BOLU ABANT IZZET BAYSAL UNIVERSITY

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

MATHEMATICAL INEQUALITIES

MASTER OF SCIENCE

SEHER İBİŞ

BOLU, AUGUST 2018

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THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

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APPROVAL OF THE THESIS

MATHEMATICAL INEQUALITIES submitted by **Seher İBİŞ** in partial fulfillment of the requirements for the degree of **Master of Science** in **Department of Mathematics, The Graduate School of Natural and Applied Sciences of BOLU ABANT IZZET BAYSAL UNIVERSITY in 03/08/2018** by

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To my family

DECLARATION

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Seher İBİŞ

ABSTRACT

MATHEMATICAL INEQUALITIES
MSC THESIS
SEHER İBİŞ,
BOLU ABANT IZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS
(SUPERVISOR: ASSIST. PROF. DR. SİBEL KILIÇARSLAN CANSU)

BOLU, AUGUST 2018

In this thesis, inequalities which are one of the most important topics in mathematics has been studied. We give the original solutions to the problems about inequalities in national and international mathematical Olympics.

KEYWORDS: Inequalities, Cauchy – Schwarz, Jensen, Arithmetic-Geometric-Harmonic- Root Mean Square , Schur.

ÖZET

MATEMATİKSEL EŞİTSİZLİKLER
YÜKSEK LISANS TEZI
SEHER İBİŞ
BOLU ABANT İZZET BAYSAL ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATIK ANABILIM DALI
(TEZ DANIŞMANI: DR.ÖĞR.ÜYESİ SİBEL KILIÇARSLAN CANSU))

BOLU, AĞUSTOS - 2018

Bu çalışmada matematikteki en önemli konulardan biri olan eşitsizlikler konusu çalışılmıştır. Ulusal ve uluslararası matematik olimpiyatlarında eşitsizliklerle ile ilgili çıkmış sorulara özgün çözümler yapılmıştır.

ANAHTAR KELİMELER Eşitsizlikler, Cauchy- Schwarz, Jensen, Aritmetik- Geometrik- Harmonik –Karesel Ortalamalar, Schur.

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LIST OF ABBREVIATIONS AND SYMBOLS

A.M	**:**	Arithmetic Mean
G.M	**:**	Geometric Mean
H.M	**:**	Harmonic Mean
R.M.S	**:**	Root Mean Square
UMO	**:**	National Mathematics Olympiad
AMO	**:**	Antalya Mathematics Olympiad
IMO	**:**	International Mathematics Olympiad
CYC	**:**	Cyclic Notation
UIMO	**:**	National Primary Mathematics Olympiad
USSR M.O	**:**	Union of Soviet Socialist Republics Mathematics Olympiad
TST	**:**	Team Selection Test
IRMO	**:**	India Regional Mathematical Olympiad
ANO	**:**	Azerbaijan National Olympiad

ACKNOWLEDGEMENTS

I would like to express my gratitude to my supervisor Assist.Prof.Dr.Sibel Kılıçarslan Cansu for constant guidance, supports, useful comments and advices during this work.

I also want to express my gratitude to my siblings for endless support and motivation. My biggest thank goes to my parents Adil İbiş and Emine İbiş who always encourage me, believe in me and never let me give up throughout my life.

1. INTRODUCTION

The national mathematics Olympiads which is made by TUBITAK in Turkey held in three stages for selecting students to participate in international mathematics Olympiad. The national mathematics Olympiad has four main subjects: Analysis, Number Theory, Finite Mathematics, Geometry. In this thesis, we deal with inequalities which are one of the most significant subject of Analysis. Inequalities which are frequently used in Olympic problems and which are useful in the solutions of the questions have been studied in four chapters.

In chapter two, basic definitions and properties of inequalities are given. Cauchy-Schwarz inequality and Jensen inequality are studied in chapter three and chapter four respectively. In chapter five, Arithmetic, Geometric, Harmonic Mean and Root Mean Square which are the most used inequalities in olympiads are investigated and the relationship between these means are discussed. At the end of each chapter, national and international mathematics Olympiad questions which are related with the subject of the chapter are solved.

We think that this thesis will be useful for the students who are preparing for the mathematics olympiads, teachers who teach these students and those who are interested in mathematics olympiads.

2. ELEMENTARY INEQUALITIES

The following inequalities are well-known and generally used to show simple inequalities.

The square of a number cannot be negative ; $u^2 \geq 0$.

If $u \geq v$ and $v \geq w$ then for any $u, v, w \in \mathbb{R}$, $u \geq w$.

If $u \geq v$ then $u + w \geq v + w$ for any $u, v, w \in \mathbb{R}$.

If $u \geq v$ and $m \geq n$ then $u + m \geq v + n$ for any $u, v, m, n \in \mathbb{R}$.

If $u \geq v$ and $m \geq n$ then $u \cdot m \geq v \cdot n$ for any $u, v, m, n \in \mathbb{R}^+$.

Question 1.(UMO 2004) If $4x^2 + 9y^2 = 8$ for real numbers x, y; what is the maximum value of

$$8x^2 + 9xy + 18y^2 + 2x + 3y\ ?$$

Solution. We can rewrite the given expression as

$$A = 8x^2 + 9xy + 18y^2 + 2x + 3y = 2(4x^2 + 9y^2) + 9xy + 2x + 3y.$$

We are looking for the greatest value of $A = 16 + 9xy + 2x + 3y$. According to the equality

$$(2x + 3y)^2 = 4x^2 + 12xy + 9y^2 = 8 + 12xy\ , \quad (*)$$

we can find the maximum value of $2x + 3y$, if xy has its maximum value. So, it is enough to find the maximum value of xy. Since

$$(2x - 3y)^2 = 4x^2 - 12xy + 9y^2 = 8 - 12xy \geq 0,$$

we get $xy \leq (2/3)$, which means that maximum value of xy is $2/3$. Thus, when we put this value in the equality (*) , we have

$$(2x + 3y)^2 = 8 + 12 \cdot (2/3) = 16.$$

So, the maximum value of $2x + 3y$ is 4. Therefore, the maximum value of A is $16 + 9 \cdot (2/3) + 4 = 26$.

Question 2.(UMO 2002) For real numbers p, q, r; if $p^2 + q^2 + r^2 = 1$, find the minimum value of

$$pq + qr + pr.$$

Solution by author. Since square of a number cannot be negative,

$$(p + q + r)^2 \geq 0.$$

After expanding the square,

$$p^2 + q^2 + r^2 + 2pq + 2qr + 2pr \geq 0$$

$$p^2 + q^2 + r^2 \geq -2(pq + qr + pr).$$

Since $p^2 + q^2 + r^2 = 1$,

$$-\frac{1}{2} \leq pq + qr + pr$$

Therefore, the minimum value of $pq + qr + pr$ is $-\frac{1}{2}$.

Question 3.(UMO 2008) For u, v real numbers; if $uv = 1$ and following inequality is satisfied, what is the maximum value of A ?

$$((u + v)^2 - 2).((u + v)^2 + 4) \geq A \cdot (u - v)^2.$$

Solution by author. When we simplify the inequality, we get

$$(u^2 + v^2 + 2uv + 4)(u^2 + v^2 + 2uv - 2) \geq A \cdot (u^2 + v^2 - 2uv) .$$

Since $uv = 1$, we have

$$(u^2 + v^2 + 6)(u^2 + v^2) - A \cdot (u^2 + v^2 - 2) \geq 0.$$

If $uv = 1$, then $u \neq 0$ and $v \neq 0$. So, we get that $u^2 + v^2 > 0$. If $u^2 + v^2 = t$, then

$$(t + 6) \cdot t - A \cdot (t - 2) \geq 0$$

$$t^2 + (6 - A)t + 2A \geq 0.$$

For u, v real numbers, if the inequality is satisfied, then $\Delta \leq 0$.

So,

$$\Delta = (6 - A)^2 - 4 \cdot 2A \leq 0$$

$$A^2 - 20A + 36 \leq 0$$

$$(A - 2)(A - 18) \leq 0$$

$$2 \leq A \leq 18.$$

Therefore, maximum value of A is 18.

2.1 Triangle Inequality

If $p, q \in \mathbb{R}$, then triangle inequality is stated as

$$|p + q| \leq |p| + |q|.$$

Proof. It is clear that for $p, q \in \mathbb{R}$,

$$p \leq |p| \text{ and } q \leq |q|.$$

If the inequalities are added, then

$$p + q \leq |p| + |q|.$$

When we take absolute value of two sides, we get

$$|p+q| \leq \big||p|+|q|\big| = |p|+|q|.$$

2.2 The Inequality $2(x^{n+m}+y^{n+m}) \geq (x^n+y^n)(x^m+y^m)$

Let $x, y \geq 0$ such that $n, m \in \mathbb{Z}^+$, the inequality

$$2(x^{n+m}+y^{n+m}) \geq (x^n+y^n)(x^m+y^m)$$

is satisfied.

Proof. If the inequality does not change when we interchange the variables, then the inequality is symmetric according to the variables. So we can assume $x \geq y \geq 0$. In this case,

$$x^n \geq y^n \text{ and } x^m \geq y^m \text{ for } n, m \in Z^+.$$

Thus, $(x^n-y^n)(x^m-y^m) \geq 0$. From this inequality, we have that

$$x^{n+m}+y^{n+m} \geq x^n y^m + x^m y^n.$$

If we add $x^{n+m}+y^{n+m}$ to the both sides, then

$$2(x^{n+m}+y^{n+m}) \geq (x^n+y^n)(x^m+y^m) \ .$$

Question 1. For positive numbers p, q and r, prove that

$$3(p^8+q^8+r^8) \geq (p^5+q^5+r^5)(p^3+q^3+r^3).$$

Solution. We know the inequality

$$2(p^{n+m}+q^{n+m}) \geq (p^n+q^n)(p^m+q^m).$$

So, we can write the following three inequalities

$$2(p^8+q^8) \geq (p^5+q^5)(p^3+q^3)$$

$$2(p^8 + r^8) \geq (p^5 + r^5)(p^3 + r^3)$$

$$2(r^8 + q^8) \geq (r^5 + q^5)(r^3 + q^3)$$

If we write the addition of above inequalities, then

$$4(p^8 + q^8 + r^8) \geq (p^5 + q^5)(p^3 + q^3) + (p^5 + r^5)(p^3 + r^3)$$

$$+(r^5 + q^5)(r^3 + q^3).$$

When we simplify the right-hand side, we get

$$3(p^8 + q^8 + r^8) \geq p^8 + q^8 + r^8 + p^3q^5 + p^5q^3 + p^3r^5 + p^5r^3 + q^3r^5 + q^5r^3$$

$$\geq (p^5 + q^5 + r^5)(p^3 + q^3 + r^3).$$

2.3 The Inequality $(\boldsymbol{u_1} + \boldsymbol{u_2} + \cdots + \boldsymbol{u_n})\left(\frac{\mathbf{1}}{\boldsymbol{u_1}} + \frac{\mathbf{1}}{\boldsymbol{u_2}} + \cdots + \frac{\mathbf{1}}{\boldsymbol{u_n}}\right) \geq \boldsymbol{n^2}$

If $u_1, u_2, \dots, u_n$ are positive numbers, then

$$(u_1 + u_2 + \cdots + u_n)\left(\frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n}\right) \geq n^2.$$

Proof. If we do the multiplication $(u_1 + u_2 + \cdots + u_n)\left(\frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n}\right)$, we obtain that

$$n + \left(\frac{u_1}{u_2} + \frac{u_2}{u_1}\right) + \left(\frac{u_1}{u_3} + \frac{u_3}{u_1}\right) + \cdots + \left(\frac{u_n}{u_{n-1}} + \frac{u_{n-1}}{u_n}\right).$$

Here, there are $\binom{n}{2}$ times $\frac{u_i}{u_k} + \frac{u_k}{u_i}$ pairs. Since $\frac{u_i}{u_k} + \frac{u_k}{u_i} \geq 2$,

we have that $(u_1 + u_2 + \cdots + u_n)\left(\frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n}\right) \geq n + 2\binom{n}{2} = n^2$

2.4 $\sum_{cyc}$ Notation (Circular Sum)

We can express the inequality in a shorter way by using $\sum_{cyc}$ symbol instead of writing similar terms in inequalities. This notation is not a requirement but this is a notation that makes our process easier. For example; the inequality

$$\frac{1}{x^3(z+y)}+\frac{1}{y^3(x+z)}+\frac{1}{z^3(y+x)}\geq\frac{3}{2}$$

is written like

$$\sum_{cyc}\frac{1}{x^3(y+z)}\geq\frac{3}{2},$$

by using above notation. The expression cyc indicates that the sum is circular.

Question 1.(ROMANIA 2008) Let $a,b,c\in\mathbb{R}^+$ such that $ba+cb+ac=3$. Prove that

$$\frac{1}{1+a^2(b+c)}+\frac{1}{1+b^2(a+c)}+\frac{1}{1+c^2(b+a)}\leq\frac{1}{abc}.$$

Solution. From the $A.M-G.M$ inequality (see Chapter 5),

$$\frac{ab+bc+ca}{3}\geq\sqrt[3]{(abc)^2}.$$

For $ba+cb+ac=3$, we have that $abc\leq 1$. So,

$$\sum_{cyc}\frac{1}{1+a^2(b+c)}\leq\sum_{cyc}\frac{1}{abc+a^2(b+c)}$$

$$=\sum_{cyc}\frac{1}{a(ba+ca+cb)}$$

$$=\sum_{cyc}\frac{1}{3a}$$

$$= \frac{1}{3a} + \frac{1}{3b} + \frac{1}{3c}$$

$$= \frac{1}{3}\left(\frac{ba + cb + ca}{abc}\right) = \frac{1}{abc}\,.$$

2.5 Schur Inequality

Let $a, b, c \geq 0$ and $n > 0$, then

$$a^n(a-b)(a-c) + b^n(b-a)(b-c) + c^n(c-a)(c-b) \geq 0.$$

Equality is possible iff $a = b = c = 0$. When $n = 1$, the following special case arises:

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + ac(a+c) + bc(b+c).$$

Proof. Since the expression of Schur's inequality is symmetric in terms of a, b and c assume that $a \geq b \geq c$ without lost of generality. We can write the left-hand side of Schur's inequality in the form of

$$(a-b)[a^n(a-c) - b^n(b-c)] + c^n(c-a)(c-b).$$

Note that $a^n \geq b^n$ and $a - c \geq b - c$. Thus, the equation inside the square bracket is non-negative. Since $(a-b)$ is also non-negative, the multiplication on the left-hand side of the above equation is non-negative. Moreover, c^n is non-negative and $(c-a)$, $(c-b)$ are both negative. So, the multiplication on the right-hand side is also non-negative above. Hence, the above statement is greater than or equal to zero.

Question 1. Let $p, q, r \geq 0$ and $p + q + r = 1$, show that

$$p^3 + q^3 + r^3 + 6pqr \geq \frac{1}{4}.$$

Solution. If $n = 1$, then Schur inequality implies that

$$p^3 + q^3 + r^3 - p^2(r+q) - q^2(r+p) - r^2(q+p) + 3pqr \geq 0$$

So,

$$p^3 + q^3 + r^3 + 3pqr \geq p^2(r+q) + q^2(r+p) + r^2(q+p).$$

Now, multiply both side of the inequality by 3 and then add $6pqr + p^3 + q^3 + r^3$ to the both sides. Thus,

$$4p^3 + 4q^3 + 4r^3 + 15pqr \geq 3p^2(r+q) + 3q^2(r+p) + 3r^2(q+p)$$

$$+p^3 + q^3 + r^3 + 6pqr.$$

On the other hand, since

$$(p+q+r)^3 = p^3 + q^3 + r^3 + 6pqr \; + 3p^2(r+q) + 3q^2(r+p) + 3r^2(q+p),$$

it implies that

$$4p^3 + 4q^3 + 4r^3 + 24pqr \geq 4p^3 + 4q^3 + 4r^3 + 15pqr \geq (p+q+r)^3 = 1.$$

Therefore,

$$p^3 + q^3 + r^3 + 6pqr \geq \frac{1}{4}.$$

3. CAUCHY- SCHWARZ INEQUALITY

The Cauchy-Schwarz inequality can be stated as follows:

Let $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ be two sequences of real numbers, then

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2. \qquad (*)$$

The equality is possible iff $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are proportional, that is, $a_k = \lambda b_k$ for some constant λ and for all $k \in \{1,2, \dots, n\}$.

The above inequality is the most common form of Cauchy-Schwarz inequality. Different forms of this formula are used in Olympiad questions. The purpose here is that the following formulas make the questions easier and more understandable;

1. $\sqrt{a_1 b_1} + \cdots + \sqrt{a_n b_n} \leq \sqrt{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b)}$

2. $\dfrac{(a_1 + a_2 + \cdots + a_n)^2}{a_1 b_1 + \cdots + a_n b_n} \leq \dfrac{a_1}{b_1} + \dfrac{a_2}{b_2} + \cdots + \dfrac{a_n}{b_n}.$

3. $\dfrac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} \leq \dfrac{{a_1}^2}{b_1} + \cdots + \dfrac{{a_n}^2}{b_n}$

4. $\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \sum_{k=1}^{n} |a_k|^2 \cdot \sum_{k=1}^{n} |b_k|^2$

3.1 Proofs of Cauchy-Schwarz Inequality

Proof 1. After doing necessary multiplications, the following equality is obtained,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_i b_j - a_j b_i\right)^2 = \sum_{i=1}^{n} a_i{}^2 \sum_{j=1}^{n} b_j{}^2 + \sum_{i=1}^{n} b_i{}^2 \sum_{j=1}^{n} a_j{}^2 - 2\sum_{i=1}^{n} a_i\, b_i \sum_{j=1}^{n} b_j a_j$$

$$= 2\left(\sum_{i=1}^{n} a_i{}^2\right)\left(\sum_{i=1}^{n} b_i{}^2\right) - 2\left(\sum_{i=1}^{n} a_i\, b_i\right)^2.$$

Since left-hand side of the equation is a sum of the squares of real numbers, it is greater than zero or equal to zero. Thus

$$\left(\sum_{i=1}^{n} a_i{}^2\right)\left(\sum_{i=1}^{n} b_i{}^2\right) \geq \left(\sum_{i=1}^{n} a_i\, b_i\right)^2.$$

Proof 2. We will prove Cauchy-Schwarz inequality by mathematical induction. For the case $n = 1$, it is trivial. Note that

$$(a_1 b_1 + a_2 b_2)^2 = a_1{}^2 b_1{}^2 + 2a_1 b_1 a_2 b_2 + a_2{}^2 b_2{}^2$$
$$\leq a_1{}^2 b_1{}^2 + a_1{}^2 b_2{}^2 + a_2{}^2 b_1{}^2 + a_2{}^2 b_2{}^2 = (a_1{}^2 + a_2{}^2)\left(b_1{}^2 + b_2{}^2\right),$$

which means that (*) holds for $n = 2$.

Assume that the inequality (*) holds for an arbitrary integer k, that is,

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{k} a_i{}^2\right)\left(\sum_{i=1}^{k} b_i{}^2\right).$$

Using the induction hypothesis, one has

$$\sqrt{\sum_{i=1}^{k+1} a_i{}^2}\,.\,\sqrt{\sum_{i=1}^{k+1} b_i{}^2} = \sqrt{\sum_{i=1}^{k} a_i{}^2 + a_{k+1}{}^2}\,.\,\sqrt{\sum_{i=1}^{k} b_i{}^2 + b_{k+1}{}^2}$$

$$\geq \sqrt{\sum_{i=1}^{k} a_i{}^2}\,.\,\sqrt{\sum_{i=1}^{k} b_i{}^2} + |a_{k+1} b_{k+1}|$$

$$\geq \sum_{i=1}^{k} |a_i b_i| + |a_{k+1} b_{k+1}| = \sum_{i=1}^{k+1} |a_i b_i|.$$

This means that (*) holds $n = k + 1$. Therefore (*) holds for all natural numbers n.

Question 1. Which one of the numbers $\sqrt[3]{4} - \sqrt[3]{10} + \sqrt[3]{25}$ and $\sqrt[3]{6} - \sqrt[3]{9} + \sqrt[3]{15}$ is greater?

Solution . Let we say $\sqrt[3]{2} = a$, $\sqrt[3]{3} = b$, $\sqrt[3]{5} = c$. Then, we get

$\sqrt[3]{4} - \sqrt[3]{10} + \sqrt[3]{25} = a^2 - ac + c^2$ and $\sqrt[3]{6} - \sqrt[3]{9} + \sqrt[3]{15} = ab - b^2 + bc$.

From Cauchy-Schwarz inequality,

$$a \cdot c + c \cdot b + b \cdot a \leq \sqrt{a^2 + b^2 + c^2}\sqrt{c^2 + a^2 + b^2} = a^2 + b^2 + c^2.$$

Above inequality implies that $ac + ba + cb \leq a^2 + b^2 + c^2$ and so, the inequality

$$ab - b^2 + bc \leq a^2 - ac + c^2$$

is satisfied. Therefore, the number $\sqrt[3]{4} - \sqrt[3]{10} + \sqrt[3]{25}$ is greater than

$\sqrt[3]{6} - \sqrt[3]{9} + \sqrt[3]{15}$.

Question 2. Let $a, b, c \in \mathbb{R}^+$. Find the minimum value of

$$\left(5a + \frac{4}{b} + \frac{1}{3c}\right)\left(\frac{20}{a} + 100b + 12c\right).$$

Solution. If we use the following version of Cauchy-Schwarz inequality

$$\left(\sqrt{p_1 q_1} + \sqrt{p_2 q_2} + \sqrt{p_3 q_3}\right)^2 \leq (p_1 + p_2 + p_3)(q_1 + q_2 + q_3),$$

then we have

$$\left(\sqrt{5a\frac{20}{a}} + \sqrt{\frac{4}{b}100b} + \sqrt{\frac{1}{3c}12c}\right)^2 \leq \left(5a + \frac{4}{b} + \frac{1}{3c}\right)\left(\frac{20}{a} + 100b + 12c\right)$$

$$(10+20+2)^2 \leq \left(5a+\frac{4}{b}+\frac{1}{3c}\right)\left(\frac{20}{a}+100b+12c\right).$$

Therefore, the minimum value of the above expression is $32^2 = 1024$.

Question 3. Let $p, q, r, s, t \in \mathbb{R}$. If $p+q+r+s+t = 15$ and

$p^2+q^2+r^2+s^2+t^2 = 45$, find the maximum value of $|p-q+r-s+t|$.

Solution. From Cauchy-Schwarz inequality

$$(p+q+r+s)^2 \leq (1^2+1^2+1^2+1^2)(p^2+q^2+s^2+t^2)$$

$$(15-t)^2 \leq 4 \cdot (45-t^2)$$

$$225-30t+t^2 \leq 180-4t^2.$$

So,

$$5t^2-30t+45 \leq 0 \text{ ,that is, } (t-3)^2 \leq 0.$$

The inequality is satisfied when $t = 3$. Similarly, $p = q = r = s = 3$. Therefore, maximum value is $|p-q+r-s+t| = 3$.

Question 4. Let a, b, c, d be positive numbers. Show that

$$\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

Solution. Cauchy-Schwarz inequality implies that

$$\left(\left(\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2+\left(\sqrt{c}\right)^2+\left(\sqrt{d}\right)^2\right)\left(\frac{1^2}{\left(\sqrt{a}\right)^2}+\frac{1^2}{\left(\sqrt{b}\right)^2}+\frac{2^2}{\left(\sqrt{c}\right)^2}+\frac{4^2}{\left(\sqrt{d}\right)^2}\right)$$
$$\geq (1+1+2+4)^2$$

This is the desired result.

Question 5.(IMO Shortlist 1987) Let $x^2+y^2+z^2 = 2$ where $x, y, z \in \mathbb{R}$. Show that

$$x + y + z \leq 2 + xyz\,.$$

Solution. Let we write $x + y + z - xyz = x(1 - yz) + (y + z)$. If Cauchy-Schwarz inequality is applied to this statement, then

$$x(1 - yz) + (y + z) \leq \sqrt{x^2 + (y + z)^2}\sqrt{1 + (1 - yz)^2}.$$

If we can show that

$$\sqrt{(2yz + x^2 + y^2 + z^2)(y^2z^2 - 2yz + 2)} \leq 2,$$

then given inequality is satisfied. So assume that

$$\sqrt{(2yz + x^2 + y^2 + z^2)(y^2z^2 - 2yz + 2)} \geq 2.$$

Since $x^2 + y^2 + z^2 = 2$, the above inequality turns out to be

$$2(yz + 1)(y^2z^2 - 2yz + 2) \geq 4.$$

Thus,

$$2(y^3z^3 - y^2z^2 + 2) \geq 4,$$

which implies that

$$y^3z^3 \geq y^2z^2.$$

Hence $yz \geq 1$, which contradicts with the fact

$$2yz \leq y^2 + z^2 \leq 2.$$

Therefore,

$$\sqrt{(2yz + x^2 + y^2 + z^2)(y^2z^2 - 2yz + 2)} \leq 2.$$

Question 6.(IMO 1995) If $p, q, r \in \mathbb{R}^+$ and $pqr = 1$, then show that

$$\frac{1}{p^3(q + r)} + \frac{1}{q^3(r + p)} + \frac{1}{r^3(p + q)} \geq \frac{3}{2}.$$

Solution. Assume that

$$a = \frac{1}{p} \qquad b = \frac{1}{q} \qquad c = \frac{1}{r}.$$

So, $pqr = abc = 1$ and the left-hand side of inequality is

$$A = \frac{a^2}{b+c} + \frac{b^2}{a+c} + \frac{c^2}{a+b}.$$

If we multiply the above expression by

$$\frac{1}{2}\big((b+c) + (c+a) + (a+b)\big) = a + b + c,$$

then we can apply the Cauchy-Schwarz inequality. Therefore,

$$(a+b+c)A = \frac{1}{2}\big((b+c) + (c+a) + (a+b)\big)\left(\frac{a^2}{b+c} + \frac{b^2}{a+c} + \frac{c^2}{a+b}\right).$$

$$\geq \frac{1}{2}(a+b+c)^2.$$

When we use $A.M - G.M$ inequality (see Chapter 5), we get

$$A \geq \frac{1}{2}(a+b+c) \geq \frac{3}{2}\sqrt[3]{abc} = \frac{3}{2}.$$

Question 7.(UMO 2014) For integer numbers p, q, r, s, u, v ; if $p + q + r + s + u + v = 9$, find the minimum value of

$$p^2 + q^2 + r^2 + s^2 + u^2 + v^2.$$

Solution. From Cauchy-Schwarz inequality,

$$(p+q+r+s+u+v)^2 \leq (1^2+1^2+1^2+1^2+1^2+1^2)(p^2+q^2+r^2+s^2+u^2+v^2)$$

$$9^2 \leq 6 \cdot (p^2+q^2+r^2+s^2+u^2+v^2).$$

So ,

$$\frac{27}{2} \le p^2 + q^2 + r^2 + s^2 + u^2 + v^2.$$

The minimum value is $\frac{27}{2}$.

Question 8.(TÜRKİYE 2005) For $a, b, c, d \in \mathbb{R}$; prove that

$$\sqrt{a^4 + c^4} + \sqrt{a^4 + d^4} + \sqrt{b^4 + c^4} + \sqrt{b^4 + d^4} \ge 2\sqrt{2}(ad + bc).$$

Solution by author. When we use the following form of Cauchy-Schwarz inequality

$$ax + by \le \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}$$

we get

$$a^2 + c^2 \le \sqrt{2} \cdot \sqrt{a^4 + c^4}$$

$$a^2 + d^2 \le \sqrt{2} \cdot \sqrt{a^4 + d^4}$$

$$b^2 + c^2 \le \sqrt{2} \cdot \sqrt{b^4 + c^4}$$

$$b^2 + d^2 \le \sqrt{2} \cdot \sqrt{b^4 + d^4}\ .$$

If we add these inequalities,

$$2(a^2 + \ b^2 + c^2 + d^2) \le \sqrt{2}\left(\sqrt{a^4 + c^4} + \sqrt{a^4 + d^4} + \sqrt{b^4 + c^4} + \sqrt{b^4 + d^4}\right).$$

This means that

$$\sqrt{2}(a^2 + \ b^2 + c^2 + d^2) \le \sqrt{a^4 + c^4} + \sqrt{a^4 + d^4} + \sqrt{b^4 + c^4} + \sqrt{b^4 + d^4} \qquad (*)$$

Also, $A.M - G.M$ inequality (see Chapter 5) implies

$$2ad \le a^2 + d^2$$

$$2bc \le b^2 + c^2$$

If we add the above inequalities and multiply by $\sqrt{2}$, then

$$2\sqrt{2}(ad+bc)\ \leq \sqrt{2}(a^2+b^2+c^2+d^2). \qquad (**)$$

From (*) and (**), we have

$$2\sqrt{2}(ad+bc)\ \leq \sqrt{2}(a^2+b^2+c^2+d^2)$$

$$\leq \sqrt{a^4+c^4}+\sqrt{a^4+d^4}+\sqrt{b^4+c^4}+\sqrt{b^4+d^4}$$

Therefore,

$$2\sqrt{2}(ad+bc) \leq \sqrt{a^4+c^4}+\sqrt{a^4+d^4}+\sqrt{b^4+c^4}+\sqrt{b^4+d^4}\,.$$

Question 9.(AMO 2000) For $p, q, r > 0$ and $p+q+r=1$, find the minimum value of

$$\frac{1}{p}+\frac{9}{q}+\frac{25}{r}\,.$$

Solution by author. We can write $(1+3+5)$ in the form of

$\left(\frac{1}{\sqrt{p}}\sqrt{p}+\frac{3}{\sqrt{q}}\sqrt{q}+\frac{5}{\sqrt{r}}\sqrt{r}\right)$. If Cauchy-Schwarz inequality is applied to $(1+3+5)$, then

$$(1+3+5)^2=\left(\frac{1}{\sqrt{p}}\sqrt{p}+\frac{3}{\sqrt{q}}\sqrt{q}+\frac{5}{\sqrt{r}}\sqrt{r}\right)^2$$

$$\leq \left[\left(\frac{1}{\sqrt{p}}\right)^2+\left(\frac{3}{\sqrt{q}}\right)^2+\left(\frac{5}{\sqrt{r}}\right)^2\right]\cdot\left[\left(\sqrt{p}\right)^2+\left(\sqrt{q}\right)^2+\left(\sqrt{r}\right)^2\right].$$

$$(9)^2 \leq \left(\frac{1}{p}+\frac{9}{q}+\frac{25}{r}\right)\cdot(p+q+r).$$

Since $p+q+r=1$,

$$81 \leq \frac{1}{p}+\frac{9}{q}+\frac{25}{r}\,.$$

Thus the minimum value of $\frac{1}{p}+\frac{9}{q}+\frac{25}{r}$ is 81.

Question 10.(UMO 2011) For integers u, v, w; if $u^2+v^2+w^2=2011$, what is the maximum value of

$$u+v+w.$$

Solution by author. By Cauchy-Schwarz inequality,

$$(u+v+w)^2 \leq (1+1+1)(u^2+v^2+w^2)$$

Since $u^2+v^2+w^2=2011$, we have

$$(u+v+w)^2 \leq 3 \cdot 2011$$

$$u+v+w \leq 77{,}67$$

So, the maximum value is 77.

4. JENSEN'S INEQUALITY

Definition 4.1. *Let $f(u)$ be a real valued function defined on the interval $I = [a,b]$. f is said to be* ***convex*** *if for every $u_1, u_2 \in [a,b]$ and $0 \geq v \geq 1$,*

$$f(vu_1 + (1-v)u_2) \leq vf(u_1) + (1-v)f(u_2).$$

A function is said to be ***strictly convex*** *if the equality is strict for $u_1 \neq u_2$.*

Definition 4.2. *A real valued function $f(u)$ is said to be* ***concave (strictly concave)*** *if $-f(u)$ is convex (strictly convex) .*

Definition 4.3. *If $f''(u)$ exits on $[a,b]$ and $f''(u) \geq 0$ on $[a,b]$, then $f(u)$ is convex on $[a,b]$.*

If $f''(u)$ exits on $[a,b]$ and $f''(u) \leq 0$ on $[a,b]$, then $f(u)$ is concave on $[a,b]$.

Theorem 4.4.(Jensen's Inequality) *Let $f(u)$ be a function defined on an interval I and $u_1, u_2, \dots, u_n \in I$, $v_1, v_2, \dots, v_n \geq 0$ with $v_1 + v_2 + \cdots + v_n = k$.*

If $f(u)$ is a convex function, then

$$f\left(\frac{v_1u_1 + v_2u_2 + \cdots + v_nu_n}{k}\right) \leq \frac{v_1f(u_1) + v_2f(u_2) + \cdots + v_nf(u_n)}{k}.$$

If $f(u)$ is a concave function, then

$$f\left(\frac{v_1u_1 + v_2u_2 + \cdots + v_nu_n}{k}\right) \geq \frac{v_1f(u_1) + v_2f(u_2) + \cdots + v_nf(u_n)}{k}.$$

Proof . We prove Jensen's inequality by mathematical induction. It will be enough to prove only the simplified version. For the case $n = 1$ we must have $u_1 = 1$. Clearly $f(1u_1) = 1f(u_1)$. For the case $n = 2$, we have to show that for any $u_1, u_2 \in I$,

$$f(v_1u_1 + v_2u_2) \leq v_1 f(u_1) + v_2 f(u_2).$$

Since $v_1 + v_2$ must be equal to 1, we can put $v_2 = 1 - v_1$, so above inequality becomes

$$f(v_1u_1 + (1 - v_1)u_2) \leq v_1 f(u_1) + (1 - v_1) f(u_2)$$

which is true by the definition of convex function.

Now, assume that Jensen's inequality holds for some t, for every $u_1, u_2, \dots, u_t \in I$ and for every $v_1, v_2, \dots, v_t \in \mathbb{R}^+$ such that $v_1 + v_2 + \cdots + v_t = 1$. So we have that

$$f(v_1u_1 + v_2u_2 + \cdots + v_tu_t) \leq v_1 f(u_1) + v_2 f(u_2) + \cdots + v_t f(u_t).$$

Let $u_{t+1} \in I$ and let $v_1, v_2, \dots, v_t, v_{t+1} \in \mathbb{R}^+$ with $v_1 + v_2 + \cdots + v_t + v_{t+1} = 1$. Then

$$f(v_1u_1 + v_2u_2 + \cdots + v_tu_t + v_{t+1}u_{t+1}) = f\left(\sum_{m=1}^{t} v_m u_m + v_{t+1}u_{t+1}\right)$$

$$= f\left((1 - v_{t+1})\frac{1}{1 - v_{t+1}}\sum_{m=1}^{t} v_m u_m + v_{t+1}u_{t+1}\right)$$

$$\leq (1 - v_{t+1}) f\left(\frac{1}{1 - v_{t+1}}\sum_{m=1}^{t} v_m u_m\right) + v_{t+1} f(u_{t+1})$$

$$\leq (1 - v_{t+1}) f\left(\sum_{m=1}^{t} \frac{v_m}{1 - v_{t+1}} u_m\right) + v_{t+1} f(u_{t+1})$$

Observe that, since $v_1 + v_2 + \cdots + v_t + v_{t+1} = 1$ we have

$v_1 + v_2 + \cdots + v_t = 1 - v_{t+1}$ and so

$$\frac{v_1}{1 - v_{t+1}} + \frac{v_2}{1 - v_{t+1}} + \cdots + \frac{v_t}{1 - v_{t+1}} = 1.$$

Thus by the induction hypothesis,

$$f(v_1u_1 + v_2u_2 + \cdots + v_tu_t + v_{t+1}u_{t+1})$$

$$\leq f\left((1 - v_{t+1})\sum_{m=1}^{t}\frac{v_m}{1 - v_{t+1}}f(u_m) + v_{t+1}f(u_{t+1})\right)$$

$$\leq v_1f(u_1) + v_2f(u_2) + \cdots + v_tf(u_t) + v_{t+1}f(u_{t+1})$$

Question 1. If $u, v, w \in \mathbb{R}^+$ with $u + v + w = 6$, find the minimum value of

$$\left(u + \frac{1}{u}\right)^5 + \left(v + \frac{1}{v}\right)^5 + \left(w + \frac{1}{w}\right)^5.$$

Solution. Choose the function $f(a) = \left(a + \frac{1}{a}\right)^5$. For $a > 0$, since

$$f''(a) = \frac{10}{a^3}\left(a + \frac{1}{a}\right)^4 + 20\left(a + \frac{1}{a}\right)^3\left(1 - \frac{1}{a^2}\right)^2 > 0,$$

the function is convex. From Jensen's inequality

$$f\left(\frac{u + v + w}{3}\right) \leq \frac{f(u) + f(v) + f(w)}{3}.$$

So,

$$f(2) = \left(2 + \frac{1}{2}\right)^5 = \left(\frac{5}{2}\right)^5 \leq \frac{f(u) + f(v) + f(w)}{3}.$$

Therefore, the minimum value of $\left(u + \frac{1}{u}\right)^5 + \left(v + \frac{1}{v}\right)^5 + \left(w + \frac{1}{w}\right)^5$ is $3\left(\frac{5}{2}\right)^5$.

Question 2. If $p, q, r \in \mathbb{R}^+$ with $p + q + r = 3 \cdot 2^7$, show that

$$\left(1 + \sqrt[7]{p}\right)^7 + \left(1 + \sqrt[7]{q}\right)^7 + \left(1 + \sqrt[7]{r}\right)^7 \leq 3^8.$$

Solution. Choose the function $f(u) = \left(1 + \sqrt[7]{u}\right)^7$. If we take second derivative of $f(u)$, we get

$$f''(u) = -\frac{6}{7u^2}\left(1 + \sqrt[7]{u}\right)^5\sqrt[7]{u}.$$

For $u > 0$, since $f''(u) < 0$, the function is concave. So, from Jensen's inequality

$$f\left(\frac{p+q+r}{3}\right) \geq \frac{f(p)+f(q)+f(r)}{3}.$$

Since $p+q+r = 3 \cdot 2^7$ and $f(2^7) = 3^7$, then

$$f(2^7) = 3^7 \geq \frac{\left(1+\sqrt[7]{p}\right)^7 + \left(1+\sqrt[7]{q}\right)^7 + \left(1+\sqrt[7]{r}\right)^7}{3}.$$

Therefore,

$$\left(1+\sqrt[7]{p}\right)^7 + \left(1+\sqrt[7]{q}\right)^7 + \left(1+\sqrt[7]{r}\right)^7 \leq 3^8.$$

Question 3. If $p, r, s \in \mathbb{R}^+$ with $p+r+s = 1$, show that

$$\left(p+\frac{1}{p}\right)^2 + \left(r+\frac{1}{r}\right)^2 + \left(s+\frac{1}{s}\right)^2 \geq \frac{100}{3}.$$

Solution by author. Let we choose the function

$$f(u) = \left(u+\frac{1}{u}\right)^2.$$

For $u > 0$ $f(u)$ is convex, since

$$f''(u) = 2\left(1-\frac{1}{u^2}\right)^2 + \frac{4}{u^3}\left(u+\frac{1}{u}\right) > 0.$$

So, Jensen's inequality implies

$$f\left(\frac{p+r+s}{3}\right) \leq \frac{f(p)+f(r)+f(s)}{3}.$$

Since $p+r+s = 1$,

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}+\frac{1}{\frac{1}{3}}\right)^2 = \frac{100}{9} \leq \frac{f(p)+f(r)+f(s)}{3}.$$

Therefore,

$$\frac{100}{3} \leq \left(p + \frac{1}{p}\right)^2 + \left(r + \frac{1}{r}\right)^2 + \left(s + \frac{1}{s}\right)^2.$$

Question 4. Let p, q, r be positive numbers. Prove that

$$\sqrt[3]{(p^2+1)(q^2+1)(r^2+1)} \leq \left(\frac{p+q+r}{3}\right)^2 + 1\,.$$

Solution by author. Choose the function

$$f(u) = \ln(1+u^2)\,.$$

If we take the second derivative of $f(u)$, then we get

$$f''(u) = \frac{-2u^2+2}{(1+u^2)^2}.$$

For $u > 0$, since $f''(u) \leq 0$, the function is concave. Then Jensen's inequality gives that

$$f\left(\frac{p+q+r}{3}\right) \geq \frac{f(p)+f(q)+f(r)}{3}.$$

Since

$$f\left(\frac{p+q+r}{3}\right) = \ln\left(1 + \left(\frac{p+q+r}{3}\right)^2\right)$$

and

$$f(p) = \ln(1+p^2)\,, \quad f(q) = \ln(1+q^2)\,, \quad f(r) = \ln(1+r)\,,$$

we have

$$\ln\left(1 + \left(\frac{p+q+r}{3}\right)^2\right) \geq \frac{\ln(1+p^2)+\ln(1+q^2)+\ln(1+r^2)}{3}.$$

This implies that

$$\ln\left(1+\left(\frac{p+q+r}{3}\right)^2\right) \geq \ln[(1+p^2)(1+q^2)(1+r^2)]^{1/3}\,.$$

So, the above expression implies that

$$1+\left(\frac{p+q+r}{3}\right)^2 \geq \sqrt[3]{(1+p^2)(1+q^2)(1+r^2)}\,.$$

5. ARITHMETIC-GEOMETRIC- HARMONIC MEAN-ROOT-MEAN SQUARE INEQUALITY

The Arithmetic Mean (A.M) –Geometric Mean (G.M)- Harmonic Mean (H.M) and Root-Mean Square (R.M.S) inequality is one of the primal inequalities in Algebra, and they are used in Mathematics Olympiads to solve many problems.

Definition 5.1. *Let* $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ *. The arithmetic mean of these* n *numbers is*

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Definition 5.2. *Let* $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ *. The geometric mean of these* n *numbers is*

$$G_n = \sqrt[n]{a_1 a_2 \dots a_n}.$$

Definition 5.3. *Let* $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ *. The harmonic mean of these* n *numbers is*

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

Definition 5.4. *Let* $a_1, a_2, \dots, a_n \in \mathbb{R}^+$*. The root-mean square of these* n *numbers is*

$$RMS_n = \sqrt{\frac{{a_1}^2 + {a_2}^2 + {a_3}^2 + \cdots + {a_n}^2}{n}}.$$

The $A.M - G.M - H.M - R.M.S$ inequalities states that

$$RMS_n \geq A_n \geq G_n \geq H_n ,$$

and equality occurs iff $a_1 = a_2 = \cdots = a_n$.

5.1 Proofs of $A.M - G.M - H.M - R.M.S$ Inequalities

Proof 1. We will prove $A_n \geq G_n$ by mathematical induction. If $n = 2$, then

$(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$ implies that

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$$

For $n = k$, let $A_k \geq G_k$, then

$$A_{2k} = \frac{a_1 + \cdots + a_{2k}}{2k} = \frac{(a_1 + \cdots + a_k)/k + (a_{k+1} + \cdots + a_{2k})/k}{2}$$

$$A_{2k} \geq \frac{\sqrt[k]{a_1 a_2 \ldots a_k} + \sqrt[k]{a_{k+1} a_{k+2} \ldots a_{2k}}}{2}$$

$$A_{2k} \geq \sqrt[2k]{a_1 a_2 \ldots a_{2k}} = G_{2k} .$$

That is, $A_n \geq G_n$ for $n = 2,4,8,16, \ldots$ Now, for $n > 2$, we will show if $A_n \geq G_n$, then $A_{n-1} \geq G_{n-1}$. If

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n},$$

we can substitute $\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}$ for a_n. In this case, we have

$$\frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}}.$$

When we simplify above inequality, we get

$$\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^n \geq (a_1 a_2 \ldots a_{n-1}) \left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)$$

and

$$\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}} .$$

To prove $G_n \geq H_n$, it is sufficient to take $1/a_i$ instead of a_i . Indeed, the inequality

$$\frac{1/a_1 + 1/a_2 + \cdots + 1/a_n}{n} \geq \sqrt[n]{\frac{1}{a_1 a_2 \ldots a_n}}$$

implies $G_n \geq H_n$ when we do the necessary multiplications.

We can prove $RMS_n \geq A_n$ by using Jensen's inequality. For $u \in R^+$, let consider the function $f(u) = u^2$. Since $f''(u) = 2 \geq 0$, the function is convex. So, Jensen's inequality says that

$$f\left(\frac{u_1 + u_2 + \cdots + u_n}{n}\right) \leq \frac{f(u_1) + f(u_2) + \cdots + f(u_{n)}}{n}.$$

$$\left(\frac{u_1 + u_2 + \cdots + u_n}{n}\right)^2 \leq \frac{{u_1}^2 + {u_2}^2 + {u_n}^2}{n}.$$

If we take the square root of both sides, then

$$\frac{u_1 + u_2 + \cdots + u_n}{n} \leq \sqrt{\frac{{u_1}^2 + {u_2}^2 + {u_n}^2}{n}}.$$

Proof 2. We can prove $A_n \geq G_n$ by using Jensen's inequality. For $u \in R^+$, let we take the function $f(u) = \ln u$. Since $f'(u) = 1/u$ and $f''(u) = -1/u^2 < 0$, for $u \in R^+$, $f(u)$ is a concave function and

$$f\left(\frac{u_1 + u_2 + \cdots + u_n}{n}\right) \geq \frac{f(u_1) + f(u_2) + \cdots + f(u_{n)}}{n}$$

is satisfied. So,

$$\ln\left(\frac{u_1 + u_2 + \cdots + u_n}{n}\right) \geq \frac{\ln u_1 + \ln u_2 + \cdots + \ln u_n}{n}$$

$$= \ln\ (u_1 u_2 \ldots u_n)^{1/n}$$

implies that

$$\frac{u_1 + u_2 + \cdots + u_n}{n} \geq \sqrt[n]{u_1 u_2 \dots u_n}.$$

5.2 Generalized Arithmetic-Geometric Mean Inequality

Let $a_1, a_2, \dots, a_n \geq 0$ and $u_1, u_2, \dots, u_n > 0$ such that $u_1 + u_2 + u_3 + \cdots + u_n = k$. Then

$$\frac{u_1\, a_1 + u_2 a_2 + \cdots + \, u_n\, a_n}{k} \geq \sqrt[k]{{a_1}^{u_1} {a_2}^{u_2} \dots {a_n}^{u_n}}$$

is satisfied. The above inequality is called Generalized Arithmetic-Geometric Mean inequality.

Proof. The proof is easily made by using Jensen's inequality. Thus, for $a \in (0, \infty)$, let we consider concave function $f(a) = \ln a$, that is, $f''(a) < 0$. So, according to Jensen's inequality, we get

$$\ln\left(\frac{u_1\, a_1 + u_2 a_2 + \cdots + \, u_n\, a_n}{k}\right) \geq \frac{u_1}{k} \ln a_1 + \frac{u_2}{k} \ln a_2 + \cdots + \frac{u_n}{k} \ln a_n$$

$$\geq \ln {a_1}^{u_1/k} + \ln {a_2}^{u_2/k} + \cdots + \ln {a_n}^{u_n/k}$$

$$= \ln\left({a_1}^{u_1/k} {a_2}^{u_2/k} \dots {a_n}^{u_n/k}\right).$$

From above inequality, we have that

$$\frac{u_1\, a_1 + u_2 a_2 + \cdots + \, u_n\, a_n}{k} \geq \sqrt[k]{{a_1}^{u_1} {a_2}^{u_2} \dots {a_n}^{u_n}}\,.$$

Question 1. For $a, b, c, d \in \mathbb{R}$, if $abcd = 4$, then find the minimum value of

$$\frac{1}{a} + \frac{1}{2b} + \frac{2}{3c} + \frac{3}{4d}\,.$$

Solution. For the numbers $\frac{1}{a}, \frac{1}{2b}, \frac{2}{3c}, \frac{3}{4d}$, $A.M - G.M$ inequality implies

$$\frac{\frac{1}{a}+\frac{1}{2b}+\frac{2}{3c}+\frac{3}{4d}}{4} \geq \sqrt[4]{\frac{1}{a}\cdot\frac{1}{2b}\cdot\frac{2}{3c}\cdot\frac{3}{4d}}.$$

Since $abcd = 4$,

$$\frac{1}{a}+\frac{1}{2b}+\frac{2}{3c}+\frac{3}{4d} \geq 4\sqrt[4]{\frac{1}{16}} = 2.$$

Question 2. Let $p, r, s > 0$ and $p + r + s = 60$. Find the minimum value of

$$\frac{4}{p}+\frac{9}{r}+\frac{25}{s}.$$

Solution 1. We will use $H.M \leq A.M$. We can write $p + r + s$ in the form of

$$p+r+s = \frac{p}{2}+\frac{p}{2}+\frac{r}{3}+\frac{r}{3}+\frac{r}{3}+\frac{s}{5}+\frac{s}{5}+\frac{s}{5}+\frac{s}{5}+\frac{s}{5}$$

such that sum of the inverse of p, r, s is $\frac{4}{p}, \frac{9}{r}$ and $\frac{25}{s}$. Thus, harmonic mean of above 10 numbers is

$$H.M = \frac{10}{\frac{2}{p}+\frac{2}{p}+\frac{3}{r}+\frac{3}{r}+\frac{3}{r}+\frac{5}{s}+\frac{5}{s}+\frac{5}{s}+\frac{5}{s}+\frac{5}{s}} = \frac{10}{\frac{4}{p}+\frac{9}{r}+\frac{25}{s}}.$$

The arithmetic mean of these numbers is

$$A.M = \frac{\frac{p}{2}+\frac{p}{2}+\frac{r}{3}+\frac{r}{3}+\frac{r}{3}+\frac{s}{5}+\frac{s}{5}+\frac{s}{5}+\frac{s}{5}+\frac{s}{5}}{10} = \frac{p+r+s}{10} = 6.$$

Since $H.M \leq A.M$,

$$\frac{10}{\frac{4}{p}+\frac{9}{r}+\frac{25}{s}} \leq 6.$$

Thus, minimum value of $\frac{4}{p}+\frac{9}{r}+\frac{25}{s}$ is $\frac{5}{3}$.

Solution 2 by author. We can write $(2 + 3 + 5)$ in the form

$$\frac{2}{\sqrt{p}}\sqrt{p}+\frac{3}{\sqrt{r}}\sqrt{r}+\frac{5}{\sqrt{s}}\sqrt{s}\,.$$

If we apply the Cauchy-Schwarz inequality, then

$$(2+3+5)^2=\left(\frac{2}{\sqrt{p}}\sqrt{p}+\frac{3}{\sqrt{r}}\sqrt{r}+\frac{5}{\sqrt{s}}\sqrt{s}\right)^2$$

$$\leq\left[\left(\frac{2}{\sqrt{p}}\right)^2+\left(\frac{3}{\sqrt{r}}\right)^2+\left(\frac{5}{\sqrt{s}}\right)^2\right]\cdot\left[\left(\sqrt{p}\right)^2+\left(\sqrt{r}\right)^2+\left(\sqrt{s}\right)^2\right].$$

$$(10)^2\leq\left(\frac{4}{p}+\frac{9}{r}+\frac{25}{s}\right)\cdot(p+r+s).$$

Since $p+r+s=60$,

$$\frac{100}{60}=\frac{5}{3}\leq\frac{4}{p}+\frac{9}{r}+\frac{25}{s}\,.$$

Therefore, minimum value of $\frac{4}{p}+\frac{9}{r}+\frac{25}{s}$ is $\frac{5}{3}$.

Question 3. For $p,q,r>0$, if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=5$, then show that minimum value of the following expression is 6.

$$\sqrt{5p-1}\sqrt{5q-1}+\sqrt{5q-1}\sqrt{5r-1}+\sqrt{5r-1}\sqrt{5p-1} \qquad *$$

Solution. If we write $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=5$ in the form of $\frac{1}{5p}+\frac{1}{5q}+\frac{1}{5r}=1$, then the question can be expressed like;

For $u,v,w>0$ and $\frac{1}{u}+\frac{1}{v}+\frac{1}{w}=1$, show that

$$A=\sqrt{u-1}\sqrt{v-1}+\sqrt{v-1}\sqrt{w-1}+\sqrt{w-1}\sqrt{u-1}\geq 6.$$

If we apply $A.M-G.M$ inequality to the left side, then

$$A\geq 3\sqrt[3]{(u-1)(v-1)(w-1)}.$$

So, we must prove that

$$(u-1)(v-1)(w-1) \geq 8.$$

If we divide both side by uvw and consider

$$1-\frac{1}{u}=\frac{1}{v}+\frac{1}{w}, \qquad 1-\frac{1}{v}= \frac{1}{u}+\frac{1}{w}, \qquad 1-\frac{1}{w}= \frac{1}{u}+\frac{1}{v},$$

then the inequality that we need to prove is

$$\left(\frac{1}{v}+\frac{1}{w}\right)\left(\frac{1}{u}+\frac{1}{w}\right)\left(\frac{1}{u}+\frac{1}{v}\right) \geq \frac{8}{uvw}.$$

This inequality is obtained by multiplying the following inequalities;

$$\frac{1}{v}+\frac{1}{w} \geq 2\sqrt{\frac{1}{vw}}, \qquad \frac{1}{u}+\frac{1}{w} \geq 2\sqrt{\frac{1}{uw}}, \qquad \frac{1}{u}+\frac{1}{v} \geq 2\sqrt{\frac{1}{uv}}.$$

Question 4.(UIMO 2002) Let $a \in \mathbb{R}$. If $n \in \mathbb{R}^+$,then find the minimum value of

$$na+\frac{1}{a^n}.$$

Solution. Since $na+\frac{1}{a^n}=\underbrace{a+a+\cdots+a}_{n}+\frac{1}{a^n}$; if we use the $A.M \geq G.M$ inequality, then

$$\frac{a+a+\cdots+a+\frac{1}{a^n}}{n+1} \geq \sqrt[n+1]{a\cdot a \cdots a\cdot\frac{1}{a^n}}=1.$$

That is, $na+\frac{1}{a^n} \geq n+1$. So, if we take $n=1$, then the minimum value of $na+\frac{1}{a^n}$ is 2.

Question 5.(USSR M.O 1988) For $x,y,z \in \mathbb{R}^+$, prove the inequality

$$\frac{x^4+y^4+z^4}{xyz} \geq \sqrt{8}.$$

Solution. For x^4 and y^4, $A.M \geq G.M$ implies that

$$x^4 + y^4 \geq 2x^2y^2.$$

Also, for the numbers $2x^2y^2$ and z^2, if we use $A.M \geq G.M$, then

$$\frac{2x^2y^2 + z^2}{2} \geq \sqrt{2x^2y^2z^2}.$$

That is, $2x^2y^2 + z^2 \geq 2\sqrt{2}xyz$. Hence from the inequality

$$x^4 + y^4 + z^4 \geq 2x^2y^2 + z^2 \geq 2\sqrt{2}xyz\ ,$$

we get

$$\frac{x^4 + y^4 + z^4}{xyz} \geq 2\sqrt{2}.$$

Question 6.(BELARUS 1999) If $x^2 + y^2 + z^2 = 3$, then show that

$$\frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \geq \frac{3}{2}.$$

Solution. Since $A.M \geq G.M$,

$$\frac{x^2+y^2}{2} \geq xy, \qquad \frac{x^2+z^2}{2} \geq xz, \qquad \frac{z^2+y^2}{2} \geq zy,$$

and we have

$$\frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \geq \frac{1}{1+\frac{x^2+y^2}{2}} + \frac{1}{1+\frac{x^2+z^2}{2}} + \frac{1}{1+\frac{z^2+y^2}{2}}\,.$$

Since $x^2 + y^2 + z^2 = 3$,

$$\frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \quad \geq \frac{2}{5-z^2} + \frac{2}{5-x^2} + \frac{2}{5-y^2}\,.$$

Now, if we use $A.M - H.M$ inequality for the right-hand side, then

$$\frac{1}{1+xy}+\frac{1}{1+yz}+\frac{1}{1+zx}\geq\frac{9}{\frac{5-z^2}{2}+\frac{5-x^2}{2}+\frac{5-y^2}{2}}$$

$$=\frac{18}{15-(x^2+y^2+z^2)}$$

$$=\frac{18}{12}=\frac{3}{2}.$$

Question 7.(AMO 2015) For $a,b,c,d\in\mathbb{R}^+$, what is the maximum value of

$$S=\frac{abc+bcd}{a^3+b^3+c^3+d^3}\ ?$$

Solution. We can write $a^3+b^3+c^3+d^3$ in the form of

$$\left(a^3+\frac{b^3}{2}+\frac{c^3}{2}\right)+\left(\frac{b^3}{2}+\frac{c^3}{2}+d^3\right).$$

If we use $A.M-G.M$ inequality, then we get the following inequalities

$$a^3+\frac{b^3}{2}+\frac{c^3}{2}\geq 3\cdot\sqrt[3]{\frac{(abc)^3}{4}}=\frac{3abc}{\sqrt[3]{4}}\quad,\qquad \frac{b^3}{2}+\frac{c^3}{2}+d^3\geq 3\cdot\sqrt[3]{\frac{(bcd)^3}{4}}=\frac{3bcd}{\sqrt[3]{4}}$$

When we add this inequalities, we have that

$$a^3+b^3+c^3+d^3\geq\frac{3}{\sqrt[3]{4}}(abc+bcd).$$

Therefore,

$$S\leq\frac{\sqrt[3]{4}}{3}.$$

For $t>0$, equality occurs when $a=d=t,\ b=c=\sqrt[3]{2}t$. For example, if we choose $a=d=1,\ b=c=\sqrt[3]{2}$, then we have

$$S=\frac{2\sqrt[3]{4}}{1+1+2+2}=\frac{\sqrt[3]{4}}{3}.$$

Question 8.(AMO 2007) For positive numbers $u_1, u_2, \dots, u_7$, if

$$\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_7}=1,$$

Then prove that

$$(2000+u_1)(2000+u_2)\dots(2000+u_7)\geq 2007^7 .$$

Solution. For $k=1,2,3,\dots,7$, if we write $\frac{1}{u_k}$ instead of u_k, then we can express the question like;

For $u_1+u_2+\cdots+u_7=1,$ prove that

$$\left(2000+\frac{1}{u_1}\right)\left(2000+\frac{1}{u_2}\right)\dots\left(2000+\frac{1}{u_7}\right)\geq 2007^7 .$$

Let write

$$2000+\frac{1}{u_1}=\frac{1}{u_1}(2000u_1+1)$$

Since $u_1+u_2+\cdots+u_7=1,$

$$=\frac{1}{u_1}\left(\underbrace{u_1+u_1+\cdots+u_1}_{2000}+u_1+u_2+\cdots+u_7\right).$$

If we apply $A.M-G.M$ inequality, then

$$\frac{1}{u_1}\left(\underbrace{u_1+u_1+\cdots+u_1}_{2000}+u_1+u_2+\cdots+u_7\right)\geq\frac{1}{u_1}\cdot 2007\cdot\sqrt[2007]{{u_1}^{2001}u_2u_3\dots u_7} .$$

That is,

$$2000+\frac{1}{u_1}\geq\frac{1}{u_1}\cdot 2007\cdot\sqrt[2007]{{u_1}^{2001}u_2u_3\dots u_7} .$$

If we do this operations for $u_2u_3\dots u_7$ and multiply, then we have

$$\left(2000+\frac{1}{u_1}\right)\left(2000+\frac{1}{u_2}\right)\dots\left(2000+\frac{1}{u_7}\right)$$

$$\geq \frac{1}{u_1u_2\dots u_7}\cdot 2007^7\cdot \sqrt[2007]{{u_1}^{2007}{u_2}^{2007}\dots {u_7}^{2007}}$$

$$= 2007^7.$$

Question 9.(AMO 2006) For $x>0$, find the minimum value of $x^7+7\cdot\frac{a^{88}}{x}$ in terms of a .

Solution by author. We can write

$$x^7+7\cdot\frac{a^{88}}{x}=x^7+\underbrace{\frac{a^{88}}{x}+\frac{a^{88}}{x}+\cdots+\frac{a^{88}}{x}}_{7}.$$

When we use $A.M\geq G.M$, we get

$$\frac{x^7+\underbrace{\frac{a^{88}}{x}+\frac{a^{88}}{x}+\cdots+\frac{a^{88}}{x}}_{7}}{8}\geq \sqrt[8]{x^7\cdot\frac{a^{88}}{x}\cdot\frac{a^{88}}{x}\cdot\ldots\cdot\frac{a^{88}}{x}}$$

$$=8\cdot\sqrt[8]{x^7\cdot\frac{a^{88\cdot 7}}{x^7}}$$

$$=8\cdot\sqrt[8]{(a^{7\cdot 11})^8}$$

Therefore,

$$x^7+7\cdot\frac{a^{88}}{x}\geq 8\cdot a^{77}$$

Question 10.(UMO 2013) If $x,y,z\in\mathbb{R}^+$, what is the minimum value of

$$\frac{(x^2+y^3+z^6)}{xyz}.$$

Solution by author. Let we rewrite $x^2+y^3+z^6$ in the form

$$\frac{x^2}{3}+\frac{x^2}{3}+\frac{x^2}{3}+\frac{y^3}{2}+\frac{y^3}{2}+z^6.$$

If we apply $A.M \geq G.M$, then we get

$$\frac{\frac{x^2}{3}+\frac{x^2}{3}+\frac{x^2}{3}+\frac{y^3}{2}+\frac{y^3}{2}+z^6}{6} \geq \sqrt[6]{\frac{x^6 \cdot y^6 \cdot z^6}{3^3 \cdot 2^2}}$$

$$x^2+y^3+z^6 \geq 6 \cdot \frac{xyz}{\sqrt{3} \cdot \sqrt[3]{2}}$$

$$\frac{x^2+y^3+z^6}{xyz} \geq \frac{2\sqrt{3}}{\sqrt[3]{2}}.$$

Therefore, the minimum value of $\frac{x^2+y^3+z^6}{xyz}$ is $\frac{2\sqrt{3}}{\sqrt[3]{2}}$.

Question 11.(AMO 1998) For $u, v, w \geq 0$; if $u+v+w \leq 3$, then prove that

$$\frac{2}{1+u}+\frac{2}{1+v}+\frac{2}{1+w} \geq 3.$$

Solution by author. When we apply $A.M \geq H.M$, we get

$$\frac{\frac{2}{1+u}+\frac{2}{1+v}+\frac{2}{1+w}}{3} \geq \frac{3}{\frac{1+u}{2}+\frac{1+v}{2}+\frac{1+w}{2}}$$

$$\frac{2}{1+u}+\frac{2}{1+v}+\frac{2}{1+w} \geq \frac{9}{\frac{u+v+w+3}{2}}$$

$$\frac{2}{1+u}+\frac{2}{1+v}+\frac{2}{1+w} \geq \frac{18}{u+v+w+3}.$$

Since $u+v+w \leq 3$, we have

$$\frac{2}{1+u}+\frac{2}{1+v}+\frac{2}{1+w} \geq \frac{18}{3+3}.$$

Therefore,

$$\frac{2}{1+u}+\frac{2}{1+v}+\frac{2}{1+w}\geq 3\,.$$

Question 12.(AMO 2010) If $u, v, w \in \mathbb{R}^+$ with $u+v+w=3$, then show that

$$\frac{1}{1+\sqrt[3]{uv}}+\frac{1}{1+\sqrt[3]{vw}}+\frac{1}{1+\sqrt[3]{uw}}\geq \frac{3}{2}.$$

Solution by author. From $A.M \geq G.M$, we have that

$$\frac{1+u+v}{3}\geq \sqrt[3]{uv}\ , \qquad \frac{1+v+w}{3}\geq \sqrt[3]{vw}\ , \qquad \frac{1+u+w}{3}\geq \sqrt[3]{uw}\,.$$

So,

$$\frac{1}{1+\sqrt[3]{uv}}+\frac{1}{1+\sqrt[3]{vw}}+\frac{1}{1+\sqrt[3]{uw}}\geq \frac{1}{1+\frac{1+u+v}{3}}+\frac{1}{1+\frac{1+v+w}{3}}+\frac{1}{1+\frac{1+u+w}{3}}$$

$$=\frac{1}{\frac{4+u+v}{3}}+\frac{1}{\frac{4+v+w}{3}}+\frac{1}{\frac{4+u+w}{3}}$$

$$=\frac{3}{4+u+v}+\frac{3}{4+v+w}+\frac{3}{4+u+w}.$$

Since $u+v+w=3$,

$$\frac{1}{1+\sqrt[3]{uv}}+\frac{1}{1+\sqrt[3]{vw}}+\frac{1}{1+\sqrt[3]{uw}}\geq \frac{3}{7-w}+\frac{3}{7-u}+\frac{3}{7-v}$$

Now, if we apply $A.M \geq H.M$ for the right-hand side, then we have

$$\frac{1}{1+\sqrt[3]{uv}}+\frac{1}{1+\sqrt[3]{vw}}+\frac{1}{1+\sqrt[3]{uw}}\geq \frac{9}{\frac{7-w}{3}+\frac{7-u}{3}+\frac{7-v}{3}}$$

$$=\frac{27}{21-(u+v+w)}$$

Since $u+v+w=3$,

$$\frac{1}{1+\sqrt[3]{uv}}+\frac{1}{1+\sqrt[3]{vw}}+\frac{1}{1+\sqrt[3]{uw}} \geq \frac{27}{18}=\frac{3}{2}.$$

Question 13.(AMO 2015) If $a > 1$ and $x^2 + ax + 10b \geq 0$, find the minimum value of

$$S = \frac{b+11}{a-1}.$$

Solution. Given inequalities satisfy that $\Delta = a^2 - 4 \cdot 10b \leq 0$, it means that $b \geq \frac{a^2}{40}$.

Therefore,

$$S = \frac{b+11}{a-1} \geq \frac{\frac{a^2}{40}+11}{a-1} = \frac{1}{40} \cdot \frac{a^2+440}{a-1}$$

$$= \frac{1}{40} \cdot \frac{(a-1)^2 + 2(a-1) + 441}{a-1}$$

$$= \frac{1}{40} \cdot \left((a-1) + \frac{441}{(a-1)} + 2\right)$$

If we use $A.M \geq G.M$ for the right-hand side, then we have

$$S = \frac{b+11}{a-1} \geq \frac{1}{40} \cdot \left(2\sqrt{441} + 2\right) = \frac{11}{10}.$$

So, the minimum value of S is $\frac{11}{10}$.

Question 14.(AMO 2008) For $a > 0$, $b > 0$ and $c \in [0,7]$, what is the minimum value of

$$(a+b) \cdot \left(\frac{1}{ca+b} + \frac{1}{cb+a}\right).$$

Solution. If we apply $A.M - G.M$ inequality, then we get

$$\frac{1}{ca+b} + \frac{1}{cb+a} \geq 2\sqrt{\frac{1}{(ca+b)(cb+a)}}.$$

Again $A.M \geq G.M$ implies that

$$\sqrt{(ac+b)(bc+a)} \leq \frac{ac+b+bc+a}{2} = \frac{(a+b)(c+1)}{2}.$$

After multiplying these inequalities, we find that

$$\left(\frac{1}{ca+b}+\frac{1}{cb+a}\right)\frac{(a+b)(c+1)}{2} \geq 2.$$

So, if we choose $c = 7$, then the minimum value of above expression is

$$\frac{4}{(c+1)} = \frac{1}{2}.$$

Question 15.(AMO 2005) For $p, q, r \in \mathbb{R}^+$; if $p+q+r=1$, then show that

$$18pqr + 7(p^2+q^2+r^2) \geq 3.$$

Solution by author. By using Cauchy-Schwarz inequality, we get

$$1 = p+q+r \leq \sqrt{3}\cdot\sqrt{p^2+q^2+r^2}.$$

That is,

$$p^2+q^2+r^2 \geq \frac{1}{3}.$$

Now, we need to find a lower bound for xyz. From $G.M \geq H.M,$ we have

$$\sqrt[3]{pqr} \geq \frac{3}{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}} \qquad (*)$$

Since $(p+q+r)\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) \geq 3^2$, we get $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \geq 9$. If we put

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 9$$

in (*), we have

$$\sqrt[3]{pqr} \geq \frac{3}{9} = \frac{1}{3}.$$

This implies that

$$pqr \geq \frac{1}{27}.$$

Therefore,

$$18pqr + 7(p^2 + q^2 + r^2) \geq 18 \cdot \frac{1}{27} + 7 \cdot \frac{1}{3} = \frac{9}{3} = 3.$$

Question 16.(IMO Shortlist 2008) Let $a, b, c, d \in \mathbb{R}^+$, if $abcd = 1$ and $a + b + c + d > \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$, then prove that

$$a + b + c + d < \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}.$$

Solution. If we apply $A.M - G.M$ inequality to the numbers $\frac{a}{b}, \frac{a}{b}, \frac{b}{c}$ and $\frac{a}{d}$, then we obtain

$$a = \sqrt[4]{\frac{a^4}{abcd}} = \sqrt[4]{\frac{a}{b} \cdot \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{a}{d}} \leq \frac{1}{4}\left(\frac{a}{b} + \frac{a}{b} + \frac{b}{c} + \frac{a}{d}\right)$$

Similarly,

$$b \leq \frac{1}{4}\left(\frac{b}{c} + \frac{b}{c} + \frac{c}{d} + \frac{b}{a}\right), \quad c \leq \frac{1}{4}\left(\frac{c}{d} + \frac{c}{d} + \frac{d}{a} + \frac{c}{b}\right), \quad d \leq \frac{1}{4}\left(\frac{d}{a} + \frac{d}{a} + \frac{a}{b} + \frac{d}{c}\right).$$

If we add these inequalities, then

$$a + b + c + d \leq \frac{3}{4}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}\right) + \frac{1}{4}\left(\frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}\right).$$

Therefore, if

$$a + b + c + d > \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a},$$

then

$$a+b+c+d<\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}.$$

Question 17.(ANO 2015) Let $a,b,c\in\mathbb{R}^+$ such that $abc={}^{1}/_{8}$. Show that

$$a^2+b^2+c^2+a^2b^2+a^2c^2+b^2c^2\geq\frac{15}{16}.$$

Solution by author. If we apply $A.M-G.M$ inequality to $a^2+b^2+c^2$ and $a^2b^2+a^2c^2+b^2c^2$, we have

$$a^2+b^2+c^2\geq 3\sqrt[3]{a^2b^2c^2}=\frac{3}{4},$$

$$a^2b^2+a^2c^2+b^2c^2\geq 3\sqrt[3]{a^4b^4c^4}=\frac{3}{16}.$$

When we add these ineaqualities, we get

$$a^2+b^2+c^2+a^2b^2+a^2c^2+b^2c^2\geq\frac{3}{4}+\frac{3}{16}=\frac{15}{16}.$$

Question 18.(IRMO 2016) Let x,y,z be non-negative real numbers. If $xyz=1$, then prove that

$$(x^3+2y)(y^3+2z)(z^3+2x)\geq 27\,.$$

Solution by author. When we use $A.M-G.M$ inequality, we have

$$x^3+y+y\geq 3\sqrt[3]{x^3y^2}=3x\sqrt[3]{y^2}\,.$$

In the same manner,

$$y^3+z+z\geq 3\sqrt[3]{y^3z^2}=3y\sqrt[3]{z^2}\,.$$

$$z^3+x+x\geq 3\sqrt[3]{z^3x^2}=3z\sqrt[3]{x^2}\,.$$

If we multiply above inequalities, we get

$$(x^3 + 2y)(y^3 + 2z)(z^3 + 2x) \geq 27xyz\sqrt[3]{(xyz)^2}.$$

Since $xyz = 1$,

$$(x^3 + 2y)(y^3 + 2z)(z^3 + 2x) \geq 27\,.$$

Question 19.(TÜRKİYE 2007) For $a, b, c \in \mathbb{R}^+$, if $a + b + c = 1$, then prove that

$$\frac{1}{ab + 2c^2 + 2c} + \frac{1}{bc + 2a^2 + 2a} + \frac{1}{ac + 2b^2 + 2b} \geq \frac{1}{ab + bc + ac}\,.$$

Solution. To prove the inequality, we have to show that

$$\frac{ab + bc + ac}{ab + 2c^2 + 2c} + \frac{ab + bc + ac}{bc + 2a^2 + 2a} + \frac{ab + bc + ac}{ac + 2b^2 + 2b} \geq 1.$$

So, we need to show

$$\frac{ab + bc + ac}{ab + 2c^2 + 2c} \geq \frac{ab}{ab + bc + ac}.$$

This inequality is equal to

$$a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) \geq a^2b^2 + 2abc^2 + 2abc.$$

Since $a + b + c = 1$, we can write the above inequality in form

$$b^2c^2 + c^2a^2 \geq 2abc^2.$$

From $A.M - G.M$ inequality, the following inequality is true

$$\frac{b^2c^2 + c^2a^2}{2} \geq \sqrt{b^2c^2c^2a^2}\,.$$

Similarly, we can show that

$$\frac{ab + bc + ac}{bc + 2a^2 + 2a} \geq \frac{bc}{ab + bc + ac}$$

and

$$\frac{ab+bc+ac}{ac+2b^2+2b} \geq \frac{ac}{ab+bc+ac}.$$

Therefore, if we add these inequalities, then we have the desired inequality.

6. CONCLUSSIONS AND RECOMMENDATIONS

Inequalities are one of most frequently used subject in mathematics olympiads. Arithmetic, Geometric, Harmonic mean and their relations, Cauchy-Schwarz and Jensen inequalities are the most common topics. Olympiad problems can have specific solutions but some problems can be solved using more than one method. In this thesis, we concern national and international olympiad problems about inequalities. We tried to figure out original solutions to these problems.

It is normal that some questions can be quite diffucult for someone who has begun to solve olympiad problems newly. The solutions of the questions are depending on a little bit of knowledge, a little experience and a little bit of key point. The aim of this thesis is to be useful to those who are working in this field by writing solutions to the unsolved olympiad problems. Other analysis subjects that are encountered in the olympiads can be also worked for future studies.

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