THE REPUBLIC OF TURKEY
BAHCESEHIR UNIVERSITY

DEVELOPMENT OF THE STRUCTURAL MODEL FOR A THIN WALLED OPEN PROFILE RECTANGULAR CANTILEVER BEAM WITH LONGITUDINALLY EMBEDDED PIEZOELECTRIC

Master of Science Thesis

ALI GOLPAYGANI

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THE REPUBLIC OF TURKEY

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MASTERS PROGRAM IN MECHATRONICS ENGINEERING

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ABSTRACT

DEVELOPMENT OF THE STRUCTURAL MODEL FOR A THIN WALLED OPEN PROFILE RECTANGULAR CANTILEVER BEAM WITH LONGITUDINALLY EMBEDDED PIEZOELECTRIC

Golpaygani, Ali

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Morphing blade can change shape and adapt to various operating conditions, resulting in more efficient wing operation. In addition, shape control can be used for active vibration suppression, thereby improving the service life of the blade. In this thesis, a thin walled, open rectangular profile beam is used as a simplified model of morphing wind turbine blades. Piezoelectric strips are assumed to be embedded to the outer surface of the beam for shape and vibration control. An analytic elastic structural model of the beam is developed based on the coupled flexural-torsional model using energy approaches. A general model for the interaction between the beam and the piezoelectric strips is assumed. Due to the shortcomings of this interaction model, two equivalent models are developed. The analytic results obtained from the structural model are compared and validated with the results obtained from a numerical finite element software.

Keywords: Morphing Blades, Open Profile, Piezoelectric, Coupled Flexural-Torsional Model

ÖZET

BOYUNA PİEZOELEKTRİK ŞERİTLİ, İNCE DUVARLI, AÇIK DİKDÖRTGEN PROFİLE SAHİP ÇIKMA KİRİŞİN YAPISAL MODELİ

Ali Golpaygani

Mekatronik Mühendisliği

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Değişken ortam koşullarına uygun olarak şekil değiştirmesi, türbin kanatlarının performansını arttırabilmektedir. Bunun yanı sıra, şekil kontrolü ile aktif titreşim sönümlemesi yapılarak kanatların ömürleri uzatılabilmektedir. Bu tezde, şekil değiştirebilen rüzgar türbin kanatlarını modellemek için ince cidarlı, açık ve dikdörtgen kesitli bir kirişin çözümsel modelin geliştirilmiştir. Şekil ve titreşim kontrolünün, kirişin yüzeyine yapıştırılan piezoelektrik şeritler vasıtası yapılacağı varsayılmıştır. Literatürde yer alan benzer çalışmalardan farklı olarak kirişin bağlaşık bükülgen-burulma kuramı dayalı elastik yapısal modeli enerji yöntemleri kullanılarak geliştirilmiştir. Kirişin yüzeyine yapıştırılan piezoelektrik şeritlerin elektromekanik davranışı ve şeritlerin kiriş ile etkileşimi için genel bir matematiksel model kullanılmıştır. Bu genel modelin eksikliklerini telafi edebilmek için iki farklı eşdeğer kiriş kesit modeli geliştirilmiştir. Matematiksel modeller üzerinden elde edilen çözümlemeler, sonlu elemanlar yazılımı ile sayısal olarak doğrulanmıştır.

Anahtar Kelimeler: Değişken Şekilli Kanat, Açık Kesitli Profil, Piezoelektrik, Bağlaşık Bükülgen-Burulma Modeli

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SYMBOLS

Aerodynamic center	:	A_e
Axial load	:	P
Aerodynamic center x coordinate	:	x_A
Average warping function	:	$\overline{\omega}_s$
Aerodynamic pressure	:	q_∞
Airfoil chord length	:	c
Angle of attack	:	α
Bending moment in x direction	:	M_x
Bending moment in *x* direction (when *x* is principal axis)	:	M_1
Bending moment in *y* direction	:	M_y
Bending moment in *y* direction (when *y* is principal axis)	:	M_2
Blade length	:	l
Blade angular speed	:	ω_b
Blade tangential speed	:	V_b
Cross sectional area	:	A
Centroid location	:	C
Curve of the cross section profile	:	s
Cross section width	:	b
Displaced centroid location	:	C'
Displaced and rotated centroid location	:	C''
Deflection of shear center in *x* direction	:	u
Deflection of shear center in *y* direction	:	v
Deflection in longitudinal direction	:	w
Displaced shear center location	:	O'
Deflection of point *B* in *x* direction	:	u_B
Deflection of point *B* in *y* direction	:	v_B

Deflection of point *B* in z direction	:	w_B
Displaced location of point *B*	:	B'
Displaced and rotated position of point *B*	:	B''
Drag coefficient	:	C_D
Drag section coefficient	:	c_d
Drag force	:	D
Drag force per unit length	:	D'
Distance between shear center O and aerodynamic axis A_e	:	e
Distributed fictitious load in x direction	:	q_x
Distributed fictitious load in y direction	:	q_y
Displacement in the direction of the tangent to the cross section profile	:	w_s
Equivalent second moment of area (*x*-*x*)	:	I_{ex}
Effective moment on the beam (beam & piezo)	:	M_1^*
Equivalent overall thickness of the cross section	:	t_{eq}
Free airstream velocity	:	U_∞
Fictitious torque per unit length	:	μ
Location of an arbitrary point *B*	:	B
Lift coefficient	:	C_L
Lift section coefficient	:	c_l
Lift force	:	L
Lift force per unit length	:	L'
Longitudinal strain at point *B*	:	ε_b
Nominal air density	:	ρ_∞
Normal stress due to P , M_1 and M_2	:	σ
Polar area moment of inertia about shear center	:	I_o

Pitching moment coefficient	:	C_M
Pitching moment section coefficient	:	c_m
Pitching moment at a distance of quarter of the chord measured from the leading edge	:	$c_{m,c/4}$
Pitching moment	:	M
Pitching moment per unit length	:	M'
Potential of external loads	:	W_{ext}
Proportionality constant between v and M_1	:	K
Proportionality constant between v and M_1^*	:	K^*
Piezoelectric Charge Constant	:	d_{31}
Reference area of the turbine	:	S
Relative airstream velocity	:	V_∞
Shear center *x* coordinate	:	x_0
Shear center *y* coordinate	:	y_0
Shear center location before cross sectional deformations	:	O
Shear force in *x* direction	:	V_x
Shear force in *y* direction	:	V_y
Shear modulus	:	G
Shear strain at the midline of the profile thickness	:	γ_{sz}
Strain induced by piezoelectric layer	:	ε_p
Second moment of area about *x*-axis	:	I_{xx}
Second moment of area about *y*-axis	:	I_{yy}
Twisting angle of the cross section	:	ϕ
Total resisting torque	:	T
Torsional constant	:	I_t
Thickness of the cross section	:	t
Total potential energy	:	Π

Total strain energy	:	U
Thickness of the piezoelectric layer	:	t_p
Torque due to uniform torsion	:	T_t
Variational operator	:	δ
Voltage of piezoelectric layer	:	ν
Warping function	:	ω_s
Warping modulus	:	I_ω
Warping torque	:	T_ω
Warping displacement	:	w_w
Warping displacement at $s=0$	:	w_0
Width of piezoelectric layer	:	b_p
Young's modulus	:	E
Young's modulus of the piezoelectric material	:	E_p

1. INTRODUCTION

Controlling the shape of a structure in a desired form is of interest to the structural engineering professionals. There are many ways to adapt the shape to a new form to cope with the new environmental and working conditions. Changing the skin shape of the structure by the help of smart and/or memory shape materials or internal mechanisms are common ways which are widely used in aerospace and other industries. Despite the fact that of huge progresses in the field, there are rarely mathematical solutions which can describe the new shape by mathematical functions. This comes from the point of high complexity of the structures and actuators in use.

Amongst many strategies, using piezoelectric actuator patches on the structure surface has the advantage of having reliable mathematical model regarding its structural and mechanical behavior. Beside the ease of use of the piezoelectric actuators to control the shape, one have to know the physics and the governing differential equations of the main structure to predict the behavior of the whole structure.

In this report an open cross-section beam-column subjected to end moments and axial forces has been studied. The coupled Flexural-Torsional behavior of the structure investigated and the governing differential equation are being introduced, by force balance and minimum potential energy methods. The piezoelectric actuation and its mathematical model as a concentrated moment at the tip of the structure due to the polarization effect has been derived. A mathematical model in static mode is built and the results are compared with a numerical model solved in COMSOL Multiphysics.

The study is not confined to static shape control and can be used in dynamic and vibration control of thin-walled structures. A special application is to control the angle of attack in wind turbine blade and airplane wing structures.

2. LITERATURE REVIEW

One of the most important points in the study of wind-turbine blade design is to consider the load alleviation and efficiency. These two important concepts are highly interrelated as both have their roots in the aerodynamic forces acting on the blade surface. Many design and manufacturing methods has been developed and used to overcome unnecessary loads and increasing the efficiency since inventing the wind-turbines. Most of the efforts done by aerospace industry are based on holistic approaches when one reviews the literature on structural perspective of the problem[1]. In other words, it is very difficult or highly time consuming to construct and solve a mathematical models which can predict the structural behavior for the moment.

From practical and experimental point of view, there are many researches carried out to improve the performance and controllability of flexible wind turbine blades based on aerospace experiences. Despite the fact that these accomplishments have had efficient improvements on blade fatigue, (Van Kuik et al, December 2006, p. 5) they have led to models which are:

a. “Too complicated: guy wires, springs, hinges, elastomeric bumpers
b. Not reliable enough
c. Too far away from commercial technology
d. In case of stall aerodynamics not yet sufficiently understood
e. Not all concepts are up-scalable”

Currently special interests are toward the using of morphing structures for wind-turbine blades. Although there are similarities between the wind-turbine blades and helicopter blades, but most research concepts in airplane wing design seems fit more to wind turbine application. In Figure 2.1: Morphing wing classification different morphing wing concepts are shown. These concepts are categorized as bellow:

1. In-plane alteration

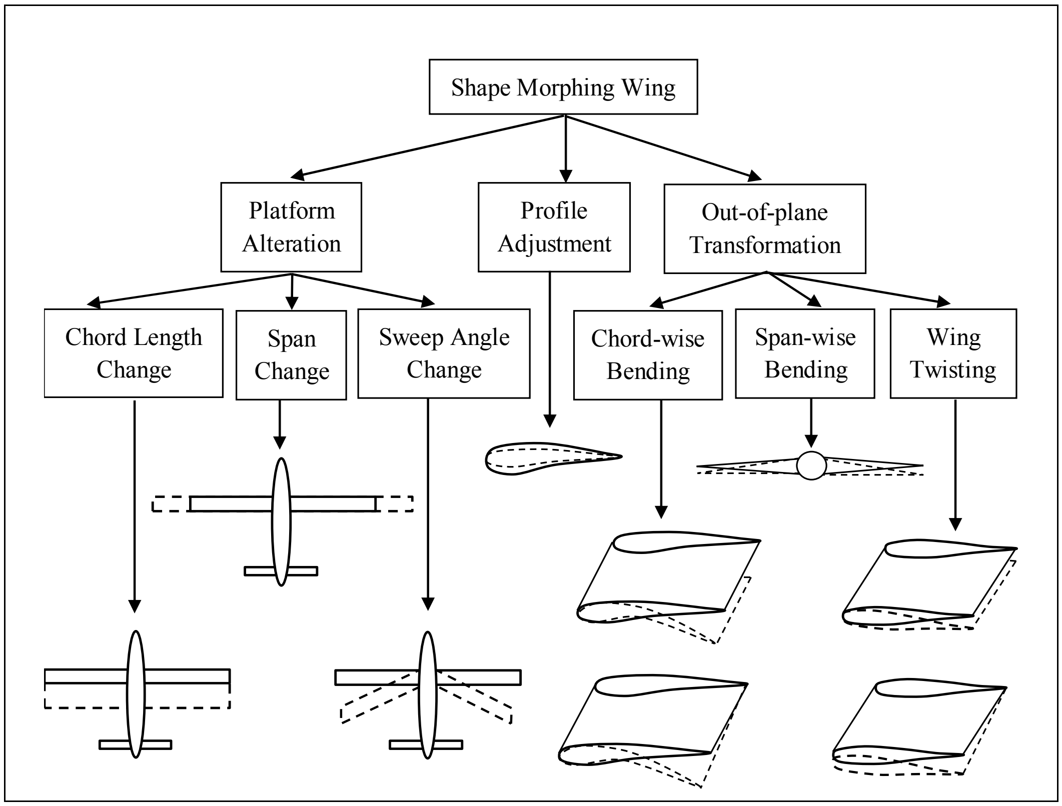
[1] Wenbin Yu, Associate Professor, Utah State University CTO, AnalySwift LLC, Dewey H. Hodges, Proffessor, Georgia Institute of Technology, Senior Consultant, AnalySwift LLC

1.1 Chord length change

1.2 Span change

1.3 Sweep angle change

Figure 2.1: Morphing wing classification



Source: Adapted from A.Y.N. Sofla et al, Journal of Materials and Design 31 (2010) 1284–1292

2. Morphing materials
3. Out-of-plane transformation

3.1 Chord-wise bending

3.2 Chord-wise twisting

3.3 Span-wise bending

Another missing concept in Figure 2.1: Morphing wing classification, which is introduced in this thesis is categorized under the third concept and characterized as coupled Flexural-Torsional model:

3.4 Span-wise bending-twisting

This concept is studied by Vladimir Fedorov (Fedorov, 2012) to investigate the effect of Bend-Twist coupling on lowering destructive loads and fatigue on the blade structure.

2.1 IN-PLANE ALTERATION

2.1.1 Span Change

This type of structural changes will expand the length of the blade by using telescopic parts hidden into the main blade. This telescopic extension part is used during lower speed wind times to collect more energy from bigger capture area (Scott J. Johnson, 2008).

Although this design has higher efficiency and load control capability compared with standard blade, the following difficulties have yet to be overcome:

a) Complex structures and mechanisms
b) Structural integrity of the blade specially when extended
c) Higher weight
d) Low response in controlling the extension and retraction

2.1.2 Chord Length Change

Chord length can be changed by means of mechanically actuated mechanisms as used in airplane wings. A concept model proposed by GE (Bonnet, 2007) in which the blade extends its chord by means of a four bar linkage mechanism. The main problem of this design is higher nose caused by discontinuity on the original airfoil, vibration, and lower efficiency due to turbulence.

Another concept using mechanisms inside the blade using two ribs for leading and trailing edge of the blade has been reported by Galanti (Vlad Paul Galantai, 2010). Both designs benefit from higher load controllability, but they suffer from complex actuation mechanisms, low response, and complicated control system.

2.1.3 Sweep Angle Change

This concept has been reported by many researchers and used in aircraft models like F14 Tomcat, and B-1B Lancer. The mechanisms used in these applications are very complicated to be applied in slender wind turbine blade. There are potential benefits using this technology in wind turbine blade design, but although they have shown potential increase in lift/drag enhancement characteristics these techniques are not applied in wind turbine blade design yet.

2.2 MORPHING MATERIALS

Morphing skins concepts have their most application on chord wise direction when strong substructures can bear span wise and aerodynamic forces. The morphing shapes can be actively controlled or adjusted via structural compounding of the skin. The stiffness of the skin might be tailored to approach the desired shape under forthcoming aerodynamic forces acting on the blade surface. There are many materials and concepts studied by the researchers for wing fabrication.

2.2.1 Stretchable Structures

2.2.1.1 Elastomers

Elastomers are a kind of polymers with high flexibility and good shape memory. Their low tensile strength let them to undergo an elongation up to 1000% before rupture (Thill et al, March 2008). This property comes from its low cross-links between the polymer chains (Kikuta, 2003).

When a design is based on using elastomers for morphing skin, the following criteria should be under consideration (Kikuta, 2003):

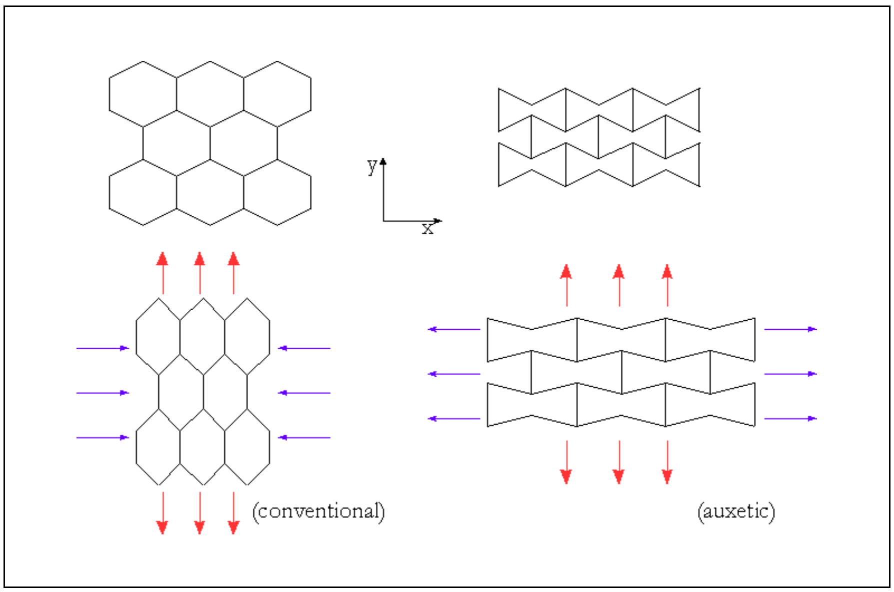
1. elastic/flexible in chord-wise direction to allow low force actuation,
2. stiff in span-wise direction to withstand aerodynamic and
3. inertial loads,
4. toughness,
5. abrasion and chemical resistant,
6. resistant to different weather conditions,
7. high strain capability,

8. high strain recovery rate,
9. environmental longevity and fatigue resistance

2.2.1.2 Auxetic materials

Auxetic materials or negative Poisson's ratio materials, have interesting property of become narrower when they are under pressure and wider when exposed to tension. In **Error! Reference source not found.** (Thill et al, March 2008) a comparison between onventional honeycomb and auxetic structures has been shown.

Figure 2.2: Auxetic material (negative Poisson's ratio)



Source: http://groups.exeter.ac.uk/auxetic/, accessed Jan 15, 2015

These materials are rarely available in nature (caw and cat skins), but can be fabricated by modifying existing foams, and polymers. Based on their high energy absorption, shear stiffness, plane strain fracture toughness, and indentation resistance they are ideal candidate core material for morphing structures. There are other materials used in aerospace industry which have been reviewed by Thill *et al* (Thill et al, March 2008).

1. Rollable and collapsible structures
2. Foldable structures
3. Inflatable structures

4. Overlapping/stacked/nested structures
5. Extreme anisotropic material
6. Multi-stable composites
7. Segmented structures
8. Folded inner skins
9. Multilayered skins
10. Shape memory alloys
11. Shape memory polymers
12. Elastic memory composites
13. Shape memory textiles
14. Magnetic shape memory materials
15. Flexible matrix composites

2.3 OUT-OF-PLANE TRANSFORMATION

2.3.1 Chord-Wise Change

Out-of-plane chord-wise change includes adding a series of flaps which can have out of the plane movements. The idea comes from airplane flaps to increase the lift force during approach and landing.

Using morphing flap has benefit of lowering maintenance costs compared to pivoted or discrete ailerons, but both concepts have great influences on performance of wind turbine according to the review study by Xavier Lachenal *et al* (Lachenal et al, 2013). A prototype wind turbine has been made and tested by UpWind project in which the effectiveness of trailing edge flaps been proved (Hulskamp et al, 2010).

2.3.2 Chord-Wise Twisting

This concept has been recently studied by Roelof Vos *et al* (Vos et al, 2010), in which an open cross section airfoil model is being loaded at its trailing edge discontinuity. The forces has been applied by means of screw system on the shell structure causing the total structure to be rotated around an aluminum bar acting as the main spar.

The concept was mainly developed for helicopter and it should be justified for wind-turbine applications. A model based on this concept has been constructed and tested at Penn State University in the US (Nagelsmit, 2014).

The structural model is based on the torsion theory of open cross-sections with the boundary conditions of free-free rotation ends. The boundary condition at the base would cause lack of structural integrity.

2.3.3 Span-Wise Bending

The benefits of using an elliptical and cambered span wing design will cause to reductions in induced drag were obtained by the use of nonplanar lifting surfaces. This aerodynamic proven concept has been used by Wiggins et al (Wiggins, 2003). Despite the fact this design benefits from aerodynamic aspect of its wing shape, it suffers mainly from structural integrity and high weight. In addition application of this concept in wind-turbine yet to be investigated.

2.3.4 Span-Wise Bending-Twisting

There are many studies on flexural-torsional coupled deformation of beam like structures. This concept has been investigated and theoretical solutions to the related problem exists for them. Among the pioneer scientists who have considered these kind of structures are elasticity theorist Saint-Venant (France), Vlasov (Russia), and Timoshenko (Ukraine). More recent studies also have been carried by other researchers like Wai Keung Tso (Tso, 1964), G.I Ioannidis et al (Ioannidis et al, 1999), Engin Emsen (Emsen, 2006), and many others.

Vladimir Fedorov (Fedorov, 2012) has studied the coupled behavior of a wind-turbine blade and has some suggestion for tailoring the structural stiffness of the blade toward the desired shape adaptation.

Special interests had been toward the open cross-section beams and columns, in which flexural –torsional coupling behavior is dominant. Because of this behavior these structures are of the great potential candidate for modeling a wind-turbine blade.

There is also many research on simple shell and smart beams in the literature. The beams with a simple rectangular cross-section is being treated as an open cross-section in theory of structure. Alvord Marques (Marques, 2008) in his master thesis investigated the behavior of a multilayer elastic beam under different loading conditions.

Another interesting application is presented by W. Raither et al (Raither et al, 2012) in which a hybrid open cross section airfoil has been analyzed. Stiffness can be manipulated by using piezoelectric actuators in the main spar, and openness-closeness are also virtually acquired by using piezoelectric layers.
In other words, by designing an appropriate controller during adaptation process, the cross-section will be open in order to minimize the energy consumption.
This concept is a semi passive, as the deformation would come from external aerodynamic forces, and you have just the structural stiffness under control.

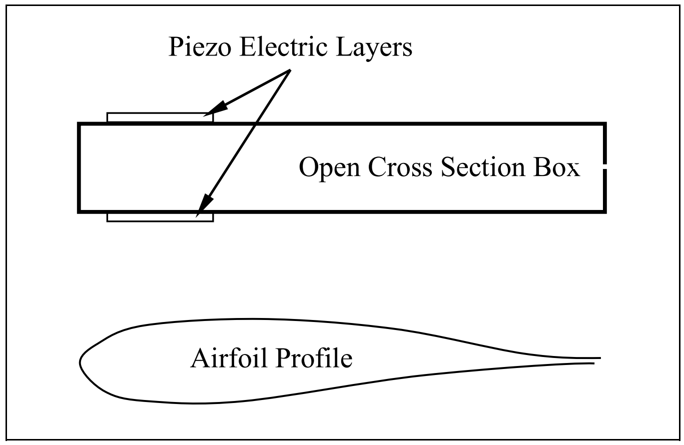
Another model based on the open-section concept is proposed by Weddingen *et al* (Van Weddingen et al, 2010). He proposed an open cross-section airfoil to be used to the benefit of its lower torsional stiffness.

He also suggested to cover the open leading and trailing sections to be sealed with soft sealant to prevent entrance of dusts and contaminants into the blade compartment. In the report the actuation mechanisms are not explained, which could be a big concerns to deform the structure to the desired shape. Neither W. Raither, nor Weddingen, have not any proposal for mathematical model of their concepts.

Presented model in this thesis is also based on an open cross section airfoil. This model is a beam column from structural point of view, and the open part is at the trailing edge of the airfoil. To the extent of this thesis a differential equation based on the theory of warping torsion under the exposure of an end moment is developed. The actuation loads are exerted by two piezoelectric layers at the top and bottom of the blade. As shown in Figure 2.3, an open profile box with similar overall dimensions to its counterpart airfoil is selected for validation purpose.

The strain induced by these layers and equivalent moment at the tip are mathematically modeled, and a solutions are derived.

Figure 2.3: Morphing blade with piezoelectric layers



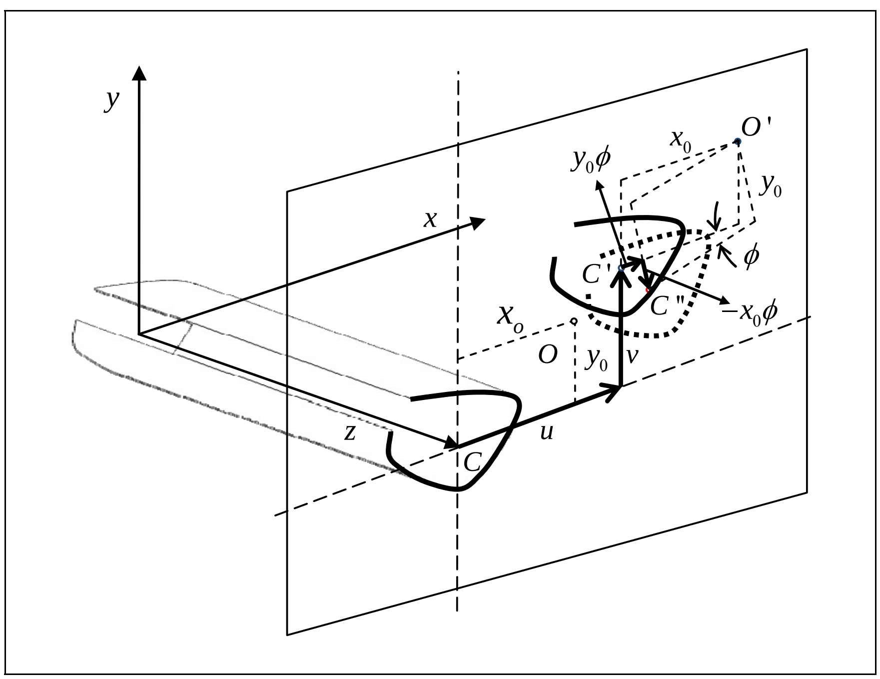
3. COUPLED FLEXURAL-TORSIONAL DEFORMATION

When a beam is under exposure of axial forces, moments, torque any combinations of these loading it will deflect in lateral directions. It might also rotates about the longitudinal axis. This kind of behavior would be much more severe when the beam has an open cross-section, as the flexural and torsional stiffness of open cross-section beams are smaller than the closed profiles. This phenomenon will happen during free vibration of the structure as well as the structures under time dependent or static forces.

3.1 KINEMATICS OF CROSS-SECTIONAL DEFORMATIONS

A complete analysis is not possible if the cross-sectional rotation which plays an important role in any combined flexural-torsional deformation is not taken into account.

Figure 3.1: Centroid displacements in coupled bending-twisting



In Figure 3.1 the cross-section of the beam will undergo into three different displacements consisting of two translations and one rotation. These displacements are represented by u, v, and ϕ and describing the deflections of shear center in x and y directions, and rotation about the shear center respectively. By definition, shear center is a fictitious point where there is no torsion if the line of action of a shear force passes through it. During any coupled deformation of the beam, the centroid of the cross section - C, will be displaced into a new position which can be defined by the coordinates of $\left(u + y_0\phi, v - x_0\phi\right)$. This new location is the combination of deflection of the centroid C due to translation from C to C', (u, v) plus deflection of C' to C'' due to rotation of C' around the displaced shear center O'.

This rotation causes $\overrightarrow{OC} = \overrightarrow{O'C'} = -x_0\vec{i} - y_0\vec{j}$ - a vector with origin at the shear center and end at the centroid - to rotate about newly displaced shear center O' to the amount of ϕ so that its end C' will reach to a new position C''. One can determine this new position by using rotation matrix as

$$\begin{bmatrix} X_{C''} \\ Y_{C''} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} -x_0 \\ -y_0 \end{bmatrix}, \qquad (3.1)$$

So the new vector $\overrightarrow{O'C''}$ pointed from O' to the centroid C'' can be shown by its components as,

$$\overrightarrow{O'C''} = (-x_0\cos\phi + y_0\sin\phi)\vec{i} + (-x_0\sin\phi - y_0\cos\phi)\vec{j}. \qquad (3.2)$$

By using small angle approximation, the vector $\overrightarrow{O'C''}$ will have the following form,

$$\overrightarrow{O'C''} = (-x_0 + y_0\phi)\vec{i} + (-x_0\phi - y_0)\vec{j} \qquad (3.3)$$

Now displacement vector $\overrightarrow{C'C''}$ can be defined as,

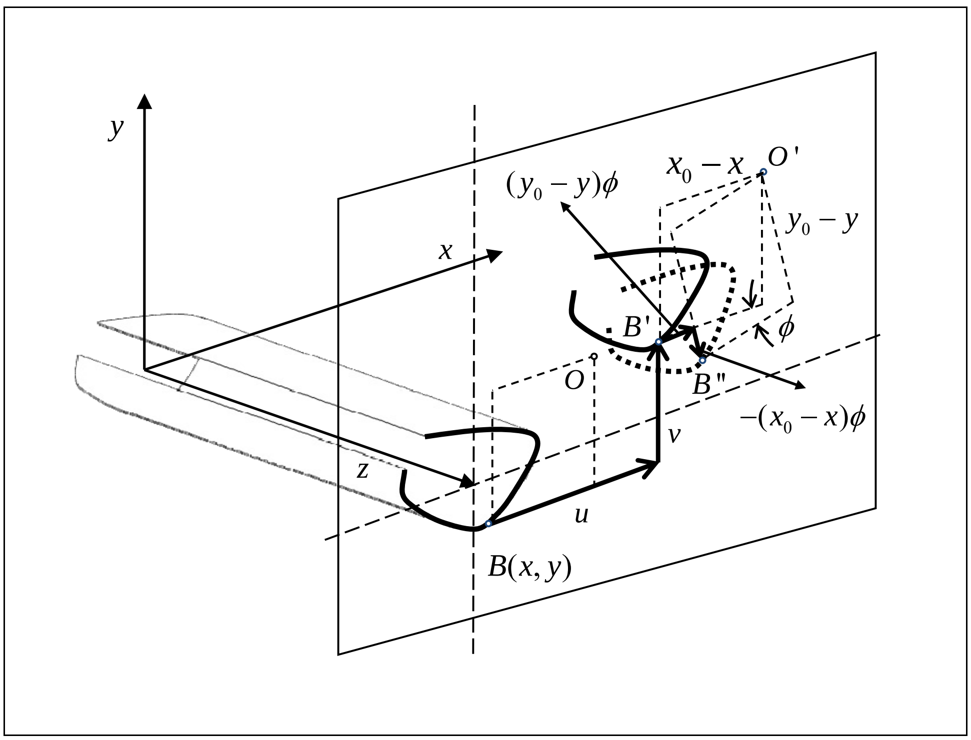
$$\overrightarrow{C'C''} = \overrightarrow{O'C''} - \overrightarrow{O'C'} \tag{3.4}$$

$$\overrightarrow{C'C''} = \left[(-x_0 + y_0\phi)\vec{i} + (-x_0\phi - y_0)\vec{j} \right] - \left[-x_0\vec{i} - y_0\vec{j} \right] = y_0\phi\vec{i} - x_0\phi\vec{j} \tag{3.5}$$

Total displacement vector $\overrightarrow{CC''}$ of the centroid from C to C'' is finally in the form of

$$\overrightarrow{CC''} = \overrightarrow{CC'} + \overrightarrow{C'C''} = \left(u + y_0\phi\right)\vec{i} + \left(v - x_0\phi\right)\vec{j} \tag{3.6}$$

Figure 3.2: Deflections of an arbitrary point B in coupled bending-twisting



In the same way one can find total deflection of any point B on the cross by adding translation and rotation counterparts. As it is described in Figure 3.2, a part of the point B deflection is due to translation of the cross section which are defined by u, and v - the same as of the centroid C.

If the coordinates of B is defined by x, and y, then the deflection of point B due to rotation around displaced shear center O' are be defined as $(y_0 - y)\phi$, and $-(x_0 - x)\phi$. Now the total deflection of an arbitrary point B during flexural-torsional deformation are defined as:

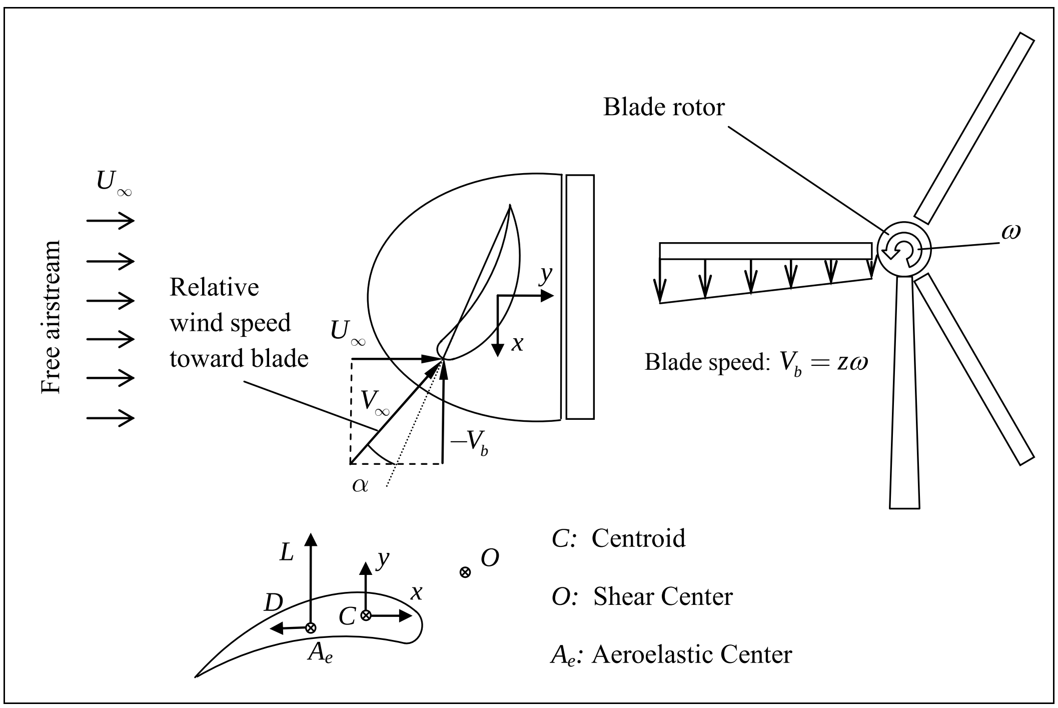
$$u_B = u + (y_0 - y)\phi \tag{3.7}$$

$$v_B = v - (x_0 - x)\phi \tag{3.8}$$

3.2 AERODYNAMIC LOADS

There are different loading forms exerted on the blade which their effects have to be taken into consideration. In this study the effects of gravity and centrifugal forces of rotating blade are ignored. A careful observation shows that the most important loads are aerodynamic forces (two lateral forces and one pitching moment), Figure 3.3: Aerodynamic loads on a typical wind turbine blade.

Figure 3.3: Aerodynamic loads on a typical wind turbine blade



The total lift force is defined by (Anderson, Jr., 2011, p. 29):

$$L = C_L q_\infty S \tag{3.9}$$

Similarly, the drag force and pitching moment can be described as:

$$D = C_D q_\infty S \tag{3.10}$$

$$M = C_M q_\infty S l \tag{3.11}$$

where S is the reference area of the turbine, l is the blade length, q_∞ is the aerodynamic pressure, C_L is the lift coefficient, C_D is the drag coefficient, and C_M is the pitching moment coefficient.

These loads are function of different variables including air density, air velocity, chord length of the airfoils, Reynolds's number, and Mach number. But for subsonic situations with low Reynolds's number these loads are treated as not being affected by these two recent factors.

The Equations (3.9), (3.10), and (3.11) represents total loads acted on the structure. A more useful form of these variables are ones which are defined for a unit length of the blade,

$$L' = c_l c q_\infty \tag{3.12}$$

$$D' = c_d c q_\infty \tag{3.13}$$

$$M' = c_m c^2 q_\infty \tag{3.14}$$

Here c is the airfoil chord length at the cross section under study. The length l and the reference area S are replaced by the chord length c and its square c^2, respectively for calculating local load coefficients. For uniform prismatic blades c is constant. These forces are functions of the local conditions at each section throughout the blade span.

Aerodynamic pressure q_∞ is the same for total and local loads and equals to,

$$q_\infty = \frac{1}{2}\rho_\infty V_\infty^2 \tag{3.15}$$

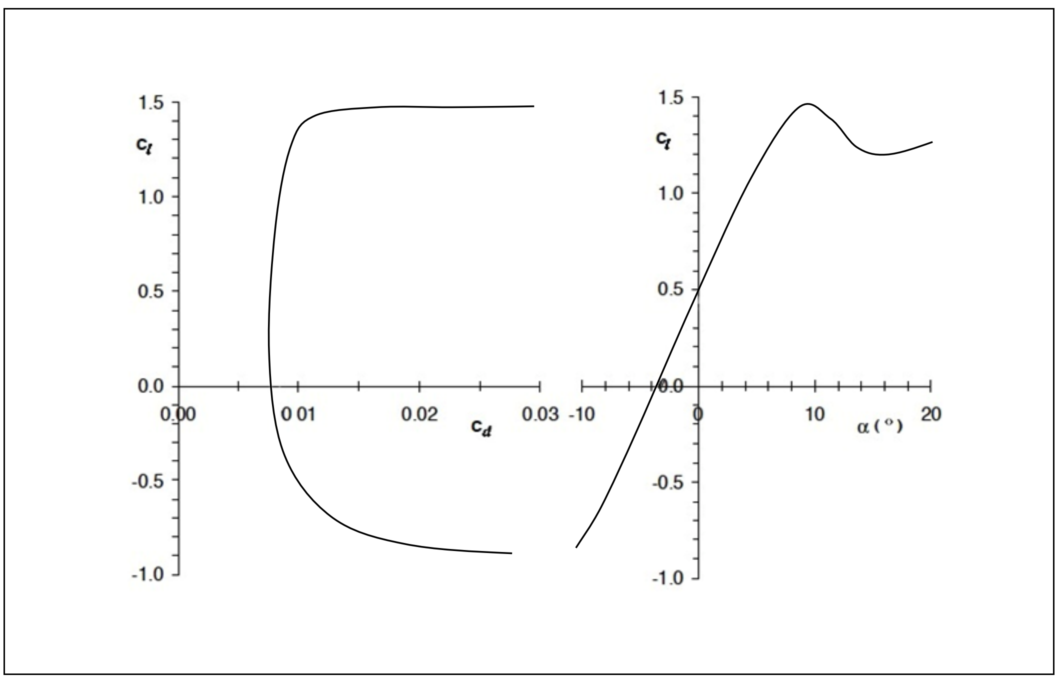
As it is shown in Figure 3.3, the wind turbine blades are rotating at an angular speed of ω, so the relative velocity should be calculated as:

$$V_\infty = (U_\infty^2 + V_b^2)^{1/2} = (U_\infty^2 + z^2\omega^2)^{1/2} \tag{3.16}$$

3.3 LIFT, PITCHING MOMENT, AND DRAG SECTION COEFFICIENT

It would be useful if one could describe lift force intensity as a function of angle of attack. Fortunately, there is a theoretical basis for treating c_l as a linear function of α, the angle of attack as shown in Figure 3.4: Typical thin airfoil.

Figure 3.4: Typical thin airfoil data



Source: Adapted from W.A. Timmer *et al*, SUMMARY OF THE DELFT UNIVERSITY WIND TURBINE DEDICATED AIRFOILS, AIAA-2003-0352

This assumption is valid especially for symmetric thin airfoil for which lift coefficient is formulated as (Anderson, Jr., 2011, p. 344):

$$c_l = 2\pi\alpha \qquad (3.17)$$

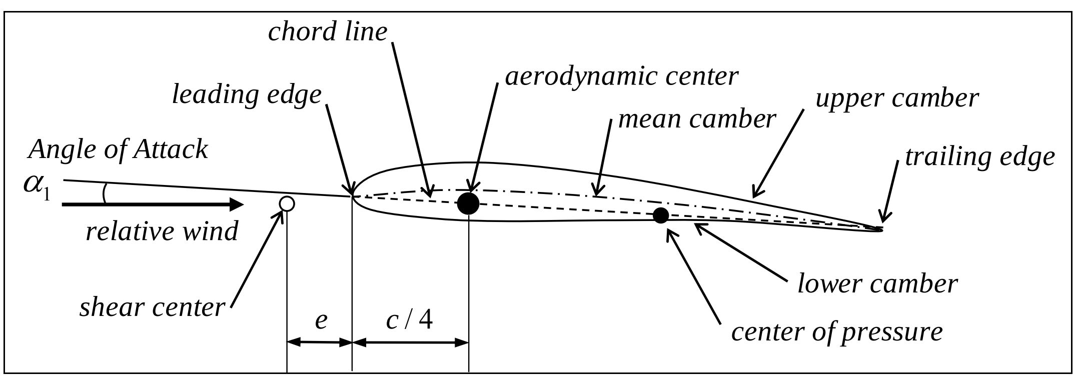
This theoretical concept also predicts that section pitching moment coefficient $c_{m,c/4}$ is zero for a wide range of α at a distance of quarter of the chord measured from the leading edge.

$$c_{m,c/4} = 0 \qquad (3.18)$$

Despite the fact that there is not any linear relationship between drag section coefficient and angle of attack, a close look at the experimentally available Lift to Drag ratio diagrams (c_l / c_d polar plots) will clarify that these ratio is in a range of 10-100 for most airfoils. These ratios are described using Figure 3.4: Typical thin airfoil data.

In Figure 3.5: Airfoil nomenclature, the quarter chord point and other important definitions of a typical airfoil are described. The center of pressure is defined as a point at the airfoil cross section at which the pitching moment due to aerodynamic loads is zero, if all forces are displaced and the resulting moments added to that point. By this definition, one can conclude also that the quarter chord point and center of pressure are coincide for thin airfoil.

Figure 3.5: Airfoil nomenclature

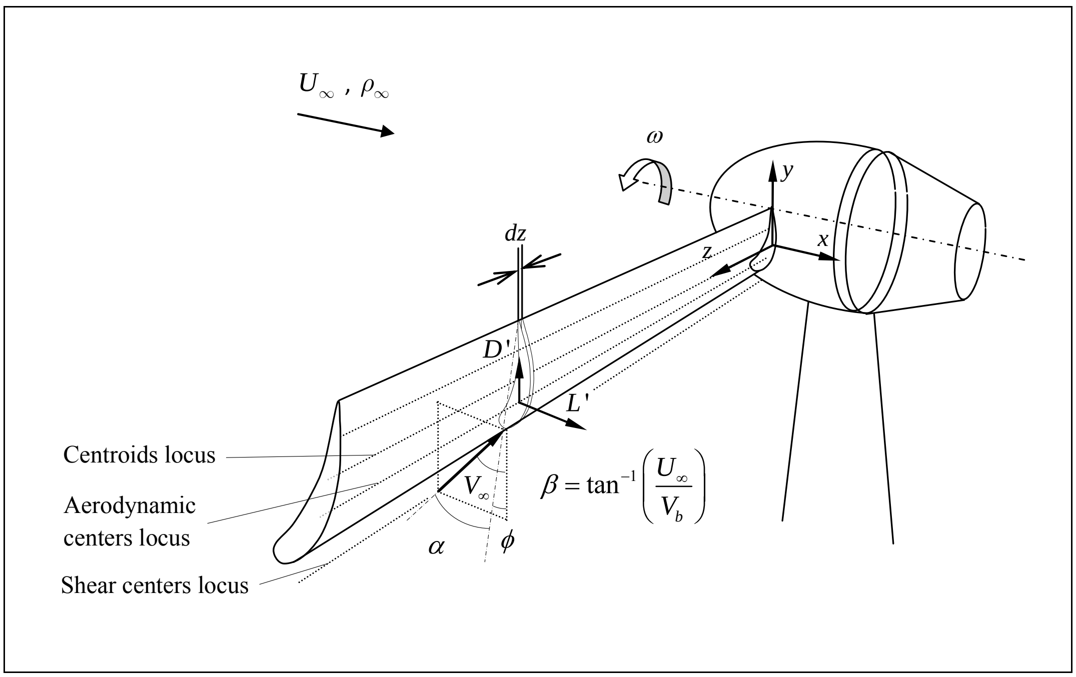


According to above discussion, by considering the drag coefficient c_d as a very small quantity compared to lift coefficient c_l in a wide range of angle of attack, and the fact that drag force is acting on a blade side with highest area moment of inertia I_{yy} with respect to I_{xx}, so for the sake of simplicity the drag force is ignored in this study.

3.4 LIFT FORCE FORMULATION OF WIND TURBINE BLADE

In previous section, the lift force intensity per unit length L' has been defined as a function of several variables, but it is assumed that all the parameters are constant at nominal design point in a steady state wind and weather condition. The blade is assumed to be prismatic non-twisting cross section, so its chord is also constant throughout its length.

Figure 3.6: Velocity analysis on wind turbine blade



Now the remaining variable affecting the lift force would be the twisting angle caused by aerodynamic and control forces acted upon the blade tip and surfaces.

The angle of attack consists of two functions β , and ϕ which are described as:

$$\alpha = \beta - \phi \tag{3.19}$$

The angle β is a function of tangential blade velocity and the main air stream velocity U_∞, and can be written as

$$\beta = \arctan\left(\frac{U_\infty}{z\omega}\right) \tag{3.20}$$

So α would be defined as ,

$$\alpha = \arctan\left(\frac{U_\infty}{z\omega}\right) - \phi \tag{3.21}$$

And if one sets the twisting angle at the root of the blade equal to zero - $\phi_0 = \phi(0) = 0$, it means that

$$\alpha_0 = \alpha(0) = \frac{\pi}{2} \tag{3.22}$$

As discussed in section 3.3, the lift coefficient per unit length is a function of angle of attack. This coefficient will be used to show the lift force and its corresponding twisting moment as,

$$L' = c_l c q_\infty = \pi c \alpha \rho_\infty (z^2\omega^2 + U_\infty^2) \ , \tag{3.23}$$

Now by inserting Equation (3.21) into (3.23) one have the lift force per unit length as:

$$L' = \pi c \rho_\infty (z^2\omega^2 + U_\infty^2)\left(\arctan\left(\frac{U_\infty}{z\omega}\right) - \phi\right) \tag{3.24}$$

The pitching moment induced by lift for is formulated as,

$$M_{L'} = eL' = \pi e c \alpha \rho_\infty (z^2\omega^2 + U_\infty^2) \tag{3.25}$$

where $e = -(x_0 - x_A)$ as shown in Figure 3.5: Airfoil nomenclature, is the distance between shear center O' and aerodynamic axis A_e.

If Equation (3.21) inserted, this pitching moment can also be shown as a function of ϕ too

$$M_{L'} = e\pi c \rho_\infty (z^2\omega^2 + U_\infty^2)\left(\arctan\left(\frac{U_\infty}{z\omega}\right) - \phi \right) \tag{3.26}$$

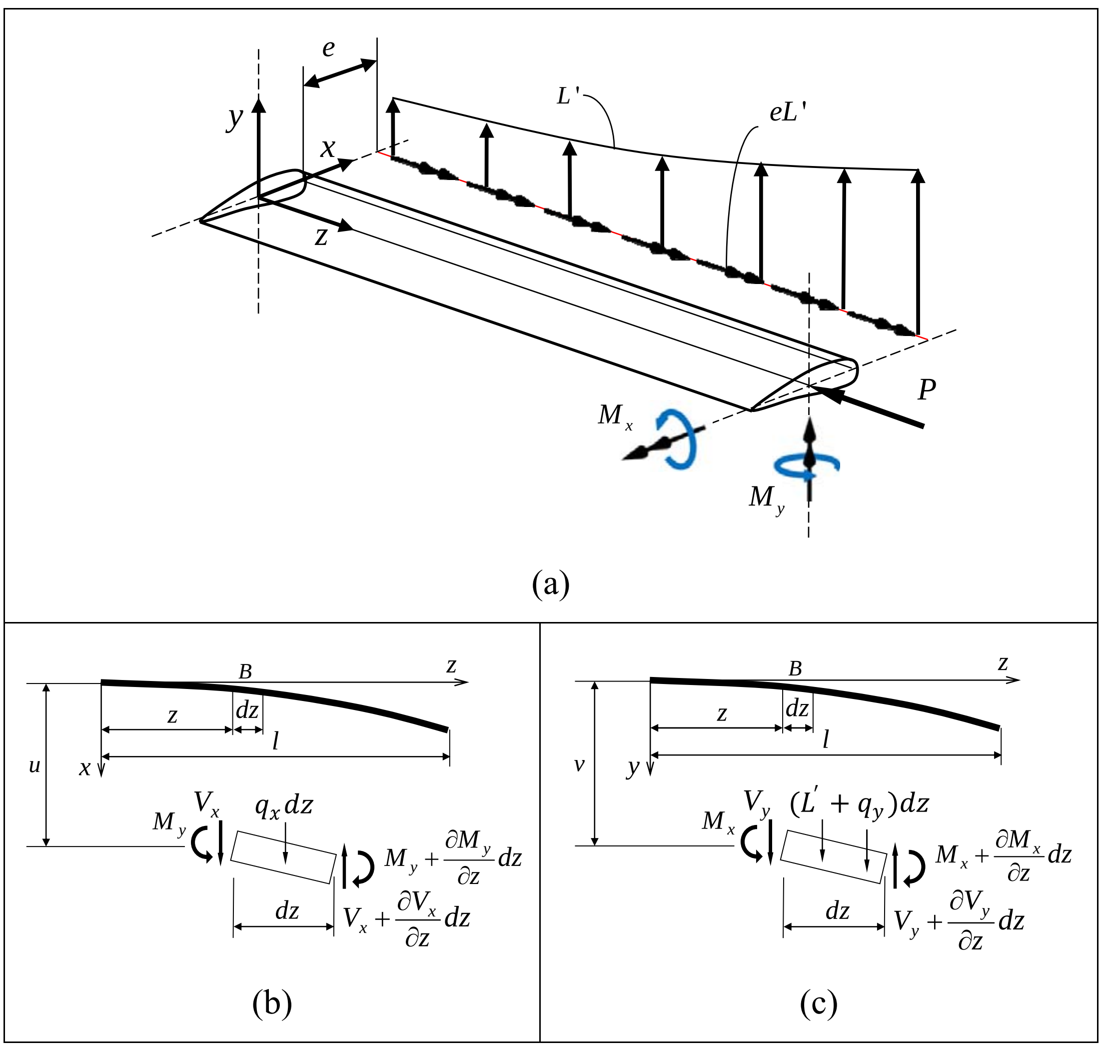
Here c is the chord length of the airfoil, ω is the angular frequency of wind turbine rotor, c_l is lift force intensity, q_∞ air kinetic energy the blade is experiencing, V_∞ is relative velocity the blade experiences, U_∞ is the nominal wind speed of the main steam, and ρ_∞ is the nominal air density.

4. DETERMINING THE EQUATIONS OF MOTIONS

4.1 FORCE BALANCE APPROACH

Under loading of any type a cantilever open cross-sections beam will undergo a combination of deflection and rotation and warping up to a new equilibrium positions that the internal forces are equal in magnitude and opposite to the direction of the external loads.

Figure 4.1: Combined loading on wind turbine blade



As shown in Figure 4.1: Combined loading on wind turbine blade(a-c), a turbine blade as beam like structure is exposed to a combined loading. Theses loads includes lateral distributed aerodynamic lift force L', distributed aerodynamic pitching moment eL'. In addition two bending moments M_x and M_y plus an axial load P are included as the control forces. Due to low rotational speed of the blade, distributed centrifugal force and inertial loads are ignored.

The deformation u in x direction and v in y direction are due to the effects of distribution of normal and shear stress throughout the structure. On studying of a deformation in the structure one needs to consider an infinitesimal element in an arbitrary point on the beam and write the equilibrium of the forces and moments in the direction of principal axis of the cross-section, as described in Figure 4.1. Taking the moment balance about point *B* (Senjanovic´ et al, 2007) in Figure 4.1(b) one finds

$$M_x - \left(M_x + \frac{\partial M_x}{\partial z} dz \right) + \left(V_y + \frac{\partial V_y}{\partial z} dz \right) dz - (q_y + L') dz \left(\frac{dz}{2} \right) = 0 \qquad (4.1)$$

And when ignoring the higher order differentials one have:

$$\frac{\partial M_x}{\partial z} = V_y \qquad (4.2)$$

Similarly as it is shown in Figure 4.1(c) one would find relationship between M_y and V_x as:

$$M_y - \left(M_y + \frac{\partial M_y}{\partial z} dz \right) + \left(V_x + \frac{\partial V_x}{\partial z} dz \right) dz - q_x dz \left(\frac{dz}{2} \right) = 0 \ , \qquad (4.3)$$

And again if ignoring the higher order differentials one would find a simplified form of Equation (4.3) as:

$$\frac{\partial M_y}{\partial z} = V_x \tag{4.4}$$

A similar explanations can be derived for shear forces in *x*, *y*, and torque distribution in *z* direction as well. From force equilibrium in *y* direction it is known that:

$$V_y - (V_y + V_y \frac{\partial V_y}{\partial z} dz) + (q_y + L')dz = 0 \tag{4.5}$$

If one disregards the higher order differentials would find:

$$\frac{\partial V_y}{\partial z} = q_y + L' \tag{4.6}$$

From force balance in *x* direction one can write an equilibrium equation defining as:

$$V_x - (V_x + V_x \frac{\partial V_x}{\partial z} dz) + (q_x)dz = 0 \tag{4.7}$$

After simplification and linearizing the Equation (4.7) it can be rewritten as:

$$\frac{\partial V_x}{\partial z} = q_x \tag{4.8}$$

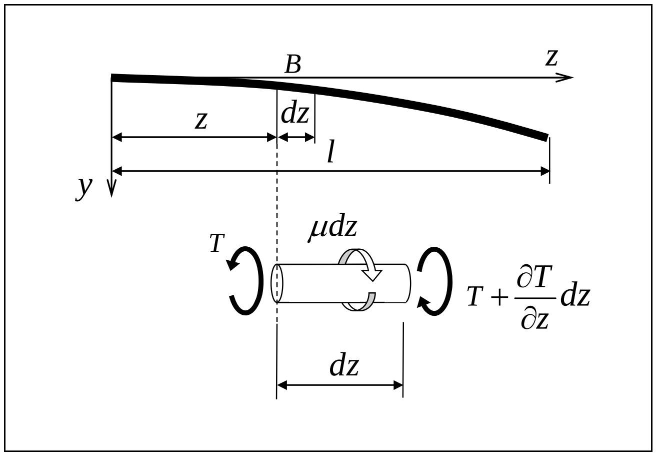
As shown in Figure 4.2 a torque balance is written for the element dz at point B as Equation (4.9) in order to find a relationship between the external torque per unit length μ and internal resisting torque T :

$$T - (T + T \frac{\partial T}{\partial z} dz) - (\mu + eL')dz = 0 \tag{4.9}$$

And its simplified form is derived as:

$$\frac{\partial T}{\partial z} = -\mu - eL' \tag{4.10}$$

Figure 4.2: Torque balance of an element *dz*



4.1.1 Combined Torsional and Flexural DEs

When a beam is loaded by axial force and end moments, some bending and torsional deformation are expected to happen throughout the beam length. This behavior and its effects on cross sectional deflections are discussed in section 4.1. The internal moments M_x and M_y which are acting on the cross-section (Timoshenko Stephen P., 1963, p. 230) can be defined by the following equations:

$$M_x = EI_{xx} \frac{\partial^2 v}{\partial z^2} \tag{4.11}$$

$$M_y = EI_{yy} \frac{\partial^2 u}{\partial z^2} \tag{4.12}$$

At the same time there is a total resisting torque T exerts on the cross-section. This torque consists of pure torsional part T_t (Timoshenko Stephen P., 1963, p. 218), and warping torque component T_ω (Timoshenko Stephen P., 1963, p. 223),

$$T = T_t + T_\omega \tag{4.13}$$

$$T_t = GI_t \frac{\partial \phi}{\partial z} \tag{4.14}$$

$$T_\omega = -EI_\omega \frac{\partial^3 \phi}{\partial z^3} \tag{4.15}$$

After substituting of sectional forces into the equilibrium Equations (4.2), (4.4), (4.6), (4.8), and (4.10) one would find the differential equations of the beam as:

$$EI_x \frac{\partial^4 v}{\partial z^4} = q_y + L' \tag{4.16}$$

$$EI_y \frac{\partial^4 u}{\partial z^4} = q_x \tag{4.17}$$

$$EI_\omega \frac{\partial^4 \phi}{\partial z^4} - GI_t \frac{\partial \phi^2}{\partial z} = \mu + eL' \tag{4.18}$$

4.1.2 External Loads q and μ

Despite the fact that there is actually no torsional force applied at the end of an open cross-section beam, one can observe rotation in the structure. This is mainly because of internal forces which act as torque. Let's assume a loading condition described by Equation (4.19), which cause a combined stress field on the structure. In order to find the load applied on the cross section due to these stress combination, one would observe an element with the length ds as shown in Figure 4.3. Here s is the profile curve of the cross section. There are six stress components acting on this typical element; two normal compressive and four shear stresses. But only the normal stress contribute into the calculation as it is assumed that there is no shear strain exists and the cross section will not distort due to shear stresses (Bazant Zdenek P, 2010, p. 392). The external compressive force and bending moments about principal axes of the cross section will produce combined normal stresses throughout the beam which is described as:

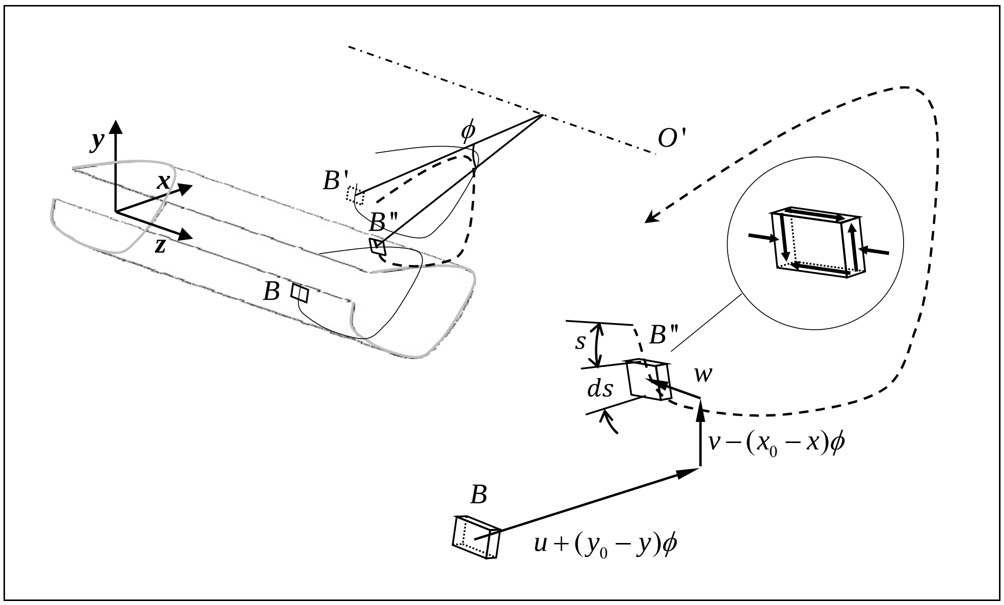
$$\sigma = -\frac{P}{A} - \frac{M_1 y}{I_{xx}} - \frac{M_2 x}{I_{yy}} \tag{4.19}$$

where M_1 and M_2 are representing M_x and M_y respectively.

One would define three fictitious loads based on this stress distribution on the element (Timoshenko Stephen P., 1963, p. 244)

$$q_x = -\int_A (\sigma dA) \frac{d^2}{dz^2}\left[u + (y_0 - y)\phi\right] \tag{4.20}$$

Figure 4.3: Displacements induced by bending moments and axial load



$$q_y = -\int_A (\sigma dA) \frac{d^2}{dz^2}\left[v - (x_0 - x)\phi\right] \tag{4.21}$$

$$\begin{aligned} \mu = &-\int_A (\sigma dA)(y_0 - y) \frac{d^2}{dz^2}\left[u + (y_0 - y)\phi\right] \\ &+\int_A (\sigma dA)(x_0 - x) \frac{d^2}{dz^2}\left[v - (x_0 - x)\phi\right] \end{aligned} \tag{4.22}$$

These forces are called fictitious as they are assumed to act on the lateral directions x, y, and as distributed torque around z axis with equivalent effects on loading condition shown in Figure 4.1: Combined loading on wind turbine blade(a).

Recalling the following sectional properties from elementary mechanics:

$$\int_A dA = A\,,\ \int_A ydA = \int_A xdA = \int_A xydA = 0\,,\ \int_A y^2 dA = I_{xx}\,,\ \int_A x^2 dA = I_{yy}$$

$$I_o = I_{xx} + I_{yy} + A({x_0}^2 + {y_0}^2)$$

where I_0 is polar moment of area using parallel axis theorem, and A is the cross section area. Now by expanding and simplifying the Equations (4.20) - (4.22) one will find:

$$q_x = -P\frac{d^2u}{dz^2} - \left(Py_0 - M_1\right)\frac{d^2\phi}{dz^2} \tag{4.23}$$

$$q_y = -P\frac{d^2v}{dz^2} + \left(Px_0 - M_2\right)\frac{d^2\phi}{dz^2} \tag{4.24}$$

$$\mu = -\left(Py_0 - M_1\right)\frac{d^2u}{dz^2} + \left(Px_0 - M_2\right)\frac{d^2v}{dz^2} - \left(M_1\beta_1 + M_2\beta_2 + P\frac{I_0}{A}\right)\frac{d^2\phi}{dz^2} \tag{4.25}$$

In which β_1 and β_2 are defined as:

$$\beta_1 = \frac{1}{I_{xx}}\left(\int_A y^2 dA + \int_A x^2 ydA\right) - 2y_0 \tag{4.26}$$

$$\beta_2 = \frac{1}{I_{yy}}\left(\int_A x^2 dA + \int_A xy^2 dA\right) - 2x_0 \tag{4.27}$$

4.1.3 Differential Equation of Flexural-Torsional of Beam

Using expressions derived for equivalent fictitious loads Equations (4.23) - (4.25) into the corresponding parts of Equations (4.16) - (4.18), one will have the final form of the coupled differential equation describing the Flexural-Torsional deformation of a prismatic beam under loading described in Figure 4.1(a):

$$EI_{yy}\frac{\partial^4 u}{\partial z^4}+P\frac{d^2u}{dz^2}+\left(Py_0-M_1\right)\frac{d^2\phi}{dz^2}=0 \tag{4.28}$$

$$EI_{xx}\frac{\partial^4 v}{\partial z^4}+P\frac{d^2v}{dz^2}-\left(Px_0-M_2\right)\frac{d^2\phi}{dz^2}-L'=0 \tag{4.29}$$

$$\begin{aligned}&EI_{\omega}\frac{\partial^4 \phi}{\partial z^4}-\left(GI_t-M_1\beta_1-M_2\beta_2-P\frac{I_0}{A}\right)\frac{d^2\phi}{dz^2}\\&+\left(Py_0-M_1\right)\frac{d^2u}{dz^2}-\left(Px_0-M_2\right)\frac{d^2v}{dz^2}-eL'=0\end{aligned} \tag{4.30}$$

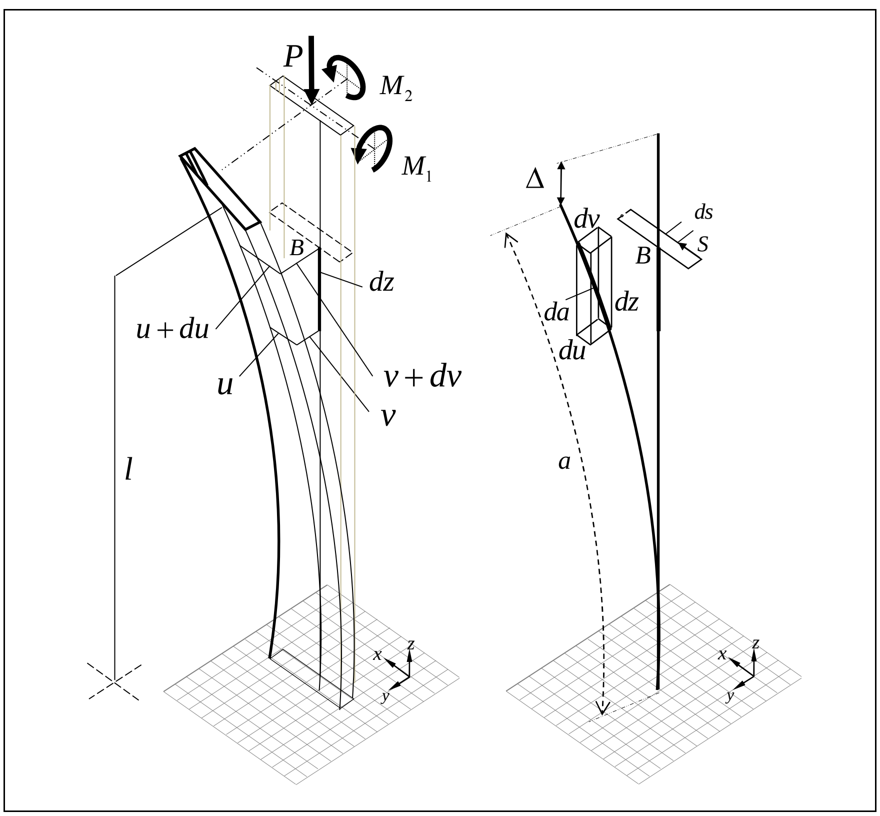
4.2 ENERGY APPROACH

Whenever a beam column is subjected to a combined loading it will deflect, and these deflections should be investigated to determine the governing differential equations of deformation. For this purpose the deflections terms u, v, and ϕ in $u-v$ planes which are parallel to the reference plane $x-y$, were studied is section 4.1. But force balance approach is not providing the boundary conditions of the problem, which are vital information for solving the differential equations. In order to overcome this deficiency the energy approach is used because this methodology will provide both differential equations and boundary conditions of the problem at the same time.

4.2.1 Longitudinal Strains of a Buckled Beam

In order to find the total potential energy of the structure, there are important longitudinal strain terms that contribute in the formulation of potential energy.

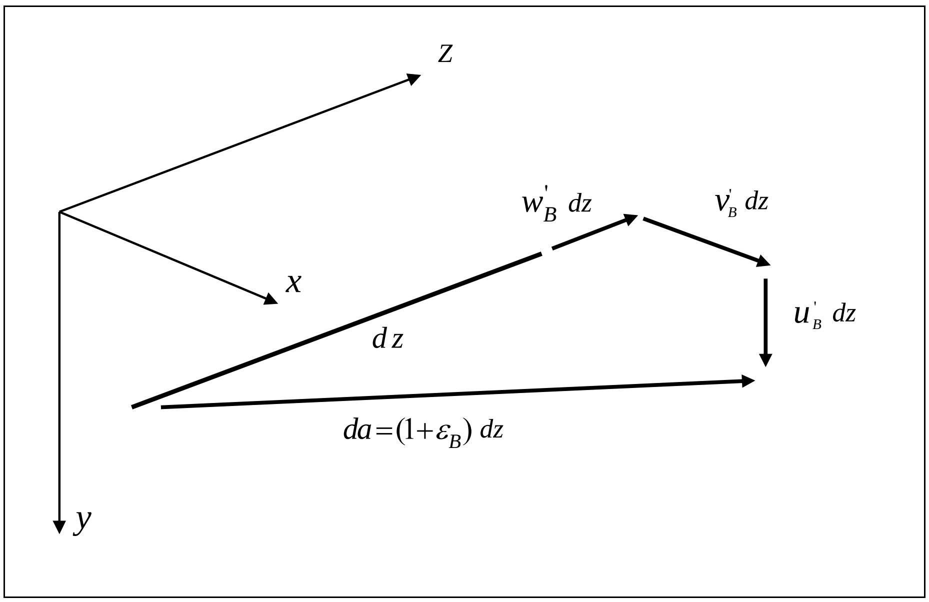
Figure 4.4: Longitudinal deflection of a column under combined loading



Theses strains are caused by lateral and axial deformations of the cross section. The deflection components of point *B* in *x* and *y* directions were described in section 3.1 and defined by $u_B = u + (y_0 - y)\phi$, $v_B = v - (x_0 - x)\phi$, where x_0 and y_0 are the coordinates of shear center. The displacement of point B in *z* direction is more complicated and is defined by (Trahair, 1993, p. 348):

$$w_B = (w - xu' - yv' + \omega\phi') + (-xv'\phi + yu'\phi) \tag{4.31}$$

Figure 4.5: Longitudinal normal strain



Source: Adapted from Trahair, July 20, 1993, p. 351

The infinitesimal arc length da of a buckled beam in Figure 4.4 is redrawn in Figure 4.5, and the longitudinal strain is formulated using Pythagorean Theorem (Trahair, 1993, p. 351) as:

$$(1+\varepsilon_B)dz = \sqrt{(u_B' dz)^2 + (v_B' dz)^2 + (dz + w_B' dz)^2} \tag{4.32}$$

or

$$(1+\varepsilon_B) = \sqrt{u_B'^2 + v_B'^2 + w_B'^2 + 2w_B' + 1} \tag{4.33}$$

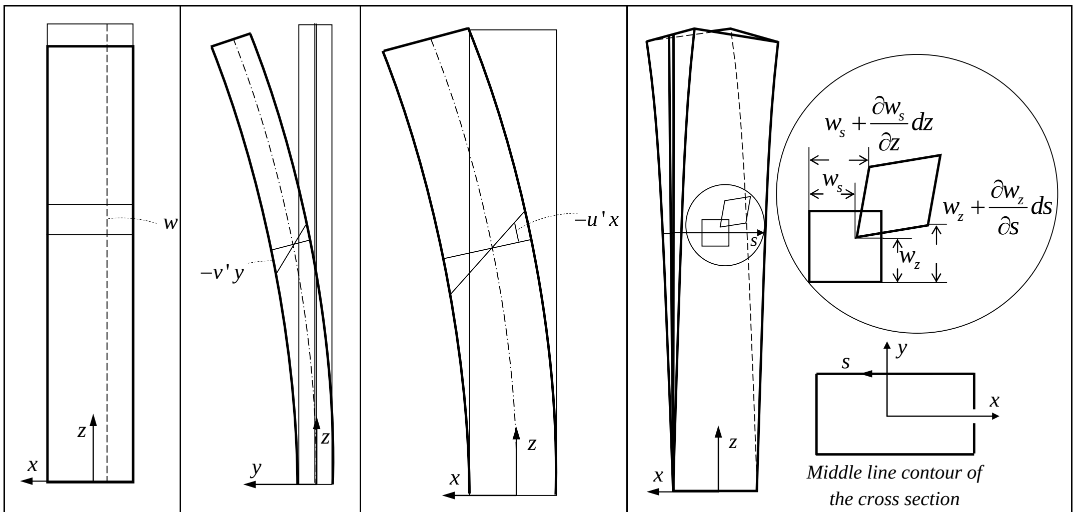
If one sets $u_B'^2 + v_B'^2 + w_B'^2 + 2w_B'$ equal to N and let it approaches to zero, after using Taylor series expansion (Thomas George B., 2005, p. 824) would have:

$$\lim_{n\to 0}\sqrt{1+N} = 1+\frac{N}{2}-\frac{N^2}{8}+\frac{N^3}{16}-\frac{5N^4}{128}+\frac{7N^5}{256}+O(N^6)\ . \tag{4.34}$$

So, if one assumes N as a small quantity and ignore the higher order terms in Equation (4.34), the final form of the Equation (4.33) would be:

$$(1+\varepsilon_B) = \frac{u_B'^2 + v_B'^2 + w_B'^2}{2} + w_B' + 1 \tag{4.35}$$

Figure 4.6: Longitudinal strain terms due to deflections in *x, y, z*, and warping



(a)	(b)	(c)	(d)
Average longitudinal strain	Longitudinal strain in y-z plane	Longitudinal strain in x-z plane	Longitudinal strain due to warping

Based on Equation (4.35) and considering the fact that $w_B^{'}$ is a very small quantity with respect to $u_B^{'}$, and $v_B^{'}$, the longitudinal strain due to lateral deflections and axial displacement can be written as:

$$\varepsilon_B = \frac{1}{2}\left(u_B^{'\,2} + v_B^{'\,2}\right) + w_B^{'} \tag{4.36}$$

for any arbitrary point B on the longitudinal fiber.

Total longitudinal strain (Trahair, 1993, p. 351) at B is defined by inserting Equations (3.7), (3.8), and (4.31) into equation (4.36)

$$\begin{aligned}\varepsilon_B &= \{w' - xu'' - yv'' + \omega\phi''\} + \{\frac{1}{2}[u'^2 + v'^2 + (x_0^2 + y_0^2)\phi'^2] \\ &\quad - x_0 v'\phi' + y_0 u'\phi' + x[-x_0\phi'^2 - \phi v''] + y[-y_0\phi'^2 + \phi u''] \\ &\quad + \frac{1}{2}[x^2 + y^2]\phi'^2\}\end{aligned} \tag{4.37}$$

The terms in the first part of Equation (4.37), w', xu'', yv'' are longitudinal strains due to deflection in *z, y*, and *x* directions; and $\omega\phi''$ is warping strain respectively.
And its variation is expressed as,

$$\begin{aligned}\delta\varepsilon_B &= \{(\delta w)' - x(\delta u)'' - y(\delta v)'' + \omega(\delta\phi)''\} \\ &\quad + \{[u'(\delta u)' + v'(\delta v)' + (x_0^2 + y_0^2)\phi'(\delta\phi)'] \\ &\quad - x_0(\delta v)'\phi' - x_0 v'(\delta\phi)' + y_0(\delta u)'\phi' + y_0 u'(\delta\phi)' \\ &\quad + x[-2x_0\phi'(\delta\phi)' - \delta\phi v'' - \phi(\delta v)''] \\ &\quad + y[-2y_0\phi'(\delta\phi)' + \delta\phi u'' + \phi(\delta u)''] \\ &\quad + [x^2 + y^2]\phi'(\delta\phi)'\}\end{aligned} \tag{4.38}$$

4.2.2 Warping Strain

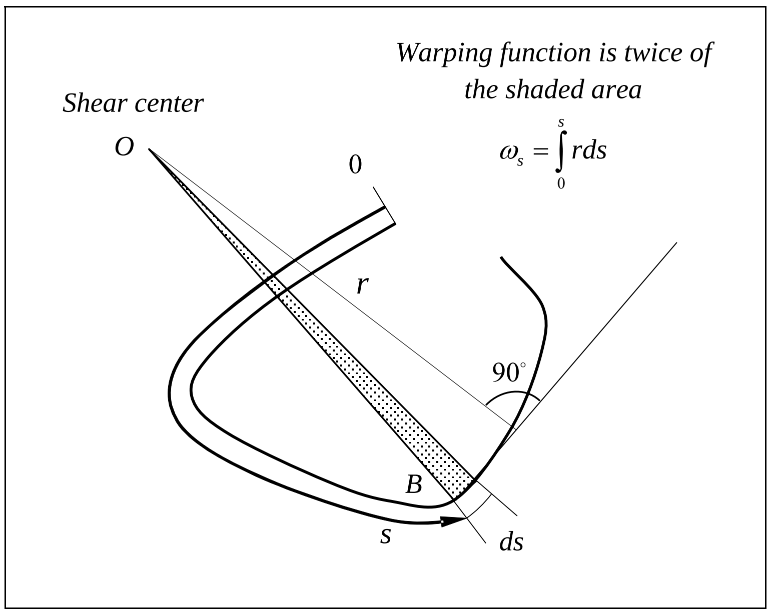
The term $\omega\phi''$ in Equation (4.37) needs some more explanation which is based on the theory of warping torsion. According to the so called Wagner's assumption, the shear strain in the middle surfaces of an open profile is negligible (Bazant Zdenek P, 2010, p.

392). Based on the theory of elasticity (Timoshenko S., 1951, p. 6) shear strain γ_{sz} can be written as shown in Figure 4.6-(d),

$$\gamma_{sz} = \frac{\partial w_s}{\partial z} + \frac{\partial w_z}{\partial s} \tag{4.39}$$

Here, w_z is the displacement in z-direction, and w_s is the displacement tangent to the cross sectional profile s.

Figure 4.7: Warping function



Now, based on the Wagner's assumption one may write,

$$\frac{\partial w_z}{\partial s} = -\frac{\partial w_s}{\partial z} \tag{4.40}$$

And after integration on curve s along the middle surface of the cross section Equation (4.40) becomes:

$$w_z - w_0 = -\int_0^s \frac{\partial w_s}{\partial z} ds = -\int_0^s \frac{\partial r\phi}{\partial z} ds \,. \tag{4.41}$$

Here r is the normal distance of the tangent line at point B on curve s to the shear center. These parameters are explained if Figure 4.7: Warping function. w_0 is the constant of integration and can be expressed as the longitudinal displacement at the origin of curve s. The integral $\int_0^s rds = \omega_s$ is defined as the warping function of the cross section. As ϕ is constant over the cross section Equation (4.41) can be re-written as:

$$w = w_0 - \phi' \omega_s \tag{4.42}$$

Here w_z is replaced by w to generalize the formulation.

Average warping displacement is represented by integrating Equation (4.42) divided by the total cross sectional area A:

$$\overline{w} = \frac{1}{A}\int_0^l w_0 dA - \frac{1}{A}\int_0^l \phi' \omega_s dA = w_0 - \frac{\phi'}{A}\int_0^l \omega_s dA \tag{4.43}$$

By subtracting Equations (4.43) from (4.42) and rearranging the terms one will find (Timoshenko Stephen P., 1963, p. 215):

$$w_w = \phi'(\frac{1}{A}\int_0^l \omega_s dA - \omega_s) = \phi'(\overline{\omega}_s - \omega_s)\,, \tag{4.44}$$

where $w_w = w - \overline{w}$ is the warping displacement.

4.2.3 Minimum Total Potential Energy

Total potential energy of the beam-column can be shown by the notation Π and is formulated as:

$$\Pi = U + W_{ext} \tag{4.45}$$

Based on the principle of minimum potential energy the variation of total potential energy Π will be equal to zero:

$$\delta\Pi = \delta(U + W_{ext}) = 0 \tag{4.46}$$

Here U is represented the total strain energy of the structure and defined as:

$$\begin{aligned} U = &\int_A \Delta_B \sigma dA + \frac{1}{2}\int_0^l EI_y (u'')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l EI_x (v'')^2 \,\mathrm{d}z \\ &+ \frac{1}{2}\int_0^l GI_t (\phi')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l EI_\omega (\phi'')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l EA(w')^2 \,\mathrm{d}z \end{aligned} \tag{4.47}$$

where

$$\sigma = -\frac{P}{A} - \frac{M_1 y}{I_{xx}} - \frac{M_2 x}{I_{yy}} \tag{4.48}$$

and

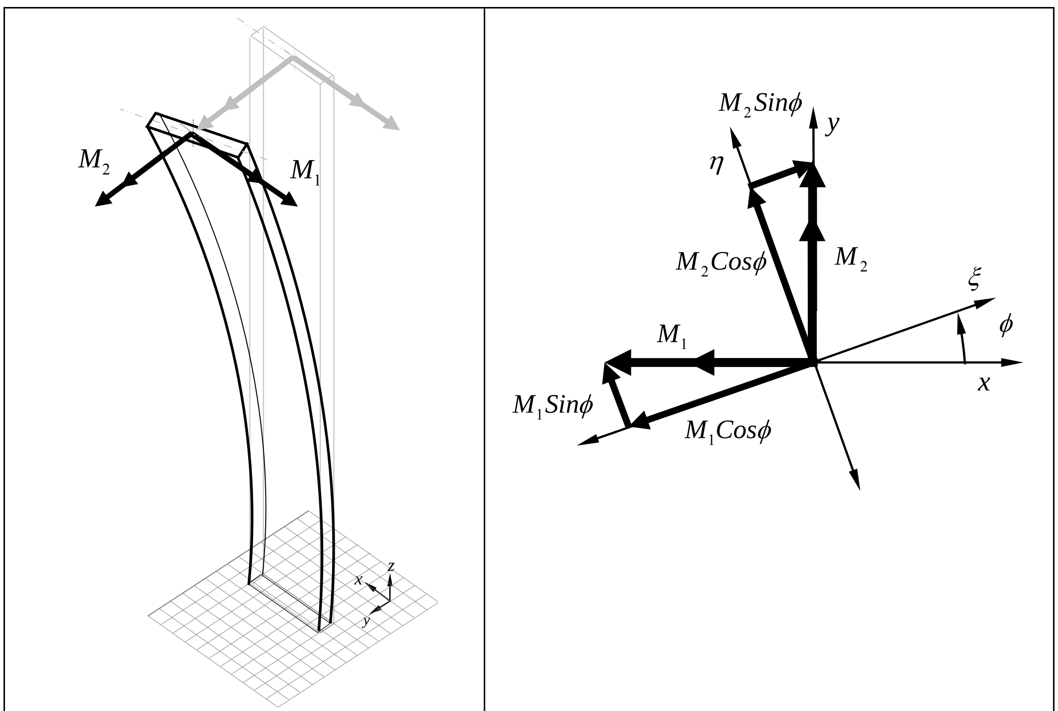
$$\Delta_B = \int_0^l \varepsilon_B dz \ , \tag{4.49}$$

The first term of the strain energy refers to the incremental strain energy of stress caused by external loads. Second and third terms of Equation (4.47) are related to flexural strain energy, and forth term represents the uniform torsion of the beam. The fifth term belongs to the warping component of strain energy in the structure, and the last term stands for strain energy due to axial strain.

4.2.3.1 Potential of external loads

When the beam undergoes deformations and reaches to a new equilibrium position, new loading condition will appear at the local coordinate of the cross section (ξ, η, ζ). In fact there is no new loads, but decomposition of the original loads onto the local coordinates are acting as the new forces. In Figure 4.8(a) and (b), it is explained how the magnitude and direction of the external forces in *x*-*y* plane are not changed, but the projections of these forces acted as intermittent loads which are responsible for the potential of the external forces.

Figure 4.8: External loads components decomposition in *x*-*y* plane

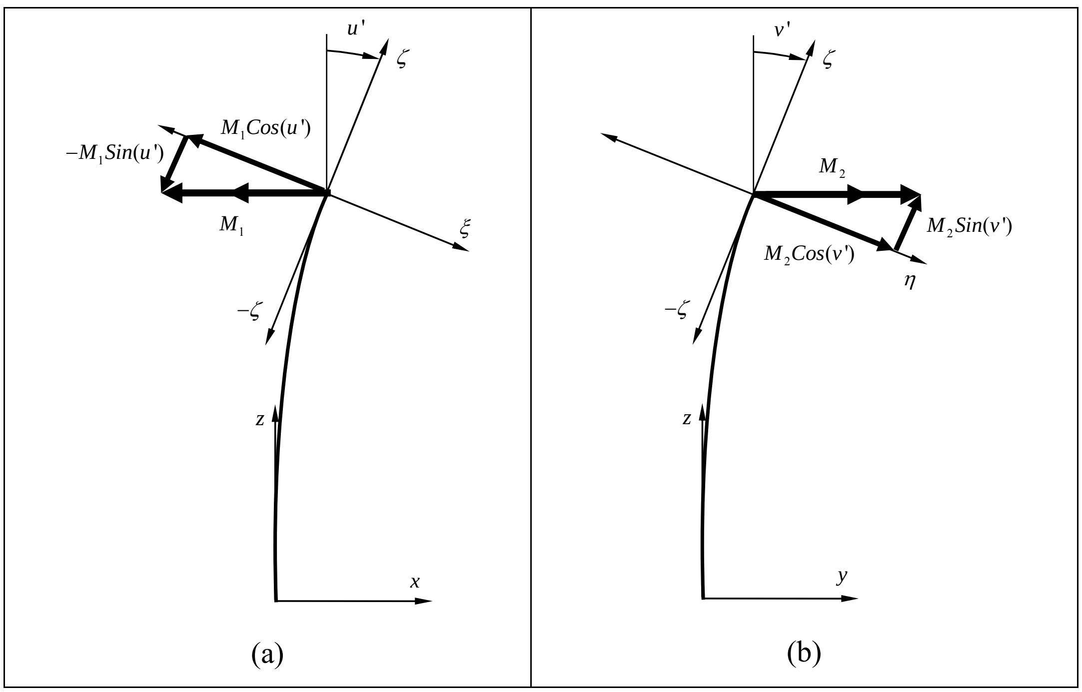


(a) (b)

Decomposition of moment M_1 and M_2 onto the η and ξ axis are $M_1 \cos\phi$ and $-M_2 \sin\phi$ respectively. By using small angle approximation one can find these moment as M_1 and $-M_2\phi$.

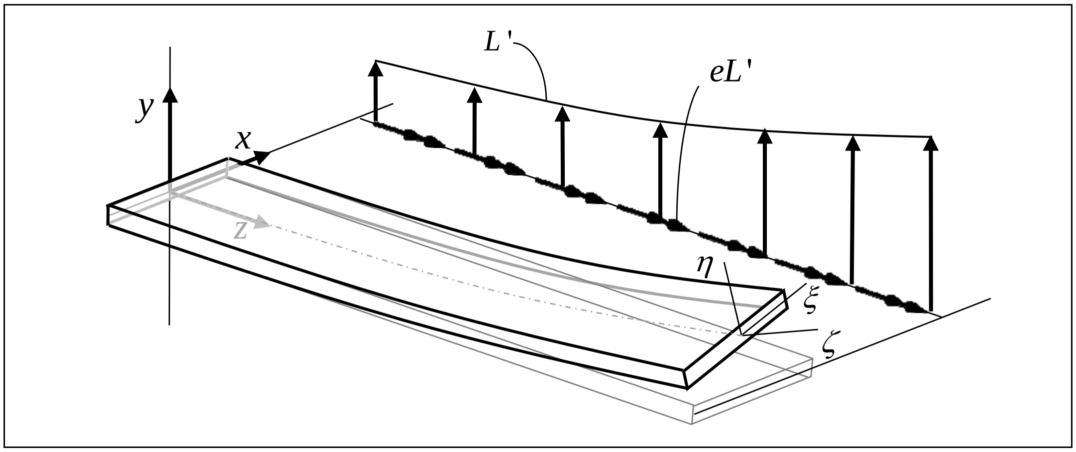
Similar explanation is valid for decomposition of moments M_1 and M_2 in $\zeta-\eta$ and $\zeta-\xi$ planes. The new equilibrium position is shown in Figure 4.9(a) and (b). The consequent projections of M_1 and M_2 are responsible for twisting the cross section as they are acting like twisting torques in ζ direction. They are explained as $-M_1 \sin u'$ and $M_2 \sin v'$ which by taking into account of small angle approximation they can be written as $-M_1 u'$ and $M_2 v'$ respectively.

Figure 4.9: External loads components decomposition in *x*-*z* and *y*-*z* planes



In Figure 4.10, the aerodynamic loads and their contribution to the deformation is shown.

Figure 4.10: Aerodynamic lift and pitching moment intensity



The corresponding deflection term for lift intensity L' and pitching moment intensity are v and ϕ respectively, and their potential is in the form of integral $-\int_0^l (L'v + e\phi L')dz$.

These loads and their counterpart deflections are addressed in Table 4.1.

Table 4.1: Projection of external loads onto the local cross sectional axis

External loads	$-M_2\phi$	$M_1\phi$	L'	M_2v'	$-M_1u'$	eL'
Corresponding deflection	v'	u'	v	ϕ	ϕ	ϕ

Potential of external loads W_{ext} in Equation (4.45) is in the form of work done by external loads which one can derive them by multiplying loads with their corresponding deflections.

In Table 4.2 the Potential of the external loads and their variations are also explained.

Table 4.2: Potential of external loads and their variations

Potential	Variation
$-\left[-M_2\phi.v'\right]_{z=l}$	$\left[M_2\phi(\delta v)'\right]_{z=l}$
$-\left[M_1\phi.u'\right]_{z=l}$	$-\left[M_1\phi(\delta u)'\right]_{z=l}$
$-\left[\left(M_2v'-M_1u'\right).\phi\right]_{z=l}$ $-\int_0^l (e\phi L')dz$	$-\left[\left(M_2v'-M_1u'\right)\delta\phi\right]_{z=l}$ $-\int_0^l (eL')\delta\phi dz$
$-\int_0^l (L'v)dz$	$-\int_0^l (L')\delta v dz$

Now according to the above derivations of the new loading one can formulate the potential of external loads as:

$$\begin{aligned} W_{ext} &= -\left[\left(M_2v'-M_1u'\right).\phi\right]_{z=l} - \left[-M_2\phi.v'\right]_{z=l} \\ &\quad -\left[M_1\phi.u'\right]_{z=l} - \int_0^l (L'v+e\phi L')dz \end{aligned} \tag{4.50}$$

4.3 TOTAL POTENTIAL ENERGY OF THE STRUCTURE

By defining the combined stress in the structure and include all the energy terms explained in section 4.2, one can formulate the total potential energy as:

$$
\begin{aligned}
\Pi &= \frac{1}{2}\int_0^l EI_y (u'')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l EI_x (v'')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l GI_t (\phi')^2 \,\mathrm{d}z \\
&+ \frac{1}{2}\int_0^l EI_\omega (\phi'')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l EA (w')^2 \,\mathrm{d}z \\
&+ \int_0^l \int_A \left(-\frac{P}{A} - \frac{M_1 y}{I_{xx}} - \frac{M_2 x}{I_{yy}} \right)
\left\{ \begin{array}{l}
\{w' - xu'' - yv'' + \omega\phi''\} \\
+\{\frac{1}{2}[u'^2 + v'^2 + (x_0^2 + y_0^2)\phi'^2] \\
-x_0 v'\phi' + y_0 u'\phi' \\
+x[-x_0\phi'^2 - \phi v''] \\
+y[-y_0\phi'^2 + \phi u''] \\
+\frac{1}{2}[x^2 + y^2]\phi'^2\}
\end{array} \right\} dAdz \\
&- \left[\left(M_2 v' - M_1 u' \right).\phi \right]_{z=l} - \left[-M_2 \phi . v' \right]_{z=l} - \left[M_1 \phi . u' \right]_{z=l} - \int_0^l (L'v + e\phi L')dz
\end{aligned}
\tag{4.51}
$$

For more effective evaluation of Equation (4.51) one would separate the terms and divide the formula into five parts as:

$$
\begin{aligned}
S_U &= \frac{1}{2}\int_0^l EI_{yy} (u'')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l EI_{xx} (v'')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l GI_t (\phi')^2 \,\mathrm{d}z \\
&+ \frac{1}{2}\int_0^l EI_\omega (\phi'')^2 \,\mathrm{d}z + \frac{1}{2}\int_0^l EA (w')^2 \,\mathrm{d}z
\end{aligned}
\tag{4.52}
$$

$$
S_{E1} = -\int_0^l \int_A \frac{P}{A}
\left\{ \begin{array}{l}
\{w' - xu'' - yv'' + \omega\phi''\} \\
+\{\frac{1}{2}[u'^2 + v'^2 + (x_0^2 + y_0^2)\phi'^2] \\
-x_0 v'\phi' + y_0 u'\phi' + x[-x_0\phi'^2 - \phi v''] \\
+y[-y_0\phi'^2 + \phi u''] + \frac{1}{2}[x^2 + y^2]\phi'^2\}
\end{array} \right\} dAdz
\tag{4.53}
$$

$$S_{E2} = -\int_0^l \int_A \frac{M_1 y}{I_{xx}} \left\{ \begin{array}{l} \{w' - xu'' - yv'' + \omega\phi''\} \\ +\{\frac{1}{2}[u'^2 + v'^2 + (x_0^2 + y_0^2)\phi'^2] \\ -x_0 v'\phi' + y_0 u'\phi' + x[-x_0\phi'^2 - \phi v''] \\ +y[-y_0\phi'^2 + \phi u''] + \frac{1}{2}[x^2 + y^2]\phi'^2\} \end{array} \right\} dAdz \tag{4.54}$$

$$S_{E3} = -\int_0^l \int_A \frac{M_2 x}{I_{yy}} \left\{ \begin{array}{l} \{w' - xu'' - yv'' + \omega\phi''\} \\ +\{\frac{1}{2}[u'^2 + v'^2 + (x_0^2 + y_0^2)\phi'^2] \\ -x_0 v'\phi' + y_0 u'\phi' + x[-x_0\phi'^2 - \phi v''] \\ +y[-y_0\phi'^2 + \phi u''] + \frac{1}{2}[x^2 + y^2]\phi'^2\} \end{array} \right\} dAdz \tag{4.55}$$

$$\begin{aligned} S_{E4} = &-\left[\left(M_2 v' - M_1 u'\right)\phi\right]_{z=l} - \left[-M_2 \phi v'\right]_{z=l} \\ &-\left[M_1 \phi u'\right]_{z=l} - \int_0^l (L'v + e\phi L')dz \end{aligned} \tag{4.56}$$

And reference to Equation (4.46) the variation of Equation (4.51) should be zero at equilibrium position:

$$\delta\Pi = \delta\left(S_U + S_{E1} + S_{E2} + S_{E3} + S_{E4}\right) = 0 \tag{4.57}$$

The evaluation of Equation (4.57) needs the variational operator to operate inside each integral signs:

$$\begin{aligned} \delta S_U = &\int_0^l EI_{yy} u''\left(\delta u\right)'' \mathrm{d}\,z + \int_0^l EI_{xx} v''\left(\delta v\right)'' \mathrm{d}\,z + \int_0^l GI_t \phi'\left(\delta\phi\right)' \mathrm{d}\,z \\ &+\int_0^l EI_\omega \phi''\left(\delta\phi\right)'' \mathrm{d}\,z + \int_0^l EAw'\left(\delta w\right)' \mathrm{d}\,z \end{aligned} \tag{4.58}$$

$$\delta S_{E1} = -\int_0^l \int_A \frac{P}{A} \begin{Bmatrix} \{(\delta w)' - x(\delta u)'' - y(\delta v)'' + \omega(\delta\phi)''\} \\ +\{[u'(\delta u)' + v'(\delta v)' + (x_0^2 + y_0^2)\phi'(\delta\phi)'] \\ -x_0(\delta v)'\phi' - x_0 v'(\delta\phi)' + y_0(\delta u)'\phi' + y_0 u'(\delta\phi)' \\ +x[-2x_0\phi'(\delta\phi)' - \delta\phi v'' - \phi(\delta v)''] \\ +y[-2y_0\phi'(\delta\phi)' + \delta\phi u'' + \phi(\delta u)''] \\ +[x^2 + y^2]\phi'(\delta\phi)'\} \end{Bmatrix} dAdz \tag{4.59}$$

$$\delta S_{E2} = -\int_0^l \int_A \frac{M_1 y}{I_{xx}} \begin{Bmatrix} \{(\delta w)' - x(\delta u)'' - y(\delta v)'' + \omega(\delta\phi)''\} \\ +\{[u'(\delta u)' + v'(\delta v)' + (x_0^2 + y_0^2)\phi'(\delta\phi)'] \\ -x_0(\delta v)'\phi' - x_0 v'(\delta\phi)' + y_0(\delta u)'\phi' + y_0 u'(\delta\phi)' \\ +x[-2x_0\phi'(\delta\phi)' - \delta\phi v'' - \phi(\delta v)''] \\ +y[-2y_0\phi'(\delta\phi)' + \delta\phi u'' + \phi(\delta u)''] \\ +[x^2 + y^2]\phi'(\delta\phi)'\} \end{Bmatrix} dAdz \tag{4.60}$$

$$\delta S_{E3} = -\int_0^l \int_A \frac{M_2 x}{I_{yy}} \begin{Bmatrix} \{(\delta w)' - x(\delta u)'' - y(\delta v)'' + \omega(\delta\phi)''\} \\ +\{[u'(\delta u)' + v'(\delta v)' + (x_0^2 + y_0^2)\phi'(\delta\phi)'] \\ -x_0(\delta v)'\phi' - x_0 v'(\delta\phi)' + y_0(\delta u)'\phi' + y_0 u'(\delta\phi)' \\ +x[-2x_0\phi'(\delta\phi)' - \delta\phi v'' - \phi(\delta v)''] \\ +y[-2y_0\phi'(\delta\phi)' + \delta\phi u'' + \phi(\delta u)''] \\ +[x^2 + y^2]\phi'(\delta\phi)'\} \end{Bmatrix} dAdz \tag{4.61}$$

$$\delta S_{E4} = \delta\left(-\left[(M_2 v' - M_1 u')\phi\right]_{z=l} - \left[-M_2\phi.v'\right]_{z=l} - \left[M_1\phi.u'\right]_{z=l} - \int_0^l (L'v + e\phi L')dz \right) \tag{4.62}$$

4.3.1 Flexural-Torsional DEs and BCs

By evaluating and inserting the Statements (4.58) - (4.62) into equation (4.57) and rearranging the coefficients of δu, δv, $\delta\phi$, δw, $(\delta u)'$, $(\delta v)'$, and $(\delta\phi)'$ one would find all the information of the beam column under combined loading. This will provide the differential equations of the structure with any boundary conditions.

4.3.1.1 Differential equations

$$EAw'' = 0 \tag{4.63}$$

$$u^{(IV)}+\frac{P}{EI_{yy}}u''+\frac{1}{EI_{yy}}\left(y_0P-M_1-M_2\frac{I_{xy}}{I_{yy}}\right)\phi''=0 \tag{4.64}$$

$$v^{(IV)}+\frac{P}{EI_{xx}}v''+\frac{1}{EI_{xx}}\left(-x_0P+M_2+M_1\frac{I_{yx}}{I_{xx}}\right)\phi''=0 \tag{4.65}$$

$$\begin{gathered}\phi^{(IV)}+\frac{1}{EI_{\omega}}\left(-GI_t+\left[\beta_1-2x_0\frac{I_{yx}}{I_{xx}}\right]M_1+\left[\beta_2-2y_0\frac{I_{xy}}{I_{yy}}\right]M_2+P\frac{I_0}{A}\right)\phi'' \\ +\frac{1}{EI_{\omega}}\left(y_0P-M_2\frac{I_{xy}}{I_{yy}}-M_1\right)u''+\frac{1}{EI_{\omega}}\left(-x_0P+M_2+M_1\frac{I_{yx}}{I_{xx}}\right)v''-eL'=0\end{gathered} \tag{4.66}$$

where

$$\beta_1=-2y_0+\frac{1}{I_{xx}}\left(\int_A yx^2dA+\int_A y^3dA\right)$$

$$\beta_2=-2x_0+\frac{1}{I_{yy}}\left(\int_A x^3dA+\int_A xy^2dA\right)$$

4.3.1.2 Boundary conditions for *z=0*

In case of cantilever beam-column, the geometrical or essential boundary conditions are all constant and known to be zero at $z=0$. It means the variations and all of their corresponding expressions will be vanished as $\delta w(0)=\delta u(0)=\delta v(0)=\delta\phi(0)=\left(\delta u(0)\right)'=\left(\delta v(0)\right)'=\left(\delta\phi(0)\right)'=0$. So the only set of the boundary conditions for z=0 would be based on the nature of the problem. In the cantilever beam case one would define these boundary conditions as:

$$w(0)=u(0)=v(0)=\phi(0)=u'(0)=v'(0)=\phi'(0)=0 \tag{4.67}$$

4.3.1.3 Boundary conditions for *z=l*

At the end point of the beam at $z=l$, the essential variables w, u, v, ϕ, w', u', v', and ϕ' are free to change, and their variations could not be set to zero. It means their corresponding coefficients should be zero; so one will find seven boundary conditions as:

$\delta w:$

$$\left[EAw'-P\right]_{z=l}=0 \tag{4.68}$$

$\delta u:$

$$\left[u'''+\frac{P}{EI_{yy}}u'-\frac{1}{EI_{yy}}\left(M_2\frac{I_{xy}}{I_{yy}}-y_0P+M_1\right)\phi'\right]_{z=l}=0 \tag{4.69}$$

$\delta v:$

$$\left[v'''+\frac{P}{EI_{xx}}v'-\frac{1}{EI_{xx}}\left(x_0P-M_1\frac{I_{yx}}{I_{xx}}-M_2\right)\phi'\right]_{z=l}=0 \tag{4.70}$$

$\delta\phi:$

$$\left[\begin{array}{l}\phi'''-\frac{GI_t}{EI_\omega}\phi'-\frac{1}{EI_\omega}\left(M_1-y_0P\right)u'-\frac{1}{EI_\omega}\left(x_0P-M_2\right)v' \\ +\frac{1}{EI_\omega}\left(\left[-2x_0\frac{I_{yx}}{I_{xx}}+\beta_1\right]M_1+\left[-2y_0\frac{I_{xy}}{I_{yy}}+\beta_2\right]M_2+P\frac{I_0}{A}\right)\phi'\end{array}\right]_{z=l}=0 \tag{4.71}$$

$(\delta u)':$

$$\left[EI_{yy}u''+\left(-2M_1-M_2\frac{I_{xy}}{I_{yy}}\right)\phi+M_1\frac{I_{yx}}{I_{xx}}+M_2\right]_{z=l}=0 \tag{4.72}$$

$(\delta v)':$

$$\left[EI_{xx}v''+\left(M_1\frac{I_{yx}}{I_{xx}}+2M_2\phi\right)\phi+M_1+M_2\frac{I_{xy}}{I_{yy}}\right]_{z=l}=0 \tag{4.73}$$

$(\delta\phi)':$

$$\left[EI_\omega\phi''-M_1\frac{I_{\omega y}}{I_{xx}}-M_2\frac{I_{\omega x}}{I_{yy}}\right]_{=l}=0 \tag{4.74}$$

4.4 BEAM-COLUMN SUBJECTED TO END MOMENTS

In this thesis the main aim is to find analytical exact solution to simplified differential Equations (4.63) - (4.66). In the simplified problem the axial load P, distributed Lift force intensity L', centrifugal force, induced aerodynamic twisting moments eL', and inertial loads are not present. Despite the fact that this is one of the simplest form of the equation set, it has complicated solution process. There would be no exact solution for combined loading form and the best way is to look for an approximate methods.

4.4.1 Differential Equation of the Problem with End Moments

If the column is loaded with two end moments in x and y directions the governing differential equations of the problem are represented as,

$$u^{(IV)} + a\phi'' = 0 \tag{4.75}$$

$$v^{(IV)} - b\phi'' = 0 \tag{4.76}$$

$$\phi^{(IV)} - c\phi'' + du'' - fv'' = 0 \ , \tag{4.77}$$

where

$$a = \left(\frac{-M_1 - M_2 \dfrac{I_{xy}}{I_{yy}}}{EI_{yy}} \right), \ b = \left(\frac{-M_2 - M_1 \dfrac{I_{yx}}{I_{xx}}}{EI_{xx}} \right), \ d = \left(\frac{-M_1 - M_2 \dfrac{I_{xy}}{I_{yy}}}{EI_{\omega}} \right), \ f = \left(\frac{-M_2 - M_1 \dfrac{I_{yx}}{I_{xx}}}{EI_{\omega}} \right)$$

and

$$c = \left(\frac{GI_t + (2x_0 \dfrac{I_{yx}}{I_{xx}} - \beta_1)M_1 + M_2(2y_0 \dfrac{I_{xy}}{I_{yy}} - \beta_2)}{EI_{\omega}} \right)$$

4.4.2 Boundary Conditions for *z=0*

The boundary conditions in the presence of end moments are similar to the one for general form except the term $\delta w(0)$ which does not make sense here; it means:

$$\delta u(0) = \delta v(0) = \delta\phi(0) = [\delta u(0)]' = [\delta v(0)]' = [\delta\phi(0)]' = 0 \tag{4.78}$$

4.4.3 Boundary Conditions for *z=l*

Like the general form, the essential variables w , u , v , ϕ , w' , u' , v' , and ϕ' are free to change at the end point ($z = l$), and their variations could not be set to zero. It means their corresponding coefficients should be zero; so one can find six boundary conditions as,

$$[u''' + a\phi']_{z=l} = 0 \tag{4.79}$$

$$[v''' - b\phi']_{=l} = 0 \tag{4.80}$$

$$[\phi''' - c\phi' + g_1 u' - g_2 v']_{z=l} = 0 \tag{4.81}$$

$$[u'' - g_3\phi + h_1]_{=l} = 0 \tag{4.82}$$

$$[v'' - g_4\phi + h_2]_{=l} = 0 \tag{4.83}$$

$$[v'' - g_4\phi + h_2]_{=l} = 0 \tag{4.84}$$

where

$$a = \frac{-M_2(I_{xy} / I_{yy}) - M_1}{EI_y} \tag{4.85}$$

$$b = \frac{-M_1(I_{yx} / I_{xx}) - M_2}{EI_{xx}} \tag{4.86}$$

$$d = \frac{-M_2(I_{xy}/I_{yy}) - M_1}{EI_\omega} \tag{4.87}$$

$$c = \frac{GI_t + \left(2x_0(I_{yx}/I_{xx}) - \beta_1\right)M_1 + \left(2y_0(I_{xy}/I_{yy}) - \beta_2\right)M_2}{EI_\omega} \tag{4.88}$$

$$f = \frac{-M_2 - M_1(I_{yx}/I_{xx})}{EI_\omega} \tag{4.89}$$

$$g_1 = -M_1 / EI_\omega \tag{4.90}$$

$$g_2 = -M_2 / EI_\omega \tag{4.91}$$

$$g_3 = \frac{2M_1 + M_2(I_{xy}/I_{yy})}{EI_{yy}} \tag{4.92}$$

$$g_4 = \frac{-M_1(I_{yx}/I_{xx}) - 2M_2}{EI_{xx}} \tag{4.93}$$

$$h_1 = \frac{M_1(I_{yx}/I_{xx}) + M_2}{EI_{yy}} \tag{4.94}$$

$$h_2 = \frac{M_1 + M_2(I_{xy}/I_{yy})}{EI_{xx}} \tag{4.95}$$

$$h_3 = \frac{M_1x_0 + M_2y_0}{EI_\omega} \tag{4.96}$$

$$x_0 = \frac{I_{\omega y}}{I_{xx}}, y_0 = \frac{I_{\omega x}}{I_{yy}} \tag{4.97}$$

4.4.4 General Solution of the Differential Equation

In order to solve these three coupled differential equations, one can assume three exponential function as follows:

$$u = Ae^{rz}$$

$$v = Be^{rz} \tag{4.98}$$

$$\phi = Ce^{rz}$$

Inserting Expressions (4.98) into the corresponding differential Equations (4.75), (4.76), and (4.77), one will have the following matrix equation,

$$\begin{pmatrix} r^4 & 0 & ar^2 \\ 0 & r^4 & -br^2 \\ dr^2 & -fr^2 & -cr^2 + r^4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.99}$$

This will give a polynomial equation of r to the power of 12 as $r^8(-ad - bf - cr^2 + r^4) = 0$; so one can rewrite the equations (4.98) as:

$$u = \sum_{i=1}^{12} A_i e^{r_i z}$$

$$v = \sum_{i=1}^{12} B_i e^{r_i z} \tag{4.100}$$

$$\phi = \sum_{i=1}^{12} C_i e^{r_i z}$$

Solving for the characteristic equation of the matrix Equation (4.99) would give us the following roots for r :

$$r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = 0 \tag{4.101}$$

$$r_9 = -im \tag{4.102}$$

$$r_{10} = im \tag{4.103}$$

$$r_{11} = -n \tag{4.104}$$

$$r_{12} = n \tag{4.105}$$

in which

$$m = \frac{\sqrt{-c + \sqrt{c^2 + 4ad + 4bf}}}{\sqrt{2}} \tag{4.106}$$

$$n = \frac{\sqrt{c + \sqrt{c^2 + 4ad + 4bf}}}{\sqrt{2}} \tag{4.107}$$

Now the general solutions for u, v, and ϕ can be constructed as,

$$\begin{aligned} u &= \widetilde{A} + A_9 e^{-im} + A_{10} e^{im} + A_{11} e^{-nz} + A_{12} e^{nz} \\ v &= \widetilde{B} + B_9 e^{-im} + B_{10} e^{-im} + B_{11} e^{-nz} + B_{12} e^{nz} \\ \phi &= \widetilde{C} + C_9 e^{-im} + C_{10} e^{-im} + C_{11} e^{-nz} + C_{12} e^{nz} \end{aligned} \tag{4.108}$$

where

$$i = \sqrt{-1}$$

$$\widetilde{A} = \sum_{i=1}^{8} A_i \ , \qquad \widetilde{B} = \sum_{i=1}^{8} B_i \ , \quad \widetilde{C} = \sum_{i=1}^{8} C_i$$

Now by using the Euler's Formula one can rewrite Equations (4.108) as:

$$u = \widetilde{A} + \left(A_9 + A_{10}\right)\cos mz + i\left(A_{10} - A_9\right)\sin mz + A_{11}e^{-nz} + A_{12}e^{nz}$$

$$v = \widetilde{B} + (B_9 + B_{10})\cos mz + i(B_{10} - B_9)\sin mz + B_{11}e^{-nz} + B_{12}e^{nz} \tag{4.109}$$

$$\phi = \widetilde{C} + (C_9 + C_{10})\cos mz + i(C_{10} - C_9)\sin mz + C_{11}e^{-nz} + C_{12}e^{nz}$$

From the first row of the matrix Equation (4.99) one can find a relationship between A_i, and C_i as:

$$C_i = -\frac{r_i^2}{a} A_i \tag{4.110}$$

this means $\widetilde{C} = 0$.

Another relationship expression between A_i and B_i would be needed for the future calculation, which can be found from the second row of Equation (4.99) as:

$$C_i = \frac{r_i^2}{b} B_i \tag{4.111}$$

Substituting Equation (4.111) into Equation (4.110), the necessary relationship between A_i and B_i will be:

$$B_i = -\frac{b}{a} A_i \tag{4.112}$$

From the third row of Equation (4.99), a relationship between three coefficients is derived as:

$$d r_i^2 A_i - f r_i^2 B_i + \left(-c r_i^2 + r_i^4\right) C_i = 0 \tag{4.113}$$

Then by factoring r_i^2, summing up first eight terms, and taking Equation (4.112) into account one have:

$$d\widetilde{A} - f\widetilde{B} = 0 \tag{4.114}$$

Comparing the expressions given for d and f in Equations (4.87) and (4.89), it can be seen that both coefficients cannot be equal to zero simultaneously (except for the trivial case where $M_1 = M_2 = 0$). Thus, it can be concluded that:

$$\widetilde{A} = \widetilde{B} = 0 \tag{4.115}$$

Using the Expressions (4.110) - (4.115) one can rewrite the Equations (43) in terms of $\widehat{A}_i$ coefficients:

$$\begin{aligned} u &= \widehat{A}_9 \cos mz + \widehat{A}_{10} \sin mz + \widehat{A}_{11} e^{-nz} + \widehat{A}_{12} e^{nz} \\ v &= \widehat{B}_9 \cos mz + \widehat{B}_{10} \sin mz + \widehat{B}_{11} e^{-nz} + \widehat{B}_{12} e^{nz} \\ \phi &= \widehat{C}_9 \cos mz + \widehat{C}_{10} \sin mz + \widehat{C}_{11} e^{-nz} + \widehat{C}_{12} e^{nz} \end{aligned} \tag{4.116}$$

where,

$\widehat{A}_9 = A_9 + A_{10}$, $\widehat{A}_{10} = i(-A_9 + A_{10})$, $\widehat{A}_{11} = A_{11}$, $\widehat{A}_{12} = A_{12}$,

$\widehat{B}_9 = B_9 + B_{10}$, $\widehat{B}_{10} = i(-B_9 + B_{10})$, $\widehat{B}_{11} = B_{11}$, $\widehat{B}_{12} = B_{12}$,

$\widehat{C}_9 = C_9 + C_{10}$, $\widehat{C}_{10} = i(-C_9 + C_{10})$, $\widehat{C}_{11} = C_{11}$, $\widehat{C}_{12} = C_{12}$.

4.4.5 Applying the Boundary Conditions

By using the general BCs (4.78) to (4.84) for a beam-column influenced by end moments, one can find the coefficients of Equation set (4.116). There are twelve different coefficients in these equations, and one have the same number of BCs as bellow:

1. $u(0) = 0$

$$\hat{A}_9 + 0 + \hat{A}_{11} + \hat{A}_{12} + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$

2. $u'(0) = 0$

$$0 + m.\hat{A}_{10} - n.\hat{A}_{11} + n.\hat{A}_{12} + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0$$

3. $v(0) = 0$

$$0 + 0 + 0 + 0 + \hat{B}_9 + 0 + \hat{B}_{11} + \hat{B}_{12} + 0 + 0 + 0 + 0 = 0$$

4. $v'(0) = 0$

$$0 + 0 + 0 + 0 + 0 + m\hat{B}_{10} - n\hat{B}_{11} + n\hat{B}_{12} + 0 + 0 + 0 + 0 = 0$$

5. $\phi(0) = 0$

$$0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + \hat{C}_9 + 0 + \hat{C}_{11} + \hat{C}_{12} = 0$$

6. $\phi'(0) = 0$

$$0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + m\hat{C}_{10} - n\hat{C}_{11} + n\hat{C}_{12} = 0$$

7. $\left[u''' + a\phi'\right]_{z=l} = 0$

$$m^3 \sin ml\,\hat{A}_9 - m^3 \cos ml\,\hat{A}_{10} - n^3 e^{-nl}\,\hat{A}_{11} + n^3 e^{nl}\,\hat{A}_{12} + 0 + 0 + 0 + 0$$
$$-am \sin ml\,\hat{C}_9 + am \cos ml\,\hat{C}_{10} - nae^{-nl}\,\hat{C}_{11} + nae^{nl}\,\hat{C}_{12} = 0$$

8. $\left[v'''-b\phi'\right]_{z=l}=0$

$$0+0+0+0+m^3\sin ml\widehat{B}_9-m^3\cos ml\widehat{B}_{10}-n^3e^{-nl}\widehat{B}_{11}+n^3e^{nl}\widehat{B}_{12}$$
$$+bm\sin ml\widehat{C}_9-bm\cos ml\widehat{C}_{10}+bne^{-nl}\widehat{C}_{11}-bne^{nl}\widehat{C}_{12}=0$$

9. $\left[\phi'''-c\phi'+g_1u'-g_2v'\right]_{z=l}=0$

$$\left(m^3+cm\right)\sin ml\widehat{C}_9-\left(m^3+cm\right)\cos ml\widehat{C}_{10}-\left(n^3-cn\right)e^{-nl}\widehat{C}_{11}+\left(n^3-cn\right)e^{nl}\widehat{C}_{12}$$
$$-g_1m\widehat{A}_9\sin ml+g_1m\widehat{A}_{10}\cos ml-g_1n\widehat{A}_{11}e^{-nl}+g_1n\widehat{A}_{12}e^{nl}$$
$$+g_2m\widehat{B}_9\sin ml-g_2m\widehat{B}_{10}\cos ml+g_2n\widehat{B}_{11}e^{-nl}-g_2n\widehat{B}_{12}e^{nl}=0$$

10. $\left[u''-g_3\phi+h_1\right]_{z=l}=0$

$$-m^2\cos ml\widehat{A}_9-m^2\sin ml\widehat{A}_{10}+n^2e^{-nl}\widehat{A}_{11}+n^2e^{nl}\widehat{A}_{12}$$
$$+0+0+0+0$$
$$+g_3\widehat{C}_9\cos mz+g_3\widehat{C}_{10}\sin mz+g_3\widehat{C}_{11}e^{-nz}+g_3\widehat{C}_{12}e^{nz}=-h_1$$

11. $\left[v''-g_4\phi+h_2\right]_{z=l}=0$

$$0+0+0+0$$
$$-m^2\cos ml\widehat{B}_9-m^2\sin ml\widehat{B}_{10}+n^2e^{-nl}\widehat{B}_{11}+n^2e^{nl}\widehat{B}_{12}$$
$$-g_4\widehat{C}_9\cos mz-g_4\widehat{C}_{10}\sin mz-g_4\widehat{C}_{11}e^{-nz}-g_4\widehat{C}_{12}e^{nz}=-h_2$$

12. $\left[\phi''-h_3\right]_{z=l}=0$

$$0+0+0+0+0+0+0+0$$
$$-m^2\cos ml\widehat{C}_9-m^2\sin ml\widehat{C}_{10}+n^2e^{-nl}\widehat{C}_{11}+n^2e^{nl}\widehat{C}_{12}=h_3$$

Constructing the matrix form of the systems of equations derived by BCs, one will have:

$$
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & m & -n & n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & m & -n & n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & -n & n \\
m^3\sin ml & -m^3\cos ml & -n^3e^{-nl} & n^3e^{nl} & 0 & 0 & 0 & 0 & -am\sin ml & am\cos ml & -ane^{-nl} & ane^{nl} \\
0 & 0 & 0 & 0 & m^3\sin ml & -m^3\cos ml & -n^3e^{-nl} & n^3e^{nl} & bm\sin ml & -bm\cos ml & bne^{-nl} & -bne^{nl} \\
-g_1m\sin ml & g_1m\cos ml & -g_1ne^{-nl} & g_1ne^{nl} & g_2m\sin ml & -g_2m\cos ml & g_2ne^{-nl} & -g_2ne^{nl} & \left(m^3+cm\right)\sin ml & -\left(m^3+cm\right)\cos ml & -\left(n^3-cn\right)e^{-nl} & \left(n^3-cn\right)e^{nl} \\
-m^2\cos ml & -m^2\sin ml & n^2e^{-nl} & n^2e^{nl} & 0 & 0 & 0 & 0 & g_3\cos mz & g_3\sin mz & g_3e^{-nz} & g_3e^{nz} \\
0 & 0 & 0 & 0 & -m^2\cos ml & -m^2\sin ml & n^2e^{-nl} & n^2e^{nl} & -g_4\cos mz & -g_4\sin mz & -g_4e^{-nz} & -g_4e^{nz} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m^2\cos ml & -m^2\sin ml & n^2e^{-nl} & n^2e^{nl}
\end{pmatrix}
\begin{pmatrix}
\hat{A}_9 \\ \hat{A}_{10} \\ \hat{A}_{11} \\ \hat{A}_{12} \\ \hat{B}_9 \\ \hat{B}_{10} \\ \hat{B}_{11} \\ \hat{B}_{12} \\ \hat{C}_9 \\ \hat{C}_{10} \\ \hat{C}_{11} \\ \hat{C}_{12}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -h_1 \\ -h_2 \\ h_3
\end{pmatrix}
$$

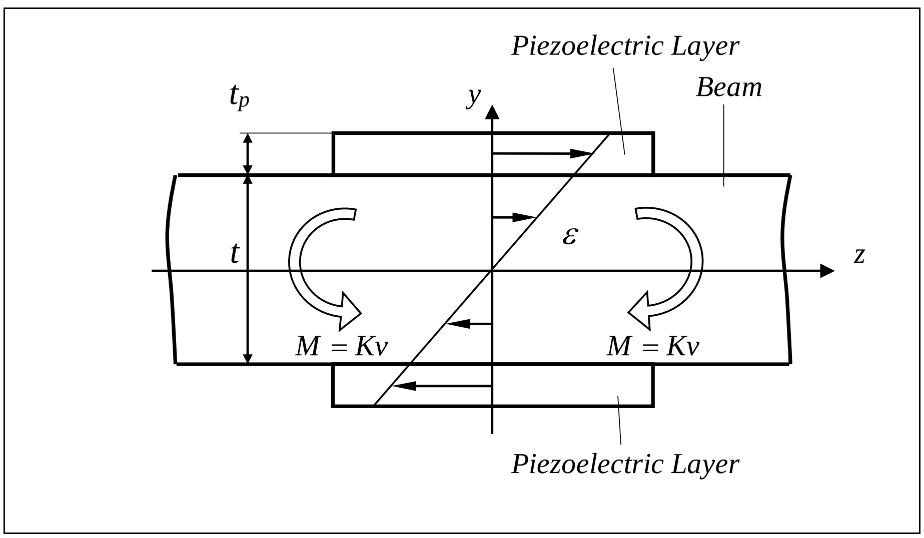
(4.117)

After symbolic solution of system of equations using Mathematica®, one would find the coefficients defined by the following functions as presented in APPENDIX I: MATHEMATICA® CODE, SOLVING FOR THE COEFFICIENTS.

5. BENDING MOMENT INDUCED BY PIEZOELECTRIC LAYER

There are some ways to exert an axial force or a moment on the beam at a certain point, but using piezoelectric layers are more practical from the installation and operation point of view. First, it is easy to embed into the beam structure or attach it by strong adhesives on the surface of the member. Second advantage is its precision for shape control and adaptation, as it can calibrate the structural deformation with the help of equivalent voltage comes over the piezoelectric layer. Third benefit is due to its fast response to voltage changes it can be used as a means of exertion of control force for vibration control systems.

Figure 5.1: Moment induced by piezoelectric layer



There are induced forces in any piezoelectric patch when subjected to a voltage. When two layers at the top and bottom actuated at the same time, a bending moment is created. One can assume that this system is equivalent to two equal and opposite moments at the two ends of the piezoelectric layer as in Figure 5.1.

Determining the angle of twist under the bending moment M_1 about the z-z axis requires to solve the coupled Equations (4.75), and (4.77) in which a, b, c, d, and f are functions of M_1, some cross-sectional and material properties. In this thesis the moment M_1 is proportional to the input voltage v (Devasia, September-October 1993):

$$M_1 = Kv \tag{5.1}$$

where, K is the proportionality constant between the voltage and moment induced by piezoelectric layer.

The moment M_1^* caused by piezoelectric actuators at the top and bottom of the beam can be defined by:

$$M_1^* = t_p b_p \left[\left(\sigma_p^t - \sigma_p^b \right) \left(t/2 + t_p/2 \right) \right] = \frac{t_p b_p E_p}{2} \left[\left(\varepsilon_p^t - \varepsilon_p^b \right) \left(t + t_p \right) \right] \tag{5.2}$$

The superscript *t* and *b* refer to top and bottom actuators, and ε_p^t, ε_p^b are the strain in piezoelectric layer due to polarization effect. The induced strain is defined as:

$$\varepsilon_p = \frac{d_{31} v}{t_p} \tag{5.3}$$

Replacing Equation (5.3) into (5.2) one will have

$$M_1^* = b_p E_p d_{31} \left(t + t_p \right) \frac{\left(v_1 - v_2 \right)}{2} \triangleq K^* \frac{\left(v_1 - v_2 \right)}{2} \ . \tag{5.4}$$

M_1^* is the effective moment acting on a beam with equivalent area moment of inertia I_{ex} . If $v_1 = -v_2 \triangleq v$, then Equation (5.4) will have the form:

$$M_1^* = K^* v \tag{5.5}$$

The value of the moment M_1 in Equation (5.1) is smaller than that of the Equation (5.5) shows for M_1^* as the later acts on the beam with equivalent area moment of inertia I_{ex} , but M_1 is acting on the beam without piezoelectric layers, means

$$\frac{M_1^*}{I_{ex}} = \frac{M_1}{I_{xx}} \quad . \tag{5.6}$$

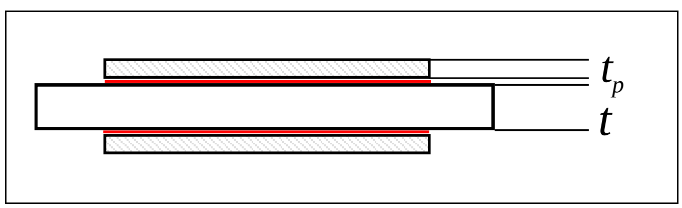
From Equations (5.5) and (5.6) one can determine the relationship between K , and K^* as

$$K = \frac{I_x}{I_{ex}} K^* \tag{5.7}$$

5.1 MODIFICATION ON EQUATION OF INDUCED MOMENT M_1

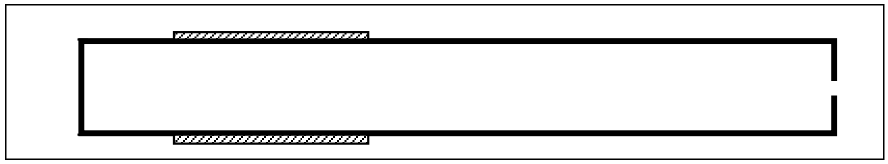
The method described by Santosh Devasia et al (Devasia, September-October 1993). has been developed for a rectangular cross section beam with piezoelectric patches attached at top and bottom faces, Figure 5.2.

Figure 5.2: Beam with adhesive and piezoelectric layer



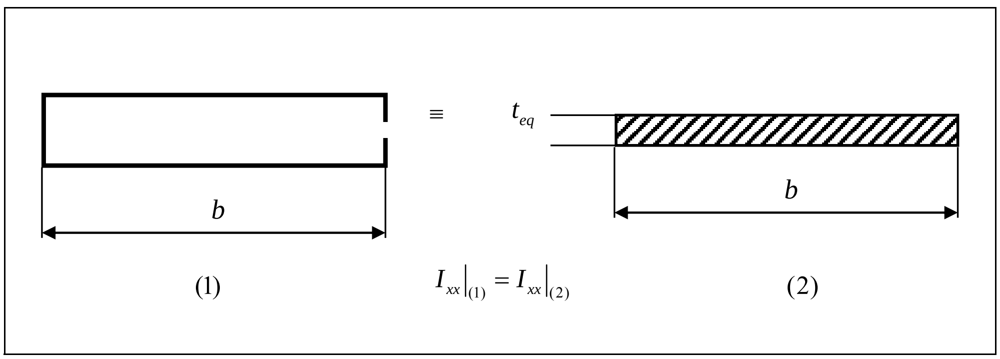
In their study the cross section of the beam is solid and the beam width is equal to that of the piezoelectric layer. In addition there is an adhesive layer between the piezo and beam surfaces. In the current study the adhesive layer does not exist, and its effect on the static behavior of the structure is assumed to be small and negligible. But the main difference between the two cases is the beam cross section, which is a thin walled open profile here, as shown in Figure 5.3.

Figure 5.3: Thin-Walled Lipped Channel schematic



In order to compensate for this difference it is proposed to construct two different cross sections with the same width and the same second area moment of inertia as the equivalent beams as shown in Figure 5.4.

Figure 5.4: Equivalent cross sections



Now one can determine the equivalent thickness t_{eq} as:

$$t_{eq} = \left(\frac{12}{b} I_{xx} \right)^{1/3} \tag{5.8}$$

and the expression (5.4) should be modified as bellow:

$$M_1^* = b_p E_p d_{31} \left(t_{eq} + t_p \right) \frac{(v_1 - v_2)}{2} \triangleq K^* \frac{(v_1 - v_2)}{2} \tag{5.9}$$

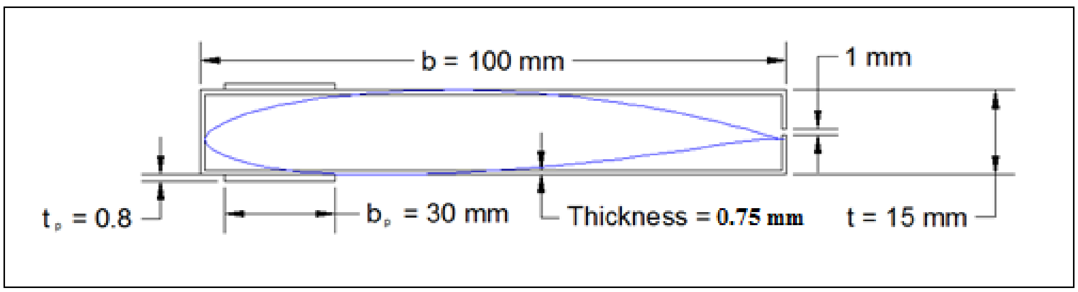
6. RESULTS

6.1 MODEL (1): THIN-WALLED LIPPED CHANNEL SECTORIAL PROPERTIES

As the final aim of this study is to investigate the effects of piezoelectric actuator layers on an open section airfoil wind-turbine blade, a Thin-Walled Lipped Channel (TWLC) is chosen such that its profile is as closer as to the target airfoil. For the first steps of the study the behavior of such a profile is investigated, as its cross sectional and sectorial properties are either theoretically known or can be calculated with the help of some available computer codes. The airfoil S834 is proposed as an appropriate airfoil for using in design of small wind-turbine blades (Selig, November 2004).

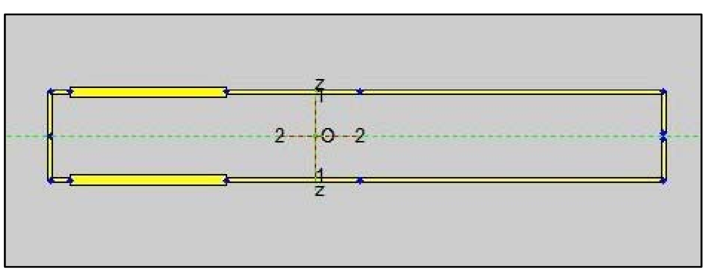
The airfoil biggest dimensions 100mmx15mm, has been used to construct TWLC with the thickness of 0.75mm as shown in Figure 6.1.

Figure 6.1: Model (1) Thin-Walled Lipped Channel



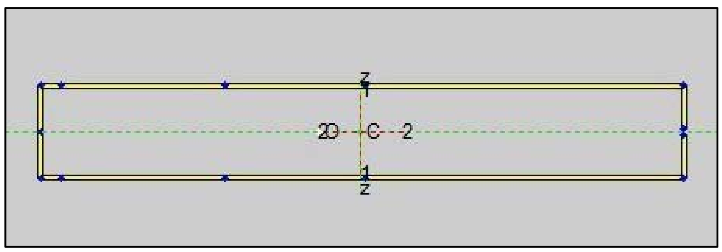
TWLC has a discontinuity with the size of 1mm at its shorter side. Besides of the boundary conditions required to derive the coefficients in Equations (4.116), there are several cross-sectional properties to be calculated.

Figure 6.2: Cross sectional properties of Model (1) – including piezoelectric layer



Calculation of some properties like centroid coordinates $\bar{x}$ and $\bar{y}$, second area moment of inertias I_{xx} and I_{yy}, and total area A are straightforward but some other sectorial properties like warping constant I_{ω}, torsional constant I_t , shear center coordinates and location x_0 and y_0 are derived by using the computer code CUFSM V4.05 (Schafer, 2014). This program is based on the theory of Finite Strip Method, which is a semi-theoretical approach with respect to FEM (Schafer, 2006).

Figure 6.3: Cross sectional properties of Model (1) – without piezoelectric layer



The sectorial properties of the cross section with and without piezoelectric layer has been determined as they are needed for calculating moment M_1, Figure 6.2, and Figure 6.3. After using data output from computer code CUFSM V4.05, and doing hand calculations the following parameters could be reported:

M_1 = 3.791 N.m

x_0 = -87.3451 x 10^{-3} m

y_0 = 0

I_{ex} = 9978.4288 x 10^{-12} m^4

I_{yy} = 212070.3587 x 10^{-12} m^4

I_t = 87.5851 x 10^{-12} m^4

I_{ω} = 16772246.5988 x 10^{-18} m^6

β_1 = 0

β_2 = -186.2302 x 10^{-3} m

6.2 COMSOL SIMULATION MODEL (1)

Preparing a 3 dimensional model in SolidWorks with all the details required was carried out and fed into the COMSOL software. The other structural and physical parameters of the beam and piezoelectric layers like shear modulus, Young Modulus for aluminum, and Piezoelectric Charge Constant (d_{31}) were entered accordingly:

d_{31} = - 264 x 10^{-12}C/N

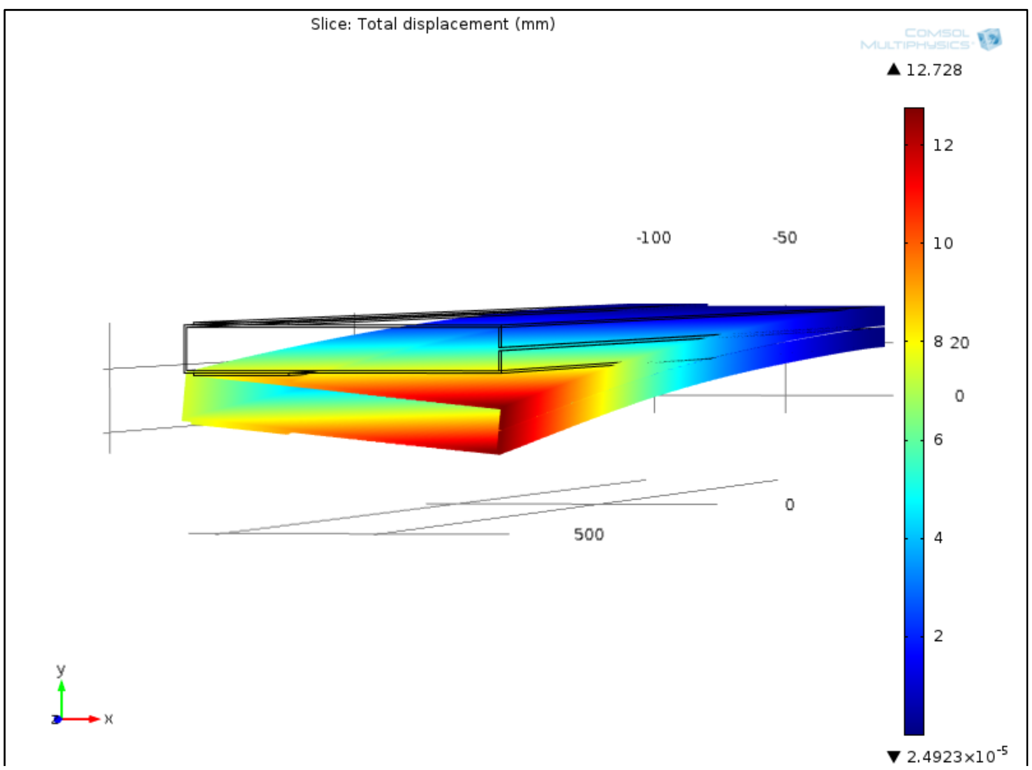
G = 27 x 10^{9} Pa

E = 69 x 10^{9} Pa

v = 1000 V

The simulation outcome can be seen in Figure 6.4.

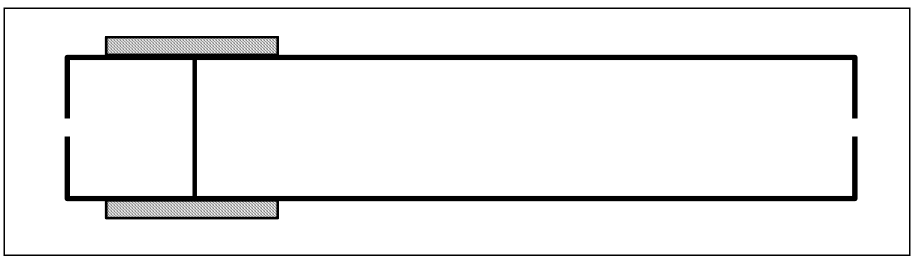
Figure 6.4: COMSOL simulation output Model (1)



6.3 MODEL (2): THIN-WALLED OPEN CROSS SECTION BEAM WITH STIFFENER

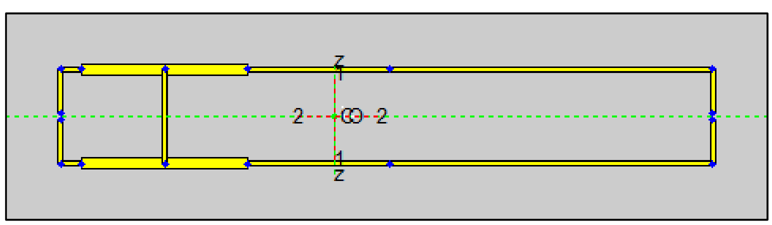
In order to verify the methodology on different beam profiles, another thin-walled open cross section beam was constructed and modeled using computer code CUFSM V4.05, and COMSOL software. The new model had two open sections in opposite directions, but with same overall dimensions (width and height). The profile thickness and piezoelectric layers were also the same. A stiffener was placed just under the piezo-layers to keep higher the structural integrity, as shown in Figure 6.5.

Figure 6.5: Model (2) Beam with two open sections profile



Using the same software and hand calculations resulted the following outcomes for this new model as shown in Figure 6.6, and Figure 6.7:

Figure 6.6: Cross sectional properties of Model (2) – including piezoelectric layer

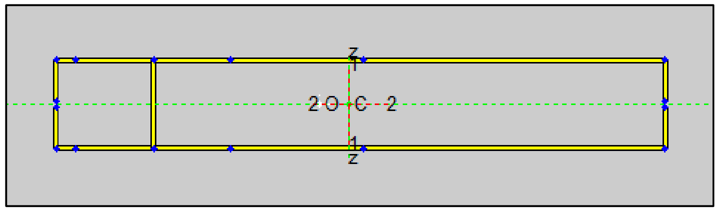


M_1 = 3.833 N.m

x_0 = -53.4606 x 10^{-3} m

y_0 = 0

Figure 6.7: Cross sectional properties of Model (2) – without piezoelectric layer



I_{ex} = 10159.2188 x 10^{-12} m^4

I_{yy} = 21823.4568 x 10^{-12} m^4

I_t = 89.4483 x 10^{-12} m^4

I_ω = 15314295.3154 x 10^{-18} m^6

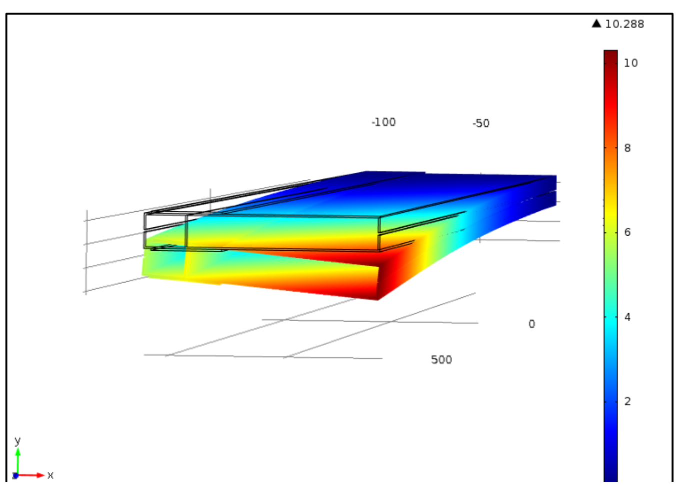
$\beta_1 = 0$

β_2 = -120.9767 x 10^{-3} m

6.4 COMSOL SIMULATION MODEL (2)

The COMSOL output for Model (2) has been shown in Figure 6.8. The deformation is overall follow the same pattern of Model (1), but the maximum deflection is smaller in this model. This might be the effect of higher structural integrity and stiffness which causes this lower deformation.

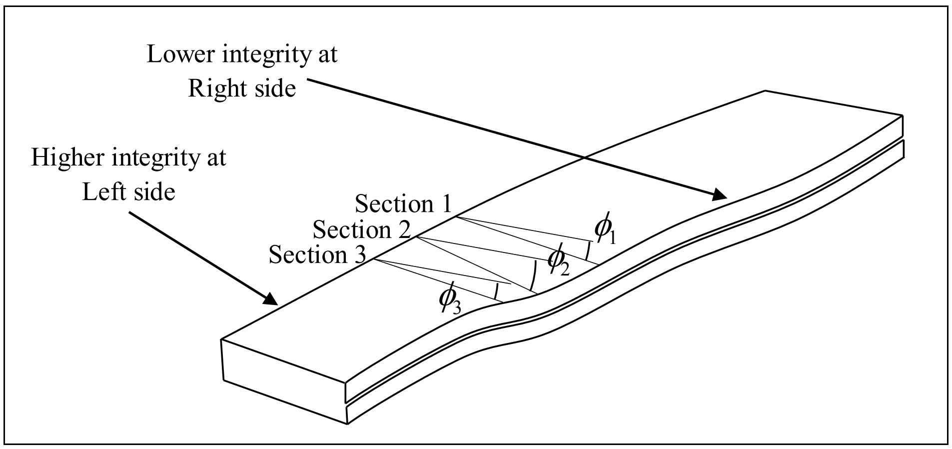
Figure 6.8: COMSOL simulation output Model (2)



6.5 COMPARISON BETWEEN THEORY AND COMSOL OUTPUT

With the help of Equations (4.116), which is the basis for constructing matrix equation (4.117), and then solving it in Mathematica software, one would find deformations u, v, and ϕ in x, y, and rotation about z axis for any point on the beam. Of course, at any cross section the rotation is the same in theory; but due to the fact that there are local deformations at some points the theory and COMSOL results might not be fitted properly. This is especially true for the structures with lower integrity, like Model (1), or when one wants to predict the deformation at a far corner of the profile, Figure 6.9.

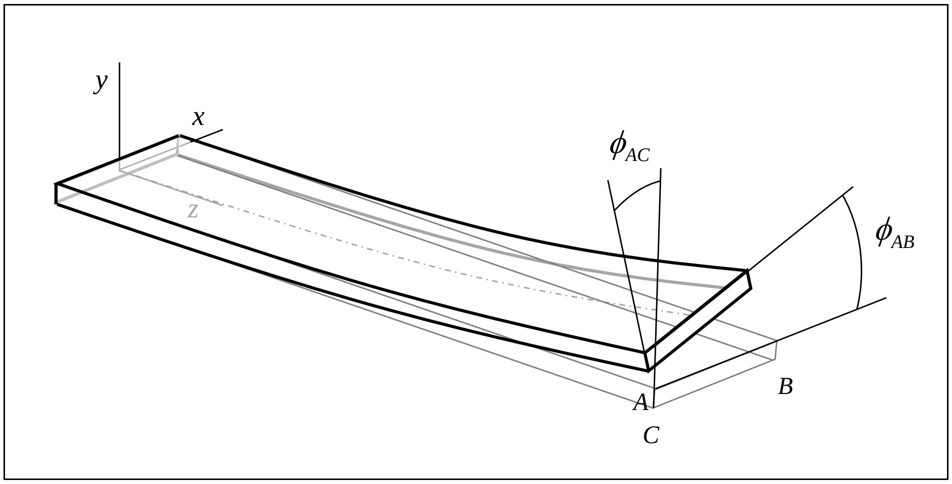
Figure 6.9: Local deformation on the beam



It seems that the twisting angle at cross sections 1, 2, and 3 should increase or decrease smoothly and but at section 2 calculation shows a locally higher twisting angle than it should be, and vice versa at section 3.

So, instead of upper edge, the data of the left edge has been used to calculate twisting angle at each cross section, because of higher local integrity. In order to avoid misinterpretation of the comparing results the COMSOL output data screened to be from the left side of the profile, as show in Figure 6.9, and Figure 6.10.

Figure 6.10: Twisting angle measured at different edges



Twisting angles ϕ_{AB}, and ϕ_{AC} should be the same, but practically due to local deformation they are different. In addition ϕ_{AB} is much more reliable to be represented as the cross sectional twist angle as it has lower deflection fluctuation.

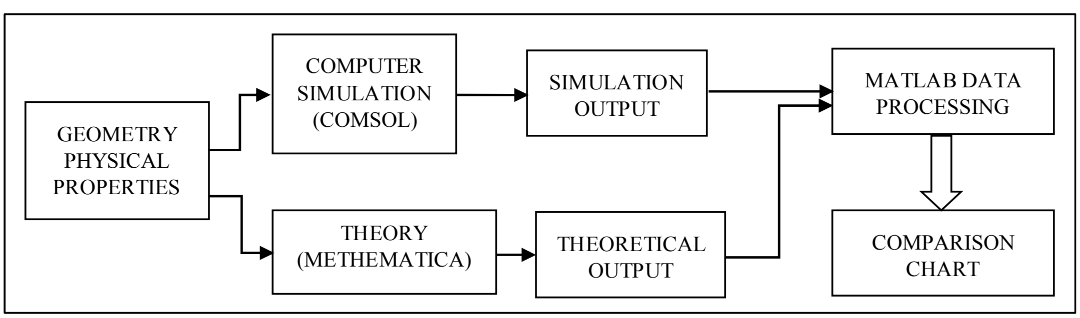
For comparing the theoretical and COMSOL results one should perform the following steps:

a. Preparing an excel file containing the data of deflections components u, v and twisting angles ϕ at specific longitudinal fiber under study.

b. Determining the parameters m, n and twelve required coefficients using Mathematica ® code,

c. Feeding the abovementioned data into MATLAB® code to draw the corresponding graphs in one frame.

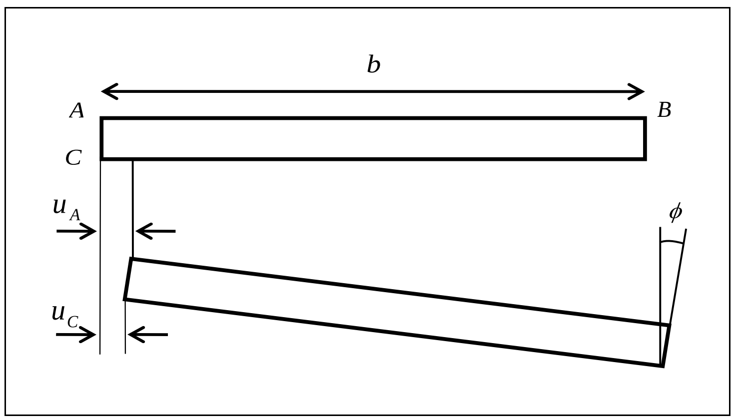
Figure 6.11: Result comparison steps flow-chart



For analyzing the models in this thesis one thousand intersection planes are passed throughout the start to the end of the beam model, then corresponding deflections at each cross section screened as one can find in APPENDIX III: COMSOL® OUTPUT FILE (modelsolutioninfo(slice-3-lr).xlsx). These data and similar one for point C has been used for calculating of twist angle at each plane using Equation (6.1). The resulting plot for longitudinal fiber passing through point A is shown in Figure 7.2 and Figure 7.4Figure 7.4: Comparing results from Theory and COMSOL, Model (2), for Model (1) and Model (2) respectively.

$$\phi = \tan^{-1}\left(\frac{u_A - u_C}{y_A - y_C}\right) \tag{6.1}$$

Figure 6.12: Calculation of twisting angle

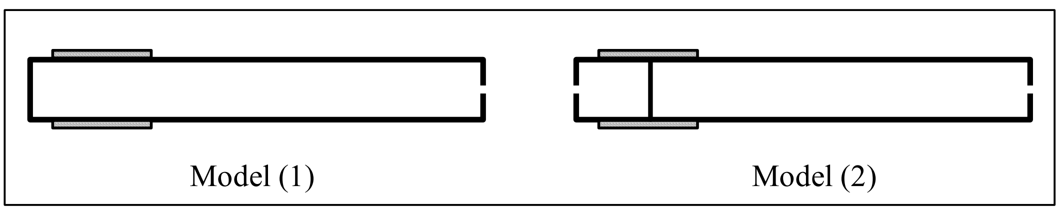


7. DISCUSSIONS AND CONCLUSIONS

One of the main goals of this study was to construct a mathematical models which were able to predict the shape of slender beam like structures similar to wind turbine blades, when changing control parameters. It needed to study different alternative parameters like solidity, hollowness, thickness, openness of the cross section among other things. The actuation system was also important regarding installation and mathematical modeling of the problem.

As target models, two rectangular open profile beams with length of 1000mm were set up with two piezoelectric layers at the top and bottom. Model (1) consisted of just one open section, and Model (2) had two open sections separated by a stiffener at the piezoelectric layers position as shown in Figure 7.1. The open profile leads to lower actuation force – lower voltage in piezoelectric layers, easier fabrication and manufacturing, but more mathematical complexity.

Figure 7.1: Comparing models



The resulting graphs in Figure 7.2 show good compliance for deflection v in y direction for Model (1), but differences in COMSOL and theory for u and ϕ. All the graphs follow the same pattern and seem very fit at the starts but gradually diffract toward the end of the beam. It means they show a combined characters of exponential and sinusoidal function which assumed in Equation set (4.116). This is observable in Figure 7.3. For u and ϕ this difference is always increasing, but for v it is increasing up to the length 800mm then decrease again.

Figure 7.2: Comparing results from Theory and COMSOL, Model (1)

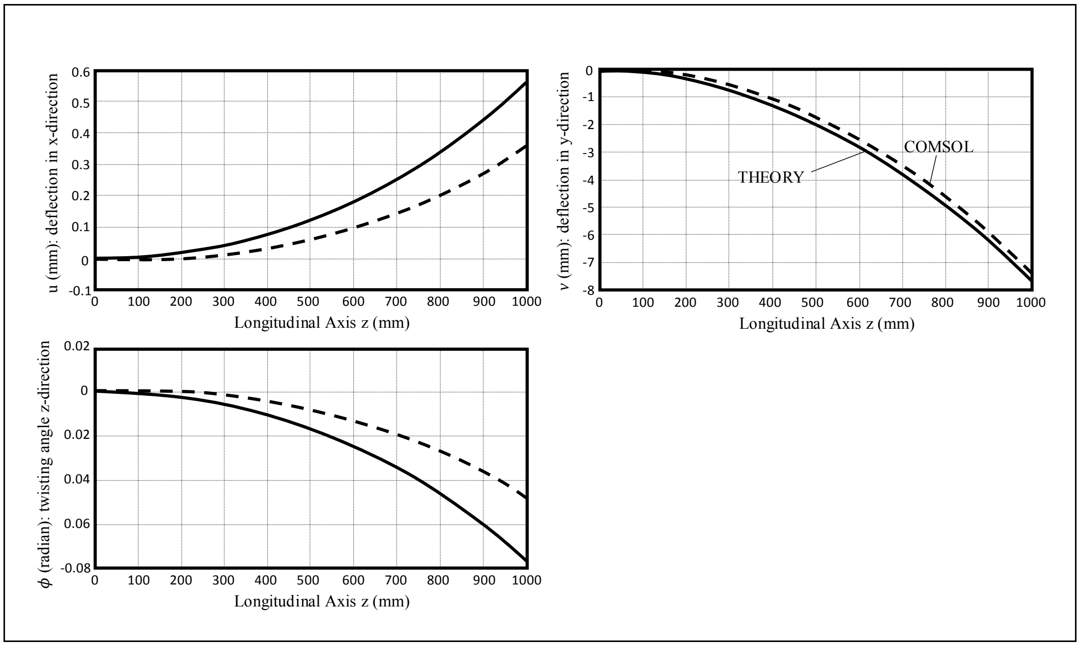
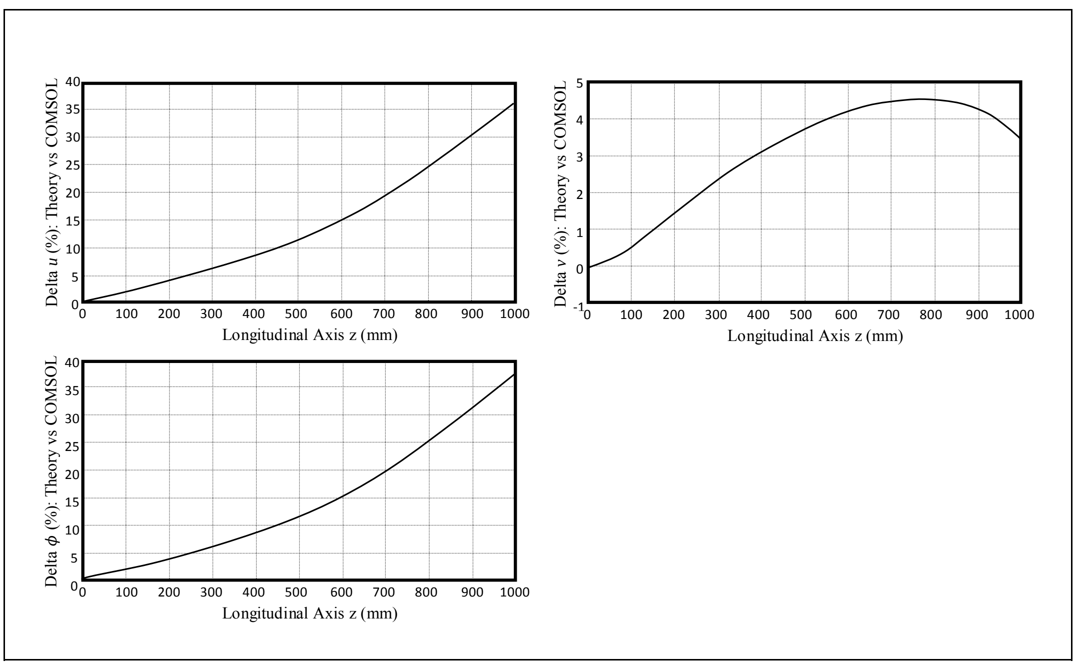
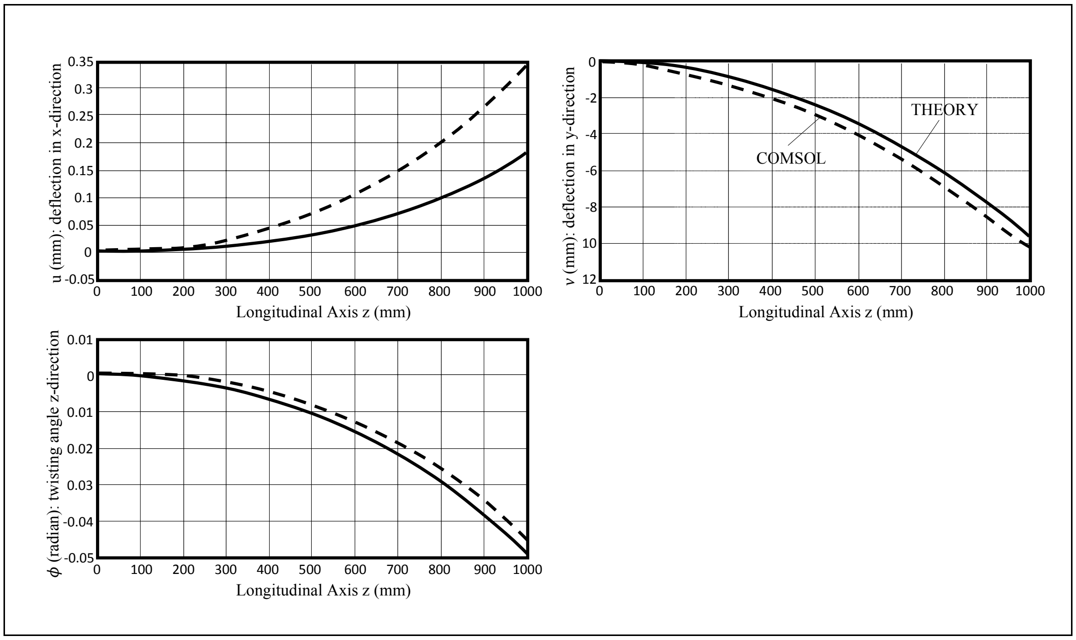


Figure 7.3: Difference of the variables (%) between theory and COMSOL Model (1)



In case of Model (2) which is equipped by a web-like stiffener, the results follow more or less the same patterns as shown in Figure 7.4.

Figure 7.4: Comparing results from Theory and COMSOL, Model (2)

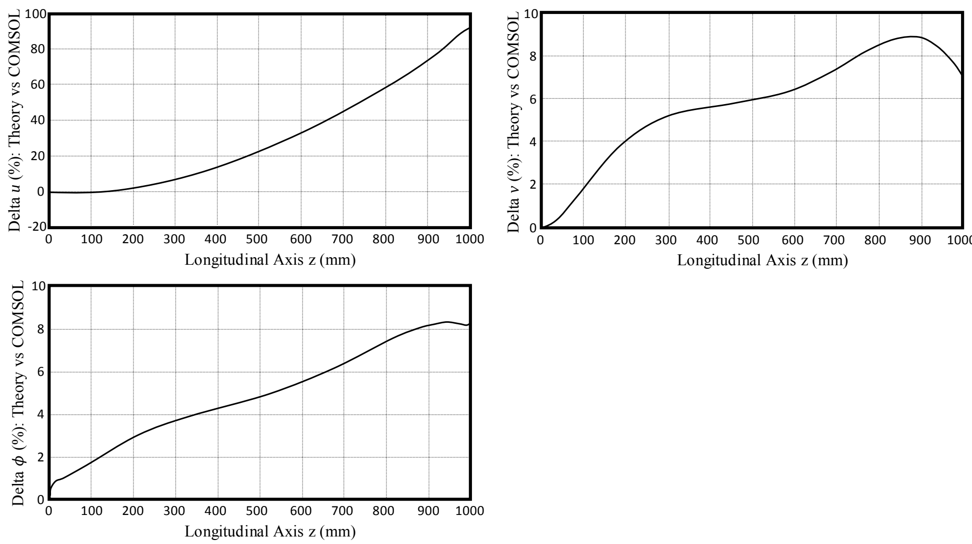


But the main difference here is that the twisting angles are nearly similar. It might mean when one have a stiffer structure it can be more predictability for theoretical calculation to foresee the behavior of the beam.

Similar difference graph can be drawn for Model (2), in order to investigate how the beam structure behaves when bend and twist at the same time. In Figure 7.5 it is shown that the difference level is lower than in Model (1) but still increasing toward the end of the beam.

There are two extreme points here instead of one in Model (1) at 250mm and 900mm in which the difference level starts to decrease. This might be for the different buckling mode shape of the structure in Model (2); as it is stiffer it tends to buckle with second or third mode.

Figure 7.5: Difference of the variables (%) between theory and COMSOL Model (2)



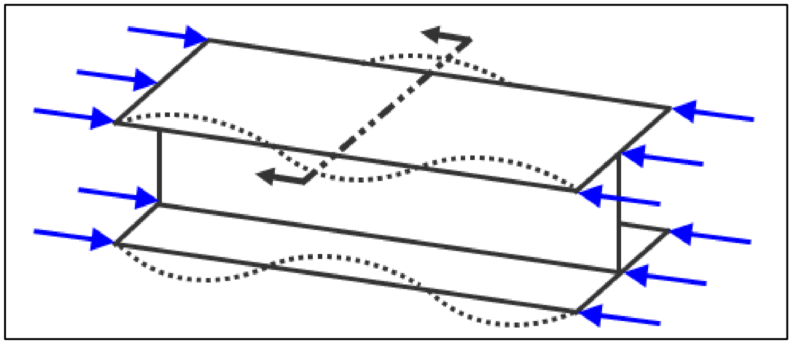
The two models have been analyzed by theoretical approach and COMSOL software and the following comments can be recited as the main conclusion points of this study:

1. There are high compliance between theory and COMSOL for deflection of the structure in the plane with minimum second moment of inertia I_{xx} - plane x-z in current study. This compliance exists for both models.

2. There is low compliance between theory and COMSOL for deflection of the structure in the plane with maximum second moment of inertia I_{yy} - plane y-z in current study. This inconsistency exists for both models.

3. In general models with higher structural integrity as Model (2), shows more conformities for all three deflection terms in theory and COMSOL analysis.

Figure 7.6: local buckling



Source:
http://download.autodesk.com/us/algor/userguides/mergedProjects/Analysis_Types/Linear/Linear_Critical_Buckl ng_Load.htm, accessed May 30, 2015)

4. It seems the theory can predict only global buckling deformations, but COMSOL can foresee local buckling, Figure 7.6: local buckling.

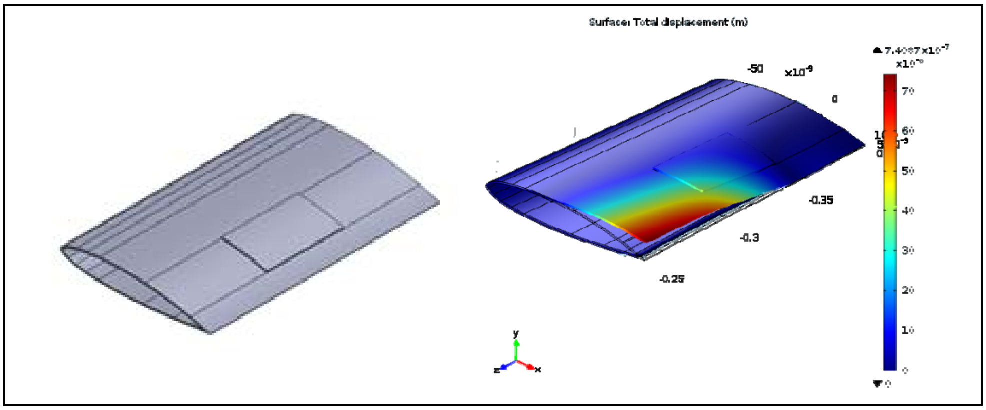
5. For the purpose of twisting analysis of the wind turbine blades, the theoretical analysis would be valuable and precise enough. Constructing a COMSOL model and performing computer simulation is a time consuming and unnecessary for each problem every time. Using the theoretical analysis results is much more reasonable for adaptive shape control system designing strategy.

6. There are sources of error exists in theoretical analysis here, like in determining the shear center coordinates x_0 and y_0 , warping modulus I_ω , and induced piezoelectric moment. This is not preventable as there are not exact solution methodologies for calculating warping modulus and shear center location when the cross section cannot be represented by a well-defined mathematical function.

The main concerns for future works are to expand the study with additional emphasis on the following points:

a. Real airfoil cross section blades. This idea is presented in Figure 7.7: Future study (Modeling wind turbine blade).

Figure 7.7: Future study (Modeling wind turbine blade)



b. Dynamic analysis of the model
c. Experimental analysis
d. Piezoelectric layer optimum shapes and dimensions
e. Piezoelectric layer optimum placement

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APPENDICES

APPENDIX I: MATHEMATICA® CODE, SOLVING FOR THE COEFFICIENTS

```
AA=(_{
    {1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, m, -n, n, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, m, -n, n, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, m, -n, n},
    {R1, -R2, -R3, R4, 0, 0, 0, 0, -(a/m2) R1, (a/m2) R2, -(a/n2) R3,
(a/n2) R4},
    {0, 0, 0, 0, R1, -R2, -R3, R4, (b/m2) R1, -(b/m2) R2, (b/n2) R3, -
(b/n2) R4},
    {-(g1/m2) R1, (g1/m2) R2, -(g1/n2) R3, (g1/n2) R4, (g2/m2) R1, -
(g2/m2) R2, (g2/n2) R3, -(g2/n2) R4, ((c m+m3)/m3) R1, -((c m+m3)/m3)
R2, -((n3-c n)/n3) R3, ((n3-c n)/n3) R4},
    {-R2/m, -R1/m, R3/n, R4/n, 0, 0, 0, 0, -(g3/m3) R1, -(g3/m3) R2, -
(g3/n3) R3, -(g3/n3) R4},
    {0, 0, 0, 0, -R2/m, -R1/m, R3/n, R4/n, -(g4/m3) R1, -(g4/m3) R2, -
(g4/n3) R3, -(g4/n3) R4},
    {0, 0, 0, 0, 0, 0, 0, 0, -R2/m, -R1/m, R3/n, R4/n}
   }_);
AA//MatrixForm;
AAinverse=Inverse[AA];
AAinverse//MatrixForm;
x={0,0,0,0,0,0,0,0,0,-h1,-h2,h3};
Aout=AAinverse.x//Simplify;
Aout//MatrixForm
   X0=-53.4606*10^-3;
```

```
Y0=0;
Ixx=10159.2188*10^-12;
Iyy=21823.4568*10^-12;
Ixy=0;
It=89.4483*10^-12;
Iw=15314295.3154*10^-18;
Beta1=0;
Beta2=-120.9767*10^-3;
A=220*10^-6;
M1=3.833;
M2=0;
P=0;
G=27*10^9;
EE=69*10^9;
L=1;
"a="
a=(-M1-M2 Ixy/Iyy)/(EE Iyy);
"b="
b=(-M2-M1 Ixy/Ixx)/(EE Ixx);
"c="
c=(G It+(2 X0 Ixy/Ixx-Beta1) M1+(2 Y0 Ixy/Iyy-Beta2) M2)/(EE Iw);
"d="
d=(-M1-M2 Ixy/Iyy)/(EE Iw);
"f="
f=(-M2-M1 Ixy/Ixx)/(EE Iw);
"m="
m=Sqrt[-c+Sqrt[c^2+4 a d+4 b f]]/Sqrt[2]
"n="
n=Sqrt[c+Sqrt[c^2+4 a d+4 b f]]/Sqrt[2]
"h1="
h1=(M2+M1 Ixy/Ixx)/(EE Iyy);
"h2="
h2=(M1+M2 Ixy/Iyy)/(EE Ixx);
"h3="
h3=(X0 M1+Y0 M2)/(EE Iw);

g1=(-M1)/(EE Iw);
g2=(-M2)/(EE Iw);
g3=(2M1+M2 Ixy/Iyy)/(EE Iyy);
g4=(-2M2-M1 Ixy/Ixx)/(EE Ixx);

R1=(m^3)Sin[m L];
R2=(m^3)Cos[m L];
R3=(n^3)*Exp[-n L];
R4=(n^3)*Exp[n L];

"A9="
A9=-(a (-g1 (2 n3 R2-m3 (R3+R4)) (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2
(R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2
R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1
R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+(m2+n2) (-2 n2 R1+m2 (R3-R4)) (n R2 (-R3+R4)+m R1 (R3+R4))
(h3 m4 n4 (-2 m n R2 (R3+R4)-n2 (2 R12+2 R22+R1 R3-R1 R4)+m2 (R1 R3-R1
R4-2 R3 R4))+g2 (h2 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3
R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-
R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))))+(2 n R2+m
(R3+R4)) (b g2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2
```

(R3+R4)-m n3 R2 (R3+R4)) (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))-m2 n2 (c (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (h1 m2 n2 (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4))+g3 h3 (4 n6 R1 R2+m4 n2 R1 (R3-R4)+2 m6 R3 R4+m2 n4 R2 (-R3+R4)-m3 n3 (R1+R2) (R3+R4)))+n (h1 (2 g2 g4 n7 R2 (R12-R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))+g2 g4 R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 g2 g4 m2 n5 R2 (R12 (R3-R4)+2 R1 R3 R4+R22 (-R3+R4))+g2 g4 m n6 (-2 R13 (R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42))+g2 g4 m3 n4 (-2 R13 (R3+R4)+R1 R2 (2 R2-R3+R4) (R3+R4)+R12 (R32-R42)+R22 (-R32+R42))+m7 R2 (R3+R4) (g2 g4 R1 (R3-R4)+4 n4 (2 R3 R4+R1 (-R3+R4)))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2+g2 g4 (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (g2 g4 (-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))))+g3 (-g2 h2 (m2+n2) (2 n5 R2 (R12-R22) (R3-R4)+4 m2 n3 R1 R2 R3 R4+m3 n2 R1 (R1-R2) (R32-R42)+m5 R1 R2 (R32-R42)+m4 n (-R22 (R3-R4)2+R12 (R3+R4)2)+m n4 (-2 R13 (R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42)))+h3 m2 n (n6 R2 (2 R12+2 R22+R1 (R3-R4)) (4 R1+R3-R4)-3 m5 n R2 (R3+R4) (2 R3 R4+R1 (-R3+R4))-m6 (2 R3 R4+R1 (-R3+R4))2+m n5 (2 R13 (R3+R4)+10 R1 R22 (R3+R4)+R12 (R32-R42)+2 R22 (R32-R42))+m4 n2 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))+m2 n4 R2 (-4 R12 (R3-R4)+2 R3 (R3-R4) R4+R1 (R32+14 R3 R4+R42))+m3 n3 (R3+R4) (-2 R23+R12 (-2 R2-R3+R4)+R1 (2 R3 R4+R2 (-R3+R4))))))))/(m n (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (-a g1 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1 g3-g2 g4) n7 R2 (-R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))-(g1 g3-g2 g4) R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 (g1 g3-g2 g4) m2 n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2 (R3+R4) (g2 g4 R1 (R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3 R4))+(g1 g3-g2 g4) m n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-R42))+(g1 g3-g2 g4) m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2-(g1 g3-g2 g4) (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-(g1 g3-g2 g4) (-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c m2 n (2 n6 (R12+R22) (2 R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1 (R3-R4)) (R3+R4)+m5 n R2 (R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2 (R3+R4) (R12+R22+3 R1 (R3-R4)-R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-R3+R4))+2 m4 n2 (3 R1 R3 R4 (-R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4 R3 R4+R42))+2 m2 n4 (-3 R13 (R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3 R4+R42)+R22 (R32+4 R3 R4+R42))))))

"A9="
A10=-(-a (g1 (2 n2 R1+m2 (-R3+R4)) (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))+(m2+n2) (2 n R2+m (R3+R4)) (n R2 (-R3+R4)+m R1 (R3+R4)) (h3 m4 n4 (2 m n R2 (R3+R4)+n2 (2 R12+2 R22+R1 R3-R1 R4)+m2 (-R1 R3+R1 R4+2 R3 R4))-g2 (h2 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))))+(2 R1+R3-R4) (b g2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4)) (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-

```
R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2
n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))-m2
n2 (c (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-
R3+R4))) (h1 m2 n2 (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-R3+R4)-m3 n
R2 (R3+R4)+m n3 R2 (R3+R4))+g3 h3 (4 n6 R1 R2+m4 n2 R1 (R3-R4)+2 m6 R3
R4+m2 n4 R2 (-R3+R4)-m3 n3 (R1+R2) (R3+R4)))+n (h1 (2 g2 g4 n7 R2
(R12-R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))+g2 g4 R12
(R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 g2 g4 m2 n5 R2 (R12
(R3-R4)+2 R1 R3 R4+R22 (-R3+R4))+g2 g4 m n6 (-2 R13 (R3+R4)+2 R1 R22
(R3+R4)+R22 (-R32+R42))+g2 g4 m3 n4 (-2 R13 (R3+R4)+R1 R2 (2 R2-R3+R4)
(R3+R4)+R12 (R32-R42)+R22 (-R32+R42))+m7 R2 (R3+R4) (g2 g4 R1 (R3-
R4)+4 n4 (2 R3 R4+R1 (-R3+R4)))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2+g2 g4 (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (g2 g4 (-
R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))))+g3 (-g2 h2
(m2+n2) (2 n5 R2 (R12-R22) (R3-R4)+4 m2 n3 R1 R2 R3 R4+m3 n2 R1 (R1-
R2) (R32-R42)+m5 R1 R2 (R32-R42)+m4 n (-R22 (R3-R4)2+R12 (R3+R4)2)+m
n4 (-2 R13 (R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42)))+h3 m2 n (n6 R2 (2
R12+2 R22+R1 (R3-R4)) (4 R1+R3-R4)-3 m5 n R2 (R3+R4) (2 R3 R4+R1 (-
R3+R4))-m6 (2 R3 R4+R1 (-R3+R4))2+m n5 (2 R13 (R3+R4)+10 R1 R22
(R3+R4)+R12 (R32-R42)+2 R22 (R32-R42))+m4 n2 (2 R13 (R3-R4)+2 R1 (R3-
R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))+m2 n4
R2 (-4 R12 (R3-R4)+2 R3 (R3-R4) R4+R1 (R32+14 R3 R4+R42))+m3 n3
(R3+R4) (-2 R23+R12 (-2 R2-R3+R4)+R1 (2 R3 R4+R2 (-R3+R4)))))))))/(m
(n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-
R3+R4))) (-a g1 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-
m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3
R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1
g3-g2 g4) n7 R2 (-R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-
R4))-(g1 g3-g2 g4) R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2
(g1 g3-g2 g4) m2 n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2
(R3+R4) (g2 g4 R1 (R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3
R4))+(g1 g3-g2 g4) m n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-
R42))+(g1 g3-g2 g4) m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4)
(R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2-(g1 g3-g2 g4) (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n
(-(g1 g3-g2 g4) (-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1
(R3-R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c
m2 n (2 n6 (R12+R22) (2 R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1
(R3-R4)) (R3+R4)+m5 n R2 (R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2
(R3+R4) (R12+R22+3 R1 (R3-R4)-R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-
R3+R4))+2 m4 n2 (3 R1 R3 R4 (-R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4
R3 R4+R42))+2 m2 n4 (-3 R13 (R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3
R4+R42)+R22 (R32+4 R3 R4+R42))))))//Simplify

"A11="
A11=-(a (-g1 (m n2 R1-n3 R2+m3 R4) (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n
R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (h1 m2 n2 (-2 n4 (R12+R22)+2 m2
n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6
R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+(m2+n2) (n2 R1+m n R2+m2 R4) (n R2 (-R3+R4)+m R1 (R3+R4))
(h3 m4 n4 (-2 m n R2 (R3+R4)-n2 (2 R12+2 R22+R1 R3-R1 R4)+m2 (R1 R3-R1
R4-2 R3 R4))+g2 (h2 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3
R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-
R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))))+(-n R2+m
(R1-R4)) (b g2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2
(R3+R4)-m n3 R2 (R3+R4)) (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-
R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2
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n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))-m2
n2 (c (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-
R3+R4))) (h1 m2 n2 (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-R3+R4)-m3 n
R2 (R3+R4)+m n3 R2 (R3+R4))+g3 h3 (4 n6 R1 R2+m4 n2 R1 (R3-R4)+2 m6 R3
R4+m2 n4 R2 (-R3+R4)-m3 n3 (R1+R2) (R3+R4)))+n (h1 (2 g2 g4 n7 R2
(R12-R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))+g2 g4 R12
(R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 g2 g4 m2 n5 R2 (R12
(R3-R4)+2 R1 R3 R4+R22 (-R3+R4))+g2 g4 m n6 (-2 R13 (R3+R4)+2 R1 R22
(R3+R4)+R22 (-R32+R42))+g2 g4 m3 n4 (-2 R13 (R3+R4)+R1 R2 (2 R2-R3+R4)
(R3+R4)+R12 (R32-R42)+R22 (-R32+R42))+m7 R2 (R3+R4) (g2 g4 R1 (R3-
R4)+4 n4 (2 R3 R4+R1 (-R3+R4)))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2+g2 g4 (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (g2 g4 (-
R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))))+g3 (-g2 h2
(m2+n2) (2 n5 R2 (R12-R22) (R3-R4)+4 m2 n3 R1 R2 R3 R4+m3 n2 R1 (R1-
R2) (R32-R42)+m5 R1 R2 (R32-R42)+m4 n (-R22 (R3-R4)2+R12 (R3+R4)2)+m
n4 (-2 R13 (R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42)))+h3 m2 n (n6 R2 (2
R12+2 R22+R1 (R3-R4)) (4 R1+R3-R4)-3 m5 n R2 (R3+R4) (2 R3 R4+R1 (-
R3+R4))-m6 (2 R3 R4+R1 (-R3+R4))2+m n5 (2 R13 (R3+R4)+10 R1 R22
(R3+R4)+R12 (R32-R42)+2 R22 (R32-R42))+m4 n2 (2 R13 (R3-R4)+2 R1 (R3-
R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))+m2 n4
R2 (-4 R12 (R3-R4)+2 R3 (R3-R4) R4+R1 (R32+14 R3 R4+R42))+m3 n3
(R3+R4) (-2 R23+R12 (-2 R2-R3+R4)+R1 (2 R3 R4+R2 (-R3+R4)))))))))/(m n
(n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-
R3+R4))) (-a g1 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-
m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3
R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1
g3-g2 g4) n7 R2 (-R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-
R4))-(g1 g3-g2 g4) R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2
(g1 g3-g2 g4) m2 n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2
(R3+R4) (g2 g4 R1 (R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3
R4))+(g1 g3-g2 g4) m n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-
R42))+(g1 g3-g2 g4) m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4)
(R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2-(g1 g3-g2 g4) (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n
(-(g1 g3-g2 g4) (-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1
(R3-R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c
m2 n (2 n6 (R12+R22) (2 R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1
(R3-R4)) (R3+R4)+m5 n R2 (R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2
(R3+R4) (R12+R22+3 R1 (R3-R4)-R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-
R3+R4))+2 m4 n2 (3 R1 R3 R4 (-R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4
R3 R4+R42))+2 m2 n4 (-3 R13 (R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3
R4+R42)+R22 (R32+4 R3 R4+R42))))))//Simplify

"A12="
A12=-(a (g1 (-m n2 R1-n3 R2+m3 R3) (-n2 (2 R12+2 R22+R1 (R3-R4))-2 m n
R2 (R3+R4)+m2 (R1 (R3-R4)-2 R3 R4)) (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2
R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1
R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))-(m2+n2) (-n2 R1+m n R2+m2 R3) (n R2 (-R3+R4)+m R1 (R3+R4))
(h3 m4 n4 (-2 m n R2 (R3+R4)-n2 (2 R12+2 R22+R1 R3-R1 R4)+m2 (R1 R3-R1
R4-2 R3 R4))+g2 (h2 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3
R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-
R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))))-(n R2+m
(R1+R3)) (b g2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2
(R3+R4)-m n3 R2 (R3+R4)) (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-
R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2
n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))-m2
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n2 (c (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-
R3+R4))) (h1 m2 n2 (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-R3+R4)-m3 n
R2 (R3+R4)+m n3 R2 (R3+R4))+g3 h3 (4 n6 R1 R2+m4 n2 R1 (R3-R4)+2 m6 R3
R4+m2 n4 R2 (-R3+R4)-m3 n3 (R1+R2) (R3+R4)))+n (h1 (2 g2 g4 n7 R2
(R12-R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))+g2 g4 R12
(R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 g2 g4 m2 n5 R2 (R12
(R3-R4)+2 R1 R3 R4+R22 (-R3+R4))+g2 g4 m n6 (-2 R13 (R3+R4)+2 R1 R22
(R3+R4)+R22 (-R32+R42))+g2 g4 m3 n4 (-2 R13 (R3+R4)+R1 R2 (2 R2-R3+R4)
(R3+R4)+R12 (R32-R42)+R22 (-R32+R42))+m7 R2 (R3+R4) (g2 g4 R1 (R3-
R4)+4 n4 (2 R3 R4+R1 (-R3+R4)))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2+g2 g4 (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (g2 g4 (-
R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))))+g3 (-g2 h2
(m2+n2) (2 n5 R2 (R12-R22) (R3-R4)+4 m2 n3 R1 R2 R3 R4+m3 n2 R1 (R1-
R2) (R32-R42)+m5 R1 R2 (R32-R42)+m4 n (-R22 (R3-R4)2+R12 (R3+R4)2)+m
n4 (-2 R13 (R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42)))+h3 m2 n (n6 R2 (2
R12+2 R22+R1 (R3-R4)) (4 R1+R3-R4)-3 m5 n R2 (R3+R4) (2 R3 R4+R1 (-
R3+R4))-m6 (2 R3 R4+R1 (-R3+R4))2+m n5 (2 R13 (R3+R4)+10 R1 R22
(R3+R4)+R12 (R32-R42)+2 R22 (R32-R42))+m4 n2 (2 R13 (R3-R4)+2 R1 (R3-
R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))+m2 n4
R2 (-4 R12 (R3-R4)+2 R3 (R3-R4) R4+R1 (R32+14 R3 R4+R42))+m3 n3
(R3+R4) (-2 R23+R12 (-2 R2-R3+R4)+R1 (2 R3 R4+R2 (-R3+R4)))))))))/(m n
(n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-
R3+R4))) (-a g1 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-
m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3
R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1
g3-g2 g4) n7 R2 (-R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-
R4))-(g1 g3-g2 g4) R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2
(g1 g3-g2 g4) m2 n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2
(R3+R4) (g2 g4 R1 (R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3
R4))+(g1 g3-g2 g4) m n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-
R42))+(g1 g3-g2 g4) m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4)
(R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2-(g1 g3-g2 g4) (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n
(-(g1 g3-g2 g4) (-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1
(R3-R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c
m2 n (2 n6 (R12+R22) (2 R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1
(R3-R4)) (R3+R4)+m5 n R2 (R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2
(R3+R4) (R12+R22+3 R1 (R3-R4)-R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-
R3+R4))+2 m4 n2 (3 R1 R3 R4 (-R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4
R3 R4+R42))+2 m2 n4 (-3 R13 (R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3
R4+R42)+R22 (R32+4 R3 R4+R42))))))//Simplify

"B9="
B9=-(-a g1 (2 n R2+m (R3+R4)) (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-
R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4)) (h2 m2 n2 (-2 n4 (R12+R22)+2
m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4
n6 R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+b (-g2 (2 n3 R2-m3 (R3+R4)) (n2 (2 R12+2 R22+R1 (R3-R4))+2 m
n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (h2 m2 n2 (-2 n4 (R12+R22)+2 m2
n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4 n6
R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+(m2+n2) (-2 n2 R1+m2 (R3-R4)) (n R2 (-R3+R4)+m R1 (R3+R4))
(h3 m4 n4 (2 m n R2 (R3+R4)+n2 (2 R12+2 R22+R1 R3-R1 R4)+m2 (-R1 R3+R1
R4+2 R3 R4))+g1 (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3
R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-
R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))))-m2 n2 (2 n
R2+m (R3+R4)) (c (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2
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R3 R4+R1 (-R3+R4))) (h2 m2 n2 (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-
R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4))+g4 h3 (4 n6 R1 R2+m4 n2 R1
(R3-R4)+2 m6 R3 R4+m2 n4 R2 (-R3+R4)-m3 n3 (R1+R2) (R3+R4)))+n (h2 (2
g1 g3 n7 R2 (-R12+R22) (R3-R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 g1 g3 m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m5 n2 (R3+R4) (4 n4 R2 (2
R12+2 R22+R1 (R3-R4))+g1 g3 R12 (-R3+R4))+g1 g3 m n6 (2 R13 (R3+R4)-2
R1 R22 (R3+R4)+R22 (R32-R42))+g1 g3 m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-
R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m7 R2 (R3+R4) (g1 g3 R1
(-R3+R4)+4 n4 (2 R3 R4+R1 (-R3+R4)))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2-g1 g3 (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-g1 g3
(-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))))+g4 (g1 h1 (m2+n2)
(2 n5 R2 (R12-R22) (R3-R4)+4 m2 n3 R1 R2 R3 R4+m3 n2 R1 (R1-R2) (R32-
R42)+m5 R1 R2 (R32-R42)+m4 n (-R22 (R3-R4)2+R12 (R3+R4)2)+m n4 (-2 R13
(R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42)))+h3 m2 n (n6 R2 (2 R12+2
R22+R1 (R3-R4)) (4 R1+R3-R4)-3 m5 n R2 (R3+R4) (2 R3 R4+R1 (-R3+R4))-
m6 (2 R3 R4+R1 (-R3+R4))2+m n5 (2 R13 (R3+R4)+10 R1 R22 (R3+R4)+R12
(R32-R42)+2 R22 (R32-R42))+m4 n2 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))+m2 n4 R2 (-4 R12
(R3-R4)+2 R3 (R3-R4) R4+R1 (R32+14 R3 R4+R42))+m3 n3 (R3+R4) (-2
R23+R12 (-2 R2-R3+R4)+R1 (2 R3 R4+R2 (-R3+R4))))))))/(m n (n2 (2 R12+2
R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (-a g1 (2
n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m
n^3 R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-
R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1 g3-g2 g4) n7 R2 (-
R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))-(g1 g3-g2 g4)
R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 (g1 g3-g2 g4) m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2 (R3+R4) (g2 g4 R1
(R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3 R4))+(g1 g3-g2 g4) m
n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-R42))+(g1 g3-g2 g4) m3 n4
(2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-
R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2-(g1 g3-g2 g4) (-R22
(R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-(g1 g3-g2 g4) (-R22 (R3-
R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12
(R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c m2 n (2 n6 (R12+R22) (2
R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1 (R3-R4)) (R3+R4)+m5 n R2
(R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2 (R3+R4) (R12+R22+3 R1 (R3-R4)-
R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-R3+R4))+2 m4 n2 (3 R1 R3 R4 (-
R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4 R3 R4+R42))+2 m2 n4 (-3 R13
(R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3 R4+R42)+R22 (R32+4 R3
R4+R42))))))//Simplify

"B10="
B10=-(-a g1 (2 R1+R3-R4) (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-
R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4)) (h2 m2 n2 (-2 n4 (R12+R22)+2
m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4
n6 R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+b (-g2 (2 n2 R1+m2 (-R3+R4)) (n2 (2 R12+2 R22+R1 (R3-R4))+2
m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (h2 m2 n2 (-2 n4 (R12+R22)+2
m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4
n6 R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+(m2+n2) (2 n R2+m (R3+R4)) (n R2 (-R3+R4)+m R1 (R3+R4)) (h3
m4 n4 (2 m n R2 (R3+R4)+n2 (2 R12+2 R22+R1 R3-R1 R4)+m2 (-R1 R3+R1
R4+2 R3 R4))+g1 (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3
R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-
R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))))-m2 n2 (2
R1+R3-R4) (c (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3
R4+R1 (-R3+R4))) (h2 m2 n2 (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-
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R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4))+g4 h3 (4 n6 R1 R2+m4 n2 R1
(R3-R4)+2 m6 R3 R4+m2 n4 R2 (-R3+R4)-m3 n3 (R1+R2) (R3+R4)))+n (h2 (2
g1 g3 n7 R2 (-R12+R22) (R3-R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 g1 g3 m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m5 n2 (R3+R4) (4 n4 R2 (2
R12+2 R22+R1 (R3-R4))+g1 g3 R12 (-R3+R4))+g1 g3 m n6 (2 R13 (R3+R4)-2
R1 R22 (R3+R4)+R22 (R32-R42))+g1 g3 m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-
R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m7 R2 (R3+R4) (g1 g3 R1
(-R3+R4)+4 n4 (2 R3 R4+R1 (-R3+R4)))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2-g1 g3 (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-g1 g3
(-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))))+g4 (g1 h1 (m2+n2)
(2 n5 R2 (R12-R22) (R3-R4)+4 m2 n3 R1 R2 R3 R4+m3 n2 R1 (R1-R2) (R32-
R42)+m5 R1 R2 (R32-R42)+m4 n (-R22 (R3-R4)2+R12 (R3+R4)2)+m n4 (-2 R13
(R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42)))+h3 m2 n (n6 R2 (2 R12+2
R22+R1 (R3-R4)) (4 R1+R3-R4)-3 m5 n R2 (R3+R4) (2 R3 R4+R1 (-R3+R4))-
m6 (2 R3 R4+R1 (-R3+R4))2+m n5 (2 R13 (R3+R4)+10 R1 R22 (R3+R4)+R12
(R32-R42)+2 R22 (R32-R42))+m4 n2 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))+m2 n4 R2 (-4 R12
(R3-R4)+2 R3 (R3-R4) R4+R1 (R32+14 R3 R4+R42))+m3 n3 (R3+R4) (-2
R23+R12 (-2 R2-R3+R4)+R1 (2 R3 R4+R2 (-R3+R4))))))))/(m (n2 (2 R12+2
R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (-a g1 (2
n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m
n^3 R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-
R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1 g3-g2 g4) n7 R2 (-
R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))-(g1 g3-g2 g4)
R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 (g1 g3-g2 g4) m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2 (R3+R4) (g2 g4 R1
(R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3 R4))+(g1 g3-g2 g4) m
n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-R42))+(g1 g3-g2 g4) m3 n4
(2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-
R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2-(g1 g3-g2 g4) (-R22
(R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-(g1 g3-g2 g4) (-R22 (R3-
R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12
(R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c m2 n (2 n6 (R12+R22) (2
R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1 (R3-R4)) (R3+R4)+m5 n R2
(R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2 (R3+R4) (R12+R22+3 R1 (R3-R4)-
R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-R3+R4))+2 m4 n2 (3 R1 R3 R4 (-
R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4 R3 R4+R42))+2 m2 n4 (-3 R13
(R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3 R4+R42)+R22 (R32+4 R3
R4+R42))))))//Simplify

"B11="
B11=-(a g1 (-n R2+m (R1-R4)) (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4
R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4)) (h2 m2 n2 (-2 n4 (R12+R22)+2 m2
n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4 n6
R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+b (-g2 (m n2 R1-n3 R2+m3 R4) (n2 (2 R12+2 R22+R1 (R3-R4))+2
m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (h2 m2 n2 (-2 n4 (R12+R22)+2
m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4
n6 R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+(m2+n2) (n2 R1+m n R2+m2 R4) (n R2 (-R3+R4)+m R1 (R3+R4))
(h3 m4 n4 (2 m n R2 (R3+R4)+n2 (2 R12+2 R22+R1 R3-R1 R4)+m2 (-R1 R3+R1
R4+2 R3 R4))+g1 (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3
R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-
R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))))-m2 n2 (-n
R2+m (R1-R4)) (c (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2
R3 R4+R1 (-R3+R4))) (h2 m2 n2 (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-
R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4))+g4 h3 (4 n6 R1 R2+m4 n2 R1
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(R3-R4)+2 m6 R3 R4+m2 n4 R2 (-R3+R4)-m3 n3 (R1+R2) (R3+R4)))+n (h2 (2
g1 g3 n7 R2 (-R12+R22) (R3-R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 g1 g3 m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m5 n2 (R3+R4) (4 n4 R2 (2
R12+2 R22+R1 (R3-R4))+g1 g3 R12 (-R3+R4))+g1 g3 m n6 (2 R13 (R3+R4)-2
R1 R22 (R3+R4)+R22 (R32-R42))+g1 g3 m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-
R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m7 R2 (R3+R4) (g1 g3 R1
(-R3+R4)+4 n4 (2 R3 R4+R1 (-R3+R4)))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2-g1 g3 (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-g1 g3
(-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))))+g4 (g1 h1 (m2+n2)
(2 n5 R2 (R12-R22) (R3-R4)+4 m2 n3 R1 R2 R3 R4+m3 n2 R1 (R1-R2) (R32-
R42)+m5 R1 R2 (R32-R42)+m4 n (-R22 (R3-R4)2+R12 (R3+R4)2)+m n4 (-2 R13
(R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42)))+h3 m2 n (n6 R2 (2 R12+2
R22+R1 (R3-R4)) (4 R1+R3-R4)-3 m5 n R2 (R3+R4) (2 R3 R4+R1 (-R3+R4))-
m6 (2 R3 R4+R1 (-R3+R4))2+m n5 (2 R13 (R3+R4)+10 R1 R22 (R3+R4)+R12
(R32-R42)+2 R22 (R32-R42))+m4 n2 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))+m2 n4 R2 (-4 R12
(R3-R4)+2 R3 (R3-R4) R4+R1 (R32+14 R3 R4+R42))+m3 n3 (R3+R4) (-2
R23+R12 (-2 R2-R3+R4)+R1 (2 R3 R4+R2 (-R3+R4))))))))/(m n (n2 (2 R12+2
R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (-a g1 (2
n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m
n^3 R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-
R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1 g3-g2 g4) n7 R2 (-
R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))-(g1 g3-g2 g4)
R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 (g1 g3-g2 g4) m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2 (R3+R4) (g2 g4 R1
(R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3 R4))+(g1 g3-g2 g4) m
n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-R42))+(g1 g3-g2 g4) m3 n4
(2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-
R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2-(g1 g3-g2 g4) (-R22
(R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-(g1 g3-g2 g4) (-R22 (R3-
R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12
(R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c m2 n (2 n6 (R12+R22) (2
R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1 (R3-R4)) (R3+R4)+m5 n R2
(R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2 (R3+R4) (R12+R22+3 R1 (R3-R4)-
R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-R3+R4))+2 m4 n2 (3 R1 R3 R4 (-
R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4 R3 R4+R42))+2 m2 n4 (-3 R13
(R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3 R4+R42)+R22 (R32+4 R3
R4+R42))))))//Simplify

"B12="
B12=-(-a g1 (n R2+m (R1+R3)) (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4
R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4)) (h2 m2 n2 (-2 n4 (R12+R22)+2 m2
n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4 n6
R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))+b (g2 (-m n2 R1-n3 R2+m3 R3) (-n2 (2 R12+2 R22+R1 (R3-R4))-2
m n R2 (R3+R4)+m2 (R1 (R3-R4)-2 R3 R4)) (h2 m2 n2 (-2 n4 (R12+R22)+2
m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g4 h3 (-4
n6 R1 R2+m2 n4 R2 (R3-R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2)
(R3+R4)))-(m2+n2) (-n2 R1+m n R2+m2 R3) (n R2 (-R3+R4)+m R1 (R3+R4))
(h3 m4 n4 (2 m n R2 (R3+R4)+n2 (2 R12+2 R22+R1 R3-R1 R4)+m2 (-R1 R3+R1
R4+2 R3 R4))+g1 (h1 m2 n2 (-2 n4 (R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3
R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))+g3 h3 (-4 n6 R1 R2+m2 n4 R2 (R3-
R4)-2 m6 R3 R4+m4 n2 R1 (-R3+R4)+m3 n3 (R1+R2) (R3+R4)))))+m2 n2 (n
R2+m (R1+R3)) (c (n2 (2 R12+2 R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2
R3 R4+R1 (-R3+R4))) (h2 m2 n2 (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-
R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4))+g4 h3 (4 n6 R1 R2+m4 n2 R1
(R3-R4)+2 m6 R3 R4+m2 n4 R2 (-R3+R4)-m3 n3 (R1+R2) (R3+R4)))+n (h2 (2
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g1 g3 n7 R2 (-R12+R22) (R3-R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 g1 g3 m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m5 n2 (R3+R4) (4 n4 R2 (2
R12+2 R22+R1 (R3-R4))+g1 g3 R12 (-R3+R4))+g1 g3 m n6 (2 R13 (R3+R4)-2
R1 R22 (R3+R4)+R22 (R32-R42))+g1 g3 m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-
R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m7 R2 (R3+R4) (g1 g3 R1
(-R3+R4)+4 n4 (2 R3 R4+R1 (-R3+R4)))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2-g1 g3 (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-g1 g3
(-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))))+g4 (g1 h1 (m2+n2)
(2 n5 R2 (R12-R22) (R3-R4)+4 m2 n3 R1 R2 R3 R4+m3 n2 R1 (R1-R2) (R32-
R42)+m5 R1 R2 (R32-R42)+m4 n (-R22 (R3-R4)2+R12 (R3+R4)2)+m n4 (-2 R13
(R3+R4)+2 R1 R22 (R3+R4)+R22 (-R32+R42)))+h3 m2 n (n6 R2 (2 R12+2
R22+R1 (R3-R4)) (4 R1+R3-R4)-3 m5 n R2 (R3+R4) (2 R3 R4+R1 (-R3+R4))-
m6 (2 R3 R4+R1 (-R3+R4))2+m n5 (2 R13 (R3+R4)+10 R1 R22 (R3+R4)+R12
(R32-R42)+2 R22 (R32-R42))+m4 n2 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3
R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42))+m2 n4 R2 (-4 R12
(R3-R4)+2 R3 (R3-R4) R4+R1 (R32+14 R3 R4+R42))+m3 n3 (R3+R4) (-2
R23+R12 (-2 R2-R3+R4)+R1 (2 R3 R4+R2 (-R3+R4))))))))/(m n (n2 (2 R12+2
R22+R1 (R3-R4))+2 m n R2 (R3+R4)+m2 (2 R3 R4+R1 (-R3+R4))) (-a g1 (2
n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m
n^3 R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-
R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1 g3-g2 g4) n7 R2 (-
R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))-(g1 g3-g2 g4)
R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 (g1 g3-g2 g4) m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2 (R3+R4) (g2 g4 R1
(R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3 R4))+(g1 g3-g2 g4) m
n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-R42))+(g1 g3-g2 g4) m3 n4
(2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-
R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2-(g1 g3-g2 g4) (-R22
(R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-(g1 g3-g2 g4) (-R22 (R3-
R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12
(R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c m2 n (2 n6 (R12+R22) (2
R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1 (R3-R4)) (R3+R4)+m5 n R2
(R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2 (R3+R4) (R12+R22+3 R1 (R3-R4)-
R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-R3+R4))+2 m4 n2 (3 R1 R3 R4 (-
R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4 R3 R4+R42))+2 m2 n4 (-3 R13
(R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3 R4+R42)+R22 (R32+4 R3
R4+R42))))))//Simplify

"C9="
C9=(m n (4 c h3 m2 n7 R12 R2+4 h3 m4 n7 R12 R2+4 c h3 m2 n7 R23+4 h3
m4 n7 R23-2 g1 h1 m5 n4 R12 R3+2 g2 h2 m5 n4 R12 R3-2 c h3 m5 n4 R12
R3-2 g1 h1 m3 n6 R12 R3+2 g2 h2 m3 n6 R12 R3+2 h3 m5 n6 R12 R3-2 g1 g3
h3 m3 n4 R1 R2 R3+2 g2 g4 h3 m3 n4 R1 R2 R3+2 g1 h1 m4 n5 R1 R2 R3-2
g2 h2 m4 n5 R1 R2 R3-2 c h3 m4 n5 R1 R2 R3-2 h3 m6 n5 R1 R2 R3-2 g1 g3
h3 m n6 R1 R2 R3+2 g2 g4 h3 m n6 R1 R2 R3+2 g1 h1 m2 n7 R1 R2 R3-2 g2
h2 m2 n7 R1 R2 R3+2 c h3 m2 n7 R1 R2 R3+2 h3 m4 n7 R1 R2 R3-2 c h3 m5
n4 R22 R3+2 g1 g3 h3 m2 n5 R22 R3-2 g2 g4 h3 m2 n5 R22 R3+4 c h3 m3 n6
R22 R3+6 h3 m5 n6 R22 R3+2 g1 g3 h3 n7 R22 R3-2 g2 g4 h3 n7 R22 R3-g1
g3 h3 m7 R1 R32+g2 g4 h3 m7 R1 R32-g1 g3 h3 m5 n2 R1 R32+g2 g4 h3 m5
n2 R1 R32+g1 h1 m7 n2 R1 R32-g2 h2 m7 n2 R1 R32+c h3 m7 n2 R1 R32+g1
h1 m5 n4 R1 R32-g2 h2 m5 n4 R1 R32-c h3 m5 n4 R1 R32-h3 m7 n4 R1
R32+h3 m5 n6 R1 R32+g1 g3 h3 m6 n R2 R32-g2 g4 h3 m6 n R2 R32+g1 g3 h3
m4 n3 R2 R32-g2 g4 h3 m4 n3 R2 R32-g1 h1 m6 n3 R2 R32+g2 h2 m6 n3 R2
R32-2 c h3 m6 n3 R2 R32-g1 h1 m4 n5 R2 R32+g2 h2 m4 n5 R2 R32+2 h3 m6
n5 R2 R32-2 g1 h1 m5 n4 R12 R4+2 g2 h2 m5 n4 R12 R4-2 c h3 m5 n4 R12
R4-2 g1 h1 m3 n6 R12 R4+2 g2 h2 m3 n6 R12 R4+2 h3 m5 n6 R12 R4-2 g1 g3
h3 m3 n4 R1 R2 R4+2 g2 g4 h3 m3 n4 R1 R2 R4-2 g1 h1 m4 n5 R1 R2 R4+2
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g2 h2 m4 n5 R1 R2 R4+2 c h3 m4 n5 R1 R2 R4+2 h3 m6 n5 R1 R2 R4-2 g1 g3
h3 m n6 R1 R2 R4+2 g2 g4 h3 m n6 R1 R2 R4-2 g1 h1 m2 n7 R1 R2 R4+2 g2
h2 m2 n7 R1 R2 R4-2 c h3 m2 n7 R1 R2 R4-2 h3 m4 n7 R1 R2 R4-2 c h3 m5
n4 R22 R4-2 g1 g3 h3 m2 n5 R22 R4+2 g2 g4 h3 m2 n5 R22 R4+4 c h3 m3 n6
R22 R4+6 h3 m5 n6 R22 R4-2 g1 g3 h3 n7 R22 R4+2 g2 g4 h3 n7 R22 R4-2
g1 g3 h3 m6 n R2 R3 R4+2 g2 g4 h3 m6 n R2 R3 R4-2 g1 g3 h3 m4 n3 R2 R3
R4+2 g2 g4 h3 m4 n3 R2 R3 R4+2 g1 h1 m6 n3 R2 R3 R4-2 g2 h2 m6 n3 R2
R3 R4-4 c h3 m6 n3 R2 R3 R4+2 g1 h1 m4 n5 R2 R3 R4-2 g2 h2 m4 n5 R2 R3
R4+4 c h3 m4 n5 R2 R3 R4+8 h3 m6 n5 R2 R3 R4-2 c h3 m7 n2 R32 R4+2 h3
m7 n4 R32 R4+g1 g3 h3 m7 R1 R42-g2 g4 h3 m7 R1 R42+g1 g3 h3 m5 n2 R1
R42-g2 g4 h3 m5 n2 R1 R42-g1 h1 m7 n2 R1 R42+g2 h2 m7 n2 R1 R42-c h3
m7 n2 R1 R42-g1 h1 m5 n4 R1 R42+g2 h2 m5 n4 R1 R42+c h3 m5 n4 R1
R42+h3 m7 n4 R1 R42-h3 m5 n6 R1 R42+g1 g3 h3 m6 n R2 R42-g2 g4 h3 m6 n
R2 R42+g1 g3 h3 m4 n3 R2 R42-g2 g4 h3 m4 n3 R2 R42-g1 h1 m6 n3 R2
R42+g2 h2 m6 n3 R2 R42-2 c h3 m6 n3 R2 R42-g1 h1 m4 n5 R2 R42+g2 h2 m4
n5 R2 R42+2 h3 m6 n5 R2 R42-2 c h3 m7 n2 R3 R42+2 h3 m7 n4 R3 R42-a g1
h3 (2 n3 R2-m3 (R3+R4)) (2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-
R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4))-b g2 h3 (2 n3 R2-m3 (R3+R4))
(2 n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-R3+R4)-m3 n R2 (R3+R4)+m n3 R2
(R3+R4))))/(a g1 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-
m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3
R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2-n (2 (g1
g3-g2 g4) n7 R2 (-R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-
R4))-(g1 g3-g2 g4) R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2
(g1 g3-g2 g4) m2 n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2
(R3+R4) (g2 g4 R1 (R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3
R4))+(g1 g3-g2 g4) m n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-
R42))+(g1 g3-g2 g4) m3 n4 (2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4)
(R3+R4)+R22 (R32-R42)+R12 (-R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-
R4))2-(g1 g3-g2 g4) (-R22 (R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n
(-(g1 g3-g2 g4) (-R22 (R3-R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1
(R3-R4) (R22-R3 R4)+R12 (R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c
m2 n (2 n6 (R12+R22) (2 R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1
(R3-R4)) (R3+R4)+m5 n R2 (R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2
(R3+R4) (R12+R22+3 R1 (R3-R4)-R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-
R3+R4))+2 m4 n2 (3 R1 R3 R4 (-R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4
R3 R4+R42))+2 m2 n4 (-3 R13 (R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3
R4+R42)+R22 (R32+4 R3 R4+R42)))))//Simplify

"C10="
C10=(m n2 (4 c h3 m2 n6 R13+4 h3 m4 n6 R13+4 c h3 m2 n6 R1 R22+4 h3 m4
n6 R1 R22+2 g1 g3 h3 m3 n3 R12 R3-2 g2 g4 h3 m3 n3 R12 R3-4 c h3 m4 n4
R12 R3-2 h3 m6 n4 R12 R3+2 g1 g3 h3 m n5 R12 R3-2 g2 g4 h3 m n5 R12
R3+2 c h3 m2 n6 R12 R3+4 h3 m4 n6 R12 R3+2 g1 h1 m5 n3 R1 R2 R3-2 g2
h2 m5 n3 R1 R2 R3-2 g1 g3 h3 m2 n4 R1 R2 R3+2 g2 g4 h3 m2 n4 R1 R2
R3+2 g1 h1 m3 n5 R1 R2 R3-2 g2 h2 m3 n5 R1 R2 R3+4 c h3 m3 n5 R1 R2
R3+4 h3 m5 n5 R1 R2 R3-2 g1 g3 h3 n6 R1 R2 R3+2 g2 g4 h3 n6 R1 R2 R3-2
g1 h1 m4 n4 R22 R3+2 g2 h2 m4 n4 R22 R3-2 c h3 m4 n4 R22 R3-2 g1 h1 m2
n6 R22 R3+2 g2 h2 m2 n6 R22 R3+2 h3 m4 n6 R22 R3-g1 g3 h3 m6 R1 R32+g2
g4 h3 m6 R1 R32-g1 g3 h3 m4 n2 R1 R32+g2 g4 h3 m4 n2 R1 R32+g1 h1 m6
n2 R1 R32-g2 h2 m6 n2 R1 R32+c h3 m6 n2 R1 R32+g1 h1 m4 n4 R1 R32-g2
h2 m4 n4 R1 R32-c h3 m4 n4 R1 R32-h3 m6 n4 R1 R32+h3 m4 n6 R1 R32+g1
g3 h3 m5 n R2 R32-g2 g4 h3 m5 n R2 R32+g1 g3 h3 m3 n3 R2 R32-g2 g4 h3
m3 n3 R2 R32-g1 h1 m5 n3 R2 R32+g2 h2 m5 n3 R2 R32-2 c h3 m5 n3 R2
R32-g1 h1 m3 n5 R2 R32+g2 h2 m3 n5 R2 R32+2 h3 m5 n5 R2 R32+2 g1 g3 h3
m3 n3 R12 R4-2 g2 g4 h3 m3 n3 R12 R4+4 c h3 m4 n4 R12 R4+2 h3 m6 n4
R12 R4+2 g1 g3 h3 m n5 R12 R4-2 g2 g4 h3 m n5 R12 R4-2 c h3 m2 n6 R12
R4-4 h3 m4 n6 R12 R4+2 g1 h1 m5 n3 R1 R2 R4-2 g2 h2 m5 n3 R1 R2 R4+2
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g1 g3 h3 m2 n4 R1 R2 R4-2 g2 g4 h3 m2 n4 R1 R2 R4+2 g1 h1 m3 n5 R1 R2
R4-2 g2 h2 m3 n5 R1 R2 R4+4 c h3 m3 n5 R1 R2 R4+4 h3 m5 n5 R1 R2 R4+2
g1 g3 h3 n6 R1 R2 R4-2 g2 g4 h3 n6 R1 R2 R4+2 g1 h1 m4 n4 R22 R4-2 g2
h2 m4 n4 R22 R4+2 c h3 m4 n4 R22 R4+2 g1 h1 m2 n6 R22 R4-2 g2 h2 m2 n6
R22 R4-2 h3 m4 n6 R22 R4-2 g1 g3 h3 m6 R1 R3 R4+2 g2 g4 h3 m6 R1 R3
R4-2 g1 g3 h3 m4 n2 R1 R3 R4+2 g2 g4 h3 m4 n2 R1 R3 R4+2 g1 h1 m6 n2
R1 R3 R4-2 g2 h2 m6 n2 R1 R3 R4-2 c h3 m6 n2 R1 R3 R4+2 g1 h1 m4 n4 R1
R3 R4-2 g2 h2 m4 n4 R1 R3 R4+6 c h3 m4 n4 R1 R3 R4+6 h3 m6 n4 R1 R3
R4-2 h3 m4 n6 R1 R3 R4-2 c h3 m6 n2 R32 R4+2 h3 m6 n4 R32 R4-g1 g3 h3
m6 R1 R42+g2 g4 h3 m6 R1 R42-g1 g3 h3 m4 n2 R1 R42+g2 g4 h3 m4 n2 R1
R42+g1 h1 m6 n2 R1 R42-g2 h2 m6 n2 R1 R42+c h3 m6 n2 R1 R42+g1 h1 m4
n4 R1 R42-g2 h2 m4 n4 R1 R42-c h3 m4 n4 R1 R42-h3 m6 n4 R1 R42+h3 m4
n6 R1 R42-g1 g3 h3 m5 n R2 R42+g2 g4 h3 m5 n R2 R42-g1 g3 h3 m3 n3 R2
R42+g2 g4 h3 m3 n3 R2 R42+g1 h1 m5 n3 R2 R42-g2 h2 m5 n3 R2 R42+2 c h3
m5 n3 R2 R42+g1 h1 m3 n5 R2 R42-g2 h2 m3 n5 R2 R42-2 h3 m5 n5 R2 R42+2
c h3 m6 n2 R3 R42-2 h3 m6 n4 R3 R42-a g1 h3 (2 n2 R1+m2 (-R3+R4)) (2
n4 (R12+R22)-2 m4 R3 R4+2 m2 n2 R1 (-R3+R4)-m3 n R2 (R3+R4)+m n3 R2
(R3+R4))-b g2 h3 (2 n2 R1+m2 (-R3+R4)) (2 n4 (R12+R22)-2 m4 R3 R4+2 m2
n2 R1 (-R3+R4)-m3 n R2 (R3+R4)+m n3 R2 (R3+R4))))/(a g1 (2 n^4
(R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3
R2 (R3+R4))2+b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-
R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2-n (2 (g1 g3-g2 g4) n7 R2 (-
R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))-(g1 g3-g2 g4)
R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 (g1 g3-g2 g4) m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2 (R3+R4) (g2 g4 R1
(R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3 R4))+(g1 g3-g2 g4) m
n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-R42))+(g1 g3-g2 g4) m3 n4
(2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-
R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2-(g1 g3-g2 g4) (-R22
(R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-(g1 g3-g2 g4) (-R22 (R3-
R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12
(R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c m2 n (2 n6 (R12+R22) (2
R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1 (R3-R4)) (R3+R4)+m5 n R2
(R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2 (R3+R4) (R12+R22+3 R1 (R3-R4)-
R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-R3+R4))+2 m4 n2 (3 R1 R3 R4 (-
R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4 R3 R4+R42))+2 m2 n4 (-3 R13
(R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3 R4+R42)+R22 (R32+4 R3
R4+R42)))))//Simplify

"C11="
C11=(m n (2 c h3 m3 n6 R13+2 h3 m5 n6 R13-2 c h3 m2 n7 R12 R2-2 h3 m4
n7 R12 R2+2 c h3 m3 n6 R1 R22+2 h3 m5 n6 R1 R22-2 c h3 m2 n7 R23-2 h3
m4 n7 R23+g1 g3 h3 m4 n3 R12 R3-g2 g4 h3 m4 n3 R12 R3+g1 h1 m5 n4 R12
R3-g2 h2 m5 n4 R12 R3-c h3 m5 n4 R12 R3-h3 m7 n4 R12 R3+g1 g3 h3 m2 n5
R12 R3-g2 g4 h3 m2 n5 R12 R3+g1 h1 m3 n6 R12 R3-g2 h2 m3 n6 R12 R3+c
h3 m3 n6 R12 R3+h3 m5 n6 R12 R3+g1 h1 m6 n3 R1 R2 R3-g2 h2 m6 n3 R1 R2
R3+3 c h3 m4 n5 R1 R2 R3+3 h3 m6 n5 R1 R2 R3-g1 h1 m2 n7 R1 R2 R3+g2
h2 m2 n7 R1 R2 R3-c h3 m2 n7 R1 R2 R3-h3 m4 n7 R1 R2 R3-g1 h1 m5 n4
R22 R3+g2 h2 m5 n4 R22 R3-g1 g3 h3 m2 n5 R22 R3+g2 g4 h3 m2 n5 R22 R3-
g1 h1 m3 n6 R22 R3+g2 h2 m3 n6 R22 R3-2 c h3 m3 n6 R22 R3-2 h3 m5 n6
R22 R3-g1 g3 h3 n7 R22 R3+g2 g4 h3 n7 R22 R3+g1 g3 h3 m4 n3 R12 R4-g2
g4 h3 m4 n3 R12 R4+g1 h1 m5 n4 R12 R4-g2 h2 m5 n4 R12 R4+3 c h3 m5 n4
R12 R4+h3 m7 n4 R12 R4+g1 g3 h3 m2 n5 R12 R4-g2 g4 h3 m2 n5 R12 R4+g1
h1 m3 n6 R12 R4-g2 h2 m3 n6 R12 R4-c h3 m3 n6 R12 R4-3 h3 m5 n6 R12
R4+g1 h1 m6 n3 R1 R2 R4-g2 h2 m6 n3 R1 R2 R4+2 g1 g3 h3 m3 n4 R1 R2
R4-2 g2 g4 h3 m3 n4 R1 R2 R4+2 g1 h1 m4 n5 R1 R2 R4-2 g2 h2 m4 n5 R1
R2 R4+c h3 m4 n5 R1 R2 R4+h3 m6 n5 R1 R2 R4+2 g1 g3 h3 m n6 R1 R2 R4-2
g2 g4 h3 m n6 R1 R2 R4+g1 h1 m2 n7 R1 R2 R4-g2 h2 m2 n7 R1 R2 R4+c h3
```

```
m2 n7 R1 R2 R4+h3 m4 n7 R1 R2 R4+g1 h1 m5 n4 R22 R4-g2 h2 m5 n4 R22
R4+2 c h3 m5 n4 R22 R4+g1 g3 h3 m2 n5 R22 R4-g2 g4 h3 m2 n5 R22 R4+g1
h1 m3 n6 R22 R4-g2 h2 m3 n6 R22 R4-2 c h3 m3 n6 R22 R4-4 h3 m5 n6 R22
R4+g1 g3 h3 n7 R22 R4-g2 g4 h3 n7 R22 R4-g1 g3 h3 m7 R1 R3 R4+g2 g4 h3
m7 R1 R3 R4-g1 g3 h3 m5 n2 R1 R3 R4+g2 g4 h3 m5 n2 R1 R3 R4+g1 h1 m7
n2 R1 R3 R4-g2 h2 m7 n2 R1 R3 R4-c h3 m7 n2 R1 R3 R4+g1 h1 m5 n4 R1 R3
R4-g2 h2 m5 n4 R1 R3 R4+3 c h3 m5 n4 R1 R3 R4+3 h3 m7 n4 R1 R3 R4-h3
m5 n6 R1 R3 R4+g1 g3 h3 m6 n R2 R3 R4-g2 g4 h3 m6 n R2 R3 R4+g1 g3 h3
m4 n3 R2 R3 R4-g2 g4 h3 m4 n3 R2 R3 R4-g1 h1 m6 n3 R2 R3 R4+g2 h2 m6
n3 R2 R3 R4+2 c h3 m6 n3 R2 R3 R4-g1 h1 m4 n5 R2 R3 R4+g2 h2 m4 n5 R2
R3 R4-2 c h3 m4 n5 R2 R3 R4-4 h3 m6 n5 R2 R3 R4-g1 g3 h3 m7 R1 R42+g2
g4 h3 m7 R1 R42-g1 g3 h3 m5 n2 R1 R42+g2 g4 h3 m5 n2 R1 R42+g1 h1 m7
n2 R1 R42-g2 h2 m7 n2 R1 R42+c h3 m7 n2 R1 R42+g1 h1 m5 n4 R1 R42-g2
h2 m5 n4 R1 R42-c h3 m5 n4 R1 R42-h3 m7 n4 R1 R42+h3 m5 n6 R1 R42-g1
g3 h3 m6 n R2 R42+g2 g4 h3 m6 n R2 R42-g1 g3 h3 m4 n3 R2 R42+g2 g4 h3
m4 n3 R2 R42+g1 h1 m6 n3 R2 R42-g2 h2 m6 n3 R2 R42+2 c h3 m6 n3 R2
R42+g1 h1 m4 n5 R2 R42-g2 h2 m4 n5 R2 R42-2 h3 m6 n5 R2 R42+2 c h3 m7
n2 R3 R42-2 h3 m7 n4 R3 R42+a g1 h3 (m n2 R1-n3 R2+m3 R4) (-2 n4
(R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2
(R3+R4))+b g2 h3 (m n2 R1-n3 R2+m3 R4) (-2 n4 (R12+R22)+2 m2 n2 R1
(R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))))/(a g1 (2 n^4
(R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3
R2 (R3+R4))2+b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-
R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2-n (2 (g1 g3-g2 g4) n7 R2 (-
R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))-(g1 g3-g2 g4)
R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 (g1 g3-g2 g4) m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2 (R3+R4) (g2 g4 R1
(R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3 R4))+(g1 g3-g2 g4) m
n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-R42))+(g1 g3-g2 g4) m3 n4
(2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-
R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2-(g1 g3-g2 g4) (-R22
(R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-(g1 g3-g2 g4) (-R22 (R3-
R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12
(R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c m2 n (2 n6 (R12+R22) (2
R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1 (R3-R4)) (R3+R4)+m5 n R2
(R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2 (R3+R4) (R12+R22+3 R1 (R3-R4)-
R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-R3+R4))+2 m4 n2 (3 R1 R3 R4 (-
R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4 R3 R4+R42))+2 m2 n4 (-3 R13
(R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3 R4+R42)+R22 (R32+4 R3
R4+R42)))))//Simplify

"C12="
C12=-(m n (-2 c h3 m3 n6 R13-2 h3 m5 n6 R13-2 c h3 m2 n7 R12 R2-2 h3
m4 n7 R12 R2-2 c h3 m3 n6 R1 R22-2 h3 m5 n6 R1 R22-2 c h3 m2 n7 R23-2
h3 m4 n7 R23-g1 g3 h3 m4 n3 R12 R3+g2 g4 h3 m4 n3 R12 R3+g1 h1 m5 n4
R12 R3-g2 h2 m5 n4 R12 R3+3 c h3 m5 n4 R12 R3+h3 m7 n4 R12 R3-g1 g3 h3
m2 n5 R12 R3+g2 g4 h3 m2 n5 R12 R3+g1 h1 m3 n6 R12 R3-g2 h2 m3 n6 R12
R3-c h3 m3 n6 R12 R3-3 h3 m5 n6 R12 R3-g1 h1 m6 n3 R1 R2 R3+g2 h2 m6
n3 R1 R2 R3+2 g1 g3 h3 m3 n4 R1 R2 R3-2 g2 g4 h3 m3 n4 R1 R2 R3-2 g1
h1 m4 n5 R1 R2 R3+2 g2 h2 m4 n5 R1 R2 R3-c h3 m4 n5 R1 R2 R3-h3 m6 n5
R1 R2 R3+2 g1 g3 h3 m n6 R1 R2 R3-2 g2 g4 h3 m n6 R1 R2 R3-g1 h1 m2 n7
R1 R2 R3+g2 h2 m2 n7 R1 R2 R3-c h3 m2 n7 R1 R2 R3-h3 m4 n7 R1 R2 R3+g1
h1 m5 n4 R22 R3-g2 h2 m5 n4 R22 R3+2 c h3 m5 n4 R22 R3-g1 g3 h3 m2 n5
R22 R3+g2 g4 h3 m2 n5 R22 R3+g1 h1 m3 n6 R22 R3-g2 h2 m3 n6 R22 R3-2 c
h3 m3 n6 R22 R3-4 h3 m5 n6 R22 R3-g1 g3 h3 n7 R22 R3+g2 g4 h3 n7 R22
R3+g1 g3 h3 m7 R1 R32-g2 g4 h3 m7 R1 R32+g1 g3 h3 m5 n2 R1 R32-g2 g4
```

```
h3 m5 n2 R1 R32-g1 h1 m7 n2 R1 R32+g2 h2 m7 n2 R1 R32-c h3 m7 n2 R1
R32-g1 h1 m5 n4 R1 R32+g2 h2 m5 n4 R1 R32+c h3 m5 n4 R1 R32+h3 m7 n4
R1 R32-h3 m5 n6 R1 R32-g1 g3 h3 m6 n R2 R32+g2 g4 h3 m6 n R2 R32-g1 g3
h3 m4 n3 R2 R32+g2 g4 h3 m4 n3 R2 R32+g1 h1 m6 n3 R2 R32-g2 h2 m6 n3
R2 R32+2 c h3 m6 n3 R2 R32+g1 h1 m4 n5 R2 R32-g2 h2 m4 n5 R2 R32-2 h3
m6 n5 R2 R32-g1 g3 h3 m4 n3 R12 R4+g2 g4 h3 m4 n3 R12 R4+g1 h1 m5 n4
R12 R4-g2 h2 m5 n4 R12 R4-c h3 m5 n4 R12 R4-h3 m7 n4 R12 R4-g1 g3 h3
m2 n5 R12 R4+g2 g4 h3 m2 n5 R12 R4+g1 h1 m3 n6 R12 R4-g2 h2 m3 n6 R12
R4+c h3 m3 n6 R12 R4+h3 m5 n6 R12 R4-g1 h1 m6 n3 R1 R2 R4+g2 h2 m6 n3
R1 R2 R4-3 c h3 m4 n5 R1 R2 R4-3 h3 m6 n5 R1 R2 R4+g1 h1 m2 n7 R1 R2
R4-g2 h2 m2 n7 R1 R2 R4+c h3 m2 n7 R1 R2 R4+h3 m4 n7 R1 R2 R4-g1 h1 m5
n4 R22 R4+g2 h2 m5 n4 R22 R4+g1 g3 h3 m2 n5 R22 R4-g2 g4 h3 m2 n5 R22
R4-g1 h1 m3 n6 R22 R4+g2 h2 m3 n6 R22 R4-2 c h3 m3 n6 R22 R4-2 h3 m5
n6 R22 R4+g1 g3 h3 n7 R22 R4-g2 g4 h3 n7 R22 R4+g1 g3 h3 m7 R1 R3 R4-
g2 g4 h3 m7 R1 R3 R4+g1 g3 h3 m5 n2 R1 R3 R4-g2 g4 h3 m5 n2 R1 R3 R4-
g1 h1 m7 n2 R1 R3 R4+g2 h2 m7 n2 R1 R3 R4+c h3 m7 n2 R1 R3 R4-g1 h1 m5
n4 R1 R3 R4+g2 h2 m5 n4 R1 R3 R4-3 c h3 m5 n4 R1 R3 R4-3 h3 m7 n4 R1
R3 R4+h3 m5 n6 R1 R3 R4+g1 g3 h3 m6 n R2 R3 R4-g2 g4 h3 m6 n R2 R3
R4+g1 g3 h3 m4 n3 R2 R3 R4-g2 g4 h3 m4 n3 R2 R3 R4-g1 h1 m6 n3 R2 R3
R4+g2 h2 m6 n3 R2 R3 R4+2 c h3 m6 n3 R2 R3 R4-g1 h1 m4 n5 R2 R3 R4+g2
h2 m4 n5 R2 R3 R4-2 c h3 m4 n5 R2 R3 R4-4 h3 m6 n5 R2 R3 R4+2 c h3 m7
n2 R32 R4-2 h3 m7 n4 R32 R4+a g1 h3 (-m n2 R1-n3 R2+m3 R3) (-2 n4
(R12+R22)+2 m2 n2 R1 (R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2
(R3+R4))+b g2 h3 (-m n2 R1-n3 R2+m3 R3) (-2 n4 (R12+R22)+2 m2 n2 R1
(R3-R4)+2 m4 R3 R4+m3 n R2 (R3+R4)-m n3 R2 (R3+R4))))/(-a g1 (2 n^4
(R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-R3+R4)-m^3 n R2 (R3+R4)+m n^3
R2 (R3+R4))2-b g2 (2 n^4 (R1^2+R2^2)-2 m^4 R3 R4+2 m^2 n^2 R1 (-
R3+R4)-m^3 n R2 (R3+R4)+m n^3 R2 (R3+R4))2+n (2 (g1 g3-g2 g4) n7 R2 (-
R12+R22) (R3-R4)+m5 n2 (4 n4 R2 (2 R12+2 R22+R1 (R3-R4))-(g1 g3-g2 g4)
R12 (R3-R4)) (R3+R4)+m8 n3 (2 R3 R4+R1 (-R3+R4))2+2 (g1 g3-g2 g4) m2
n5 R2 (R22 (R3-R4)-2 R1 R3 R4+R12 (-R3+R4))+m7 R2 (R3+R4) (g2 g4 R1
(R3-R4)+g1 g3 R1 (-R3+R4)+4 n4 (-R1 R3+R1 R4+2 R3 R4))+(g1 g3-g2 g4) m
n6 (2 R13 (R3+R4)-2 R1 R22 (R3+R4)+R22 (R32-R42))+(g1 g3-g2 g4) m3 n4
(2 R13 (R3+R4)-R1 R2 (2 R2-R3+R4) (R3+R4)+R22 (R32-R42)+R12 (-
R32+R42))+m4 n3 (n4 (2 R1^2+2 R2^2+R1 (R3-R4))2-(g1 g3-g2 g4) (-R22
(R3-R4)2+4 R1 R2 R3 R4+R12 (R3+R4)2))+m6 n (-(g1 g3-g2 g4) (-R22 (R3-
R4)2+R12 (R3+R4)2)-2 n4 (2 R13 (R3-R4)+2 R1 (R3-R4) (R22-R3 R4)+R12
(R32-6 R3 R4+R42)-2 R22 (R32+4 R3 R4+R42)))+c m2 n (2 n6 (R12+R22) (2
R12+2 R22+R1 (R3-R4))+m n5 R2 (6 R12+6 R22+R1 (R3-R4)) (R3+R4)+m5 n R2
(R3+R4) (R1 (R3-R4)-6 R3 R4)-2 m3 n3 R2 (R3+R4) (R12+R22+3 R1 (R3-R4)-
R3 R4)-2 m6 R3 R4 (2 R3 R4+R1 (-R3+R4))+2 m4 n2 (3 R1 R3 R4 (-
R3+R4)+R12 (R32-4 R3 R4+R42)-R22 (R32+4 R3 R4+R42))+2 m2 n4 (-3 R13
(R3-R4)+3 R1 R22 (-R3+R4)-R12 (R32-4 R3 R4+R42)+R22 (R32+4 R3
R4+R42)))))//Simplify
```

APPENDIX II: MATLAB® CODE FOR COMPARING THEORY AND COMSOL® RESULTS

```
m=zeros(5);
m(1)=0.0203774;

n=zeros(5);
n(1)=1.42962;

L=0;
A9=zeros(5);
A9(1)=0.0000117251;

A10=zeros(5);
A10(1)=-0.000230756;

A11=zeros(5);
A11(1)=-7.50709*10^-6;

A12=zeros(5);
A12(1)=-4.21797*10^-6;

B9=zeros(5);
B9(1)=0.0059477;

B10=zeros(5);
B10(1)=-0.372003;

B11=zeros(5);
B11(1)=-0.00562506;

B12=zeros(5);
B12(1)=-0.000322642;

C9=zeros(5);
C9(1)=0.0633688;

C10=zeros(5);
C10(1)=0.0016617;

C11=zeros(5);
C11(1)=-0.0316726;

C12=zeros(5);
C12(1)=-0.0316963;
```

```
MaxNodes=10000;
ca=20;
V=zeros(MaxNodes,ca);
U=zeros(MaxNodes,ca);
Phi=zeros(MaxNodes,ca);
ca=1;
nn=1;

while ca<=1
    while L<=1

U(nn,ca) = A9(ca)*cos(L*m(ca)) + A10(ca)*sin(L*m(ca)) +
A11(ca)*exp(-L*n(ca)) + A12(ca)*exp(L*n(ca));
V(nn,ca) = B9(ca)*cos(L*m(ca)) + B10(ca)*sin(L*m(ca)) +
B11(ca)*exp(-L*n(ca)) + B12(ca)*exp(L*n(ca));
Phi(nn,ca) = C9(ca)*cos(L*m(ca)) + C10(ca)*sin(L*m(ca)) +
C11(ca)*exp(-L*n(ca)) + C12(ca)*exp(L*n(ca));
L=L+0.001;
nn=nn+1;

    end
    ca=ca+1;
    nn=1;
    L=0;
end


filename='ModelSolutionInfo(slice-3-LR).xlsx';
ZZdef= xlsread(filename);
ZZstep=1:1:1000;
subplot(2,2,1), plot(ZZdef(ZZstep,1),ZZdef(ZZstep,2),'r')
hold
subplot(2,2,2), plot(ZZdef(ZZstep,1),ZZdef(ZZstep,3),'r')

text(700,-3, 'COMSOL OUTPUT','fontsize',6, 'Color', 'r');

hold
subplot(2,2,3), plot(ZZdef(ZZstep,1),ZZdef(ZZstep,4),'r')
hold



X0=-87.3451*10^-3;
X=-43.2965*10^-3;
Y0=0;
Y=7.5*10^-3;
Cstep=1:1:5;
```

```
Ustep=1:1:1000;

subplot(2,2,1),plot((U(Ustep,Cstep)+(Y0-
Y)*Phi(Ustep,Cstep))*1000,'g')
xlabel('Longitudinal Axis z (mm)')
ylabel('u (mm): deflection in x-direction')
subplot(2,2,2),plot((V(Ustep,Cstep)-(X0-
X)*Phi(Ustep,Cstep))*1000,'g')
xlabel('Longitudinal Axis z (mm)')
ylabel('v (mm): deflection in y-direction')

text(430,.5, 'Theory (M1=0 N.m)','fontsize',10, 'Color',
'g');
text(430,-4, 'Theory (M1=3.8 N.m)','fontsize',8, 'Color',
'g');

subplot(2,2,3),plot(Phi(Ustep,Cstep),'g')
xlabel('Longitudinal Axis z (mm)')
ylabel('\phi (radian): twisting angle z-direction ')
hold
```

APPENDIX III: COMSOL® OUTPUT FILE (modelsolutioninfo(slice-3-lr).xlsx)

Node #	u	v	ϕ		Node #	u	v	ϕ
1	-0.00017	-3.34E-05	2.17E-05		501	0.059401	-1.71325	-0.00803
2	-0.00051	-9.94E-05	6.66E-05		502	0.059737	-1.72053	-0.00807
3	-0.00087	-1.66E-04	1.14E-04		503	0.06007	-1.72785	-0.00812
4	-0.00121	-0.00027	1.59E-04		504	0.060402	-1.73518	-0.00816
5	-0.00153	-0.00037	2.02E-04		505	0.060735	-1.74251	-0.00821
6	-0.00184	-0.00048	2.42E-04		506	0.061077	-1.74986	-0.00825
7	-0.00209	-0.00053	2.75E-04		507	0.061428	-1.75723	-0.0083
8	-0.00232	-0.00056	3.05E-04		508	0.061779	-1.7646	-0.00835
9	-0.00252	-0.00061	3.31E-04		509	0.062128	-1.77199	-0.00839
10	-0.00271	-0.00067	3.56E-04		510	0.062474	-1.77941	-0.00844
11	-0.00288	-0.00074	3.77E-04		511	0.06282	-1.78684	-0.00849
12	-0.00303	-0.00081	3.96E-04		512	0.063166	-1.79427	-0.00853
13	-0.00315	-0.00084	4.14E-04		513	0.063508	-1.80174	-0.00858
14	-0.00327	-0.00087	4.31E-04		514	0.063849	-1.80923	-0.00862
15	-0.00338	-0.00093	4.46E-04		515	0.064191	-1.81671	-0.00867
16	-0.00349	-0.00099	4.61E-04		516	0.064543	-1.82421	-0.00872
17	-0.00361	-0.00106	0.000475		517	0.064902	-1.83174	-0.00876
18	-0.00372	-0.00116	0.000488		518	0.065261	-1.83926	-0.00881
19	-0.00383	-0.00126	0.0005		519	0.065619	-1.8468	-0.00886
20	-0.00391	-0.00133	0.00051		520	0.065974	-1.85438	-0.00891
21	-0.00398	-0.00139	0.000519		521	0.066328	-1.86195	-0.00896
22	-0.00404	-0.00146	0.000528		522	0.066683	-1.86953	-0.009
23	-0.00409	-0.00155	0.000535		523	0.067033	-1.87717	-0.00905
24	-0.00414	-0.00167	0.000542		524	0.067382	-1.8848	-0.0091
25	-0.00419	-0.00178	0.000549		525	0.067732	-1.89243	-0.00914
26	-0.00424	-0.00193	0.000555		526	0.068093	-1.90009	-0.00919
27	-0.00428	-0.0021	0.000561		527	0.068461	-1.90777	-0.00924
28	-0.00432	-0.00227	0.000567		528	0.068829	-1.91545	-0.00929
29	-0.00436	-0.00243	0.000574		529	0.069195	-1.92314	-0.00934
30	-0.00443	-0.00259	0.000582		530	0.069558	-1.93087	-0.00939
31	-0.00449	-0.00274	0.000589		531	0.069921	-1.93861	-0.00943
32	-0.00455	-0.0029	0.000597		532	0.070283	-1.94634	-0.00948
33	-0.0046	-0.00313	0.000603		533	0.070641	-1.95413	-0.00953
34	-0.00466	-0.00335	0.00061		534	0.070999	-1.96191	-0.00958
35	-0.00471	-0.00357	0.000616		535	0.071357	-1.9697	-0.00963
36	-0.00476	-0.00386	0.000622		536	0.071727	-1.97752	-0.00968
37	-0.0048	-0.00416	0.000627		537	0.072103	-1.98534	-0.00973
38	-0.00485	-0.00445	0.000633		538	0.072478	-1.99317	-0.00978

39	-0.00489	-0.00475	0.000638		539	0.072851	-2.00103	-0.00983
40	-0.00493	-0.00505	0.000644		540	0.073222	-2.00891	-0.00988
41	-0.00498	-0.00534	0.000649		541	0.073593	-2.01679	-0.00993
42	-0.00502	-0.00566	0.000654		542	0.073963	-2.02468	-0.00997
43	-0.00506	-0.00601	0.000659		543	0.074329	-2.03262	-0.01002
44	-0.0051	-0.00636	0.000664		544	0.074695	-2.04056	-0.01007
45	-0.00514	-0.00671	0.000669		545	0.075061	-2.0485	-0.01012
46	-0.00518	-0.00712	0.000674		546	0.07544	-2.05648	-0.01017
47	-0.00522	-0.00752	0.000679		547	0.075824	-2.06446	-0.01022
48	-0.00526	-0.00793	0.000683		548	0.076208	-2.07244	-0.01027
49	-0.00529	-0.00833	0.000688		549	0.076589	-2.08045	-0.01033
50	-0.00533	-0.00874	0.000692		550	0.076968	-2.08849	-0.01038
51	-0.00536	-0.00914	0.000696		551	0.077347	-2.09652	-0.01043
52	-0.00539	-0.00957	0.000701		552	0.077725	-2.10457	-0.01048
53	-0.00543	-0.01003	0.000705		553	0.078099	-2.11266	-0.01053
54	-0.00546	-0.01049	0.000709		554	0.078473	-2.12076	-0.01058
55	-0.0055	-0.01095	0.000713		555	0.078847	-2.12885	-0.01063
56	-0.00553	-0.01146	0.000718		556	0.079236	-2.13698	-0.01068
57	-0.00557	-0.01198	0.000722		557	0.079628	-2.14512	-0.01073
58	-0.0056	-0.01249	0.000726		558	0.08002	-2.15325	-0.01078
59	-0.00563	-0.01301	0.00073		559	0.080409	-2.16142	-0.01084
60	-0.00566	-0.01354	0.000734		560	0.080796	-2.16961	-0.01089
61	-0.00569	-0.01406	0.000737		561	0.081183	-2.1778	-0.01094
62	-0.00572	-0.01461	0.000741		562	0.081569	-2.186	-0.01099
63	-0.00575	-0.01519	0.000745		563	0.081952	-2.19425	-0.01104
64	-0.00579	-0.01576	0.000749		564	0.082334	-2.2025	-0.01109
65	-0.00582	-0.01635	0.000753		565	0.082716	-2.21074	-0.01114
66	-0.00586	-0.01698	0.000758		566	0.083114	-2.21903	-0.0112
67	-0.00589	-0.01761	0.000762		567	0.083514	-2.22732	-0.01125
68	-0.00593	-0.01824	0.000766		568	0.083915	-2.23561	-0.0113
69	-0.00595	-0.01888	0.000769		569	0.084312	-2.24393	-0.01136
70	-0.00598	-0.01953	0.000772		570	0.084707	-2.25228	-0.01141
71	-0.00601	-0.02017	0.000775		571	0.085102	-2.26062	-0.01146
72	-0.00603	-0.02084	0.000779		572	0.085496	-2.26898	-0.01151
73	-0.00606	-0.02154	0.000782		573	0.085886	-2.27738	-0.01157
74	-0.00609	-0.02224	0.000786		574	0.086277	-2.28578	-0.01162
75	-0.00612	-0.02295	0.000789		575	0.086667	-2.29418	-0.01167
76	-0.00615	-0.02371	0.000793		576	0.087075	-2.30262	-0.01173
77	-0.00618	-0.02446	0.000797		577	0.087483	-2.31107	-0.01178
78	-0.00621	-0.02522	0.0008		578	0.087891	-2.31951	-0.01183
79	-0.00623	-0.02598	0.000803		579	0.088296	-2.32799	-0.01189
80	-0.00626	-0.02675	0.000805		580	0.0887	-2.33649	-0.01194

81	-0.00628	-0.02752	0.000808		581	0.089103	-2.34499	-0.012
82	-0.0063	-0.02833	0.000811		582	0.089505	-2.35351	-0.01205
83	-0.00632	-0.02915	0.000813		583	0.089904	-2.36206	-0.0121
84	-0.00635	-0.02998	0.000816		584	0.090302	-2.37062	-0.01216
85	-0.00637	-0.03082	0.000819		585	0.090701	-2.37917	-0.01221
86	-0.0064	-0.0317	0.000822		586	0.091118	-2.38777	-0.01227
87	-0.00642	-0.03258	0.000825		587	0.091535	-2.39637	-0.01232
88	-0.00645	-0.03346	0.000828		588	0.091951	-2.40497	-0.01238
89	-0.00647	-0.03436	0.00083		589	0.092364	-2.4136	-0.01243
90	-0.00648	-0.03527	0.000832		590	0.092776	-2.42226	-0.01249
91	-0.0065	-0.03617	0.000834		591	0.093188	-2.43091	-0.01254
92	-0.00652	-0.03711	0.000836		592	0.093597	-2.43959	-0.0126
93	-0.00654	-0.03806	0.000838		593	0.094004	-2.4483	-0.01265
94	-0.00655	-0.03902	0.00084		594	0.094411	-2.457	-0.01271
95	-0.00657	-0.04	0.000842		595	0.094819	-2.46572	-0.01276
96	-0.0066	-0.04101	0.000845		596	0.095242	-2.47447	-0.01282
97	-0.00662	-0.04203	0.000847		597	0.095666	-2.48323	-0.01287
98	-0.00664	-0.04304	0.000849		598	0.096089	-2.49199	-0.01293
99	-0.00665	-0.04408	0.00085		599	0.096511	-2.50078	-0.01299
100	-0.00666	-0.04511	0.000851		600	0.096932	-2.50959	-0.01304
101	-0.00667	-0.04615	0.000852		601	0.097353	-2.5184	-0.0131
102	-0.00668	-0.04723	0.000854		602	0.097773	-2.52723	-0.01315
103	-0.00669	-0.04832	0.000855		603	0.098192	-2.53609	-0.01321
104	-0.00671	-0.04942	0.000856		604	0.09861	-2.54495	-0.01327
105	-0.00672	-0.05053	0.000858		605	0.099031	-2.55382	-0.01332
106	-0.00673	-0.05169	0.000859		606	0.099461	-2.56273	-0.01338
107	-0.00675	-0.05284	0.000861		607	0.099891	-2.57165	-0.01344
108	-0.00676	-0.05399	0.000862		608	0.100322	-2.58056	-0.01349
109	-0.00676	-0.05517	0.000862		609	0.100751	-2.58951	-0.01355
110	-0.00677	-0.05635	0.000862		610	0.101179	-2.59848	-0.01361
111	-0.00677	-0.05752	0.000862		611	0.101608	-2.60744	-0.01367
112	-0.00678	-0.05874	0.000863		612	0.102036	-2.61643	-0.01372
113	-0.00678	-0.05998	0.000863		613	0.102463	-2.62545	-0.01378
114	-0.00679	-0.06121	0.000863		614	0.10289	-2.63447	-0.01384
115	-0.00679	-0.06248	0.000864		615	0.10332	-2.6435	-0.01389
116	-0.0068	-0.06377	0.000864		616	0.103759	-2.65256	-0.01395
117	-0.00681	-0.06506	0.000865		617	0.104198	-2.66163	-0.01401
118	-0.00681	-0.06636	0.000865		618	0.104637	-2.6707	-0.01407
119	-0.00681	-0.06768	0.000864		619	0.105074	-2.67981	-0.01413
120	-0.00681	-0.069	0.000863		620	0.105512	-2.68894	-0.01419
121	-0.0068	-0.07032	0.000863		621	0.105949	-2.69806	-0.01424
122	-0.0068	-0.07169	0.000862		622	0.106385	-2.70721	-0.0143

123	-0.0068	-0.07307	0.000861		623	0.106821	-2.71638	-0.01436
124	-0.0068	-0.07444	0.000861		624	0.107257	-2.72555	-0.01442
125	-0.0068	-0.07585	0.00086		625	0.107696	-2.73474	-0.01448
126	-0.0068	-0.07729	0.00086		626	0.108143	-2.74396	-0.01454
127	-0.0068	-0.07873	0.000859		627	0.108591	-2.75319	-0.0146
128	-0.00679	-0.08017	0.000858		628	0.109039	-2.76241	-0.01466
129	-0.00678	-0.08164	0.000856		629	0.109485	-2.77169	-0.01472
130	-0.00677	-0.0831	0.000855		630	0.109931	-2.78097	-0.01478
131	-0.00676	-0.08457	0.000853		631	0.110377	-2.79024	-0.01484
132	-0.00675	-0.08609	0.000851		632	0.110822	-2.79955	-0.01489
133	-0.00674	-0.08762	0.00085		633	0.111266	-2.80888	-0.01495
134	-0.00673	-0.08914	0.000848		634	0.11171	-2.81821	-0.01501
135	-0.00672	-0.0907	0.000846		635	0.112159	-2.82756	-0.01507
136	-0.00671	-0.09229	0.000845		636	0.112615	-2.83694	-0.01513
137	-0.0067	-0.09387	0.000843		637	0.113071	-2.84632	-0.01519
138	-0.00669	-0.09546	0.000841		638	0.113528	-2.85571	-0.01526
139	-0.00667	-0.09708	0.000838		639	0.113982	-2.86514	-0.01532
140	-0.00665	-0.09869	0.000835		640	0.114437	-2.87458	-0.01538
141	-0.00663	-0.10031	0.000832		641	0.114892	-2.88401	-0.01544
142	-0.00661	-0.10198	0.00083		642	0.115346	-2.89348	-0.0155
143	-0.0066	-0.10366	0.000827		643	0.115799	-2.90297	-0.01556
144	-0.00658	-0.10533	0.000824		644	0.116252	-2.91246	-0.01562
145	-0.00656	-0.10704	0.000822		645	0.11671	-2.92197	-0.01568
146	-0.00654	-0.10878	0.000819		646	0.117176	-2.9315	-0.01574
147	-0.00653	-0.11051	0.000816		647	0.117641	-2.94104	-0.0158
148	-0.00651	-0.11226	0.000813		648	0.118106	-2.95059	-0.01587
149	-0.00648	-0.11403	0.000809		649	0.11857	-2.96018	-0.01593
150	-0.00645	-0.11579	0.000805		650	0.119034	-2.96977	-0.01599
151	-0.00642	-0.11756	0.000801		651	0.119498	-2.97937	-0.01605
152	-0.00639	-0.11939	0.000797		652	0.119961	-2.989	-0.01611
153	-0.00637	-0.12122	0.000793		653	0.120423	-2.99865	-0.01618
154	-0.00634	-0.12304	0.000789		654	0.120885	-3.00829	-0.01624
155	-0.00631	-0.12491	0.000785		655	0.121353	-3.01796	-0.0163
156	-0.00629	-0.1268	0.000781		656	0.121828	-3.02766	-0.01636
157	-0.00626	-0.12868	0.000778		657	0.122303	-3.03736	-0.01643
158	-0.00623	-0.13058	0.000773		658	0.122778	-3.04706	-0.01649
159	-0.00619	-0.1325	0.000768		659	0.123251	-3.05681	-0.01655
160	-0.00616	-0.13442	0.000762		660	0.123724	-3.06656	-0.01662
161	-0.00612	-0.13635	0.000757		661	0.124198	-3.07632	-0.01668
162	-0.00608	-0.13833	0.000752		662	0.12467	-3.08611	-0.01674
163	-0.00605	-0.14031	0.000747		663	0.125141	-3.09592	-0.0168
164	-0.00601	-0.14229	0.000742		664	0.125613	-3.10572	-0.01687

165	-0.00598	-0.14431	0.000737		665	0.12609	-3.11555	-0.01693
166	-0.00594	-0.14635	0.000732		666	0.126572	-3.12541	-0.017
167	-0.00591	-0.14839	0.000727		667	0.127054	-3.13527	-0.01706
168	-0.00587	-0.15044	0.000721		668	0.127535	-3.14514	-0.01712
169	-0.00582	-0.15252	0.000715		669	0.128017	-3.15505	-0.01719
170	-0.00577	-0.1546	0.000708		670	0.128499	-3.16496	-0.01725
171	-0.00573	-0.15668	0.000702		671	0.128981	-3.17487	-0.01732
172	-0.00568	-0.15881	0.000696		672	0.129464	-3.18483	-0.01738
173	-0.00563	-0.16095	0.000689		673	0.129946	-3.1948	-0.01745
174	-0.00559	-0.16308	0.000683		674	0.130429	-3.20477	-0.01751
175	-0.00555	-0.16527	0.000677		675	0.130916	-3.21477	-0.01757
176	-0.0055	-0.16746	0.000671		676	0.131407	-3.22478	-0.01764
177	-0.00546	-0.16965	0.000664		677	0.131898	-3.2348	-0.01771
178	-0.00541	-0.17187	0.000657		678	0.132389	-3.24483	-0.01777
179	-0.00535	-0.1741	0.000649		679	0.132881	-3.25491	-0.01784
180	-0.00529	-0.17633	0.000642		680	0.133372	-3.26498	-0.0179
181	-0.00524	-0.17857	0.000634		681	0.133863	-3.27505	-0.01797
182	-0.00518	-0.18087	0.000627		682	0.134355	-3.28518	-0.01803
183	-0.00513	-0.18316	0.000619		683	0.134847	-3.29531	-0.0181
184	-0.00507	-0.18545	0.000612		684	0.135339	-3.30544	-0.01816
185	-0.00502	-0.18779	0.000604		685	0.135836	-3.3156	-0.01823
186	-0.00497	-0.19014	0.000597		686	0.136337	-3.32578	-0.0183
187	-0.00491	-0.19249	0.000589		687	0.136837	-3.33596	-0.01836
188	-0.00485	-0.19486	0.000581		688	0.137338	-3.34615	-0.01843
189	-0.00479	-0.19725	0.000572		689	0.137839	-3.35639	-0.0185
190	-0.00472	-0.19964	0.000563		690	0.13834	-3.36662	-0.01857
191	-0.00466	-0.20205	0.000554		691	0.138841	-3.37686	-0.01863
192	-0.00459	-0.20449	0.000545		692	0.139343	-3.38715	-0.0187
193	-0.00453	-0.20694	0.000536		693	0.139845	-3.39744	-0.01877
194	-0.00446	-0.20939	0.000527		694	0.140346	-3.40774	-0.01883
195	-0.0044	-0.21189	0.000519		695	0.140854	-3.41806	-0.0189
196	-0.00434	-0.2144	0.00051		696	0.141364	-3.42841	-0.01897
197	-0.00427	-0.21691	0.000501		697	0.141874	-3.43875	-0.01904
198	-0.0042	-0.21944	0.000491		698	0.142385	-3.44911	-0.0191
199	-0.00413	-0.22199	0.000481		699	0.142896	-3.45951	-0.01917
200	-0.00405	-0.22453	0.000471		700	0.143407	-3.46991	-0.01924
201	-0.00398	-0.2271	0.000461		701	0.143918	-3.48031	-0.01931
202	-0.0039	-0.22971	0.000451		702	0.144429	-3.49077	-0.01938
203	-0.00383	-0.23231	0.000441		703	0.144941	-3.50122	-0.01945
204	-0.00376	-0.23492	0.00043		704	0.145452	-3.51168	-0.01951
205	-0.00368	-0.23758	0.00042		705	0.14597	-3.52217	-0.01958
206	-0.00361	-0.24025	0.000411		706	0.14649	-3.53268	-0.01965

207	-0.00354	-0.24291	0.0004		707	0.147011	-3.54319	-0.01972
208	-0.00345	-0.24561	0.000389		708	0.147531	-3.55372	-0.01979
209	-0.00337	-0.24831	0.000377		709	0.148052	-3.56428	-0.01986
210	-0.00329	-0.25102	0.000366		710	0.148573	-3.57485	-0.01993
211	-0.0032	-0.25375	0.000355		711	0.149094	-3.58542	-0.02
212	-0.00312	-0.25651	0.000343		712	0.149615	-3.59604	-0.02007
213	-0.00303	-0.25927	0.000332		713	0.150137	-3.60666	-0.02014
214	-0.00295	-0.26204	0.00032		714	0.150658	-3.61728	-0.02021
215	-0.00287	-0.26487	0.000309		715	0.151187	-3.62794	-0.02028
216	-0.00278	-0.26769	0.000298		716	0.151718	-3.63861	-0.02035
217	-0.0027	-0.27051	0.000286		717	0.152248	-3.64928	-0.02042
218	-0.00261	-0.27337	0.000274		718	0.152779	-3.65999	-0.02049
219	-0.00251	-0.27623	0.000261		719	0.15331	-3.67071	-0.02056
220	-0.00242	-0.2791	0.000248		720	0.153842	-3.68144	-0.02063
221	-0.00233	-0.28199	0.000235		721	0.154373	-3.69218	-0.0207
222	-0.00223	-0.28491	0.000223		722	0.154904	-3.70297	-0.02077
223	-0.00214	-0.28784	0.00021		723	0.155436	-3.71376	-0.02084
224	-0.00205	-0.29077	0.000198		724	0.155968	-3.72454	-0.02092
225	-0.00195	-0.29375	0.000185		725	0.156508	-3.73538	-0.02099
226	-0.00186	-0.29673	0.000172		726	0.157049	-3.74622	-0.02106
227	-0.00177	-0.29971	0.00016		727	0.15759	-3.75706	-0.02113
228	-0.00167	-0.30273	0.000146		728	0.158131	-3.76793	-0.0212
229	-0.00156	-0.30575	0.000132		729	0.158673	-3.77882	-0.02128
230	-0.00146	-0.30878	0.000117		730	0.159215	-3.78972	-0.02135
231	-0.00135	-0.31183	0.000103		731	0.159756	-3.80063	-0.02142
232	-0.00125	-0.31491	8.95E-05		732	0.160299	-3.81158	-0.02149
233	-0.00115	-0.31799	7.55E-05		733	0.160841	-3.82254	-0.02157
234	-0.00104	-0.32109	6.17E-05		734	0.161383	-3.83349	-0.02164
235	-0.00094	-3.24E-01	4.78E-05		735	0.161934	-3.8445	-0.02171
236	-0.00084	-3.27E-01	3.4E-05		736	0.162486	-3.8555	-0.02178
237	-0.00074	-3.31E-01	1.98E-05		737	0.163037	-3.86651	-0.02186
238	-0.00062	-3.34E-01	4.53E-06		738	0.16359	-3.87755	-0.02193
239	-0.00051	-3.37E-01	-1.1E-05		739	0.164142	-3.88862	-0.02201
240	-0.0004	-3.40E-01	-2.6E-05		740	0.164694	-3.89968	-0.02208
241	-0.00028	-3.43E-01	-4.2E-05		741	0.165247	-3.91077	-0.02215
242	-0.00017	-3.47E-01	-5.7E-05		742	0.1658	-3.92189	-0.02223
243	-5.8E-05	-3.50E-01	-7.2E-05		743	0.166353	-3.93301	-0.0223
244	5.49E-05	-3.53E-01	-8.7E-05		744	0.166906	-3.94414	-0.02237
245	0.000167	-3.56E-01	-0.0001		745	0.167468	-3.95531	-0.02245
246	0.000279	-3.60E-01	-0.00012		746	0.168031	-3.96649	-0.02252
247	0.000391	-3.63E-01	-0.00013		747	0.168594	-3.97767	-0.0226
248	0.000514	-3.66E-01	-0.00015		748	0.169157	-3.98888	-0.02267

249	0.000638	-3.70E-01	-0.00017		749	0.16972	-4.00012	-0.02275
250	0.000761	-3.73E-01	-0.00018		750	0.170283	-4.01136	-0.02282
251	0.000884	-3.76E-01	-0.0002		751	0.170847	-4.02261	-0.0229
252	0.001007	-0.37971	-0.00022		752	0.171411	-4.03391	-0.02297
253	0.001129	-0.38311	-0.00023		753	0.171975	-4.0452	-0.02305
254	0.001252	-0.38653	-0.00025		754	0.17254	-4.0565	-0.02312
255	0.001374	-0.38999	-0.00027		755	0.173114	-4.06785	-0.0232
256	0.001495	-0.39344	-0.00028		756	0.173688	-4.07919	-0.02328
257	0.001617	-0.3969	-0.0003		757	0.174262	-4.09054	-0.02335
258	0.00175	-0.4004	-0.00032		758	0.174836	-4.10193	-0.02343
259	0.001883	-0.4039	-0.00033		759	0.175411	-4.11334	-0.02351
260	0.002016	-0.40741	-0.00035		760	0.175985	-4.12475	-0.02358
261	0.002148	-0.41095	-0.00037		761	0.17656	-4.13618	-0.02366
262	0.00228	-0.41451	-0.00039		762	0.177136	-4.14765	-0.02374
263	0.002413	-0.41806	-0.00041		763	0.177711	-4.15912	-0.02381
264	0.002545	-0.42164	-0.00042		764	0.178288	-4.17059	-0.02389
265	0.002676	-0.42526	-0.00044		765	0.178874	-4.18211	-0.02397
266	0.002808	-0.42888	-0.00046		766	0.179459	-4.19363	-0.02405
267	0.00294	-0.4325	-0.00048		767	0.180045	-4.20515	-0.02413
268	0.003082	-0.43616	-0.0005		768	0.180631	-4.21672	-0.0242
269	0.003225	-0.43982	-0.00052		769	0.181217	-4.2283	-0.02428
270	0.003367	-0.44348	-0.00053		770	0.181803	-4.23988	-0.02436
271	0.003509	-0.44718	-0.00055		771	0.18239	-4.2515	-0.02444
272	0.003651	-0.4509	-0.00057		772	0.182977	-4.26314	-0.02452
273	0.003793	-0.45462	-0.00059		773	0.183564	-4.27478	-0.02459
274	0.003934	-0.45836	-0.00061		774	0.184153	-4.28643	-0.02467
275	0.004075	-0.46213	-0.00063		775	0.184751	-4.29812	-0.02475
276	0.004216	-0.46591	-0.00065		776	0.185348	-4.30982	-0.02483
277	0.004359	-0.46969	-0.00067		777	0.185945	-4.32152	-0.02491
278	0.00451	-0.47351	-0.00069		778	0.186544	-4.33326	-0.02499
279	0.004662	-0.47733	-0.00071		779	0.187142	-4.34502	-0.02507
280	0.004814	-0.48115	-0.00073		780	0.18774	-4.35678	-0.02515
281	0.004966	-0.48501	-0.00075		781	0.188339	-4.36857	-0.02523
282	0.005117	-0.48889	-0.00077		782	0.188938	-4.38038	-0.02531
283	0.005268	-0.49276	-0.00079		783	0.189537	-4.3922	-0.02539
284	0.005419	-0.49667	-0.00081		784	0.190139	-4.40403	-0.02547
285	0.00557	-0.5006	-0.00083		785	0.190748	-4.41591	-0.02555
286	0.00572	-0.50453	-0.00085		786	0.191358	-4.42778	-0.02563
287	0.005873	-0.50847	-0.00087		787	0.191967	-4.43965	-0.02571
288	0.006034	-0.51245	-0.00089		788	0.192578	-4.45158	-0.0258
289	0.006195	-0.51643	-0.00092		789	0.193188	-4.46351	-0.02588
290	0.006356	-0.52041	-0.00094		790	0.193799	-4.47545	-0.02596

291	0.006517	-0.52443	-0.00096		791	0.19441	-4.48742	-0.02604
292	0.006677	-0.52847	-0.00098		792	0.195021	-4.49941	-0.02612
293	0.006838	-0.5325	-0.001		793	0.195633	-4.51141	-0.0262
294	0.006998	-0.53657	-0.00102		794	0.196248	-4.52342	-0.02628
295	0.007159	-0.54066	-0.00104		795	0.19687	-4.53547	-0.02637
296	0.007319	-0.54475	-0.00107		796	0.197492	-4.54752	-0.02645
297	0.007481	-0.54885	-0.00109		797	0.198114	-4.55958	-0.02653
298	0.007652	-0.55298	-0.00111		798	0.198737	-4.57169	-0.02662
299	0.007822	-0.55712	-0.00113		799	0.19936	-4.5838	-0.0267
300	0.007992	-0.56125	-0.00116		800	0.199983	-4.59592	-0.02678
301	0.008162	-0.56544	-0.00118		801	0.200607	-4.60807	-0.02687
302	0.008332	-0.56963	-0.0012		802	0.201231	-4.62025	-0.02695
303	0.008502	-0.57382	-0.00122		803	0.201856	-4.63242	-0.02703
304	0.008671	-0.57805	-0.00125		804	0.202484	-4.64462	-0.02712
305	0.008841	-0.5823	-0.00127		805	0.203119	-4.65685	-0.0272
306	0.00901	-0.58655	-0.00129		806	0.203754	-4.66908	-0.02728
307	0.009183	-0.59081	-0.00132		807	0.204389	-4.68132	-0.02737
308	0.009362	-0.5951	-0.00134		808	0.205026	-4.69361	-0.02745
309	0.009542	-0.59939	-0.00136		809	0.205662	-4.70591	-0.02754
310	0.009721	-0.60369	-0.00139		810	0.206298	-4.7182	-0.02762
311	0.0099	-0.60803	-0.00141		811	0.206935	-4.73055	-0.02771
312	0.01008	-0.61238	-0.00144		812	0.207573	-4.7429	-0.02779
313	0.010259	-0.61673	-0.00146		813	0.20821	-4.75526	-0.02788
314	0.010437	-0.62112	-0.00148		814	0.208852	-4.76764	-0.02796
315	0.010616	-0.62552	-0.00151		815	0.209501	-4.78006	-0.02805
316	0.010794	-0.62993	-0.00153		816	0.210149	-4.79247	-0.02814
317	0.010977	-0.63435	-0.00156		817	0.210798	-4.80489	-0.02822
318	0.011165	-0.6388	-0.00158		818	0.211447	-4.81737	-0.02831
319	0.011354	-0.64325	-0.00161		819	0.212097	-4.82985	-0.0284
320	0.011543	-0.6477	-0.00163		820	0.212746	-4.84233	-0.02848
321	0.011731	-0.65221	-0.00166		821	0.213397	-4.85486	-0.02857
322	0.011919	-0.65671	-0.00168		822	0.214048	-4.8674	-0.02866
323	0.012107	-0.66122	-0.00171		823	0.214699	-4.87995	-0.02874
324	0.012295	-0.66576	-0.00173		824	0.215355	-4.89252	-0.02883
325	0.012482	-0.67032	-0.00176		825	0.216017	-4.90511	-0.02892
326	0.01267	-0.67489	-0.00178		826	0.21668	-4.91771	-0.02901
327	0.012862	-0.67947	-0.00181		827	0.217342	-4.93032	-0.0291
328	0.013059	-0.68407	-0.00184		828	0.218005	-4.94299	-0.02918
329	0.013257	-0.68868	-0.00186		829	0.218669	-4.95565	-0.02927
330	0.013454	-0.69329	-0.00189		830	0.219332	-4.96832	-0.02936
331	0.013651	-0.69795	-0.00192		831	0.219997	-4.98104	-0.02945
332	0.013848	-0.70261	-0.00194		832	0.220662	-4.99377	-0.02954

333	0.014045	-0.70728	-0.00197		833	0.221326	-5.0065	-0.02963
334	0.014241	-0.71198	-0.00199		834	0.221998	-5.01926	-0.02972
335	0.014438	-0.7167	-0.00202		835	0.222674	-5.03204	-0.02981
336	0.014634	-0.72141	-0.00205		836	0.22335	-5.04483	-0.0299
337	0.014836	-0.72615	-0.00207		837	0.224027	-5.05763	-0.02999
338	0.015044	-0.73092	-0.0021		838	0.224705	-5.07048	-0.03008
339	0.015252	-0.73568	-0.00213		839	0.225382	-5.08334	-0.03017
340	0.01546	-0.74046	-0.00216		840	0.22606	-5.09619	-0.03026
341	0.015666	-0.74528	-0.00219		841	0.226739	-5.1091	-0.03035
342	0.015871	-0.75009	-0.00221		842	0.227418	-5.12202	-0.03044
343	0.016077	-0.75491	-0.00224		843	0.228097	-5.13494	-0.03053
344	0.016281	-0.75978	-0.00227		844	0.228784	-5.14789	-0.03062
345	0.016484	-0.76465	-0.00229		845	0.229474	-5.16087	-0.03071
346	0.016687	-0.76952	-0.00232		846	0.230165	-5.17384	-0.0308
347	0.016898	-0.77442	-0.00235		847	0.230857	-5.18684	-0.0309
348	0.017115	-0.77934	-0.00238		848	0.231549	-5.19988	-0.03099
349	0.017331	-0.78426	-0.00241		849	0.232241	-5.21292	-0.03108
350	0.017548	-0.78919	-0.00244		850	0.232934	-5.22597	-0.03117
351	0.017762	-0.79416	-0.00247		851	0.233628	-5.23908	-0.03126
352	0.017977	-0.79914	-0.00249		852	0.234321	-5.25219	-0.03136
353	0.018192	-0.80411	-0.00252		853	0.235015	-5.2653	-0.03145
354	0.018404	-0.80913	-0.00255		854	0.235717	-5.27845	-0.03154
355	0.018617	-0.81416	-0.00258		855	0.236423	-5.29161	-0.03164
356	0.018829	-0.81919	-0.00261		856	0.237129	-5.30478	-0.03173
357	0.01905	-0.82424	-0.00264		857	0.237835	-5.31797	-0.03182
358	0.019275	-0.82932	-0.00267		858	0.238543	-5.33121	-0.03192
359	0.019501	-0.8344	-0.0027		859	0.23925	-5.34444	-0.03201
360	0.019725	-0.83949	-0.00273		860	0.239958	-5.35768	-0.03211
361	0.019949	-0.84462	-0.00276		861	0.240668	-5.37099	-0.0322
362	0.020172	-0.84974	-0.00279		862	0.241377	-5.38429	-0.0323
363	0.020395	-0.85487	-0.00282		863	0.242087	-5.39759	-0.03239
364	0.020616	-0.86005	-0.00285		864	0.242805	-5.41094	-0.03249
365	0.020838	-0.86524	-0.00288		865	0.243525	-5.4243	-0.03258
366	0.021059	-0.87042	-0.00291		866	0.244246	-5.43766	-0.03268
367	0.021288	-0.87563	-0.00294		867	0.244968	-5.45105	-0.03277
368	0.021523	-0.88086	-0.00297		868	0.245691	-5.46448	-0.03287
369	0.021757	-0.88609	-0.003		869	0.246413	-5.47792	-0.03297
370	0.02199	-0.89134	-0.00303		870	0.247136	-5.49135	-0.03306
371	0.022222	-0.89662	-0.00306		871	0.247861	-5.50486	-0.03316
372	0.022454	-0.90191	-0.00309		872	0.248586	-5.51836	-0.03326
373	0.022685	-0.90719	-0.00312		873	0.249311	-5.53186	-0.03335
374	0.022915	-0.91253	-0.00316		874	0.250046	-5.5454	-0.03345

375	0.023145	-0.91786	-0.00319		875	0.250783	-5.55896	-0.03355
376	0.023374	-0.9232	-0.00322		876	0.25152	-5.57252	-0.03365
377	0.023613	-0.92857	-0.00325		877	0.252257	-5.58611	-0.03374
378	0.023856	-0.93396	-0.00328		878	0.252996	-5.59974	-0.03384
379	0.024098	-0.93934	-0.00331		879	0.253735	-5.61337	-0.03394
380	0.02434	-0.94475	-0.00335		880	0.254474	-5.62701	-0.03404
381	0.02458	-0.95019	-0.00338		881	0.255215	-5.64071	-0.03414
382	0.024821	-0.95562	-0.00341		882	0.255955	-5.65441	-0.03424
383	0.025061	-0.96107	-0.00344		883	0.256696	-5.66811	-0.03434
384	0.025299	-0.96656	-0.00347		884	0.257448	-5.68186	-0.03444
385	0.025537	-0.97205	-0.00351		885	0.258201	-5.69562	-0.03454
386	0.025775	-0.97754	-0.00354		886	0.258955	-5.70938	-0.03464
387	0.026023	-0.98307	-0.00357		887	0.259709	-5.72318	-0.03474
388	0.026274	-0.9886	-0.00361		888	0.260464	-5.73701	-0.03484
389	0.026525	-0.99414	-0.00364		889	0.26122	-5.75084	-0.03494
390	0.026775	-0.99971	-0.00367		890	0.261975	-5.76469	-0.03504
391	0.027024	-1.0053	-0.00371		891	0.262732	-5.77859	-0.03514
392	0.027273	-1.01089	-0.00374		892	0.263489	-5.79249	-0.03524
393	0.027521	-1.01649	-0.00377		893	0.264246	-5.80639	-0.03534
394	0.027768	-1.02213	-0.0038		894	0.265015	-5.82035	-0.03544
395	0.028015	-1.02778	-0.00384		895	0.265786	-5.83431	-0.03554
396	0.028261	-1.03342	-0.00387		896	0.266557	-5.84827	-0.03565
397	0.028519	-1.03911	-0.0039		897	0.267329	-5.86227	-0.03575
398	0.028778	-1.0448	-0.00394		898	0.268101	-5.87631	-0.03585
399	0.029037	-1.05049	-0.00397		899	0.268873	-5.89034	-0.03596
400	0.029295	-1.05621	-0.00401		900	0.269646	-5.9044	-0.03606
401	0.029551	-1.06196	-0.00404		901	0.27042	-5.91851	-0.03616
402	0.029808	-1.0677	-0.00408		902	0.271194	-5.93262	-0.03626
403	0.030063	-1.07346	-0.00411		903	0.271968	-5.94673	-0.03637
404	0.030317	-1.07926	-0.00415		904	0.272756	-5.96089	-0.03647
405	0.03057	-1.08506	-0.00418		905	0.273544	-5.97506	-0.03658
406	0.030824	-1.09086	-0.00421		906	0.274333	-5.98922	-0.03668
407	0.031092	-1.0967	-0.00425		907	0.275122	-6.00344	-0.03679
408	0.031363	-1.10254	-0.00429		908	0.275912	-6.01768	-0.03689
409	0.031633	-1.10838	-0.00432		909	0.276702	-6.03192	-0.037
410	0.0319	-1.11426	-0.00436		910	0.277492	-6.04619	-0.0371
411	0.032166	-1.12016	-0.00439		911	0.278284	-6.06051	-0.03721
412	0.032432	-1.12606	-0.00443		912	0.279075	-6.07483	-0.03731
413	0.032696	-1.13197	-0.00446		913	0.279867	-6.08915	-0.03742
414	0.032957	-1.13793	-0.0045		914	0.280674	-6.10352	-0.03752
415	0.033218	-1.14388	-0.00453		915	0.281481	-6.1179	-0.03763
416	0.033479	-1.14984	-0.00457		916	0.282287	-6.13227	-0.03774

417	0.033757	-1.15583	-0.00461		917	0.283095	-6.1467	-0.03785
418	0.034035	-1.16183	-0.00464		918	0.283903	-6.16116	-0.03795
419	0.034313	-1.16783	-0.00468		919	0.284711	-6.17561	-0.03806
420	0.034589	-1.17386	-0.00472		920	0.28552	-6.1901	-0.03817
421	0.034862	-1.17991	-0.00475		921	0.28633	-6.20463	-0.03828
422	0.035136	-1.18596	-0.00479		922	0.28714	-6.21916	-0.03839
423	0.035408	-1.19204	-0.00483		923	0.287952	-6.2337	-0.03849
424	0.035677	-1.19814	-0.00486		924	0.288778	-6.24828	-0.0386
425	0.035947	-1.20425	-0.0049		925	0.289603	-6.26287	-0.03871
426	0.036216	-1.21036	-0.00493		926	0.290429	-6.27745	-0.03882
427	0.036503	-1.21651	-0.00497		927	0.291256	-6.29211	-0.03893
428	0.036789	-1.22266	-0.00501		928	0.292083	-6.30678	-0.03904
429	0.037076	-1.22881	-0.00505		929	0.292911	-6.32144	-0.03915
430	0.037359	-1.235	-0.00509		930	0.29374	-6.33615	-0.03926
431	0.037641	-1.2412	-0.00512		931	0.294569	-6.35091	-0.03937
432	0.037923	-1.24741	-0.00516		932	0.295399	-6.36566	-0.03948
433	0.038203	-1.25364	-0.0052		933	0.296231	-6.38042	-0.03959
434	0.038481	-1.2599	-0.00524		934	0.297077	-6.39522	-0.03971
435	0.038758	-1.26616	-0.00527		935	0.297923	-6.41002	-0.03982
436	0.039037	-1.27242	-0.00531		936	0.298768	-6.42483	-0.03993
437	0.039332	-1.27873	-0.00535		937	0.299615	-6.4397	-0.04004
438	0.039627	-1.28503	-0.00539		938	0.300462	-6.45459	-0.04016
439	0.039921	-1.29133	-0.00543		939	0.30131	-6.46948	-0.04027
440	0.040212	-1.29768	-0.00547		940	0.302158	-6.48441	-0.04038
441	0.040503	-1.30404	-0.00551		941	0.303007	-6.49938	-0.04049
442	0.040793	-1.31039	-0.00555		942	0.303856	-6.51435	-0.04061
443	0.041081	-1.31678	-0.00558		943	0.30471	-6.52934	-0.04072
444	0.041367	-1.32319	-0.00562		944	0.305577	-6.54436	-0.04083
445	0.041652	-1.3296	-0.00566		945	0.306444	-6.55939	-0.04095
446	0.041941	-1.33602	-0.0057		946	0.307311	-6.57441	-0.04107
447	0.042243	-1.34248	-0.00574		947	0.30818	-6.58952	-0.04118
448	0.042546	-1.34893	-0.00578		948	0.309049	-6.60463	-0.0413
449	0.042849	-1.35539	-0.00582		949	0.309918	-6.61974	-0.04141
450	0.043148	-1.36189	-0.00586		950	0.310788	-6.63491	-0.04153
451	0.043446	-1.3684	-0.0059		951	0.311659	-6.65011	-0.04164
452	0.043745	-1.37492	-0.00594		952	0.31253	-6.66531	-0.04176
453	0.044041	-1.38146	-0.00598		953	0.313408	-6.68053	-0.04188
454	0.044335	-1.38802	-0.00602		954	0.314299	-6.69578	-0.04199
455	0.044629	-1.39459	-0.00606		955	0.31519	-6.71103	-0.04211
456	0.044926	-1.40116	-0.0061		956	0.316082	-6.72628	-0.04223
457	0.045237	-1.40777	-0.00614		957	0.316976	-6.74162	-0.04235
458	0.045548	-1.41438	-0.00618		958	0.317871	-6.75697	-0.04247

459	0.045859	-1.42099	-0.00622		959	0.318765	-6.77231	-0.04259
460	0.046166	-1.42765	-0.00626		960	0.319662	-6.78773	-0.04271
461	0.046473	-1.43431	-0.0063		961	0.32056	-6.80317	-0.04283
462	0.046779	-1.44097	-0.00635		962	0.321458	-6.81861	-0.04295
463	0.047083	-1.44767	-0.00639		963	0.322363	-6.83406	-0.04307
464	0.047385	-1.45439	-0.00643		964	0.32328	-6.84954	-0.04319
465	0.047687	-1.46111	-0.00647		965	0.324197	-6.86502	-0.04331
466	0.047994	-1.46784	-0.00651		966	0.325115	-6.88051	-0.04343
467	0.048313	-1.4746	-0.00655		967	0.326038	-6.89609	-0.04355
468	0.048632	-1.48136	-0.00659		968	0.32696	-6.91168	-0.04368
469	0.048951	-1.48812	-0.00664		969	0.327883	-6.92726	-0.0438
470	0.049266	-1.49493	-0.00668		970	0.328811	-6.94293	-0.04392
471	0.049581	-1.50175	-0.00672		971	0.329739	-6.95863	-0.04405
472	0.049895	-1.50856	-0.00676		972	0.330667	-6.97432	-0.04417
473	0.050207	-1.51542	-0.0068		973	0.3316	-6.99002	-0.0443
474	0.050518	-1.52229	-0.00684		974	0.332537	-7.00572	-0.04442
475	0.050828	-1.52916	-0.00689		975	0.333474	-7.02143	-0.04455
476	0.051144	-1.53604	-0.00693		976	0.334414	-7.03718	-0.04467
477	0.051471	-1.54296	-0.00697		977	0.335357	-7.05298	-0.0448
478	0.051799	-1.54987	-0.00702		978	0.336301	-7.06877	-0.04492
479	0.052126	-1.55678	-0.00706		979	0.337247	-7.08461	-0.04505
480	0.052449	-1.56375	-0.0071		980	0.338198	-7.1005	-0.04517
481	0.052771	-1.57072	-0.00715		981	0.339148	-7.11639	-0.0453
482	0.053094	-1.57769	-0.00719		982	0.340102	-7.13229	-0.04543
483	0.053414	-1.5847	-0.00723		983	0.341061	-7.14819	-0.04555
484	0.053732	-1.59172	-0.00727		984	0.342021	-7.1641	-0.04568
485	0.054051	-1.59874	-0.00732		985	0.342975	-7.18006	-0.04581
486	0.054376	-1.60578	-0.00736		986	0.343929	-7.19605	-0.04593
487	0.054713	-1.61285	-0.0074		987	0.344881	-7.21205	-0.04606
488	0.055049	-1.61991	-0.00745		988	0.345829	-7.22811	-0.04619
489	0.055385	-1.62698	-0.00749		989	0.346776	-7.24417	-0.04631
490	0.055719	-1.6341	-0.00754		990	0.347719	-7.2602	-0.04644
491	0.056052	-1.64121	-0.00758		991	0.348658	-7.27621	-0.04657
492	0.056386	-1.64833	-0.00763		992	0.349585	-7.29227	-0.04669
493	0.056717	-1.65549	-0.00767		993	0.350509	-7.30835	-0.04681
494	0.057048	-1.66266	-0.00771		994	0.35142	-7.32449	-0.04694
495	0.057379	-1.66983	-0.00776		995	0.352372	-7.34056	-0.04706
496	0.057713	-1.67703	-0.0078		996	0.353371	-7.35656	-0.0472
497	0.058052	-1.68425	-0.00785		997	0.354349	-7.37259	-0.04732
498	0.058391	-1.69147	-0.00789		998	0.355302	-7.38865	-0.04745
499	0.058729	-1.6987	-0.00794		999	0.356335	-7.40468	-0.04759
500	0.059065	-1.70598	-0.00798		1000	0.357274	-7.42074	-0.04771