BOLU ABANT IZZET BAYSAL UNIVERSITY

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SOME GENERALIZATIONS OF UNBOUNDED ORDER CONVERGENCE TYPES IN RIESZ SPACES AND RELATED TOPICS

DOCTOR OF PHILOSOPHY

MEHMET VURAL

BOLU, SEPTEMBER 2018

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APPROVAL OF THE THESIS

SOME GENERALIZATIONS OF UNBOUNDED ORDER CONVERGENCE TYPES IN RIESZ SPACES AND RELATED TOPICS submitted by **Mehmet VURAL** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in **Department of Mathematics, The Graduate School of Natural and Applied Sciences of BOLU ABANT IZZET BAYSAL UNIVERSITY in 07/09/2018** by

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Mehmet VURAL

ABSTRACT

SOME GENERALIZATIONS OF UNBOUNDED ORDER CONVERGENCE TYPES IN RIESZ SPACES AND RELATED TOPICS

PHD THESIS

MEHMET VURAL

ABANT IZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS

(SUPERVISOR: PROF. DR. ZAFER ERCAN)

BOLU, SEPTEMBER 2018

One of the main aim of this thesis is to generalize the the notion of multi-normed spaces to multi-pseudonormed spaces by replacing seminorms with pseudoseminorms and the fundemental properties of this generalized space were investigated and the notion of continuous operators between multi-pseudonormed spaces was elaborated. The other main thing is defined unbounded locally solid Riesz space and investigate its fundamental properties. In the Rest of the thesis, apart from the generalizations, we focused on the problem if topological space structure can be characterized in some real-valued maps; the answer is affirmative : 0-1-valued quasimetrics and we reproves that if an inequality is valid in reals then it is valid in any Riesz space(need not to be Archimedean) without using Kakutani Representation theorem.

KEYWORDS: Multi-pseudonormed space, Unbounded locally solid Riesz space, 0-1-valued generalized quasimetrics, inequalities in Riesz spaces.

ÖZET

RİESZ UZAYLARDA SINIRSIZ SIRA YAKINSAMANIN BAZI GENELLEMELERİ VE İLİŞKİLİ KONULAR
DOKTORA TEZİ
MEHMET VURAL
ABANT İZZET BAYSAL ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI
(TEZ DANIŞMANI: PROF. DR. ZAFER ERCAN)
BOLU, EYLÜL 2018

Bu tezin temel amaçlarından biri yarınormları yarınormsularla değiştirerek çoklu-normlu uzay kavramını çoklu-normsu uzaylara genellemek ve bu yeni uzayın temel özelliklerini incelemektir ve çoklu-normsular arasındaki sürekli operatörleri ele almaktır. Diğer ana amaç ise sınırsız yerel solid Riesz uzay kavramını tanımlamak ve bu uzayın temel özelliklerini incelemektir. Tezin geri kalanında ise, bu genellemelerden ayrı olarak topolojik uzay yapısı bir takım reel değerli fonksiyonlar tarafından karakterize edilebilir mi problemine odaklanılmıştır; cevap ise olumludur: 0-1-değerli quasimetrikler ve Kakutani gösterim teoremi kullanmaksızın ve Riesz uzayın Arşimedyan olup olmadığına bakılmaksızın reel sayılarda geçerli olan her eşitsizliğin herhangi bir Riesz uzayda da geçerli olduğu ispatlanmıştır.

ANAHTAR KELİMELER: Çoklu-normsu uzaylar, sınırsız yerel solid uzaylar 0-1 değerli sözde metrikler, Riesz uzaylarda eşitsizlikler.

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LIST OF ABBREVIATIONS

A^d	Disjoint complement of A
A^{dd}	Band generated by A
E^+	Positive cone of E
E_x	Ideal generated by x
E	Order dual of E
E	Second order dual of E
E^δ	Dedekind complition of E
$\langle E, E' \rangle$	Riesz dual system
$\mathscr{L}(E,F)$	The vector space of operators E to F
$\mathscr{L}_b(E,F)$	The order bounded operators E to F
$\sigma(E,E')$	Weak topology on E
$\lvert\sigma\rvert(E,E')$	Absolute weak topology on E
x^+	Positive part of vector x
x^-	Negative part of vector x
$\lvert x\rvert$	Absolute value of the vector x
$x_\alpha \uparrow x$	Increasing net to x
$x_\alpha \downarrow x$	Decreasing net to x
$x_\alpha \xrightarrow{o} x$	Order convergence
$x_\alpha \xrightarrow{uo} x$	Unbounded order convergence
$x_\alpha \xrightarrow{p} x$	p convergence
$x_\alpha \xrightarrow{up} x$	Unbounded p convergence
$x_\alpha \xrightarrow{\tau} x$	Topological convergence
$x \wedge y$	Infumum of x and y
$x \vee y$	Supremum of x and y
$x \perp y$	Disjoint vectors
X^*	Algebraic dual of X
X'	Topological dual of X
X''	Second dual of X

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1. INTRODUCTION

One of the main convergence in Riesz space [1] is *order convergence*. In a Riesz space,say E, a net $(x_\alpha)_{\alpha\in A}$ is said to be order convergent to $x \in E$ (briefly; $x_\alpha \xrightarrow{o} x$, x_α o-converges to x) if another net $(y_\beta)_{\beta\in B}$ exists in E such that:

i. $y_\beta \downarrow 0$, that is, $(y_\beta)_{\beta\in B}$ is decreasing to 0.

ii. For each $\beta_0 \in B$ there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq y_{\beta_0}$ for all $\alpha \geq \alpha_0$.

Unbounded order convergence in a Riesz space has been defined and studied in Nakano (1948) and in Wickstead (1977). Recently many authors have started to work on this topic in Gao et al (2017),Gao and Xanthos (2014),Gao (2014).Namely in a Riesz space,say E, a net (x_α) [2] is called *unbounded order convergent* if the net $(|x_\alpha - x| \wedge u)$ is order convergent to zero for any vector in E_+ (briefly; $x_\alpha \xrightarrow{uo} x$, x_α *uo*-converges to x). In general every order convergent net is unbounded order convergent but the converse is not valid (Consider c_0 as a Riesz space,the standard unit vectors (e_n) uo-converges but not o-converges). It is obvious that order convergence and unbounded order converge are coincide for order bounded nets. Although in general unbounded order convergence is not topological convergence in Gao et al (2017), in atomic Riesz space it is (see Theorem 2,Dabboorasad et al (2017)).

In a normed vector lattice, say E, *unbounded norm convergence* is defined as an analogy of unbounded order convergence as;a net (x_α) in E is said to be unbounded order convergent to $x \in E$ if $(|x_\alpha - x| \wedge u) \xrightarrow{\|\cdot\| 0}$ for any vector $u \in E^+$ (briefly; $x_\alpha \xrightarrow{un} x$, x_α *un*-converges to x). The notion of unbounded norm convergence has been defined in Troitsky (2004) and many papers have been written on it, (i.e, Deng et al (2017), Kandic et al (2017)) and it has been extended to the locally solid Riesz space. (see Dabboorasad et al (2018)). In Kandic et al (2017) it has been noticed that unbounded norm convergence defines a topology, that is, there exists a new topology on the normed Riesz space E so that the unbounded norm convergence and topological convergence coincide with respect to this new topology which is called *un-topology* in Kandic et al (2017). In the same paper it is also proved that in Banach lattices, the norm convergence and unbounded norm converge coincide if and only if it has (strong)order unit (Theorem 2.3).

[1] In this paper all Riesz spaces be assumed Archimedean
[2] The index is not written unless it is necessary

In a locally solid Riesz space, the notion of unbounded topological convergence has been defined and studied in Dabboorasad et al (2018) and Taylor (2017). The definition of unbounded topological convergence is given as follows ; a net (x_α) in a locally solid Riesz space (E,τ) is said to be unbounded topological convergent to $x \in E$ if $(|x_\alpha - x| \wedge u) \xrightarrow{\tau} 0$ for any vector $u \in E^+$ (briefly; $x_\alpha \xrightarrow{u\tau} x$, x_α $u\tau$-converges to x). Unbounded topological convergence in a locally solid Riesz space not only defines a topology but also it has a locally solid topology.

Let X be a vector space and E be a Riesz space. A map p from X into E is called lattice norm or $E-$valued lattice norm if the following conditions are satisfied: (i) $p(x) = 0 \leftrightarrow x = 0$, (ii) $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and for all $\lambda \in \mathbb{R}$, and (iii) $p(x+y) \leq p(x) + p(y)$ for each $x, y \in X$, in this case the triple (X, p, E) is called *lattice normed space*, abbreviated as LNS. Let us consider any Riesz space, the map $p(x) = |x|$ is an $E-$valued lattice norm and for any normed space $(X, \|.\|$, the map $N(x) = \|x\|$ is an $\mathbb{R}-$valued lattice norm. If X, E are Riesz spaces and p is a map from X into E with the monotonicity property (i.e; $x \leq y$ implies $p(x) \leq p(y)$), then the triple (X, p, E) is called *lattice normed vector lattice*, abbreviated as LNVL. In a lattice-normed vector lattice (X,p,E), a net (x_α) in X is said to be p-convergent to $x \in X$ if $p(|x_\alpha - x|) \xrightarrow{o} 0$ in E (briefly; $x_\alpha \xrightarrow{p} x$, x_α p-converges to x).In [?] the notion of unbounded p-convergence is defined as follows ; a net (x_α) in X is said to be unbounded p-convergent to $x \in X$ if $p(|x_\alpha - x| \wedge u) \xrightarrow{o} 0$ in E for any vector $u \in X^+$ (briefly; $x_\alpha \xrightarrow{up} x$, x_α up-converges to x)

From this point on the basic definitions concerning unity and convenience are going to be surveyed.

A *binary relation* $\mathscr{R}$ on a non-void set E is a subset of $E \times E$,the elements of a binary relation are written as $x\mathscr{R}y$ instead of $(x,y) \in \mathscr{R}$. If a binary relation $\mathscr{R}$ on E is reflexive ($x\mathscr{R}x\ \forall x \in E$), antisymmetric($x\mathscr{R}y$ and $y\mathscr{R}x$ implies that $x = y \quad \forall x, y \in E$) and transitive ($x\mathscr{R}y$ and $y\mathscr{R}z$ implies that $x\mathscr{R}z \quad \forall x, y, z \in E$), then it is called *order relation* and it is written as $x \leq y$ or $y \geq x$ instead of $x\mathscr{R}y$. The pair $(E, \leq)$ is called *partially ordered set*. An element $x \in E$ is called an *upper bound* for a given non-empty subset A of E if $a \leq x$ for all $a \in A$. If the set of all upper bounds of $A \subseteq E$ is non-empty, then it is said that A is *bounded above*. An upper bound x of A is called *least upper bound* or *supremum* if $x \leq y$ holds for any element y of the set of all upper bounds. *Lower* *bound, bounded below and greatest lower bound(infimum)* can be easily defined as an analogy of upper bound, bounded above and supremum, respectively. $A \subseteq E$ is said to be *order bounded* if A is bounded below and bounded above. For any $a, b \in E$, the set $\{x : a \leq x \leq b\}$ is called *order interval*, denoted by $[a, b]$.

A partially ordered set E is said to be *lattice* if the supremum and infimum of the set $\{x, y\}$ exists for any $x, y \in E$. The supremum and infimum of two elements denoted by $x \vee y$ and $x \wedge y$, respectively. If a nonempty subset A of E satisfies the statement $x, y \in A \Rightarrow x \wedge y, x \vee y \in E$, then it is called *sublattice*. The supremum and infimum of a set $A \subseteq E$ are denoted by $\sup(A)$ or $\bigvee A$ and $\inf(A)$ or $\bigwedge A$,if they exist.

A vector space over $\mathbb{R}$,say E, which is equipped with an order relation $\leq$ is said to be *ordered vector space* if the following two axioms are satisfied:

i. If $x, y, z \in E$ and $x \leq y$, then $x + z \leq y + z$,

ii. If $x, y \in E$, $x \leq y$ and $\alpha \in \mathbb{R}^+$ then $\alpha x \leq \alpha y$.

In an ordered vector space E, E^+ denotes the set of all positives elements of E, that is $E^+ = \{x : x \in E \text{ and } 0 \leq x\}$ and it is called *positive cone*.

An ordered vector space is called *Riesz space* or *vector lattice* if the order relation on itself is also a lattice. In a Riesz space E, some special vectors lie in E which is related with a fixed vector $x \in E$: *positive part* of x as $x^+ := x \vee 0$, *negative part* of x as $x_- := (-x) \vee 0$ and *absolute value* of x as $|x| := x \vee (-x)$. For any fixed $x, y \in E$, if $|x| \wedge |y| = 0$, they are called *disjoint* and $x \perp y$ refers to the disjointness of x and y. In this sense, let $A \subset E$ be given, the set $A^d := \{x : x \in E, x \perp a \forall a \in A\}$ denotes the *disjoint complement* of A.

In a Riesz space, a net $(x_\alpha)_{\alpha \in I}$ is called increasing if $\alpha, \beta \in I$ and $\alpha \leq \beta$, then $x_\alpha \leq x_\beta$. it is denoted by $x_\alpha \uparrow$. Also the notion $x_\alpha \uparrow x$ for an $x \in E$ indicates that x_α is increasing and $\sup_{\alpha \in I} x_\alpha = x$. Analogously, the decreasing net and the notions $x_\alpha \downarrow, x_\alpha \downarrow x$ are defined.

For any element,say x, of a positive cone of a vector lattice,say E, if the statement $\frac{1}{n}x \downarrow 0$ holds then E is called Archimedean vector lattice.

A vector lattice is said to be *Dedekind complete* or *order complete* if every nonempty subset which is bounded above has a supremum.

If a subset A of a vector lattice E is a vector subspace and sublattice, then it is called *vector sublattice* and if for any $0 < x \in E$ there exists $0 < a \in A$ such that $0 < a < x$, then it is called *order dense* vector sublattice and also if for any $x \in E_+$ there is $a \in A$ such that $x \leq a$, then it is called *majorizing* vector sublattice.

A subset $A \subset E$ is called *solid* if $y \in A$ whenever $|y| \leq |x|$ in E for some $x \in A$. A solid subset A of a vector lattice E is said to be *ideal* if it is also a vector subspace. For any subset A; I_A denotes the intersection of all ideals containing A, is called *ideal generated by* A and is formulated as follow:

$$I_A := \{x \in E : \exists a_1, a_2 ..., a_n \in A \text{ and } \lambda \in \mathbb{R}^+ \text{ with } |x| \leq \lambda \sum_{i=1}^{n} |x_i|\}$$

If $A = \{a\}$, then I_A is called *principal ideal.* A subset A of a vector lattice E is called *order closed* if $\{a_\alpha\} \subseteq A$ and $a_\alpha \xrightarrow{o} a$ in E implies that $a \in A$. Order closed ideal is called as a *band*.For any subset A; B_A denotes the intersection of all bands containing A, is called *band generated by* A and is formulated as follow:

$$B_A := \{x \in E : \exists \{a_\alpha\} \subseteq A \text{ with } 0 \leq a_\alpha \uparrow |x|\}$$

Let E and F be vector lattices and T be a linear operator from E into F. T is called *lattice homomorphism* if the equality $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in E$. If it is one-to-one, it is called *lattice isomorphism*.

For any vector lattice E, there exists unique Dedekind complete vector lattice F and a lattice isomorphism $T : E \to F$ such that $T(E)$ is a majorizing and order dense sublattice of F, hence F is called Dedekind complition of E and denoted by E^δ.

A topology τ on a vector space E is called *linear topology* if it makes both addition and scalar multiplication continuous.A linear topology τ on a Riesz space is called *locally solid* if it has a neighborhood system at zero consisting of solid sets. In this case, the pair (E, τ) is called locally solid Riesz space and if the neigbourhood system at zero consists of convex and solid sets, then τ is called *locally convex-solid topology*, so the pair (E, τ) is called locally convex-solid Riesz space.

2. AIM AND SCOPE OF THE STUDY

In **chapter 3**: We introduce the multi-pseudonormed spaces as a generalization of multi-normed spaces by replacing seminorms with pseudoseminorms.

In **chapter 4**:We introduce the notion of unbounded locally solid Riesz space and investigate its some fundamental properties. Especially in the last two parts of this chapter, we define the product of unbounded locally solid Riesz space and unbounded absolute weakly solid Riesz space.

In **chapter 5**: We prove that all topologies come from a family of 0-1-valued quasimetrics. Therefore we refine the main theorem of Kopperman (1988) by taking 0-1-valued generalized quasimetrics and also reprove that every topological space is induced by a quasi-uniformity.

In **chapter 6**: We prove that an elementary inequality is true in $\mathbb{R}$ then it is true in any Riesz space that is not to be Archimedean without using Kakutani representation theorem and Stone–Weierstrass theorem.

3. LATTICE MULTI-RIESZ PSEUDONORMED VECTOR LATTICE

3.1 Introduction

Given a set X, a convergence "$\xrightarrow{\mathbb{C}}$" for nets in X is defined by the following two conditions: 1) $x_\alpha \equiv x \Rightarrow x_\alpha \xrightarrow{\mathbb{C}} x$; 2) $x_\alpha \xrightarrow{\mathbb{C}} x \Rightarrow x_\beta \xrightarrow{\mathbb{C}} x$ for every subnet x_β of the net x_α. Let "$\xrightarrow{\mathbb{C}}$" be a *convergence* on a vector space X which agrees with linear operations, i.e.:

$$X \ni x_\alpha \xrightarrow{\mathbb{C}} x,\ X \ni y_\alpha \xrightarrow{\mathbb{C}} y,\ \mathbb{R} \ni r_\alpha \to r \tag{1}$$

implies

$$r_\alpha \cdot x_\alpha + y_\alpha \xrightarrow{\mathbb{C}} r \cdot x + y. \tag{2}$$

In this case, we say that $X = (X, \mathbb{C})$ is a *convergence vector space*. Basic examples of such convergence vector spaces are: a topological vector space $X = (X, \tau)$ with τ-*convergence* and a space X of measurable functions on a measure space with *almost everywhere convergence*. If in addition, we assume that X is a vector lattice in which the convergence agrees with lattice operations in the sense that (1) implies

$$r_\alpha \cdot x_\alpha \wedge y_\alpha \xrightarrow{\mathbb{C}} r \cdot x \wedge y, \tag{3}$$

we say that $X = (X, \mathbb{C})$ is a *convergence vector lattice*.

In this chapter, we introduce several new convergence structures on vector spaces and vector lattices. But we deeply focus on the section of lattice multi-Riesz pseudonormed vector lattice.

3.2 Multi-Pseudonormed Spaces (MPNS)

Definition 1. *We say that a collection $\mathscr{M} = \{m_\alpha\}_{\alpha \in A}$ of seminorms on a $(complex)$ vector space X is a* multi-norm *if for any $0 \neq x \in X$ there is $m_\alpha \in \mathscr{M}$ such that $m_\alpha(x) > 0$. In this case, we say that $X = (X, \mathscr{M})$ is a* multi-normed space $($*abbreviated by* MNS$)$ *with the multi-norm $\mathscr{M}$.*

Multi-normed spaces in the sense of Definition 1 (see, for example Kutateladze (1996)[p.94]) are also known as Hausdorff locally convex vector spaces. Notice that nowadays the name *multi-normed space* becomes popular for quite different class of spaces Dales and Polyakov (2012).

Definition 2. *Given a vector space X, a function $p : X \to \mathbb{R}$ is called* pseudoseminorm *whenever:*

(a) $p(x) \geq 0$ *for all* $x \in X$*;*

(b) $p(x+y) \leq p(x)+p(y)$ *for all* $x,y \in X$*; and*

(c) $\lim_{n\to\infty} p(\alpha_n x) = 0$ *for all* $x \in X$ *and for all* $\mathbb{C} \ni \alpha_n \to 0$.

If, additionally,

(d) $p(x) > 0$ *implies* $x \neq 0$,

we say that p *is a* pseudonorm.

Example 1. *Let us consider the set of all real-valued continuous functions on* $[0,1]$, *the map*

$$p : C([0,1]) \to \mathbb{R}$$
$$f \mapsto p(f) := |f(x_0)|$$

where f *assigns the maximum value at* x_0. *The map* p *is a pseudonorm.*

Remark 1. *Let us consider the all bounded real-valued functions on* $[a,b]$ *as a vector space. For a fixed* $x_0 \in [a,b]$, *we can define a map*

$$p : B([a,b]) \to \mathbb{R}$$
$$f \mapsto p(f) := max\{f(x_0), 0\}$$

p *is a pseudoseminorm on* $B([a,b])$. *The topology generated by* p *does not define a linear topology since consider the sequence of functions, defined by*

$$f_n(x) := \begin{cases} -n & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

f_n *converges to zero function with respect to the topology generated by* p, *but* $-f_n$ *does* *not converges to zero function.*

Definition 3. *A collection $\mathscr{P} = \{p_\alpha\}_{\alpha \in A}$ of pseudoseminorms on X is said to be a* multi-pseudonorm *if, for any $0 \neq x \in X$ there is $p_\alpha \in \mathscr{P}$ such that $p_\alpha(x) > 0$. In this case we say that $X = (X, \mathscr{P})$ is a* multi-pseudonormed space (*abbreviated by* MPNS) *with the multi-pseudonorm $\mathscr{P}$.*

Example 2. *Let $(\mathbb{P}_i)_{i \in I}$ be the family of the partitions of $[0,1]$, then for each $i \in I$, consider the map:*

$$p : B([0,1]) \to \mathbb{R}$$
$$f \mapsto p(f)$$

given by $p(f) = |\sum_{\mathbb{P}_{i_k}} f(x_k)|$ where $f(x_k)$ is the maximum value of f on k^{th} interval of $\mathbb{P}_i$. For each $i \in I$, the map p_i is a pseudoseminorm, and $\mathscr{P} = \{p_i\}_{i \in I}$ is a multi-pseudonorm on $B([0,1])$.

Remark 2. *The example given in (Remark 1.) can be used for giving an example to multi-pseudonormed space. Let us consider $B([a,b])$, and the family of pseudoseminorms $\mathscr{P} = \{p_x\}_{x \in [a,b]}$, the family $\mathscr{P}$ satisfies the condition that for any $0 \neq f \in B([a,b])$ there exists $p_{x_0} \in \mathscr{P}$ such that $p_{x_0}(f) > 0$. Hence $(B([a,b]), \mathscr{P})$ is a multi-pseudonormed space.The topology generated by the family $\mathscr{P}$ does not define a linear topology too but the sets $U_{p,\varepsilon}\{x \in X : p(x) \leq \varepsilon\}$ where $p \in \mathscr{P}$ and $0 < \varepsilon \in \mathbb{R}$ as a neighbourhood subbase at zero defines a linear topology. If the each member of the family $\mathscr{P}$ satisfies $(i) p(\lambda x) \leq p(x)$ for any $|\lambda| \leq 1$ and $(ii) p(x_n) \to 0$ implies $p(\lambda x_n) \to 0$ for all $\lambda \in \mathbb{R}$, then the topology $< \mathscr{P} >$ turns a linear topology. (schaefer (1966))*

3.3 Lattice Multi-Pseudonormed Spaces (LMPNS)

3.3.1 Lattice normed spaces (LNS)

Definition 4. *Let us consider a complex or real vector space,say X and a real vector lattice ,say E. A map p from X into E_+ is said to be E-*valued norm*, whenever*:
(a) $p(\alpha x) = |\alpha| \cdot p(x)$ *for all $x \in X, \alpha \in \mathbb{C}$;*

(b) $p(x+y) \leq p(x)+p(y)$ *for all* $x,y \in X$*; and*

(c) $x \neq 0$ *implies* $p(x) \neq 0$.

If, additionally,

(d) $p(x) = e_1 + e_2$ *with* $x \in X$, $e_1, e_2 \in E_+$ *implies that* $x_1 + x_2 = x$ *with* $p(x_1) = e_1$ *and* $p(x_2) = e_2$ *for some* $x_1, x_2 \in X$*, we say that the* E*-valued norm* p *is said to be* decomposable. *A vector space* (X, p, E) *equipped with an* E*-valued norm* p *is called* lattice normed space (*abbreviated by* LNS).

Remark 3. *In some sense lattice normed space structures can be understood as 'super-structure' since any vector lattice* X *can be written as LNS* $(X, |.|, X)$ *and also any normed space* $(X, \|.\|)$ *as* $(X, \|.\|, \mathbb{R})$.

3.3.2 Lattice-valued pseudonorms

Definition 5. *Let us consider a complex or real vector space,say* X *and a real vector lattice ,say* E*. A map* p *from* X *into* E_+ *is said to be* E-valued pseudonorm*, whenever*:

(a) $p(x+y) \leq p(x)+p(y)$ *for all* $x,y \in X$*;*

(b) $x \neq 0$ *implies* $p(x) \neq 0$*;*

(c) $p(\alpha_n x) \xrightarrow{o} 0$ *for all* $x \in X$ *and* $\mathbb{C} \ni \alpha_n \to 0$.

If condition (b) *is dropped,* p *is said to be an* E-valued pseudoseminorm.

Decomposable E-valued pseudonorm *and* lattice pseudonormed space (*shortly,* LPNS) *are defined similarly to corresponding ones in Definition 4.*

Example 3. *Let us consider* $C([0,1])$ *as a vector lattice,* $x_0 \in (0,1)$ *and* $0 < \varepsilon < \min\{1-x_0, x_0\}$, $\varepsilon \in \mathbb{R}$ *be fixed. Consider the map:*

$$p_{x_0,\varepsilon} : C([0,1]) \to C([0,1])$$
$$f \mapsto p_{x_0,\varepsilon}(f)$$

given by

$$p_{x_0,\varepsilon}(f)(x) := \begin{cases} |f|(x) & x \notin [x_0-\varepsilon, x_0+\varepsilon] \\ l^-_{|f|}(x) & x \in [x_0-\varepsilon, x_0] \\ l^+_{|f|}(x) & x \in [x_0, x_0+\varepsilon] \end{cases}$$

where $l^-_{|f|}(x)$ *and* $l^+_{|f|}(x)$ *are the lines passing through the points* x_0 *and* $|f|(x_0-\varepsilon)$ *and the points* x_0 *and* $|f|(x_0+\varepsilon)$*,respectively.* $p_{x_0,\varepsilon}$ *is a* $C([0,1])-$*valued pseudoseminorm.*

Example 4. *Let us consider* $c_0 := \{(x_n) : x_n \to 0\}$ *as a vector space and* $c_{00} := \{(x_n) : x_n \to 0$ *and* $\{x_n \neq 0 : n \in \mathbb{N}\}$ *is finite*$\}$ *as a vector lattice, then consider the map p:*

$$p : c_0 \to c_{00}$$
$$(x_n) \mapsto p(x_n) = \lceil |x_n| \rceil$$

It is $c_{00}-$*valued pseudonorm.*

3.3.3 Lattice multi-normed spaces (LMNS)

Definition 6. *Let us consider a complex or real vector space,say X and a real vector lattice ,say E. A map p from X into* E_+ *is said to be E-*valued seminorm *if:*
(a) $m(\alpha x) = |\alpha| \cdot m(x)$ *for all* $x \in X$*,* $\alpha \in \mathbb{C}$*; and*
(b) $m(x+y) \leq m(x) + m(y)$ *for all* $x, y \in X$*.*

Example 5. *Let us consider* $l_\infty = \{(x_n) : \sup_n |x_n| < \infty\}$ *as a vector lattice, the map*

$$p : l_\infty \to l_\infty$$
$$x = (x_n) \mapsto p(x) := (y_n)$$

where $y_n = \frac{y_1+y_2+\ldots+y_n}{n}$ *for each n. p is a* $l_\infty-$*valued seminorm.*

The following definition is similar to Definition 1

Definition 7. *We say that a collection* $\mathscr{L} = \{l_\alpha\}_{\alpha\in A}$ *of E-valued seminorms on X is a* lattice multi-norm *if for any* $0 \neq x \in X$ *there is* $l_\alpha \in \mathscr{L}$ *such that* $l_\alpha(x) \neq 0$*. In this case* *we say that* $X = (X, \mathscr{L}, E)$ *is a* lattice multi-normed space (*abbreviated by* LMNS) *with the* E*-valued lattice multi-norm* $\mathscr{L}$.

Example 6. *Let consider the map* $\pi_n : C(\mathbb{R}) \to C_0(\mathbb{R})$ *defined by* $f \mapsto fe_n$ *where*

$$e_n = \begin{cases} \frac{1}{n}x + 1 & x \in [-n, 0] \\ \frac{-1}{n}x + 1 & x \in [0, n] \\ 0 & \text{otherwise} \end{cases}$$

For each $n \in \mathbb{N}$, π_n *is a* $C_0(\mathbb{R})-$*valued seminorm so the triple* $(C(\mathbb{R}), \mathscr{L}, C_0(\mathbb{R}))$ *is a LMNS where* $\mathscr{L}$ *denotes the family of seminorms* $\{\pi_n\}_{n \in \mathbb{N}}$.

3.3.4 Lattice multi-pseudonorms

Definition 8. *We say that a collection* $\mathscr{G} = \{g_\alpha\}_{\alpha \in A}$ *of* E*-valued pseudoseminorms on* X *is a* lattice multi-pseudonorm *if for any* $0 \neq x \in X$ *there is* $g_\alpha \in \mathscr{L}$ *such that* $g_\alpha(x) \neq 0$. *In this case we say that* $X = (X, \mathscr{G}, E)$ *is a* lattice multi-pseudonormed space (*abbreviated by* LMPNS) *with the* E*-valued multi-pseudonorm* $\mathscr{G}$.

Example 7. *Let* X *be a normed space, consider the map* $\pi_n : X^{\mathbb{N}} \to \mathbb{R}$, *given by* $\pi_n(f) = \|f(n)\|$. π_n *is a real-valued pseudoseminorm for each* $n \in \mathbb{N}$, *so* $(X^{\mathbb{N}}, \mathscr{G}, \mathbb{R})$ *is a LMPNS where* $\mathscr{G} = \{\pi_n\}_{n \in \mathbb{N}}$.

3.4 Lattice Multi-Pseudonormed Vector Lattices (MLPNVL)

In the last section of the paper, we apply concepts developed above to the case when X is a vector lattice.

3.4.1 Multi-pseudonormed vector lattices (MPNVL)

Definition 9. *Given a vector lattice* X, *a pseudoseminorm* $r : X \to \mathbb{R}$ *is called a* Riesz pseudoseminorm *whenever* $|x| \leq |y|$ *implies* $r(x) \leq r(y)$.

We say that a collection $\mathscr{R} = \{r_\alpha\}_{\alpha \in A}$ *of Riesz pseudoseminorms on* X *is a* multi-Riesz pseudonorm *if for any* $0 \neq x \in X$ *there is* $r_\alpha \in \mathscr{R}$ *such that* $r_\alpha(x) > 0$.

In this case we say that $X = (X, \mathscr{R})$ is a multi-Riesz pseudonormed space *with the multi-Riesz pseudonorm $\mathscr{R}$.*

Hausdorff locally solid vector lattices (cf. Aliprantis and Burkinshaw (1978)) are exactly multi-pseudonormed vector lattices with multi-pseudonorms consisting of so-called Riesz pseudoseminorms. They have been investigated recently from the point of view of multi-Riesz pseudonorms in Ercan and Vural (2018).

Proposition 3.4.1. *Let $(X, \mathscr{R})$ be multi-Risz pseudonormed space. The collection $\mathscr{R}^u = \{r^u_\alpha\}_{\alpha \in A, u \in X_+}$ of functions defined by*

$$r^u_\alpha(x) := r_\alpha(|x| \wedge u) \quad (x \in X)$$

is a multi-Riesz pseudonorm. Moreover, the $u\tau$-topology is exactly the topology of multi-Riesz pseudonormed space $(X, \mathscr{R}^u)$.

Proof. Let $\alpha \in A$ and $u \in X_+$ be fixed elements. It is easily seen that for any $x \in X$, $r^u_\alpha(x) > 0$, since $r^u_\alpha(x) = r_\alpha(|x| \wedge u) > 0$. For the condition (b);

$$\begin{aligned} r^u_\alpha(x+y) = r_\alpha(|x+y| \wedge u) &\leq r_\alpha((|x| + |y|) \wedge u) \\ &\leq r_\alpha(|x| \wedge u + |y| \wedge u) \\ &\leq r_\alpha(|x| \wedge u) + r_\alpha(|y| \wedge u) \\ &= r^u_\alpha(x) + r^u_\alpha(y) \end{aligned}$$

For the condition (c); let $\{\lambda_n\} \subset \mathbb{C}$ be any sequence such that $\lambda_n \to 0$, the inequality

$$r^u_\alpha(\lambda_n x) = r_\alpha(|\lambda_n x| \wedge u) = r_\alpha(|\lambda_n||x| \wedge u) \leq r_\alpha(|\lambda_n||x|)$$

gives that $\lim_{n\to\infty} r^u_\alpha(\lambda_n x) = 0$. And also for a given $0 \neq x \in X$, there exist $\alpha_0 \in A$ such that $r_\alpha(x) > 0$ and if we choose $u_0 \in X_+$ such that $|x| \leq |u_0|$,then $r^{u_0}_{\alpha_0}(x) = r_{\alpha_0}(|x| \wedge u_0) = r_{\alpha_0}(x) > 0$. □

Theorem 3.4.2. *Let $(X, \mathscr{R})$ be Dedekind complete multi-Riesz pseudonormed space. Then the multi-Riesz pseudonorm $\mathscr{R}^u$ is metrizable iff $\mathscr{R}$ is metrizable and X has a countable orthogonal system.*

Proof. Assume $\mathscr{R}^u$ is metrizable so there must be a countable family $(\alpha_i, u_i)_{i\in I}$ where $\alpha_i \in A, u_i \in X_+$ for each $i \in I$ such that $U_{p^{u_i}_{\alpha_i},\varepsilon_i} = \{x \in X : p^{u_i}_{\alpha_i} \leq \varepsilon_i, 0 < \varepsilon_i \in \mathbb{R}\}$ is the neigbourhood subbase of zero for $(X, \mathscr{R}^u)$. It is easy to see that $U_{p_{\alpha_i},\varepsilon_i} = \{x \in X : p_{\alpha_i} \leq \varepsilon_i, 0 < \varepsilon_i \in \mathbb{R}\}$ is the neigbourhood subase of zero $(X, \mathscr{R}^u)$. 'only if' direction is easily seen.

□

3.4.2 Lattice multi-Riesz pseudonormed vector lattices (LM-RPNVL)

Definition 10. *Let X and E be vector lattices. An E-valued seminorm r is called E-*valued Riesz seminorm *if $r(x) \leq r(y)$ whenever $|x| \leq |y|$.*

Remark 4. *The map in the Example 5 is not $l_\infty-$valued Riesz seminorm since $x = (1, -1, 1, -1, 1, -1...)$ and $x = (1, 0, 1, 0, 1, 0...)$ is in l_∞ and $|y| \leq |x|$ but $p(y) \nleq p(x)$, and also the map in the Example 6 fails to be $C_0(\mathbb{R})-$valued Riesz seminorm. But the map $\pi_n : X^{\mathbb{N}} \to \mathbb{R}$ in the Example 4 is a real-valued Riesz seminorm.*

Definition 11. *Given vector lattices X and E. We say that a collection $\mathscr{L} = \{l_\alpha\}_{\alpha\in A}$ of E-valued Riesz seminorms on X is a* lattice multi-Riesz norm *if for any $0 \neq x \in X$ there is $l_\alpha \in \mathscr{L}$ such that $l_\alpha(x) \neq 0$. In this case we say that $X = (X, \mathscr{L}, E)$ is a* lattice multi-Riesz normed space *with the lattice multi-Riesz norm $\mathscr{L}$.*

Example 8. *Consider the real-valued bounded functions on $[0, 1]$, $\mathbb{P} = \{x_0 = 0, x_1, ..., x_n = 1\}$ be a partition of $[0, 1]$, the map*

$$T : B([0, 1]) \to C([0, 1])$$
$$f \mapsto T(f)$$

given by $T(f) := l_i(x)$ whenever $x \in (x_{i-1}, x_i)$ where $1 \leq i \leq n$ and $l_i(x)$ is the line passing through x_{i-1} and $f(x_i)$.

Definition 12. *Let X and E be vector lattices. An E-valued pseudoseminorm r is called an E-*valued Riesz pseudoseminorm *if $r(x) \leq r(y)$ whenever $|x| \leq |y|$. If additionally $r(x) \neq 0$ for any nonzero vector $x \in X$, we say that r is an E-*valued Riesz pseudonorm.

We say that a collection $\mathscr{R} = \{r_\alpha\}_{\alpha \in A}$ *of E-valued Riesz pseudoseminorms on X is an E-*valued multi-Risz pseudonorm *if for any* $0 \neq x \in X$ *there is* $r_\alpha \in \mathscr{R}$ *such that* $r_\alpha(x) \neq 0$*. In this case we say that* $X = (X, \mathscr{R}, E)$ *is a* lattice multi-Riesz pseudonormed lattice (*abbreviated by* LMRPNVL) *with the E-valued multi-Riesz pseudonorm* $\mathscr{R}$*.*

Proposition 3.4.3. *Every Hausdorff locally solid vector lattice is a LMRPNVL.*

Proof. Let (X, τ) be a locally solid Riesz space, it is known by Fremlin's theorem that τ is generated by a family of Riesz pseudoseminorms $(p_i)_{i \in I}$. And $E : \mathbb{R}^I$ is the vector lattice of all real-valued functions on I . Define a map

$$r : X \longrightarrow E$$
$$x \longmapsto r(x)$$

given by $r(x)[i] = p_i(x)$.Clearly that the map r satisfies the conditions (a),(b),(d)(?) and the monotonicity. we check condition (c);
Suppose (λ_n) is a sequence $\mathbb{R}$ such that $\lim_{n\to\infty} \lambda_n = 0$. Our aim to show that $r(\lambda_n x) \xrightarrow{o} 0$ in E. Note that $\lim_{n\to\infty} \lambda_n = 0$ in $\mathbb{R}$ so it is bounded in $\mathbb{R}$, hence the sequence $(\lambda_n x)$ is order bounded in X. since the sequence $r(\lambda_n x)$ is order bounded in E. so

$$r(\lambda_n x) \xrightarrow{o} 0 \text{ if and only if } r(\lambda_n x) \to 0 \text{ pointwise in } E$$

Let $i \in I$, then $r(\lambda_n x)[i] = p_i(\lambda_n x)$, so $p_i(\lambda_n x) \to 0$ as $n \to \infty$ since p_i is a Riesz Pseudoseminorm. Thus r is lattice-valued Riesz Pseudoseminorm and $(X, \{r\}, E)$ is a LMRPNVL. □

Definition 13. *In an LMRPNVL* $(X, \mathscr{R}, E)$*, A net* (x_α) *is said to be* $\mathscr{R}-$*converges to* $x \in X$ *if* $r_\lambda(x_\alpha - x) \xrightarrow{o} 0$ *in E for each* $r_\lambda \in \mathscr{R}$ *and this convergence abbreviated as* $x_\alpha \xrightarrow{\mathscr{R}} x$*.*

Theorem 3.4.4. *In an LMRPNVL* $(X, \mathscr{R}, E)$*, let consider the nets* $(x_\alpha)_{\alpha \in A}, (y_\beta)_{\beta \in B}$*. If* $x_\alpha \xrightarrow{\mathscr{R}} x$ *and* $y_\alpha \xrightarrow{\mathscr{R}} y$ *then* $(x_\alpha \vee y_\beta)_{(\alpha,\beta) \in (A \times B)} \xrightarrow{\mathscr{R}} x \vee y$*. Moreover,* $x_\alpha \xrightarrow{\mathscr{R}} x$ *implies that* $x_\alpha^- \xrightarrow{\mathscr{R}} x^-$*.*

Proof. Let $\lambda \in \Lambda$ be fixed. If $x_\alpha \xrightarrow{\mathscr{R}} x$ and $y_\beta \xrightarrow{\mathscr{R}} y$ then there exist two net $(z_{\alpha'})_{\alpha' \in A'}$ and $(w_{\beta'})_{\beta' \in B'}$ in E such that $(z_{\alpha'}) \downarrow 0$ and $(w_{\beta'}) \downarrow 0$ and also for a given $(\alpha', \beta') \in$

$A^{'} \times B^{'}$ there are $\alpha_{\alpha'} \in A$ and $\beta_{\beta'} \in B$ such that $r_\lambda(x_\alpha - x) \le z_{\alpha'}$ and $r_\lambda(y_\beta - y) \le w_{\beta'}$ for all $(\alpha, \beta) \ge (\alpha_{\alpha'}, \beta_{\beta'})$. By using the inequality $|a \vee b - a \vee v| \le |b - c|$:

$$\begin{aligned} r_\lambda(x_\alpha \vee y_\beta - x \vee y) &= r_\lambda(|x_\alpha \vee y_\beta - x_\alpha \vee y + x_\alpha \vee y - x \vee y|) \\ &\le r_\lambda(|x_\alpha \vee y_\beta - x_\alpha \vee y| + r_\lambda(|x_\alpha \vee y - x \vee y|) \\ &\le r_\lambda(|y_\beta - y| + r_\lambda(|x_\alpha - x|) \\ &\le w_{\beta'} + z_{\alpha'} \end{aligned}$$

for all $\alpha \ge \alpha_{\alpha'}$ and $\beta \ge \beta_{\beta'}$. Since $(w_{\beta'} + z_{\alpha'}) \downarrow 0$, then $r_\lambda(x_\alpha \vee y_\beta - x \vee y) \xrightarrow{o} 0$ in E. □

Definition 14. *Let $(X, \mathscr{R}, E)$ be a LMRPNVL and $Y \subset X$. Y is called $\mathscr{R}-$closed in X if, for any net (x_α) in Y that $\mathscr{R}-$convergent to $x \in X$, it implies that $x \in Y$.*

Lemma 3.4.5. *The positive cone X_+ is $\mathscr{R}-$closed.*

Proof. Let $\{x_\alpha\} \subset X_+$ and $x_\alpha \xrightarrow{\mathscr{R}} x$, so $r_\lambda(x_\alpha - x)$ goes to zero for any $\lambda \in \Lambda$. By the previous theorem $r_\lambda(x_\alpha)^- - (x)^-$ goes to zero in E, for all α we have $x_\alpha^- = 0$, it follows $r_\lambda(x)^- = 0$ for all $\lambda \in \Lambda$, then $(x)^- = 0$, it means $x \in X_+$. □

Proposition 3.4.6. *Any monotone $\mathscr{R}-$convergent net in an LMRPNVL $(X, \mathscr{R}, E)$ $o-$converges to its $\mathscr{R}-$limit.*

Proof. Let $x_\alpha \uparrow$ be a net in X and $x_\alpha \xrightarrow{\mathscr{R}} x$.Fix arbitrary α, $x_\beta - x_\alpha \in X_+$ for $\beta \ge \alpha$, by Theorem 1 $x_\beta - x_\alpha \xrightarrow{\mathscr{R}} x - x_\alpha$ and by Lemma 1 $x - x_\alpha \in X_+$ so $x \ge x_\alpha$ for any α, so x is an upper bound of $\{x_\alpha\}$ since α is arbitrary.And now let assume $y \ge x_\alpha$ for all α, then again $y - x_\alpha \xrightarrow{\mathscr{R}} y - x$ implies $y - x \in X_+$, then $y \ge x$.Thus $x_\alpha \uparrow x$. □

Definition 15. *Let $(X, \mathscr{R}, E)$ be an LMRPNVL. Then*

i. A net $(x_\alpha)_{\alpha \in A}$ in X is said to be $\mathscr{R}-$Cauchy if the net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha') \in (A \times A)}$ $\mathscr{R}-$converges to zero.

ii. X is called $\mathscr{R}-$complete if every $\mathscr{R}-$Cauchy net in X is $\mathscr{R}-$convergent.

iii. A subset $Y \subset X$ is called $\mathscr{R}-$bounded if there exists $e \in E$ such that $\pi(y) \le e$ for all $y \in Y$.

iv. X is called $o\mathscr{R}-$continuous if $x_\alpha \xrightarrow{o} 0$ implies that $\pi(x_\alpha) \xrightarrow{o} 0$.

v. X is called $\mathscr{R}-$KB-space if every $\mathscr{R}-$bounded increasing net in X_+ is $\mathscr{R}-$convergent.

Theorem 3.4.7. *For a $\mathscr{R}-$complete LMRPNVL $(X,\mathscr{R},E)$,TFAE.*

1. *X is $o\mathscr{R}-$continuous*

2. *If $0 \leq x_\alpha \uparrow \leq x$ holds in X, then (x_α) is $\mathscr{R}-$Cauchy.*

3. *$x_\alpha \downarrow 0$ in X implies $r_\lambda(x_\alpha) \downarrow 0$ for each $r_\lambda \in \mathscr{R}$.*

Proof. $(i) \Rightarrow (ii)$ Let $0 \leq x_\alpha \uparrow \leq x$ in X, by [2,lemma 4.8] there exists a net in X such that $(y_\beta - x_\alpha)_{\alpha,\beta} \downarrow 0$ so X is $o\mathscr{R}-$continuous then $r_\lambda(y_\beta - x_\alpha) \to 0$ for any $r_\lambda \in \mathscr{R}$ and so $r_\lambda(y_{\beta_k} - x_\alpha) \to 0$ where $y_{\beta_k} = y_\beta$, hence x_α is $\mathscr{R}-$Cauchy.
$(ii) \Rightarrow (iii)$ Assume that $x_\alpha \downarrow 0$ in X. Fix arbitrary α_0, for $\alpha \leq \alpha_0$, $x_\alpha \leq x_{\alpha_0}$, and $(x_\alpha - x_{\alpha_0})_{\alpha \leq \alpha_0} \uparrow \leq x_{\alpha_0}$, by the assumption the net $(x_\alpha - x_{\alpha_0})$ is $\mathscr{R}-$Cauchy since X is $\mathscr{R}-$complete then there exists $x \in X$ such that $(x_\alpha - x) \xrightarrow{\mathscr{R}} 0$ as $\alpha_0 < \alpha \to \infty$,so by proposition 2.5., $x_\alpha \downarrow x$ and hence $x = 0$. As a result $x_\alpha \xrightarrow{\mathscr{R}} 0$ and by the monotonicity of r_λ, $r_\lambda x_\alpha \downarrow 0$.
$(iii) \Rightarrow (i)$ Let $x_\alpha \to 0$, then there exists a net $z_\beta \downarrow 0$ such that, for any β there exist α_β so that $|x_\alpha| \leq z_\beta$ for all $\alpha \geq \alpha_\beta$. Hence $\pi(x_\alpha) \leq \pi(z_\beta)$ for all $\alpha \geq \alpha_\beta$, by (ii), $\pi(z_\beta) \downarrow 0$ therefore $\pi(x_\alpha) \xrightarrow{o} 0$ or $x_\alpha \xrightarrow{\pi} 0$. □

Theorem 3.4.8. *Let $(X,\mathscr{R},E)$ be an $o\mathscr{R}-$continuous and $\mathscr{R}-$complete LMRPNVL, then X is order complete.*

Proof. Assume $0 \leq x_\alpha \uparrow \leq u$, then by Theorem 2 x_α is a $\mathscr{R}-$Cauchy net, then there exists an $x \in X$ such that $x_\alpha \xrightarrow{\mathscr{R}} x$ by the $\mathscr{R}-$completeness of X.And from the Proposition 3 $x_\alpha \uparrow x$,this completes the proof. □

Theorem 3.4.9. *If a LMRPNVL $(X,\mathscr{R},E)$ is $\mathscr{R}-$KB-space then it is $o\mathscr{R}-$continuous.*

Proof. Let we assume $x_\alpha \downarrow 0$, we define $y_\alpha := x_{\alpha_0} - x_\alpha (\alpha \geq \alpha_0)$ for a fixed α_0. y_α is an increasing net and it is $\mathscr{R}-$bounded, so by the assumption there exists $y \in X$ such that $y_\alpha \xrightarrow{\mathscr{R}} y$, then by proposition 3 $y_\alpha \xrightarrow{o} y$ in X. Actually $y_\alpha \uparrow y$, then $y = \sup_{\alpha \geq \alpha_0} y_\alpha = \sup_{\alpha \geq \alpha_0}(x_{\alpha_0} - x_\alpha) = x_{\alpha_0}$, so $x_\alpha \downarrow_{\mathscr{R}} 0$. Hence by theorem 2 X is $o\mathscr{R}-$continuous. □

Theorem 3.4.10. *If a LMRPNVL $(X,\mathscr{R},E)$ is $\mathscr{R}-$KB-space then X is order complete.*

Proof. Let we assume $0 \leq x_\alpha \uparrow \leq y \in X$, then $r_\lambda(x_\alpha) \leq r_\lambda(y)$ for all $r_\lambda \in \mathscr{R}$, then by the assumption there exists $x \in X$ such that $x_\alpha \xrightarrow{\mathscr{R}} x$, then by proposition 3 $x_\alpha \xrightarrow{o} x$. □

Theorem 3.4.11. *Let $(X,\mathscr{R},E)$ be a $\mathscr{R}-$KB-space and $Y \subseteq X$ be a order closed sublattice, then $(Y,\mathscr{R},E)$ is also $\mathscr{R}-$KB-space.*

Proof. Let we assume $y_\alpha \uparrow$ be a $\mathscr{R}-$bounded net in Y_+, then X is a $\mathscr{R}-$KB-space, there exists $x \in X$ such that $y_\alpha \xrightarrow{\mathscr{R}} x$. By proposition 3 $y_\alpha \uparrow x$, so by the closedness of Y; $x \in Y$. Therefore $(Y,\mathscr{R},E)$ is a $\mathscr{R}-$KB-space. □

In a LMRPNVL $X,\mathscr{R},E$, the elements $x,y \in X$ is said to be $\mathscr{R}-$disjoint if $r_\lambda(x) \perp r_\lambda(y)$ for all $r_\lambda \in \mathscr{R}$ and a subset B of X is said to be $\mathscr{R}-$band if for some non-empty subset $M \subseteq X$, $B = \{x \in X : m \perp_{\mathscr{R}} x \;\; \forall m \in M\}$. Neither a $\mathscr{R}-$band is a band nor a band is a $\mathscr{R}-$band in general, see Aydin et al (2017) for examples. But the property in the following definition guarantee that every $\mathscr{R}-$band is a band in X.

Definition 16. *Let $(X,\mathscr{R},E)$ be an LMRPNVL, $(X,\mathscr{R},E)$ is called $\mathscr{R}-$fatou space if $0 \leq x_\alpha \uparrow x$ in X implies $x_\alpha \uparrow_{\mathscr{R}} x$*

Proposition 3.4.12. *In a $\mathscr{R}-$fatou space, every $\mathscr{R}-$band is a band.*

Proof. Let $B \subseteq X$ be a $\mathscr{R}-$band, so by the definition there exists a subset M of X such that $B = \{x \in X : m \perp_{\mathscr{R}} x \;\; \forall m \in M\}$. It is easy to see that B is an order ideal. Let we assume $0 \leq x_\alpha \uparrow x$ in X where $\{x_\alpha\} \subseteq B$, by the assumption $x_\alpha \uparrow_{\mathscr{R}} x$, so by theorem 1 ; $r_\lambda(x) \wedge r_\lambda(m) = 0$ for all $m \in M$, hence B is a band. □

Definition 17. *Let $(X,\mathscr{R},E)$ be a LMRPNVL and $A \subseteq X$. A subset $B \subseteq A$ is said to be $\mathscr{R}-$dense in A if for a fixed $r_\lambda \in \mathscr{R}$, for every $a \in A$ and for any $0 \neq u \in r_\lambda(X)$ there is $b \in B$ such that $r_\lambda(a-b) \leq u$.*

Remark 5. *Let consider the set of all polynomials defined on $[0,1]$, denoted by $P([0,1])$ as a vector space. $P([0,1])$ is $C([0,1])-$dense with respect to the map*

$$p : P([0,1] \to C([0,1]$$
$$f \mapsto p(f)$$

given by

$$p(f)=\begin{cases}|f| & f\in P([0,1])\\ 0 & \text{otherwise}\end{cases}$$

But it is not $C([0,1])-$dense with respect to the map, given in example 3, since if we choose $x_0=\frac{1}{2}$ and $\varepsilon=\frac{1}{10^5}$ and

$$f(x)=\begin{cases}1 & x=0\\ 2 & x=\sum_{n=1}^{k}\frac{1}{4n^2} \text{ for some } k\\ l_f(x) & \text{otherwise}\end{cases}$$

l_f denotes the lines passing through $(x,1)$ and $(x,2)$. We cannot find a polynomial p such that $p(f-p_{x_0,\varepsilon})\leq g$. On the other hand the set $\{(x_n)\in c_0 : x_n=(n,n,n,n,0,0,0...)\}$ (first $n-1$ term is equal to n). This set is not dense in c_0 but it is $c_{00}-$dense in c_0.

Definition 18. *Let $(X,\mathscr{R},E)$ be a LMRPNVL, a vector $e\in X$ is called $\mathscr{R}$-unit if, for any $x\in X_+$ we have $\pi(x-x\wedge ne)\xrightarrow{o}0$ in E.*

Remark 6. *Let $(X,\mathscr{R},E)$ be a LMRPNVL,*

i. $\mathscr{R}-$unit need not to be positive, consider the example in Remark 1; the function

$$f(x)=\begin{cases}1 & x=x_0\\ -1 & x\neq x_0\end{cases}$$

is a $\mathscr{R}-$unit since for any $g\in B([a,b])^+$, $p(g-g\wedge nf)$ is equal to $p(h)$ where

$$h(x)=\begin{cases}0 & x=x_0\\ g(x)-nf(x) & x\neq x_0\end{cases}$$

so $p(h) = 0$.

ii. *If* e *is an* $\mathscr{R}-$*unit and* $0 < \alpha \in \mathbb{R}$, *then* αe *is also a* $\mathscr{R}-unit$. *Since for a fixed* $r_\lambda \in \mathscr{R}$, *and for any* $x \in X_+$

$$r_\lambda(x - x \wedge n\alpha e) = r_\lambda(\alpha(\frac{x}{\alpha} - \frac{x}{\alpha} \wedge ne)) \leq \overline{[\alpha]} r_\lambda(\frac{x}{\alpha} - \frac{x}{\alpha} \wedge ne) \to 0$$

where $\overline{[\alpha]} = \min\{m \in \mathbb{Z} : \alpha \leq m\}$

iii. *If* e_1 *and* e_2 *are* $\mathscr{R}-$*units, then* $e_1 + e_2$ *is also a* $\mathscr{R}-$*unit.*

iv. *If* e *is a* $\mathscr{R}-$*unit, then* e *need not to be a weak unit in* X. *In example 4, the sequence* $(1,1,1,0,...)$ *is an* $\mathscr{R}-$*unit, but* $(0,0,0,1,0,...) \neq 0$ *and* $(1,1,1,0,...) \wedge (0,0,0,1,0,...) = 0$.

v. *If* $e \in X$ *is a strong unit, then* e *is a* $\mathscr{R}$*-unit. Since For any* $x \in X_+$ *there exists a* $k \in \mathbb{N}$ *such that* $x \leq ke$, *Let* $r_\lambda \in \mathscr{R}$ *be fixed, then* e *is a strong unit, so* $r_\lambda(x - x \wedge ne) = r_\lambda(x - x) = r_\lambda(0) = 0$ *for all* $n \leq k$.

vi. *If* X *is* $o\mathscr{R}$*-continuous , then every weak unit of* X *is* $\mathscr{R}$*-unit. Since Let* e *be a weak unit. For each* $x \in X_+$, $x \wedge ne \uparrow x$ *so it means* $x - x \wedge ne \downarrow 0$, *from the* $o\mathscr{R}$*-continuity* $r_\lambda(x - x \wedge ne) \to 0$ *for any* $r_\lambda \in \mathscr{R}$.

Proposition 3.4.13. *Let* $(X, \mathscr{R}, E)$ *be a LMRPNVL and* $e \in X_+$. *If the ideal generated by* e *in* X *is* $\mathscr{R}-$*dense in* X *then* e *is* $\mathscr{R}-$*unit.*

Proof. Firstly let we assume $r_\lambda(0) = u > 0$ for some $r_\lambda \in \mathscr{R}$. For $0 \in X$ and $0 \neq u \in r_\lambda(X)$, there must be $b \in X$ such that $r_\lambda(-b) \leq r_\lambda(0)$ but we know that $r_\lambda(0) \leq r_\lambda(-b)$, hence $r_\lambda(-b) = r_\lambda(0)$, moreover for each $x \in [-|b|, |b|]$, $r_\lambda(x) = u$, and $r_{\lambda}{}_{\frac{1}{n}|b|} \to 0$ but it is impossible unless $u = 0$. Then $r_\lambda(x - x \wedge ne)$ goes to zero for each $x \in X_+$. $\square$

3.4.3 Mixed LMRPNVLs

Let $(X, \mathscr{R}, E)$ be a LMRPNVL and $(E, \|.\|)$ be a normed lattice, then $\mathscr{R}^* = \{\|r_\lambda\|\}_{\lambda \in \Lambda}$ is also an $\mathbb{R}-$valued pseudoseminorm on X, and $(X, \mathscr{R}^*, E)$ is called mixed-LMRPNVL

Remark 7. *Let* $(X, \mathscr{R}^*, \mathbb{R})$ *be a mixed-LMRPNVL:*

i. If $(X,\mathscr{R},E)$ is $o\mathscr{R}-$continuous and $\|.\|$ is order continuous then $(X,\mathscr{R}^*,\mathbb{R})$ is $o\mathscr{R}^*-$continuous.Since $x_\alpha \xrightarrow{o} 0$ implies $r_\alpha(x_\alpha) \xrightarrow{o} 0$ implies $\|r_\alpha(x_\alpha)\| \to 0$.

ii. If a subset Y of X is $\mathscr{R}-$bounded in $(X,\mathscr{R},E)$, then Y is $\mathscr{R}^*-$bounded in $(X,\mathscr{R}^*,\mathbb{R})$. Let $r_\lambda \in \mathscr{R}$ be fixed so there exists $e \in E$ such that $r_\lambda(y) \leq e$ for all $y \in Y$ so by the monotonicity of the norm; $\|r_\lambda(y)\| \leq \|e\|$.

iii. If $Y \subseteq X$ is $\mathscr{R}-$dense in $(X,\mathscr{R},E)$ then Y is $\mathscr{R}^*-$dense in $(X,\mathscr{R}^*,\mathbb{R})$. Since Let $r_\lambda \in \mathscr{R}$ be fixed. $a \in X$ and $0 \neq u \in r_\lambda(X)$ be given, we know that there exists $y \in Y$ such that $r_\lambda(a-y) \leq u$, so by the monotonicity of the norm $\|r_\lambda(a-y)\| \leq \|u\|$.

iv. If $(X,\mathscr{R},E)$ is $\mathscr{R}-$fatou space and $\|.\|$ is order continuous then $(X,\mathscr{R}^*,\mathbb{R})$ is also $\mathscr{R}^*-$fatou space. Since $0 \leq x_\alpha \uparrow x$ implies $x_\alpha \uparrow_{\mathscr{R}} x$, it means $r_\lambda(x_\alpha - x) \downarrow 0$ for any $r_\lambda \in \mathscr{R}$ so by the order continuity $\|r_\lambda(x_\alpha - x)\| \downarrow 0$.

v. If $(X,\mathscr{R}_1,E)$ and $(E,\mathscr{R}_2,F)$ be two $\mathscr{R}_1-$KB and $\mathscr{R}_2-$KB spaces, then $\mathscr{R}_2 \circ \mathscr{R}_1$ is a $F-$valued pseudoseminorm on X and $(X,\mathscr{R}_2 \circ \mathscr{R}_1,F)$ is $\mathscr{R}-$KB-space. Since let we assume $\{x_\alpha\}$ be an $\mathscr{R}_2 \circ \mathscr{R}_1 = \mathscr{R}-$bounded increasing net, so for a fixed $r_\lambda \in \mathscr{R}_2$ and $r_\beta \in \mathscr{R}_1$, there exists $f \in F_+$ such that $r_\lambda(r_\beta(x_\alpha)) \leq f$ for all α, $\mathscr{R}_2$ is an KB-space so ...As a consequences of this remark it is easily seen that if $(E,\|.\|)$ is a KB-space and $(X,\mathscr{R},E)$ is a $\mathscr{R}-$KB-space then $(X,\mathscr{R}^*,\mathbb{R})$ is a $\mathscr{R}^*-$KB-space.

3.4.4 $\mathscr{R}-$continuous operators on LMRPNVLs

An operator $T: E \to F$ between two Riesz spaces is said to be order continuous if $x_\alpha \xrightarrow{o} 0$ in E implies $T(x_\alpha) \xrightarrow{o} 0$ in F. We refer to Aliprantis and Burkinshaw (1985) and Aliprantis and Burkinshaw (1978) for the basic properties of the class of order continuous operators. Recently in Bahramnezhad and Azar (2017) , Bahramnezhad and Azar (2017) and Aydin (2018) new classes of operators defined with different type order continuity on operators. In this section we generalize these classes of operators.

Definition 19. *Let $X_1 = (X_1,\mathscr{R}_1,E_1)$ and $X_2 = (X_2,\mathscr{R}_2,E_2)$ be two lattice multi-Riesz pseudonormed lattice. A positive operator $T: X_1 \to X_2$ is said to be $\mathscr{R}-$continuous* *operator if* $x_\alpha \xrightarrow{\mathscr{R}_1} 0$ *implies* $T(x_\alpha) \xrightarrow{\mathscr{R}_2} 0$.

Example 9. *Let* $T : X \to E$ *be any order continuous positive operator between two Riesz spaces,so* $T : (X, |.|, X) \to (E, |.|E)$ *is* $\mathscr{R}-$*continuous operator.*

Example 10. *Let* $T : X \to E$ *be any unbounded order continuous positive operator between two Riesz spaces,so* $T : (X, \{r_u\}_{u\in X_+}, X) \to (E, \{l_v\}_{v\in E_+}, E)$ *is* $\mathscr{R}-$*continuous operator where* $r_u(x) := |x| \wedge u$ *and* $l_u(x) := |x| \wedge v$.

Example 11. *Let* $T : X \to E$ *be any strongly unbounded order continuous positive operator between two Riesz spaces,so* $T : (X, |.|, X) \to (E, \{l_v\}_{v\in E_+}, E)$ *is* $\mathscr{R}-$*continuous operator*

Example 12. *A lattice valued locally solid Riesz space (LNLS) is a triple* (X, p, E) *where X is a vector lattice, p is a* $E-$*valued vector norm and E be a locally solid Riesz space. In a LNLS, a net* x_α *is said to be* $p_\tau-$*converges to x if* $p(x_\alpha - x) \xrightarrow{\tau} 0$, *and an operator between two LNLS* $T : X_1 \to X_2$ *is said to be* $p_\tau-$*continuous if* $x_\alpha \xrightarrow{p_\tau} 0$ *in* X_1 *implies* $T(x_\alpha) \xrightarrow{p_\tau} 0$ *in* X_2. *Now consider the family of real-valued Riesz pseudonorms* $\mathscr{R} := (r_i \circ p)_{i\in I}$ *where* $(r_i)_{i\in I}$ *be the family of Riesz pseudonorms which generates the locally solid topology.* $\mathscr{R}-$*continuity coincides with* $p_\tau-$*continuity.*

Proposition 3.4.14. *If a positive* $\mathscr{R}-$*continuous operator* $T : X_1 \to X_2$ *dominates* $S : X_1 \to X_2$, *then S is also* $\mathscr{R}-$*continuous operator.*

Proof. Let $(x_\alpha) \xrightarrow{\mathscr{R}_1} 0$, and we know that $r^2_\lambda(T(x_\alpha)) \xrightarrow{o} 0$ in E_2 for each $r^2_\lambda \in \mathscr{R}_2$, by the monotonicity of r^2_λ, $r^2_\lambda(S(x_\alpha)) \leq r^2_\lambda(T(x_\alpha))$, hence $S(x_\alpha) \xrightarrow{\mathscr{R}_2} 0$. □

Theorem 3.4.15. *Let* $(X,\mathscr{R},E)$ *be an LMRPNVL, and f be an order bounded linear functional on* $(X,\mathscr{R},E)$, *the following statements are equivalent:*

i. f *is* $\mathscr{R}-$*continuous*

ii. f^+ *and* f^- *are both* $\mathscr{R}-$*continuous*

iii. $|f|$ *is* $\mathscr{R}-$*continuous*

Proof. $(i) \Longrightarrow (ii)$ Let (x_α) be a net in X_+ such that $x_\alpha \xrightarrow{\mathscr{R}} 0$, $f^+ = \sup\{f(y) : 0 \leq y \leq x\}$ so we can choose a net y_α such that $0 \leq y_\alpha \leq x_\alpha$ and $f^+(x_\alpha) - \lambda_\alpha \leq f(y_\alpha)$ where

$\lambda_\alpha \downarrow 0$ be a net in $\mathbb{R}$. Hence we have that $y_\alpha \xrightarrow{\mathscr{R}} 0$, then by the assumption $f(y_\alpha) \to 0$ in $\mathbb{R}$,So by the inequality $f^+(x_\alpha) \leq \lambda_\alpha + f(y_\alpha)$, we get $f^+(x_\alpha) \to 0$. Therefore f^+ is $\mathscr{R}-$continuous and $f^- = (-f)^+$ is also $\mathscr{R}-$continuous. $(ii) \implies (iii)$ It is clear that $|f|$ is $\mathscr{R}-$continuous since $|f| = f^+ + f^-.(iii) \implies (i)$ $|f|$ dominates f,then follows from Proposition 4, f is $\mathscr{R}-$continuous. □

Theorem 3.4.16. *Let $(X,\mathscr{R},E)$ be an LMRPNVL, then $L_{\mathscr{R}}(X,\mathbb{R})$ is a band of E .*

Proof. It is easily seen that $L_{\mathscr{R}}(X,\mathbb{R})$ is an ideal as a consequence of Theorem 3. Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net in $L_{\mathscr{R}}(X,\mathbb{R})$ such that $0 \leq f_\lambda \uparrow f$ in E . Let assume $0 \leq x_\alpha \xrightarrow{\mathscr{R}} 0$ in X, then for any $\lambda \in \Lambda$ we have $0 \leq f(x_\alpha) = (f - f_\lambda)(x_\alpha) + f_\lambda(x_\alpha)$,then $f_\lambda(x_\alpha) \to 0$ and $(f - f_\lambda)(x_\alpha) \to 0$ since $f - f_\lambda \downarrow 0$. □

Proposition 3.4.17. *Let $X_1 = (X_1,\mathscr{R}_1,E_1)$ and $X_2 = (X_2,\mathscr{R}_2,E_2)$ be two lattice multi-Riesz pseudonormed lattice,then we have the following:*

i. If $0 \leq T \in L_{\mathscr{R}}(X_1,X_2)$ and X_1 is $O\mathscr{R}-$continuous then $x_\alpha \downarrow 0$ in X_1 implies $T(x_\alpha) \downarrow 0$ in X_2.

ii. If $T : x_1 \to X_2$ is an onto lattice homomorphism and $x_\alpha \downarrow 0$ in X_1 implies $T(x_\alpha) \downarrow 0$ in X_2, then $T \in L_{\mathscr{R}}(X_1,X_2)$

Proof. i. Assume $0 \leq T \in L_{\mathscr{R}}(X_1,X_2)$ and $x_\alpha \downarrow 0$ in X_1, X_1 is $O\mathscr{R}-$ continuous then $x_\alpha \downarrow_{\mathscr{R}_1} 0$, by the assumption $T(x_\alpha) \downarrow_{\mathscr{R}_2} 0$ and by proposition 3 $T(x_\alpha) \downarrow 0$.

ii. It is trivial.

□

4. TOWARDS A THEORY OF UNBOUNDED LOCALLY SOLID RIESZ SPACES

4.1 Introduction

Recall that a linear topology τ on E is called locally solid if it has a neighborhood system at zero consisting of solid sets. One can easily show that given a set P of Riesz pseudoseminorms [1] defines a solid topology with a subbase of zero which is $\{p^{-1}(-\varepsilon,\varepsilon) : p \in P, \varepsilon > 0\}$. This topology is denoted by $< P >$, and it is called locally solid topology generated by P. Conversely, Fremlin's Theorem says that every locally solid topology is a generated by a family of Riesz pseudoseminorm. That is, a linear topology τ is locally solid if and only if $\tau =< P >$ for some set P of Riesz pseudoseminorms.(see Fremlin (1974))

Theorem 4.1.1. *Let $(E, ||.||)$ be a normed vector lattice. For any $u \in E^+$, the map $P_u : E \longrightarrow \mathbb{R}^+$ defined by $P_u(x) = |||x| \wedge u||$, is a Riesz pseudoseminorm. Moreover, the un-topology and the topology which is generated by the family $(P_u)_{u\in E^+}$ coincide.*

Proof. Let $u \in E^+$ be given. Obviously, the conditions (1),(2) and (5) hold. For condition (3): Let $x,\ y \in E$ be given. Since $|x+y| \leq |x| + |y|$, we have $|x+y| \wedge u \leq (|x| + |y|) \wedge u \leq |x| \wedge u + |y| \wedge u$ and since $||.||$ is a lattice norm, we get the inequality $P_u(x+y) \leq P_u(x) + P_u(y)$ by the monotonicity and the triangle inequality properties of lattice norm. For condition(4): Let $\{\lambda_n\} \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} \lambda_n = 0$ and $x \in E$, the inequality

$$P_u(\lambda_n x) = |||\lambda_n x| \wedge u|| \leq ||\lambda_n x|| = |\lambda_n|||x||$$

implies that $\lim_{n\to\infty} P_u(\lambda_n x) = 0$. Hence P_u is a Riesz pseudoseminorm.

Let (x_α) be a net converging to x in *un*-topology, that is, $||\ |x_\alpha - x| \wedge u\ || \to 0$ for each $u \in E^+$. By definition, $P_u(x_\alpha - x)$ converges to zero for each $u \in E^+$, so it converges to x in the topology generated by the family $(P_u)_{u\in E^+}$. Converse direction is also true. This completes the proof. □

[1] For the convenience of the definition of $E-$valued Riesz pseudoseminorm in the previous chapter, the term 'Riesz pseudoseminorm' will used instead of 'Riesz pseudonorm'.

This theorem shows that $un-$topology is locally solid. Following theorem is on a short characterization of Riesz pseudonorm.

Theorem 4.1.2. *Let E be a Riesz space and $p : E \to \mathbb{R}$ be a map. The followings are equivalent:*

i. p is a Riesz pseudoseminorm;

ii. $p(x) = p(|x|)$ for all $x \in E$ and for each $u \in E^+$, the map $p_u : E \to \mathbb{R}$, defined by $p_u(x) = p(|x| \wedge u)$, is a Riesz pseudoseminorm.

Proof. If (i) holds, following the proof of Theorem(2.1), we can get (ii). Suppose that (ii) holds. Since $p(x) = p(|x|) = p_{|x|}(x) \geq 0$, it is obvious that $p(x) = p(|x|) = p_{|x|}(x) = 0$ whenever $x = 0$. Let $x, y \in E$ be given. Then

$$\begin{aligned} p(x+y) =& p(|x+y| \wedge (|x|+|y|) \\ =& p_{|x|+|y|}(|x+y|) \\ \leq& p_{|x|+|y|}(|x|) + p_{|x|+|y|}(|y|) \\ =& p(|x| \wedge (|x|+|y|)) + p(|x| \wedge (|x|+|y|) \\ =& p(|x|) + p(|y|) \end{aligned}$$

so that p satisfies the triangle inequality. Let $x \in E$ be given. Then

$$\lim_{n\to\infty} p(\lambda_n x) = \lim_{n\to\infty} p(|\lambda_n x|) = \lim_{n\to\infty} p_{|x|}(|\lambda_n||x|) = 0.$$

If $|x| \leq |y|$ then

$$p(x) = p(|x|) = p(|x| \wedge |y|) = p_{|y|}(|x|) = p_{|y|}(x) \leq p_{|y|}(y) = p(|y|) = p(y).$$

This completes the proof. □

4.2 The main definition and its motivation

Let p be a Riesz pseudoseminorm on E. For each $u \in E_+$, the map $p_u : E \to \mathbb{R}$ is also a Riesz pseudoseminorm defined by $p_u(x) = p(|x| \wedge u)$. Let (E, τ) be a locally solid Risz space. So there exists a family of Riesz pseudoseminorms $(p_i)_{i \in I}$

such that $\tau =< (p_i)_{i\in I} >$. For any $A \subset E_+$, there exists a different family of Riesz pseudoseminorms $(p_{i,a})_{i\in I, a\in A}$ where $p_{i,a}(x) = p(|x|\wedge a)$ for each $i \in I$ and $a \in A$. This related family defines a locally solid topology. This fact coincides with the Mitchell A. Taylor's definition of "unbounded $\tau-$convergence with respect to A" in Taylor (2017). Here is the Mitchell A. Taylor's definition.

Definition 4.2.1. *Let X be a vector lattice, $A \subseteq X$ be an ideal and τ be a locally solid topology on A. Let (x_α) be a net in X and $x \in X$. We say that (x_α) unbounded τ-converges to x with respect to A if $|(x_\alpha) - x| \wedge |a| \xrightarrow{\tau}$ for all $a \in A_+$.*

In Taylor (2017), the topology corresponding to the convergence in the above definition is denoted by $u_A\tau$.

Observation:

Let E be a Riesz space, and $p : E \to \mathbb{R}$ be a Riesz pseudoseminorm. For a given nonempty set $A \subset E_+$, consider the map $p_A : E \to \mathbb{R}$ defined by

$$p_A(x) = \sup_{a\in A} p(|x|\wedge a)$$

It is obvious that the map p_A satisfies the conditions $(1)-(3)$ and (5), we must check condition (4): Let $\{\lambda_n\} \subset \mathbb{R}$ be any sequence converging to zero. Then

$$\begin{aligned} p_A(\lambda_n x) = \sup_{a\in A} p(|\lambda_n x|\wedge a) &= \sup_{a\in A} p(|\lambda_n||x|\wedge a) \\ &\leq \sup_{a\in A} p(|\lambda_n||x|) \\ &= p(|\lambda_n||x|) \longrightarrow 0 \end{aligned}$$

so that p_A is a Riesz pseudoseminorm.

Let $P = (p_i)_{i\in I}$ be a family of Riesz pseudoseminorms and $\mathscr{A} \subset \mathscr{P}(E_+)$ that does not contain the empty set. This family generates a topology, say τ. The locally solid topology which is generated by the family $\{p_{i,A} : i \in I, A \in \mathscr{A}\}$ will be denoted by $u < \tau, \mathscr{A} >$. Actually, if $\mathscr{A}$ contains the empty set, then $u < \tau, \mathscr{A} >$ is nothing but a discrete topology. For any $\{A\} \in \mathscr{P}(E_+)$, $u < \tau, \{A\} >\neq u_A\tau$, but $u < \tau, \bigcup A >= u_A\tau$ holds. Moreover, $u < \tau, \{A\} >\subset u_A\tau$. As an example let consider $\mathbb{R}^2$ with Euclidean norm,and take the set of non-negative part of $x-$axis as A, then the sequence (x_n) $(x_n := 2+\sin n)$ does not converges in $u_A\tau$,but converges in $u<\tau,\{A\}>$.

Some remarks:

Let (E,τ) be a locally solid Riesz space, $(p_i)_{i\in I}$ be the family of Riesz pseudoseminorms such that $\tau =<(p_i)_{i\in I}>$. Then

(1) For any $\mathscr{A}\subset\mathscr{P}(E_+)$, $u<\tau,\mathscr{A}>\subset\tau$ holds.
Proof: $x_\alpha\xrightarrow{\tau} x \Longleftrightarrow p_i(x_\alpha - x)\to 0 \Longleftrightarrow p_i(|x_\alpha - x|)\to 0$, and for each $a\in E_+$ we have $p_i(|x_\alpha - x|\wedge a)\le p_i(|x_\alpha - x|)$, hence

$$\sup_{a\in A} p_i(|x_\alpha - x|\wedge a)\le p_i(|x_\alpha - x|) \text{ for each } A\in\mathscr{A}.$$

So $x_\alpha \xrightarrow{u<\tau,\mathscr{A}>} x$.

(2) If $\mathscr{A}=\{\{E_+\}\}$, then $u<\tau,\{\{E_+\}\}>=\tau$.
Proof: It is clear that $\sup_{a\in E_+} p_i(|x|\wedge a)=p_i(|x|)=p_i(x)$.

(3) If $\mathscr{A}\subset\mathscr{B}$ then $u<\tau,\mathscr{A}>\subset u<\tau,\mathscr{B}>$ for all $\mathscr{A}\subset\mathscr{B}\subset\mathscr{P}(E_+)$.
Proof: $x_\alpha\xrightarrow{u<\tau,\mathscr{B}>} x\Longleftrightarrow \sup_{b\in B} p_i(|x_\alpha - x|\wedge a)\to 0$ for each $B\in\mathscr{B}$

$$\Longrightarrow \sup_{a\in A} p_i(|x_\alpha - x|\wedge a)\to 0 \text{ for each } A\in\mathscr{A}\subset\mathscr{B}.$$

Hence, $x_\alpha\xrightarrow{u<\tau,\mathscr{A}>} x$.

(4) For each $\mathscr{A}\subset\mathscr{P}(E_+)$ $u<\tau,\bigcup\mathscr{A}>\subset u<\tau,\mathscr{A}>$ holds.
Proof: Let $x_\alpha\xrightarrow{u<\tau,\mathscr{A}>} x$. So for a fixed $i\in I$ and $A\in\mathscr{A}$, $p_{i,A}(x_\alpha - x)\to 0\Longleftrightarrow$ $\sup_{a\in A} p_i(|x_\alpha - x|\wedge a)\to 0$, and it is obvious that

$$p_i(|x_\alpha - x|\wedge a)\le \sup_{a\in A} p_i(|x_\alpha - x|\wedge a) \text{ for each } a\in A.$$

Hence, $p_{i,\{a\}}(x_\alpha - x)=p_{i,\{a\}}(|x_\alpha - x|)=\sup_{a\in\{a\}} p_i(|x_\alpha - x|\wedge a)=p_i(|x_\alpha - x|\wedge a)\to 0$.

(5) For each $\mathscr{A} \subset \mathscr{P}(E_+)$ $u<\tau,\bigcup\mathscr{A}>=u<\tau,I(\bigcup\mathscr{A})>$ holds where $I(\bigcup\mathscr{A})$ is the ideal generated by $\bigcup\mathscr{A}$.

Proof: Since $\bigcup\mathscr{A} \subset I(\bigcup\mathscr{A})$, we have $u<\tau,\bigcup\mathscr{A}>\subset u<\tau,I(\bigcup\mathscr{A})>$ from (3). Let $x_\alpha \xrightarrow{u<\tau,\mathscr{A}>} x$ and $b \in I(\bigcup\mathscr{A})_+$ be given, there exists $a_1,...,a_n \in \bigcup\mathscr{A}$ and $k \geq 0$ such that $0 \leq b \leq k(a_1+...+a_n)$. Then

$$|x_\alpha - x| \wedge b \leq |x_\alpha - x| \wedge k(a_1+...+a_n) \leq \sum_{i=1}^n |x_\alpha - x| \wedge ka_i,$$

$$= k\sum_{i=1}^n \frac{1}{k}|x_\alpha - x| \wedge a_i$$

$$\leq km\sum_{i=1}^n |x_\alpha - x| \wedge a_i$$

where m is the smallest positive integer greater than $\frac{1}{k}$. Then by the monotonicity of p_i,

$$p_i(|x_\alpha - x| \wedge b) \leq p_i(km\sum_{i=1}^n |x_\alpha - x| \wedge a_i) \to 0.$$

Hence, $p_i(|x_\alpha - x| \wedge b) = \sup_{b\in\{b\}} p_i(|x_\alpha - x| \wedge b)$. This completes the proof.

(6) For each $\mathscr{A} \subset \mathscr{P}(E_+)$ $u<\tau,\bigcup\mathscr{A}>=u<\tau,\overline{\bigcup\mathscr{A}}>$ holds.

Proof: Suppose that $x_\alpha \xrightarrow{u<\tau,\mathscr{A}>} x$ and $b \in (\overline{\bigcup\mathscr{A}})_+$ be given. Choose a net $(b_\beta) \in \bigcup\mathscr{A}$ with $b_\beta \xrightarrow{u<\tau,\mathscr{A}>} b$. Let $i \in I$ be fixed and $\varepsilon \geq 0$ be given. Choose β_0 such that $p_i(b_{\beta_0} - b) < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}|x_\alpha - x| \wedge b &= |x_\alpha - x| \wedge (b - b_{\beta_0} + b_{\beta_0})\\ &\leq |x_\alpha - x| \wedge (|b - b_{\beta_0}| + |b_{\beta_0}|)\\ &\leq |x_\alpha - x| \wedge |b - b_{\beta_0}| + |x_\alpha - x| \wedge |b_{\beta_0}|\end{aligned}$$

Applying p_i to this inequality, one can show the existence of α_0 such that $p_{i,\{b\}}(x_\alpha - x) < \varepsilon$. This completes the proof.

(7) If $0 \leq a \leq b$, then $u<\tau,\{a\}>u<\tau,\{b\}>$.

Proof: It is clear that;

$$\begin{aligned}\sup_{a\in\{a\}} p_i(|x_\alpha - x| \wedge a) &= p_i(|x_\alpha - x| \wedge a)\\ &\leq p_i(|x_\alpha - x| \wedge b)\\ &= \sup_{b\in\{b\}} p_i(|x_\alpha - x| \wedge b)\end{aligned}$$

(8) If $e \in E$ is a strong order unit, then $u < \tau, \{\{e\}\} >= u < \tau, \bigcup E_+ >$. But the converse of this statement is not true in general. For example, consider c_0 as a Banach lattice with supremum norm, with norm topology τ and $e = (\frac{1}{n})$. Then $u < \tau, \{\{e\}\} >= u < \tau, \bigcup E_+ >$, but e is not an order unit.

(9) If e is a quasi-interior point, then $u < \tau, \{\{e\}\} >= u < \tau, \bigcup E_+ >$ from (5) and (6).

(10) For any $\mathscr{A} \subset \mathscr{P}(E_+)$, $u < \tau, \mathscr{A} >= u < u < \tau, \mathscr{A} >, \mathscr{A} >$ holds.

4.3 Unbounded locally solid Riesz space

From the motivation of the above observation, we give the following definition.

Definition 4.3.1. *A real valued map q on a Riesz space E is said to be unbounded Riesz pseudoseminorm if there exists a Riesz pseudoseminorm p on E and $A \subset E^+$ satisfying $q(x) = \sup_{a\in A} p(|x| \wedge a)$. In this case, we say that q is generated by p and the subset A.*

It is obvious that every unbounded Riesz pseudoseminorm is a Riesz pseudoseminorm. So the topology which is generated by unbounded Riesz pseudoseminorm is a locally solid topology. If unbounded Riesz pseudoseminorm q is generated by Riesz pseudoseminorm p and $A \subset E^+$, then the topology generated by q is weaker than the topology which is generated by p. Recall that every family of Riesz pseudoseminorms defines a locally solid topology. Conversely, every locally solid topology is determined by a family of Riesz pseudoseminorms.

Definition 4.3.2. *Let (E,τ) be a locally solid Resz space generated by the family $(p_i)_{i\in I}$ of Riesz pseudoseminorms . The locally solid Riesz space on E generated by* *the family of unbounded Riesz pseudoseminorm on E is called unbounded locally solid Riesz space generated by τ, and denoted by $\tau^{'}$*

Proposition 4.3.3. *Let (E,τ) be a locally solid Riesz space. If τ is a Hausdorff locally solid topology, then the unbounded locally solid topology is also Hausdorff.*

Proof. Let $(p_i)_{i\in I}$ be a family of Riesz pseudoseminorms such that $\tau =< (p_i)_{i\in I} >$ and $x \neq 0$ be given, then there exists some $i_0 \in I$ such that $p_{i_0}(x) > 0$.Then,

$$q_{i_0,\{|x|\}} := \sup_{a\in\{|x|\}} p_{i_0}(|x| \wedge a) = p_{i_0}(|x| \wedge |x|) = p_{i_0}(|x|) = p_{i_0}(x) > 0.$$

It is obvious that $q_{i_0,\{|x|\}}$ is an unbounded Riesz pseudoseminorm, so $\tau^{'}$ is a Hausdorff topology. □

Definition 4.3.4. *A net (x_α) in a locally solid Riesz space (E,τ) is unbounded topological convergent if it is convergent in unbounded locally solid Riesz space $(E,\tau^{'})$.*

Theorem 4.3.5. *Let (E,τ) be a Hausdorff locally solid Riesz space and (x_α) be an increasing net. Then the followings are equivalent:*

1. $(x_\alpha) \xrightarrow{\tau} x$ *in* (E,τ);
2. $(x_\alpha) \xrightarrow{\tau^{'}} x$ *in* $(E,\tau^{'})$.

Proof. Since $\tau^{'} \subset \tau$, it is easy to see that (1) implies (2). Now suppose (2) holds. Since $\tau^{'}$ is a Hausdorff locally solid Riesz space by the Proposition 4.3, we have $x_\alpha \uparrow x$. Thus $|x|$ is an upper bound for the net (x_α) and $2|x|$ is an upper bound for the net $(|x_\alpha - x|)$. Now suppose that $(p_i)_{i\in I}$ is the family of Riesz pseudoseminorms such that $\tau =< (p_i)_{i\in I} >$. Let $i \in I$ be arbitrary. Then,

$$\begin{aligned} p_i(x_\alpha - x) = p_i(|x_\alpha - x|) = p_i(|x_\alpha - x| \wedge 2|x|) & \\ = \sup_{a\in\{2|x|\}} p_i(|x_\alpha - x| \wedge a) & \\ := q_{i,\{2|x|\}}(x_\alpha - x) \to 0. & \end{aligned}$$

This completes the proof. □

Theorem 4.3.6. *Let (E,τ) be a Hausdorff locally solid Riesz space, and $\tau^{'}$ be the unbounded locally solid topology generated by τ. Then τ has Lebesgue property if and only if $\tau^{'}$ has Lebesgue property.*

Proof. One side of the implication is clear. Let us assume that $x_\alpha \downarrow 0$ implies $x_\alpha \xrightarrow{\tau'} 0$. Then , it is easy to see that $x_\alpha \xrightarrow{\tau} 0$ by using the Theorem 4.5. This completes the proof. □

4.3.1 Product of unbounded locally solid Riesz space

Theorem 4.3.7. *Let $(E_i, \tau_i)_{i\in I}$ be a family of locally solid Riesz spaces. Then the product space $\prod_{i\in I} E_i$ is unbounded locally solid Riesz space if and only if for each i, E_i is an unbounded locally solid Riesz space.*

Proof. Suppose that for each $i \in I$, (E_i, τ_i) is an unbounded locally solid Riesz space, and τ_i is generated by a family Q_i of the unbounded Riesz pseudoseminorms on E_i. So for each $q \in Q_i$, there exists a Riesz pseudoseminorm p on E_i and $A_i \subset E_i^+$, depending on q, such that

$$q(x) = \sup_{a\in A_i} p(|x| \wedge a) \text{ for all } x \in E_i.$$

Let $j \in I$ and $q \in Q_j$ be given. Choose p and A_j as above. Let P_j be the projection from $E = \prod_i E_i$ into E_j and f_j be vector space embedding of E_j into E, that is, f_j sends $x \in E_j$ to (x_i) where $x_j = x$ and $x_i = 0$ for all $i \neq j$. One can show that for each Riesz pseudoseminorm on E_j, $p \circ P_j$ is a Riesz pseudoseminorm on E. We note that for each $q \in Q_j$,

$$q \circ P_j((x_i)) = q(P_j(x_i)) = q(x_j) = \sup_{a\in A} p(|x_j| \wedge a) = sup_{a\in A_j}\ p \circ P_j(|(x_i)| \wedge f_j(a)).$$

Thus, $q \circ P_j$ is an unbounded Resz pseudoseminorm on E. And the the topology of $\prod_i E_i$ is the topology generated by $\{q \circ P_j : j \in I, q \in Q_j\}$. Hence, the locally solid Riesz space $\prod_i E_i$ is an unbounded locally solid Riesz space.

Now suppose that $E = \prod_i E_i$ is an unbounded locally solid Riesz space, and i_0 is given. Suppose that the topology of E is generated by the family Q of unbounded Riesz pseudoseminorm on E. Let $q \in Q$ be given. There exists $A = (A_i) \in E_+$ and Riesz pseudoseminorm p on E such that $q(x) = \sup_{a\in A} p(|x| \wedge a)$ for all $x \in E$. It is obvious that for each i_0, $p \circ f_{i_0}$ is a Riesz pseudoseminorm on E_{i_0} and

$$q \circ f_{i_0}(x) = \sup_{a\in A_{i_0}} p \circ f_{i_0}(|x| \wedge a).$$

Hence q_{i_0} is an unbounded Riesz pseudoseminorm on E_{i_0}. Now one can show that the topology of E_{i_0} is generated by $\{q \circ f_{i_0} : q \in Q\}$. Hence, E_{i_0} is an unbounded locally solid Riesz space. This completes the proof. □

Let X be a product space of topological spaces $(X_i)_{i \in I}$. A net (x_α) converges to x in X if and only if $x^i_\alpha \to x_i$ in X_i for each $i \in I$, where $x_\alpha = (x^i_\alpha)_{i \in I}$ and $x = (x_i)$. By using this fact, the proof of the following theorem is easy.

Theorem 4.3.8. *Let $(E_i, \tau_i)_{i \in I}$ be a family of locally solid Riesz spaces. For each $\mathscr{A}_i \subset \mathscr{P}(E_i^+)$, we have*

$$u < \prod_i \tau_i, \prod_i \mathscr{A}_i >= \prod_i u < \tau_i, \mathscr{A}_i >.$$

4.3.2 Unbounded absolute weakly locally solid Riesz space

The concept of unbounded absolute weak convergence (briefly uaw-convergence) was considered and studied in Zabeti (2017). Let E and F be vector spaces. If there exists a bilinear map $T : E \times F \to \mathbb{R}$ satisfying

$T(x, F) = 0 \Longrightarrow x = 0,$

$T(E, y) = 0 \Longrightarrow x = 0$, then the pair $\langle E, F \rangle$ is called a dual pair. In this case, E can be considered as a vector subspace of $\mathbb{R}^F$, by embedding $x \to x^*$, $x^*(y) = T(x, y)$. We can consider $\mathbb{R}^F$ as a topological space with product topology $\prod_{y \in F} \mathbb{R}$ and restriction of this topology on E is the topology generated by the family $(p_y)_{y \in F}$ of seminorms, where $p_y : E \to \mathbb{R}$ defined by $p_y(x) = |T(x, y)|$. This topology is independent of T and is denoted by $\sigma(E, F)$. Similarly, $\sigma(F, E)$ can be defined. One of the main results is that the topological dual of E with respect to $\sigma(E, F)$ is a vector space which is isomorphic to F, this is denoted by $(E, \sigma(E, F))' \cong F$.

Definition 4.3.9. *If $\langle E, F \rangle$ is a dual pair of Riesz spaces with respect to a positive linear map $T : E \times F \to \mathbb{R}$, then we call that as a positive dual pair (with respect to T).*

We note that if $\langle E, F \rangle$ is a positive dual pair with bilinear map T, then one can show that the embedding $x \to x^*$, $x^*(y) = T(x, y)$ is bipositive. The order dual of a Riesz space E is the vector space of order bounded functionals from E into $\mathbb{R}$ and denoted by $E^\sim$, which is a Dedekind complete Riesz space. Throughout the paper we suppose that $E^{\sim}$ separates the points of E, that is, for each nonzero $x \in E$, there exists $f \in E^{\sim}$ with $f(x) \neq 0$. So, $\langle E, E^{\sim} \rangle$ is a positive dual pair via the map $(x, f) \to f(x)$. If τ is a Hausdorff locally solid topology on E, then the topological dual $E^{'}$ is an ideal of $E^{\sim}$. Let $A \subset E^{\sim}$ be given. For each $f \in A$, the map $p_{|f|} : E \to \mathbb{R}$. $p_{|f|}(x) = |f|(|x|)$ is a Riesz seminorm. The locally convex-solid topology generated by $(p_{|f|})_{f \in E^{\sim}}$ is called *absolute weak topology* and denoted by $|\sigma|(E, A)$.

Now we are going to define an unbounded absolute locally solid topology. For this, first we need the following Lemma.

Lemma 4.3.10. *Let $\langle E, F \rangle$ be a positive dual pair with respect to T. For each $a \in E$ and $y \in F$, the map $p : E \to \mathbb{R}$ defined by*

$$p(x) = T(|x| \wedge |a|, |y|)$$

is a Riesz pseudoseminorm on E.

Proof. Without loss of the generality, we can suppose that a and y are positive. Obviously the conditions $(1), (2)$ and (5) are satisfied . For the condition (3): for a given pair $x, y \in E$,

$$\begin{aligned} p(x+y) =& T(|x+y| \wedge a, y) \\ \leq& T(|x| + |y| \wedge a, y) \text{ by positivity} \\ \leq& T(|x| \wedge a, y) + T(|y| \wedge a, y) \text{ by linearity and positivity} \\ =& p(x) + p(y) \end{aligned}$$

hence, the condition (3) holds. For the condition (4), let $\{\lambda_n\} \subset \mathbb{R}$ be a sequence such that $\lim_{n \to \infty} \lambda_n = 0$ and $x \in E$, we have

$$\begin{aligned} p(\lambda_n x) = T(|\lambda_n x| \wedge a, y) = T(|\lambda_n||x| \wedge a, y) &= T(|\lambda_n|(|x| \wedge \tfrac{1}{|\lambda_n|} a), y) \\ &= |\lambda_n| T(|x| \wedge \tfrac{1}{|\lambda_n|} a), y) \\ &\leq |\lambda_n| T(|x|, y) \end{aligned}$$

So, $T(|x|, y)$is a real number,$|\lambda_n| T(|x|, y) \to 0$,thus the condition (4)also holds. □

By using the same motivation, for a given $A \subset E_+, e_0 \in E$ and $f_0 \in F$, the map $\sup_{a \in A} T(|x| \wedge a \wedge |e_0|, |f_0|)$ is also a Riesz pseudoseminorm, and it will be denoted by p_{A,e_0,f_0}

Definition 4.3.11. *Let $\langle E, F \rangle$ be a positive dual pair. Let $E_0 \subset E$,$F_0 \subset F$ and $\mathscr{A} \subset \mathscr{P}(E_+)$ be nonempty sets. Then the topology generated by $(p_{A,e_0,f_0})_{A \in \mathscr{A}, e_0 \in E_0, f_0 \in F_0}$ is called unbounded locally solid Riesz space on the positive pair $\langle E, F \rangle$ with respect to E_0, F_0 and $\mathscr{A}$. This topology is denoted by $u|\sigma|((E,F), E_0, F_0, \mathscr{A})$.*

By using some routine arguments, the proof of the above theorem can be given.

Theorem 4.3.12. *Let (E, F) be a positive dual pair. Let Let $E_0 \subset E$,$F_0 \subset F$ and $\mathscr{A} \subset \mathscr{P}(E_+)$ be nonempty sets. Then*

$$u|\sigma|((E,F), E_0, F_0, \mathscr{A}) = u|\sigma|((E,F), I(E_0), I(F_0), \mathscr{A}).$$

Definition 4.3.13. *A net (x_α) in E is called* ***unbounded absolutely weakly convergent to*** *x with respect to $(E_0, F_0, \mathscr{A})$, $x \in E$, if and only if the net (x_α) converges to x in the the topology $u|\sigma|((E,F), E_0, F_0, \mathscr{A})$*

Remark 4.3.14. *These observations and results can be extended into locally solid lattice-ordered groups studied in Hong (2015).*

5. ALL TOPOLOGIES COME FROM A FAMILY OF 0 1-VALUED QUASIMETRICS

Topological spaces are natural extensions of metric topologies. A topological space whose topology is a metric topology is called a ***metrizable space***. Most of the fundamental examples of topological spaces are not metrizable (for general definitions and examples, see Engelking (1989)), therefore one of the fundamental research topics in General Topology has been to find conditions under which a topological space is metrizable. In Kopperman (1988), despite the fact that not all topologies taken into account are metrizable, types of such conditions are shown to be obtained in terms of generalized quasi-metrics. To prove this, Kopperman introduced the notion of continuity spaces in Kopperman (1988), which reads as follows.

A semigroup $(A,+)$ with identity 0 and absorbing element $\infty \neq 0$ is called a ***value semigroup*** if the following conditions are satisfied:

(i) If $a+x=b$ and $b+y=a$, then $a=b$ (in this case, if $a \leq b$ is defined as $b=a+x$ for some x, then $\leq$ defines a partial order on A).

(ii) For each a, there is a unique b such that $b+b=a$ (in this case, one writes $b=\frac{1}{2}a$).

(iii) For each a, b, the element $a \wedge b := \inf\{a,b\}$ exists.

(iv) For each a, b, c, the equality $a \wedge b + c = (a+c) \wedge (b+c)$ holds.

A ***set of positives*** in a value semigroup A is a subset $P \subset A$ satisfying the following:

(i) if a, $b \in P$, then $a \wedge b \in P$;

(ii) $r \in P$ and $r \leq a$, then $a \in P$;

(iii) $r \in P$, then $\frac{r}{2} \in P$;

(iv) if $a \leq b+r$ for each $r \in P$, then $a \leq b$.

Let X be a non-empty set, A a value semigroup, P a set of positives of A, and $d : X \times X \to A$ a function such that $d(x,x)=0$ and $d(x,z) \leq d(x,y)+d(y,z)$ for all x, y,

$z \in X$. Then $\mathscr{A} = (X,d,A,P)$ is called a ***continuity space***. For each $x \in X$ and $r \in P$, we write

$$B[x,r] = \{y \in X : d(x,y) \leq r\}$$

Theorem 5.0.1 (Kopperman Kopperman (1988)). *Let $\mathscr{A} = (X,d,A,P)$ be a continuity space. Then*

$$\mathrm{To}(\mathscr{A}) := \{U \subset X : \textit{ for each } x \in U \textit{ there exists } r \in P \textit{ such that } B[x,r] \subset U\}$$

is a topology on X. Moreover, every topology on X is of this form.

5.1 The Main Result

The main issue of the present note is to reveal the fact that Kopperman's theorem can be refined by taking $0-1$-valued generalized quasi-metric spaces instead of continuity spaces. We will first define related notions which will be used in the sequel.

A function $d : X \times X \to [0,\infty)$ is called a ***quasi-metric*** if $d(x,x) = 0$ and $d(x,z) \leq d(x,y) + d(y,z)$ for all x, y, $z \in X$. A $0-1$***-valued generalized quasi-metric*** on a set X is a function from $X \times X$ into $\{0,1\}^I$ for some non-empty set I if for each $i \in I$ the function $d_i : X \times X \to \{0,1\}$ defined by $d_i(x,y) = d(x,y)(i)$ is a quasi-metric. In such a case, we will refer to $(d_i)_{i\in I}$ as a ***partition*** of d. A set X equipped with a $0-1$-valued generalized quasi-metric d is called a $0-1$***-valued generalized quasi-metric space***. It is easily seen that the set $\{0,1\}^I$ is a semigroup with respect to the operation $+$, given by $(f+g)(i) = \max\{f(i),g(i)\}$, moreover $(A,+)$ is a value semigroup with identity element ($0(i) = 0$ for each $i \in I$), absorbing element ($\infty(i) = 1$ for each $i \in I$) and the pointwise order.

Theorem 5.1.1. *The set*

$$P = \{r \in \{0,1\}^I : \{j \in I : r(j) = 0\} \text{ is finite}\}$$

is a set of positives in the value semigroup $(A,+)$.

Proof. i. $f,g \in P$, then $(f \wedge g) \in P$ since the number of zeros of the function $(f \wedge g)(i) = \inf\{f(i),g(i)\}$ is at most $|\{j \in I : f(j) = 0\}| + |\{j \in I : g(i) = 0\}|$,

ii. If $f \in P$ and $f \leq g$, then $f(i) \leq g(i)$ for each $i \in I$, it leads us; set of zeros of g is a subset of the set of zeros of f,

iii. For each $r \in A$, $r + r = r$, it is idempotent, hence if $r \in P$ then $r = \frac{r}{2} \in P$.

□

Following the usual custom, we denote this space by (X,d,I). A subset $U \subset X$ is called ***open*** if for each $x \in U$ there exists a finite set $J \subset I$ such that

$$\bigcap_{i \in J} \{y \in X : d(x,y)(i) = 0\} \subset U.$$

The set of open sets with respect to (X,d,I) is denoted by $\mathrm{To}(X,d,I)$.

Lemma 5.1.2. *Let (X,d,I) be a $0-1$-valued generalized quasi-metric space. Then, for each $x \in X$ and $i \in I$, the set $\{y \in X : d_i(x,y) = 0\}$ is open.*

Proof. Let $U := \{y \in X : d_i(x,y) = 0\}$ and let $y \in U$ be given. Then $d_i(x,y) = 0$. If $d_i(y,z) = 0$, then we have

$$0 \leq d_i(x,z) \leq d_i(x,y) + d_i(y,z) = 0,$$

so that

$$\{z : d_i(y,z) = 0\} \subset U.$$

It follows that U is open. □

The proof of the following is elementary and is therefore omitted.

Theorem 5.1.3. *Let (X,d,I) be a $0-1$-valued generalized quasi-metric space. Then* $\mathrm{To}(X,d,I)$ *is a topological space. If $(d_i)_{i \in I}$ is the partition corresponding to d, then the family*

$$\{\{y \in X : d_i(x,y) = 0\} : i \in I, x \in X\}$$

is a subbase of $\mathrm{To}(X,d,I)$.

Let us denote the truth value of a proposition p by $t(p)$; that is, $t(p) = 1$ if p is true, $t(p) = 0$ if p is false. Let (X,τ) be a topological space and $U \in \tau$ be given. For each $U \in \tau$, the map $d_U : X \times X \to \mathbb{R}$ defined by

$$\begin{cases} 0, & \text{if } t(x \in U \implies y \in U) = 1; \\ 1, & \text{if } t(x \in U \implies y \in U) = 0. \end{cases}$$

is a quasi-metric. Indeed, let x, y and $z \in X$ be given. If $x \notin U$ then, $t(x \in U \implies y \in U) = 1$ so $d_U(x,z) = 0$. If $x \in U$, $d_U(x,y) = 0$ and $d_U(y,z) = 0$, then $y \in U$ and $z \in U$. Thus $t(x \in U \implies z \in U) = 1$, so $d_U(x,z) = 0$. This shows that d_U is a quasi-metric. Also, for each $U \in \tau$, one can define a function $p_U : X \times X \to \mathbb{R}$ as $p_U(x,y) = \chi_{\{U\}}(x)\chi_{\{U^c\}}(y)$ where $\chi_{\{U\}}$ denotes the characteristic function of U, it is also a quasi-metric,which is equivalent to the quasi-metric d_U. In particular, we have the following.

Lemma 5.1.4. *Let (X,τ) be a topological space and $U \in \tau$ be given. Then, for each $x \in U$, one has*

$$U = \{y \in X : d_U(x,y) = 0\}.$$

Interestingly enough, the converse of the above fact is also true.

Theorem 5.1.5. *Every topological spaces comes from a $0-1$-valued generalized quasi-metric space. That is, if (X,τ) is a topological space, then there exists a $0-1$-valued generalized quasi-metric on X such that $\tau = \mathrm{To}(X,d,I)$.*

Proof. For each x, $y \in X$ and $U \in \tau$, if the proposition "$x \in U \implies y \in U$" is true let $d(x,y)(U) = 0$, and otherwise let it be 1. Then we have a function $d : X \times X \to \{0,1\}^\tau$. One can easily show that it is indeed a $0-1$-valued generalized quasi-metric. Now we show that $\tau = \mathrm{To}(X,d,I)$. Let $U \in \tau$ and $x \in U$ be given. Since $\{y \in X : d_U(x,y) = 0\} \in \mathrm{To}(X,d,I)$ it directly follows that

$$U = \{y \in X : d_U(x,y) = 0\},$$

whence $U \in To(X,d,I)$. Now, let $V \in To(X,d,I)$ be given. If $V = X$, then obviously $V \in \tau$. Suppose that $V \neq X$. Let $x \in V$. Then there exists $U_1, \ldots U_n \in \tau$ such that

$$\bigcap_{i=1}^{n} \{y \in X : d_{U_i}(x,y) = 0\} \subset V.$$

By Lemma 2.3 we have

$$x \in \bigcap_{i=1}^{n} U_i \subset V,$$

so that $V \in \tau$. The proof that $\tau = To(X,d,I)$ is now complete. □

Theorem 2.4 shows that each topology is a $To(X,d,I)$, and Kopperman's result (in Kopperman (1988)) that each topology arises from a continuity space follows. It is obvious that a subbase of the space $To(X,d,I)$ is

$$\mathscr{B} = \{\{y : d_i(x,y) = 0\} : x \in X, i \in I\},$$

where $(d_i)_{i \in I}$ is a partition of d.

5.2 Pervin quasi-uniformity

For a non-empty set X, a subset $\mathscr{U} \subset \mathscr{P}(X \times X)$ is called a ***quasi-uniformity*** if it satisfies the following axioms.

(i) For each $U \in \mathscr{U}$, $\Delta \subset U$.

(ii) If $U \in \mathscr{U}$, $U \subset V \subset X \times X$ then $V \in \mathscr{U}$.

(iii) If $U, V \in \mathscr{U}$ then $U \cap V \in \mathscr{U}$.

(iv) For each $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V \circ V \subset U$,

where $\Delta = \{(x,x) : x \in X\}$ and $\circ$ denotes the usual composition of binary relations, that is; For any binary relations $V, W \subset X \times X$, the composition of V and W is

$$V \circ W = \{(x,z) \in X \times X : (x,y) \in V \text{ and } (y,z) \in W \text{ for some } y \in X\}.$$

The pair $(X, \mathscr{U})$ is called a ***quasi-uniform*** space. Kelley's proof (in Kelley (1995)) that every uniformity $\mathscr{U}$ on X defines a topology can easily be modified to show that every quasi-uniformity $\mathscr{U}$ on X defines a topology.

$$\tau_{\mathscr{U}} = \{U \subset X : \text{ for each } x \in X \text{ there exists } V \in \mathscr{U} \text{ such that } V(x) \subset U\},$$

where

$$V(x) = \{y \in X : (x,y) \in V\}.$$

It should be noted that uniformities are symmetric quasi-uniformities, here is the definition of symmetricity; if $V \in \mathscr{U}$ then $V^{-1} = \{(y,x) : (x,y \in V)\}$. PervinPervin (1962) has proved that every topological space is quasi-uniformizable; that is, for a given topology τ on X there exists a a quasi-uniformity $\mathscr{U}$ on X such that $\tau = \tau_{\mathscr{U}}$, namely $\mathscr{U}$ is the intersection of all quasi-uniformities which contains,

$$\mathscr{B} = \{(V \times V) \cup ((X \setminus V) \times X) : V \in \tau\}.$$

Here $\mathscr{U}$ is called ***Pervin quasi-uniformity***. As an application of Theorem 2.4 we can reprove the following theorem.

Theorem 5.2.1 (Pervin Pervin (1962)). *Every topological space is quasi-uniformizable.*

Note that the above theorem improves on the result proved in Kopperman (1988), that each generalized quasi-metric space induces a quasi-uniformity, and the topology induced by this quasi-uniformity is that induced by any generalized quasi-metric space that induces this quasi-uniformity. Thus each topology is induced by a quasi-uniformity.

5.3 Some Remarks

Through lack of symmetry, categorizing the notion of convergence as right convergence and left convergence is reasonable in a $0-1$-valued generalized quasi-metric space (X,d,I). The definition is as follows.

Definition 5.3.1. *A net $(x_\alpha)_{\alpha\in A}$ right converges to x in (X,d,I), denoted by $(x_\alpha) \xrightarrow{r} x$, if for each $i \in I$ there exists $\alpha_0 \in A$ such that $d_i(x,x_\alpha) = 0$ for all $\alpha \geq \alpha_0$. A net $(x_\alpha)_{\alpha\in A}$ is called right Cauchy (or, r-Cauchy) if for each $i \in I$ there exist $\alpha_0 \in A$ such that $d_i(x_\alpha,x_\beta) = 0$ for all $\beta \geq \alpha \geq \alpha_0$.*

The definitions of left convergence and left Cauchyness are given similarly: for the sake of simplicity, only 'right' versions of them are used in the rest of the note.

Remark 8. *Several familiar topological notions can be derived using the structure of $0-1$-valued generalized quasi-metric spaces. We list some of them below.*

(1) Let (X,d,I) be a $0-1$-valued generalized quasi-metric space. Then the following are equivalent:

(a) The net $(x_\alpha)_{\alpha\in A}$ *right converges to* x *in* (X,d,I).

(b) The net $(x_\alpha)_{\alpha\in A}$ *converges to* x *in* $\mathrm{To}(X,d,I)$.

(c) The net $d(x,x_\alpha)$ *converges to zero in the product topological space* $\{0,1\}^I$.

(2) Let (X,d,I) *and* (Y,p,J) *be* $0-1$*-valued generalized quasi-metric spaces, and* f *a function from* X *into* Y. *Then* f *is continuous at a point if and only if for each* $j\in J$ *there exists* $i\in I$ *such that* $d_i(x,y)=0$ *implies* $p_j(f(x),f(y))=0$.

(3) Let (X,d,I) *be a* $0-1$*-valued generalized quasi-metric space. Then* $\mathrm{To}(X,d,I)$ *is a* T_0 *space if and only if for every distinct pair* $x,y\in X$ *there exists* $i\in I$ *such that* $d_i(x,y)=1$ *or* $d_i(y,x)=1$

(4) $\mathrm{To}(X,d,I)$ *is* T_1*-space if and only if for every distinct pair* $x,y\in X$ *there exists* $i\in I$ *such that* $d_i(x,y)=1$ *and* $d_i(y,x)=1$.

(5) $\mathrm{To}(X,d,I)$ *is a* T_2*-space if and only if for every distinct pair* $x,y\in X$ *there exists* $i,j\in I$ *satisfying the following;*

i. $d_i(x,y)=d_j(y,x)=1$,

ii. $d_i(x,w)=d_j(y,w)=0$ *implies* $d_i(w,y)=0$ *or* $d_j(w,x)=0$.

Proof. ($\Longleftarrow$) We consider the open sets $U=\{z\in X: d_i(x,z)=0\}$ and $V=\{z\in X: d_j(y,z)=0\}$, it is obvious that $x\in U$ and $y\in V$ and by the assumption (i), $x\notin V$ and $y\notin U$. Assuming $U\cap V\neq\emptyset$, so $U\cap V$ has at least one element, say w. Here is the contradiction;

$$d_i(x,y)\le d_i(x,w)+d_i(w,y)$$
$$d_i(x,y)\le 0$$
$$1\le 0$$

($\Longrightarrow$) Assuming that $\mathrm{To}(X,d,I)$ is a T_2 topology, for any distinct pair $x,y\in X$ there exist open sets U and V such that $x\in U, y\in V$ and $U\cap V=\emptyset$. By the definition of open set, there exist $i,j\in I$ such that

$$\{z\in X: d_i(x,z)=0\}\subset U \text{ and } \{z\in X: d_j(y,z)=0\}\subset V$$

it is easily seen that $d_i(x,y) = d_j(y,x) = 1$, and (ii) is logically true since there is no $w \in X$ such that $d_i(x,w) = d_j(y,w) = 0$. $\square$

(6) The notion of statistical convergence of a sequence of real numbers is as follows: A sequence (x_n) of real numbers is said to ***converge statistically*** *to the real number x if for each $\varepsilon > 0$ one has $\delta(A_\varepsilon) = 0$, where $A_\varepsilon = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$ and*

$$\delta(A_\varepsilon) = \lim_{n\to\infty} \frac{\sum_{a \in A_\varepsilon, a \leq n} 1}{n}.$$

In Maio (2000), it is defined for topological spaces as well. Here is its variant using the aforementioned arguments: A sequence (x_n) in (X,d,I) is said to convergence statistically to x if

$$\lim_{n\to\infty} \frac{|n \in \mathbb{N} : d_i(x,x_n) = 1|}{n} = 0$$

holds for each $i \in I$.

6. A SHORT NOTE ON INEQUALITIES IN RIESZ SPACES

6.1 Introduction

We will follow standart notations in Riesz space theory (see,e.g.Aliprantis and Burkinshaw (1985)). As noted in Wickstead (2007), textbooks on vector lattices abound in equalities and inequalities which often take quite a lot of proving. By using the Kakutani representation theorem, it can easily be proved that any elementary equality or inequality which holds in the reals also holds in Archimedean Riesz spaces (see: theorem 1.4 Wickstead (2007), p.66 (Meyer-Nieberg)The terms elementary equality or inequality are defined as follows:

Definition 6.1.1. *Let E be a Riesz space and $n \in \mathbb{N}$. We call a function $f : \prod_{i=1}^{n} E \to E$ is elementary if f is in the following form:*

$$f(x_1, x_2 ... x_n) = y_1 y_2 y_3 ... y_{k_n}$$

where

$$y_i \in \{(,),|,\vee,\wedge,+,-,^+,^-\} \cup \{x_i : i = 1,2,...n\} \cup \mathbb{R}$$

and the sequence $y_1 y_2 y_3 ... y_{k_n}$ valid in E. In this case we define a function

$$f' : \prod_{i=1}^{n} \mathbb{R} \to \mathbb{R}$$

given by

$$f(x'_1, x'_2 ... x'_n) = y'_1 y'_2 y'_3 ... y'_{k_n}$$

such that if $y_i \in \{(,),|,\vee,\wedge,+,-,^+,^-\} \cup \mathbb{R}$, then $y'_i = y_i$, otherwise $y'_i = x'_k$ such that $y_i = x_k$.

6.2 The main result

The main result of this chapter is the following:

Theorem 6.2.1. *Let E be a Riesz space and $f, g : \prod_{i=1}^{n} E \to E$ be elementary functions. We have the following:*

i. $f(x_1,x_2...x_n) \leq g(x_1,x_2...x_n) \Leftarrow f'(x'_1,x'_2...x'_n) \leq g'(x'_1,x'_2...x'_n)$

ii. $f(x_1,x_2...x_n) < g(x_1,x_2...x_n) \Leftarrow f'(x'_1,x'_2...x'_n) < g'(x'_1,x'_2...x'_n)$

iii. $f(x_1,x_2...x_n) = g(x_1,x_2...x_n) \Leftarrow f'(x'_1,x'_2...x'_n) = g'(x'_1,x'_2...x'_n)$

Notice that any of the above inequality (or equality) is called *elementary inequality* (or *elementary equality*) in literature. The above theorem says, for instance, that since for $x,y,z \in \mathbb{R}$ we have the validity of

$$x+(y \vee z) = (x+y) \vee (x+z)$$

in $\mathbb{R}$, we also have the equality

$$x+(y \vee z) = (x+y) \vee (x+z)$$

in a Riesz space E, for any $x,y,z \in E$ where E is not necessarily Archimedean Riesz space. The proof of the above fact depends on some heavy representation theorems that are valid in ZFC (see Luxemburg and Zaanen (1971)). On the other hand, we can prove theorem 1.2 for (not necessarily Archimedean) Riesz spaces in ZF. In Aliprantis (1996) the belief that the above theorem can be proved for any Riesz space is mentioned. In what follows, we not only prove that claim but give the proof of it in ZF.

Proof. Let E be a Riesz space (not necessarily Archimedean) and X be a non-empty set. Consider the Riesz space $\mathbb{R}^X$ under the pointwise algebraic operations and pointwise ordering. Let V be an order ideal of $\mathbb{R}^X$, it is known that the quotient vector space $\mathbb{R}^X/V$ is a Riesz space with respect to the following order:

$$[f] \leq [g] \text{ if and only if } f \leq f+v \text{ in } \mathbb{R}^X \text{ for some } v \in V$$

It is obvious that if an elementary equality or inequality is true in $\mathbb{R}$ then it is true in $\mathbb{R}^X$. hence it is true in $\mathbb{R}^X/V$ since the map $\mathbb{R}^X \to \mathbb{R}^X/V$, $f \mapsto [f]$ is a Riesz homomorphism (by the corollary 1.4). One of the Fremlin's theorem states that in ZFC for any ordered vector space F there exists a nonempty set X, and an order ideal V of $\mathbb{R}^X$ such that $F \to \mathbb{R}^X/V$ is a injective Riesz homomorphism. This statement is true in ZF, for the proof of this theorem of Fremlin, see Buskes et al (2008). Hence The Riesz space E is Riesz isomorphic to the Riesz subspace of $\mathbb{R}^X/V$ for some X and for some order ideal V of $\mathbb{R}^X$. Therefore If an equality or inequality is true in $\mathbb{R}$ then it is also true in E. □

Being immediate applications of this theorem, many equalities and inequalities in Riesz spaces can be shownd to hold, some of which are the following:

1. $x \vee y = -((-x) \wedge (-y))$ and $x \wedge y = -((-x) \vee (-y))$

2. $x + y = x \wedge y + x \vee y$

3. $x + (y \vee z) = (x+y) \vee (x+z)$ and $x + (y \wedge z) = (x+y) \wedge (x+z)$

4. $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$ and $\alpha(x \wedge y) = (\alpha x) \wedge (\alpha y)$

5. $x \vee y = \frac{1}{2}(x+y+|x-y|)$ and $x \wedge y = \frac{1}{2}(x+y-|x-y|)$

6. $|x-y| = x \vee y - x \wedge y$

7. $|x| \vee |y| = \frac{1}{2}(|x+y| + |x-y|)$

8. $|x| \wedge |y| = \frac{1}{2}(|x+y| - |x-y|)$

9. $||x| - |y|| \leq |x+y| \leq |x| + |y|$

10. $|x \vee z - y \vee z| \leq |x-y|$ and $|x \wedge z - y \wedge z| \leq |x-y|$

11. If x and y are positive, then $x \wedge (y+z) \leq x \wedge y + x \wedge z$

12. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

13. $|x| + |y| - |x+y| = 2(x^+ \wedge y^- + x^- \wedge y^+)$

The proof of the following theorem immediately follows the definition.

Theorem 6.2.2. *Let E be a Riesz space and $f : \prod_{i=1}^{n} E \to E$ be elementary functions. Then, for any $a = (a_1, a_2, ...a_n)$ and $b = (b_1, b_2, ...b_n)$ in $\prod_{i=1}^{n} E$ we have*

$$|f(a) - f(b)| \leq K \sum_{i=1}^{n} |a_i - b_i|$$

for some positive real number K depending on a and b.

As an application of the above theorem we have the following. Let E be a Riesz space and f be an elementary function. For each $i = 1, 2, 3, ..., n$ suppose that $(x^i_{a(i)})_{a(i) \in A(i)}$ be nets. Then

1. If for each i, $(x_{a(i)})$ order converges to x_i, then $f(x^1_{a(1)}, x^2_{a(2)}, ..., x^n_{a(n)})$ is order convergent to $f(x_1, x_2, ..., x_n)$.

2. If for each i, $(x_{a(i)})$ unbounded order converges to x_i, then $f(x^1_{a(1)}, x^2_{a(2)}, ..., x^n_{a(n)})$ is unbounded order convergent to $f(x_1, x_2, ..., x_n)$.

3. If (E, τ) is a locally solid Riesz space and for each i, $(x_{a(i)})$ $\tau-$converges to x_i, then $f(x^1_{a(1)}, x^2_{a(2)}, ..., x^n_{a(n)})$ is $\tau-$convergent to $f(x_1, x_2, ..., x_n)$.

The above observations generalizes many well-known results on convergence of nets.

7. CONCLUSIONS AND OUTLOOK

On the third chapter: several new convergence spaces introduced on vector spaces and vector lattices. In the main part of this chapter lattice multi-Riesz pseudonormed vector lattices have been studied. This space has mainly two importances; one of both is that every Hausdorff locally solid vector lattice is a LMRPNVL, the other one is the definition of $\mathscr{R}-$continuous operators since it generalizes the other type continuous operators.

On the fourth chapter: Firstly, by using the observation on this chapter, the newly LMRPNVL spaces can be defined as follows: Let we choose E order complete, for a nonempty $A \subseteq E_+$ the map

$$\pi_A : X \longrightarrow E$$
$$x \longmapsto \pi_A(x) = \sup_{a \in A} \pi(|x| \wedge a)$$

is also a lattice-valued Riesz pseudoseminorm. secondly let d be an translation-invariant lattice pseudometric, and for a fixed $A \subseteq G_+$ consider the map;

$$d_A : G \times G \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto d_A(x,y) = \sup_{a \in A} d(|x-y| \wedge a, 0)$$

Similarly, one can show that it is a translation-invariant lattice pseudometric.Hence by using the fact that a group topology τ on an l-group G is locally solid if and only if it is generated by a family of translation-invariant lattice pseudometrics, we can study on new convergence types.

On the fifth chapter: A new characterization of topological spaces was done. The equality between the new characterization and two other characterizations, which have been already known, was stated. Since 0-1 valued maps are useful, to carry some definitions to topological spaces will be easier : uniform continuity and statistically convergent.

To emphasize its importance, it can be argued that even though the definition of solid topology is clear, generally we apply to a family of pseudoseminorms to describe them. This characterization can offer us that. Also to ponder if further studies in this field is possible, the following question was put: Can the topological properties be characterized in terms of the index set?

On the sixth chapter: We can call what we had done as simplification because the standart proof of the claimed fact Kakutanı represantation theorem, stone weistress theorem were utilized. But there are some difficulties conscerning its structure. Also my study is axiomatic because the proof was done without axiom of choice. Moreover the claim was applied in all Riesz Spaces, by not taking into account if it has Arcihimedean property, or not.

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