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DEPARTMENT OF MATHEMATICS

BANDS IN PARTIALLY ORDERED SPACE WITH UNIT ORDER

MASTER OF SCIENCE

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APPROVAL OF THE THESIS

PARTIALLY ORDERED SPACE WITH ORDER UNIT submitted by **OKAN ODABAŞI** in partial fulfillment of the requirements for the degree of **Master of Science** in **DEPARMENT OF MATHEMATICS, The Graduate School of Natural and Applied Sciences of ABANT IZZET BAYSAL UNIVERSITY in 09/07/2018** by,

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To my family and my supervisor...

DECLARATION

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OKAN ODABAŞI

ABSTRACT

BANDS IN PARTIALLY ORDERED SPACE WITH UNIT ORDER
M.S. THESIS
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ABANT İZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
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This thesis consists of four chapters. In chapter I,II it is given some neccesary definitions,which is used in other chapters. In chapter III, we deal with some basic properties of Riesz spaces, and characterizations of bands of C(Ω)-spaces, which is one of the fundemental example of Riesz spaces, are given. In the last chapter, the notion of directed partially ordered vector spaces are introduced and also the notion of bands are given in terms of disjointness. Bands are studing via Riesz space cover X of Y on the condition X with order unitt Y can be express as C(X), here X compact and T_2 space. one can express bands in X, seperate complement as subset of X

KEYWORDS: Bands, Seperatness, Functionals representation, Order Unitt, Partially ordered vector space, Polyhedral cone, vector completion

ÖZET

KISMİ SIRALI SIRA BİRİMİ OLAN VEKTÖR UZAYINDAKİ BANDLAR
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Bu tez dört bölümden oluşmaktadır. İlk iki bölümde diğer bölümlerde kullanılmak üzere, gerekli tanımlar verildi. Üçüncü bölümde Riesz Uzayların'nın bazı temel özellikleriyle ve Riesz Uzaylarının önemli örneklerinden biri olan, C(X) uzayındaki bandlar ile ilgilenildi. Son bölümde ise, Archimedian sıralı vektör uzaylarında band kavramı diklik vasıtasıyla ifade edilebilir. Şöyle ki; bandlar X'in vector lattice cover Y'yi kullanarak çalışılabilir. Eğer X Sıra Birim'e sahip ise, Ω compact Hausdorff olduğu durumda lattice cover Y C(X) tarafından temsil edilebilir. Biz Ω'nın alt kümeleri vasıtasıyla X'teki bandları ve onun disjoint complements karakterize edilir. Dahası biz X'teki bu bandları C(X) genişleten iki methodu analiz ettik ve band'ın carrier'nin onun genişlemesiyle nasıl ilgili olduğu gösterildi.

ANAHTAR KELİMELER: Band, Diklik, Foksiyonel Temsil, Sıra Birim,Kısmi Sıralı Vektör Uzayı, Kone, Riesz Tamlanışı

TABLE OF CONTENTS

Page

LIST OF ABBREVIATIONS AND SYMBOLS

$\mathbf{A^d}$	: Disjoint complement of A.
$\mathbf{C}\,(\mathbf{X})$	: Space of continous functions on X.
$\mathbf{sat}\,(\mathbf{M})$	: Saturation of M.
$\mathbf{Aff}(\mathbf{M})$	: Affine hull of M.
$\mathbf{Z}\,(\mathbf{M})$	: Zero set induced by M.
$\mathbf{C}\,[\mathbf{I}]$	: Functions which are defined on I and continuous.
$\mathbf{M^u}$	: All upper bounds M.
$\mathbf{U} \perp \mathbf{V}$	: Disjoint sets.
$\mathbf{u}^+$	: Positive side of u.
$\mathbf{u}^-$	: Negative side of u.
$\|\mathbf{u}\|$	: Absolute value of u.
$\mathbf{a} \vee \mathbf{b}$	: Least upper bound of a and b.
$\mathbf{a} \wedge \mathbf{b}$	: Greatest upper bound of a and b.
$\mathbf{Car}(\mathbf{X})$	: Carrier of X.

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1. INTRODUCTION

Partially ordered vector space without lattice property is a generalization of vector lattices in some sense. For example, space of operators between Riesz spaces has mostly no lattice property. To over come this problem, one use assume codomain is Dedekind complete.

Another way, lattice theorey notions could be extended to partially ordered vector spaces. There are two type aproach in this notion. One can be given necessarry definations in vector lattice for partially ordered vector space such as upper bound, lower bound, disjointness etc.

Actually, definitions in vector lattice could be express in partially ordered vector space, However, it is really hard to construct properities of this notions in partially ordered vector space.

Second approch is that find a proper embeding which is from partially ordered vector space to Riesz space. Later Riesz space properties can be used for the space which is embeded. The second approch was studied (anke Kalauch and O. vann gans) in terms of disjointness. To be sucessful at embeding technic, necessary condition is that our embeding must be order dense. Van handel in his P.h.D thesis characterize partially ordered vector spaces which are possible to embed order dense in a Riesz space. This kind of space is called pre-Riesz space.

Not only directed Archemedean spaces are pre-Riesz space, but also some non-Archemedan pre-Riesz spaces. Formulation of lattice notion in partialy ordered vector spaces may provide different notions for lattice. One can be said that the most useful generalizations will be those where the two approaches. But we will prefer second approach in this thesis.

Disjointness property has so many exciting consequence In Riesz space theory. It is quite meaningful how we can state disjointness in partially ordered vector space which has not lattice property. x,y disjoint elements of Riesz space $\iff$ absolute value of x+y and x-y must be equal. This equality is the motivation of disjointness notion for partially ordered vector spaces. Here it is also needed to be define some basic properties of Riesz space such as absolute value. This many basic properties is hold for directed Archemedean space.

Maris van Haandel in his P.h.d thesis (Completions In Riesz space Theory) has developed a theory on Riesz completions of partially ordered vector spaces In this thesis we used pre-Riesz space to embeded our space to extendt a Riesz space. and this embeding should preserve our sv-disjointness. Because bands are charterize via disjointness. Our extension will be excatly C(X) and we know how bands in C(X) as a result our sv-disjointness and disjointness in Riesz spaces will coincede each other. After we need only find a correponce with bands in C(X) the we will achive our aim. At last, we will explain how many possible band can be in our space and show a bound for our partially ordered vector space.

2. PRELIMINARY

In this chapter, necessarily and sufficient results are given, which are needed for the next chapter.

2.1 Topological Spaces

Definition 2.1.1. A non-empty set $\mathcal{T} \subseteq \mathcal{P}(\mathcal{X})$ is called **topology** on $X \iff \mathcal{T}$ satisfies the following conditions;

1. X and $\emptyset$ contained by $\mathcal{T}$.
2. The union of any number of sets in $\mathcal{T}$ contained by $\mathcal{T}$.
3. The intersection of finite number sets in $\mathcal{T}$ contained by $\mathcal{T}$.

Any set X equipped with $\mathcal{T}$ is called **topological space**. In other words, A topology space is a tuple $(X, \mathcal{T})$ having of a set X and a topology $\mathcal{T}$ on X, For convenience, we will abbreviate Topology as Top

A **indiscrete top** $\mathcal{T}$ on X is $\{\emptyset, X\}$; that means, it's the topology in which only $\emptyset$ and X are open.

A **metric** on a set Y is a $\mathbb{R}$-valued function $m : Y \times Y \to \mathbb{R}$ such thatt forr all u, v, $t \in Y$,

1. $m(u, v) = 0 \iff u = v$
2. $m(u, v) = m(v, u)$
3. $m(u, t) \leq m(u, v)$+ $m(v, t)$

The tuple (Y, m) is called **metric space**

Let (Y, m) be metric space forr each u_0, $r_0 > 0$,

$$B(u_0, r_0) = \{u \in Y : d(u_0, y) < r_0\}$$

it is an O-ball with center u_0 and radiu $r_0 > 0$. Then

$$\mathcal{T}_d = \{U \quad \subset \quad X| \quad U \quad is \quad union \quad some \quad O - balls\}$$

is a topology on X. So, in this sense every metric space can be considered as topological space, and such topological spaces is called **metrizable**.

Let X be any set. The **discrete topology** on X is the collection of all subset of X. it is induced by a metric, the so-called **discrete** metric d on X, which is defined by,

$$d(x, y) = \begin{cases} 0 & \text{if x=y} \\ 1 & \text{otherwise} \end{cases}$$

Definition 2.1.2. If $(X, \mathcal{T})$ is top space , one can say that a subset O of X is an **open set** of X if O contained by $\mathcal{T}$.

Definition 2.1.3. If X is top space with $\mathcal{T}$, one can be called that subset K of X is **closed sett** if K^c is open in X

Definition 2.1.4. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ are top spaces. A map from X to Y is **continuous** if for each open subsett V of Y, $f^{-1}(V)$ is an open subsett of X

Definition 2.1.5. A collection $\mathcal{A}$ of open subsett of a space X is **open covering** if the union of the elements of $\mathcal{A}$ is equal to X

Definition 2.1.6. A top space X is **compact** if every covering $\mathcal{A}$ of X contain a finite subcover of X and this is also cover X

Let $m(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ...(x_n - y_n)^2}$, In $\mathbb{R}^n$. Then define a $naturel$ $topology$ that is, $U \subseteq \mathbb{R}^n$ open if forr each $x = x_1, x_2, ..., x_n \in U$, there exists open $(a_1, b_1), ...(a_n, b_n)$ such that $x \in (a_1, b_1) \times .... \times (a_n, b_n) \subset U$ known that w.r.t this topology, a subset K is compact $\iff$ K is closed and bounded.

2.2 Normed Spaces

Definition 2.2.1. (Megginson, 1998) Let V be real lineer space . A **norm** is $\mathbb{R}$-valued function $\|.\| : V \to \mathbb{R}$ defined on a lineer space is if the following condition holds,

(1) $\|y\| \geq 0$ for each $y \in L$ and $\|y\| = 0$ if and only if $y = 0$.

(2) $\|\alpha y\| = |\alpha| \cdot \|y\|$.

(3) $\|y + z\| \leq \|y\| + \|z\|$ (the triangle in equality).

A [1] **normed space** is a lineer space which is endowed with a norm. Every normed space has a natural metric defined by means of its norm via this relation $d(x, y) = \|x - y\|$. This natural metric is called the metric induced by the norm. The topology generated by d is called **norm topology**.

Definition 2.2.2. If T is a mapping from the lineer space V into the lineer space W(notation $T : V \to W$), then T is called **linear operator** if

$$T(\alpha f + \beta g) = \alpha T_f + \beta T_g$$

for all f and g in V and all ($\mathbb{C}$ or $\mathbb{R}$) numbers α and β .

Theorem 2.2.3. If V and W are normed spaces and $T : V \to W$ is a lineer operator, then the following conditions for T are equivalent.

1. T is continuous at one point $f_0 \in V$.
2. T is continuous on V.
3. T is norm bounded.

T is called a **linear form** if it is taken $\mathbb{R}$ rather than W.

In particular, therefore, a linear form on V is normed bounded $\iff$ is continuous.

[1] Normed space is an integral part of mathematics, hence it is important to know that the cardinality of infinite vector spaces. There is an article on Cardinality of infinite dimensional vector space published by Zafer Ercan, entitled “Maşallah, Ne Kadar çok Norm Varmiş"

Theorem 2.2.4. Let $(X, \|.\|)$ be a normed space.

$$X^* = \{f|X \to \mathbb{R} \quad continous \quad linear \quad forms\},$$

X^* is a normed space with norm $\|f\| = Sup_{\|x\| \leq 1} \|f(x)\|$ then,

$$B^* = \{f \in X^* : \|f\| \leq 1\}$$

B^* is w^***-compact**, that means following:

Each net (f_α) in B^* has a subnet $(f_{\alpha ...})$ and $f \in B^*$ such thatt,

$$f_\alpha \to f(x)$$

forr each $x \in X$

3. CHAPTER 1

3.1 Ordered Vector Spaces

Thought the exposition we shall deal with real vector spaces.

Definition 3.1.1. Let P has at least one element and $\geq$ be a relation on P such thatt $\leq \subseteq P \times P$. Let us write $x \leq y$ whenever $(x, y) \in \leq$. If the followings are satisfying then the relation $\leq$ is said to be *partial ordering* on P where u,v,z $\in P$,

1. $u \leq u$ for all $u \in P$ (reflexivity)
2. If $u \leq v$ and $u \leq v$, then $u = v$ (anti-symmetry)
3. If $u \leq v$ and $v \leq z$, then $u \leq z$ (transitivity)

A sett X endowed with an order relation $\leq$ is called a partially ordered sett,and it is denoted by (X,$\leq$). we will abbreviate partially ordered set as P.O.S.

A non-empty subsett of a partially ordered set $(X, \leq)$ and $x \in X$. One define the followings;

x yields an **lower bound** of A if for each $a \in A$, $a \geq x$. In this case, one can write $x \leq A$. A is called **bounded from below** if $x \leq A$ for some $x \in X$

x yields **greatest lower bound** of A if x is greater than all lower boundd of A. Namely, if $A \geq x$ and $A \geq y$ then $x \geq y$. In this case, one can write as $x =$inf$A$

x yields an **upper bound** of A if for each $a \in A$, $a \leq x$. In this case, one may write $x \geq A$. A is called **boundedd from above** if $x \geq A$ for some $x \in X$

x yields **least upper bound** of A if x is greater than all upper boundd of A. namely, if $A \leq x$ and $A \leq y$ then $x \leq y$ and we write as $x =$sup$A$

If $x =$sup$A$ and $x \in A$ then x is called **maximum** of A. In this case, one write $x =$max$A$.

For example, in $\mathbb{R}$, max(0,1]=1, but there is no maximum of open interval $(0,1)$.

If $x =$inf$A$ and $x \in A$ then x is called **minimum** of A. In this case, one may write $x =$min$A$.

For instance, min[0,1]=0. But there is no minimum of open interval $(0,1)$.

Definition 3.1.2. An non-empty set X with an order $\leq$ is called **lattice** if every pair of elements $x, y \in X$ has both infimum and supremum

Definition 3.1.3. A vector space E, which is equipped with a partial order $\leq$, is called **ordered vector space** or ordered linear space if the following two conditions satisfies;

1. $u \geq v$, then $u + w \geq v + w$ for all $u, v, w \in E$
2. $u \geq v$, then $\alpha u \geq \alpha v$ for all $\alpha \in \mathbb{R}^+$

Ordered vector space is shown as $(E, \leq)$ pair.

Example 3.1.4. $\mathbb{R}$ with the usual order is an ordered vector space.

Example 3.1.5. Let X be a top space. The set of real valued continous funtions on X is represented by $C(X)$. It can be seen that $C(X)$ is an lineer space with pointwise algebraic operation. Moreover, it is an ordered lineer space in respect of the relation;

we define a pointwise relation $\leq$ on $C(X)$ as follows,

$$f \leq g :\Longleftrightarrow f(x) \leq g(x) \text{ forr all } x \in X$$

One can easily check that the following hold for each f, g, $h \in C(X)$ and $\alpha \in \mathbb{R}$

1. $f \leq f$, $\forall\, x \in X$,
2. $f \leq g$ and $g \leq f$ then $f = g$,
3. $f \leq g$, $g \leq h$ then $f \leq h$,
4. $f \leq g$ then $f + h \leq g + h$,
5. $f \leq g$ then $\alpha\, f \leq \alpha\, g$ for each $\alpha \geq 0$

Thus, $C(X)$ is vector space.

3.2 Archimedean Vector Spaces

Definition 3.2.1. (Zaanen, 1997)Let W is ordered vector space. W is **Archimedean** if forr every $x, y \in E$ such thatt $nx \leq y$, $\forall n \in \mathbb{N}$ than x≤0.

Note that Riesz Space will define in the next chapter. Now it will be given Archemedean space definition for Riesz spaces.

Definition 3.2.2. The Riesz space W is **Archimedean** if

$$inf(\frac{1}{n}u) = 0$$

hold for every $u \in W^+$

Example 3.2.3. The space of continous functions, from $[0, 1]$ to $\mathbb{R}$, is denoted by $C([0, 1])$ and it is is an Archimedan space. W.r.t following ordering;

$$f \leq g :\Leftrightarrow f(c) \leq g(c), \forall c \in [0, 1],$$

Example 3.2.4. Assume that $n \leq 2$. We define the lexicographical order on $\mathbb{R}^n$ as follows.

$$x = (x_1, ..., x_n) \leq (y_1, ..., y_n) = y$$

if there exists $k \in 0, ..., n$ such thatt

$$x_1 = y_1, ..., x_k = y_k \text{ and } x_{k+1} < y_{k+1}$$

It can easily be checked that $\mathbb{R}^n$ equipped with this order is a Riesz space. Further, it is totally ordered such that the order is non-Archimeadean. If

$$x = (0, 1, 0, ...0) \text{ and } y = (1, 0, ..., 0),$$

then $nx \leq y$ for every $n \in \mathbb{N}$

3.3 Order Unit In Ordered Vector Spaces

Definition 3.3.1. An element $e \geq 0$ in ordered lineer space E is an **order unit** whenever forr each $x \in E$ there exists some $\lambda \geq 0$ with $|x| \leq \lambda\, e$

Definition 3.3.2. An element $e \geq 0$ of an Archimedian vector space E is a **weak order** unit if

$$|f| \wedge e = 0 \quad implies \quad f = 0$$

Every order unit is weak unitt but the reverse inclusion is not true.

Example 3.3.3. $C(0, 1)$ has no order unit.

Note that C(K) has order unit where K is compact.

3.4 Directed Ordered Vector Spaces

Even if cone will be used in this chapter, it will define later.

Definition 3.4.1. A partially ordered set W is **directed** if for every $x, y \in W$ there exist $z \in W$ such thatt $z \geq x$ and $z \geq y$

The relation between cone and order is given in the following definiton.

Definition 3.4.2. If K is a cone of a lineer space E then E is an ordered lineer space in respect of the relation

$$x \leq y :\Longleftrightarrow y - x \in K$$

A vector space E equipped with cone is directed if $X = K - K$; In this case, K is called **generating** conee

This definition is so importart because it provides us to define a new disjointness.

Definition 3.4.3. Let $[a, b]$ be a closed interval on $\mathbb{R}$ and $x_0 = a < ... < x_n = b$, we define

$$P = \{x_i, i = 0, ..., n\}$$

The set P is called *partition* of $[a, b]$.

Example 3.4.4. Let P is a partition of [a,b]. The relation is given by

$$P \leq Q \Leftrightarrow Q \subset P$$

with respect to this relation P is directed.

Example 3.4.5. Let I be the set of all polynomials from $[0, 1]$ to $\mathbb{R}$, equipped with the following relation,

$$f_1 \leq g_1 :\Leftrightarrow f(c) \leq g(c), \forall c \in [0, 1]$$

I is directed.

4. VECTOR LATICES

In this chapter we introduce the notion of the Riesz subspaces, order ideal and bands

4.1 Riesz Spaces

Definition 4.1.1. A lineer space E is called **Riesz space** or **vector lattice** if two following condition satisfy,

i) L is an ordered lineer space.

ii) L is lattice.

Example 4.1.2. $C[0, 1]$ is a Riesz space under pointwise ordering.

There are of course many vector subspaces which also vector lattices under the same order. In general, ordered lineer space may not be vector lattice.

Theorem 4.1.3. (Aliprantis, 1985) Let E be a vector lattice for each $b, c, d \in E$ of a vector lattice, $\alpha \in \mathbb{R}$. One has the following facts,

(i). b $\vee$ c = -[(-b) $\wedge$(-c)] and b $\wedge$ c = -[(-b) $\vee$ (-c)].

(ii). b+c=(b $\wedge$ c)+ (b $\vee$ c).

(iii). b+(c $\vee$ d)= (b+c) $\vee$ (b+c) and b+(c $\wedge$ d)= (b+c) $\wedge$ (b+c).

(iv). α (b $\vee$ c) = (α b) $\vee$ (α c) and α (b $\wedge$ c) =(α b) $\wedge$ (α c) for all $\alpha \geq 0$

The Proof of the ttheorem 4.1.3 i, ii, iv can be easily done.

Proof of iii. it is obvious that $b + c \leq b + c \vee d$ and $b + d \leq b + c \vee d$, and
hence $(b + c) \vee (b + d) \leq b + c \vee d$ ... (1)

Also one has $c = -b + (b + c) \leq -b + (b + c) \vee (b + d)$ ... (2)
and similarly, $d = -b + (b + d) \leq -b + (b + d) \vee (b + c)$ ... (3)

Thus the equation iii is obtained from 1, 2, 3. □

If B is a subset of a vector lattice where sup B exist, then

(a) inf(-B) exist and moreover,

$$\sup(x + B) = x + \sup B;$$

$$\inf(-B) = -\sup B$$

(b) the sup of the set $x + B := \{x + b : b \in B\}$ exist and

(c) for each $\alpha \geq 0$ the sup of the set $\alpha B := \{\alpha b : b \in B\}$ exists and

$$\sup(\alpha B) = \alpha \sup B$$

Let E be a vector lattice. The set $\{u \in E : u \geq 0\}$ is called **positive cone** of E it is denoted by E_+. In particular for each $u \in E$, we define

$$u^+ := u \vee 0,\ u^- := (-u) \vee 0, \text{ and } |u| := u \vee (-u)$$

The element u^+ is called the **positive side**, u^- the **negative side**, and $|u|$ the **absolute value** of u. The main relations between $u, u^+, u^- and |u|$ are in the following theorem. Morever, u, u^+, u^- and $|u|$ are positive.

Theorem 4.1.4. (Aliprantis, 1985) If t is an element of vector lattice, then we have;

1. $t = t^+ - t^-$.
2. $|t| = t^+ + t^-$.
3. $t^+ \wedge t^- = 0$.

Besides, the representation in (1) is unique in such manner that if $t = u - v$ holds with $u \wedge v = 0$, then $u = t^+$ and $v = t^-$.

(1). From Theorem 3.1.4 can be seen that $t = t + 0 = t \vee 0 + t \wedge 0 = t \vee 0 - (-t) \wedge 0 = t^+ - t^-$.

(2) Using Theorem 3.1.4 and (1), we get

$|t| = t \vee (-t) = (2t) \vee 0 - t = 2\,(t \vee 0) - t = 2t^+ - t = 2t^+ - (t^+ - t^-)$
$= t^+ - t^-$

(3) Note that
$t^+ \wedge t^- = (t^+ - t^-) \wedge 0 + t^- = -[(-t) \vee 0] + t^- = -t^- + t^- = 0.$ ☐

Definition 4.1.5. A linear map $T : E \to F$ is called **positive operator** if $T(E^+) \subseteq F^+$.

Observe that if $T : E \to F$ is a positive map between two vector lattice, then from $\pm x \leq |x|$ we see that $\pm Tx \leq T|x|$, and so

$$|Tx| \leq T|x|$$

holds for all $x \in E$

In terms of positive part, identity in Theorem **??**(2) takes the following useful form:

$$x = (x - y)^+ + x \wedge y.$$

Regarding the absolute value, we have the following useful identities.

Theorem 4.1.6. (Aliprantis, 1985)If t and z are elements in a vector lattice, then we get some result as follow,

(1) $t \vee z = \frac{1}{2}(t + z + |t - z|)$ and $t \wedge z = \frac{1}{2}(t + z - |t - z|)$.

(2) $|t - z| = t \vee t - z \wedge z$.

(3) $|t| \vee |z| = \frac{1}{2}(|t + z| + |t - z|)$.

(4) $|t| \wedge |z| = \frac{1}{2}\,||t + z| - |t - z||$.

Proof. Although we skipped proof of the identities 1,2,3, They will be used in the proof of identity 4.

(4) by using (1) and (3), we can get,
$\big|\, |t+z| - |t-z| \,\big| = 2\,(|t+z| \vee |t-z|) - (|t+z| + |t-z|)$
$= 2(|t| + |z|) - 2\,(|t| \vee |z|)$
$= 2\,(|t| \wedge |z|)$ □

Corollary 4.1.7. Let E be ordered lineer space and for all $t \in E$. Then following are equivalent;

i) E is Vector lattice.

ii) if $t \vee 0$ exist ,then $t^+ \in E$

iii) if $(-t) \vee 0$, then $x^- \in E$

iv) if $t \vee (-t)$, then $|t| \in E$

Definition 4.1.8. Let E is Riesz space and $x, y \in E$. If $|x| \wedge |y| = 0$ then x and y is perpendicular each other. And,it is denoted by $x \perp y$

Note that by Theorem 4.1.6 (4). We obtain an altarnative definition as $t \perp z$ if and only if $|t+z| = |t-z|$. Two subsetts T and Z of a vector lattice are **disjoint**(denoted $T \perp Z$) if t $\perp$ z holds for all $t \in T$ and all $z \in T$.

If T is a non-empty subset of a vector lattice E, then its **disjoint complement** T^d is defined by

$$T^d := \{t \in E : t \perp z \text{ for all } z \in T\}$$

T^{dd} is written for $(T^d)^d$. Note that $T \cap T^d = \{0\}$.

If T and Z are subsetts of a vector lattice, then we will write

$$\begin{aligned}
|T| &:= \{|t| : t \in T\};\\
T^+ &:= \{t^+ : t \in T\};\\
T^- &:= \{t^- : t \in T\};\\
T \vee Z &:= \{t \vee z : t \in T \text{ and } z \in Z\};\\
T \wedge Z &:= \{t \wedge z : t \in T \text{ and } z \in T\};\\
u \vee T &:= \{u \vee T : t \in T\};\\
u \wedge T &:= \{u \wedge t : t \in T\}.
\end{aligned}$$

The following theorem show us that every vector lattice satisfies infinite distributive law

Theorem 4.1.9. (Aliprantis, 1985) Let T be non-empty subsett of a vector lattice. If sup$T$ exists, then sup$(t \wedge T)$ exists for each t and

$$\text{sup}(t \wedge T) = x \wedge \text{ sup } T.$$

Anagously, if inf$T$ exists, then infimum$(t \wedge T)$ exists for each x and

$$infimum(t \vee T) = t \vee infimum(T).$$

The following result has lots of important inequalites thatt are commonly used .

Theorem 4.1.10. (Aliprantis, 1985) For a,b,c in a Riesz space E following inequalities hold:

1. $||a| - |b|| \leq |a + b| \leq |a| + |b|$. (the triangle inequality)
2. $|a \vee c - b \vee c| \leq |a - b|$ and $|a \wedge c - b \wedge c| \leq |a - b|$.
3. if in addition a, b and a are all positive, then

$$a \wedge (b + c) \leq a \wedge b + a \wedge c.$$

In particular, note that we have where E is Riesz space

$$|a^+ - b^+| \leq |a - b|.$$

Definition 4.1.11. The linear subspace V of E is called a **vector lattice subspace** of E if for all members f and g of V the elements $f \vee g$ and $f \wedge g$ are likewise members of V

We make some remarks concerning these definitions. In the definition of a Riesz subspace it is sufficient to say that V is a Riesz subspace of E if V is a linear subspace of E such that $f \in V, g \in V$ implies $f \vee g \in V$ because this implies already that $f \wedge g \in V$. Indeed, from $f \in V, g \in V$ it follows that $-f \in V, -g \in V$, so

$$f \wedge g = -\{(-f) \vee (-g)\} \in V$$

Definition 4.1.12. (Zaanen, 1997)The subset T of E is said to be **solid** if it follows from $f \in S$ and $|g| \leq |f|$ that $g \in S$

Notice that a solid sett and lineer subspace is not need solid when it is intersected. To illusturate, close-unitt ball of L_1 norm on $C[0, 1]$ is solid but affin functions and its intersect is not solid in subspace.

4.2 Ideal in Riesz Spaces

Definition 4.2.1. A vector subspace F of a vector lattice E is said to be an **order ideal** whenever $|x| \leq |y|$ and $y \in F$ imply $x \in F$

Definition 4.2.2. The subset T of a Riesz space E is called **ideal** in E if A is a solid linear subspace of E.

Sometimes this is called an order ideal if it necessary to distinguish it from an algebraic ideal in ring.

The definition of an ideal may be reformulated by saying that an ideal A in E is a linear subspace such that $f \in A, |g| \leq |f|$ implies $g \in A$. This can be reformulated one more by saying that an ideal A in E is a linear subspace in E such that

(i) $f \in A$ if and only if $|f| \in A$,

(ii) $0 \leq g \leq f \in A$ implies $g \in A$.

(iii) $0 \leq f \in A$ and $g \in E^+$ implies $f \wedge g \in A$

The last condition clearly show the analogy with the definition of a ring ideal in a commutative ring R, where I is an ideal in R whenever I is a subring satisfying the condition that $f \in I$, $g \in R$ implies $fg \in I$

Example 4.2.3. Let W be a Riesz space $C([0, 1])$ and V be the linear subspace of W consisting of all constant function on $[0, 1]$. Then V is a Riesz subspace but not ideal.

Theorem 4.2.4. Let A_1 and A_2 be ideals in Riesz space E. Then algebraic sum $A_1 + A_2$ is an ideal in E.

Proof. Let $f \in A_1 + A_2$ and $|g| \leq |f|$. we have to prove that $g \in A_1 + A_2$. The element f can be written as $f = f' + f''$ with $f' \in A_1$ and $f'' \in A_2$. Note now that

$$g^+ \leq |g| \leq |f| \leq |f'| + |f''|,$$

so in view of the Riesz decomposition theorem there exist element g' and g'' such that $g^+ = g' + g''$ with $0 \leq g' \leq |f'|$ and $0 \leq g'' \leq |f''|$. Since $f' \in A_1$ and $f'' \in A_2$, it follows that $g' \in A_1$ and $g'' \in A_2$, so $g^+ = g' + g'' \in A_1 + A_2$. Similarly $g^- \in A_1 + A_2$. Hence $g = g^+ - g^- \in A_1 + A2$

□

4.3 Band in Riesz Spaces

Definition 4.3.1. An Order ideal B of Riesz space E is called **band** if it follows from $D \subset B$, $D \neq \emptyset$ and $f_0 = supD$ existing in E that $f_0 \in B$

Remark 4.3.2. it is obvious that a band is an ideal and that ideal is a Riesz subspace. However converse of these statements do not hold.

Let E be a Riesz space $C([0, 1])$,

Example 4.3.3. Let A be linear subspace of E consisting of all functions f satisfying $f(1/2) = 0$. Then A is an ideal but not a band of E

Example 4.3.4. Let B be the linear subspace of E consisting of all f satisfy $f \equiv 0$ on $[0, 1/2]$. Then B is a band in E.

Example 4.3.5. Let E be a Riesz space if B and V are band in E than algebraic sum of B+V is not band in E.

Theorem 4.3.6. (Jonge and Roijj, 1977) The following statements are equivalent.

(a) L is Archimedean.

(b) $u, v \in L^+$ and $0 \leq nv \leq u$ for n=1,2,3 ... implies $v = 0$.

(c) If $v \in L^+$, $v = 0$, then nv: n=1,2,3... is not bounded above.

(d) $A = (A^d)^d$ for every band A in L.

5. CHAPTER 4

In this chapter, the notions of bands of Riesz spaces are generalized to partially ordered lineer space which has order unitt, and the fundamental results are given.

Definition 5.0.1. (Aliprantis and Tourky, 2007) A non empty subset K of a lineer space is said to be a **cone** if it satiesfies the following three properties:

1. K+K $\subseteq$ K,
2. α K $\subseteq$ K for all $\alpha \geq 0$,
3. K $\cap$ (-K) =0,

Recall in Riesz Spaces disjointnes is defined as $|x| \wedge |y| = 0$ another defination which gets from first one $|x + y| = |x - y|$ it gives motivation for how should define disjointnes in partial ordered space.

$t, z \in E$ is *disjoint*, in symbols $t \perp z$,if

$$\{t + z, t - z\}^u = \{t - z, -t - z\}^u$$

A subset T of E of The *disjoint complement* is given by

$$T^d := \{z \in E : t \perp z \text{ for all } t \in T\}.$$

A lineer subspace B of E is called a *band* if $(B^d)^d = B$

This characterization is valid for Archimedian Riesz space. In partially ordered space, it is not simply define bands. For this porpose, one should define disjointnes and for better results it is required some extra properties on partially ordered space.

From now on, we will suppose that T is Archimedean P.O.V.S and has an *order unit* $o \in K$. In other words ,for each $t \in T$ There exists $0 \leq \beta$ such that

$$-\beta o \leq t \leq \beta t$$

Then T is directed.

Naturally, an order unit $o \in K$ produce a norm $\|.\|_o$ on X by

$$\|x\|_o = inf\{\beta > 0 \ -\beta o \leq x \leq \beta o\}.$$

via this norm each positive lineer functional $\Psi : T \to \mathbb{R}$ is continous, as $|\Psi(t)| \leq \Psi(o) \ \forall \ t \in T$ with $\|t\|_o \leq 1$.

In the norm dual $T^{'}$ the set

$$K^* := \{\Psi \in T^{'} : \Psi(K) \subseteq [0, \infty)\}$$

is a cone. We remember the functional representation of T. Denote

$$\Sigma := \{\Psi \in K^* : \Psi(o) = 1\}.$$

By using Banach-Alaoglu Ttheorem close unit ball $B^{'}$ of $T^{'}$ is weak*-compact. also Σ subset of $B^{'}$ is weak*-closed ,and hence weak*-compact.Let see this set

$$\Lambda := \{\Psi \in \Lambda : \Psi \text{ is extrem point of } \Lambda\}$$

Definition 5.0.2 (Extreme). $\Psi \in \Sigma$ is called an extreme point of Σ if $\varphi_1, \varphi_2 \in \Sigma$, $\alpha\varphi_1 + (1-\alpha)\varphi_2 = \varphi \Rightarrow \varphi_1 = \varphi_2 = \varphi$

Theorem 5.0.3. If a convex set C of a lineer space X is compact for normed topology τ on X, then C has an extrem point. Moreover, C is the τ - closed convex hull of its extrem points.

By using Krein-Milman $\Sigma \neq \emptyset$. Generally, Σ is not necessary to be weak*-closed, even T is not required finite dimensional condition.For $M \subseteq X'$ the weak*-closure of M is denoted by $\overline{M}$.Thus $\overline{\Lambda}$

Theorem 5.0.4. Let $\Phi : T \to C(\overline{\Lambda})$ is map.It is defined as

$$(\Phi(t))(\Psi) = \Psi(t) \text{ for } \Psi \in \overline{\Lambda}$$

(Φ, Λ) is a functional representation of T. That is, Φ is lineer, bipositive, unit goes to constant 1 function under Φ (this indicate isometry in respect of the o-norm) and the image of Φ separetes the points of $\overline{\Lambda}$.

Definition 5.0.5 (order dense)**.** A lineer subspace D of a P.O.V.S X is called **order dense** in X if $\forall\, x \in X$ one get

$$t = inf\{z \in D : z \geq t\}$$

In other words, x is the sup of the set $\{z \in D : z \geq t\}$ in X

5.1 Riesz completion

Definition 5.1.1 (Haandel, 1993)**.** Let E be a Riesz space and X is partialy ordered lineer space, then Y is called vector lattice cover if the following satiesfied,

1. there is a bipositive lineer map from X to E
2. $i(X)$ is order dense in E

Moreover, The pair (E, i) is called *Riesz completion* of X if E has no proper vector sublattice of including $i(X)$

Remark 5.1.2. If G has a Riesz complation,then it is unique up to Riesz isomorphisms. This follows from the fact that if both (E_1, φ_1) and (E_2, φ_2) are Riesz complations of G, there is Riesz homomorphisms $\tilde{\varphi_1} : E_2 \to E_1$ and $\tilde{\varphi_1} : E_1 \to E_2$ such that $\tilde{\varphi_1} \circ \varphi_2 = \varphi_1$ and $\tilde{\varphi_2} \circ \varphi_1 = \varphi_2$. Since $\tilde{\varphi_1} \circ \tilde{\varphi_2} \circ \varphi_1 = \varphi_1$ and $\tilde{\varphi_2} \circ \tilde{\varphi_1} \circ \varphi_2 = \varphi_2$ we can conclude that $\tilde{\varphi_1} \circ \tilde{\varphi_2} =id_{E_1}$ and $\tilde{\varphi_2} \circ \tilde{\varphi_1} =id_{E_2}$. This means that $\tilde{\varphi_1}$ and $\tilde{\varphi_2}$ are each other's inverses, and therefore, E_1 and E_2 are Riesz isomorphic.

Theorem 5.1.3. [Haandel, 1993] Let X and Y be Poset and let $t, z \in X$

(i) If Y has a subspace X, then $t \perp_{sv} z$ in Y indicate that $t \perp_{sv} z$ in X.

(ii) If X is order dense subspace of Y , then $t \perp_{sv} z$ in $Y \iff x \perp_{sv} y$ in X.

Definition 5.1.4 (Haandel, 1993)**.** E is called **pre-Riesz** if for every $x \in E$ and every finite non-empty subsett S of E such thatt every upper bound of $x + S$ is an upper bound of S one has that x is positive.

The following theorem will give us an alternative and more practical definition of pre-Riesz space.

Theorem 5.1.5. (Haandel, 1993) Let X be a partialy ordered lineer space. The following statements are equivalent

1. X is pre-Riesz.
2. There exists a Riesz space E a bipositive lineer map from X to E such thatt $i(X)$ is order dense in E and generates E as a Riesz space. Furthermore, whole spaces E are iso-morphic as Riesz spaces.

By using Theorem 5.1.3 and Theorem 5.1.5 one get following theorem,

Theorem 5.1.6. (Kalauch, Lemmans and Gaans, 2015) Let (E, i) is a vector lattice cover of X, then $t \perp_{sv} z$ if and only if $i(t) \perp_{sv} i(z)$

Theorem 5.1.7. (Kalauch, Lemmans and Gaans, 2015) Let X be an Archimeadian partialy ordered lineer space and has an order unit o. Then $C(\overline{\Lambda})$ is a lineer lattice cover of X. Furthermore, Riesz subspace of $C(\overline{\Lambda})$ generated by $\Phi(X)$ is the Riesz complation of X.

Proposition 5.1.8. (Kalauch, Lemmans and Gaans, 2015) Let X be pre-Riesz space and T is lineer subsett of X then,

1. $(T^{d_{sv}})^{d_{sv}} \supseteq T$.
2. $T^{d_{sv}}$ is sv-band in X

Proof. (i) Let $t \in T$. Then $t \perp_{sv} z$ for every $z \in T^{d_{sv}}$, Thus $t \in (T^{d_{sv}})^{d_{sv}}$

(ii) By (i), $((T^{d_{sv}})^{d_{sv}})^{d_{sv}} \subseteq M^{d_{sv}}$, and by (i) $T^{d_{sv}}$ this yields reverse inclusion hence $T^{d_{sv}}$ is band. □

Proposition 5.1.9. Let E be a pre-Riesz space, X an order dense subspace of E and I a band in E. Then there is a band J in Y such thatt $I = J \cap X$.

Proof. Let $D = I^d$ in X , so $D^d = (I^d)^d = I$, Since I is band in X. So, I is the disjoint complement of D in X. Let J be the disjoint complement of D in Y. J is a band in Y due to proposition 5.1.8, and it can be observed in Maris Vaan Handel Phd Thesis. □

Theorem 5.1.10. (Haandel, 1993) Every directed partialy ordered vector space G has a Riesz complation.

Definition 5.1.11. Let S be a subsett of $C(\Omega)$, Then the **carrier** of S is defined by

$$carr(S) = \{\omega \in \Omega : \text{there is } s \in S \text{ such thatt } s(\omega) \neq 0\ \}$$

The carier of an ideal in $C(\Omega)$ is an open subset of Ω. Different ideals in $C(\Omega)$ may have same carier.

Definition 5.1.12. An open subsett O of Ω is called **regularly open** if O=int($\overline{O}$)

Example 5.1.13. An example of a non-regular open subsett of $[0, 1]$ is $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$

Definition 5.1.14. For a open subsett O of Ω

$$I_O := \{s \in C(\Omega) : \forall\omega \in \Omega\backslash O \text{ one has } s(\omega) = 0\}$$

is the greatest ideal containing O as its carier.

Proposition 5.1.15. If P is an open subsett of Ω as well, then $I_O = I_P$ if and only if $O = P$

Theorem 5.1.16. Let B a band in $C(\Omega)$

1. For every band B in $C(\Omega)$ there is a open subsett O such thatt $B = I_O$,(thus whenever its carier is known, a band is determined)

2. For every open subsett O of Ω the ideal I_O is a band if and only if O is regularly open.

Recall that x and $y \in C(\Omega)$ are disjoint if and only if for every $\omega \in \Omega$ one has $x(\omega) = 0$ or $y(\omega) = 0$.

We will intersect properties of disjointnes in vector lattice covers and disjointnes in $C(X)$. If X is an Archimedian P.O.V.S which has unit and Φ is its embedding in to $C(\overline{\Lambda})$,then theorem 5.1.7 yield that for $x, y \in X$

$x \perp y$ in $X \iff \Phi(x) \perp (y)$ in $C(\overline{\Lambda}) \iff$ for each $\Psi \in \overline{\Lambda}$ one has $\Psi(x) = 0$ or $\Psi(y)$

Furthermore, If B is a band in X, then one can find a regular open sett $O \subset \overline{\Lambda}$ such thatt

$$B = \{x \in X : \Psi(x) = 0 \text{ for all } \Psi \in \overline{\Lambda}\backslash O\}$$

5.2 Characterization of bands

We dealed with an Archimedian P.O.V.S X with order unitt and its Ψ-embedding in to $C(\overline{\lambda})$ as given previous section.

For $B \subset X$ we will work with carr($\Psi(B)$) , and it will be abbreviated by carr(B)

The following remark as a consequence of Banach-Stone Theorem.

Remark 5.2.1. If $X = C(\Omega)$, here Ω compct Hausdorf space, then $\lambda = \overline{\lambda}$ and Ω homemorphic, such thatt the carier of $B \subseteq C(\Omega)$ defined as a subsett of $\overline{\lambda}$ has correspondence to the carier of B seen as a subsett of Ω. In that meaning the definition of the carier in X is compitable with the definition of the carier in previous section. It is denoted

$$N(B) := \overline{\lambda} \backslash carr(B) = \{\Psi \in \overline{\lambda} : \Psi(b) = 0 \ \forall \ b \in B\}$$

For a set $T \subset \overline{\lambda}$, we fix the notions $T^c := \overline{\lambda}$ and spanned(T) denotes the lineer subspace of X' spanned by T and the affin hull is defined by

$$\text{affine}(T) = \left\{ \sum_{i=1}^{n} \alpha_i t_i : \sum_{i=1}^{n} \alpha_i = 1, \ n \in \mathbb{N}, \ \alpha_i \in \mathbb{R}, t_i \in T \right\}$$

For $T \subseteq \overline{\Lambda}$ denote the **zero set induced by**

$$Zero(M) = \{x \in X : \Psi(x) = 0 \ \text{ for all } \ \Psi \in M\}$$

Now, we shall characterize those set $T \subset \overline{\Lambda}$ that belongs B in X in a manner that $B = Zero(T)$ and $T = N(B)$. The next notion is required.

Definition 5.2.2. (Kalauch, Lemmans and Gaans, 2015) For $T \subset \overline{\Lambda}$ we define the **saturation** of T by

$$\text{saturation}(T) := \overline{\Lambda} \cap \overline{\text{spanned}(T)}$$

A sett $T \subseteq \overline{\Lambda}$ is **saturated** if T = saturation(T)

Proposition 5.2.3. Let B a band in X, then $N(B)$ is saturated.

In the following proposition it will be charterized the saturation in terms of zero sets. Before this one, Two properties should be stated.

Lemma 5.2.4. (Kalauch, Lemmans and Gaans, 2015) Let $T \subset \overline{\Lambda}$. Then the next is satified,

(i) saturation(saturation(T)) = saturation(T);

(ii) $Zero$(saturation(T)) = $Zero(T)$.

Proposition 5.2.5. Let T $\subset \overline{\Lambda}$. Then we have,

$$\text{saturation}(T) = N(Zero(T))$$

Proof. If $\Psi \in \overline{\Lambda} \cap \text{spanned}(T)$ and $x \in Zero(T)$, then $\Psi(t) = 0$, so

$$\overline{\Lambda} \cap \text{spanned}(T) \subseteq \{\Psi \in \overline{\Lambda} : \Psi(t) = 0 \text{ for all } t \in Zero(T)\}$$

$$= \bigcap \{\Psi \in \overline{\Lambda} : \Psi(t) = 0\}.$$

For the reason that the right hand side of the equality is weak* -closed sett, it brings us that

$$\overline{\Lambda} \cap \overline{\text{spanned}(T)} \subseteq \{\Psi \in \overline{\Lambda} : \Psi(x) = 0 \text{ for all } x \in Zero(T)\}$$

Vice versa, let

$$\Psi \in \overline{\Lambda} \backslash \overline{\text{spanned(T)}}$$

By using Hahn-Banach Ttheorem, there is $t \in X$ such thatt $\psi(t) \neq 0$ and $\forall$
$\Psi \in \overline{\text{spanned}(T)}$ one has $\Psi(t) = 0$. Spesificaly, $x \in Zero(T)$. Thus,

$$\psi \notin \{\Psi \in \overline{\Lambda} : \Psi(x) = 0 \text{ for all } x \in Zero(T)\}$$

□

It can be seen that the saturation of $T \subset \overline{\Lambda}$ is the biggest subsett N of $\overline{\Lambda}$ such thatt $Zero(N) = Zero(T)$.

In the consideration the following corollary it is quite natural to use the affine hull rather than the lineer span while describing saturations. As a result of Proposition 5.2.5 that the same sets are obtained.

Corollary 5.2.6. Let $M \subset \overline{\Lambda}$. Then saturation$(T) = \overline{\Lambda} \cap \overline{\text{affine}(T)}$.

Our final result will be related with understanding bands in terms of subsets of $\overline{\Lambda}$, The following llemmas are required for this purpose.

Lemma 5.2.7. (Kalauch, Lemmans and Gaans, 2015) Let $B \subseteq X$. Then $B \subseteq Zero(N(B)) \subseteq B^{dd}$

As a direct result of Lemma 5.2.7, one have the following result,

Proposition 5.2.8. Let B be a band in X. Then $B = Zero(N(B))$

Thus, in analogy to bands in $C(\Omega)$(we discused in previous section Band in C(X)), whenever its carier is known , a band in X is determined

Lemma 5.2.9. (Kalauch, Lemmans and Gaans, 2015) Let $B \subseteq X$ be such thatt B=Zero(N(B)). Then

$$B^d = \{x \in X : \Psi(x) = 0 \,\forall\, \Psi \in \text{saturation}((N(B))^c\}$$

The following definition will introduce those saturated subsetts of $\overline{\Lambda}$ the zero sets of which take place to be bands. Like a band and its disjoint complemant. The next definition is also meaningful in the more general case of a top vector space E (rather than dual of X) and a subset V of E (rather than $\overline{\Lambda}$). With hindsight and in the light of Corollary 5.2.6, a subset T of V is called *saturated* (in V) if

$$T = saturation(T) := V \cap \overline{\text{affine}(T)}$$

Definition 5.2.10. (Kalauch, Lemmans and Gaans, 2015) Let E be a top vector space and let V be subset of E. A subset T of V is called *bisaturated* (in V) if

$$T = V \cap \overline{\text{affine}(V \backslash (V \cap \overline{\text{affine}(V \backslash T)}))}.$$

For two subset T_1 and T_2 of V the set $\{T_1, T_2\}$ is called a *bisaturatted pair* (in V). If

$$T_1 = V \cap \overline{\text{affine}(V \backslash T_2)} \text{ and } T_2 = V \cap \overline{\text{affine}(V \backslash T_1)}$$

.

For a subset T of V we simply write $T^c := V\backslash T$. Note that a set $T \subseteq V$ is bisaturatted if and only if

$$T = \text{saturation}((\text{saturation}(T^c))^c),$$

and for two subsets $T_1, T_2 \subseteq V$ the set $\{T_1, T_2\}$ is a bisaturatted pair if and only if T_1 = saturation$({T_2}^c)$ and T_2=saturation$({T_1}^c)$

Obviously, every bisaturatted subset of V is saturated. But the converse is not true. Next it will be given an example of a bisaturatted set.

Example 5.2.11. Consider continous function from $[0, 1]$ to $\mathbb{R}$(so $\overline{\Lambda}$ contain the calulation points and can be correpondeced with $[0, 1]$), and

$$B = \{x \in C[0, 1] : x(t) = 0 \,\forall\, t \in [\frac{1}{2}, 1]\}.$$

it is claimed that $N(B) = [\frac{1}{2}, 1]$ is bisaturatted. Indeed,

$$\text{saturation}((N(B))^c) = [0, \frac{1}{2}]$$

,

so

$$\text{saturation}[(\text{saturation}[(N(B))^c])^c] = [\frac{1}{2}, 1] = N(B)$$

it is easy to see that $(N(B))^c = [0, \frac{1}{2})$ is not saturated, because of the fact that every saturated set is closed.

The following lemma is connected with bisaturatted sets and bisaturatted tuple in V and try to find out a geometric depict of bisaturatted sets. we will that for a bisaturatted tuple $\{T_1, T_2\}$ in V we have

$$T_1 \cup T_2 = V,$$

Because $T_1 \cup T_2 = T_1 \cup \text{saturation}({T_1}^c) \supseteq T_1 \cup {T_1}^c = V$

Lemma 5.2.12. (Kalauch, Lemmans and Gaans, 2015) For two saturated sets $T_1, T_2 \subseteq V$ the next statements are equivalent to each other,

(a) T_1 is bisaturatted and M_2 = saturation$({T_1}^c)$;

(b) $\{T_1, T_2\}$ is a bisaturatted tuple;

(c) $\overline{\text{affine}(T_1)} = \overline{\text{affine}(V\backslash T_2)}$ and $\overline{\text{affine}(T_2)} = \overline{\text{affine}(T_1\backslash V)}$

(d) there is affine subspaces S_1 and S_2 of E such thatt

$$V \subseteq S_1 \cup S_2,\, S_1 = \overline{\text{affine}(V\backslash S_2)},\, S_2 = \overline{\text{affine}(V\backslash S_1)},\, T_i = V \cap S_i,\, i = 1, 2.$$

Proof. (a) ⇒ (b): T_1 is bisaturatted, so T_1 = saturation((saturation$({T_1}^c))^c)$ = saturation$({T_2}^c)$

(b) ⇒ (c): we have

$$T_2 = V \cap \overline{\text{affine}({T_1}^c)} \subseteq \overline{\text{affine}({T_1}^c)}$$

so $\overline{\text{affine}(T_2)} \subseteq \overline{\text{affine}({T_1}^c)}$. Also,

$$T_2 = V \cap \overline{\text{affine}({T_1}^c)} \supseteq V \cap {T_1}^c = {T_1}^c,$$

so $\overline{\text{affine}(T_2)} \supseteq \overline{\text{affine}({T_1}^c)}$. Hence $\overline{\text{affine}(T_1)} = \overline{\text{affine}({T_2}^c)}$. By similar method, $\overline{\text{affine}(T_1)} = \overline{\text{affine}({T_2}^c)}$.

$(c) \Rightarrow (d)$: Choose $S_i = \overline{\text{affine}(T_i)}$, $i = 1, 2$. Since T_i is saturated, $T_i = V \cap \overline{\text{affine}(T_i)} = V \cap S_i$. Hence

$$S_1 = \overline{\text{affine}(T_1)} = \overline{\text{affine}(V\backslash M_2)} = \overline{\text{affine}(V\backslash(V \cap S_2))} = \overline{\text{affine}(V\backslash S_2)}$$

and,similarly, $S_2 = \overline{\text{affine}(V\backslash S_1)}$.

(d) ⇒ (a): Since $V\backslash T_1 = V\backslash(V \cap S_1) = V\backslash S_1$, we have

$$T_1 = V \cap S_1 = V \cap \overline{\text{affine}(V\backslash S_2)} = V \cap \overline{\text{affine}(V\backslash \overline{\text{affine}(V\backslash S_1)})}$$
$$= V \cap \overline{\text{affine}(V\backslash \overline{\text{affine}(V\backslash T_1)})} = V \cap \overline{\text{affine}(V\backslash (V \cap \overline{\text{affine}(V\backslash T_1)})}$$
$$=\text{saturation}((\text{saturation}({T_1}^c))^c,$$

thus T_1 is bisaturatted. □

For $E = X'$ and $V = \overline{\Lambda}$, it is described bisaturatted pairs in $\overline{\Lambda}$ to disjoint complements in X.

Proposition 5.2.13. Let $T_1, T_2 \subseteq \overline{\Lambda}$ be saturated sets and $B_1 = Zero(T_1)$ and $B_2 = Zero(T_2)$. Then $\{T_1, T_2\}$ be a bisaturatted pair if and only if $B1 = {B_2}^d$ and $B_2 = {B_1}^d$.

Proof. Let $\{T_1, T_2\}$ be a bisaturatted pair. To show that $B_2 \subseteq {B_1}^d$, let $x_1 \in B_1$, $x_2 \in B_2$. For all $\Psi \in M_1$ one has $\Phi(x_1)(\Psi) = 0$ and for all $\Psi \in M_2$ one has $\Phi(x_2)(\Psi) = 0$. Since $M_1 \cup M_2 \supseteq \overline{\Lambda}$, it follows that $\Phi(x_1) \perp \Psi(x_2)$, which implies $x_1 \perp x_2$. Hence $B_2 \subseteq {B_1}^d$.

Likewise, let $x \in B_1^d$. Then forr every $y \in B_1$ one can get $x \perp y$ and $\Phi(x) \perp \Phi(y)$. Hence, $\Phi(x)(\Psi) = 0$ for every

$$\Psi \in \text{carr}(B_1) = \overline{\Lambda} \backslash N(B_1) = \overline{\Lambda} \backslash M_1,$$

due to Proposition 5.2.5. Thus, $\Phi(x)(\Psi) = \Psi(x) = 0$ for every

$$\Psi \in \text{saturation(carr}(B_1)) = \text{saturation}(\overline{\Lambda} \backslash M_1) = M_2$$

Hence $x \in Zero(M_2) = B_2$. Therefore ${B_1}^d = B_2$ and, by symmetry, ${B_2}^d = B_1$.

If $B_2 = {B_1}^d$, then due to Proposition 5.2.5 ,

$$\overline{\Lambda} \backslash M_1 = \overline{\Lambda} \backslash N(Zero(M_1)) = \overline{\Lambda} \backslash N(B_1) = \text{carr}(B_1)$$

so, by

$$Zero(\overline{\Lambda} \backslash M_1) = Zero(\text{carr}(B_1)) = {B_1}^d.$$

Again by Proposition 5.2.5 .

$$M_2 = N(Zero(M_2)) = N(B_2) = N({B_1}^d) = N(Zero(\overline{\Lambda} \backslash M_1)) = \text{saturation}(\overline{\Lambda} \backslash M1).$$

By symmetry, $\{M_1, M_2\}$ is a bisaturatted pair

We reach at the desired result. The assumption in ttheorem is natural because of proposition 5.2.5

□

Theorem 5.2.14. (Kalauch, Lemmans and Gaans, 2015) Let $B \subset X$ be such thatt $B = Zero(N(B))$. Then B is a band if and only if $N(B)$ is bisaturatted.

Proof. Assume, that $B_1 := B = Zero(N(B))$ is a band. Define as $M_1 := N(B)$ and $M_2 := \text{saturation}({M_1}^c)$. Then by LLemma 5.2.9 , $\{M_1, M_2\}$ is a bisaturatted tuple. Hence $N(B)$ is a bisaturatted set, by using Lemma 5.2.12

Coversely, let $M_1 : N(B)$ be bisaturatted and let $B_1 := B = Zero(N(B))$. Let $M_2 =$ saturation$({M_1}^c)$. Then, by Lemma 5.2.12 , $\{M_1, M_2\}$ is a bisaturatted pair. Let $B_2 := Zero(M_2)$. Proposition 5.2.13 yields that $B_1 = {B_2}^d = {B_1}^{dd}$, hence $B = B_1$ is a band. $\square$

Theorem 5.2.15. (Kalauch, Lemmans and Gaans, 2015) The cardinality of bands in an n-dimensionals partialy order vector space X order by generate polyhedral cone K is bounded above by $\frac{1}{4}2^{2^n}$, provided $n \geq 2$

A combination of Theorem 4.2.17 and Lemma 4.2.18 yields Theorem 4.2.19. it is believed that this bound in Theorem 4.2.19 is far from optimal. The next example indicates that there is (n+1)-dimensionals partialy order lineer space with $\binom{2n}{n} + 2$ band. Specifically, it follows that there is P.O.V.S with dimensions $n \geq 4$ that have more band than an n-dimensionals Archimedian vector lattice, which has 2^n bands.

Example 5.2.16. (Kalauch, Lemmans and Gaans, 2015) Let P be a polytop in $\mathbb{R}^n$ with vertex set $V = \{v_1, ...v_{2n}\}$ that is in generel lineer position, In other words, for each affin subspace $U \subset \mathbb{R}^n$ with dim$U < v$, we get that $|V \cap U| \leq$ dimension $U+$ 1.That is, no three points on straight line in V exist, no four point in V in two dimensional plane, etc. To prove existence of such a set V, let n ≥ 1 and S_{n-1} be unit sphere in $\mathbb{R}^n$. Pick with equal probability at random $v_1, ..., v_{2n}$ from S_{n-1}. Then the convex hull of $\{v_1, ..., v_{2n}\}$ is , with probability one, a polytop, P. If neccessary make a translation, in order to P can contain 0. By this way constitute with the vertex set $V = \{v_1, ..., v_{2n}\}$ of P is in generel lineer position.

For $i = 1, ..., 2n$ let $w_i = (v_i, 1) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ and define

$$K_V = \{x \in \mathbb{R}^{n+1} : \langle w_i, x\rangle \geq 0 \quad for\ all \quad .ßcontent = 1, ...2n\}$$

Here $\langle \cdot, \cdot\rangle$ we use natural inner product on $\mathbb{R}^{n+1}$. As $w_1, ..., w_2n$ span $\mathbb{R}^{n+1}$, K_V is a close generate polyhedral cone in $\mathbb{R}^{n+1}$ with $u = (0, 1) \in \mathbb{R}^n \times \mathbb{R}$ in its interior. At this moment write that

$$\Sigma = \{z \in {K_V}' : \langle z, u\rangle = 1\} = \{z \in {K_V}' : z = (v, 1) \quad and \quad v \in P\}$$

and $\Lambda = \{w_1, ..., w_{2n}\}$, as V is in generel lineer position. Furthermore, if U is an affin subspace of $\mathbb{R}^{n+1}$ contained in Σ with dimension U is greater than n, then $|\Lambda \cap U| \leq$ dimension $U + 1$.

Next it will be shown that $(\mathbb{R}^{n+1}, K_V)$ has $\binom{2n}{n} + 2$ bends in virtue of showing that $S \subset \Lambda$ is bisaturatted $\iff$ $S = \emptyset$, $S = \Lambda$, $|S| = n$. Obviously, if S is a bisatrated set in Λ and cardinality of S greater then n , then $dimS > n - 1$, as V is in generel lineer position. Hence, $S = \Lambda \cap$ affine $S = \Lambda$. In contrast, if S is bisaturatted in Λ and cardinality of S less than n, then $|\Lambda \setminus S| > n$, and thus $S = \emptyset$

On the other hand, if Cardinality of S is n, then dimen$(S) = n - 1$ and dimen($\Lambda \setminus S$) , as V is in generel lineer position. This indicates that

$$|\Lambda \cap affineS| \leq dimen(affine(\Lambda \cap affineS)) + 1 = dimen(S) + 1 = n = |S| \text{ ,}$$

and because $\Lambda \cap$ affine $S \supseteq S$ it follows that $\Lambda \cap$ affine $S = S$. In the same manner $\Lambda \cap$ affine $(\Lambda \setminus S)$ = $\Lambda \setminus S$. Thus $\{S, \Lambda \setminus S\}$ is bisaturatted tuple in Λ. it follows that $(\mathbb{R}^{n+1}, K_V)$ has $\binom{2n}{n} + 2$ bands.

it is conjuctured that the optimal bound from above for the cardinality of bands in an $(n+1)$ dimensionals partialy ordered lineer space (X, K) with close generate cone K, is $\binom{2n}{n} + 2$.

Example 5.2.17. (Kalauch, Lemmans and Gaans, 2015) Let $Y = \mathbb{R}^4$, we will define,

$$\varphi_1 = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 2 \end{pmatrix} \quad \varphi_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 3 \end{pmatrix} \quad \varphi_4 = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 0 \end{pmatrix} \quad \varphi_5 = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and,

let us focus on the set K

K = $\{x \in \mathbb{R}^4 : \varphi_i(x) \geq 0$ for all $i \in \{1, ..., 5\}.\}$

$$\begin{bmatrix} 0 & 4 & 2 & 2 \\ 4 & 0 & 2 & 2 \\ 2 & 2 & 0 & 4 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 2 & 2 \\ 4 & 0 & 2 & 2 \\ 2 & 2 & 0 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 2 & 2 \\ 2 & 2 & 0 & 4 \\ 0 & 4 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 4 \\ 0 & 4 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since matrices of $\varphi_1, \varphi_2, \varphi_3, \varphi_5$ rank=4, K is cone in $\mathbb{R}^4$.

$$\varphi_1 u = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix} * \frac{1}{8}(1,1,1,1) = 1$$

$$\varphi_2 u = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 2 \end{pmatrix} * \frac{1}{8}(1,1,1,1) = 1$$

$$\varphi_3 u = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix} * \frac{1}{8}(1,1,1,1) = 1$$

$$\varphi_4 u = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 0 \end{pmatrix} * \frac{1}{8}(1,1,1,1) = 1$$

$$\varphi_5 u = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \end{pmatrix} * \frac{1}{8}(1,1,1,1) = 1$$

$For v = \frac{1}{4}(1,1,1,1)^{Tranpose}$. We have $\varphi_i(v) = 1$ for all $i \in 1,2,3,4,5$, thus v is an inner point of K, and we set $\Sigma := \{\varphi \in K'\colon \varphi_u = 1\}$. A simple evaluation give us that φ_i is an extrem point of $\Sigma \; \forall \; i$, thus

$$\Lambda = \{\varphi, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$$

Observe that span $\{\varphi, \varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ has dimension 3. Indeed,

$$\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 = 0$$

and $\varphi_1, \varphi_2, \varphi_3$ are not lineerly dependent. We embed $(\mathbb{R}^4, K)$ into $(\mathbb{R}^5, \mathbb{R}^5_+)$ under Φ.

Now we would determine all bands in $(\mathbb{R}^4, K)$. Because of proposition every band equals $Zero(M)$ for some saturatted set $M \subseteq \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$.

Theorem indicate that for a saturatted set $M \subseteq \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$ one has that $Zero(M)$ is a band if and only if M is bisaturatted.

here list of the sets $N \subset N \subseteq \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$ for which { φ: $i \in N$} is bisaturatted: $\emptyset$, $\{5\}$, $\{1,4\}$, $\{1,3\}$, $\{1,2\}$, $\{2,4\}$, $\{2,3\}$, $\{3,4\}$, $\{3,4,5\}$, $\{2,3,5\}$, $\{1,3,5\}$, $\{1,2,3,4\}$, $\{1,2,3,4,5\}$. Hence we get 16 bands in $(\mathbb{R}^4, K)$.

It is known that in Riesz spaces the whole bands is directed. Here it is an interesting result bands in partialy ordered lineer space can not be directed. Above example shows us this fact.

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