}) + \frac{\alpha^{+}}{4k}(\tfrac{x+t}{2})^{-2l}q(\tfrac{x+t}{2}) + \frac{\alpha^{-}}{4k}(d-\tfrac{x-t}{2})^{-2l}q(d-\tfrac{x-t}{2})$$

$$+\frac{\alpha^{+}k}{2}\int_{\frac{x-\zeta}{2}}^{d} K_{11}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta - \frac{\alpha^{+}k}{2}\int_{\frac{x+t}{2}}^{d} K_{11}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta$$

$$-\frac{\alpha^{-}k}{2}\int_{d-\frac{x-t}{2}}^{d} K_{11}(\zeta,t+2d-x-\zeta)u_1(\zeta)d\zeta + \frac{\alpha^{-}k}{2}\int_{d-\frac{x+t}{2}}^{d} K_{11}(\zeta,t+x+\zeta-2d)u_1(\zeta)d\zeta$$

$$+\frac{\alpha^{+}k}{2}\int_{\frac{x-t}{2}}^{d} K_{22}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta + \frac{\alpha^{+}k}{2}\int_{\frac{x+t}{2}}^{d} K_{22}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta$$

$$+\frac{\alpha^{-}k}{2}\int_{d-\frac{x-t}{2}}^{d} K_{22}(\zeta,t+2d-x-\zeta)u_2(\zeta)d\zeta + \frac{\alpha^{-}k}{2}\int_{d-\frac{x+t}{2}}^{d} K_{22}(\zeta,t+x+\zeta-2d)u_2(\zeta)d\zeta$$

$$+\frac{\alpha^{+}}{2}\int_{0}^{d} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{12}(s,\zeta)d\zeta ds - \frac{\alpha^{-}}{2}\int_{0}^{d} q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d} K_{12}(s,\zeta)d\zeta ds$$

$$+\frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{11}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta + \frac{k}{2}\int_{d}^{x} K_{11}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta$$

$$-\frac{k}{2}\int_{d}^{x} K_{11}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta + \frac{k}{2}\int_{d}^{x} K_{12}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta$$

$$+\frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{22}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta + \frac{k}{2}\int_{d}^{x} K_{22}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta$$

$$+\frac{k}{2}\int_{d}^{x} K_{22}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta + \frac{1}{2}\int_{d}^{x} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{12}(s,\zeta)d\zeta ds$$

$$K_{21}(x,t) = -\frac{\alpha^+ k}{4} u_1(\tfrac{x+t}{2}) a_1(\tfrac{x+t}{2}) - \frac{\alpha^- k}{4} u_1(d - \tfrac{x-t}{2}) a_1(d - \tfrac{x-t}{2})$$

$$+ \frac{\alpha^+}{4k}(\tfrac{x+t}{2})^{-2l} q(\tfrac{x+t}{2}) a_1(\tfrac{x+t}{2}) + \frac{\alpha^-}{4k}(d - \tfrac{x-t}{2})^{-2l} q(d - \tfrac{x-t}{2}) a_1(d - \tfrac{x-t}{2})$$

$$- \frac{\alpha^+ k}{4} u_1(\tfrac{x+t}{2}) + \frac{\alpha^- k}{4} u_1(d - \tfrac{x-t}{2}) + \frac{\alpha^+ k}{4} u_2(\tfrac{x+t}{2}) - \frac{\alpha^- k}{4} u_2(d - \tfrac{x-t}{2}) + \frac{\alpha^+ k}{4} u_2(\tfrac{x+t}{2}) a_1(\tfrac{x+t}{2})$$

$$- \frac{\alpha^- k}{4} u_2(d - \tfrac{x-\zeta}{2}) a_1(d - \tfrac{x-\zeta}{2}) + \frac{\alpha^+}{4k}(\tfrac{x+t}{2})^{-2l} q(\tfrac{x+t}{2}) - \frac{\alpha^-}{4k}(d - \tfrac{x-t}{2})^{-2l} q(d - \tfrac{x-t}{2})$$

$$- \frac{\alpha^+ k}{2} \int_{\frac{x-t}{2}}^{d} K_{11}(\zeta, t+\zeta-x) u_1(\zeta) d\zeta - \frac{\alpha^+ k}{2} \int_{\frac{x+t}{2}}^{d} K_{11}(\zeta, t+x-\zeta) u_1(\zeta) d\zeta$$

$$+ \frac{\alpha^- k}{2} \int_{d-\frac{x-t}{2}}^{d} K_{11}(\zeta, t+2d-x-\zeta) u_1(\zeta) d\zeta + \frac{\alpha^- k}{2} \int_{d-\frac{x+t}{2}}^{d} K_{11}(\zeta, t+x+\zeta-2d) u_1(\zeta) d\zeta$$

$$- \frac{\alpha^+ k}{2} \int_{\frac{x-t}{2}}^{d} K_{22}(\zeta, t+\zeta-x) u_2(\zeta) d\zeta + \frac{\alpha^+ k}{2} \int_{\frac{x+t}{2}}^{d} K_{22}(\zeta, t+x-\zeta) u_2(\zeta) d\zeta$$

$$- \frac{\alpha^- k}{2} \int_{d-\frac{x-t}{2}}^{d} K_{22}(\zeta, t+2d-x-\zeta) u_2(\zeta) d\zeta + \frac{\alpha^- k}{2} \int_{d-\frac{x+t}{2}}^{d} K_{22}(\zeta, t+x+\zeta-2d) u_2(\zeta) d\zeta$$

$$+ \frac{\alpha^+}{2k} \int_{\frac{x-t}{2}}^{d} K_{11}(\zeta, t+\zeta-x) q(\zeta) \zeta^{-2\ell} d\zeta + \frac{\alpha^+}{2k} \int_{\frac{x+t}{2}}^{d} K_{11}(\zeta, t+x-\zeta) q(\zeta) \zeta^{-2\ell} d\zeta$$

$$- \frac{\alpha^-}{2k} \int_{d-\frac{x-t}{2}}^{d} K_{11}(\zeta, t+2d-x-\zeta) \zeta^{-2\ell} q(\zeta) d\zeta - \frac{\alpha^-}{2k} \int_{d-\frac{x+t}{2}}^{d} K_{11}(\zeta, t+x+\zeta-2d) \zeta^{-2\ell} q(\zeta) d\zeta$$

$$- \frac{k}{2} \int_{\frac{x-t}{2}}^{x} K_{11}(\zeta, t+\zeta-x) u_1(\zeta) d\zeta - \frac{k}{2} \int_{d}^{x} K_{11}(\zeta, t+\zeta-x) u_1(\zeta) d\zeta$$

$$- \frac{k}{2} \int_{d}^{x} K_{11}(\zeta, t+x-\zeta) u_1(\zeta) d\zeta - \frac{k}{2} \int_{\frac{x-t}{2}}^{x} K_{22}(\zeta, t+\zeta-x) u_2(\zeta) d\zeta$$

$$- \frac{k}{2} \int_{d}^{x} K_{22}(\zeta, t+\zeta-x) u_2(\zeta) d\zeta + \frac{k}{2} \int_{d}^{x} K_{22}(\zeta, t+x-\zeta) u_2(\zeta) d\zeta$$

$$+ \frac{1}{2k} \int_{\frac{x-t}{2}}^{x} K_{11}(\zeta, t+\zeta-x) \zeta^{-2\ell} q(\zeta) d\zeta + \frac{1}{2k} \int_{d}^{x} K_{11}(\zeta, t+\zeta-x) \zeta^{-2\ell} q(\zeta) d\zeta$$

$$+\frac{1}{2k}\int_d^x K_{11}(\zeta, t+x-\zeta)\zeta^{-2\ell} q(\zeta)d\zeta$$

$$K_{22}(x,t) = -\frac{\alpha^+ k}{4} u_1(\tfrac{x+t}{2}) a_2(\tfrac{x+t}{2}) + \frac{\alpha^- k}{4} u_1(d-\tfrac{x-t}{2}) a_2(d-\tfrac{x-t}{2})$$

$$+\frac{\alpha^+}{4k}(\tfrac{x+t}{2})^{-2l} q(\tfrac{x+t}{2}) a_2(\tfrac{x+t}{2}) - \frac{\alpha^-}{4k}(d-\tfrac{x-t}{2})^{-2l} q(d-\tfrac{x-t}{2}) a_2(d-\tfrac{x-t}{2})$$

$$\frac{\alpha^+ k}{4} u_2(\tfrac{x+t}{2}) a_2(\tfrac{x+t}{2}) - \frac{\alpha^- k}{4} u_2(d-\tfrac{x-t}{2}) a_2(d-\tfrac{x-t}{2})$$

$$-\frac{\alpha^+ k}{2}\int_{\frac{x-t}{2}}^d K_{12}(\zeta, t+\zeta-x)u_1(\zeta)d\zeta - \frac{\alpha^+ k}{2}\int_{\frac{x+t}{2}}^d K_{12}(\zeta, t+x-\zeta)u_1(\zeta)d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{d-\frac{x-t}{2}}^d K_{12}(\zeta, t+2d-x-\zeta)u_1(\zeta)d\zeta + \frac{\alpha^- k}{2}\int_{d-\frac{x+t}{2}}^d K_{12}(\zeta, t+x+\zeta-2d)u_1(\zeta)d\zeta$$

$$+\frac{\alpha^+ k}{2}\int_{\frac{x-t}{2}}^d K_{21}(\zeta, t+\zeta-x)u_2(\zeta)d\zeta - \frac{\alpha^+ k}{2}\int_{\frac{x+t}{2}}^d K_{21}(\zeta, t+x-\zeta)u_2(\zeta)d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{d-\frac{x-t}{2}}^d K_{21}(\zeta, t+2d-x-\zeta)u_2(\zeta)d\zeta - \frac{\alpha^- k}{2}\int_{d-\frac{x+t}{2}}^d K_{21}(\zeta, t+x+\zeta-2d)u_2(\zeta)d\zeta$$

$$+\frac{\alpha^+}{2k}\int_{\frac{x-t}{2}}^d K_{12}(\zeta, t+\zeta-x)q(\zeta)\zeta^{-2\ell}d\zeta + \frac{\alpha^+}{2k}\int_{\frac{x+t}{2}}^d K_{12}(\zeta, t+x-\zeta)q(\zeta)\zeta^{-2\ell}d\zeta$$

$$-\frac{\alpha^-}{2k}\int_{d-\frac{x-t}{2}}^d K_{12}(\zeta, t+2d-x-\zeta)\zeta^{-2\ell}q(\zeta)d\zeta - \frac{\alpha^-}{2k}\int_{d-\frac{x+t}{2}}^d K_{12}(\zeta, t+x+\zeta-2d)\zeta^{-2\ell}q(\zeta)d\zeta$$

$$-\frac{k}{2}\int_{\frac{x-t}{2}}^x K_{12}(\zeta, t+\zeta-x)u_1(\zeta)d\zeta - \frac{k}{2}\int_d^x K_{12}(\zeta, t+\zeta-x)u_1(\zeta)d\zeta$$

$$-\frac{k}{2}\int_d^x K_{12}(\zeta, t+x-\zeta)u_1(\zeta)d\zeta + \frac{k}{2}\int_{\frac{x-t}{2}}^x K_{21}(\zeta, t+\zeta-x)u_2(\zeta)d\zeta$$

$$+\frac{k}{2}\int_d^x K_{21}(\zeta, t+\zeta-x)u_2(\zeta)d\zeta - \frac{k}{2}\int_d^x K_{21}(\zeta, t+x-\zeta)u_2(\zeta)d\zeta$$

$$-\frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{22}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta-\frac{k}{2}\int_{d}^{x} K_{22}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta$$

$$+\frac{1}{2k}\int_{\frac{x-t}{2}}^{x} K_{12}(\zeta,t+\zeta-x)\zeta^{-2\ell}q(\zeta)d\zeta+\frac{1}{2k}\int_{d}^{x} K_{12}(\zeta,t+\zeta-x)\zeta^{-2\ell}q(\zeta)d\zeta$$

$$+\frac{1}{2k}\int_{d}^{x} K_{12}(\zeta,t+x-\zeta)\zeta^{-2\ell}q(\zeta)d\zeta$$

integral denklemleri elde edilir. Benzer şekilde diğer bölgeler için de integral denklemleri kolayca alınabilir.

2.2 Çözüm İçin İntegral Gösterilimin Varlığı ve Özellikleri

Bu bölümde 2.1 alt bölümünde alınan integral denklemlerinin her bölge için çözümünün varlığı ve tekliği gösterilecektir. Ayrıca çevirme operatörünün çekirdeğinin sağladığı özellikler incelenecektir. Bunun için ardışık yaklaşımlar yöntemi uygulanırsa

$1^{-})$ $d < x < 2d, \; -x < t < x - 2d < 2d - x$ aralığı için:

$$K_{11}^{(0)}(x,t) = \frac{\alpha^{+}k}{4}u_1\left(\tfrac{x+t}{2}\right)a_2\left(\tfrac{x+t}{2}\right) + \frac{\alpha^{-}k}{4}u_1\left(d-\tfrac{x-t}{2}\right)a_2\left(d-\tfrac{x-t}{2}\right)$$
$$-\frac{\alpha^{+}}{4k}\left(\tfrac{x+t}{2}\right)^{-2l}q\left(\tfrac{x+t}{2}\right)a_2\left(\tfrac{x+t}{2}\right) - \frac{\alpha^{-}}{4k}\left(d-\tfrac{x-t}{2}\right)^{-2l}q\left(d-\tfrac{x-t}{2}\right)a_2\left(d-\tfrac{x-t}{2}\right)$$
$$-\frac{\alpha^{+}k}{4}u_2\left(\tfrac{x+t}{2}\right)a_2\left(\tfrac{x+t}{2}\right) - \frac{\alpha^{-}k}{4}u_2\left(d-\tfrac{x-t}{2}\right)a_2\left(d-\tfrac{x-t}{2}\right)$$

$$K_{11}^{(n)}(x,t) = -\frac{\alpha^{+}k}{2}\int_{\frac{x-t}{2}}^{d} K_{12}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta + \frac{\alpha^{+}k}{2}\int_{\frac{x+t}{2}}^{d} K_{12}^{(n-1)}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta$$
$$+\frac{\alpha^{-}k}{2}\int_{d-\frac{x-t}{2}}^{d} K_{12}^{(n-1)}(\zeta,t+2d-x-\zeta)u_1(\zeta)d\zeta - \frac{\alpha^{-}k}{2}\int_{d-\frac{x+t}{2}}^{d} K_{12}^{(n-1)}(\zeta,t+x+\zeta-2d)u_1(\zeta)d\zeta$$
$$+\frac{\alpha^{+}k}{2}\int_{\frac{x-t}{2}}^{d} K_{21}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta + \frac{\alpha^{+}k}{2}\int_{\frac{x+t}{2}}^{d} K_{21}^{(n-1)}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta$$
$$+\frac{\alpha^{-}k}{2}\int_{d-\frac{x-t}{2}}^{d} K_{21}^{(n-1)}(\zeta,t+2d-x-\zeta)u_2(\zeta)d\zeta + \frac{\alpha^{-}k}{2}\int_{d-\frac{x+t}{2}}^{d} K_{21}^{(n-1)}(\zeta,t+x+\zeta-2d)u_2(\zeta)d\zeta$$
$$+\frac{\alpha^{+}}{2}\int_{0}^{d} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{11}^{(n-1)}(s,\zeta)d\zeta ds - \frac{\alpha^{-}}{2}\int_{0}^{d} q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d} K_{11}^{(n-1)}(s,\zeta)d\zeta ds$$
$$-\frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{12}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta - \frac{k}{2}\int_{d}^{x} K_{12}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta$$
$$+\frac{k}{2}\int_{d}^{x} K_{12}^{(n-1)}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta + \frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{21}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta$$
$$+\frac{k}{2}\int_{d}^{x} K_{21}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta + \frac{k}{2}\int_{d}^{x} K_{21}^{(n-1)}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta$$

$$+\frac{1}{2}\int_{d}^{x} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{11}(s,\zeta)d\zeta ds$$

$$K_{12}^{(0)}(x,t) = -\frac{\alpha^{+}k}{4}u_1\left(\tfrac{x+t}{2}\right)a_1\left(\tfrac{x+t}{2}\right) - \frac{\alpha^{-}k}{4}u_1\left(d-\tfrac{x-t}{2}\right)a_1\left(d-\tfrac{x-t}{2}\right)$$

$$+\frac{\alpha^{+}}{4k}\left(\tfrac{x+t}{2}\right)^{-2l}q\left(\tfrac{x+t}{2}\right)a_1\left(\tfrac{x+t}{2}\right) + \frac{\alpha^{-}}{4k}\left(d-\tfrac{x-t}{2}\right)^{-2l}q\left(d-\tfrac{x-t}{2}\right)a_1\left(d-\tfrac{x-t}{2}\right)$$

$$-\frac{\alpha^{+}k}{4}u_1\left(\tfrac{x+t}{2}\right) - \frac{\alpha^{-}k}{4}u_1\left(d-\tfrac{x-t}{2}\right) + \frac{\alpha^{+}k}{4}u_2\left(\tfrac{x+t}{2}\right) + \frac{\alpha^{-}k}{4}u_2\left(d-\tfrac{x-t}{2}\right) + \frac{\alpha^{+}k}{4}u_2\left(\tfrac{x+t}{2}\right)a_1\left(\tfrac{x+t}{2}\right)$$

$$+\frac{\alpha^{-}k}{4}u_2\left(d-\tfrac{x-t}{2}\right)a_1\left(d-\tfrac{x-t}{2}\right) + \frac{\alpha^{+}}{4k}\left(\tfrac{x+t}{2}\right)^{-2l}q\left(\tfrac{x+t}{2}\right) + \frac{\alpha^{-}}{4k}\left(d-\tfrac{x-t}{2}\right)^{-2l}q\left(d-\tfrac{x-t}{2}\right)$$

$$K_{12}^{(n)}(x,t) = \frac{\alpha^{+}k}{2}\int_{\frac{x-\zeta}{2}}^{d} K_{11}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta - \frac{\alpha^{+}k}{2}\int_{\frac{x+t}{2}}^{d} K_{11}^{(n-1)}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta$$

$$-\frac{\alpha^{-}k}{2}\int_{d-\frac{x-t}{2}}^{d} K_{11}^{(n-1)}(\zeta,t+2d-x-\zeta)u_1(\zeta)d\zeta + \frac{\alpha^{-}k}{2}\int_{d-\frac{x+t}{2}}^{d} K_{11}^{(n-1)}(\zeta,t+x+\zeta-2d)u_1(\zeta)d\zeta$$

$$+\frac{\alpha^{+}k}{2}\int_{\frac{x-t}{2}}^{d} K_{22}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta + \frac{\alpha^{+}k}{2}\int_{\frac{x+t}{2}}^{d} K_{22}^{(n-1)}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta$$

$$+\frac{\alpha^{-}k}{2}\int_{d-\frac{x-t}{2}}^{d} K_{22}^{(n-1)}(\zeta,t+2d-x-\zeta)u_2(\zeta)d\zeta + \frac{\alpha^{-}k}{2}\int_{d-\frac{x+t}{2}}^{d} K_{22}^{(n-1)}(\zeta,t+x+\zeta-2d)u_2(\zeta)d\zeta$$

$$+\frac{\alpha^{+}}{2}\int_{0}^{d} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{12}^{(n-1)}(s,\zeta)d\zeta ds - \frac{\alpha^{-}}{2}\int_{0}^{d} q(s)s^{-2\ell}\int_{t-x-s+2d}^{t+x+s-2d} K_{12}^{(n-1)}(s,\zeta)d\zeta ds$$

$$+\frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{11}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta + \frac{k}{2}\int_{d}^{x} K_{11}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta$$

$$-\frac{k}{2}\int_{d}^{x} K_{11}^{(n-1)}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta + \frac{k}{2}\int_{d}^{x} K_{12}^{(n-1)}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta$$

$$+\frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{22}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta + \frac{k}{2}\int_{d}^{x} K_{22}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta$$

$$+\frac{k}{2}\int_{d}^{x} K_{22}^{(n-1)}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta + \frac{1}{2}\int_{d}^{x} q(s)s^{-2\ell}\int_{t-x+s}^{t+x-s} K_{12}^{(n-1)}(s,\zeta)d\zeta ds$$

$$K_{21}^{(0)}(x,t) = -\frac{\alpha^+ k}{4} u_1(\tfrac{x+t}{2}) a_1(\tfrac{x+t}{2}) - \frac{\alpha^- k}{4} u_1(d - \tfrac{x-t}{2}) a_1(d - \tfrac{x-t}{2})$$

$$+\frac{\alpha^+}{4k}(\tfrac{x+t}{2})^{-2l} q(\tfrac{x+t}{2}) a_1(\tfrac{x+t}{2}) + \frac{\alpha^-}{4k}(d - \tfrac{x-t}{2})^{-2l} q(d - \tfrac{x-t}{2}) a_1(d - \tfrac{x-t}{2})$$

$$-\frac{\alpha^+ k}{4} u_1(\tfrac{x+t}{2}) + \frac{\alpha^- k}{4} u_1(d - \tfrac{x-t}{2}) + \frac{\alpha^+ k}{4} u_2(\tfrac{x+t}{2}) - \frac{\alpha^- k}{4} u_2(d - \tfrac{x-t}{2}) + \frac{\alpha^+ k}{4} u_2(\tfrac{x+t}{2}) a_1(\tfrac{x+t}{2})$$

$$-\frac{\alpha^- k}{4} u_2(d - \tfrac{x-\zeta}{2}) a_1(d - \tfrac{x-\zeta}{2}) + \frac{\alpha^+}{4k}(\tfrac{x+t}{2})^{-2l} q(\tfrac{x+t}{2}) - \frac{\alpha^-}{4k}(d - \tfrac{x-t}{2})^{-2l} q(d - \tfrac{x-t}{2})$$

$$K_{21}^{(n)}(x,t) = -\frac{\alpha^+ k}{2}\int_{\frac{x-t}{2}}^{d} K_{11}^{(n-1)}(\zeta, t+\zeta-x) u_1(\zeta)\,d\zeta - \frac{\alpha^+ k}{2}\int_{\frac{x+t}{2}}^{d} K_{11}^{(n-1)}(\zeta, t+x-\zeta) u_1(\zeta)\,d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{d-\frac{x-t}{2}}^{d} K_{11}^{(n-1)}(\zeta, t+2d-x-\zeta) u_1(\zeta)\,d\zeta + \frac{\alpha^- k}{2}\int_{d-\frac{x+t}{2}}^{d} K_{11}^{(n-1)}(\zeta, t+x+\zeta-2d) u_1(\zeta)\,d\zeta$$

$$-\frac{\alpha^+ k}{2}\int_{\frac{x-t}{2}}^{d} K_{22}^{(n-1)}(\zeta, t+\zeta-x) u_2(\zeta)\,d\zeta + \frac{\alpha^+ k}{2}\int_{\frac{x+t}{2}}^{d} K_{22}^{(n-1)}(\zeta, t+x-\zeta) u_2(\zeta)\,d\zeta$$

$$-\frac{\alpha^- k}{2}\int_{d-\frac{x-t}{2}}^{d} K_{22}^{(n-1)}(\zeta, t+2d-x-\zeta) u_2(\zeta)\,d\zeta + \frac{\alpha^- k}{2}\int_{d-\frac{x+t}{2}}^{d} K_{22}^{(n-1)}(\zeta, t+x+\zeta-2d) u_2(\zeta)\,d\zeta$$

$$+\frac{\alpha^+}{2k}\int_{\frac{x-t}{2}}^{d} K_{11}^{(n-1)}(\zeta, t+\zeta-x) q(\zeta)\zeta^{-2\ell}\,d\zeta + \frac{\alpha^+}{2k}\int_{\frac{x+t}{2}}^{d} K_{11}^{(n-1)}(\zeta, t+x-\zeta) q(\zeta)\zeta^{-2\ell}\,d\zeta$$

$$-\frac{\alpha^-}{2k}\int_{d-\frac{x-t}{2}}^{d} K_{11}^{(n-1)}(\zeta, t+2d-x-\zeta)\zeta^{-2\ell} q(\zeta)\,d\zeta - \frac{\alpha^-}{2k}\int_{d-\frac{x+t}{2}}^{d} K_{11}^{(n-1)}(\zeta, t+x+\zeta-2d)\zeta^{-2\ell} q(\zeta)\,d\zeta$$

$$-\frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{11}(\zeta, t+\zeta-x) u_1(\zeta)\,d\zeta - \frac{k}{2}\int_{d}^{x} K_{11}(\zeta, t+\zeta-x) u_1(\zeta)\,d\zeta$$

$$-\frac{k}{2}\int_{d}^{x} K_{11}^{(n-1)}(\zeta, t+x-\zeta) u_1(\zeta)\,d\zeta - \frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{22}^{(n-1)}(\zeta, t+\zeta-x) u_2(\zeta)\,d\zeta$$

$$-\frac{k}{2}\int_{d}^{x} K_{22}^{(n-1)}(\zeta, t+\zeta-x) u_2(\zeta)\,d\zeta + \frac{k}{2}\int_{d}^{x} K_{22}^{(n-1)}(\zeta, t+x-\zeta) u_2(\zeta)\,d\zeta$$

$$+\frac{1}{2k}\int_{\frac{x-t}{2}}^{x} K_{11}^{(n-1)}(\zeta, t+\zeta-x)\zeta^{-2\ell} q(\zeta)\,d\zeta + \frac{1}{2k}\int_{d}^{x} K_{11}^{(n-1)}(\zeta, t+\zeta-x)\zeta^{-2\ell} q(\zeta)\,d\zeta$$

$$+\frac{1}{2k}\int_d^x K_{11}^{(n-1)}(\zeta, t+x-\zeta)\zeta^{-2\ell} q(\zeta)d\zeta$$

$$K_{22}^{(0)}(x,t) = -\frac{\alpha^+ k}{4} u_1(\tfrac{x+t}{2}) a_2(\tfrac{x+t}{2}) + \frac{\alpha^- k}{4} u_1(d-\tfrac{x-t}{2}) a_2(d-\tfrac{x-t}{2})$$

$$+\frac{\alpha^+}{4k}(\tfrac{x+t}{2})^{-2\ell} q(\tfrac{x+t}{2}) a_2(\tfrac{x+t}{2}) - \frac{\alpha^-}{4k}(d-\tfrac{x-t}{2})^{-2\ell} q(d-\tfrac{x-t}{2}) a_2(d-\tfrac{x-t}{2})$$

$$\frac{\alpha^+ k}{4} u_2(\tfrac{x+t}{2}) a_2(\tfrac{x+t}{2}) - \frac{\alpha^- k}{4} u_2(d-\tfrac{x-t}{2}) a_2(d-\tfrac{x-t}{2})$$

$$K_{22}^{(n)}(x,t) = -\frac{\alpha^+ k}{2}\int_{\frac{x-t}{2}}^d K_{12}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta - \frac{\alpha^+ k}{2}\int_{\frac{x+t}{2}}^d K_{12}^{(n-1)}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{d-\frac{x-t}{2}}^d K_{12}^{(n-1)}(\zeta,t+2d-x-\zeta)u_1(\zeta)d\zeta + \frac{\alpha^- k}{2}\int_{d-\frac{x+t}{2}}^d K_{12}^{(n-1)}(\zeta,t+x+\zeta-2d)u_1(\zeta)d\zeta$$

$$+\frac{\alpha^+ k}{2}\int_{\frac{x-t}{2}}^d K_{21}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta - \frac{\alpha^+ k}{2}\int_{\frac{x+t}{2}}^d K_{21}^{(n-1)}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta$$

$$+\frac{\alpha^- k}{2}\int_{d-\frac{x-t}{2}}^d K_{21}^{(n-1)}(\zeta,t+2d-x-\zeta)u_2(\zeta)d\zeta - \frac{\alpha^- k}{2}\int_{d-\frac{x+t}{2}}^d K_{21}^{(n-1)}(\zeta,t+x+\zeta-2d)u_2(\zeta)d\zeta$$

$$+\frac{\alpha^+}{2k}\int_{\frac{x-t}{2}}^d K_{12}^{(n-1)}(\zeta,t+\zeta-x)q(\zeta)\zeta^{-2\ell}d\zeta + \frac{\alpha^+}{2k}\int_{\frac{x+t}{2}}^d K_{12}^{(n-1)}(\zeta,t+x-\zeta)q(\zeta)\zeta^{-2\ell}d\zeta$$

$$-\frac{\alpha^-}{2k}\int_{d-\frac{x-t}{2}}^d K_{12}^{(n-1)}(\zeta,t+2d-x-\zeta)\zeta^{-2\ell}q(\zeta)d\zeta - \frac{\alpha^-}{2k}\int_{d-\frac{x+t}{2}}^d K_{12}^{(n-1)}(\zeta,t+x+\zeta-2d)\zeta^{-2\ell}q(\zeta)d\zeta$$

$$-\frac{k}{2}\int_{\frac{x-t}{2}}^x K_{12}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta - \frac{k}{2}\int_d^x K_{12}^{(n-1)}(\zeta,t+\zeta-x)u_1(\zeta)d\zeta$$

$$-\frac{k}{2}\int_d^x K_{12}^{(n-1)}(\zeta,t+x-\zeta)u_1(\zeta)d\zeta + \frac{k}{2}\int_{\frac{x-t}{2}}^x K_{21}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta$$

$$+\frac{k}{2}\int_d^x K_{21}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta - \frac{k}{2}\int_d^x K_{21}^{(n-1)}(\zeta,t+x-\zeta)u_2(\zeta)d\zeta$$

$$-\frac{k}{2}\int_{\frac{x-t}{2}}^{x} K_{22}^{(n-1)}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta-\frac{k}{2}\int_{d}^{x} K_{22}(\zeta,t+\zeta-x)u_2(\zeta)d\zeta$$

$$+\frac{1}{2k}\int_{\frac{x-t}{2}}^{x} K_{12}^{(n-1)}(\zeta,t+\zeta-x)\zeta^{-2\ell}q(\zeta)d\zeta+\frac{1}{2k}\int_{d}^{x} K_{12}^{(n-1)}(\zeta,t+\zeta-x)\zeta^{-2\ell}q(\zeta)d\zeta$$

$$+\frac{1}{2k}\int_{d}^{x} K_{12}^{(n-1)}(\zeta,t+x-\zeta)\zeta^{-2\ell}q(\zeta)d\zeta$$

olarak alınsın. Şimdi, Yukarıdaki denklemlerde mutlak değer alınıp eşitliğin her iki tarafı $[-x,x]$ aralığında t'ye göre integrallenirse;

$$\int_{-x}^{x}\left|K_{11}^{(0)}(x,t)\right|dt \le \frac{\alpha^{+}|k|}{2}\int_{0}^{x}|u_1(\zeta)a_2(\zeta)|\,d\zeta+\frac{\alpha^{-}|k|}{2}\int_{d-x}^{d}|u_1(\zeta)a_2(\zeta)|\,d\zeta$$

$$+\frac{\alpha^{+}}{2|k|}\int_{0}^{x}\zeta^{-2l}|q(\zeta)a_2(\zeta)|\,d\zeta+\frac{|\alpha^{-}|}{2|k|}\int_{d-x}^{d}(\zeta)^{-2l}|q(\zeta)a_2(\zeta)|\,d\zeta$$

$$+\frac{\alpha^{+}|k|}{2}\int_{0}^{x}|u_2(\zeta)a_2(\zeta)|\,d\zeta+\frac{|\alpha^{-}||k|}{2}\int_{d-x}^{d}|u_2(\zeta)a_2(\zeta)|\,d\zeta$$

$$\int_{-x}^{x}\left|K_{12}^{(0)}(x,t)\right|dt \le \frac{\alpha^{+}|k|}{2}\int_{0}^{x}|u_1(\zeta)a_1(\zeta)|\,d\zeta+\frac{|\alpha^{-}||k|}{2}\int_{d-x}^{d}|u_1(\zeta)a_1(\zeta)|\,d\zeta$$

$$+\frac{\alpha^{+}}{2|k|}\int_{0}^{x}(\zeta)^{-2l}|q(\zeta)a_1(\zeta)|\,d\zeta+\frac{|\alpha^{-}|}{2|k|}\int_{d-x}^{d}(\zeta)^{-2l}|q(\zeta)a_1(\zeta)|\,d\zeta$$

$$+\frac{\alpha^{+}|k|}{2}\int_{0}^{x}|u_1(\zeta)|\,d\zeta+\frac{|\alpha^{-}||k|}{2}\int_{d-x}^{d}|u_1(\zeta)|\,d\zeta+\frac{\alpha^{+}|k|}{2}\int_{0}^{x}|u_2(\zeta)|\,d\zeta+\frac{|\alpha^{-}||k|}{2}\int_{d-x}^{d}|u_2(\zeta)|\,d\zeta$$

$$+\frac{\alpha^{+}|k|}{2}\int_{0}^{x}|u_2(\zeta)a_1(\zeta)d\zeta|+\frac{|\alpha^{-}||k|}{2}\int_{d-x}^{d}|u_2(\zeta)a_1(\zeta)|\,d\zeta+\frac{\alpha^{+}}{2|k|}\int_{0}^{x}\zeta^{-2l}|q(\zeta)|\,d\zeta+\frac{|\alpha^{-}|}{2|k|}\int_{d-x}^{d}\zeta^{-2l}|q(\zeta)|\,d\zeta$$

$$\int_{-x}^{x}\left|K_{21}^{(0)}(x,t)\right|dt \le \frac{\alpha^{+}|k|}{2}\int_{0}^{x}|u_1(\zeta)a_1(\zeta)|\,d\zeta+\frac{|\alpha^{-}||k|}{2}\int_{d-x}^{d}|u_1(\zeta)a_1(\zeta)|\,d\zeta$$

$$+\frac{\alpha^{+}}{2|k|}\int_{0}^{x}(\zeta)^{-2l}|q(\zeta)a_1(\zeta)|\,d\zeta+\frac{|\alpha^{-}|}{2|k|}\int_{d-x}^{d}(\zeta)^{-2l}|q(\zeta)a_1(\zeta)|\,d\zeta+\frac{\alpha^{+}|k|}{2}\int_{0}^{x}|u_1(\zeta)|\,d\zeta$$

$$+\frac{|\alpha^-||k|}{2}\int_{d-x}^{d}|u_1(\zeta)|\,d\zeta+\frac{\alpha^+|k|}{2}\int_0^x|u_2(\zeta)|\,d\zeta+\frac{|\alpha^-||k|}{2}\int_{d-x}^{d}|u_2(\zeta)|\,d\zeta+\frac{\alpha^+|k|}{2}\int_0^x|u_2(\zeta)a_1(\zeta)|\,d\zeta$$

$$+\frac{|\alpha^-||k|}{2}\int_0^x|u_2(\zeta)a_1(\zeta)|\,d\zeta+\frac{\alpha^+}{2|k|}\int_0^x\zeta^{-2l}|q(\zeta)|\,d\zeta+\frac{|\alpha^-|}{2|k|}\int_{d-x}^{d}\zeta^{-2l}|q(\zeta)|\,d\zeta$$

$$\int_{-x}^{x}\left|K_{22}^{(0)}(x,t)\right|dt\leq\frac{\alpha^+|k|}{2}\int_0^x|u_1(\zeta)a_2(\zeta)|\,d\zeta+\frac{|\alpha^-||k|}{2}\int_{d-x}^{d}|u_1(\zeta)a_2(\zeta)|\,d\zeta$$

$$+\frac{\alpha^+}{2|k|}\int_0^x(\zeta)^{-2l}|q(\zeta)a_2(\zeta)|\,d\zeta+\frac{|\alpha^-|}{2|k|}\int_{d-x}^{d}(\zeta)^{-2l}|q(\zeta)a_2(\zeta)|\,d\zeta$$

$$+\frac{\alpha^+|k|}{2}\int_0^x|u_2(\zeta)a_2(\zeta)|\,d\zeta+\frac{|\alpha^-||k|}{2}\int_{d-x}^{d}|u_2(\zeta)a_2(\zeta)|\,d\zeta$$

olur. $a_1(x)$ ve $a_2(x)$ fonksiyonları mutlak sürekli fonksiyonlar ve dolayısıyla sınırlı olduklarından, yani $|a_1(x)|,|a_2(x)|<M$ olacak şekilde bir $M>0$ sayısı var olduğundan,

$$\int_{-x}^{x}\left|K_{11}^{(0)}(x,t)\right|dt\leq M|k|\left(\alpha^++|\alpha^-|\right)\int_0^x\left(|u_1(t)|+|u_2(t)|\right)dt+\frac{M}{2|k|}\left(\alpha^++|\alpha^-|\right)\int_0^x\left|\zeta^{-2l}q(\zeta)\right|d\zeta$$

$$\int_{-x}^{x}\left|K_{12}^{(0)}(x,t)\right|dt\leq(M+1)|k|\left(\alpha^++|\alpha^-|\right)\int_0^x\left(|u_1(t)|+|u_2(t)|\right)dt$$

$$+\frac{M+1}{|k|}\left(\alpha^++|\alpha^-|\right)\int_0^x\left|\zeta^{-2l}q(\zeta)\right|d\zeta$$

$$\int_{-x}^{x}\left|K_{21}^{(0)}(x,t)\right|dt\leq(M+1)|k|\left(\alpha^++|\alpha^-|\right)\int_0^x\left(|u_1(t)|+|u_2(t)|\right)dt$$

$$+\frac{M+1}{|k|}\left(\alpha^++|\alpha^-|\right)\int_0^x\left|\zeta^{-2l}q(\zeta)\right|d\zeta$$

$$\int_{-x}^{x}\left|K_{22}^{(0)}(x,t)\right|dt\leq M|k|\left(\alpha^++|\alpha^-|\right)\int_0^x\left(|u_1(t)|+|u_2(t)|\right)dt+\frac{M}{2|k|}\left(\alpha^++|\alpha^-|\right)\int_0^x\left|\zeta^{-2l}q(\zeta)\right|d\zeta$$

eşitsizlikleri elde edilir. $\max\left((M+1)|k|\left(\alpha^++|\alpha^-|\right),\frac{M+1}{|k|}\left(\alpha^++|\alpha^-|\right)\right)=M_1$ alınırsa;

$$\int_{-x}^{x}\left|K_{ij}^{(0)}(x,t)\right|dt \le M_1\int_0^x\left(|u_1(t)|+|u_2(t)|+t^{-2l}|q(t)|\right)dt = M_1\sigma_1(x)$$

eşitsizlikleri elde edilir. Burada $\sigma_1(x)=\int_0^x\left(|u_1(t)|+|u_2(t)|+t^{-2l}|q(t)|\right)dt$ ve $i,j=1,2$ dir.

$K_{ij}^{(n)}(x,t)$ $(i,j=1,2)$ ifadelerinin mutlak değeri alınır, eşitliğin her iki tarafı $[-x,x]$ aralığında t ye göre integrallenir ve gerekli işlemler yapılırsa ,

$$\begin{aligned}
\int_{-x}^{x}\left|K_{11}^{(n)}(x,t)\right|dt \le\ & \frac{\alpha^+|k|}{2}\int_0^d|u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|dsd\zeta+\frac{\alpha^+|k|}{2}\int_0^d|u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|dsd\zeta\\
&+\frac{\alpha^-|k|}{2}\int_{d-x}^{x}|u_1(\zeta)|\int_{2d-2x-\zeta}^{\zeta}|K(\zeta,s)|\,dsd\zeta+\frac{|\alpha^-||k|}{2}\int_{d-x}^{x}|u_1(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{12}^{(n-1)}(\zeta,s)\right|dsd\zeta\\
&+\frac{\alpha^+|k|}{2}\int_0^d|u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right|dsd\zeta+\frac{\alpha^+|k|}{2}\int_0^d|u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right|dsd\zeta\\
&+\frac{|\alpha^-||k|}{2}\int_{d-x}^{x}|u_2(\zeta)|\int_{2d-2x-\zeta}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right|dsd\zeta+\frac{|\alpha^-||k|}{2}\int_{d-x}^{x}|u_2(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{21}^{(n-1)}(\zeta,s)\right|dsd\zeta\\
&+\frac{\alpha^+}{2}\int_0^d|q(s)|s^{-2\ell}\left\{\int_{s-2x}^{2x-s}\int_{\zeta-x+s}^{x-s}\left|K_{11}^{(n-1)}(s,\zeta)\right|dtd\zeta\right\}ds\\
&+\frac{|\alpha^-|}{2}\int_0^d|q(s)|s^{-2\ell}\left\{\int_{2d-2x-s}^{2x+s-2d}\int_{\zeta-x-s+2d}^{\zeta+x+s-2d}\left|K_{11}^{(n-1)}(s,\zeta)\right|dtd\zeta\right\}ds\\
&+\frac{|k|}{2}\int_0^x|u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|dsd\zeta+\frac{|k|}{2}\int_d^x|u_1(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|dsd\zeta\\
&+\frac{|k|}{2}\int_d^x u_1(\zeta)\int_{-\zeta}^{2x-\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|dsd\zeta+\frac{|k|}{2}\int_0^x u_2(\zeta)\int_{-\zeta}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right|dsd\zeta\\
&+\frac{|k|}{2}\int_d^x|u_2(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right|dsd\zeta+\frac{|k|}{2}\int_d^x|u_2(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right|dsd\zeta\\
&+\frac{1}{2}\int_d^x|q(s)|s^{-2\ell}\left\{\int_{s-2x}^{2x-s}\int_{\zeta-x+s}^{x-s}\left|K_{11}^{(n-1)}(s,\zeta)\right|dtd\zeta\right\}ds
\end{aligned}$$

$$\int_{-x}^{x}\left|K_{12}^{(n)}(x,t)\right|dt \le \frac{\alpha^+ k}{2}\int_0^d |u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{\alpha^+ k}{2}\int_0^d |u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{|\alpha^-||k|}{2}\int_{d-x}^{d} u_1(\zeta)\int_{2d-2x-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{|\alpha^-| k}{2}\int_{d-x}^{x} |u_1(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{\alpha^+ |k|}{2}\int_0^d |u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{\alpha^+ k}{2}\int_0^d |u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{|\alpha^-||k|}{2}\int_{d-x}^{d} |u_2(\zeta)|\int_{2d-2x-\zeta}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{|\alpha^-||k|}{2}\int_{d-x}^{d} |u_2(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{22}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{\alpha^+}{2}\int_0^d q(s)s^{-2\ell}\left\{\int_{s-2x}^{2x-s}\int_{\zeta-x+s}^{\zeta+x-s}\left|K_{12}^{(n-1)}(s,\zeta)\right|dtd\zeta\right\}ds$$

$$-\frac{|\alpha^-|}{2}\int_0^d |q(s)|\, s^{-2\ell}\left\{\int_{2d-2x-s}^{2x-s-2d}\int_{\zeta-x-s+2d}^{\zeta+x+s-2d}\left|K_{12}^{(n-1)}(s,\zeta)\right|dtd\zeta\right\}ds$$

$$+\frac{|k|}{2}\int_0^x |u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{|k|}{2}\int_d^x |u_1(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{|k|}{2}\int_d^x |u_1(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{|k|}{2}\int_d^x |u_1(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{|k|}{2}\int_0^x |u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{|k|}{2}\int_d^x |u_2(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{|k|}{2}\int_d^x |u_2(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{1}{2}\int_d^x |q(s)|\, s^{-2\ell}\left\{\int_{s-2x}^{2x-s}\int_{\zeta-x+s}^{\zeta+x-s}\left|K_{12}^{(n-1)}(s,\zeta)\right|dtd\zeta\right\}ds$$

$$\int_{-x}^{x}\left|K_{21}^{(n)}(x,t)\right|dt \le \frac{\alpha^+ |k|}{2}\int_0^d |u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{\alpha^+ |k|}{2}\int_0^d |u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{|\alpha^-||k|}{2}\int_{d-x}^{d} |u_1(\zeta)|\int_{2d-2x-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta + \frac{|\alpha^-||k|}{2}\int_{d-x}^{d} |u_1(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{11}^{(n-1)}(\zeta,s)\right|dsd\zeta$$

$$+\frac{\alpha^+|k|}{2}\int_0^d |u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{\alpha^+|k|}{2}\int_0^d |u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{|\alpha^-||k|}{2}\int_{d-x}^d |u_2(\zeta)|\int_{2d-2x-\zeta}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{|\alpha^-||k|}{2}\int_{d-x}^d |u_2(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{22}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{\alpha^+}{2|k|}\int_0^d |q(\zeta)|\,\zeta^{-2\ell}\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{\alpha^+}{2|k|}\int_0^d |q(\zeta)|\,\zeta^{-2\ell}\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{|\alpha^-|}{2|k|}\int_{d-x}^d \zeta^{-2\ell}|q(\zeta)|\int_{2d-2x-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{|\alpha^-|}{2|k|}\int_{d-x}^d \zeta^{-2\ell}|q(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{|k|}{2}\int_0^x |u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{|k|}{2}\int_d^x |u_1(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| d\zeta dt$$

$$+\frac{|k|}{2}\int_d^x |u_1(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{|k|}{2}\int_0^x |u_2(\zeta)|\int_{-\zeta}^{\zeta}|K_{22}(\zeta,s)|\, ds d\zeta$$

$$+\frac{|k|}{2}\int_d^x |u_2(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{|k|}{2}\int_d^x |u_2(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{22}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{1}{2|k|}\int_0^x \zeta^{-2\ell}|q(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{1}{2|k|}\int_d^x \zeta^{-2\ell}|q(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{1}{2|k|}\int_d^x \zeta^{-2\ell}|q(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{11}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$\int_{-x}^{x}\left|K_{22}^{(n)}(x,t)\right| dt \le \frac{\alpha^+|k|}{2}\int_0^d |u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{\alpha^+|k|}{2}\int_0^d |u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{|\alpha^-||k|}{2}\int_{d-x}^d |u_1(\zeta)|\int_{2d-2x-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{|\alpha^-||k|}{2}\int_{d-x}^d |u_1(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{12}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{\alpha^+|k|}{2}\int_0^d |u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{\alpha^+|k|}{2}\int_0^d |u_2(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right| ds d\zeta+\frac{|\alpha^-||k|}{2}\int_{d-x}^d |u_2(\zeta)|\int_{2d-2x-\zeta}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right| ds d\zeta$$

$$+\frac{|\alpha^-||k|}{2}\int_{d-x}^{d}|u_2(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{21}^{(n-1)}(\zeta,s)\right|ds\,d\zeta+\frac{\alpha^+}{2|k|}\int_{0}^{d}|q(\zeta)|\zeta^{-2\ell}\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta$$

$$+\frac{\alpha^+}{2|k|}\int_{0}^{d}|q(\zeta)|\zeta^{-2\ell}\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta-\frac{|\alpha^-|}{2|k|}\int_{d-x}^{d}\zeta^{-2\ell}|q(\zeta)|\int_{2d-2x-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta$$

$$+\frac{|\alpha^-|}{2|k|}\int_{d-x}^{d}\zeta^{-2\ell}|q(\zeta)|\int_{-\zeta}^{2x-\zeta-2d}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta+\frac{|k|}{2}\int_{0}^{x}|u_1(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta$$

$$+\frac{|k|}{2}\int_{d}^{x}|u_1(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta+\frac{|k|}{2}\int_{d}^{x}|u_1(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta$$

$$+\frac{|k|}{2}\int_{0}^{x}|u_2(\zeta)|\int_{-\zeta}^{\zeta}|K_{21}(\zeta,s)|\,ds\,d\zeta+\frac{|k|}{2}\int_{d}^{x}|u_2(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right|ds\,d\zeta$$

$$+\frac{|k|}{2}\int_{d}^{x}|u_2(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{21}^{(n-1)}(\zeta,s)\right|ds\,d\zeta+\frac{|k|}{2}\int_{0}^{x}|u_2(\zeta)|\int_{-\zeta}^{\zeta}|K_{22}(\zeta,s)|\,ds\,d\zeta$$

$$+\frac{|k|}{2}\int_{d}^{x}|u_2(\zeta)|\int_{\zeta-2x}^{\zeta}|K_{22}(\zeta,s)|\,ds\,d\zeta+\frac{1}{2|k|}\int_{0}^{x}\zeta^{-2\ell}|q(\zeta)|\int_{-\zeta}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta$$

$$+\frac{1}{2|k|}\int_{d}^{x}\zeta^{-2\ell}|q(\zeta)|\int_{\zeta-2x}^{\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta+\frac{1}{2|k|}\int_{d}^{x}\zeta^{-2\ell}|q(\zeta)|\int_{-\zeta}^{2x-\zeta}\left|K_{12}^{(n-1)}(\zeta,s)\right|ds\,d\zeta$$

olur. Ardışık yaklaşımlar metodu kullanılırsa;

$n=1$ için:

$$\int_{-x}^{x}\left|K_{11}^{(1)}(x,t)\right|dt\le\frac{\alpha^+|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^+|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^-||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^-||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{\alpha^+|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^+|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^-||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^-||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^+}{2}M_1c_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|\alpha^-|}{2}M_1c_2\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{1}{2}M_1c_1\frac{\sigma_1^2(x)}{2!}=\left(2\alpha^+|k|+2|\alpha^-||k|+3|k|+\frac{1+\alpha^+}{2}c_1+\frac{|\alpha^-|}{2}c_2\right)M_1\frac{\sigma_1^2(x)}{2!}$$

$$\int_{-x}^{x}\left|K_{12}^{(1)}(x,t)\right|dt\le\frac{\alpha^+|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^+|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^-||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^-||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{\alpha^{+}|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^{+}|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^{+}}{2}M_1c_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|\alpha^{-}|}{2}M_1c_2\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{k}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{k}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{1}{2}M_1c_1\frac{\sigma_1^2(x)}{2!}$$

$$=\left(2\alpha^{+}|k|+2|\alpha^{-}||k|+\frac{7|k|}{2}+\frac{1+\alpha^{+}}{2}c_1+\frac{|\alpha^{-}|}{2}c_2\right)M_1\frac{\sigma_1^2(x)}{2!}$$

$$\int_{-x}^{x}\left|K_{21}^{(1)}(x,t)\right|dt\le\frac{\alpha^{+}|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^{+}|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{\alpha^{+}}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^{+}}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}|}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}|}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{1}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{1}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{1}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}$$

$$=\left(\alpha^{+}|k|+2|\alpha^{-}||k|+\frac{\alpha^{+}}{|k|}+\frac{|\alpha^{-}|}{|k|}+3|k|+\frac{3}{2|k|}\right)M_1\frac{\sigma_1^2(x)}{2!}$$

$$\int_{-x}^{x}\left|K_{22}^{(1)}(x,t)\right|dt\le\frac{\alpha^{+}|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^{+}|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}||k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^{+}}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{\alpha^{+}}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|\alpha^{-}|}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|\alpha^{-}|}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}$$

$$+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{|k|}{2}M_1\frac{\sigma_1^2(x)}{2!}+\frac{1}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{1}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}+\frac{1}{2|k|}M_1\frac{\sigma_1^2(x)}{2!}$$

$$=\left(\alpha^{+}|k|+2|\alpha^{-}||k|+\frac{\alpha^{+}}{|k|}+\frac{|\alpha^{-}|}{|k|}+4|k|+\frac{3}{2|k|}\right)M_1\frac{\sigma_1^2(x)}{2!}$$

$$\left(2\alpha^{+}|k|+2|\alpha^{-}||k|+4|k|+\frac{1+\alpha^{+}}{2}c_1+\frac{|\alpha^{-}|}{2}c_2+\frac{\alpha^{+}}{|k|}+\frac{|\alpha^{-}|}{|k|}+\frac{3}{2|k|}\right)M_1=C$$

alınırsa $\int_{-x}^{x}\left|K_{ij}^{(1)}(x,t)\right|dt\le C^2\frac{\sigma_1^2(x)}{2!}$ **eşitsizliği elde edilir. Aynı şekilde** $n=2$ **için** $\int_{-x}^{x}\left|K_{ij}^{(2)}(x,t)\right|dt\le C^3\frac{\sigma_1^3(x)}{3!}$ **elde edilir.** $\int_{-x}^{x}\left|K_{ij}^{(n)}(x,t)\right|dt\le C^{n+1}\frac{\sigma_1^{n+1}(x)}{(n+1)!}$ **eşitsizliğinin**

doğruluğunu göstermek için tümevarım yöntemi kullanılırsa kolayca

$$\int_{-x}^{x}\left|K_{ij}^{(n)}(x,t)\right|dt \le C^{n+1}\frac{\sigma_1^{n+1}(x)}{(n+1)!}$$

eşitsizliğinin doğru olduğu gösterilebilir. Aynı işlemler yapılırsa diğer aralıklar için de bu eşitsizlik elde edilir. Bu eşitsizliklerden $\sum\limits_{n=0}^{\infty}\int_{-x}^{x}\left|K_{ij}^{(n)}(x,t)\right|dt$ serisinin $L_1(0,\pi)$ uzayında düzgün yakınsak olduğu açıktır ve serinin toplamı $K_{ij}(x,t)\in L_1(0,\pi)$ fonksiyonu aşağıdaki eşitsizliği sağlar;

$$\sum_{n=0}^{\infty}\int_{-x}^{x}\left|K_{ij}^{(n)}(x,t)\right|dt \le e^{C\sigma_1(x)}-1$$

$x>d$ ise,

$$y_1=y_{10}+a(x)e^{ikx}+b(x)e^{ik(2d-x)}+\int_{-x}^{x}K_{11}(x,t)e^{ikt}dt+i\int_{-x}^{x}K_{12}(x,t)e^{ikt}dt$$

$$y_3=y_{30}+ia(x)e^{ikx}-ib(x)e^{ik(2d-x)}+\int_{-x}^{x}K_{21}(x,t)e^{ikt}dt+i\int_{-x}^{x}K_{22}(x,t)e^{ikt}dt$$

şeklinde gösterimin varlığı gösterildi. Burada $\begin{pmatrix}y_{10}\\ y_{30}\end{pmatrix}=\begin{pmatrix}\alpha^{+}e^{ikx}+\alpha^{-}e^{ik(2d-x)}\\ i\alpha^{+}e^{ikx}-i\alpha^{-}e^{ik(2d-x)}\end{pmatrix}$.

O halde aşağıdaki teorem ispatlanmış olur.

Teorem 2.2.1 $\int_0^{\pi}|q(t)|\,t^{-2l}dt<+\infty$ olsun. (2.1.1) diferansiyel denklemleri sisteminin $\begin{pmatrix}y_1\\ y_3\end{pmatrix}'(0)=\begin{pmatrix}1\\ i\end{pmatrix}$ başlangıç ve (2.1.4) süreksizlik koşullarını sağlayan her bir çözümü

$$\begin{pmatrix}y_1\\ y_3\end{pmatrix}=\begin{pmatrix}y_{10}+a(x)e^{ikx}+b(x)e^{ik(2d-x)}+\int_{-x}^{x}K_{11}(x,t)e^{ikt}dt+i\int_{-x}^{x}K_{12}(x,t)e^{ikt}dt\\ y_{30}+ia(x)e^{ikx}-ib(x)e^{ik(2d-x)}+\int_{-x}^{x}K_{21}(x,t)e^{ikt}dt+i\int_{-x}^{x}K_{22}(x,t)e^{ikt}dt\end{pmatrix}$$

şeklinde gösterime sahiptir, öyleki $\sigma(x)=\int_0^x\left(|u_1(t)|+|u_2(t)|+t^{-2l}|q(t)|\right)dt$ olmak üzere

$$\int_{-x}^{x}|K_{ij}(x,t)|\,dt=\sum_{n=0}^{\infty}\int_{-x}^{x}\left|K_{ij}^{(n)}(x,t)\right|dt\le e^{C\sigma_1(x)}-1$$

eşitsizliği sağlanır. Burada $a(x), b(x) \in AC(0,\pi]$, $\begin{pmatrix} y_{10} \\ y_{30} \end{pmatrix} = \begin{pmatrix} \alpha^{+} e^{ikx} + \alpha^{-} e^{ik(2d-x)} \\ i\alpha^{+} e^{ikx} - i\alpha^{-} e^{ik(2d-x)} \end{pmatrix}$,

$$\max\left(\frac{M+2}{2}|k|\left(\alpha^{+}+|\alpha^{-}|\right), \frac{1}{2|k|}\left(\alpha^{+}+|\alpha^{-}|\right)\right)\left(2\alpha^{+}|k| + 2|\alpha^{-}||k| + 4|k| + \frac{1+\alpha^{+}}{2}c_1 + \frac{|\alpha^{-}|}{2}c_2 + \frac{\alpha^{+}}{|k|} + \frac{|\alpha^{-}|}{|k|} + \frac{3}{2|k|}\right) = C$$ şeklindedir.

III. BÖLÜM

3.1. Karakteristik Fonksiyon ve Özellikleri

Bu bölümde L probleminin spektrumunun özellikleri araştırılacaktır. $q(x) = 0$ olması durumunda L problemi L_0 ile gösterilsin. $\varphi(x,k) = \begin{pmatrix} \varphi_1(x,k) \\ \varphi_3(x,k) \end{pmatrix}$ fonksiyonu $\varphi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ başlangıç koşulu ile (2.1.4) süreksizlik koşulunu sağlayan çözüm olsun. $q(x) = 0$ durumunda bu çözüm $\varphi_0(x,k)$ ile, $a(x),\ b(x),$ $y_1(x,k)$ ve $K_{ij}(x,t)$ fonksiyonları sırasıyla $a_0(x),\ b_0(x),\ y_1^0(x,k)$ ve $K_{ij0}(x,t)$ $(i,j = 1,2)$ şeklinde gösterilsin. öyle ki, $k \in R$ için

$x < d$ iken

$$\varphi_{01}(x,k) = \frac{y_1^0(x,k) - \overline{y_1^0(x,k)}}{2i} = \sin kx + a_{10}(x)\sin kx + a_{20}(x)\cos kx$$

$$+ \int_0^x \tilde{K}_{110}(x,t)\sin kt\,dt + \int_0^x \tilde{K}_{120}(x,t)\cos kt\,dt$$

ve

$$\varphi_{03}(x,k) = \frac{y_3^0(x,k) - \overline{y_3^0(x,k)}}{2i} = \cos kx + a_{10}(x)\cos kx - a_{20}(x)\sin kx$$

$$+ \int_0^x \tilde{K}_{210}(x,t)\sin kt\,dt + \int_0^x \tilde{K}_{220}(x,t)\cos kt\,dt$$

$x > d$ iken

$$\varphi_{01}(x,k) = \frac{y_1^0(x,k) - \overline{y_1^0(x,k)}}{2i} = \frac{y_{10}(x,k) - \overline{y_{10}(x,k)}}{2i} + \frac{a_0(x)e^{ikx} - \overline{a_0(x)e^{ikx}}}{2i}$$

$$+ \frac{b_0(x)e^{ik(2d-x)} - \overline{b_0(x)e^{ik(2d-x)}}}{2i} + \int_{-x}^x K_{110}(x,t)\frac{e^{ikt} - e^{-ikt}}{2i}dt + \int_{-x}^x K_{120}(x,t)\frac{e^{ikt} + e^{-ikt}}{2}dt$$

$$= \alpha^+ \sin kx + \alpha^- \sin k(2d-x) + a_{10}(x)\sin kx + a_{20}(x)\cos kx + b_{10}(x)\sin k(2d-x)$$

$$+ b_{20}(x)\cos k(2d-x) + \int_0^x \tilde{K}_{110}(x,t)\sin kt\,dt + \int_0^x \tilde{K}_{120}(x,t)\cos kt\,dt$$

ve

$$\varphi_{03}(x,k) = \frac{y_3^0(x,k) - \overline{y_3^0(x,k)}}{2i} = \frac{y_{30}(x,k) - \overline{y_{30}(x,k)}}{2i} + \frac{ia_0(x)e^{ikx} - \overline{ia_0(x)e^{ikx}}}{2i}$$

$$- \frac{ib_0(x)e^{ik(2d-x)} - \overline{ib_0(x)e^{ik(2d-x)}}}{2i} + \int_{-x}^x K_{210}(x,t)\frac{e^{ikt} - e^{-ikt}}{2i}dt + \int_{-x}^x K_{220}(x,t)\frac{e^{ikt} + e^{-ikt}}{2}dt$$

$$= \alpha^+ \cos kx - \alpha^- \cos k(2d-x) + a_{10}(x)\cos kx - a_{20}(x)\sin kx - b_{10}(x)\cos k(2d-x)$$

$$+b_{20}(x)\sin k(2d-x)+\int_0^x \tilde{K}_{210}(x,t)\sin kt\,dt+\int_0^x \tilde{K}_{220}(x,t)\cos kt\,dt$$

şeklindedir. Burada, $\tilde{K}_{110}(x,t)=K_{110}(x,t)-K_{110}(x,-t)$ ve $\tilde{K}_{120}(x,t)=K_{120}(x,t)-K_{120}(x,-t)$ dir.

$\Delta_0(k)$ ile L_0 probleminin karakteristik fonksiyonu gösterilecek olursa;

$$\varphi_{01}(\pi,k)=\Delta_0(k)=(a_{10}(\pi)+\alpha^+)\sin k\pi+(b_{10}(\pi)+\alpha^-)\sin k(2d-\pi)+a_{20}(\pi)\cos k\pi$$
$$+b_{20}(\pi)\cos k(2d-\pi)+\int_0^\pi \tilde{K}_{110}(\pi,t)\sin kt\,dt+\int_0^\pi \tilde{K}_{120}(\pi,t)\cos kt\,dt$$

olduğu açıktır. $\Delta_0(k)=0$ denkleminin $n\in N$ için k_n^0 kökleri L_0 probleminin özdeğerleridir. Ayrıca $n=0$ için $k_0^0=0$ dır.

Aşağıdaki lemma doğrudur.

Lemma 3.1.1: $\inf\limits_{n\neq m}\left|k_n^0-k_m^0\right|=\beta>0$ yani $\Delta_0(k)=0$ karakteristik denkleminin kökleri ayrıktır.

İspat: Tersi kabul edilecek olursa, yani $\{k_n^0\}$ dizisinin $\left\{k_{n_p}^0\right\}$ ve $\left\{\tilde{k}_{n_p}^0\right\}$ alt dizileri vardır, öyleki $k_{n_p}^0\neq\tilde{k}_{n_p}^0$ ve $p\to\infty$ iken $k_{n_p}^0,\tilde{k}_{n_p}^0\to\infty$ ve ayrıca

$$\lim_{p\to\infty}\left|k_{n_p}^0-\tilde{k}_{n_p}^0\right|=0$$

dır. $L_2(0,\pi;R^2)$ uzayında L_0 probleminin $\varphi_0(x,k_{n_p}^0)$ ve $\varphi_0(x,\tilde{k}_{n_p}^0)$ özfonksiyonlarının ortogonallik koşulundan yararlanılırsa;

$$0=\int_0^\pi \varphi_0(x,k_{n_p}^0)\overline{\varphi_0(x,\tilde{k}_{n_p}^0)}dx=\int_0^\pi \varphi_0(x,k_{n_p}^0)\overline{\varphi_0(x,k_{n_p}^0)}dx$$
$$+\int_0^\pi \varphi_0(x,k_{n_p}^0)\left[\overline{\varphi_0(x,\tilde{k}_{n_p}^0)-\varphi_0(x,k_{n_p}^0)}\right]dx$$

veya

$$0=\int_0^\pi \left\|\varphi_0(x,k_{n_p}^0)\right\|^2 dx+\int_0^\pi \varphi_0(x,k_{n_p}^0)\left[\overline{\varphi_0(x,\tilde{k}_{n_p}^0)-\varphi_0(x,k_{n_p}^0)}\right]dx \qquad (3.1.1)$$

Şimdi,

$$\int_0^\pi <\varphi_0(x,k_{n_p}^0),\ \left[\overline{\varphi_0(x,\tilde{k}_{n_p}^0)-\varphi_0(x,k_{n_p}^0)}\right]>dx=$$

$$\int_0^d <\varphi_0(x,k^0_{n_p}),\ \overline{\left[\varphi_0(x,\tilde{k}^0_{n_p})-\varphi_0(x,k^0_{n_p})\right]}> dx$$

$$+\int_d^\pi <\varphi_0(x,k^0_{n_p}),\ \overline{\left[\varphi_0(x,\tilde{k}^0_{n_p})-\varphi_0(x,k^0_{n_p})\right]}> dx = I_1 + I_2$$

şeklinde yazılsın.

I_1 integrali için $\varphi_0(x,k)$ çözümünün ifadesinden ve $a(x)$, $K(x,t)$ fonksiyonları için elde edilen integral eşitliklerden görüldüğü gibi her $x \in [0,\pi]$ için x'e göre düzgün olarak $\lim\limits_{p\to\infty}\left\|\varphi_0(x,\tilde{k}^0_{n_p})-\varphi_0(x,k^0_{n_p})\right\|_{R^2}=0$ olduğu kolayca gösterilebilir. Bunun için,

$$\begin{aligned}
\lim_{p\to\infty}\Bigg[&\Bigg((1+a_{10}(x))(\sin\tilde{k}^0_{n_p}x-\sin k^0_{n_p}x)+a_{20}()(\cos\tilde{k}^0_{n_p}x-\cos k^0_{n_p}x)\\
&+\int_0^x\tilde{K}_{110}(x,t)(\sin\tilde{k}^0_{n_p}t-\sin k^0_{n_p}t)dt+\int_0^x\tilde{K}_{120}(x,t)(\cos\tilde{k}^0_{n_p}t-\cos k^0_{n_p}t)dt\Bigg)^2\\
&+\Bigg((1+a_{10}(x))(\cos\tilde{k}^0_{n_p}x-\cos k^0_{n_p}x)-a_{20}(x)(\sin\tilde{k}^0_{n_p}x-\sin k^0_{n_p}x)\\
&+\int_0^x\tilde{K}_{210}(x,t)(\sin\tilde{k}^0_{n_p}t-\sin k^0_{n_p}t)dt+\int_0^x\tilde{K}_{220}(x,t)(\cos\tilde{k}^0_{n_p}t-\cos k^0_{n_p}t)dt\Bigg)^2\Bigg]^{\frac{1}{2}}=
\end{aligned}$$

$$\begin{aligned}
=\lim_{p\to\infty}\Bigg[&\Bigg(2(1+a_{10}(x))\sin\Big(\frac{\tilde{k}^0_{n_p}-k^0_{n_p}}{2}\Big)x\cos\Big(\frac{\tilde{k}^0_{n_p}+k^0_{n_p}}{2}\Big)x-2a_{20}(x)\sin\Big(\frac{\tilde{k}^0_{n_p}-k^0_{n_p}}{2}\Big)x\sin\Big(\frac{\tilde{k}^0_{n_p}+k^0_{n_p}}{2}\Big)x\\
&+2\int_0^x\tilde{K}_{110}(x,t)\sin\Big(\frac{\tilde{k}^0_{n_p}-k^0_{n_p}}{2}\Big)t\cos\Big(\frac{\tilde{k}^0_{n_p}+k^0_{n_p}}{2}\Big)tdt\\
&-2\int_0^x\tilde{K}_{120}(x,t)\sin\Big(\frac{\tilde{k}^0_{n_p}-k^0_{n_p}}{2}\Big)t\sin\Big(\frac{\tilde{k}^0_{n_p}+k^0_{n_p}}{2}\Big)tdt\Bigg)^2+\\
&\Bigg(-2(1+a_{10}(x))\sin\Big(\frac{\tilde{k}^0_{n_p}-k^0_{n_p}}{2}\Big)x\sin\Big(\frac{\tilde{k}^0_{n_p}+k^0_{n_p}}{2}\Big)x-2a_{20}(x)\sin\Big(\frac{\tilde{k}^0_{n_p}-k^0_{n_p}}{2}\Big)x\cos\Big(\frac{\tilde{k}^0_{n_p}+k^0_{n_p}}{2}\Big)x\\
&+2\int_0^x\tilde{K}_{210}(x,t)\sin\Big(\frac{\tilde{k}^0_{n_p}-k^0_{n_p}}{2}\Big)t\cos\Big(\frac{\tilde{k}^0_{n_p}+k^0_{n_p}}{2}\Big)tdt\\
&-2\int_0^x\tilde{K}_{220}(x,t)\sin\Big(\frac{\tilde{k}^0_{n_p}-k^0_{n_p}}{2}\Big)t\sin\Big(\frac{\tilde{k}^0_{n_p}+k^0_{n_p}}{2}\Big)tdt\Bigg)^2\Bigg]^{\frac{1}{2}}=0
\end{aligned}$$

Benzer şekilde I_2 integrali için,

$$\lim_{p\to\infty}\left\|\varphi_0(x,\tilde{k}^0_{n_p})-\varphi_0(x,k^0_{n_p})\right\|_{R^2}$$

$$
\begin{aligned}
= \lim_{p\to\infty} \Bigg[\Bigg(& (\alpha^{+} + a_{10}(x))(\sin \tilde{k}^{0}_{n_p} x - \sin k^{0}_{n_p} x) + (\alpha^{-} + b_{10}(x))(\sin \tilde{k}^{0}_{n_p}(2d - x) - \sin k^{0}_{n_p}(2d - x)) \\
& + a_{20}(x)(\cos \tilde{k}^{0}_{n_p} x - \cos k^{0}_{n_p} x) + b_{20}(x)(\cos \tilde{k}^{0}_{n_p}(2d - x) - \cos k^{0}_{n_p} 2d - x)) \\
& + \int_0^x \tilde{K}_{110}(x,t)(\sin \tilde{k}^{0}_{n_p} t - \sin k^{0}_{n_p} t)dt + \int_0^x \tilde{K}_{120}(x,t)(\cos \tilde{k}^{0}_{n_p} t - \cos k^{0}_{n_p} t)dt \Bigg)^2 + \\
\Bigg(& (\alpha^{+} + a_{10}(x))(\cos \tilde{k}^{0}_{n_p} x - \cos k^{0}_{n_p} x) - (\alpha^{-} + b_{10}(x))(\cos \tilde{k}^{0}_{n_p}(2d - x) - \cos k^{0}_{n_p} 2d - x)) \\
& - a_{20}(x)(\sin \tilde{k}^{0}_{n_p} x - \sin k^{0}_{n_p} x) + b_{20}(x)(\sin \tilde{k}^{0}_{n_p}(2d - x) - \sin k^{0}_{n_p}(2d - x) \\
& + \int_0^x \tilde{K}_{210}(x,t)(\sin \tilde{k}^{0}_{n_p} t - \sin k^{0}_{n_p} t)dt + \int_0^x \tilde{K}_{220}(x,t)(\cos \tilde{k}^{0}_{n_p} t - \cos k^{0}_{n_p} t)dt \Bigg)^2 \Bigg]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
= \lim_{p\to\infty} \Bigg[\Bigg(& 2(\alpha^{+} + a_{10}(x)) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})x \cos(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})x \\
& + 2(\alpha^{-} + b_{10}(x)) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})(2d - x) \cos(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})^{0}_{n_p}(2d - x) \\
& - 2a_{20}(x) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})x \sin(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})x - 2b_{20}(x) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})(2d - x) \sin(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})(2d - x)) \\
& + 2\int_0^x \tilde{K}_{210}(x,t) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})t \cos(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})t dt \\
& - 2\int_0^x \tilde{K}_{220}(x,t) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})t \sin(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})t dt \Bigg)^2 \\
+ \Bigg(& 2(\alpha^{+} + a_{10}(x)) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})x \sin(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})x \\
& + 2(\alpha^{-} + b_{10}(x)) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})(2d - x) \sin(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})(2d - x)) \\
& - 2a_{20}(x) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})x \cos(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})x - 2b_{20}(x) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})(2d - x) \cos(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})^{0}_{n_p}(2d - x) \\
& + 2\int_0^x \tilde{K}_{210}(x,t) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})t \cos(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})t dt \\
& - 2\int_0^x \tilde{K}_{220}(x,t) \sin(\tfrac{\tilde{k}^{0}_{n_p} - k^{0}_{n_p}}{2})t \sin(\tfrac{\tilde{k}^{0}_{n_p} + k^{0}_{n_p}}{2})t dt \Bigg)^2 \Bigg]^{\frac{1}{2}} = 0
\end{aligned}
$$

olması yeterlidir. Burada $|\cdot|_{R^2} = \sqrt{<\cdot,\cdot>}$ dir.

Bu nedenle (3.1.1) eşitsizliğinde $p \to \infty$ iken limite geçilirse $\int_0^\pi \left\| \varphi_0(x, k^{0}_{n_p}) \right\|^2 dx = 0$ olduğu elde edilir. Bu ise bir çelişkidir. Bu çelişki Lemma'nın doğru olduğunu gösterir.

$\Delta(k)$, $\{k_n\}$ ve $\{\alpha_n\}$ 'ler sırasıyla L probleminin karakteristik fonksiyonunu, özdeğer dizisini ve normalleştirici sayılar dizisini göstersin.

$$\varphi(x,k) = \begin{pmatrix} \varphi_1(x,k) \\ \varphi_3(x,k) \end{pmatrix}$$

ile (2.1.1) denkleminin $\varphi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ başlangıç koşulunu ve (2.1.4) süreksizlik koşullarını sağlayan çözümü gösterilsin

$\varphi(x,k)$ matris çözümünün $\varphi_1(x,k)$ ve $\varphi_3(x,k)$ bileşeni için

$$\varphi_1(x,k) = \alpha^+ \sin kx + \alpha^- \sin k(2d - x) + a_1(x)\sin kx + a_2(x)\cos kx + b_1(x)\sin k(2d - x)$$
$$+ b_2(x)\cos k(2d - x) + \int_0^x K_{11}(x,t)\sin kt\,dt + \int_0^x K_{12}(x,t)\cos kt\,dt$$

$$\varphi_3(x,k) = \alpha^+ \cos kx - \alpha^- \cos k(2d - x) + a_1(x)\cos kx - a_2(x)\sin kx - b_1(x)\cos k(2d - x)$$
$$+ b_2(x)\sin k(2d - x) + \int_0^x K_{21}(x,t)\sin kt\,dt + \int_0^x K_{22}(x,t)\cos kt\,dt$$

ifadeleri kullanılırsa,

$$\varphi_1(x,k) = \varphi_{10}(x,k) + \frac{\sin kx}{2k}\int_d^x t^{-2l} q(t) a_2(t)\,dt + \frac{\alpha^+ \sin kx}{2k}\int_0^d t^{-2\ell} q(t) a_2(t)\,dt$$
$$- \frac{\sin k(2d - x)}{2k}\int_d^x t^{-2l} q(t) b_2(t)\,dt + \frac{\alpha^- \sin k(2d - x)}{2k}\int_0^d t^{-2\ell} q(t) a_2(t)\,dt$$
$$- \frac{\alpha^+ \cos kx}{2k}\int_d^x t^{-2l} q(t)\,dt - \frac{\cos kx}{2k}\int_d^x t^{-2l} q(t) a_1(t)\,dt - \frac{\alpha^+ \cos kx}{2k}\int_0^d t^{-2\ell} q(t)\,dt$$
$$- \frac{\alpha^+ \cos kx}{2k}\int_0^d t^{-2\ell} q(t) a_1(t)\,dt + \frac{\alpha^- \cos k(2d - x)}{2k}\int_d^x t^{-2l} q(t)\,dt$$
$$+ \frac{\cos k(2d - x)}{2k}\int_d^x t^{-2l} q(t) b_1(t)\,dt - \frac{\alpha^- \cos k(2d - x)}{2k}\int_0^d t^{-2l} q(t)\,dt$$
$$- \frac{\alpha^- \cos k(2d - x)}{2k}\int_0^d t^{-2\ell} q(t) a_1(t)\,dt + \int_0^x \left(K_{11}(x,t) - K_{110}(x,t) \right) \sin kt\,dt$$
$$+ \int_0^x \left(K_{12}(x,t) - K_{120}(x,t) \right) \cos kt\,dt$$

ve

$$\varphi_3(x,k) = \varphi_{30}(x,k) + \frac{\cos kx}{2k}\int_d^x t^{-2l} q(t) a_2(t)\,dt + \frac{\alpha^+ \cos kx}{2k}\int_0^d t^{-2\ell} q(t) a_2(t)\,dt$$

$$+ \frac{\cos k(2d-x)}{2k}\int_d^x t^{-2l} q(t) b_2(t)\,dt - \frac{\alpha^- \cos k(2d-x)}{2k}\int_0^d t^{-2\ell} q(t) a_2(t)\,dt$$

$$+ \frac{\alpha^+ \sin kx}{2k}\int_d^x t^{-2l} q(t)\,dt + \frac{\sin kx}{2k}\int_d^x t^{-2l} q(t) a_1(t)\,dt + \frac{\alpha^+ \sin kx}{2k}\int_0^d t^{-2\ell} q(t)\,dt$$

$$+ \frac{\alpha^+ \sin kx}{2k}\int_0^d t^{-2\ell} q(t) a_1(t)\,dt + \frac{\alpha^- \sin k(2d-x)}{2k}\int_d^x t^{-2l} q(t)\,dt$$

$$+ \frac{\sin k(2d-x)}{2k}\int_d^x t^{-2l} q(t) b_1(t)\,dt - \frac{\alpha^- \sin k(2d-x)}{2k}\int_0^d t^{-2l} q(t)\,dt$$

$$- \frac{\alpha^- \sin k(2d-x)}{2k}\int_0^d t^{-2\ell} q(t) a_1(t)\,dt + \int_0^x \left(\tilde{K}_{21}(x,t) - \tilde{K}_{210}(x,t)\right) \sin kt\,dt$$

$$+ \int_0^x \left(\tilde{K}_{22}(x,t) - \tilde{K}_{220}(x,t)\right) \cos kt\,dt$$

elde edilir.

Böylece L probleminin karakteristik denklemi

$$\Delta(k) = \Delta_0(k) + \frac{\sin k\pi}{2k}\int_d^\pi t^{-2l} q(t) a_2(t)\,dt + \frac{\alpha^+ \sin k\pi}{2k}\int_0^d t^{-2\ell} q(t) a_2(t)\,dt$$

$$- \frac{\sin k(2d-\pi)}{2k}\int_d^\pi t^{-2l} q(t) b_2(t)\,dt + \frac{\alpha^- \sin k(2d-\pi)}{2k}\int_0^d t^{-2\ell} q(t) a_2(t)\,dt$$

$$- \frac{\alpha^+ \cos k\pi}{2k}\int_d^\pi t^{-2l} q(t)\,dt - \frac{\cos k\pi}{2k}\int_d^\pi t^{-2l} q(t) a_1(t)\,dt - \frac{\alpha^+ \cos k\pi}{2k}\int_0^d t^{-2\ell} q(t)\,dt$$

$$- \frac{\alpha^+}{2k}\int_0^d t^{-2\ell} q(t) a_1(t)\,dt + \frac{\alpha^- \cos k(2d-\pi)}{2k}\int_d^\pi t^{-2l} q(t)\,dt + \frac{\cos k(2d-\pi)}{2k}\int_d^\pi t^{-2l} q(t) b_1(t)\,dt$$

$$- \frac{\alpha^- \cos k(2d-\pi)}{2k}\int_0^d t^{-2l} q(t)\,dt - \frac{\alpha^- \cos k(2d-\pi)}{2k}\int_0^d t^{-2\ell} q(t) a_1(t)\,dt$$

$$+ \int_0^\pi \left(\tilde{K}_{11}(x,t) - \tilde{K}_{110}(x,t)\right) \sin kt\,dt + \int_0^\pi \left(\tilde{K}_{12}(x,t) - \tilde{K}_{120}(x,t)\right) \cos kt\,dt$$

şeklindedir.

Lemma 3.1.2: L probleminin özdeğerleri basittir. Yani $\dot{\Delta}(k_n) \neq 0$ dır.

İspat:

$$\begin{cases} y_1' - k_n u_2(x) y_3 = k_n y_3 \\ -y_3' - k u_1(x) y_1 + \frac{1}{k} q(x) x^{-2\ell} y_1 = k y_1 \end{cases} \Rightarrow \begin{cases} (k_n y_1)' - k_n^2 u_2(x) y_3 = k_n^2 y_3 \\ -(k_n y_3)' - k_n^2 u_1(x) y_1 + q(x) x^{-2\ell} y_1 = k_n^2 y_1 \end{cases}$$

denklem sistemi

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k_n y_3 \\ k_n y_1 \end{pmatrix}' + \begin{pmatrix} -k_n u_2 & 0 \\ 0 & -k_n u_1 + q(x) x^{-2\ell} \end{pmatrix} \begin{pmatrix} k_n y_3 \\ k_n y_1 \end{pmatrix} = k_n \begin{pmatrix} k_n y_3 \\ k_n y_1 \end{pmatrix}$$

şeklinde yazılsın.

$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Omega(x,k_n) = \begin{pmatrix} -k_n u_2 & 0 \\ 0 & -k_n u_1 + q(x) x^{-2\ell} \end{pmatrix}$ ve $\varphi = \begin{pmatrix} \tilde{y}_3 \\ \tilde{y}_1 \end{pmatrix} = \begin{pmatrix} k_n y_3 \\ k_n y_1 \end{pmatrix}$ olarak alınırsa,

$$B\varphi'(x,k_n) + \Omega(x,k_n)\varphi(x,k_n) = k_n \varphi(x,k_n)$$

Dirac diferansiyel denklemleri sistemi tipinde bir sistem elde edilmiş olur.

Bu sistemin $\varphi(0,k_n) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ başlangıç koşulunu sağlayan çözümü $\varphi(x,k_n) = \begin{pmatrix} \varphi_3 \\ \varphi_1 \end{pmatrix}$ olsun.

$B\varphi'(x,k_n) + \Omega(x,k_n)\varphi(x,k_n) = k_n\varphi(x,k_n)$ denkleminde her iki tarafın k_n' ye göre türevi alınırsa;

$$\begin{cases} B\varphi'(x,k_n) + \Omega(x,k_n)\varphi(x,k_n) = k_n\varphi(x,k_n) \\ B\dot{\varphi}'(x,k_n) + \Omega(x,k_n)\dot{\varphi}(x,k_n) + \dot{\Omega}(x,k_n)\varphi(x,k_n) = k_n\dot{\varphi}(x,k_n) + \varphi(x,k_n) \end{cases}$$

sistemi alınır R^2 öklid uzayında birinci denklem $\dot{\varphi}(x,k_n)$, ikinci denklem ise $\varphi(x,k_n)$ ile skaler olarak çarpılır ve ikinciden birinci çıkarılırsa,

$$B\dot{\varphi}'(x,k_n)\,\varphi(x,k_n) - B\varphi'(x,k_n)\dot{\varphi}(x,k_n) + \dot{\Omega}(x,k_n)\varphi(x,k_n)\,\varphi(x,k_n) = \varphi^2(x,k_n)$$

olur. Yani,

$$< \psi(x,k_n),\psi(x,k_n) > = < B\dot{\psi}'(x,k_n)),\psi(x,k_n) > - < B\psi'(x,k_n),\dot{\psi}(x,k_n) >$$

$$+ < \dot{\Omega}(x,k_n)\psi(x,k_n),\psi(x,k_n) >$$

elde edilir.

Son eşitlik $[0,\pi]$ aralığı üzerinde integrallenirse ve

$$\alpha_n^0 = \int_0^\pi \psi^2(x,k_n)dx = \int_0^\pi \left[\psi_1^2(x,k_n) + \psi_3^2(x,k_n)\right] dx$$

olduğu gözönünde bulundurulursa,

$$< B\dot{\psi}'(x,k_n),\psi(x,k_n) > = \dot{\varphi}_1'(x,k_n)\psi_3(x,k_n) - \dot{\psi}_3'(x,k_n)\psi_1(x,k_n)$$

$$< B\psi'(x,k_n),\dot{\psi}(x,k_n) > = \varphi_1'(x,k_n)\dot{\varphi}_3(x,k_n) - \varphi_3'(x,k_n)\dot{\varphi}_1(x,k_n)$$

$< \dot{\Omega}(x,k_n)\psi(x,k_n),\psi(x,k_n) > = -u_2\varphi_3^2 - u_1\varphi_1^2$ olmak üzere;

$$\int_0^\pi \left(\dot{\varphi}_1'(x,k_n)\psi_3(x,k_n) - \dot{\psi}_3'(x,k_n)\psi_1(x,k_n) - \varphi_1'(x,k_n)\dot{\varphi}_3(x,k_n) + \varphi_3'(x,k_n)\dot{\varphi}_1(x,k_n)\right) dx$$

$$+ \int_0^\pi \left(-u_2\varphi_3^2 - u_1\varphi_1^2\right) dx = \int_0^\pi \left[\psi_1^2(x,k_n) + \psi_3^2(x,k_n)\right] dx$$

$$\int_0^\pi \left[x^{2l}\psi_1^2(x,k_n) + x^{-2l}\psi_3^2(x,k_n)\right] dx = \left(\dot{\varphi}_1(x,k_n)\varphi_3(x,k_n) - \dot{\varphi}_3(x,k_n)\varphi_1(x,k_n)\right)\Big|_0^\pi$$

$$= \dot{\varphi}_1(\pi,k_n)\varphi_3(\pi,k_n) - \dot{\varphi}_1(0,k_n)\varphi_3(0,k_n) - \dot{\varphi}_3(\pi,k_n)\varphi_1(\pi,k_n) + \dot{\varphi}_3(0,k_n)\varphi_1(0,k_n)$$

$$= \dot{\varphi}_1(\pi,k_n)\varphi_3(\pi,k_n) - \dot{\varphi}_3(\pi,k_n)\varphi_1(\pi,k_n)$$

ayrıca $\varphi_1(\pi,k_n) = \Delta(k_n) = 0$ ve $\dot{\varphi}_1(\pi,k_n) = \varphi_1(\pi,k_n) + k_n\dot{\varphi}_1(\pi,k_n)$ olacağından;

$$k_n\dot{\Delta}(k_n)\varphi_3(\pi,k_n) = \int_0^\pi \left[x^{2l}\psi_1^2(x,k_n) + x^{-2l}\psi_3^2(x,k_n)\right] dx$$

olu. $\psi(\pi,k_n) = k_n\varphi(\pi,k_n)$ olduğundan,

$\dot{\Delta}(k_n)\varphi_3(\pi,k_n) = \int_0^\pi \left[x^{2l}\psi_1^2(x,k_n) + x^{-2l}\psi_3^2(x,k_n)\right] dx$ elde edilir. Başka bir gösterimle

$\dot{\Delta}(k_n)\varphi_3(\pi,k_n) = \alpha_n = \alpha_n^0 + \int_0^\pi \left(u_2\varphi_3^2 + u_1\varphi_1^2\right) dx$ olur. Yani,

$$\alpha_n = \dot{\Delta}(k_n)\varphi_3(\pi,k_n)$$

elde edilir. Buradan $\Delta(k_n) \neq 0$ olduğu açıktır, burada $\alpha_n^0 = \int_0^{\pi} \left[\varphi_1^2(x, k_n) + \varphi_3^2(x, k_n) \right] dx$

şeklindedir

3.2. Özdeğerler ve Normalleştirici Sayıların Asimptotik İfadeleri

Bu alt bölümde L probleminin özdeğerleri ve normalleştirici sayıları için n'in yeterince büyük değerlerinde asimtotik ifadeler elde edilecektir.

Lemma 3.2.1 : L probleminin özdeğerleri için $k_n = k_n^0 + \varepsilon_n$ asimptotik eşitliği doğrudur. Burada $\{\varepsilon_n\} \in \ell_2$ dir.

İspat: δ yeterince küçük pozitif sayı olmak üzere ($\delta < \frac{\beta}{2}$)

$$\begin{aligned}\Gamma_n &= \left\{ k : |k| = \left|k_n^0\right| + \frac{\beta}{2},\ n = 0, \pm 1, \pm 2, \ldots \right\}\\ G_\delta &= \left\{ k : \left|k - k_n^0\right| \geq \delta,\ n = 0, \pm 1, \pm 2, \ldots \right\}\end{aligned}$$

olsun. $k \in \overline{G_\delta}$ için

$$\begin{aligned}\Delta_0(k) &= (a_{10}(\pi) + \alpha^+) \sin k\pi + (b_{10}(\pi) + \alpha^-) \sin k(2d - \pi) + a_{20}(\pi) \cos k\pi\\ &+ b_{20}(\pi) \cos k(2d - \pi) + \int_0^\pi K_{110}(\pi, t) \sin kt\,dt + \int_0^\pi K_{120}(\pi, t) \cos kt\,dt\end{aligned}$$

olduğundan

$$\begin{aligned}|\Delta_0(k)| &\geq \frac{|a_{10}(\pi) + \alpha^+|}{2} \left|e^{ik\pi} - e^{-ik\pi}\right| - \frac{|b_{10}(\pi) + \alpha^-|}{2} \left|e^{ik(2d-\pi)} - e^{-ik(2d-\pi)}\right| -\\ &- \frac{|a_{20}(\pi)|}{2} \left|e^{ik\pi} + e^{-ik\pi}\right| - \frac{|b_{20}(\pi)|}{2} \left|e^{ik(2d-\pi)} + e^{-ik(2d-\pi)}\right|\\ &- \frac{1}{2}\left|\int_0^\pi K_{110}(\pi, t) \left(e^{ikt} - e^{-ikt}\right) dt\right| - \frac{1}{2}\left|\int_0^\pi K_{120}(\pi, t) \left(e^{ikt} + e^{-ikt}\right) dt\right|\end{aligned}$$

şimdi $k = x + iy$ alırsak;

$$\begin{aligned}&= |a_{10}(\pi) + \alpha^+| \left|\sin(\pi x + i\pi y)\right| - |b_{10}(\pi) + \alpha^-| \left|\sin(x(2a - \pi) + i(2a - \pi)y)\right| -\\ &- |a_{20}(\pi)| \left|\cos(\pi x + i\pi y)\right| - |b_{20}(\pi)| \left|\cos((2d - \pi)x + i(2d - \pi)y)\right|\\ &- \left|\int_0^\pi K_{110}(\pi, t) \left|\sin(t\pi x + ity)\right| dt\right| - \left|\int_0^\pi K_{120}(\pi, t) \left|\cos(tx + ity)\right| dt\right|\\ &\geq |a_{10}(\pi) + \alpha^+| \left|\sinh(\pi y)\right| - |b_{10}(\pi) + \alpha^-| \left|\cosh(2d - \pi)y\right| -\\ &- |a_{20}(\pi)| \left|\sinh(\pi y)\right| - |b_{20}(\pi)| \left|\cosh(2d - \pi)y\right|\\ &- \left|\int_0^\pi K_{110}(\pi, t) \left|\cosh ty\right| dt\right| - \left|\int_0^\pi K_{120}(\pi, t) \left|\sinh ty\right| dt\right| \geq e^{|\mathrm{Im}\,k|\pi} C_\delta\end{aligned}$$

olur. Diğer taraftan

$$\Delta(k) = \varphi_1(\pi, k)$$

ve

$$\Delta_0(k) = \varphi_{01}(\pi, k)$$

olduğundan

$$\begin{aligned}
\Delta(k) = {} & \Delta_0(k) + \frac{\sin k\pi}{2k}\int_d^\pi t^{-2l} q(t) a_2(t)dt + \frac{\alpha^+ \sin k\pi}{2k}\int_0^d t^{-2\ell} q(t) a_2(t)dt \\
& - \frac{\sin k(2d-\pi)}{2k}\int_d^\pi t^{-2l} q(t) b_2(t)dt + \frac{\alpha^- \sin k(2d-\pi)}{2k}\int_0^d t^{-2\ell} q(t) a_2(t)dt \\
& - \frac{\alpha^+ \cos k\pi}{2k}\int_d^\pi t^{-2l} q(t)dt - \frac{\cos k\pi}{2k}\int_d^\pi t^{-2l} q(t) a_1(t)dt \\
& - \frac{\alpha^+ \cos k\pi}{2k}\int_0^d t^{-2\ell} q(t)dt - \frac{\alpha^+}{2k}\int_0^d t^{-2\ell} q(t) a_1(t)dt + \frac{\alpha^- \cos k(2d-\pi)}{2k}\int_d^\pi t^{-2l} q(t)dt \\
& + \frac{\cos k(2d-\pi)}{2k}\int_d^\pi t^{-2l} q(t) b_1(t)dt - \frac{\alpha^- \cos k(2d-\pi)}{2k}\int_0^d t^{-2l} q(t)dt \\
& - \frac{\alpha^- \cos k(2d-\pi)}{2k}\int_0^d t^{-2\ell} q(t) a_1(t)dt + \int_0^\pi \left(\tilde{K}_{11}(\pi,t) - \tilde{K}_{110}(\pi,t)\right)\sin kt\,dt \\
& + \int_0^\pi \left(\tilde{K}_{12}(\pi,t) - \tilde{K}_{120}(\pi,t)\right)\cos kt\,dt
\end{aligned}$$

olur (Marchenko, 1977, lemma 1.3.1 'den)

Yani n 'nin yeterince büyük değerlerinde $k \in \Gamma_n$ için

$$|\Delta(k) - \Delta_0(k)| < \frac{C_\delta}{2} e^{|\operatorname{Im} k|\pi}$$

eşitsizliği sağlanır. Böylece n yeterince büyük doğal sayı olmak üzere $k \in \Gamma_n$ için

$$|\Delta_0(k)| \geq C_\delta e^{|\operatorname{Im} k|\pi} > \frac{C_\delta}{2} e^{|\operatorname{Im} k|\pi} > |\Delta(k) - \Delta_0(k)|$$

eşitsizliği elde edilir.

Bu durumda Rouché teoremi uygulanırsa, n'nin yeterince büyük değerlerinde Γ_n yörüngesinin iç kısmında $\Delta_0(k)$ ve $\Delta_0(k)+\{\Delta(k)-\Delta_0(k)\}=\Delta(k)$ fonksiyonlarının eşit sayıda sıfırları yani $(n+1)$ sayıda $k_0,\dots,k_n$ sıfırları vardır.

Benzer şekilde Rouché teoreminden yararlanarak gösterilir ki; yeterince büyük n'ler için $\left|k-k_n^0\right|<\delta$ çemberlerinin herbirinde $\Delta(k)$ fonksiyonunun yalnızca bir sıfırı vardır.

Bu durumda $\lim\limits_{n\to\infty}\varepsilon_n=0$ olmak üzere $k_n=k_n^0+\varepsilon_n$ yazılabilir. k_n sayıları, $\Delta(k)$ karakteristik denkleminin kökleri olduğundan

$$\Delta(k_n)=\Delta_0(k_n^0+\varepsilon_n)+\frac{\sin(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2\ell}q(t)a_2(t)dt$$

$$+\frac{\alpha^+\sin(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_0^d t^{-2\ell}q(t)a_2(t)dt-\frac{\sin(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)b_2(t)dt$$

$$+\frac{\alpha^-\sin(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_0^d t^{-2\ell}q(t)a_2(t)dt-\frac{\alpha^+\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)dt$$

$$-\frac{\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)a_1(t)dt-\frac{\alpha^+\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_0^d t^{-2\ell}q(t)dt$$

$$-\frac{\alpha^+\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_0^d t^{-2\ell}q(t)a_1(t)dt+\frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)dt$$

$$+\frac{\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)b_1(t)dt-\frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_0^d t^{-2l}q(t)dt$$

$$-\frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_0^d t^{-2\ell}q(t)a_1(t)dt$$

$$+\int_0^\pi\left(\tilde{K}_{11}(\pi,t)-\tilde{K}_{110}(\pi,t)\right)\sin(k_n^0+\varepsilon_n)tdt$$

$$+\int_0^\pi\left(\tilde{K}_{12}(\pi,t)-\tilde{K}_{120}(\pi,t)\right)\cos(k_n^0+\varepsilon_n)tdt=0$$

dır. Diğer taraftan $\Delta_0(k_n^0)=0$ olduğundan

$$\Delta_0(k_n)=\Delta_0(k_n^0+\varepsilon_n)=\Delta_0(k_n^0)+\Delta_0(k_n^0)\varepsilon_n+\ddot{\Delta}_0(k_n^0)\frac{\varepsilon_n^2}{2!}+\dots$$

$$\Delta_0(k_n^0+\varepsilon_n) = \Delta_0(k_n^0)\varepsilon_n + o(\varepsilon_n) = (\Delta_0(k_n^0) + o(1))\varepsilon_n$$

olur. Eğer $\Delta_0(k_n^0+\varepsilon_n)$ ifadesi $\Delta(k_n)$ ifadesinde yerine yazılırsa,

$$(\Delta_0(k_n^0)+o(1))\varepsilon_n + \frac{\sin(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)a_2(t)dt - \frac{\sin(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)b_2(t)dt$$

$$-\frac{\alpha^+\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)dt - \frac{\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)a_1(t)dt$$

$$+\frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)dt + \frac{\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)b_1(t)dt$$

$$-\frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_0^{d} t^{-2l}q(t)dt + \int_0^{\pi}\left(\tilde{K}_{11}(\pi,t) - \tilde{K}_{110}(\pi,t)\right)\sin(k_n^0+\varepsilon_n)tdt$$

$$+\int_0^{\pi}\left(\tilde{K}_{12}(\pi,t) - \tilde{K}_{120}(\pi,t)\right)\cos(k_n^0+\varepsilon_n)tdt = 0$$

olur. Buradan,

$$\varepsilon_n = -\left(\frac{1}{\Delta_0(k_n^0)+o(1)}\right)\left\{\frac{\sin(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)a_2(t)dt\right.$$

$$-\frac{\sin(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)b_2(t)dt - \frac{\alpha^+\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)dt$$

$$-\frac{\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)a_1(t)dt + \frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)dt$$

$$+\frac{\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^{\pi} t^{-2l}q(t)b_1(t)dt - \frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_0^{d} t^{-2l}q(t)dt$$

$$+\int_0^{\pi}\left(\tilde{K}_{11}(\pi,t) - \tilde{K}_{110}(\pi,t)\right)\sin(k_n^0+\varepsilon_n)tdt$$

$$\left.+\int_0^{\pi}\left(\tilde{K}_{12}(\pi,t) - \tilde{K}_{120}(\pi,t)\right)\cos(k_n^0+\varepsilon_n)tdt\right\} \qquad (3.2.1)$$

elde edilir. $\Delta_0(k)$ fonksiyonu sinüs tipli (Levin, 1971) olduğundan her n doğal sayısı için

$\left|\Delta_0(k_n^0)\right| \geq \gamma > 0$ eşitliği sağlanacak şekilde $\gamma > 0$ sayısı vardır. Zhdanovich (1960) ve

Krein'in (1948) çalışmalarından yararlanılırsa $\sup_n |h_n| \le M$ olmak üzere

$$k_n^0 = n + h_n$$

olduğu açıktır. O halde (3.2.1) eşitliğinde

$$-\left(\frac{1}{\Delta_0(k_n^0)+o(1)}\right)\left\{\frac{\sin(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)a_2(t)dt\right.$$

$$-\frac{\sin(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)b_2(t)dt-\frac{\alpha^+\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)dt$$

$$-\frac{\cos(k_n^0+\varepsilon_n)\pi}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)a_1(t)dt+\frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)dt$$

$$\left.+\frac{\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_d^\pi t^{-2l}q(t)b_1(t)dt-\frac{\alpha^-\cos(k_n^0+\varepsilon_n)(2d-\pi)}{2(k_n^0+\varepsilon_n)}\int_0^d t^{-2l}q(t)dt\right\}=O(\frac{1}{n})$$

olduğunu kullanırsak,

$$\varepsilon_n=\int_0^\pi\left(\hat{K}_{11}(\pi,t)-\hat{K}_{110}(\pi,t)\right)\sin(k_n^0+\varepsilon_n)t\,dt$$

$$+\int_0^\pi\left(\hat{K}_{12}(\pi,t)-\hat{K}_{120}(\pi,t)\right)\cos(k_n^0+\varepsilon_n)t\,dt+O(\frac{1}{n})$$

elde edilir. Ayrıca (Marchenko, 1977, p.67),

$\left\{\int_0^\pi\left(\hat{K}_{11}(\pi,t)-\hat{K}_{110}(\pi,t)\right)\sin(k_n^0+\varepsilon_n)t\,dt\right\}\in \ell_2$ ve

$\left\{\int_0^\pi\left(\hat{K}_{12}(\pi,t)-\hat{K}_{120}(\pi,t)\right)\cos(k_n^0+\varepsilon_n)t\,dt\right\}\in \ell_2$ olduğundan $\{\varepsilon_n\}\in \ell_2$ elde edilir.

Lemma 3.2.2: L probleminin normalleştirici sayıları için $\alpha_n = \alpha_{n0} + \delta_n$ asimptotik eşitliği geçerlidir. Burada $\{\delta_n\} \in \ell_2$ dir.

İspat:

$$\Delta(k)=\Delta_0(k)+\frac{\sin k\pi}{2k}\int_d^\pi t^{-2l}q(t)a_2(t)dt+\frac{\alpha^+\sin k\pi}{2k}\int_0^d t^{-2\ell}q(t)a_2(t)dt$$

$$-\frac{\sin k(2d-\pi)}{2k}\int_d^\pi t^{-2l}q(t)b_2(t)dt+\frac{\alpha^-\sin k(2d-\pi)}{2k}\int_0^d t^{-2\ell}q(t)a_2(t)dt$$

$$-\frac{\alpha^+\cos k\pi}{2k}\int_d^\pi t^{-2l}q(t)dt-\frac{\cos k\pi}{2k}\int_d^\pi t^{-2l}q(t)a_1(t)dt-\frac{\alpha^+\cos k\pi}{2k}\int_0^d t^{-2\ell}q(t)dt$$

$$-\frac{\alpha^{+}\cos k\pi}{2k}\int_0^d t^{-2\ell}q(t)a_1(t)dt+\frac{\alpha^{-}\cos k(2d-\pi)}{2k}\int_d^\pi t^{-2l}q(t)dt$$

$$+\frac{\cos k(2d-\pi)}{2k}\int_d^\pi t^{-2l}q(t)b_1(t)dt-\frac{\alpha^{-}\cos k(2d-\pi)}{2k}\int_0^d t^{-2l}q(t)dt$$

$$-\frac{\alpha^{-}\cos k(2d-\pi)}{2k}\int_0^d t^{-2\ell}q(t)a_1(t)dt$$

$$+\int_0^\pi\left(\tilde{K}_{11}(\pi,t)-\tilde{K}_{110}(\pi,t)\right)\sin kt\,dt+\int_0^\pi\left(\tilde{K}_{12}(\pi,t)-\tilde{K}_{120}(\pi,t)\right)\cos kt\,dt$$

denkleminde her iki tarafın k'ya göre türevini alırsak; ve k yerine k_n yazarsak;

$$\Delta^{\cdot}(k_n)=\Delta^{\cdot}_0(k_n)-\frac{\sin k_n\pi}{2k_n^2}\int_d^\pi t^{-2l}q(t)a_2(t)dt+\frac{\pi\cos k_n\pi}{2k_n}\int_d^\pi t^{-2l}q(t)a_2(t)dt$$

$$-\frac{\alpha^{+}\sin k_n\pi}{2k_n^2}\int_0^d t^{-2\ell}q(t)a_2(t)dt+\frac{\alpha^{+}\pi\cos k_n\pi}{2k_n}\int_0^d t^{-2\ell}q(t)a_2(t)dt$$

$$+\frac{\sin k_n(2d-\pi)}{2k_n^2}\int_d^\pi t^{-2l}q(t)b_2(t)dt-\frac{(2d-\pi)\cos k_n(2d-\pi)}{2k_n}\int_d^\pi t^{-2l}q(t)b_2(t)dt$$

$$-\frac{\alpha^{-}\sin k_n(2d-\pi)}{2k_n^2}\int_0^d t^{-2\ell}q(t)a_2(t)dt+\frac{\alpha^{-}(2d-\pi)\cos k_n(2d-\pi)}{2k_n}\int_0^d t^{-2\ell}q(t)a_2(t)dt$$

$$+\frac{\alpha^{+}\cos k_n\pi}{2k_n^2}\int_d^\pi t^{-2l}q(t)dt-\frac{\alpha^{+}\pi\sin k_n\pi}{2k_n}\int_d^\pi t^{-2l}q(t)dt+\frac{\cos k_n\pi}{2k_n^2}\int_d^\pi t^{-2l}q(t)a_1(t)dt$$

$$+\frac{\pi\cos k_n\pi}{2k_n}\int_d^\pi t^{-2l}q(t)a_1(t)dt+\frac{\alpha^{+}\cos k_n\pi}{2k_n^2}\int_0^d t^{-2l}q(t)dt$$

$$-\frac{\alpha^{+}\pi\sin k_n\pi}{2k_n}\int_0^d t^{-2l}q(t)dt+\frac{\alpha^{+}\cos k\pi}{2k_n^2}\int_0^d t^{-2\ell}q(t)a_1(t)dt$$

$$-\frac{\alpha^{+}\pi\sin k\pi}{2k_n}\int_0^d t^{-2\ell}q(t)a_1(t)dt$$

$$-\frac{\cos k_n(2d-\pi)}{2k_n^2}\left(\alpha^{-}\int_d^\pi t^{-2l}q(t)dt+\int_d^\pi t^{-2l}q(t)b_1(t)dt-\alpha^{-}\int_0^d t^{-2l}q(t)dt\right.$$

$$
-\alpha^{-}\int_0^d t^{-2\ell}q(t)a_1(t)dt\Bigg) + \frac{(2d-\pi)\cos k_n(2d-\pi)}{2k_n}\Bigg(\alpha^{-}\int_d^\pi t^{-2l}q(t)dt
$$

$$
+\int_d^\pi t^{-2l}q(t)b_1(t)dt - \alpha^{-}\int_0^d t^{-2l}q(t)dt - \alpha^{-}\int_0^d t^{-2\ell}q(t)a_1(t)dt\Bigg)
$$

$$
+\int_0^\pi\left(\dot{\tilde{K}}_{11}(\pi,t)-\dot{\tilde{K}}'_{11}(\pi,t)\right)\sin k_n t dt + \int_0^\pi t\left(\tilde{K}_{11}(\pi,t)-\tilde{K}'_{11}(\pi,t)\right)\cos k_n t dt
$$

$$
+\int_0^\pi\left(\dot{\tilde{K}}_{12}(\pi,t)-\dot{\tilde{K}}'_{12}(\pi,t)\right)\cos k_n t dt - \int_0^\pi t\left(\tilde{K}_{12}(\pi,t)-\tilde{K}'_{12}(\pi,t)\right)\sin k_n t dt
$$

$$
= \Delta_0(k_n) + b_n
$$

olur. Burada, Lemma 3.2.1'in ispatında $\{\varepsilon_n\} \in \ell_2$ olduğunun gösterilmesine benzer olarak $\{b_n\} \in \ell_2$ olduğu açıktır.

$$
\Delta_0(k_n) = \Delta_0(k_n^0) + \dot{\Delta}_0(k_n^0)\varepsilon_n + \ddot{\Delta}_0(k_n^0)\frac{\varepsilon_n^2}{2!} + \ldots
$$

$$
\dot{\Delta}_0(k_n) = \dot{\Delta}_0(k_n^0) + \ddot{\Delta}_0(k_n^0)\varepsilon_n + \dddot{\Delta}_0(k_n^0)\frac{\varepsilon_n^2}{2!} + \ldots
$$

$$
\Delta_0(k_n) = \Delta_0(k_n^0 + \varepsilon_n) = \Delta_0(k_n^0) + o(\varepsilon_n)
$$

Ayrıca,

$$
\varphi_3(\pi,k) = \varphi_{30}(\pi,k) + \frac{\cos k\pi}{2k}\int_d^\pi t^{-2l}q(t)a_2(t)dt + \frac{\alpha^{+}\cos k\pi}{2k}\int_0^d t^{-2\ell}q(t)a_2(t)dt
$$

$$
+\frac{\cos k(2d-\pi)}{2k}\int_d^\pi t^{-2l}q(t)b_2(t)dt - \frac{\alpha^{-}\cos k(2d-\pi)}{2k}\int_0^d t^{-2\ell}q(t)a_2(t)dt
$$

$$
+\frac{\alpha^{+}\sin k\pi}{2k}\int_d^\pi t^{-2l}q(t)dt + \frac{\sin k\pi}{2k}\int_d^\pi t^{-2l}q(t)a_1(t)dt + \frac{\alpha^{+}\sin k\pi}{2k}\int_0^d t^{-2\ell}q(t)dt
$$

$$
+\frac{\alpha^{+}\sin k\pi}{2k}\int_0^d t^{-2\ell}q(t)a_1(t)dt + \frac{\alpha^{-}\sin k(2d-\pi)}{2k}\int_d^\pi t^{-2l}q(t)dt + \frac{\sin k(2d-\pi)}{2k}\int_d^\pi t^{-2l}q(t)b_1(t)dt
$$

$$
-\frac{\alpha^{-}\sin k(2d-\pi)}{2k}\int_0^d t^{-2l}q(t)dt - \frac{\alpha^{-}\sin k(2d-\pi)}{2k}\int_0^d t^{-2\ell}q(t)a_1(t)dt
$$

$$
+\int_0^\pi\left(\tilde{K}_{21}(\pi,t)-\tilde{K}'_{21}(\pi,t)\right)\sin ktdt + \int_0^\pi\left(\tilde{K}_{22}(\pi,t)-\tilde{K}'_{22}(\pi,t)\right)\cos ktd
$$

olduğundan $\varphi_3(\pi, k_n) = \varphi_{30}(\pi, k_n^0) + \tilde{\vartheta}_n$, $\left\{\tilde{\vartheta}_n\right\} \in \ell_2$ olduğu açıktır.

$\Delta(k_n) = \Delta_0(k_n) + b_n = \Delta_0(k_n^0) + o(\varepsilon_n) + b_n$

$\varphi_3(\pi, k_n) = \varphi_{30}(\pi, k_n^0) + \tilde{\vartheta}_n$

ifadelerinden yararlanılırsa,

$$\alpha_n = \Delta(k_n)\varphi_3(\pi, k_n)$$

olduğundan,

$$\alpha_n = \Delta(k_n)\varphi_3(\pi, k_n) = \left(\Delta_0(k_n^0) + o(\varepsilon_n) + b_n\right)\left(\varphi_{30}(\pi, k_n^0) + \tilde{\vartheta}_n\right)$$

$$= \Delta_0(k_n)\varphi_{30}(\pi, k_n^0) + \Delta_0(k_n^0)\tilde{\vartheta}_n + (o(\varepsilon_n) + b_n)\left(\varphi_{30}(\pi, k_n) + \tilde{\vartheta}_n\right)$$

olur. Böylece,

$$\alpha_n = \alpha_n^0 + \delta_n, \{\delta_n\} \in \ell_2$$

olduğu açıktır.Burada $\delta_n = \Delta_0(k_n)\tilde{\vartheta}_n + (o(\varepsilon_n) + b_n)\left(\varphi_{30}(\pi, k_n) + \tilde{\vartheta}_n\right)$ şeklindedir.

3.3. Weyl Çözümü ve Weyl Fonksiyonunun Özellikleri

$\Phi(x,k) = \begin{pmatrix} \Phi_1(x,k) \\ \Phi_3(x,k) \end{pmatrix}$ vektör fonksiyonu, (2.1.1) denkleminin $\Phi_1(0,k) = -1$ ve $\Phi_3(\pi,k) = 0$ koşullarını ve (2.1.4) süreksizlik koşullarını sağlayan çözümü olsun. $\Phi(x,k)$ fonksiyonuna L sınır değer probleminin Weyl çözümü denir.

$$\psi(x,k) = \begin{pmatrix} \psi_1(x,k) \\ \psi_3(x,k) \end{pmatrix},\ \varphi(x,k) = \begin{pmatrix} \varphi_1(x,k) \\ \varphi_3(x,k) \end{pmatrix} \text{ ve } C(x,k) = \begin{pmatrix} C_1(x,k) \\ C_3(x,k) \end{pmatrix}$$

fonksiyonları, (2.1.1) denkleminin

$$\psi(\pi,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \varphi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ ve } C(0,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

başlangıç koşullarını ve (2.1.4) süreksizlik koşullarını sağlayan çözümleri olsun. $\psi(x,k)$ ve $C(x,k)$ fonksiyonlarının k 'ya göre tam oldukları açıktır.

Ayrıca

$$\psi(x,k) = c_1(k)\varphi(x,k) + c_2(k)C(x,k)$$

şeklinde yazılır.Buradan

$$\langle \psi(x,k), \varphi(x,k)\rangle = \tilde{\psi}(x,k)B\varphi(x,k) = \tilde{\psi}(x,k)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\varphi(x,k)$$

$$= (\psi_1(x,k), \psi_3(x,k))\begin{pmatrix} \varphi_3(x,k) \\ -\varphi_1(x,k) \end{pmatrix} = \psi_1(x,k)\varphi_3(x,k) - \psi_3(x,k)\varphi_1(x,k)$$

ve,

$$\langle \psi(x,k), \varphi(x,k)\rangle = c_1(k)\langle \varphi(x,k), \varphi(x,k)\rangle + c_2(k)\langle C(x,k), \varphi(x,k)\rangle$$

$$= c_2(k)\langle C(x,k), \varphi(x,k)\rangle = c_2(k)\tilde{C}(x,k)B\varphi(x,k)$$

$$= c_2(k)\left(C_1(x,k)\varphi_3(x,k) - C_2(x,k)\varphi_1(x,k)\right)$$

eşitlikleri elde edilir. Başlangıç koşulları kullanılırsa,

$$\langle \psi(x,k), \varphi(x,k)\rangle(0) = \psi_1(0,k)\varphi_3(0,k) - \psi_3(0,k)\varphi_1(0,k) = \psi_1(0,k) = \Delta(k)$$

ve

$$\langle \psi(x,k), \varphi(x,k)\rangle(0) = c_2(k)\left(C_1(0,k)\varphi_3(0,k) - C_3(0,k)\varphi_1(0,k)\right) = c_2(k)$$

eşitliklerinden

$c_2(k) = \psi_1(0,k) = \Delta(k)$

olarak bulunur. Aynı şekilde,

$$\langle \psi(x,k), C(x,k)\rangle = \psi(x,k) B C(x,k) = \psi(x,k) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C(x,k)$$

$$= (\psi_1(x,k), \psi_3(x,k)) \begin{pmatrix} C_3(x,k) \\ -C_1(x,k) \end{pmatrix} = \psi_1(x,k) C_3(x,k) - \psi_3(x,k) C_1(x,k)$$

ve,

$$\langle \psi(x,k), C(x,k)\rangle = c_1(k)\langle \varphi(x,k), C(x,k)\rangle + c_2(k)\langle C(x,k), C(x,k)\rangle$$

$$= c_1(k)\langle \varphi(x,k), C(x,k)\rangle = c_1(k)\varphi(x,k) B C(x,k)$$

$$= c_1(k)\left(\varphi_1(x,k) C_3(x,k) - \varphi_3(x,k) C_1(x,k)\right)$$

eşitlikleri elde edilir. Başlangıç koşulları kullanılırsa,

$$\langle \psi(x,k), C(x,k)\rangle(0) = \psi_1(0,k) C_3(0,k) - \psi_3(0,k) C_1(0,k) = -\psi_3(0,k)$$

ve

$$\langle \psi(x,k), C(x,k)\rangle(0) = c_1(k)\left(\varphi_1(0,k) C_3(0,k) - \varphi_3(0,k) C_1(0,k)\right) = -c_1(k)$$

eşitliklerinden

$c_1(k) = \psi_3(0,k)$

bulunur.Buradan,

$\psi(x,k) = \psi_3(0,k)\varphi(x,k) + \Delta(k) C(x,k)$

veya

$$\frac{\psi(x,k)}{\Delta(k)} = C(x,k) + \frac{\psi_3(0,k)}{\Delta(k)}\varphi(x,k) \qquad (3.3.1)$$

bulunur. Böylece $\Phi(x,k)$ weyl çözümü ve ,$\Phi_3(0,k) = M(k)$ weyl fonksiyonları

$$\Phi(x,k) = \frac{\psi(x,k)}{\Delta(k)}$$
$$M(k) = \frac{\psi_3(0,k)}{\Delta(k)}$$

olarak elde edilir.

Teorem 3.3.1:

$$M(k) = \frac{1}{\alpha_0(k-k_0)} + \sum_{n=1}^{\infty}\left\{\frac{1}{\alpha_n(k-k_n)} + \frac{1}{\alpha_n^0 k_n^0}\right\} \qquad (3.3.2)$$

gösterilimi doğrudur.

İspat: Öncelikle $\psi(x,k)$ çözümü için $\varphi(x,k)$ çözümüne benzer bir gösterim elde edilsin.

$$\begin{cases} y_3' + ky_1 = 0 \\ y_1' - ky_3 = 0 \end{cases}$$ denklem sisteminin homojen kısmının iki lineer bağımsız çözümü:

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} \text{ ve } \begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} -\cos kx \\ \sin kx \end{pmatrix}$$ şeklindedir.

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_1(x)e^{ikx} + c_2(x)e^{-ikx} \\ ic_1(x)e^{ikx} - ic_2(x)e^{-ikx} \end{pmatrix}$$ ve

$$\begin{pmatrix} y_1' \\ y_3' \end{pmatrix} = \begin{pmatrix} c_1'(x)e^{ikx} + c_2'(x)e^{-ikx} + ikc_1(x)e^{ikx} - ikc_2(x)e^{-ikx} \\ ic_1'(x)e^{ikx} - ic_2'(x)e^{-ikx} - kc_1(x)e^{ikx} - kc_2(x)e^{-ikx} \end{pmatrix}$$ alınır (2.1.1) denkleminde yerine yazılıp parametrelerin değişimi yöntemi uygulanırsa;

$$c_1(x) = -\frac{ik}{2}\int_x^\pi u_1(t)y_1(t)e^{-ikt}dt - \frac{k}{2}\int_x^\pi u_2(t)y_3(t)e^{-ikt}dt + \frac{i}{2k}\int_x^\pi t^{-2\ell}q(t)y_1(t)e^{-ikt}dt + c_0$$

$$c_2(x) = -\frac{k}{2}\int_x^\pi u_1(t)y_1(t)e^{ikt}dt + \frac{ik}{2}\int_x^\pi t^{-2\ell}q(t)y_1(t)e^{ikt}dt - \frac{i}{2k}\int_x^\pi u_2(t)y_3(t)e^{ikt}dt + c_1$$

bulunur. $c_1(x)$ ve $c_2(x)$ ifadeleri denklemde yerine yazılır ve gerekli işlemler yapılırsa;

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_0e^{ikx} + c_1e^{-ikx} + k\int_x^\pi u_1(t)y_1(t)\sin k(x-t)dt \\ -\frac{1}{k}\int_x^\pi t^{-2\ell}q(t)y_1(t)\sin k(x-t)dt - k\int_x^\pi u_2(t)y_3(t)\cos k(x-t)dt \\ ic_0e^{ikx} - ic_1e^{-ikx} + k\int_x^\pi u_1(t)y_1(t)\cos k(x-t)dt \\ -\frac{1}{k}\int_0^x t^{-2\ell}q(t)y_1(t)\cos k(x-t)dt + k\int_0^x u_2(t)y_3(t)\sin k(x-t)dt \end{pmatrix}$$

bulunur.

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}(\pi) = \begin{pmatrix} 1 \\ i \end{pmatrix}$$ başlangıç koşulunu sağlayan çözüm; $$\begin{pmatrix} y_{01} \\ y_{03} \end{pmatrix} = \begin{pmatrix} e^{ik(x-\pi)} \\ ie^{ik(x-\pi)} \end{pmatrix}$$

şeklinde olup $x > d$ iken

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} e^{ik(x-\pi)} + k\int_x^{\pi} u_1(t)y_1(t)\sin k(x-t)dt \\ -\frac{1}{k}\int_x^{\pi} t^{-2\ell}q(t)y_1(t)\sin k(x-t)dt - k\int_x^{\pi} u_2(t)y_3(t)\cos k(x-t)dt \\ ie^{ik(x-\pi)} + k\int_x^{\pi} u_1(t)y_1(t)\cos k(x-t)dt \\ -\frac{1}{k}\int_0^{x} t^{-2\ell}q(t)y_1(t)\cos k(x-t)dt + k\int_0^{x} u_2(t)y_3(t)\sin k(x-t)dt \end{pmatrix}$$

olarak elde edilir. $x < d$ iken çözüm

$$\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} A(k)e^{ik(x-\pi)} + B(k)e^{-ik(x-\pi)} - \frac{1}{k}\int_x^{\pi} t^{-2\ell}q(t)y_1(t)\sin k(x-t)dt \\ +k\int_x^{\pi} \{u_1(t)y_1(t)\sin k(x-t) - u_2(t)y_3(t)\cos k(x-t)\}\, dt \\ iA(k)e^{ik(x-\pi)} - iB(k)e^{-ik(x-\pi)} - \frac{1}{k}\int_0^{x} t^{-2\ell}q(t)y_1(t)\cos k(x-t)dt \\ +k\int_0^{x} \{u_1(t)y_1(t)\cos k(x-t) + u_2(t)y_3(t)\sin k(x-t)\}\, dt \end{pmatrix}$$

şeklinde aransın. (2.1.4) süreksizlik koşulu kullanılarak $A(k)$ ve $B(k)$ fonksiyonları elde edilir ve denklemde yerine yazılır gerekli düzenlemeler yapılırsa,

$x < d$ iken çözüm

$$\begin{aligned} y_1(x,k) = {} & \alpha^{+}e^{ik(x-\pi)} - \alpha^{-}e^{-ik(x+\pi-2d)} \\ & +k\int_0^{d} \left(\alpha^{+}\sin k(x-t) + \alpha^{-}\sin k(x+t-2d)\right) u_1(t)y_1(t)dt \\ & -k\int_0^{d} \left(\alpha^{+}\cos k(x-t) - \alpha^{-}\cos k(x+t-2d)\right) u_2(t)y_3(t)dt \\ & -\frac{1}{k}\int_0^{d} \left(\alpha^{+}\sin k(x-t) + \alpha^{-}\sin k(x+t-2d)\right) t^{-2\ell}q(t)y_1(t)dt \end{aligned}$$

$$+k\int_d^x \left(\sin k(x-t)u_1(t)y_1(t)-\cos k(x-t)u_2(t)y_3(t)\right)dt$$
$$-\frac{1}{k}\int_d^x \sin k(x-t)t^{-2\ell}q(t)y_1(t)dt \qquad (3.3.3)$$

$$\begin{aligned}
y_3(x,k) = {} & i\alpha^+e^{ik(x-\pi)}-i\alpha^-e^{-ik(x+\pi-2d)} \\
& +k\int_0^d \left(\alpha^+\cos k(x-t)-\alpha^-\cos k(x+t-2d)\right)u_1(t)y_1(t)dt \\
& +k\int_0^d \left(\alpha^+\sin k(x-t)-\alpha^-\sin k(x+t-2d)\right)u_2(t)y_3(t)dt \\
& -\frac{1}{k}\int_0^d \left(\alpha^+\cos k(x-t)-\alpha^-\cos k(x+t-2d)\right)t^{-2\ell}q(t)y_1(t)dt \\
& +k\int_d^x \left(\cos k(x-t)u_1(t)y_1(t)+\sin k(x-t)u_2(t)y_3(t)\right)dt \\
& -\frac{1}{k}\int_d^x \cos k(x-t)t^{-2\ell}q(t)y_1(t)dt
\end{aligned}$$

şeklinde bulunur.

$\psi(x,k)=\dfrac{y(x,k)-\overline{y(x,k)}}{2i}$ alınırsa $\varphi(x,k)$ gösterilimine benzer olarak $x<d$ iken

$$\psi_1(x,k) = -\alpha^+\sin k(\pi-x)+\alpha^-\sin k(x+\pi-2d)-a_1(x)\sin k(\pi-x)+a_2(x)\cos k(\pi-x)$$
$$-b_1(x)\sin k(x+\pi-2d)+b_2(x)\cos k(x+\pi-2d)+\int_0^{-x}\tilde{N}_{11}(x,t)\sin kt\,dt+\int_0^{x}\tilde{N}_{12}(x,t)\cos kt\,dt$$

ve

$$\psi_3(x,k) = \alpha^+\cos k(\pi-x)-\alpha^-\cos k(x+\pi-2d)+a_1(x)\cos k(\pi-x)+a_2(x)\sin k(\pi-x)$$
$$-b_1(x)\cos k(x+\pi-2d)-b_2(x)\sin k(x+\pi-2d)+\int_0^{-x}\tilde{N}_{21}(x,t)\sin kt\,dt+\int_0^{-x}\tilde{N}_{22}(x,t)\cos kt\,dt$$

gösterilimi vardır. Burada $i,j=1,2$ olmak üzere $\tilde{N}_{ij}(x,t)=N_{ij}(x,t)-N_{ij}(x,-t)$ şeklindedir.ve $\tilde{N}(x,t)=\left\{\tilde{N}_{ij}(x,t)\right\}_{i,j=1}^{2}$ matris fonksiyonunun $\tilde{N}_{ij}(x,t)$ elemanları her tutturulmuş $x\in[0,\pi]$ için t değişkenine göre $L_2(0,\pi)$ uzayına aittir.$q(x)=0$ durumuna karşılık gelen $\psi_i(x,k), a(x)$, $b(x)$ ve $\tilde{N}_{ij}(x,t)$ fonksiyonları sırasıyla $\psi_{0i}(x,k)$, $a_0(x)$, $b_0(x)$ ve $\tilde{N}_{ij0}(x,t)$ olarak gösterilirse

$x < d$ iken

$$\Phi_{01}(x,k) = -\alpha^{+}\sin k(\pi - x) + \alpha^{-}\sin k(x+\pi-2d) - a_{10}(x)\sin k(\pi - x) + a_{20}(x)\cos k(\pi - x)$$
$$- b_{10}(x)\sin k(x+\pi-2d) + b_{20}(x)\cos k(x+\pi-2d) + \int_0^{-x} K_{110}(x,t)\sin ktdt + \int_0^{-x} K_{120}(x,t)\cos ktdt$$

ve

$$\Phi_{03}(x,k) = \alpha^{+}\cos k(\pi - x) - \alpha^{-}\cos k(x+\pi-2d) + a_{10}(x)\cos k(\pi - x) + a_{20}(x)\sin k(\pi - x)$$
$$- b_{10}(x)\cos k(x+\pi-2d) - b_{20}(x)\sin k(x+\pi-2d) + \int_0^{x} K_{210}(x,t)\sin ktdt + \int_0^{-x} K_{220}(x,t)\cos ktdt$$

elde edilir. Buradan,

$$\Phi_{1}(x,k) = \Phi_{01}(x,k) + (a_1(x) - a_{10}(x))\sin k(\pi - x) + (a_2(x) - a_{20}(x))\cos k(\pi - x)$$
$$- (b_1(x) - b_{10}(x))\sin k(x+\pi-2d) + (b_2(x) - b_{20}(x)\cos k(x+\pi-2d)$$
$$+ \int_0^{-x} (K_{11}(x,t) - K_{110}(x,t))\sin ktdt + \int_0^{x} (K_{12}(x,t) - K_{120}(x,t)\cos ktdt$$

ve

$$\Phi_{3}(x,k) = \Phi_{03}(x,k) + (a_1(x) - a_{10}(x))\cos k(\pi - x) + (a_2(x) - a_{20}(x))\sin k(\pi - x)$$
$$- (b_1(x) - b_{10}(x))\cos k(x+\pi-2d) - (b_2(x) - b_{20}(x)\sin k(x+\pi-2d)$$
$$+ \int_0^{-x} (K_{21}(x,t) - K_{210}(x,t))\sin ktdt + \int_0^{-x} (K_{22}(x,t) - K_{220}(x,t))\cos ktdt$$

şeklinde gösterimler elde edilir.

$$f_1 = (a_1(x) - a_{10}(x))\sin k(\pi - x) + (a_2(x) - a_{20}(x))\cos k(\pi - x)$$
$$- (b_1(x) - b_{10}(x))\sin k(x+\pi-2d) + (b_2(x) - b_{20}(x)\cos k(x+\pi-2d)$$
$$+ \int_0^{-x} (K_{11}(x,t) - K_{110}(x,t))\sin ktdt + \int_0^{x} (K_{12}(x,t) - K_{120}(x,t)\cos ktdt$$

ve

$$f_2 = (a_1(x) - a_{10}(x))\cos k(\pi - x) + (a_2(x) - a_{20}(x))\sin k(\pi - x)$$
$$- (b_1(x) - b_{10}(x))\cos k(x+\pi-2d) - (b_2(x) - b_{20}(x)\sin k(x+\pi-2d)$$
$$+ \int_0^{-x} (K_{21}(x,t) - K_{210}(x,t))\sin ktdt + \int_0^{-x} (K_{22}(x,t) - K_{220}(x,t))\cos ktdt$$

olarak alınırsa

$$\psi_1(x,k) = \psi_{01}(x,k) + f_1,$$

$$\psi_3(x,k) = \psi_{03}(x,k) + f_2$$

şeklinde olur. Bu eşitlikler kullanılırsa

$$M(k) - M_0(k) = \frac{\psi_3(0,k)}{\psi_1(0,k)} - \frac{\psi_{03}(0,k)}{\psi_{01}(0,k)} = \frac{\psi_{03}(x,k) + f_2}{\psi_{01}(x,k) + f_1} - \frac{\psi_{03}(0,k)}{\psi_{01}(0,k)}$$

$$= \frac{(\psi_{03}(x,k) + f_2)\psi_{01}(0,k) - (\psi_{01}(x,k) + f_1)\psi_{03}(0,k)}{(\psi_{01}(x,k) + f_1)\psi_{01}(0,k)}$$

$$= \frac{f_3}{\Delta(k)} - \frac{f_1}{\Delta(k)} M_0(k)$$

elde edilir. Burada $k \in G_\delta$ için $\lim\limits_{|k| \to \infty} e^{-|\mathrm{Im}\, k|\pi} |f_i(k)| = 0$ ve $|\Delta(k)| > C_\delta e^{|\mathrm{Im}\, k|\pi}$ olduğu gözönünde bulundurulursa,

$$\lim_{|k| \to \infty} \sup_{k \in G_\delta} |M(k) - M_0(k)| = 0 \qquad (3.3.4)$$

olduğu alınır.

Diğer taraftan $\varphi(x,k_n)(\varphi_0(x,k_n^0))$ ve $\psi(x,k_n)(\psi_0(x,k_n^0))$ vektör fonksiyonları $L(L_0)$ probleminin özfonksiyonlarıdır. O yüzden de $\beta_n(\beta_n^0)$ sabitleri mevcuttur öyleki $\psi(x,k_n) = \beta_n \varphi(x,k_n)(\psi_0(x,k_n^0) = \beta_n^0 \varphi_0(x,k_n^0))$eşitliği sağlanır. O halde,

$$\psi_3(0,k_n) = \beta_n \varphi_3(0,k_n)$$

olduğundan

$$\beta_n = \psi_3(0,k_n) = \frac{1}{\varphi_3(\pi,k_n)}$$

$$\beta_n^0 = \psi_{03}(0,k_n^0) = \frac{1}{\varphi_{03}(\pi,k_n)}$$

eşitlikleri elde edilir.

$$\alpha_n = \Delta(k_n)\varphi_3(\pi,k_n)$$

$$\alpha_n^0 = \Delta_0(k_n^0)\varphi_{03}(\pi,k_n^0)$$

eşitliklerinden yararlanılır, $\psi_3(0,k)$ ve $\Delta(k)$, $k = k_n$ 'de analitik ve $\psi_3(0,k_n) \neq 0$, $\Delta(k_n) = 0$, $\Delta(k_n) \neq 0$ olduğundan $M(k)$ve $M_0(k)$,$k = k_n$ de basit kutup noktasına sahiptir. Buradan

$$\begin{cases} \operatorname*{Res}\limits_{k=k_n} M(k) = \dfrac{\psi_3(0,k_n)}{\Delta(k_n)} = \dfrac{1}{\Delta(k_n)\varphi_3(\pi,k_n)} = \dfrac{1}{\alpha_n} \\ \operatorname*{Res}\limits_{k=k_n} M_0(k) = \dfrac{\psi_{30}(0,k_n)}{\Delta_0(k_n)} = \dfrac{1}{\Delta_0(k_n)\varphi_{03}(\pi,k_n)} = \dfrac{1}{\alpha_n^0} \end{cases} \qquad (3.3.5)$$

elde edilir.

$$I_n(x)=\frac{1}{2\pi i}\int_{\Gamma_n}\frac{M(\mu)-M_0(\mu)}{k-\mu}d\mu,\ k\in \operatorname{int}\Gamma_n$$ eğrisel integrali ele alındığında

$\lim\limits_{|k|\to\infty}\sup\limits_{k\in G_\delta}|M(k)-M_0(k)|=0$ olduğundan $\lim\limits_{n\to\infty}I_n(x)=0$ dır.

$M(\mu)$'nün Γ_n 'deki aykırılıkları $k_0,k_1,\ldots,k_n,\ldots$ şeklinde sıralanmış kutup yerleri ve buralardaki rezidüleri sırasıyla $\frac{1}{\alpha_0},\frac{1}{\alpha_1},\ldots,\frac{1}{\alpha_n},\ldots$ dir. Γ_n, hiç bir kutup yerinden geçmeyen, üzerinde $|M(\mu)|<M$ eşitsizliğinin gerçeklendiği R_n yarıçaplı çember ve $n\to\infty$ iken $R_n\to\infty$ dur. k, $M(\mu)$'nün bir kutbu olmadığından $\frac{M(\mu)}{\mu-k}$, $\mu=k_n$, $n=0,1,\ldots$ ve k noktalarında kutup yerlerine sahiptir. Bu durumda (3.3.5) 'den de faydalanılırsa,

$$\operatorname{Re}s\left(\frac{M(\mu)}{\mu-k},k_n\right)=\lim_{\mu\to k_n}(\mu-k_n)\frac{M(\mu)}{\mu-k}=\frac{1}{\alpha_n(k_n-k)}$$

$$\operatorname{Re}s\left(\frac{M(\mu)}{\mu-k},k\right)=\lim_{\mu\to k}(\mu-k)\frac{M(\mu)}{\mu-k}=M(k)$$

olur. Rezidü teoremine göre,

$$\frac{1}{2\pi i}\int_{\Gamma_n}\frac{M(\mu)}{\mu-k}d\mu=M(k)+\sum_{k_n\in \operatorname{int}\Gamma_n}\frac{1}{\alpha_n(k_n-k)}$$

yazılır.

Benzer şekilde,

$$\frac{1}{2\pi i}\int_{\Gamma_n}\frac{M_0(\mu)}{\mu-k}d\mu=M_0(k)+\sum_{k_n^0\in \operatorname{int}\Gamma_n}\frac{1}{\alpha_n^0(k_n^0-k)}$$

elde edilir. Buna göre,

$$\frac{1}{2\pi i}\int_{\Gamma_n}\frac{M(\mu)-M_0(\mu)}{k-\mu}d\mu=-M(k)+M_0(k)-\sum_{k_n\in \operatorname{int}\Gamma_n}\frac{1}{\alpha_n(k_n-k)}+\sum_{k_n^0\in \operatorname{int}\Gamma_n}\frac{1}{\alpha_n^0(k_n^0-k)}$$

olur.O halde

$$I_n(x)=-M(k)+M_0(k)-\sum_{k_n\in \operatorname{int}\Gamma_n}\frac{1}{\alpha_n(k_n-k)}+\sum_{k_n^0\in \operatorname{int}\Gamma_n}\frac{1}{\alpha_n^0(k_n^0-k)}$$

olduğu elde edilir. Buradan da $n\to\infty$ iken limite geçildiğinde; $\lim\limits_{n\to\infty}I_n(x)=0$ olduğundan,

$$M(k)=M_0(k)+\sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n(k-k_n)}-\frac{1}{\alpha_n^0(k-k_n^0)}\right)\qquad(3.3.6)$$

elde edilir.

Mittag-Leffler açılımına göre,

$$M'(k) = \frac{1}{\alpha_0' k} + \sum_{n=1}^{\infty} \frac{1}{\alpha_n'}\left(\frac{1}{k - k_n'} + \frac{1}{k_n'}\right)$$

olur. $M(k)$ ve $M'(k)$ eşitlikleri kullanılırsa,

$$M(k) = \frac{1}{\alpha_0' k} + \sum_{n=1}^{\infty} \frac{1}{\alpha_n'}\left(\frac{1}{k - k_n'} + \frac{1}{k_n'}\right) + \sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n(k - k_n)} - \frac{1}{\alpha_n'(k - k_n')}\right)$$

$$= \frac{1}{\alpha_0' k} + \frac{1}{\alpha_0(k - k_0)} - \frac{1}{\alpha_0' k} + \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n'(k - k_n')} + \frac{1}{\alpha_n' k_n'} + \frac{1}{\alpha_n(k - k_n)} - \frac{1}{\alpha_n'(k - k_n')}\right)$$

olur. Buradan

$$M(k) = \frac{1}{\alpha_0(k - k_0)} + \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n' k_n'} + \frac{1}{\alpha_n(k - k_n)}\right)$$

eşitliği elde edilir.

3.4. Ters Problemler

Bu bölümde L probleminin belirlenmesi için Weyl fonksiyonuna ve diskret spektral verilere göre ters problemin çözümü verilmiştir.

L problemi ile beraber, $\tilde{q}(x)$ potansiyele sahip $\tilde{L}$ problemi ele alınsın ve herhangi α sembolü L problemine ait ise $\tilde{\alpha}$ sembolünün de $\tilde{L}$ problemine ait olduğu kabul edilsin.

Teorem 3.4.1: **Eğer $M(k) = \tilde{M}(k)$ ise $L = \tilde{L}$ dır. Dolayısıyla Weyl fonksiyonu L sınır değer problemini tek olarak belirtmektedir.**

İspat: $P(x,k) = [P_{jk}(x,k)]_{j,k=1,2}$ **matrisini alalım.**

$$P(x,k)\begin{pmatrix}\tilde{\varphi}_1 & \Phi_1\\ \tilde{\varphi}_3 & \Phi_2\end{pmatrix} = \begin{pmatrix}\varphi_1 & \Phi_1\\ \varphi_3 & \Phi_2\end{pmatrix},$$

eşitliği ile verilen

$$\tilde{\varphi} = \begin{pmatrix}\tilde{\varphi}_1\\ \tilde{\varphi}_3\end{pmatrix}, \quad \Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$$

çözümlerinin Wronsky determinantı için

$$W\left\{\tilde{\varphi}(x,k), \Phi(x,k)\right\} \equiv \tilde{\varphi}_1(x,k)\Phi_3(x,k) - \tilde{\varphi}_3(x,k)\Phi_1(x,k) = 1$$

olduğu gözönünde bulundurulursa,

$$\begin{pmatrix}P_{11}(x,k) & P_{12}(x,k)\\ P_{21}(x,k) & P_{22}(x,k)\end{pmatrix}\begin{pmatrix}\tilde{\varphi}_1 & \Phi_1\\ \tilde{\varphi}_3 & \Phi_2\end{pmatrix} = \begin{pmatrix}\varphi_1 & \Phi_1\\ \varphi_3 & \Phi_2\end{pmatrix}$$

eşitliğinin her iki tarafını soldan $\begin{pmatrix}\Phi_2 & -\Phi_1\\ -\tilde{\varphi}_3 & \tilde{\varphi}_1\end{pmatrix}$ **matrisi ile çarparsak,**

$$\begin{pmatrix}P_{11}(x,k) & P_{12}(x,k)\\ P_{21}(x,k) & P_{22}(x,k)\end{pmatrix}\begin{pmatrix}\tilde{\varphi}_1 & \Phi_1\\ \tilde{\varphi}_3 & \Phi_2\end{pmatrix}\begin{pmatrix}\Phi_2 & -\Phi_1\\ -\tilde{\varphi}_3 & \tilde{\varphi}_1\end{pmatrix} =$$

$$= \begin{pmatrix}\varphi_1 & \Phi_1\\ \varphi_3 & \Phi_2\end{pmatrix}\begin{pmatrix}\Phi_2 & -\Phi_1\\ -\tilde{\varphi}_3 & \tilde{\varphi}_1\end{pmatrix}$$

olur. Buradan,

$$\begin{pmatrix} P_{11}(x,k) & P_{12}(x,k) \\ P_{21}(x,k) & P_{22}(x,k) \end{pmatrix} = \begin{pmatrix} \varphi_1\tilde{\psi}_2 - \psi_1\tilde{\varphi}_3 & -\varphi_1\tilde{\psi}_1 + \psi_1\tilde{\varphi}_1 \\ \varphi_3\tilde{\psi}_2 - \psi_2\tilde{\varphi}_3 & -\varphi_3\tilde{\psi}_1 + \psi_2\tilde{\varphi}_1 \end{pmatrix}$$

elde edilir. Böylece,

$$\begin{cases} P_{11}(x,k) = \varphi_1(x,k)\tilde{\psi}_3(x,k) - \psi_1(x,k)\tilde{\varphi}_3(x,k) \\ P_{12}(x,k) = -\varphi_1(x,k)\tilde{\psi}_1(x,k) + \psi_1(x,k)\tilde{\varphi}_1(x,k) \\ P_{21}(x,k) = \varphi_3(x,k)\tilde{\psi}_3(x,k) - \psi_3(x,k)\tilde{\varphi}_3(x,k \\ P_{22}(x,k) = -\varphi_3(x,k)\tilde{\psi}_1(x,k) + \psi_3(x,k)\tilde{\varphi}_1(x,k \end{cases} \tag{3.4.1}$$

$$\begin{cases} \varphi_1(x,k) = P_{11}(x,k)\tilde{\varphi}_1(x,k) + P_{12}(x,k)\tilde{\varphi}_3(x,k) \\ \varphi_3(x,k) = P_{21}(x,k)\tilde{\varphi}_1(x,k) + P_{22}(x,k)\tilde{\varphi}_3(x,k) \\ \psi_1(x,k) = P_{11}(x,k)\tilde{\psi}_1(x,k) + P_{12}(x,k)\tilde{\psi}_3(x,k) \\ \psi_3(x,k) = P_{21}(x,k)\tilde{\psi}_1(x,k) + P_{22}(x,k)\tilde{\psi}_3(x,k) \end{cases} \tag{3.4.2}$$

olduğu alınır. (3.4.1) ve $\psi(x,k) = \dfrac{\chi(x,k)}{\Delta(k)}$ ifadelerinden yararlanılırsa,

$$\psi_1(x,k) = \frac{\chi_1(x,k)}{\Delta(k)}, \quad \psi_3(x,k) = \frac{\chi_3(x,k)}{\Delta(k)}$$

$$\tilde{\psi}_1(x,k) = \frac{\tilde{\chi}_1(x,k)}{\Delta(k)}, \quad \tilde{\psi}_3(x,k) = \frac{\tilde{\chi}_3(x,k)}{\Delta(k)}$$

yazılır. Buradan,

$$P_{11}(x,k) = \frac{1}{\Delta(k)}\left(\varphi_1(x,k)\tilde{\chi}_3(x,k) - \chi_1(x,k)\tilde{\varphi}_3(x,k)\right) =$$

$$= 1 + \frac{1}{\Delta(k)}\left[\varphi_1(x,k)\left(\tilde{\chi}_3(x,k) - \chi_3(x,k)\right) - \chi_1(x,k)(\tilde{\varphi}_3(x,k) - \varphi_3(x,k))\right]$$

$$P_{12}(x,k) = \frac{1}{\Delta(k)}\left(-\varphi_1(x,k)\tilde{\chi}_1(x,k) + \chi_1(x,k)\tilde{\varphi}_1(x,k)\right)$$

$$P_{21}(x,k) = \frac{1}{\Delta(k)}\left(\varphi_3(x,k)\tilde{\chi}_3(x,k) - \chi_3(x,k)\tilde{\varphi}_3(x,k)\right)$$

$$P_{22}(x,k) = \frac{1}{\Delta(k)}\left(-\varphi_3(x,k)\tilde{\chi}_1(x,k) + \chi_3(x,k)\tilde{\varphi}_1(x,k)\right) =$$

$$= 1 + \frac{1}{\Delta(k)}\left[\chi_3(x,k)(\tilde{\varphi}_1(x,k) - \varphi_1(x,k)) - \varphi_3(x,k)(\tilde{\chi}_1(x,k) - \chi_1(x,k))\right]$$

eşitlikleri elde edilir.

$$\varphi_1(x,k) = \varphi_{01}(x,k) + (a_1(x) - a_{10}(x))\sin kx + (a_2(x) - a_{20}(x))\cos kx$$

$$+ (b_1(x) - b_{10}(x))\sin k(2d - x) + (b_2(x) - b_{20}(x))\cos k(2d - x)$$

$$+ \int_0^x (\tilde K_{11}(x,t) - \tilde K_{110}(x,t))\sin kt\,dt + \int_0^x (\tilde K_{12}(x,t) - \tilde K_{120}(x,t))\cos kt\,dt$$

$$\varphi_3(x,k) = \varphi_{03}(x,k) + (a_1(x) - a_{10}(x))\cos kx - (a_2(x) - a_{20}(x))\sin kx$$

$$- (b_1(x) - b_{10}(x))\cos k(2d - x) + (b_2(x) - b_{20}(x))\sin k(2d - x)$$

$$+ \int_0^x (\tilde K_{21}(x,t) - \tilde K_{210}(x,t))\sin kt\,dt + \int_0^x (\tilde K_{22}(x,t) - \tilde K_{220}(x,t))\cos kt\,dt$$

ve

$$\psi_1(x,k) = \psi_{01}(x,k) + (a_1(x) - a_{10}(x))\sin k(\pi - x) + (a_2(x) - a_{20}(x))\cos k(\pi - x)$$

$$- (b_1(x) - b_{10}(x))\sin k(x + \pi - 2d) + (b_2(x) - b_{20}(x)\cos k(x + \pi - 2d)$$

$$+ \int_0^{\pi - x} (\hat K_{11}(x,t) - \hat K_{110}(x,t))\sin kt\,dt + \int_0^x (\hat K_{12}(x,t) - \hat K_{120}(x,t)\cos kt\,dt$$

$$\psi_3(x,k) = \psi_{03}(x,k) + (a_1(x) - a_{10}(x))\cos k(\pi - x) + (a_2(x) - a_{20}(x))\sin k(\pi - x)$$

$$- (b_1(x) - b_{10}(x))\cos k(x + \pi - 2d) - (b_2(x) - b_{20}(x)\sin k(x + \pi - 2d)$$

$$+ \int_0^{\pi - x} (\hat K_{21}(x,t) - \hat K_{210}(x,t))\sin kt\,dt + \int_0^{\pi - x} (\hat K_{22}(x,t) - \hat K_{220}(x,t))\cos ktd$$

eşitlikleri ve $|\Delta(k)| > C_\delta e^{|\mathrm{Im}\,k|\pi}$ olması dikkate alınarak, Lebesgue lemmasından,

$$\left\{\begin{array}{l} \lim\limits_{\substack{|k|\to\infty\\ k\in G_\delta}} \max\limits_{0\le x\le\pi} |P_{11}(x,k) - 1| = \lim\limits_{\substack{|k|\to\infty\\ k\in G_\delta}} \max\limits_{0\le x\le\pi} |P_{22}(x,k) - 1| = \\ = \lim\limits_{\substack{|k|\to\infty\\ k\in G_\delta}} \max\limits_{0\le x\le\pi} |P_{12}(x,k)| = \lim\limits_{\substack{|k|\to\infty\\ k\in G_\delta}} \max\limits_{0\le x\le\pi} |P_{21}(x,k)| = 0 \end{array}\right. \tag{3.4.3}$$

yazılır. (3.3.1) ve (3.4.1) den,

$$P_{11}(x,k) = \varphi_1(x,k)\tilde C_2(x,k) - C_1(x,k)\tilde\varphi_3(x,k) + (\tilde M(k) - M(k))\varphi_1(x,k)\tilde\varphi_3(x,k),$$

$$P_{12}(x,k) = C_1(x,k)\tilde\varphi_1(x,k) - \varphi_1(x,k)\tilde C_1(x,k) + (M(k) - \tilde M(k))\varphi_1(x,k)\tilde\varphi_1(x,k),$$

$$P_{21}(x,k) = \varphi_3(x,k)\tilde C_2(x,k) - C_2(x,k)\tilde\varphi_3(x,k) + (\tilde M(k) - M(k))\varphi_3(x,k)\tilde\varphi_3(x,k),$$

$$P_{22}(x,k) = C_2(x,k)\tilde\varphi_1(x,k) - \tilde C_1(x,k)\varphi_3(x,k) + (M(k) - \tilde M(k))\varphi_3(x,k)\tilde\varphi_3(x,k)$$

eşitlikleri elde edilir.

Böylece eğer $M(k) = \widetilde{M}(k)$ ise, her fikse edilmiş x için $P_{jk}(x,k)$ fonksiyonları k 'ya göre tamdırlar.

Ayrıca (3.4.3) 'den yararlanılırsa,

$$P_{11}(x,k) \equiv 1, P_{12}(x,k) \equiv 0, \; P_{22}(x,k) \equiv 1, \; P_{21}(x,k) \equiv 0$$

olduğu çıkar. Bunlar (3.4.2) eşitliklerinde gözönünde bulundurulursa, her x ve her k için,

$$\varphi_1(x,k) \equiv \tilde{\varphi}_1(x,k), \; \varphi_3(x,k) \equiv \tilde{\varphi}_3(x,k), \; \Psi_1(x,k) \equiv \tilde{\Psi}_1(x,k), \; \Psi_3(x,k) \equiv \tilde{\Psi}_3(x,k)$$

eşitlikleri elde edilir. Dolayısıyla $L = \tilde{L}$ dır.

Teorem 3.4.2: Eğer her $n \in Z$ için $k_n = \tilde{k}_n$, $\alpha_n = \tilde{\alpha}_n$ ise o halde $L = \tilde{L}$ dır. Dolayısıyla discrete spektral veriler, L problemini tek olarak belirtmektedir.

İspat:

$$M(k) = \frac{1}{\alpha_0(k - k_0)} + \sum_{n=1} \left(\frac{1}{\alpha_n^0 k_n^0} + \frac{1}{\alpha_n(k - k_n)} \right)$$

$$\widetilde{M}(k) = \frac{1}{\tilde{\alpha}_0(k - \tilde{k}_0)} + \sum_{n=1} \left(\frac{1}{\tilde{\alpha}_n^0 \tilde{k}_n^0} + \frac{1}{\tilde{\alpha}_n(k - \tilde{k}_n)} \right)$$

ve her $n \in Z$ için $k_n = \tilde{k}_n$, $\alpha_n = \tilde{\alpha}_n$ olduğundan,

$$M(k) = \widetilde{M}(k)$$

olur.

O halde $\widetilde{M}(k) = M(k)$ olur ki Teorem 3.4.1 den $L = \tilde{L}$ dır.

Teorem 3.4.3: Eğer her $n \in Z$ için $k_n = \tilde{k}_n$, $\mu_n = \tilde{\mu}_n$ ise $L = \tilde{L}$ dır, yani $\{k_n\}$ ve $\{\mu_n\}$ dizileri L problemini tek olarak belirtir.

İspat: $\Delta(k)$ ve $\tilde{\Delta}(k)$ fonksiyonlarının özelliklerinden,

$\lim_{k \to \infty} \frac{\Delta(k)}{\tilde{\Delta}(k)} = 1$ olduğu açıktır. $k_n = \tilde{k}_n$ ve $\Delta(k)$ ile $\tilde{\Delta}(k)$ fonksiyonları analitik olduğundan analitik fonksiyonların tekliği teoreminden $\Delta(k) = \tilde{\Delta}(k)$ olduğu çıkar.

$\Omega(x, k_n) = \beta_n \varphi(x, k_n)$ olduğundan

$\tilde{\Omega}(x, \tilde{k}_n) = \tilde{\beta}_n \varphi(x, \tilde{k}_n) = \tilde{\beta}_n \tilde{\varphi}(x, k_n)$ ve

$\tilde{\varphi}(x,\tilde{k}_n) = \tilde{\varphi}(x,k_n) = \beta_n \varphi(x,k_n)$ eşitlikleri elde edilir. Bu eşitliklerinden yararlanılırsa, $\beta_n = \tilde{\beta}_n$ olur. Ayrıca $\Delta(k) \equiv \tilde{\Delta}(k)$ olduğundan $\dot{\Delta}(k) \equiv \dot{\tilde{\Delta}}(k)$ olur. Dolayısıyla $\alpha_n = \dot{\Delta}(k_n)\varphi_3(\pi,k_n) = \dot{\Delta}(k_n).\frac{1}{\beta_n}$ olduğundan $\alpha_n = \tilde{\alpha}_n$ elde edilir.

O halde Teorem 3.4.2 den $L = \tilde{L}$ olur.

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ÖZGEÇMİŞ

Baki Keskin 1975 yılında Kangal'da doğdu. İlk öğrenimini Kangal'ın Kızılleniş köyünde, orta okulu Isparta'da ve Liseyi İzmir'de tamamladı. 1991 yılında Orta Doğu Teknik Üniversitesi Fen Edebiyat Fakültesi Matematik bölümünü kazandı ve 1996 yılında mezun oldu. 1997 yılında Cumhuriyet Üniversitesi Fen Edebiyat Fakültesi Matematik bölümünde Araştırma Görevlisi olarak çalışmaya başladı. 1997 yılında Cumhuriyet Üniversitesi Fen Bilimleri Enstitüsü'nde yüksek lisans öğrenimine başladı ve 2000 yılında mezun oldu. Halen Cumhuriyet Üniversitesi Fen Edebiyat Fakültesi Matematik bölümünde Araştırma Görevlisi olarak çalışmaktadır.