BOLU ABANT IZZET BAYSAL UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

ON CURVES AND SURFACES IN LORENTZ SPACE

MASTER OF SCIENCE

SAFİYE DİLAN CEYLAN

BOLU, AUGUST 2018

APPROVAL OF THE THESIS

ON CURVES AND SURFACES IN LORENTZ SPACE submitted by **Safiye Dilan CEYLAN** in partial fulfillment of the requirements for the degree of **Master of Science** in **Department of Mathematics, The Graduate School of Natural and Applied Sciences of BOLU ABANT IZZET BAYSAL UNIVERSITY in 28/08/2018** by

Examining Committee Members	**Signature**
Supervisor Assist. Prof. Dr. Serpil Karagöz Bolu Abant İzzet Baysal University	....
Member Prof. Dr. M. Kemal Sağel Burdur Mehmet Akif Ersoy University	....
Member Assist. Prof. Dr. İsmail U. Tiryaki Bolu Abant İzzet Baysal University	....

Graduation Date :

Assoc. Prof. Dr. Ömer ÖZYURT ....

Director of Graduate School of Natural and Applied Sciences

To my family

DECLARATION

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

SAFİYE DİLAN CEYLAN

ABSTRACT

ON CURVES AND SURFACES IN LORENTZ SPACE
MSC THESIS
SAFİYE DİLAN CEYLAN
BOLU ABANT İZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS
(SUPERVISOR: ASSIST. PROF. DR. SERPİL KARAGÖZ)

BOLU, AUGUST 2018

This thesis consists of five chapters.

The first chapter is devoted to the introduction.

In the second chapter, 3 dimensional Lorentz-Minkowski space and its fundamental definitions, properties and theorems related to subject are given.

In the third chapter, Curves are defined and explained in detail on 3 dimensional Lorentz- Minkowski space and Frenet vectors are defined then some examples are given.

In the forth chapter, Surfaces are defined and explained in detail on 3 dimensional Lorentz-Minkowski space, curvatures of a surface on 3 dimensional Lorentz- Minkowski space and umbilical surfaces are defined and some examples are given.

Finally in the fifth chapter, minimal surfaces on 3 dimensional Lorentz-Minkowski space and maximal surfaces are given and some theorems for maximal surfaces are examined and some examples are given.

KEYWORDS: 3 Dimensional Lorentz-Minkowski Space, Minimal Surfaces, Maximal Surfaces, Umbilical Surfaces, Frenet Equation, Weierstrass-Enneper Representation

ÖZET

LORENTZ UZAYINDA EĞRİLER VE YÜZEYLER ÜZERİNE
YÜKSEK LİSANS TEZİ
SAFİYE DİLAN CEYLAN
BOLU ABANT İZZET BAYSAL ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI
(TEZ DANIŞMANI: DR. ÖĞR. ÜYESİ SERPİL KARAGÖZ)

BOLU, AĞUSTOS - 2018

Bu tez beş bölümden oluşmaktadır.

İlk bölüm giriş kısmına ayrılmıştır.

İkinci bölümde, konu ile ilgili temel kavramlara yer verilmiştir. 3 boyutlu Lorentz – Minkowski uzayı tanımlanıp, bu uzayın özelliklerinden bahsedilmiştir.

Üçüncü bölümde, 3 boyutlu Lorentz – Minkowski uzayında eğriler ayrıntılı olarak anlatılmıştır. Frenet vektörleri tanımlanmış olup örnekler verilmiştir.

Dördüncü bölümde, 3 boyutlu Lorentz – Minkowski uzayında yüzeyler ayrıntılı olarak anlatılmıştır. Lorentz uzayında bir yüzeyin eğrilikleri ve umbilik yüzeyler tanımlanmış daha sonra da yüzeylere örnekler verilmiştir.

Son olarak beşinci bölümde, 3 boyutlu Lorentz – Minkowski uzayında minimal ve maksimal yüzeyler incelendi ve bununla ilişkili teoremler ve örnekler verilmiştir.

ANAHTAR KELİMELER: 3 boyutlu Lorentz-Minkowski Uzayı, Minimal Yüzeyler, Maksimal Yüzeyler, Umbilik Yüzeyler, Frenet Denklemleri, Weierstrass-Enneper Gösterimi

TABLE OF CONTENTS

Page

LIST OF FIGURES

Page

LIST OF ABBREVIATIONS AND SYMBOLS

$<,>_e$	: Euclidean Inner Product
$<,>_L$	: Lorentz Inner Product
E_1^3	: 3 Dimensional Lorentz-Minkowski Space
E^3	: 3 Dimensional Euclidean Space
E, F, G	: Minkowski First Fundamental Form Coeffients
L, M, N	: Minkowski Second Fundamental Form Coeffients
T_pM	: $p \in M$ Point of Tangent Space
I	: Minkowski First Fundamental Form
II	: Minkowski Second Fundamental Form
t, n, b	: Frenet Trihedron
H	: Mean Curvature
K	: Gauss Curvature
κ, τ	: Curvature, Torsion
$\mathbb{S}_1^2$	: De Sitter Surface
$\mathbb{H}^2$	: Hyperbolic Surface
C	: Light Cone

ACKNOWLEDGEMENTS

The author wishes to express her deepest gratitude to her supervisor. Assist. Prof. Dr. Serpil Karagöz for hers guidance, advice, criticism, encouragements and insight throughout the research.

1. INTRODUCTION

Lorentz space is the space furnished with the pseudo-Riemannian metric of mark (+, +, +, …, −). This is in certain the state of affairs the theory of physics entitled "special relativity theory", and it is locally in the state of affairs the “general relativity theory ".

By studying the subject Lorentz space, the basic structures in 3 dimensional Minkowski space were established. Euclidean space and Lorentz space were encountered. First of all, the important metrics for its establishment have been examined. Then, by constructing the vector structures, the concept of angle was specified.

Furthermore, we frequently compare and construct the results and techniques of (**t, n, b**, $\boldsymbol{\tau}$, κ) the 3 dimensional Lorentzian geometry to those of the 3 dimensional Riemennian geometry. The basic differences between these geometries have been compiled in this research.

2. AIM AND SCOPE OF THE STUDY

The principal aim of this study is to examine the 3 dimensional Euclidean space and establish the basic structures in 3 dimensional Lorentz-Minkowski space. With this is consciousness, we also think about the distinction and similarities between them.

In particular a comparison is made between the minimal surfaces in 3 dimensional Euclidean space is and maximal surfaces in 3 dimensional Lorentz space.

2.1 Preliminaries

Semi-Riemannian geometry involves a particular kind of (0,2) tensor on tangent spaces. Let V be an arbitrary vector space of dimension $n \geq 1$ over IR. Then bilinear form on V is an $IR-$ bilinear function $b_s: V \times V \rightarrow IR$. The form b_s is symmetric if $b_s(u, v) = b_s(v, u)\ \forall\, u, v \in V$. (O'Neill, 1983)

2.1.1 Definition

A symmetric bilinear form b_s on V is

a) Positive definite provided $u \neq 0$ implies $b_s(u, u) > 0$.

b) Negative definite provided $u \neq 0$ implies $b_s(u, u) < 0$.

c) Positive semi-definite provided $u \neq 0$ implies $b_s(u, u) \geq 0, \quad \forall\, u \in V$.

d) Negative semi-definite provided $u \neq 0$ implies $b_s(u, u) \leq 0, \quad \forall\, u \in V$.

e) Non-degenerate provided $b_s(u, v) = 0\ \forall\, v \in V$ implies $u = 0$.

Also, b_s is definite (semi-definite) provided either alternative in a), b), c), d) holds. If b_s is definite then it is clearly both semi-definite and non-degenerate. (O'Neill, 1983)

2.1.2 Definition

Let V be a vector space. The index ν of a symmetric bilinear form b_s on V is the dimension of a $U \subset V$ such that

(1) $b_s \mid U$ is negative definite.

(2) $U' \subset V$ is another subspace such that $b_s \mid U'$ is negative $\Rightarrow$ $\dim U' \leq \dim U$. (O'Neill, 1983)

2.1.3 Definition

A symmetric non-degenerate $(0,2)$ tensor field g on M of constant index is called a metric tensor. So $\forall\, p \in M,\ g \in T_2^0(M)$ smoothly assigns to each p a scalar product g_p and each g_p has the same index. Non-degenerate means that for any $u \in T_p(M)$, there is some $v \in T_p(M)$ such that $g_p\,(u, v) \neq 0$. If $(g_p)_{ij}$ are components of g_p in local coordinates, then non-degeneracy is equivalent to the condition that $\det((g_p)_{ij}) \neq 0$. (O'Neill, 1983)

2.1.4 Definition

A Lorentz manifold is a smooth manifold M furnished with a metric tensor g that the index of M is 1. Sometimes we use $\langle\, ,\rangle$ as an alternative notation for g, writing $g\,(a, b) = \langle a, b\rangle \in IR$ for tangent vectors and $g(V, W) = \langle V, W\rangle \in \mathfrak{F}(M)$ for vector fields. If $x^1, \ldots, x^n$ is a coordinate system on $\mathcal{U} \subset V$ the components of g on $\mathcal{U}$ are$(g)_{ij} = \langle \partial_{\mathrm{i}}, \partial_{\mathrm{j}}\rangle,\ 1 \leq i, j \leq n$ where ∂_{i} denotes the vector field $\frac{\partial}{\partial_{\mathrm{x_i}}}$ on $\mathcal{U}$. Since g is non-degenerate, the matrix $\left((g)_{ij}(t)\right)$ is invertible for each t in $\mathcal{U}$. The inverse matrix is denoted by $\left((g)^{ij}(t)\right)$, the formula for inverse matrix shows that the functions $\left(((g)^{ij}t)\right)$ is smooth. Finally since b_s is symmetric.

$(g)_{ij} = (g)_{ji}$ and $(g)^{ij} = (g)^{ji}$ for each $1 \leq i, j \leq n$. (O'Neill, 1983)

2.1.5 Definition

A tangent vector x to M is

(1) Space-like if $\langle x, x\rangle_L\ is\ positive$ or $x = 0$,

(2) Time-like if $\langle x, x\rangle_L\ is\ negative$,

(3) Light-like if $\langle x, x\rangle_L = 0$ and $x \neq 0$.

The set of light-like vectors in $T_p(M)$ is called the light-cone at P. The category into which a given tangent vector falls is called its casual character. Light-like vectors are also said to be null.

Let N be a submanifold of a Lorentz manifold M with metric tensor b_s, let $j: P \subset M$ be the inclusion map. The pullback $j^*(g)$ is again a smooth symmetric $(0,2)$ tensor field on P, if in addition $j^*(g)$ is non-degenerate on P and the index of $T_p(N)$ is the same for all $p \in N$, we say N is a Lorentz submanifold of M. (O'Neill, 1983)

2.1.6 Definition

Let M and N be Lorentz manifold with metric $(g)_M$ and $(g)_N$. An isometry from M to N is a diffeomorphism $\phi: M \to N$ that preserves metric tensors $\phi^*((g)_M) = (g)_N$. (O'Neill, 1983)

2.1.7 Definition

Let $x^1, \ldots, x^n$ be a coordinate system on a neighborhood $\mathcal{U}$ in a Lorentz manifold. The Christoffel symbols for this coordinate system are the real-valued functions Γ_{ij}^{k} on $\mathcal{U}$ such that

$$D_{\partial_i}(\partial_j) = \sum_k \Gamma_{ij}^{k}\, \partial_k \text{ where } 1 \leq i, j \leq n. \text{ (O'Neill, 1983)}$$

2.1.8 Proposition

Let $x^1, \ldots, x^n$ be a coordinate system on $\mathcal{U}$

(1) $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$

(2) $D_{\partial_i}\left(\sum W^j\, \partial_j\right) = \sum_k \{\frac{\partial W^k}{\partial x^i} + \sum_j \Gamma_{ij}^{k} W^j\}\partial_k$

(3) $\Gamma_{ij}^{k} = \frac{1}{2}\sum_m g^{km}\{\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^i}\}$. (O'Neill, 1983)

2.1.9 Definition

The 3 dimensional Lorentz-Minkowski space with index 1 is $E_1^3 = (IR^3, <,>_L)$ where the metric $<,>_L$ is $< u, w >_L = u_1 w_1 + u_2 w_2 - u_3 w_3$, $u = (u_1, u_2, u_3)$, $w = (w_1, w_2, w_3)$, which is called the Lorentzian metric.

The non-degenerate metric with index 1 is called as Lorentzian metric. This metric can be written as,

$$< u, w >_L = u^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} w := u^t G\, w.$$

We denoted by the 3- dimensional Euclidean space as $E^3 = (IR^3, <,>_e)$ to separate from Lorentz – Minkowski space. (Lopez, 2014)

2.1.10 Definition

A vector $x \in E_1^3$ is

(1) Space-like if $< x, x >_L$ *is positive* or $x = 0$,

(2) Time-like if $< x, x >_L$ *is negative* and

(3) Light-like if $< x, x >_L = 0$ and $x \neq 0$. (O'Neill, 1983)

The light-cone of E_1^3:

$$C = \{(x, y, z) \in E_1^3 : x^2 + y^2 - z^2 = 0\} - \{(0,0,0)\}.$$

The set of time-like vector is

$$\tau = \{(x, y, z) \in E_1^3 : x^2 + y^2 - z^2 < 0\}. \text{(Lopez, 2014)}$$

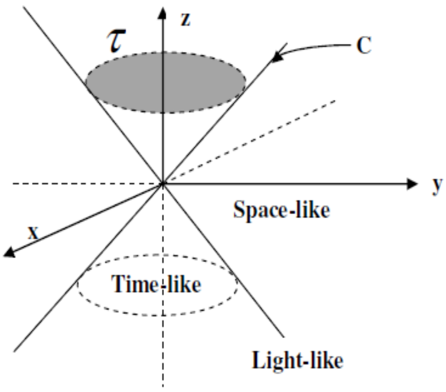


Figure 2.1. The Causal Character in Lorentz - Minkowski Space

Given $W \subset IR^3$ a vector subspace. The induced metric is

$$<,>_W : < x , y >_W = < x , y >_L , \quad x, y \in W.$$

The induced metric on W is classified in 3 cases:

a) When the metric is positive definite, W is called space-like subspace.

b) When the metric has index 1, W is called time-like subspace.

c) When the metric is degenerate then W is called light-like subspace. (O'Neill, 1983)

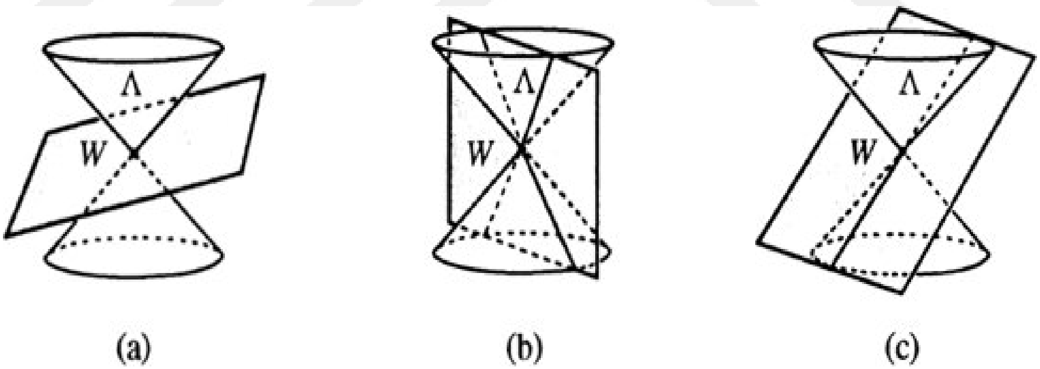


Figure 2.2. The Causal Character of Subspace in Lorentz - Minkowski Space

2.1.11 Example

Let $\vec{a} = (1, 0, 0)$, $\vec{d} = (0 ,1, 0)$, $\vec{f} = (0 ,0, 1)$. The causal character of $\vec{a}, \vec{d}, \vec{f}$ and $\vec{d} + \vec{f}$ are

$< \vec{a}, \vec{a} >_L = 1^2 + 0^2 - 0^2 = 1 \quad \rightarrow \vec{a}$ is space-like.

$< \vec{d}, \vec{d} >_L = 0^2 + 1^2 - 0^2 = 1 \quad \rightarrow \vec{d}$ is space-like.

$< \vec{f}, \vec{f} >_L = 0^2 + 0^2 - 1^2 = -1 \quad \rightarrow \vec{f}$ is time-like.

$< \vec{d}+\vec{f},\ \vec{d}+\vec{f} >_L = < \vec{d},\vec{d} >_L + 2\ < \vec{d},\vec{f} >_L + < \vec{f},\vec{f} >_L$

$< \vec{d}+\vec{f},\ \vec{d}+\vec{f} >_L = \ 0 \ \rightarrow\ \vec{d}+\vec{f}$ is light-like.

2.1.12 Example

a) The plane $span\{\vec{a},\vec{b}\}$ is space-like

b) The plane $span\{\vec{a},\vec{c}\}$ is time-like

c) The plane $span\{\vec{a},\vec{b}+\vec{c}\}$ is light-like

where $\vec{a}=(1,0,0)$, $\vec{b}=(0,1,0)$, $\vec{c}=(0,0,1)$.

a) $< \vec{a},\vec{a} >_L = 1^2+0^2-0^2=1 \rightarrow \vec{a}$ is space-like.

$$< \vec{b},\vec{b} >_L = 0^2+1^2-0^2=1 \rightarrow \vec{b} \text{ is space-like.}$$

$$< \vec{a},\vec{b} >_L = 1.0+0.1-0.0=0$$

Let $\vec{x}=c_1\vec{a}+c_2\vec{b}$.

$$< \vec{x},\vec{x} >_L = < c_1\vec{a}+c_2\vec{b}, c_1\vec{a}+c_2\vec{b} >_L$$

By direct calculation

$$< \vec{x},\vec{x} >_L = (c_1)^2+(c_2)^2>0.$$

Span $\{\vec{a},\vec{b}\}$ is space-like subspace.

b) $< \vec{a},\vec{a} >_L = 1^2+0^2-0^2=1 \rightarrow \vec{a}$ is space-like

$$< \vec{c},\vec{c} >_L = 0^2+0^2-1^2=-1 \rightarrow \vec{c} \text{ is time-like}$$

$$< \vec{a},\vec{c} >_L = 1.0+0.0-0.1=0.$$

Let $\vec{x}=c_1\vec{a}+c_2\vec{c}$.

$$< \vec{x},\vec{x} >_L = < c_1\vec{a}+c_2\vec{c}, c_1\vec{a}+c_2\vec{c} >_L$$

By direct calculation

$$< \vec{x},\vec{x} >_L = (c_1)^2-(c_2)^2.$$

If $\vec{x} = (c_1, c_2)$ then $< \vec{x}, \vec{x} >_L = (c_1)^2 - (c_2)^2$ so $\vec{x} \in \mathbf{E_1^2}$.

Span $\{ \vec{a}, \vec{c} \}$ is time-like subspace.

c) $< \vec{a}, \vec{a} >_L = 1^2 + 0^2 - 0^2 = 1 \rightarrow \vec{a}$ is space-like

$$< \vec{a}, \vec{b} + \vec{c} >_L = < \vec{a}, \vec{b} >_L + < \vec{a}, \vec{c} >_L$$

By direct calculation

$$< \vec{a}, \vec{b} + \vec{c} >_L = 0 + 0 = 0$$

$$< \vec{b} + \vec{c}, \vec{b} + \vec{c} >_L = < \vec{b}, \vec{b} >_L + 2 < \vec{b}, \vec{c} >_L + < \vec{c}, \vec{c} >_L$$

By direct calculation

$< \vec{b} + \vec{c}, \vec{b} + \vec{c} >_L = 1 + 2.0 + (-1) = 0 \rightarrow \vec{b} + \vec{c}$ is light-like.

Let $\vec{x} = c_1\vec{a} + c_2(\vec{b}+\vec{c})$ and $\vec{x} \neq 0$.

$$< \vec{x}, \vec{x} >_L = < c_1\vec{a} + c_2(\vec{b} + \vec{c}), c_1\vec{a} + c_2(\vec{b} + \vec{c}) >_L$$

By direct calculation

$$< \vec{x}, \vec{x} >_L = =(c_1)^2$$

If $\vec{x} = (0, 1)$ then $< \vec{x}, \vec{x} >_L = 0$ so span $\{ \vec{a}, \vec{b} + \vec{c} \}$ is light-like subspace and $\vec{b} + \vec{c} \neq \vec{0}$.

The causality of a vector is the character space-like, time-like and light-like Now we give some properties of subspace of E_1^3.

2.1.13 Proposition

Let (V, g) be a metric space where g is non-degenerate metric.

a) $W \subset V$ is a subspace $\Rightarrow \dim(W^{\perp}) = \dim(V) - \dim(W)$.

b) $W \subset V$ is a subspace $\Rightarrow (W^{\perp})^{\perp} = W$.

c) $W \subset V$ is a non-degenerate subspace $\Rightarrow W^{\perp}$ is a non-degenerate subspace. (Lopez, 2008)

Proof:

a) Let $\{ e_1 ,\ldots, e_m \}$ a base of W and a base $O = \{ e_1 ,\ldots, e_n\}$ of V. If $w = \sum_i x_i e_i \; \varepsilon \; W^{\perp}$, then

$$0 = < \sum_{i=1}^{n} x_i e_i , e_j > = \sum_{i=1}^{n} g_{ij} x_i = 0, \; 1 \le j \le m.$$

In a matrical expression, these m- equations writen as

$$\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{m1} & \cdots & g_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ or}$$

$AX = 0$ and A= $(g_{ij})_{mxn}$. The range of A is m because there is a sub-matrix with range exactly n. As consequence of this, the solutions of AX = 0 generate a n- m dimensional subspace.

b) Because $(W^{\perp})^{\perp} \subset W$, as a consequence dim $(W^{\perp})^{\perp}$ = dim (W).

c) Let O={e_1,…,e_m} be an orthonormal base of W. The matrix of the metric $g_{/W}$ is diagonal with only 1 and -1. The base to get an orthonormal base of V, namely O ={e_1,…,e_n}. Since dim($W^{\perp}$)= n-m, then $\{e_{m+1}, \ldots, e_n\}$ is a base of $W^{\perp}$ and this end the proof.

2.1.14 Proposition

a) Let $x \in E_1^3$. Then x is a timelike vector ⟺ $span\{ x \}^{\perp}$ is space-like and so, $E_1^3 = span\{ x \} \oplus span\{ x \}^{\perp}$. For space-like vectors, we have: x is space-like ⟺ $span\{ x \}^{\perp}$ is time-like.
b) Let $W \subset V$ be a subspace. Then W is space-like ⟺ $W^{\perp}$ is time-like.
c) W is subspace implies W is light-like ⟺ $W^{\perp}$ is light-like. (Lopez, 2008)

Proof:

a) If x is time-like vector, and by multiplying by a number if it is necessary, we put x as a part an orthonormal base of E_1^3, $B = \{ e_1, e_2, x \}$. Then

$span\{ x \}^{\perp} = < e_1, e_2 >_L$, which is a space-like subspace. For converse, let $\{ e_1, e_2\}$ be a orthonormal base of $span\{ x \}^{\perp}$, where $< , >_{span\{ x \}^{\perp}}$ is a positive definite metric. Then $\{ e_1, e_2, x \}$ is a base where diagonalizes the metric. As $g_{11} = g_{22} = 1$, then $g_{33} < 0$. x is time-like vector.

b) If W is a time-like subspace, let $x \in W$ be a time-like vector. Then $W^{\perp} \subset span\{ x \}^{\perp}$. (As $span\{ x \}^{\perp}$ is space-like, then $W^{\perp}$ is space-like. As a consequence $(W^{\perp})^{\perp} = W$.

c) Combining a) and b) then we get the required result.

2.1.15 Proposition

a) If x and y are two null vectors, then x, y are obviously linearly dependent $\Leftrightarrow < x, y >_L = 0$.

b) If x and y are two time-like or null vectors with $< x, y >_L = 0$, then they are null vectors.

c) W is a light-like subspace $\Rightarrow \dim(W \cup W^{\perp}) = 1$. (Lopez, 2008)

Proof:

a) x and y are proportional $\Rightarrow$ they are orthogonal. We suppose that they are orthogonal. In the decomposition

$$E_1^3 = span\{ e_3 \}^{\perp} \oplus span\{ e_3 \}.$$

We write $x = a + w$ and $y = b + w$ assuming that the vector w is the same in both decompositions in order to show what is wanted. As $< x, y >_L = 0$ and both are null vectors, then

$$< a, b >_L + < w, w >_L + < a, w >_L + < b, w >_L = 0.$$

$$< a, a >_L + < w, w >_L + 2 < a, w >_L = 0.$$

$$< b, b >_L + < w, w >_L + 2 < b, w >_L = 0.$$

We get,

$|a|^2 + |b|^2 - 2 < a, b >_L = 0$, that is $|a - b|^2 = 0$. Thus $a = b$, because $a - b$ is space-like vector ($a - b \in span\{ w \}^{\perp}$). So we deduce $x = y$.

b) The two vectors are time-like $\Rightarrow <x, y>_L \neq 0$. By using,

$$E_1^3 = span\{\,y\,\}^{\perp} \oplus span\{\,y\,\}$$

where $span\{\,y\,\}^{\perp}$ is a space-like subspace, we write $x = a + \lambda y$; then

$$<x, y>_L = <y, a>_L + \lambda <y, y>_L = \lambda \; <y, y>_L.$$

$<x, y>_L = 0 \;\Rightarrow\; \lambda = 0$. x and y would be equal and space-like. Similar case is valid for null or time-like vectors. For this reason x and y are null vectors.

c) If $x\,, y \in W \cup W^{\perp}$, then $<x, y>_L = 0$. Then they are linear dependent. This proves that $\dim(W \cup W^{\perp}) \leq 1$. The dimension is exactly $0 \Rightarrow$ $E_1^3 = W \oplus W^{\perp}$, and so any vector of E_1^3 would be null.

2.1.16 Proposition

Let $W \subset E_1^3$ be a 2-dimensional subspace. The followings are equivalent:

a. W is time-like subspace.

b. W contains two independent linear null vectors.

c. W contains a time-like vector. (O'Neill, 1983)

Proof:

a. **(a⇒b)** Let $\{e_1, e_2, e_3\}$ be an orthonormal base of E_1^3. Then $e_2 + e_3$ and $e_2 - e_3$ are linear independent, null vectors.

b. **(b⇒c)** If x and y are the two linear independent, null vectors, then $x + y$ or $x - y$ is a time-like vector because $<x \pm y, x \pm y>_L = \pm 2 <x, y>_L$ and $<x\,, y>_L \neq 0$ due to both vectors being time-like.

c. **(c⇒a)** Let y be a time-like vector W. Then $W^{\perp} \subset span\{\,y\,\}^{\perp}$, and $span\{\,y\,\}^{\perp}$ is a space-like subspace. So, $W^{\perp}$ is space-like, and so W is timelike. The above result can generalize to high-handed dimensions by thinking that W is hyperplane.

2.1.17 Proposition

Let W be a vector subspace of E_1^3. $a \Rightarrow b \Rightarrow c \Rightarrow a$

a. W is a light-like subspace.

b. There exists a null vector in W but not a time-like one.

c. $W \cap C = \mathrm{J} - \{0\}$ and then $\dim J = 1$ where J is a one – dimensional subspace and C is the light cone of V. (O'Neill, 1983)

Proof:

a. **(a⇒b)** Because $< , >_L$ is a degenerate metric, there is a null vector. By the 2.1.5 Proposition, there are not time-like vectors.

b. **(b⇒c)** Because there exist null vectors $W \cap C$ is a non–empty set. By using 2.1.5 Proposition again. There are two linear independent, null vectors ⇒ there would be a time-like vector.

c. **(c⇒a)** 2.1.5 Proposition say that W is neither space-like nor time-like subspace.

2.1.18 Proposition

Let $D \subset E_1^3$ be a vector plane and $\overrightarrow{n_E}$ represents an orthogonal Euclidean vector. Then D is a space-like (respectively time-like, null) plane ⇔ $\overrightarrow{n_E}$ is a time-like (respectively space-like, null) vector. (Lopez, 2014)

Proof:

If D writes as $D = \{(x, y, z) \in IR^3 : dx + ey + fz = 0\}$, then $\overrightarrow{n_E}$ is $\lambda(a, b, c)$ where $\lambda \in IR$. We write D as

$$D = \{(x, y, z) \in IR^3 : dx + ey - (-f)z = 0\} = span\,\{(d, e, -f)\}^{\perp}$$

$$< (d, e, -f), (d, e, -f) >_L = d^2 + e^2 - f^2$$

$$< (d, e, f), (d, e, f) >_L = d^2 + e^2 - f^2$$

The causal character of $(d, e, -f)$ is the same then $\overrightarrow{n_E}$.

2.1.19 Example

Find the causal character of plane $x + y - 2z = 0$.

$$\overrightarrow{n_E} = (1,1,-2)$$

$$<\overrightarrow{n_E}, \overrightarrow{n_E}>_L = 1^2 + 1^2 - 2^2 = -2 \rightarrow \overrightarrow{n_E}\ is\ time - like.$$

So plane is space-like.

2.1.20 Definition

Given $x \in E_1^3$, the norm of x is $||x||_L = \sqrt{|<x,x>_L|}$. The vector x is called unit vector if its norm is 1. (O'Neill, 1983)

2.1.21 Proposition

If $D = span\{x\}^{\perp}$ is a space-like plane, then $||x||_E \geq ||x||_L$. (Lopez, 2014)

Proof:

It sufficies if $||x||_L = 1$. Assume $\overrightarrow{n_E} = (d,e,f)$, with

$$<(d,e,f),(d,e,f)>_E = d^2 + e^2 + f^2 = 1$$

$$D = \{(x,y,z) \in IR^3 : dx + ey + fz = 0\} \text{ and } \overrightarrow{n_E} = (d,e,f)$$

$$D = \{(x,y,z) \in IR^3 : dx + ey - (-f)z = 0\} = span\{x\}^{\perp} \Rightarrow x = (d,e,-f)$$

$$<(d,e,-f),(d,e,-f)>_L = d^2 + e^2 - f^2$$

$$||x||_L = \sqrt{|<x,x>_L|} = \sqrt{|d^2 + e^2 - f^2|} = 1$$

D is space-like plane then x is time-like vector. $<x,x>_L < 0, x \neq 0$.

$$d^2 + e^2 - f^2 < 0 \rightarrow |d^2 + e^2 - f^2| = f^2 - d^2 - e^2$$

$$x = \pm \frac{(d,e,-f)}{\sqrt{f^2 - d^2 - e^2}}$$

The Euclidean norm $||x||_E$ is

$$||x||_E^2 = \frac{d^2 + e^2 + f^2}{f^2 - d^2 - e^2} = \frac{1}{f^2 - d^2 - e^2} \geq 1$$

because $f^2 - d^2 - e^2 = d^2 + e^2 + f^2 - 2d^2 - 2e^2 = 1 - 2(d^2 + e^2) \leq 1$.

So $||x||_E \geq ||x||_L$.

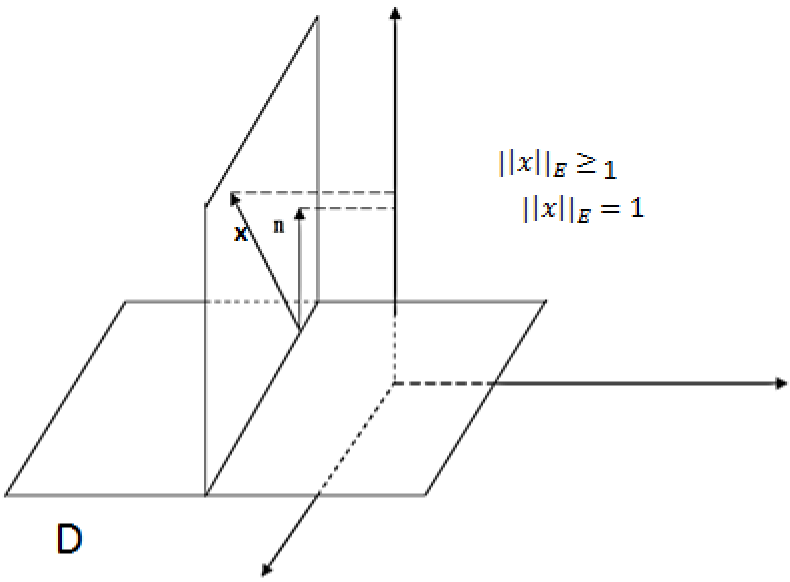


Figure 2.3. The vector $\boldsymbol{x}$ orthogonal to a space-like plane $\boldsymbol{D}$ bigger than the Euclidean normal vector $\boldsymbol{n}$ to $\boldsymbol{D}$

2.1.22 Definition

Let τ be a set of time-like vectors of E_1^3. For each $x \in \tau$ we define the time cone x as

$$\mathrm{C}(x) = \{y \in \tau: <x, y>_L < 0\}.$$

This set is non–empty set since $<x, x>_L < 0 \Rightarrow x \in \mathrm{C}(x)$. Furthermore τ the disjoint union of $\mathrm{C}(x)$ and $\mathrm{C}(-x)$. If $y \in \tau$ then $<x, y>_L \neq 0$ and so either $<x, y>_L < 0$ or $<x, y>_L > 0$ this means that $y \in \mathrm{C}(x)$ or $y \in \mathrm{C}(-x)$ and $C(x) \cap C(-x) = \emptyset$. (O'Neill, 1983)

2.1.23 Proposition

a) Two time-like vectors x, y are in this same time-like cone $\Leftrightarrow$ $<x, y>_L < 0$.

b) $x \in \mathrm{C}(y) \Leftrightarrow \mathrm{C}(x) = \mathrm{C}(y)$.

c) The time-like cones are convex sets. (O'Neill, 1983)

Proof:

a) If $< x, y >_L < 0$, then $x \in C(y)$. Let assume that $x, y \in C(t)$. We can suppose that $< t, t >_L = -1$. We write $x = d + at$ and $y = e + bt$, with $d, e \in span\{ t \}^{\perp}$. As $span\{ t \}^{\perp}$ is a space-like subspace, then $|< d, e >_L| \leq |d|.|e|$, and

$$< x, y >_L = -a.b + < d, e >_L \leq -a.b + |d|.|e|$$

Since $< d, d >_L < a^2$ and $< e, e >_L < b^2$ and $< x, y >_L < 0$.

b) If $x \in C(y)$ then $< x, y >_L < 0$, that is $y \in C(x)$.

c) Assume that $x, y \in C(t)$ and let $m \in [0,1]$. Then

$$< mx + (1 - m)y, t >_L = m < x, t >_L + (1 - m) < y, t >_L < 0.$$

And this means that $mx + (1 - m)y \in C(t)$.

2.1.24 Theorem

Let x and y be time-like vectors in Lorentz vector space. Then

1. $|< x, y >_L| \geq ||x||_L.||y||_L$, with equality $\Leftrightarrow x$ and y are collinear.
2. If x and y are in the same time cone, there exist only one non-negative number $\theta \geq 0$, which called hyperbolic angle between x and y such that

$$< x, y >_L = -||x||_L.||y||_L \cosh \theta.$$ (O'Neill, 1983)

Proof:

1. Write $x = ay + \vec{x}$ with $\vec{x} \in y^{\perp}$. Since $\vec{x}$ is space-like.

$$< x, x >_L = < ay + \vec{x}, ay + \vec{x} >_L$$

$$< x, x >_L = < ay + \vec{x}, ay >_L + < ay + \vec{x}, \vec{x} >_L$$

$$< x, x >_L = < ay, ay >_L + < \vec{x}, ay >_L + < ay, \vec{x} >_L + < \vec{x}, \vec{x} >_L$$

$$< x, x >_L = a^2 < y, y >_L + a < \vec{x}, y >_L + a < y, \vec{x} >_L + < \vec{x}, \vec{x} >_L$$

$$< x, x >_L = a^2 < y, y >_L + < \vec{x}, \vec{x} >_L$$

$$< x, x >_L = a^2 < y, y >_L + < \vec{x}, \vec{x} >_L < 0.$$ (Since x is time-like)

$$< x, x >_L - < \vec{x}, \vec{x} >_L = a^2 < y, y >_L \quad *$$

Then

$$< y,x >_L^2 = < y, ay + \vec{x} >_L . < y, ay + \vec{x} >_L$$

$$< y,x >_L^2 = (< y, ay >_L + < y, \vec{x} >_L)^2$$

$$< y,x >_L^2 = a^2 < y,y >_L^2$$

$$< y,x >_L^2 = a^2 < y,y >_L . < y,y >_L$$

$$< y,x >_L^2 = (< x,x >_L - < \vec{x},\vec{x} >_L) < y,y >_L$$

$$< y,x >_L^2 \geq (< x,x >_L . < y,y >_L)$$

$$< y,x >_L^2 \geq ||x||_L^2 . ||y||_L^2$$

$| < x,y >_L | \geq ||x||_L . ||y||_L$ (Cauchy-Schwarz backwards)

Since $< \vec{x},\vec{x} >_L \geq 0$ and $< y,y >_L < 0$.

Evidently equality holds if and only if $< \vec{x},\vec{x} >_L = 0$, which is equivalent to $\vec{x}$=0, that is, to $x = ay$.

2. So we get inequality

$$\frac{(< x,y >_L)^2}{(||x||_L . ||y||_L)^2} \geq 1.$$

If x and y lie in the same time cone, then $< x,y >_L < 0$ implies

$$\frac{| < x,y >_L |}{||x||_L \, ||y||_L} \geq 1$$

$$\frac{- < x,y >_L}{||x||_L . ||y||_L} \geq 1.$$

cosh: $[0,\infty) \rightarrow [1,\infty)$ is 1-1, there exists a unique number $\theta \in [0,\infty)$ such that

$$\cosh\theta = \frac{- < x,y >_L}{||x||_L . ||y||_L}.$$

So $< x,y >_L = -\big| |x||_L \big| |y||_L \cosh\theta$.

2.1.25 Corollary

Let x and y be time-like vectors in Lorentz vector space. If x and y are in the same time cone then

$||x||_L + ||y||_L \leq ||x + y||_L$. (O'Neill, 1983)

Proof:

Since $< x, y >_L < 0$ the backwards Cauchy-Schwarz inequality this

$$|< x, y >_L| \geq ||x||_L . \; ||y||_L$$

We know that x and y are time-like vectors

(1) $||x||_L^2 = | < x, x >_L |$

(2) $||x||_L = \sqrt{|< x, x >_L|}$

(3) $|< x, y >_L| = -< x, y >_L$

(4) $< x, y >_L = < y, x >_L$ is symmetric.

$$\begin{aligned}(||x||_L + ||y||_L)^2 &= ||x||_L^2 + 2\,||x||_L . ||y||_L + ||y||_L^2 \\ &= -< x, x >_L - < x, y >_L - < x, y >_L - < y, y >_L \\ &= -< x, x + y >_L - < x + y, y >_L \\ &= -< x, x + y >_L - < y, x + y >_L \\ &= -< x + y, x + y >_L \\ &= | < x + y, x + y >_L |\end{aligned}$$

$$(||x||_L + ||y||_L)^2 \leq ||x + y||^2$$

$$||x||_L + ||y||_L \leq ||x + y||_L .$$

2.1.26 Definition

Let $e_3 = (0,0,1)$. For a time-like vector x, we call that x as future - directed (respectively past-directed) if $x \in \mathrm{C}(e_3)$, which is $< x, e_3 >_L < 0$ (respectively $x \in \mathrm{C}(-e_3)$ or $< x, e_3 >_L > 0$). It is also equivalent to say that $x = (x_1, x_2, x_3)$ is future directed if $x_3 > 0$. We always orient by time-like cone $\mathrm{C}(e_3)$, that is, $(E_1^3, [B_u])$, where B_u is usual base if R^3. (Lopez, 2008)

2.2 The Lorentz - Minkowski Vector Product

The definition of Lorentz – Minkowski vector product is the same as the given one in the Euclidean ambient.

2.2.1 Definition

If $a, b \in E_1^3$, the Lorentz – Minkowski vector product of a and b is express by $a\, x_L\, b$ which is unique vector. It satisfies the equation

$< a\, x_L\, b, t >_L = \det(a, b, t)$. By taking t each one of the vectors of the usual base, we obtain

$$a\, x_L\, b = \begin{vmatrix} i & j & -k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Since the metric is bilinear the vector is exist and unique. Thus, if we denote $a\, x_E\, b$ the Euclidean vector product, we have that $a\, x_L\, b$ is the reflection of $a\, x_E\, b$ with respect to the plane $\{z = 0\}$. (Lopez, 2014)

2.2.2 Example

Let u and v be time-like vectors in a Lorentz vector space. If a and b are in the same time cone then

$||a\, x_L\, b||_L^2 = ||a||_L^2 . ||b||_L^2 . (\sinh\theta)^2$, $\theta = angle(a, b)$. (Lopez, 2014)

Proof:

We know that a and b are time-like vectors and both are in the same time cone

(1) $||a||_L = \sqrt{| < a, a >_L |}$

(2) $||a||_L^2 = | < a, a >_L |$

(3) $|< a, a >_L| = -< a, a >_L$

(4) $< a, b >_L = < b, a >_L$

(5) $|< a, b >_L| \geq ||a||_L . ||b||_L$

(6) $< a, b >_L^2 > ||a||_L^2 . ||b||_L^2$

(7) $< a, b >_L^2 = ||a||_L^2 . ||b||_L^2 (\cosh\theta)^2$

(8) $(\cosh\theta)^2 - (\sinh\theta)^2 = 1$

$$||a\, x_L\, b||_L^2 = | < a\, x_L b, a\, x_L b >_L |$$

$$||a\, x_L\, b||_L^2 = | < a, a >_L < b, b >_L - < a, b >_L < b, a >_L |$$

$$||a\, x_L\, b||_L^2 = | < a, a >_L < b, b >_L - < a, b >_L^2 |$$

$$||a\, x_L\, b||_L^2 = < a, b >_L^2 - < a, a >_L < b, b >_L$$

$$||a\, x_L\, b||_L^2 = < a, b >_L^2 - ||a||_L^2 . ||b||_L^2$$

$$||a\, x_L\, b||_L^2 = ||a||_L^2.\,||b||_L^2(\cosh\theta)^2 - ||a||_L^2.\,||b||_L^2$$
$$||a\, x_L\, b||_L^2 = ||a||_L^2.\,||b||_L^2[(\cosh\theta)^2 - 1]$$
$$||a\, x_L\, b||_L^2 = ||a||_L^2.\,||b||_L^2\,(\sinh\theta)^2$$

2.2.3 Proposition

The vector product have properties:

a) $a\, x_L b = -b\, x_L\, a$.

b) $a\, x_L b$ is orthogonal to a and b.

c) $a\, x_L b = 0 \Leftrightarrow \{a\,, b\}$ are not proportional.

d) $a\, x_L b \neq 0$ lies in the plane $D = <a, b>_L$ $\Leftrightarrow$ the plane D is null. (Lopez, 2008)

2.3 Isometries of Lorentz-Minkowski Space

Here we give the isometries of Minkowski space E_1^3. The set of all vector isometries of E_1^3 is denoted by $I_1(3)$. If F and F' are different orthonormal bases, the matrix M satisfies $M^t DM = D$ where

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus $I_1(3) = \{\, M \in Gl(3, IR);\ M^t DM = D\}$.

1) $\det(M) = \pm 1$.

2) $I_1(3)$ has at least two connected components.

3) $SI_1(3)$ is denoted by the set of isometries with $\det(M) = 1$.

4) $SI_1(3)$ is called the *special Lorentz group*.

5) $F \in SI_1(3)$ $\Leftrightarrow$ F is positive oriented.

We define the *ortocrone group* by

$$I_1^+(3) = \{M \in I_1(3); M \text{ maintains the time} - \text{like orientation}\}.$$

M maintains the time-like orientation. A future-directed orthonormal base F $\Rightarrow$ the base obtained by $F' = M.F$ is also future-directed. We also have the next characterization of $I_1^+(3)$: $M \in I_1^+(3)$ if and only if $m_{33} > 0$. The set $I_1^+(3)$ is a group with two components: One of them $I_1^+(3) \cap SI_1(3)$ and the other one is $I_1^+(3) - (I_1^+(3) \cap SI_1(3))$. We define the *special Lorentz ortocrone group* as the set

$$I_1^{++}(3) = SI_1(3) \cap I_1^+(3) = \{M \in I_1(3); \det(M) = 1, M \text{ maintains time} - \text{like orientation}\}.$$

This set is a group and $identity\ \ belongs\ to\ I_1^{++}(3)$. $I_1^{++}(3)$ is not a compact set because the subset

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\theta) & \sinh(\theta) \\ 0 & \sinh(\theta) & \cosh(\theta) \end{pmatrix}; \theta \in \mathrm{R}\right\}$$

is not bounded. (Lopez, 2014)

2.3.1 Theorem

The connected components of $I_1(3)$ are $I_1^{++}(3)$ and

1. $I_1^{+-}(3) = \{ M \in SI_1(3);\ m_{33} < 0\}$
2. $I_1^{-+}(3) = \{ M \in I_1^+(3)\,; \det(M) = -1\}$
3. $I_1^{--}(3) = \{ M \in I_1(3)\ ; \det(M) = -1\,,\ m_{33} < 0\}$.

If we denote by D_1 and D_2 the isometries given by $D_1 = \mathrm{diag}[1,1,-1]$ and $D_2 = \mathrm{diag}[1,-1,1]$ then the three last components correspond, respectively, with $D_1.\ D_2.I_1^{++}(3)$, $D_2.I_1^{++}(3)$, and $D_1.I_1^{++}(3)$. The rigid motion of E_1^3 are the composition of a vector isometries and a translations of E_1^3.

Next we study the isometries of the two-dimensional Lorentz-Minkowski space E_1^2. Let M be a matrix by

$$M = \begin{pmatrix} p & r \\ s & t \end{pmatrix}.$$

Then M $\in I_1(2)$ and $\Leftrightarrow\ D = M^tDM$, that is,

$$p^2 - s^2 = 1, pr - st = 0, t^2 - r^2 = 1.$$

From the first equation, we have two possibilities:

1. There exists θ such that $p = \cosh(\theta)$ and $s = \sinh(\theta)$. Equalities $t^2 - r^2 = 1$, there appear two cases again:

(a) There exists φ such that $t = \cosh(\varphi)$ and $r = \sinh(\varphi)$. With the second equation, we conclude that $\varphi = \theta$

(b) There exists φ such that $t = -\cosh(\varphi)$ and $r = \sinh(\varphi)$. We get $\varphi = -\theta$.

2. $\exists\, \theta$ such that $p = -\cosh(\theta)$ and $s = \sinh(\theta)$. Equalities $t^2 - r^2 = 1$ implies the following possibilities:

   (a) $\exists\, \varphi$ such that $t = \cosh(\varphi)$ and $r = \sinh(\varphi)$. By second equation we get $\varphi = -\theta$.

   (b) $\exists\, \varphi$ such that $t = -\cosh(\varphi)$ and $r = \sinh(\varphi)$. From $pr - st = 0$ we have $\varphi = \theta$.

As a result, we get 4 kinds of isometries.

$$\begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}, \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ -\sinh(\theta) & -\cosh(\theta) \end{pmatrix}$$

$$\begin{pmatrix} -\cosh(\theta) & \sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix}, \begin{pmatrix} -\cosh\,(\theta) & \sinh(\theta) \\ \sinh\,(\theta) & -\cosh(\theta) \end{pmatrix}.$$

With the same notation as in 2.3.1 Theorem, each one of the matrices that have appeared belong to $I_1^{++}(2), I_1^{--}(2), I_1^{-+}(2) I_1^{+-}(2)$, respectively. We see which is the difference with the isometries of E^2. It appears as equations of type $m^2 + n^2 = 1$, whose solutions can be written as $x = \cos\gamma$ and $y = \sin\gamma$.

This distinguishes the equation $m^2 - n^2 = 1$, where it is necessary to seperate the case that x is positive or negative.

We end this chapter with a study of isometries of $I_1^{++}(3)$ that leave a straightline J fixed pointwise. These kind of isometries are called *boosts.* Three types of such isometries will appear, depending on the causal character of L. Let $e_1 = (1,0,0), e_2 = (0,1,0)\ and\ e_3 = (0,0,-1)$

1. <u>J is time-like:</u> Assume that $J = span\{e_3\}$ Since $M.e_3 = e_3$, we obtain that $m_{13} = m_{23} = 0$ and$m_{33} = 1$. By using the equality $D = M^t DM$, we have $m_{31} = m_{32} = 0$ and

$$m_{11}^2 + m_{21}^2 = 1\ ,\ m_{11}.m_{12} + m_{21}.m_{22} = 0, m_{12}^2 + m = 1$$

Thus the matrix M is written as

$$M = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. <u>J is space-like:</u> Let $J = span\,\{e_1\}$. Then

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & \sinh\beta \\ 0 & \sinh\beta & \cosh\beta \end{pmatrix}.$$

3. <u>J is light-like:</u> We suppose that $J = span\,\{e_2 + e_3\}$. Then

$$M = \begin{pmatrix} 1 & \omega & -\omega \\ -\omega & 1 - \frac{\omega^2}{2} & \frac{\omega^2}{2} \\ -\omega & -\frac{\omega^2}{2} & 1 + \frac{\omega^2}{2} \end{pmatrix}.$$

In all above cases, the isometries belong to $I_1^{++}(3)$. By using boosts we can define a circle in E_1^3. In usual Euclidean space E^3, we can define a circle in several ways.

a) A circle is the set of points equidistant from a fix point.

b) Constant curvature curve.

c) The orbit of a point under a group of rotations of E^3.

In Lorentz-Minkowski space we will define a curve like in c) but replacing rotation by boosts. Let J be a fixed straight-line of E_1^3 and let $D_J = \{\phi_\omega : \omega \in R\}$ by the group of boosts which fix J. A circle is orbit $\{\phi_\omega(p_0) : \beta_\omega \in D_J\}$ of a point $p_0 \notin J, p_0 = (x_0, y_0, z_0)$. We have 3 possibilities because of the causal character of J. we have;

1. <u>J is time-like:</u> We consider $J = span\{e_3\}$. Then

$$D_J = \left\{ \phi_\omega = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}; \omega \in IR \right\}.$$

The set $\{\phi_\omega(p_0) : \omega \in R\}$ is the circle in $z = z_0$ with radius $\sqrt{x_0^2 + y_0^2}$.

2. <u>J is space-like:</u> We take $J = span\,\{e_1\}$. Then

Suppose $y_0^2 - z_0^2 \neq 0$ otherwise, it is aa straight line. Then the orbit p_0 is a branch of the hyperbola $y^2 - z^2 = y_0^2 - z_0^2$ in the plane $x = x_0$. According to $y_0^2 - z_0^2 < 0$ or $y_0^2 - z_0^2 > 0$, we will have 4 cases.

3. <u>J is light-like: </u> We assume that $J = span\{e_2 + e_3\}$ and we consider the plane $J^{\perp} = span\ \{e_1, e_2 + e_3\}$.

$$D_J = \left\{ \phi_\omega = \begin{pmatrix} 1 & \omega & -\omega \\ -\omega & 1 - \dfrac{\omega^2}{2} & \dfrac{\omega^2}{2} \\ -\omega & -\dfrac{\omega^2}{2} & 1 + \dfrac{\omega^2}{2} \end{pmatrix} ; \omega \in IR \right\}.$$

The orbit of a point $p_0 = (x_0, y_0, z_0) \notin J^{\perp}$ is a plane curve which lies in $y - z = y_0 - z_0$.

$X = x_0 + \omega(y_0 - z_0)$ and $Y = y_0 - x_0\omega - (y_0 - z_0)\omega^2/2 \Rightarrow$ the orbit of p_0 satisfies

$$Y = \frac{-X^2 + 2y_0(y_0 - z_0) + x_0^2}{2(z_0 - y_0)}$$

This means that the circle $\{\phi_\omega(p_0): \omega \in R\}$ is a parabola.

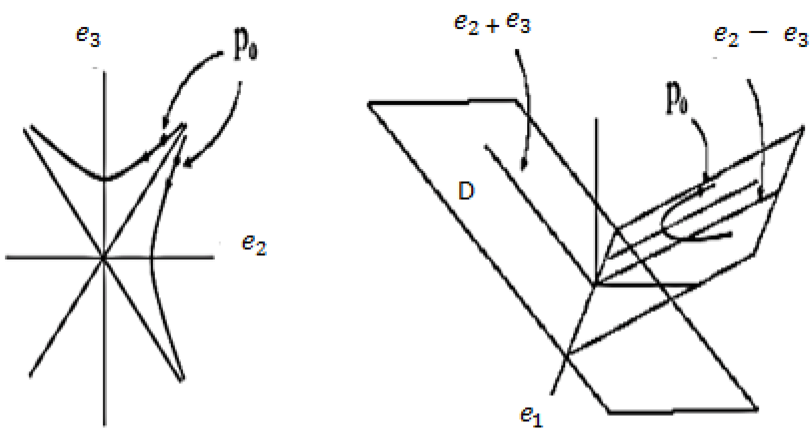


Figure 2.4. Hyperbola and Parabola

We point out that the orbits are Euclidean circles, hyperbolas and parabolas only in the case of the axis of the group of boosts is one of the above ones. They are generally affine ellipse, hyberbola or parabola, depending on the status. For instance, we consider the rotations with respect to the time-like line $J = span\{(0,1,2)\}$.

$$J^{\perp} = Span\,\{e_1, (0,2,1)/\sqrt{3}\}.$$

$$p = (1,0,0) \in J^{\perp} \Rightarrow \phi_{\omega}(p) = \cos\omega\, e_1 + \sin\omega \frac{1}{\sqrt{3}}(0,2,1)$$

that is an affine ellipse in $J^{\perp}$. (O'Neill, 1983)

3. CURVES IN LORENTZ SPACE

In this chapter we will give Frenet vectors of curves in 3 dimensional Lorentz-Minkowski space. A smooth curve is differentiable map $\beta: J \subset IR \rightarrow E_1^3$ where J is an open interval. β is parametrized curve. A curve is regular if $\beta'(t) \neq 0\ \forall\, t \in J$. We will take E_1^3 as a 3- dimensional manifold. A regular curve will be defined as immersion between the (1-dim) manifold J and the (3-dim) manifold IR^3.

3.1 The Fundamental Local Theory of Curves

Let $\beta: J \rightarrow E_1^3$ be a regular curve. When $p \in J$, the tangent space T_pJ identifies with *IR*. The differential map $(d\beta)_p: T_pJ \equiv R \rightarrow T_{\beta(p)}E_1^3 \equiv IR^3$ is

$$(d\beta)_p(s) = \frac{d}{du}|_{u=0}\ \beta(p+su) = s.\beta'(p).$$

It is also linear map $\beta'(t)$.

$\frac{\partial}{\partial p}$ is the unit tangent vector on $T_pJ \Rightarrow (d\beta)_p\left(\frac{\partial}{\partial p}\right) = \alpha'(p)$.

We now take R^3 with the Lorentzian metric $<,>_L$. By the map β we can define the induced metric of E_1^3 on J.

$$\beta: (J, \beta^* <,>_L) \rightarrow E_1^3 = (IR^3, <,>_L)$$

Obviously it is an isometric immersion.

$\beta^* <,>_{L^p}(a,b) =< (d\beta)_p(a), (d\beta)_p(b) >_L = ab < \beta'(t), \beta'(t) >_L,\ a,b \in IR$,
$\beta^* <,>_L$ is defined above pullback metric.

If we take the basis $\{\frac{\partial}{\partial p}\}$ in T_pJ,

$$\beta^* <,>_{L^p} \left(\frac{\partial}{\partial p}\right), \left(\frac{\partial}{\partial p}\right) = < \beta'(p), \beta'(p) >_L.$$

In order to classify the manifold $(J, \beta^* <,>_L)$ and since *I* is a one-dimensional manifold, we need to know the sign of $< \beta'(p), \beta'(p) >_L$. Thus

a. If $<\beta'(p),\beta'(p)>_L$ $is\ positive$ then $(J,\beta^*<,>_L)$ is a Riemannian manifold.

b. If $<\beta'(p),\beta'(p)>_L$ $is\ negative$ then $(J,\beta^*<,>_L)$ is a Lorentzian manifold.

c. If $<\beta'(p),\beta'(p)>_L=0$ then $(J,\beta^*<,>_L)$ is a degenerate manifold.

(Lopez, 2014)

3.1.1 Definition

A smooth curve in $\alpha: I \rightarrow E_1^3$ is

(1) Space-like, if for any $t \in I$, $\alpha'(t)$ is space-like ;

(2) Time-like , if for any $t \in I$ $\alpha'(t)$ is time-like ;

(3) Light-like , if for any $t \in I$ $\alpha'(t)$ is light-like. (O'Neill, 1983)

3.1.2 Proposition

Any time-like or null curve is regular. (O'Neill, 1983)

Proof:

Suppose that the curve is time-like, and we write $\alpha(s) = \big(x(s), y(s), z(s)\big)$, where the function x, y and z are differentiable functions on s. Then $<\alpha'(s),\ \alpha'(s)\ >_L = [x'(s)]^2 + [y'(s)]^2 - [z'(s)]^2 < 0, z'(s) \neq 0$, that is , α is regular curve.

If the curve is null, we have $z'(s) \neq 0$, however $x'(s) = y'(s) = 0$ and $\alpha'(s) = 0$. But means that α is space-like at s.

3.1.3 Example

$\alpha: IR \rightarrow E_1^3$

t $\rightarrow$ α(t) = $(\cosh t, \frac{t^2}{2}, \sinh t)$

a) Is α(t) regular curve?

b) Determine the causal character of α. (Lopez, 2014)

$\alpha'(t) = (\sinh t,\ t, \cosh t\ ) \neq 0\ \forall\ t \in IR$. α is regular. Note that

$$|| \alpha'(t) ||_L = (\sinh t)^2 + t^2 - (\cosh t\ )^2 = (\sinh t)^2 - (\cosh t\ )^2 + t^2 = t^2 - 1$$

$$|| \alpha'(t) ||_L^2 = |t^2 - 1|$$

$$t = \mp 1 \,.\, || \alpha'(\mp 1) ||_L = 0 \text{ but}$$

$$\alpha'(1) = \left(\frac{e^2-1}{2e}, 1, \frac{e^2+1}{2e}\right) \neq 0 \text{ and } \alpha'(-1) = \left(\frac{1-e^2}{2e}, 1, \frac{e^2+1}{2e}\right) \neq 0$$

$$< \alpha'(t), \alpha'(t) >_L = t^2 - 1$$

On the interval (-1,-1), α is a time-like curve.

On the interval $(-\infty, -1) \cup (1, \infty)$, α is space-like curve.

$\{-1,1\}$ at $t = \mp 1$, α is a light-like curve.

3.1.4 Example

(1) $\alpha(s) = a + bs, a, b \in IR^3, s \neq 0$ is the straight-line with $\alpha'(s) = s$. The causal character of α is same with vector s.

(2) $\alpha(t) = r(\cos t, \sin t, 0)$ is the circle. $\alpha'(t) = r(-\sin t,\ \cos t, 0)$ since $< \alpha'(t), \alpha'(t) >_L = r^2 > 0$ is a space-like curve. Also it lies $xy-$ plane where space-like plane.

(3) $\alpha(t) = (t, t^2, t^2)$ is the parabola. $\alpha'(t) = (1, 2t, 2t)$. $< \alpha'(t), \alpha'(t) >_L = 1 > 0$ is a space-like curve. It lies in null plane $y = z$.

(4) $\alpha(t) = r(0, \sinh t, \cosh t)$ is the hyperbola. $\alpha'(t) = r(0\ , \cosh t\ \ , \sinh t\ )$ since $< \alpha'(t), \alpha'(t) >_L = r^2 > 0$ is a space-like curve. Also the time-like plane of equation $yz -$ plane.

(5) $\alpha(t) = r(0, \cosh t, \sinh t\ )$ is the hyperbola. $\alpha'(t) = r(0, \sinh t, \cosh t)$ since $< \alpha'(t), \alpha'(t) >_L = -r^2 < 0$ is a time-like curve. Also the time-like plane of equation $yz -$ plane.

3.1.5 Example

$\alpha: IR \rightarrow E_1^3$

t $\rightarrow$ α(t)=$(r\cos t, r\sin t, ht)$ $h \neq 0, r > 0$

Find the causal character of α.

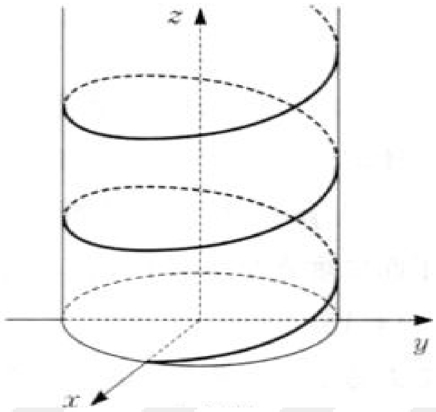


Figure 3.1. The Helix

α lies on the cylinder $x^2 + y^2 = r^2$ and pitch $2\pi h$.

$\alpha'(t) = (-r\sin t, r\cos t, h) \neq 0 \; \forall \; t \in IR.$ α is a regular curve .

$$< \alpha'(t), \alpha'(t) >_L = r^2 - h^2$$

a) If $r^2 - h^2 > 0$ then α is a space-like curve.

b) If $r^2 - h^2 < 0$ then α is a time-like curve.

c) If $r^2 - h^2 = 0$ then α is a light-like curve.

3.1.6 Example

$\alpha: IR \rightarrow E_1^3$

t $\rightarrow$ α(t) = $(ht, r\sinh t, r\cosh t)$, $h \neq 0, r > 0$

Find the causal character of α.

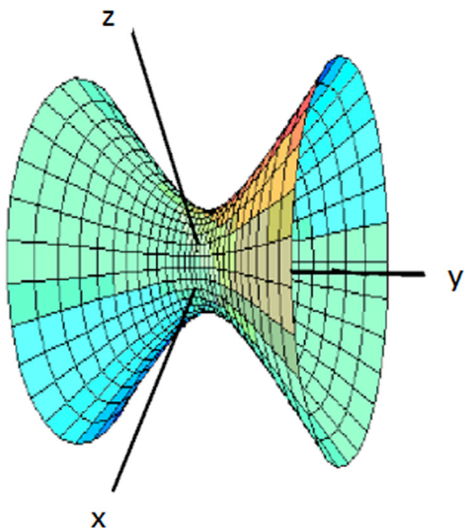


Figure 3.2. Hyperbolic Cylinder $\boldsymbol{y^2 - z^2 = -r^2}$

α lies on the hyperbolic cylinder of equation $y^2 - z^2 = -r^2$.

$\alpha'(t) = (h, r\cosh t, r\sinh t) \neq 0 \ \forall\, t \in IR$. α is a regular curve.

$< \alpha'(t), \alpha'(t) >_L = r^2 + h^2 > 0$. α is a space-like curve.

3.1.7 Example

$$\alpha: IR \to E_1^3$$

$$t \to \alpha(t) = (ht, r\cosh t, r\sinh t),\ h \neq 0, r > 0.$$

Find the causal character of α.

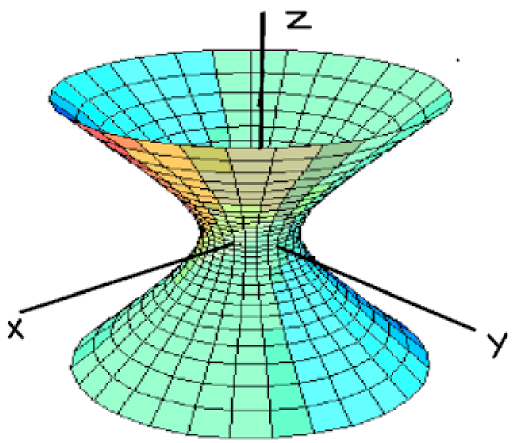


Figure 3.3. Hyperbolic Cylinder $\boldsymbol{y^2 - z^2 = r^2}$

α lies on the hyperbolic cylinder of equation $y^2 - z^2 = r^2$.

$\alpha'(t) = (h, r\sinh t\ , r\cosh t) \neq 0\ \forall\ t \in IR$. α is a regular curve.

$$< \alpha'(t), \alpha'(t) >_L = h^2 - r^2$$

a) If $h^2 - r^2 > 0$ then α is a space-like curve.

b) If $h^2 - r^2 < 0$ then α is a time-like curve.

c) If $h^2 - r^2 = 0$ then α is a light-like curve.

3.1.8 Proposition

Let $\gamma: J \to E_1^3$ be non-space-like curve and $s_0 \in J$. $\exists\ \varepsilon > 0$ and C^∞ the function $f_1, f_2: I \subset IR \to IR$ such that $s = \varphi(t)$ and

$$\beta(t) = \gamma(\varphi(t)) = (f_1(t), f_2(t), t).$$ (Carmo, 1976)

3.1.9 Theorem

Let β be a closed regular curve in $D \subset E_1^3$. β is space-like $\Rightarrow D$ is a space-like plane. (Carmo, 1976)

Proof:

Case 1: Let D is time-like plane. Take D as $x = 0$. $\beta(t) = \big(0, y(t), z(t)\big)$. Because the function $y: IR \to IR$ is periodic, it achieves a maximum at some point $t_0 \Rightarrow y'(t_0) = 0$ and so $\beta'(t_0) = \big(0, y'(t_0), z'(t_0)\big) = (0, 0, z'(t_0))$. We know that α is a regular curve $z'(t_0) \neq 0$, $< \beta'(t_0), \beta'(t_0) >_L = -[z'(t_0)]^2$. α is timelike at $t = t_0$ .which is a contradiction.

Case 2: Let D is null plane. Take D as $y = z$.

$\beta(t) = \beta(t)\big(x(t), y(t), z(t)\big)$. Let the maximum of $x(t)$be at t_0. Because the function $x: IR \to IR$ is periodic this implies $x'(t_0) = 0 \Rightarrow \beta'(t_0) = \big(0, y'(t_0), y'(t_0)\big)$. $\beta'(t_0) \neq 0 \Rightarrow y'(t_0) \neq 0$ by regularity but $< \beta'(t_0), \beta'(t_0) >_L = [y'(t_0)]^2 - [y'(t_0)]^2 = 0$. β is null at $t = t_0$ which is a contaradiction.

3.1.10 Theorem

There are not closed curves in 3 dimensional Lorentz space that are time-like or null. (Carmo, 1976)

Proof:

Suppose that the curve is closed then

$$\alpha: I \rightarrow E_1^3$$

$$\mathrm{t} \rightarrow \alpha(\mathrm{t}) = (x(t), y(t), z(t))$$

$z = z(t)$ is periodic there exists $t = t_0$ such that $z'(t_0) = 0$

$< \alpha'(t_0), \alpha'(t_0) >_L = [x'(t_0)]^2 + [y'(t_0)]^2 \geq 0$. This is a contradiction if α is time-like. If α is null then $x'(t_0) = y'(t_0) = 0 \Rightarrow \alpha'(t_0) = (0,0,0) \Rightarrow \alpha$ is regular at $t = t_0$, which is a contradiction.

3.1.11 Proposition

Let $\alpha: J \rightarrow E_1^3$ be a non-null curve. Given $x_0 \in J, \delta, \varepsilon > 0$ and a diffeomorphism $\phi: (-\varepsilon, \varepsilon) \rightarrow (x_0 - \delta, x_0 + \delta)$ such that the curve

$\beta: (-\varepsilon, \varepsilon) \rightarrow E_1^3$ given by $\beta = \alpha \, o \, \phi$ satisfies $||\beta'(s)|| = 1$ for all $s \in (-\varepsilon, \varepsilon)$. (Lopez, 2014)

3.1.12 Lemma

Let $\gamma: I \rightarrow E_1^3$ be a null curve such that the trace of γ is not a straight-line. There exist a parametrization of γ given by $\beta(s) = (\gamma \, o \, \phi)(s)$ such that $||\beta''(s)|| = 1. \gamma$ is pseudo – parametrized by arclength. (Lopez, 2014)

Proof:

We can write $\beta(s) = \gamma(\phi(s))$.

Then $\beta'(s) = \gamma'(\phi(s)). \phi'(s) = \gamma'(t). \phi'(s)$.

$$\beta''(s) = \gamma''(t). \phi'(s). \phi'(s) + \phi''(s). \gamma'(t) = \gamma''(t). [\phi'(s)]^2 + \phi''(s). \gamma'(t) \Rightarrow$$

$$< \beta''(s), \beta''(s) >_L = [\phi'(s)]^4 . ||\gamma''(t)||_L^2 \Rightarrow ||\beta''(t)||_L^2 = [\phi'(s)]^4 .$$

$$||\gamma''(t)||_L^2 \Rightarrow ||\beta''(t)||_L = [\phi'(s)]^2 . ||\gamma''(t)||_L \Rightarrow [\phi'(s)]^2 = \frac{||\beta''(t)||_L}{||\gamma''(t)||_L} \Rightarrow$$

$$[\phi'(s)] = \frac{1}{\sqrt{||\gamma''(t)||_L}} = \frac{1}{\sqrt{||\gamma''(\phi(s))||_L}}$$

Φ is the solution of the above differential equation.

3.2 Frenet Equations For Lorentzian Curves

We will assign a basis of E_1^3 at each point of a regular curve (s). So we can study the geometry of the curve.

Let α be unit velocity curve or parametrized by pseudo-arclength parameter. $\boldsymbol{t}(\boldsymbol{s})$ is tangent vector of α. In Minkowski space some problems appear.

a) The curve is null ⇒ $\boldsymbol{t}(\boldsymbol{s})$ is a null vector. We will use null frame because $\boldsymbol{t}$ is null we don't have an orthonormal basis.

b) If the curve is space-like ⇒ $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is an orthonormal basis of E_1^3. The binomial vector $\boldsymbol{b}$ is always defined by $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$. $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is not necessarily positive oriented. Such as if $\boldsymbol{t}, \boldsymbol{n}$ are space-like vectors ⇒ $\boldsymbol{b}$ is time-like. So $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is negatively oriented**.**When $\boldsymbol{t}, \boldsymbol{n}$ have not same causal characters , $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is positively oriented. $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is future directed if $\boldsymbol{b}$ is future directed.

$e_3 = (0, 0, 1)$. If $w \in$ C(e_3) then w is future directed $< w, e_3 >_L < 0$.

c) We prefer that $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is an orthonormal basis, it is future directed. α is a time-like curve ⇒ $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is not future-directed. (since $< \boldsymbol{b}, \boldsymbol{b} >_L > 0$).

If $b \in E_1^3$ and $t \neq 0$, the straight-line owing to point p has parametric equation $\alpha (t) = b + at$ where t is direction vector. $\alpha''(t) = 0$. The curvature is 0.

Conversely, if α is a regular curve that satisfies $\alpha''(t) = 0$ for any t, an integration gives $\alpha (t) = b + at$, for some values of $b, t \in E_1^3, t \neq 0$. α parametric equation of the straight-line owing to the point b with direction vector t.

When we deal with a straight-line, there are other parametrizations. For instance, $\alpha(t) = (t^3 + t, 0, 0)$ is a parametric equation of the staright-line span $\{e_1\}$ where $\alpha''(t) \neq 0$.

Consider $\alpha: I \to E_1^3$ a regular unit velocity curve or parametrized by pseudo-arclength. We call $\alpha'(s) = \boldsymbol{t}(s)$ as the tangent vector s.

Because $< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = \pm 1 \; or \; 0$. Differentiating, we get $< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_L = 0$ which means and $\boldsymbol{t}'(\boldsymbol{s})$ is perpendicular to $\boldsymbol{t}(\boldsymbol{s})$**.** We will take the curves such that $\boldsymbol{t}'(\boldsymbol{s}) \neq 0 \; \forall \; s$ and for each s $\boldsymbol{t}'(\boldsymbol{s}) \neq c \; \boldsymbol{t}(\boldsymbol{s}) \quad c \in IR$.

We have 3 possibilities on the causal character of $\boldsymbol{t}(\boldsymbol{s})$**.**

3.2.1 Definition

Let $\alpha: I \to E_1^3$ be a curve and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ be an orthonormal basis of E_1^3. The function defined by

$$\boldsymbol{\kappa}: I \to E_1^3$$

$$s \to \boldsymbol{\kappa}(s) = < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_L \text{ is called curvature function of } \alpha.$$

Real numbers $\kappa(s)$ at $\alpha(s)$ is called curvature of α at $\alpha(s)$.

$$\boldsymbol{\kappa}(s) = < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_L \Rightarrow \boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s}).$$

3.2.2 Definition

Let $\alpha: I \to E_1^3$ be a curve and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is positively oriented orthonormal basis E_1^3. The function defined by

$$\tau: I \to E_1^3$$

$$s \to \tau(s) = < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L \text{ is called torsion of } \alpha.$$

3.2.3 Definition

Let $\alpha: I \rightarrow E_1^3$ be a curve and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is not orthonormal basis E_1^3, null frame. The function defined by

$$\tau: I \rightarrow E_1^3$$

$$s \rightarrow \tau(s) = -< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L \text{ is called pseudo-torsion of } \alpha.$$

3.2.4 Definition

Let $\{ e_1, e_2, e_3 \}$ be a null frame if e_1 is a unit space-like vector and e_2, e_3 are lightlike vectors space sp$\{e_1\}^{\perp}$ such that $< e_2, e_3 >_L = -1$. e_2 *and* e_3 are in this same time cone.

Curvature, torsion and Frenet equations calculation for 3 types.

3.2.5 The Time-like Case

Let α be a time-like curve that is $\boldsymbol{t}(\boldsymbol{s})$ is a time-like vector then $\boldsymbol{t}'(\boldsymbol{s})$ is space-like vector (since span $\{ \boldsymbol{t}(\boldsymbol{s}) \}^{\perp}$ is space-like subspace).

$$< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = -1$$

$$< \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L + < \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_L = 0$$

$$2 < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0$$

$$< \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0$$

$$E_1^3 = \underbrace{\underbrace{\text{sp}\{\boldsymbol{t}(\boldsymbol{s})\}}_{t}}_{time-like} \oplus \underbrace{\underbrace{\text{sp}\{ \boldsymbol{t}(\boldsymbol{s}) \}^{\perp}}_{t'}}_{space-like\ subspace}. \quad \text{By 2.1.3 Proposition}$$

$$\boldsymbol{t}(\boldsymbol{s}) = \alpha'(s)$$

The normal vector $\boldsymbol{t}(\boldsymbol{s})$ is defined by

$$\boldsymbol{n}(\boldsymbol{s}) = \frac{\boldsymbol{t}'(\boldsymbol{s})}{\boldsymbol{\kappa}(s)}$$

$$\boldsymbol{n}(\boldsymbol{s}) = \frac{1}{\boldsymbol{\kappa}(s)}.\ \boldsymbol{t}'(\boldsymbol{s}).$$

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s}) \Rightarrow ||\boldsymbol{t}'(\boldsymbol{s})||_L = ||\boldsymbol{\kappa}(s)||_L.||\boldsymbol{n}(\boldsymbol{s})||_L \Rightarrow$$

$$||\boldsymbol{\kappa}(s)||_L = ||\boldsymbol{t}'(\boldsymbol{s})||_L \text{ (since } ||\boldsymbol{n}(\boldsymbol{s})||_L = 1 \text{ is a space-like unit vector).}$$

The curvature function $\boldsymbol{\kappa}(s)$ is

$$\boldsymbol{\kappa}(s) = < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_L.$$

We take the binomial vector $\boldsymbol{b}(\boldsymbol{s})$ as,

$$\boldsymbol{b}(\boldsymbol{s}) = \boldsymbol{t}(\boldsymbol{s}) \times \boldsymbol{n}(\boldsymbol{s}).$$

$\boldsymbol{b}(\boldsymbol{s})$ is unit and space-like. For each $s, \{ \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$ is an orthonormal basis for E_1^3. It is called the, frenet trihedron of α at s. The basis $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is positively oriented because

$$\det(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}) = < \boldsymbol{t} \times \boldsymbol{n}, \boldsymbol{b} >_{\boldsymbol{L}} = < \boldsymbol{b}, \quad \boldsymbol{b} >_{\boldsymbol{L}} = 1.$$

$\boldsymbol{b}$ is a space-like vector. We define the torsion τ of α at s.

$$\boldsymbol{\tau}(s) = < \boldsymbol{n}'(s), \boldsymbol{b}(s) >_L$$

$$\boldsymbol{n}'(\boldsymbol{s}) \in sp\, \{ \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -a$$

On the other hand $\boldsymbol{\kappa}(s) = < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_L$

$$< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -\underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-\boldsymbol{\kappa}(s)} \Rightarrow \quad < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -\boldsymbol{\kappa}(s)$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -\boldsymbol{\kappa}(s) = -a \Rightarrow \boldsymbol{\kappa}(s) = a$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 1 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$2 < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \Rightarrow < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b = 0$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = c$$

We know the definiton of $\boldsymbol{\tau}(s) = < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = c = \boldsymbol{\tau}(s)$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + 0.\boldsymbol{n}\,(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}\,(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) \in sp\,\{\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{b}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -a$$

On the other hand $< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$ (we differentiate both sides)

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{t}'(s), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \kappa(s).\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + \boldsymbol{\kappa}(s) \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 = -a \Rightarrow a = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b$$

$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$ (we differentiate both sides)

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}}$$

We know the definition of $\boldsymbol{\tau}(s) = < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -\tau(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b = -\boldsymbol{\tau}(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{B}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = c$$

$$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 1 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$2 < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 = c$$

$$\boldsymbol{b}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = 0.\boldsymbol{t}(\boldsymbol{s}) - \boldsymbol{\tau}(s)\, \boldsymbol{n}(\boldsymbol{s}) + 0.\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}(\boldsymbol{s}) = -\boldsymbol{\tau}(\boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})$$

Conversely ,

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = -\boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\kappa} & 0 \\ \boldsymbol{\kappa} & 0 & \boldsymbol{\tau} \\ 0 & -\boldsymbol{\tau} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

3.2.6 The Space-like Case

Let α be a space-like curve in E_1^3. That is $\boldsymbol{t}(\boldsymbol{s}) = \alpha'(s)$ is a space-like vector. sp{ $\boldsymbol{t}(\boldsymbol{s})$ }$^{\perp}$ is time-like subspace E_1^2.

$$< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 1$$

$$< \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L + < \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_L = 0$$

$$2 < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0$$

$$< \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0$$

$$E_1^3 = \underbrace{\underbrace{\text{sp}\{\boldsymbol{t}(\boldsymbol{s})\}}_{T}}_{space-like} \oplus \underbrace{\underbrace{\text{sp}\{\ \boldsymbol{t}(\boldsymbol{s})\ \}^{\perp}}_{T'}}_{time-like\ subspace}. \quad \text{By 2.1.3 Proposition}$$

(1) $\boldsymbol{t}'(\boldsymbol{s})$ may be space-like.

(2) $\boldsymbol{t}'(\boldsymbol{s})$ may be time-like.

(3) $\boldsymbol{t}'(\boldsymbol{s})$ may be light-like**.**

<u>Case 1</u>: $\boldsymbol{t}'(\boldsymbol{s})$ is a space-like vector.

$$\boldsymbol{\kappa}(s) = <\boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L$$

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}(\boldsymbol{s}) = \frac{1}{\boldsymbol{\kappa}(s)}.\ \boldsymbol{t}'(\boldsymbol{s}) \Rightarrow \boldsymbol{n}(\boldsymbol{s}) \text{ is a space-like vector.}$$

(since $\boldsymbol{t}'(\boldsymbol{s})$ and $\boldsymbol{n}(\boldsymbol{s})$ have the same causal character)

$$\boldsymbol{b}(\boldsymbol{s}) = \boldsymbol{t}(\boldsymbol{s}) \times \boldsymbol{n}(\boldsymbol{s}).$$

$\boldsymbol{t}(\boldsymbol{s})$ and $\boldsymbol{n}(\boldsymbol{s})$ have the same causal character, $\boldsymbol{b}(\boldsymbol{s})$ is time-like.

$$\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\} \text{ is negatively oriented.}$$

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s}) \Rightarrow ||\boldsymbol{t}'(\boldsymbol{s})||_L = ||\boldsymbol{\kappa}(s)||_L.||\boldsymbol{n}(\boldsymbol{s})||_L \Rightarrow$$

$$||\boldsymbol{\kappa}(s)||_L = ||\boldsymbol{t}'(\boldsymbol{s})||_L \text{ (since } ||\boldsymbol{n}(\boldsymbol{s})||_L = 1 \text{ is a space-like unit vector)}$$

$$\boldsymbol{n}'(\boldsymbol{s}) \in sp\{\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s})$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L = <a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L = a\underbrace{<\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L}_{1} + b\underbrace{<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L}_{0} + c\underbrace{<\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L}_{0}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L = a$$

On the other hand $\boldsymbol{\kappa}(s) = <\boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L$

$$<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L = 0 \text{ (we differentiate both sides)}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L + <\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s})>_L = 0$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L = -\underbrace{<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s})>_L}_{-\kappa(s)} \Rightarrow \quad <\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L = -\boldsymbol{\kappa}(s)$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_L = -\boldsymbol{\kappa}(s) = a \Rightarrow -\boldsymbol{\kappa}(s) = a$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L = <a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L$$

$$<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L = 1 \text{ (we differentiate both sides)}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L + <\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s})>_L = 0$$

$$2 <\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L = 0 \Rightarrow <\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L = 0$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L = a\underbrace{<\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L}_{0} + b\underbrace{<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L}_{1} + c\underbrace{<\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L}_{0}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L = b$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L = b = 0$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_L = <a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_L$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_L = a\underbrace{<\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_L}_{0} + b\underbrace{<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_L}_{0} + c\underbrace{<\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_L}_{-1}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = -c$$

$\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is negatively oriented. So

$$\boldsymbol{\tau}(s) = -< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{L}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = -c = -\boldsymbol{\tau}(s) \Rightarrow c = \boldsymbol{\tau}(s)$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{N}'(\boldsymbol{s}) = -\boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + 0.\boldsymbol{n}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{N}'(\boldsymbol{s}) = -\boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) \in sp\,\{\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{b}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a$$

On the other hand $< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$ (we differentiate both sides)

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + \boldsymbol{\kappa}(s) \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 = a \Rightarrow a = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b$$

$$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is negatively oriented. So

$$\boldsymbol{\tau}(s) = -< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{L}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = \boldsymbol{\tau}(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b = \boldsymbol{\tau}(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L}_{-1}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -c$$

$$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -1 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L + < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = 0$$

$$2 < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = 0 = -c \Rightarrow 0 = c$$

$$\boldsymbol{b}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = 0.\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s)\, \boldsymbol{n}(\boldsymbol{s}) + 0.\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = \boldsymbol{\tau}(\boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})$$

Conversely,

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = -\boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\kappa} & 0 \\ -\boldsymbol{\kappa} & 0 & \boldsymbol{\tau} \\ 0 & \boldsymbol{\tau} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

<u>Case 2</u>: $\boldsymbol{t}'(\boldsymbol{s})$ is a time-like vector.

$$\boldsymbol{\kappa}(s) = < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_L$$

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}(\boldsymbol{s}) = \frac{1}{\boldsymbol{\kappa}(s)}.\ \boldsymbol{t}'(\boldsymbol{s}) \Rightarrow \boldsymbol{n}(\boldsymbol{s}) \text{ is a time-like vector.}$$

(since $\boldsymbol{t}'(\boldsymbol{s})$ and $\boldsymbol{n}(\boldsymbol{s})$ have the same causal character.)

$$\boldsymbol{b}(\boldsymbol{s}) = \boldsymbol{t}(\boldsymbol{s}) \times \boldsymbol{n}(\boldsymbol{s}).$$

$\boldsymbol{t}(\boldsymbol{s})$ and $\boldsymbol{n}(\boldsymbol{s})$ have different causal characters, $\boldsymbol{b}(\boldsymbol{s})$ is a space-like.

$\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is positively oriented.

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s}) \Rightarrow ||\boldsymbol{t}'(\boldsymbol{s})||_L = ||\boldsymbol{\kappa}(s)||_L.||\boldsymbol{n}(\boldsymbol{s})||_L \Rightarrow$$

$$||\boldsymbol{\kappa}(s)||_L = ||\boldsymbol{t}'(\boldsymbol{s})||_L \text{ (since } ||\boldsymbol{n}(\boldsymbol{s})||_L = 1 \text{ is a time-like unit vector)}$$

$$\boldsymbol{n}'(\boldsymbol{s}) \in sp\,\{\, \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s})$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} = < a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} = a\underbrace{<\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}}_{1} + b\underbrace{<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0} + c\underbrace{<\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} = a$$

On the other hand $\kappa(s) = -<\boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_L$

$$<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} = 0 \text{ (we differentiate both sides)}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} + <\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s})>_{\boldsymbol{L}} = 0$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} = -\underbrace{<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s})>_{\boldsymbol{L}}}_{\boldsymbol{\kappa}(s)} \Rightarrow \quad <\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} = \boldsymbol{\kappa}(s)$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} = \boldsymbol{\kappa}(s) = a \;\Rightarrow \boldsymbol{\kappa}(s) = a$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}} = < a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}$$

$$<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}} = -1 \text{ (we differentiate both sides)}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}} + <\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s})>_{\boldsymbol{L}} = 0$$

$$2<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}} = 0 \;\Rightarrow <\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}} = 0$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}} = a\underbrace{<\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0} + b\underbrace{<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}}_{-1} + c\underbrace{<\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}} = -b$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}} = -b = 0 \Rightarrow b = 0$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}} = < a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}} = a\underbrace{<\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0} + b\underbrace{<\boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0} + c\underbrace{<\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}}_{1}$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}} = c$$

$\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is positively oriented. So

$$\boldsymbol{\tau}(s) = <\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_L$$

$$<\boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}} = c = \boldsymbol{\tau}(s) \Rightarrow c = \boldsymbol{\tau}(s)$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + 0.\boldsymbol{n}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) \in sp\,\{\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{b}'(\boldsymbol{s}) = a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s})$$

$$<\boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}} = < a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a$$

On the other hand $< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$ (we differentiate both sides)

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{\kappa}(s). \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + \boldsymbol{\kappa}(s) \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 = a \Rightarrow a = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -b$$

$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$ (we differentiate both sides)

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = - < \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is positively oriented. So

$$\boldsymbol{\tau}(s) = < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{L}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -\tau(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -b = -\boldsymbol{\tau}(s) \Rightarrow = b = \boldsymbol{\tau}(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \quad \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = c$$

$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 1$ (we differentiate both sides)

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$2 < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 = c \Rightarrow 0 = c$$

$$\boldsymbol{b}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = 0. \boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s)\, \boldsymbol{n}(\boldsymbol{s}) + 0. \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = \boldsymbol{\tau}(\boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})$$

Conversely,

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\kappa} & 0 \\ \boldsymbol{\kappa} & 0 & \boldsymbol{\tau} \\ 0 & \boldsymbol{\tau} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}$$

Case 3: $\boldsymbol{t}'(\boldsymbol{s})$ is a light-like vector.

$$< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 1$$

$$< \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L + < \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_L = 0$$

$$2 < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0$$

$$< \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0 \; (\boldsymbol{t}'(\boldsymbol{s}) \perp \boldsymbol{t}(\boldsymbol{s}))$$

We take the normal vector $\boldsymbol{n}(\boldsymbol{s}) = \boldsymbol{t}'(\boldsymbol{s})$

$\boldsymbol{t}(\boldsymbol{s})$ and $\boldsymbol{n}(\boldsymbol{s})$ are linearly independent vectors.

Let $\boldsymbol{b}(\boldsymbol{s})$ be the unique light-like vector such that $< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -1$. (By 3.2.3 Definition) and $\boldsymbol{b}(\boldsymbol{s})$ is orthogonal to $\boldsymbol{t}(\boldsymbol{s})$**.**

$\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is not an orthogonal basis of E_1^3. It is null frame.

$$\boldsymbol{t}'(\boldsymbol{s}) = 1.\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) \in sp\,\{ \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s})$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\,\boldsymbol{t}(\boldsymbol{s}) + b\,\boldsymbol{n}(\boldsymbol{s}) + c\,\boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a$$

$$< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \Rightarrow \quad < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

Since $\boldsymbol{n}(\boldsymbol{s})$ is a light-like vector.

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}=0=a\ \Rightarrow 0=a$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}=<a\,\boldsymbol{t}(\boldsymbol{s})+b\,\boldsymbol{n}(\boldsymbol{s})+c\,\boldsymbol{b}(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})\ >_{\boldsymbol{L}}$$

$$<\boldsymbol{n}(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}=0 \text{ (we take both sides differentiating)}$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}+<\boldsymbol{n}(\boldsymbol{s}),\boldsymbol{n}'(\boldsymbol{s})>_{\boldsymbol{L}}=0$$

$$2<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}=0\ \Rightarrow <\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}=0$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}=a\underbrace{<\boldsymbol{t}(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0}+b\underbrace{<\boldsymbol{n}(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0}+c\underbrace{<\boldsymbol{b}(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}}_{-1}$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}=-c$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}=-c=0\Rightarrow c=0$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}=<a\,\boldsymbol{t}(\boldsymbol{s})+b\,\boldsymbol{n}(\boldsymbol{s})+c\,\boldsymbol{b}(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})\ >_{\boldsymbol{L}}$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}=a\underbrace{<\boldsymbol{t}(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0}+b\underbrace{<\boldsymbol{n}(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}}_{-1}+c\underbrace{<\boldsymbol{b}(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0}$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}=-b.$$

Define the pseudo-torsion

$$\boldsymbol{\tau}(s)=\ -<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{L}$$

$$<\boldsymbol{n}'(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}=-b=-\boldsymbol{\tau}(s)\Rightarrow b=\boldsymbol{\tau}(s)$$

$$\boldsymbol{n}'(\boldsymbol{s})=a\,\boldsymbol{t}(\boldsymbol{s})+b\,\boldsymbol{n}(\boldsymbol{s})+c\,\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s})=\ 0.\boldsymbol{t}(\boldsymbol{s})+\ \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})+\ 0.\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s})=\ \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s})\in sp\,\{\,\boldsymbol{t}(\boldsymbol{s}),\boldsymbol{n}(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{b}'(\boldsymbol{s})=a\,\boldsymbol{t}(\boldsymbol{s})+b\,\boldsymbol{n}(\boldsymbol{s})+c\,\boldsymbol{b}(\boldsymbol{s})$$

$$<\boldsymbol{b}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}=<a\,\boldsymbol{t}(\boldsymbol{s})+b\,\boldsymbol{n}(\boldsymbol{s})+c\,\boldsymbol{b}(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})\ >_{\boldsymbol{L}}$$

$$<\boldsymbol{b}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}=a\underbrace{<\boldsymbol{t}(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}}_{1}+b\underbrace{<\boldsymbol{n}(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0}+c\underbrace{<\ \boldsymbol{b}(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}}_{0}$$

$$<\boldsymbol{b}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}=a$$

On the other hand $<\boldsymbol{b}(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}=0$ (we differentiate both sides)

$$<\boldsymbol{b}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}+<\boldsymbol{t}'(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}=0$$

$$<\boldsymbol{b}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}+<\ \boldsymbol{n}(\boldsymbol{s}),\boldsymbol{b}(\boldsymbol{s})>_{\boldsymbol{L}}=0$$

$$<\boldsymbol{b}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}-\quad 1=0$$

$$<\boldsymbol{b}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}=1$$

$$<\boldsymbol{b}'(\boldsymbol{s}),\boldsymbol{t}(\boldsymbol{s})>_{\boldsymbol{L}}=1=a\Rightarrow a=1$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -c$$

$$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -1 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = - < \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}}$$

Define the pseudo-torsion

$$\boldsymbol{\tau}(s) = - < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{L}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = \boldsymbol{\tau}(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -c = \boldsymbol{\tau}(s) \Rightarrow c = -\boldsymbol{\tau}(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = -b$$

$$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$2 < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 = -b \Rightarrow 0 = b$$

$$\boldsymbol{b}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = 1.\boldsymbol{t}(\boldsymbol{s}) + 0.\boldsymbol{n}(\boldsymbol{s}) - \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = 1.\boldsymbol{t}(\boldsymbol{s}) - \boldsymbol{\tau}(\boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})$$

Conversely ,

$$\boldsymbol{t}'(\boldsymbol{s}) = 1.\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = 1.\boldsymbol{t}(\boldsymbol{s}) - \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \boldsymbol{\tau} & 0 \\ 1 & 0 & -\boldsymbol{\tau} \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

We don’t know if it is positively oriented or not.

3.2.7 The Light-like Case

Let α be a light-like curve parametrized by pseudo-arclength. $\boldsymbol{t}(\boldsymbol{s}) = \alpha'(s)$ is a light-like vector.

$$< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0$$

$$< \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L + < \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_L = 0$$

$$2 < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0$$

$$< \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_L = 0 \ (\boldsymbol{t}'(\boldsymbol{s}) \perp \boldsymbol{t}(\boldsymbol{s}))$$

We take the normal vector $\boldsymbol{n}(\boldsymbol{s}) = \boldsymbol{t}'(\boldsymbol{s})$ is space-like vector. $\boldsymbol{b}(\boldsymbol{s})$ is the unit light-like vector orthogonal to $\boldsymbol{n}(\boldsymbol{s})$**.**

$$< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -1 \, , < \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_L = 0 \, , < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = 0$$

Thus $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is not an orthogonal basis of E_1^3. It is null frame of E_1^3.

$$\boldsymbol{t}'(\boldsymbol{s}) = 1.\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) \in sp \{ \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a \, \boldsymbol{t}(\boldsymbol{s}) + b \, \boldsymbol{n}(\boldsymbol{s}) + c \, \boldsymbol{b}(\boldsymbol{s})$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a \, \boldsymbol{t}(\boldsymbol{s}) + b \, \boldsymbol{n}(\boldsymbol{s}) + c \, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -c$$

$$< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \Rightarrow \quad < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + 1 = 0$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -1$$

Since $\boldsymbol{n}(\boldsymbol{s})$ is a space-like vector.

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -1 = c \ \Rightarrow -1 = c$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a \, \boldsymbol{t}(\boldsymbol{s}) + b \, \boldsymbol{n}(\boldsymbol{s}) + c \, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 1 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$2 < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \ \Rightarrow < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = \boldsymbol{b}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b = 0 \Rightarrow b = 0$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = -a$$

Define the pseudo-torsion

$$\boldsymbol{\tau}(s) = -< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{L}$$

$$< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = -a = -\boldsymbol{\tau}(s) \Rightarrow a = \boldsymbol{\tau}(s)$$

$$\boldsymbol{n}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \tau(s).\boldsymbol{t}(\boldsymbol{s}) + 0.\boldsymbol{n}(\boldsymbol{s}) + 1.\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\tau}(s).\boldsymbol{t}(\boldsymbol{s}) + 1.\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) \in sp\,\{ \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s})\}$$

$$\boldsymbol{b}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -c$$

On the other hand $< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = -1$ (we differentiate both sides)

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} + \quad 0 \quad = \quad 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 = -c \Rightarrow c = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{1} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b$$

$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$ (we differentiate both sides)

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = -< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{n}'(\boldsymbol{s}) >_{\boldsymbol{L}}$$

Define the pseudo-torsion

$$\boldsymbol{\tau}(s) = -< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = \boldsymbol{\tau}(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_{\boldsymbol{L}} = b = \boldsymbol{\tau}(s) \Rightarrow b = \boldsymbol{\tau}(s)$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = < a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = a \underbrace{< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{-1} + b \underbrace{< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0} + c \underbrace{< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}}}_{0}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = -a$$

$$< \boldsymbol{b}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 \text{ (we differentiate both sides)}$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} + < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$2 < \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0$$

$$< \boldsymbol{b}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_{\boldsymbol{L}} = 0 = -a \Rightarrow 0 = a$$

$$\boldsymbol{B}'(\boldsymbol{s}) = a\, \boldsymbol{t}(\boldsymbol{s}) + b\, \boldsymbol{n}(\boldsymbol{s}) + c\, \boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = 0.\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s}) + 0.\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = \boldsymbol{\tau}(\boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})$$

Conversely ,

$$\boldsymbol{t}'(\boldsymbol{s}) = 1.\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{n}'(\boldsymbol{s}) = \boldsymbol{\tau}(s).\boldsymbol{t}(\boldsymbol{s}) + 1.\boldsymbol{b}(\boldsymbol{s})$$

$$\boldsymbol{b}'(\boldsymbol{s}) = \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \boldsymbol{\tau} & 0 & 1 \\ 0 & \boldsymbol{\tau} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

(Bonnor 1969, Graves 1979, Inoguchi 2008)

Time-like curves and space-like curves with space-like or time-like normal vectors are called Frenet curves. The Frenet equations are written as follows.

If $< \boldsymbol{t}, \boldsymbol{t} >_{\boldsymbol{L}} = \boldsymbol{\varepsilon}$ and $< \boldsymbol{n}, \boldsymbol{n} >_{\boldsymbol{L}} = \boldsymbol{\delta}$ then

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\kappa} & 0 \\ -\delta\boldsymbol{\kappa} & 0 & \tau \\ 0 & \varepsilon\boldsymbol{\tau} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

The curvature κ is given by the function in such a way that $\boldsymbol{t}' = c.\boldsymbol{n}$ where $c \in IR$. The torsion $\boldsymbol{\tau}$ is defined as the 3th coordinate of $\boldsymbol{n}'$.

For space-like curves with light-like normal vector or light-like curves, the Frenet equations are as follows: let $< \boldsymbol{t}, \boldsymbol{t} >_L = \boldsymbol{\varepsilon}, < \boldsymbol{n}, \boldsymbol{n} >_L = \boldsymbol{\delta}$ where $\varepsilon, \delta \in \{0, 1\}$ and $\varepsilon \neq \delta$. Then

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \delta\boldsymbol{\tau} & \varepsilon\boldsymbol{\tau} & \delta \\ \varepsilon & \delta\boldsymbol{\tau} & -\varepsilon\boldsymbol{\tau} \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

The torsion is

$$\boldsymbol{\tau}(s) = -\varepsilon\delta < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L .$$

In Euclidean space, the curve is planar $\Longleftrightarrow$ its torsion is zero. In Minkowski space, we will obtain the similar results for Frenet curves.

3.2.8 Example

$$\alpha(s) = \left(\cos(s) + s\sin(s), \sin(s) - s\cos(s), \frac{1}{2}\left(s\sqrt{s^2-1} - In\left(s + \sqrt{s^2-1}\right)\right)\right)$$

for $s \in (1, \infty)$. Find the causal character of α, κ and $\boldsymbol{\tau}$. (Lopez, 2014)

We will find $\alpha'(s)$.

$$\alpha'(s) = \boldsymbol{t}(\boldsymbol{s}) = (s\cos(s), s\sin(s), \sqrt{s^2-1})$$

$$< \alpha'(s), \alpha'(s) >_L = s^2(\cos(s))^2 + s^2(\sin(s))^2 - (s^2-1)$$

$$< \alpha'(s), \alpha'(s) >_L = s^2 - s^2 + 1 = 1.$$

α is a space-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = \left(cos\,(s) - s\sin(s), \sin(s) + s\cos(s), \frac{s}{\sqrt{s^2-1}}\right).$$

$$< \alpha''(s), \alpha''(s) >_L = cos^2(s) + sin^2(s) + s^2cos^2(s) + s^2sin^2(s) - \left(\frac{s^2}{s^2-1}\right)$$

$$< \alpha''(s), \alpha''(s) >_L = 1 + s^2 - \left(\frac{s^2}{s^2-1}\right) = \frac{s^4 - s^2 - 1}{s^2-1}.$$

The causal character of $\boldsymbol{t}'(\boldsymbol{s})$ is given by the sign of $s^4 - s^2 - 1$ since

$$< \alpha''(s), \alpha''(s) >_L = \frac{s^4 - s^2 - 1}{s^2-1}.$$

If $s > \sqrt{1+\sqrt{5}}/2$ then $\boldsymbol{t}'(\boldsymbol{s})$ is space-like.

If $1 < s < \sqrt{1+\sqrt{5}}/2$ then $\boldsymbol{t}'(\boldsymbol{s})$ is time-like.

In both cases, the curvatures and the torsions are

$$\kappa(s) = \sqrt{\frac{|s^4 - s^2 - 1|}{s^2 - 1}} \quad , \qquad \boldsymbol{\tau}(s) = \frac{s^6 - 2s^4 - 2s^2 + 2}{(s^4 - s^2 - 1)\sqrt{s^2 - 1}}.$$

3.2.9 Example

(1) Let $\alpha(s) = r(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right), 0\,)$. Then

$$\alpha'(s) = \boldsymbol{t}(\boldsymbol{s}) = (-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right), 0\,)$$

$$< \alpha'(s), \alpha'(s) >_L = sin^2\left(\frac{s}{r}\right) + cos^2\left(\frac{s}{r}\right) - 0 = 1.$$

α is a space-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = \frac{1}{r}\,(-\cos\left(\frac{s}{r}\right), -\sin\left(\frac{s}{r}\right), 0\,)$$

$$< \alpha''(s), \alpha''(s) >_L = \frac{1}{r^2} > 0, \forall\, r \in IR.$$

$\boldsymbol{t}'$ is a space-like. We know that

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\boldsymbol{\kappa}(s) = \frac{1}{r} \quad , \qquad \boldsymbol{n}(\boldsymbol{s}) = (-\cos\left(\frac{s}{r}\right), -\sin\left(\frac{s}{r}\right), 0\,)$$

$\boldsymbol{n}$ is a space-like vector. $\boldsymbol{t}, \boldsymbol{n}$ have same causal character, $\boldsymbol{b}$ is a time-like vector. $\{\, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\, \}$ is negatively oriented.

$$\boldsymbol{b} = \boldsymbol{txn}.\ \boldsymbol{b}(\boldsymbol{s}) = (0, 0, -1).\boldsymbol{b}'(\boldsymbol{s}) = 0, \tau(s) = 0.$$

This basis is either positively oriented nor future directed.

(2) Let $\alpha(s) = r(0, \sinh\left(\frac{s}{r}\right), \cosh\left(\frac{s}{r}\right)\,)$. Then

$$\alpha'(s) = \boldsymbol{t}(\boldsymbol{s}) = \left(\,0, \cosh\left(\frac{s}{r}\right), \sinh\left(\frac{s}{r}\right)\,\right)$$

$$< \alpha'(s), \alpha'(s) >_L = 0 + (\cosh\left(\frac{s}{r}\right))^2 - (\sinh\left(\frac{s}{r}\right))^2 = 1.$$

α is a space-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = \frac{1}{r}(0, \sinh\left(\frac{s}{r}\right), \cosh\left(\frac{s}{r}\right)) \ .$$

$$< \alpha''(s), \alpha''(s) >_L = -\frac{1}{r^2} < 0, \forall\, r \in IR.$$

$\boldsymbol{t}'$ is a time-like. We know that

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\kappa(s) = \frac{1}{r} \ , \qquad \boldsymbol{n}(\boldsymbol{s}) = (0, \sinh\left(\frac{s}{r}\right), \cosh\left(\frac{s}{r}\right)) \ .$$

$\boldsymbol{n}$ is a timel-ike vector. $\boldsymbol{t}, \boldsymbol{n}$ have different causal characters, $\boldsymbol{b}$ is a space-like vector. $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b} \}$ is positively oriented.

$$\boldsymbol{b} = \boldsymbol{txn}.\ \boldsymbol{b}(\boldsymbol{s}) = (1, 0, 0).\boldsymbol{b}'(\boldsymbol{s}) = 0, \tau(s) = 0.$$

(3) Let $\alpha(s) = r(0,\ \cosh\left(\frac{s}{r}\right), \sinh\left(\frac{s}{r}\right))$. Then

$$\alpha'(s) = \boldsymbol{t}(\boldsymbol{s}) = (0, \sinh\left(\frac{s}{r}\right), \cosh\left(\frac{s}{r}\right)).$$

$$< \alpha'(s), \alpha'(s) >_L = 0 + (\sinh\left(\frac{s}{r}\right))^2 - \left(\cosh(\frac{s}{r})\right)^2 = -1.$$

α is a time-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = \frac{1}{r}(0,\ \cosh\left(\frac{s}{r}\right), \sinh\left(\frac{s}{r}\right)).$$

$$< \alpha''(s), \alpha''(s) >_L = \frac{1}{r^2} > 0, \forall\, r.$$

$\boldsymbol{t}'$ is a space-like. We know that

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\kappa(s) = \frac{1}{r} \ , \qquad \boldsymbol{n}(\boldsymbol{s}) = (0,\ \cosh\left(\frac{s}{r}\right), \sinh\left(\frac{s}{r}\right)).$$

$\boldsymbol{n}$ is a space-like vector. $\boldsymbol{t}, \boldsymbol{n}$ have different causal characters, $\boldsymbol{b}$ is a space-like vector. $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b} \}$ is positively oriented.

$$\boldsymbol{b} = \boldsymbol{txn}.\ \boldsymbol{b}(\boldsymbol{s}) = (-1, 0, 0).\boldsymbol{b}'(\boldsymbol{s}) = 0, \tau(s) = 0.$$

(4) Let

$$\alpha(s) = (\frac{hs}{\sqrt{r^2 - h^2}}, r\ \cosh(\frac{s}{\sqrt{r^2 - h^2}}), r\ \sinh(\frac{s}{\sqrt{r^2 - h^2}}))$$

where $r^2 - h^2 > 0$. Then

$$\alpha'(s) = \boldsymbol{t}(\boldsymbol{s}) = \frac{1}{\sqrt{r^2 - h^2}}\left(h, r\ \sinh(\frac{s}{\sqrt{r^2 - h^2}}\right), r\ \cosh(\frac{s}{\sqrt{r^2 - h^2}})).$$

$$< \alpha'(s), \alpha'(s) >_L = \frac{h^2 - r^2}{r^2 - h^2} = -1.$$

α is a time-like curve and future directed. We have

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = \frac{r}{\sqrt{r^2 - h^2}}\ (0, \cosh(\frac{s}{\sqrt{r^2 - h^2}}), \sinh(\frac{s}{\sqrt{r^2 - h^2}})).$$

$$< \alpha''(s), \alpha''(s) >_L = \frac{r^2}{r^2 - h^2} > 0, \forall\, r \in IR \text{ since } r^2 - h^2 > 0.$$

$\boldsymbol{t}'$ is a space-like. We know that

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\kappa(s) = \frac{r}{\sqrt{r^2 - h^2}}\ , \qquad \boldsymbol{n}(\boldsymbol{s}) = (0, \cosh(\frac{s}{\sqrt{r^2 - h^2}}), \sinh(\frac{s}{\sqrt{r^2 - h^2}})).$$

$\boldsymbol{n}$ is a space-like vector. $\boldsymbol{t}, \boldsymbol{n}$have different causal characters, $\boldsymbol{b}$ is a space-like vector. $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is positively oriented.

$$\boldsymbol{b} = \boldsymbol{txn}.$$

$$\boldsymbol{b}(\boldsymbol{s}) = \frac{1}{\sqrt{r^2 - h^2}}\ (-r, -h\ \sinh(\frac{s}{\sqrt{r^2 - h^2}}), -h \cosh(\frac{s}{\sqrt{r^2 - h^2}})).$$

We know that $\boldsymbol{\tau}(s) = < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L$

$$\boldsymbol{n}'(\boldsymbol{s}) = \frac{1}{\sqrt{r^2 - h^2}}\ (0, \sinh(\frac{s}{\sqrt{r^2 - h^2}}), \cosh(\frac{s}{\sqrt{r^2 - h^2}})).$$

$$\tau(s) = \frac{h}{r^2 - h^2}.$$

(5) Let

$$\alpha(s) = r(\frac{s}{r}, (\frac{s}{r})^2, (\frac{s}{r})^2)$$

$$\alpha'(s) = \boldsymbol{T}(\boldsymbol{s}) = \left(1, \frac{2s}{r}, \frac{2s}{r}\right)$$

$$< \alpha'(s), \alpha'(s) >_L = 1.$$

α is a space-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = \left(0, \frac{2}{r}, \frac{2}{r}\right).$$

$$< \alpha''(s), \alpha''(s) >_L = 0.$$

$\boldsymbol{t}'$ is a light-like. So we defined the normal vector $\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{n}(\boldsymbol{s})$.

$$\boldsymbol{n}(\boldsymbol{s}) = \left(0, \frac{2}{r}, \frac{2}{r}\right).$$

$\boldsymbol{t}(\boldsymbol{s})$ and $\boldsymbol{n}(\boldsymbol{s})$ are linearly independent vectors. Let $\boldsymbol{b}(\boldsymbol{s})$ be the unique light-like vector and $< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -1$ and also $\boldsymbol{b}(\boldsymbol{s})$ is the perpendicular to $\boldsymbol{t}(\boldsymbol{s})$. $\{\, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b} \,\}$ is not an orthonormal basis of E_1^3.

$$\boldsymbol{b}(\boldsymbol{s}) = \left(0, \frac{-r}{4}, \frac{r}{4}\right) \quad \text{and} \quad \boldsymbol{\tau}(s) = 0.$$

(6) Consider a curve constructed by the boosts about the light-like axis span$\{(0,1,1)\}$. Take the orbit β of the point (0, 1, -1). Then

$\beta(s) = (\, 2s, 1 - s^2, -1 - s^2)$. Hence $\beta'(s) = (2, -2s, -2s\,)$.

$< \beta'(s), \beta'(s) >_L = 2$. β is a space-like curve. As $|\beta'(s)|_L = 2$, we change the parameter as s by $\frac{s}{2}$. So it has pseudo-arclength parameter. Thus let

$$\alpha(s) = \left(s, \frac{1 - s^2}{4}, \frac{-1 - s^2}{4} \right).$$

$$\alpha'(s) = \boldsymbol{T}(\boldsymbol{s}) = \left(1, \frac{-s}{2}, \frac{s}{2} \right).$$

$$< \alpha'(s), \alpha'(s) >_L = 1 > 0.$$

α is a space-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = \left(0, \frac{-1}{2}, \frac{1}{2}\right).$$

$$< \alpha''(s), \alpha''(s) >_L = 0.$$

$\boldsymbol{t}'$ is a light-like. So we defined the normal vector $\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{n}(\boldsymbol{s})$. Thus

$$\boldsymbol{n}(\boldsymbol{s}) = \left(0, \frac{-1}{2}, \frac{1}{2}\right).$$

$\boldsymbol{t}(\boldsymbol{s})$ and $\boldsymbol{n}(\boldsymbol{s})$ are linearly independent vectors. Let $\boldsymbol{b}(\boldsymbol{s})$ be the unique light-like vector and $< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -1$ and also $\boldsymbol{b}(\boldsymbol{s})$ is the perpendicular to $\boldsymbol{t}(\boldsymbol{s})$. $\{ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b} \}$ is not an orthonormal basis of E_1^3.

$$\boldsymbol{b}(\boldsymbol{s}) = \left(s, 1 - \frac{s^2}{4}, -1 - \frac{s^2}{4}\right).$$

$\boldsymbol{\tau}(s) = 0.$ α is contained in the plane $y - z = 2$.

(7) Let

$$\alpha(s) = \frac{1}{r^2} (\cosh(rs), rs, \sinh(rs)).$$

$$\alpha'(s) = \boldsymbol{t}(\boldsymbol{s}) = \frac{1}{r}(\sinh(rs), 1, \cosh(rs)).$$

$$< \alpha'(s), \alpha'(s) >_L = 0.$$

α is a light-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = (\cosh(rs), 0, \sinh(rs)).$$

$$< \alpha''(s), \alpha''(s) >_L = 1 > 0.$$

$\boldsymbol{t}'$ is a space-like. We know that $\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{n}(\boldsymbol{s})$.

$\boldsymbol{n}(\boldsymbol{s}) = (\cosh(rs), 0, \sinh(rs))$. Thus α is pseduo-arclength. $\boldsymbol{b}(\boldsymbol{s})$ is unit null perpendicular to $\boldsymbol{n}(\boldsymbol{s})$**.** We know that

$$< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -1 .$$

$$\boldsymbol{b}(\boldsymbol{s}) = \frac{r}{2}(\sinh(rs), -1, \cosh(rs)) \ , \ \boldsymbol{n}'(\boldsymbol{s}) = r(\sinh(rs), 0, \cosh(rs)) .$$

The pseudo-torsion is $\boldsymbol{\tau}(s) = < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L$. We deduce that

$$\boldsymbol{\tau}(s) = \frac{-r^2}{2}.$$

(8) Let

$\alpha(s) = \frac{1}{r^2} (\cos(rs), \sin(rs), rs)$. Then

$$\alpha'(s) = \boldsymbol{T}(\boldsymbol{s}) = \frac{1}{r}(-\sin(rs), \cos(rs), 1).$$

$$< \alpha'(s), \alpha'(s) >_L = 0.$$

α is a light-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = (-\cos(rs), -\sin(rs), 0).$$

$$< \alpha''(s), \alpha''(s) >_L = 1 > 0.$$

$\boldsymbol{t}'$ is a space-like. We know that $\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{n}(\boldsymbol{s})$.

$\boldsymbol{n}(\boldsymbol{s}) = (-\cos(rs), -\sin(rs), 0)$ .Thus α is pseduo-arclength. $\boldsymbol{b}(\boldsymbol{s})$ is unit null vector perpendicular to $\boldsymbol{n}(\boldsymbol{s})$.

$$< \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -1.$$

$$\boldsymbol{b}(\boldsymbol{s}) = \frac{r}{2}(\sin(rs), -\cosh(rs), 1), \ \boldsymbol{n}'(\boldsymbol{s}) = r(\sin(rs), -\cos(rs), 0).$$

The pseudo-torsion is $\boldsymbol{\tau}(s) = < \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L$. We deduce that

$$\boldsymbol{\tau}(s) = \frac{-r^2}{2}.$$

3.3 Some Theorems About Curves In $\boldsymbol{E}_1^3$

3.3.1 Theorem

Let $\alpha: I \rightarrow E_1^3$ be unit velocity curve. α is contained in an affine plane $\Leftrightarrow$ the $\boldsymbol{\tau}$ vanishes.

The proof is the same and we omit it. However, there are more curves to consider. (Carmo, 1976)

3.3.2 Theorem

Let α be a space-like curve with light-like normal vector or a null curve.

a) $\boldsymbol{\lambda} = 0$ is $\Rightarrow$ the curve lies in plane.($\boldsymbol{\lambda}$ is pseudo-torsion)

b) When null curve lies in a plane then it means is a straight–line. There exists space-like plane curves with light-like normal vector with $\boldsymbol{\lambda} \neq 0$. (Carmo, 1976)

3.3.3 Example

Let

$$\alpha(s) = \left(s, \frac{s^3}{3}, \frac{s^3}{3}\right),\ s > 0 \text{ that lies in } y = z. \text{ We get}$$

$$\alpha'(s) = \boldsymbol{t}(\boldsymbol{s}) = (\,1, s^2, s^2).$$

$$< \alpha'(s), \alpha'(s)\ >_L = 1.$$

α is a space-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = (\,0, 2s, 2s\,).$$

$$< \alpha''(s), \alpha''(s)\ >_L = 0.$$

$\boldsymbol{t}'$ is a light-like. So we defined the normal vector $\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{n}(\boldsymbol{s})$.
$\boldsymbol{n}(\boldsymbol{s}) = (0, 2s, 2s)$.

$\boldsymbol{t}(\boldsymbol{s})$ and $\boldsymbol{n}(\boldsymbol{s})$ are linearly independent vectors. Let $\boldsymbol{b}(\boldsymbol{s})$ be the unique null vector and $< \boldsymbol{n}(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L = -1$ and also $\boldsymbol{b}(\boldsymbol{s})$ is the perpendicular to $\boldsymbol{t}(\boldsymbol{s})$. $\{\ \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\ \}$ is not an orthonormal basis of E_1^3.

$$\boldsymbol{b}(\boldsymbol{s}) = \left(\frac{s}{2}, \frac{-1}{4s}, \frac{1}{4s}\right) \quad \text{and the pseudo} - \text{torsion} \quad \boldsymbol{\tau}(s) = -< \boldsymbol{n}'(\boldsymbol{s}), \boldsymbol{b}(\boldsymbol{s}) >_L.$$

$$\boldsymbol{\tau}'(\boldsymbol{s}) = (0, 2, 2\,)$$

$$\boldsymbol{\tau}(s) = \frac{1}{s}.$$

3.3.4 Theorem

For a Frenet curve , $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$ are invariant under a rigid motion ($\boldsymbol{\tau}$ is invariant up a sign). When the curve is space-like with light-like normal vector or it is light-like, pseudo-torsion is invariant. (Carmo, 1976)

3.3.5 Example

The curves $\alpha(s) = (\cos(s), \sin(s), 0)$ and $\beta(s) = (0, \cosh(s), \sinh(s))$

$$\alpha'(s) = \boldsymbol{t}(\boldsymbol{s}) = (-\sin(s), \cos(s), 0) \text{ and } \beta'(s) = \boldsymbol{t}(\boldsymbol{s}) = (0, \sinh(s), \cosh(s))$$

$< \alpha'(s), \alpha'(s) >_L = 1$. α is a space-like curve.

$< \beta'(s), \beta'(s) >_L = -1$. β is a time-like curve.

$$\alpha''(s) = \boldsymbol{t}'(\boldsymbol{s}) = (-\cos(s), -\sin(s), 0).$$

$< \alpha''(s), \alpha''(s) >_L = 1$. $\boldsymbol{t}'$ is a space-like.

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\kappa(s) = 1, \ \boldsymbol{n}(\boldsymbol{s}) = (-\cos(s), -\sin(s), 0).$$

$\boldsymbol{n}$ is a space-like vector. $\boldsymbol{t}, \boldsymbol{n}$ have the same causal characters, $\boldsymbol{b}$ is a time-like vector. $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is negatively oriented.

$$\boldsymbol{b} = \boldsymbol{txn}. \ \boldsymbol{b}(\boldsymbol{s}) = (0, 0, -1). \ \boldsymbol{b}'(\boldsymbol{s}) = 0, \tau(s) = 0.$$

$$\beta''(s) = \boldsymbol{t}(\boldsymbol{s}) = (0, \cosh(s), \sinh(s))$$

$< \beta''(s), \beta''(s) >_L = 1$. $\boldsymbol{t}'$ is a space-like.

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\kappa(s) = 1, \ \boldsymbol{n}(\boldsymbol{s}) = (0, \cosh(s), \sinh(s)) \ .$$

$\boldsymbol{n}$ is a space-like vector. $\boldsymbol{t}, \boldsymbol{n}$ have the same causal characters, $\boldsymbol{b}$ is a time-like vector. $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is negatively oriented.

$$\boldsymbol{b} = \boldsymbol{txn}. \ \boldsymbol{b}(\boldsymbol{s}) = (-1, 0, 0). \ \boldsymbol{b}'(\boldsymbol{s}) = 0, \tau(s) = 0.$$

Although $\alpha(s)$ *and* $\beta(s)$ have $\boldsymbol{\kappa} = 1$ and $\boldsymbol{\tau} = 0$ their causal chracter of different α is space-like and β is time-like. Even so two curves with the same causal character , we should pay attention to the causal character of Frenet vectors.

For instance , the curve $\gamma(s) = (0, \sinh(s), \cosh(s))$

$$\gamma'(s) = (0, \cosh(s), \sinh(s))$$

$< \gamma'(s), \gamma'(s) >_L = 1$. γ is a space-like curve.

$$\gamma''(s) = \boldsymbol{t}'(\boldsymbol{s}) = (0, \sinh(s), \cosh(s)).$$

$< \gamma''(s), \gamma''(s) >_L = -1$. $\boldsymbol{t}'$ is a time-like.

$$\boldsymbol{t}'(\boldsymbol{s}) = \boldsymbol{\kappa}(s).\boldsymbol{n}(\boldsymbol{s})$$

$$\kappa(s) = 1, \ \boldsymbol{n}(\boldsymbol{s}) = (0, \sinh(s), \cosh(s)).$$

$\boldsymbol{n}$ is a time-like vector. $\boldsymbol{t}, \boldsymbol{n}$ have different causal characters, $\boldsymbol{b}$ is a space-like vector. $\{\, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b} \,\}$ is positively oriented.

$$\boldsymbol{b} = \boldsymbol{txn}.\ \boldsymbol{b}(\boldsymbol{s}) = (1,0,0).\ \boldsymbol{b}'(\boldsymbol{s}) = 0, \tau(s) = 0.$$

$\gamma(s)$ has $\kappa = 1, \boldsymbol{\tau} = 0$, but there does not exist a rigid motion between $\alpha\ and\ \gamma$. For α, $\boldsymbol{t}$ and $\boldsymbol{n}$ are space-like however γ is a space-like curve with time-like normal vector.

In Lorentz-Minkowski space, there exists three different Frenet curves with curvature κ and torsion $\boldsymbol{\tau}$.

3.3.6 Theorem

If $\kappa(s) > 0$ and $\boldsymbol{\tau}(s)$, $s \in I$, two differentiable maps then there exists three different regular parametrized curves $\alpha: I \to E_1^3$, $\alpha = \alpha(s)$, with curvature $\boldsymbol{\kappa}$ and torsion $\boldsymbol{\tau}$. (Lopez, 2014)

3.3.7 Theorem

Let $\boldsymbol{\tau}: I \to R$ be a smooth function. There is a space-like curve with null normal vector and a null curve with pseudo-torsion $\boldsymbol{\lambda}$. (Lopez, 2014)

3.3.8 Definition

Let α, $\beta: I \to E_1^3$ be two unit velocity curve or parametrized by the pseudo-arclength. We say that α and β have the same causal character of the Frenet frame if $\boldsymbol{t}_\alpha, \boldsymbol{n}_\alpha$ and $\boldsymbol{b}_\alpha$ have the same causal character than $\boldsymbol{t}_\beta, \boldsymbol{n}_\beta$ and $\boldsymbol{b}_\beta$, respectively. (Lopez, 2014)

3.3.9 Theorem

Let α, $\beta: I \to E_1^3$ be two regular curves that have the same causal character of the Frenet frame. They have the same $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$, or they have same pseudo - torsion depending on the case ⇒ there exist a rigid motion M of E_1^3 such that $\beta = M\ o\ \alpha$. (Carmo, 1976)

3.3.10 Example

Consider the curve $\alpha(s) = (s^2, \sinh(s^2), \cosh(s^2)), s > 0$. Then

$$\alpha'(s) = (\ 2s, 2s\cosh(s^2).\sinh(s^2)\,, 2s\ \sinh(s^2).\ \cosh(s^2)\ )$$

$$< \alpha'(s), \alpha'(s)\ >_L = 4s^2\ > 0\,, \qquad \alpha(s) \text{ is spacelike curve} .$$

$$\alpha''(s) = (2, 2\cosh(s^2) + 2s\sinh(s^2)\,, 2\sinh(s^2) + 2s\cosh(s^2)\ ).$$

Thus

$$\alpha''(s) \text{ is } \begin{cases} space - like & s \in (0, \sqrt{2}) \\ light - like & s = \sqrt{2} \\ time - like & s > \sqrt{2} \end{cases} .$$

However the parametrization by the arclength is

$$\beta(s) = \left(\frac{s}{\sqrt{2}}\,, \sinh\left(\frac{s}{\sqrt{2}}\right), \cosh\left(\frac{s}{\sqrt{2}}\right)\right)\,,$$

which it is space-like.

We also examine curves in Lorentz –Minkowski Plane.

3.4 Curves In Lorentz-Minkowski Plane

We study plane curves in Minkowski space E_1^3 giving a sign to the curvature $\boldsymbol{\kappa}$. A problem appears in a first moment showing a difference with the Euclidean context. We have two options. First, consider the two dimensional case of Lorentz-Minkowski space, the Lorentz-Minkowski plane E_1^2. The second possibility is to consider a curve of E_1^3 included in an affine plane. There are three possibilities depending on whether the plane is a space-like, time-like of light-like. If the plane is a space-like, the theory corresponds to curves in a Riemannian surface.The plane is isometric to the Euclidean plane E^2 and hence the theory is known ; the plane is time-like ⇒ it is isometric to E_1^2.

Firstly denote $E_1^2 = (IR^2,\ (dx)^2 - (dy)^2)$ the Lorentz-Minkowski plane. We describe the Frenet dihedron such that the curvature has a mark. Let $\alpha : I \to E_1^2$ be a curve parametrized by arclength. Describe the tangent vector

$$\boldsymbol{t}(s) = \alpha'(s).$$

We get away light-like curves since in E_1^2 there are two linearly independent directions of light-like vectors. Hence $\boldsymbol{t}(s)$ would be commensurate to a given direction, obtaining that the curve is a straight-line. We assume that α is space-like or time-like. The vector $\boldsymbol{t}'(s)$ is perpendicular to $\boldsymbol{t}(s)$. $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ will have different causal character**.**

In E^2, the unit normal $\boldsymbol{n}_e(s)$ is chosen so $\{\ \boldsymbol{t}(s), \boldsymbol{n}_e(s)\ \}$ has a positively oriented basis. In E_1^2 we will again choose the Frenet frame as positively oriented however the sequence of the vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ is chosen under the stipulation that the first vector is space-like and the second one is time-like. The situations are:

a) The curve is space-like. Decribe the normal vector $\boldsymbol{n}(s)$, $\{\ \boldsymbol{t}(s), \boldsymbol{n}(s)\ \}$ is positively oriented.
b) The curve is time-like. Describe the normal vector $\boldsymbol{n}(s)$, $\{\ \boldsymbol{n}(s)\ , \boldsymbol{t}(s)\}$ is positively oriented.

Let $< \boldsymbol{t}, \boldsymbol{t} >_L = \ \epsilon\ \in \{-1\ ,1\}$ depending on whether the curve is a space-like or time-like. $< \boldsymbol{n}, \boldsymbol{n} >_L = -\epsilon$. We describe the curvature of α as the function $\kappa(s)$ such that

$$\boldsymbol{t}'(\boldsymbol{s}) = \kappa(s).\boldsymbol{n}(s).$$

Hence

$$\kappa(s) = \ -\epsilon < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_L.$$

The Frenet equations are

$$\boldsymbol{t}'(\boldsymbol{s}) = \kappa(s).\boldsymbol{n}(s)$$
$$\boldsymbol{n}'(\boldsymbol{s}) = \kappa(s).\boldsymbol{t}(s).$$

We have two equations in

$$\begin{pmatrix}\boldsymbol{t}'\\ \boldsymbol{n}'\end{pmatrix} = \begin{pmatrix}0 & \kappa\\ \kappa & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\end{pmatrix} \ and\ \begin{pmatrix}\boldsymbol{t}'\\ \boldsymbol{n}'\end{pmatrix} = \begin{pmatrix}0 & \kappa\\ -\kappa & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\end{pmatrix}.$$

3.4.1 Example

(1) The set $A = \{(x, y) \in IR^2 : x^2 - y^2 = -r^2\}$consist of two components

$$A^+ = \{(x, y) \in A : y > 0\}, \quad A^- = \{(x, y) \in A : y < 0\}.$$

For A^+,

$$\text{Let } \alpha(s) = \left(r\sinh(\tfrac{s}{r}),\ r\ \cosh(\tfrac{s}{r})\right)$$

$$\alpha'(s) = \boldsymbol{t}(s) = \left(\cosh(\frac{s}{r})\ ,\sinh(\frac{s}{r})\ \right)$$

$$< \alpha'(s), \alpha'(s) >_L = 1 > 0.$$

α is a space-like curve. $\boldsymbol{t}$ is a space-like vector.

$$\alpha''(s) = \boldsymbol{t}'(s) = \left(\frac{1}{r}\sinh(\frac{s}{r})\ ,\frac{1}{r}\cosh(\frac{s}{r})\right)$$

$$\alpha''(s) = \boldsymbol{t}'(s) = \frac{1}{r}\left(\sinh(\frac{s}{r})\ ,\cosh(\frac{s}{r})\right)$$

$$\boldsymbol{n}(s) = \left(\ \sinh(\frac{s}{r}), \cosh(\frac{s}{r})\right)$$

$$< \boldsymbol{n}(s), \boldsymbol{n}(s) >_L = -1.$$

$\boldsymbol{n}$ is a time-like vector.

$$\kappa(s) = \frac{1}{r}.$$

For A^-,

$$\text{Let } \beta(s) = \left(r\sinh(\tfrac{s}{r}), -r\ \cosh(\tfrac{s}{r})\right)$$

$$\beta'(s) = \boldsymbol{t}(s) = \left(\cosh(\frac{s}{r})\ , -\sinh(\frac{s}{r})\ \right)$$

$$< \beta'(s), \beta'(s) >_L = 1.$$

β is a space-like curve. $\boldsymbol{t}$ is a space-like vector.

$$\beta''(s) = \boldsymbol{t}'(s) = \left(\frac{1}{r}\sinh(\frac{s}{r})\ , -\frac{1}{r}\cosh(\frac{s}{r})\right)$$

$$\beta''(s) = \boldsymbol{t}'(s) = -\frac{1}{r}\left(-\sinh(\frac{s}{r})\ ,\cosh(\frac{s}{r})\right)$$

$$\boldsymbol{n}(s) = \left(-\sinh(\frac{s}{r}), \cosh(\frac{s}{r})\right)$$

$$< \boldsymbol{n}(s), \boldsymbol{n}(s) >_L = -1 < 0.$$

$\boldsymbol{n}$ is a time-like vector.

$$\kappa(s) = -\frac{1}{r}.$$

(2) The set $B = \{(x, y) \in IR^2 : x^2 - y^2 = r^2\}$ consist of two components

$$B^+ = \{(x, y) \in B : x > 0\}, \quad B^- = \{(x, y) \in B : x < 0\}.$$

<u>For B^+</u>,

$$\text{Let } \alpha(s) = \left(r \cosh(\tfrac{s}{r}), r \sinh(\tfrac{s}{r}) \right)$$

$$\alpha'(s) = \boldsymbol{t}(s) = \left(\sinh(\frac{s}{r}), \cosh(\frac{s}{r}) \right)$$

$$< \alpha'(s), \alpha'(s) >_L = -1.$$

α is a time-like curve. $\boldsymbol{t}$ is a time-like vector.

$$\alpha''(s) = \boldsymbol{t}'(s) = \left(\frac{1}{r}\cosh(\frac{s}{r}), \frac{1}{r}\sinh(\frac{s}{r}) \right)$$

$$\alpha''(s) = \boldsymbol{t}'(s) = \frac{1}{r}\left(\cosh(\frac{s}{r}), \sinh(\frac{s}{r}) \right)$$

$$\boldsymbol{n}(s) = \left(\cosh(\frac{s}{r}), \sinh(\frac{s}{r}) \right)$$

$$< \boldsymbol{n}(s), \boldsymbol{n}(s) >_L = 1.$$

$\boldsymbol{n}$ is a space-like vector.

$$\kappa(s) = \frac{1}{r}.$$

<u>For B^-</u>

$$\text{Let } \beta(s) = \left(-r \cosh(\tfrac{s}{r}), r \sinh(\tfrac{s}{r}) \right)$$

$$\beta'(s) = \boldsymbol{t}(s) = \left(-\sinh(\frac{s}{r}), \cosh(\frac{s}{r}) \right)$$

$$< \beta'(s), \beta'(s) >_L = -1.$$

β is a time-like curve. $\boldsymbol{t}$ is a time-like vector.

$$\beta''(s) = \boldsymbol{t}'(s) = \Big(-\frac{1}{r}\cosh(\frac{s}{r}), \frac{1}{r}\sinh(\frac{s}{r})\Big)$$

$$\beta''(s) = \boldsymbol{t}'(s) = -\frac{1}{r}\Big(\cosh(\frac{s}{r}), -\sinh(\frac{s}{r})\Big)$$

$$\boldsymbol{n}(s) = \Big(\cosh(\frac{s}{r}), -\sinh(\frac{s}{r})\Big)$$

$$< \boldsymbol{n}(s), \boldsymbol{n}(s) >_L = 1.$$

$\boldsymbol{n}$ is a space-like vector.

$$\kappa(s) = -\frac{1}{r}.$$

For κ, let

$$\theta(s) = \int_{s_0}^{s} \kappa(t)dt.$$

Describe two curves α and β and curvature κ, where α is space-like and β is time-like:

$$\alpha(s) = \left(\int_{s_0}^{s} \cosh\theta\,(t)dt, \int_{s_0}^{s} \sinh\theta\,(t)dt\right)$$

$$\beta(s) = \left(\int_{s_0}^{s} \sinh\theta\,(t)dt\,, \int_{s_0}^{s} \cosh\theta\,(t)dt\right).$$

3.4.2 Theorem

Let $\alpha : I \rightarrow E_1^2$ be a time-like curve parametrized by arclength. Assume that there exist a unit time-like vector $v \in E_1^2$ and $\boldsymbol{t}(s)$ and v lie in this same time-like cone $\forall s$. θ is the angle between the tangent vector of α and v $\Rightarrow$ $\kappa\,(s) = \pm\theta'(s)$. (Lopez, 2014)

Proof:

We know that $-\cosh\big(\theta(s)\big) = < \boldsymbol{t}(s), v >_L$.

By differentiating both sides ,

$$-\theta'(s)\sinh\big(\theta(s)\big) = <\boldsymbol{t}'(s), v>_L + \underbrace{<\boldsymbol{t}(s), \underbrace{v'}_{0}>_L}_{0}$$

$$-\theta'(s)\sinh\big(\theta(s)\big) = <\boldsymbol{t}'(s), v>_L$$

We know that $\boldsymbol{t}'(s) = \kappa(s).\boldsymbol{n}(s)$

$$-\theta'(s)\sinh\big(\theta(s)\big) = <\kappa(s).\boldsymbol{n}(s), v>_L = \kappa(s) <\boldsymbol{n}(s), v>_L \qquad (*)$$

$$v = a\boldsymbol{t}(s) + b\boldsymbol{n}(s)$$

$$<v, \boldsymbol{t}(s)>_L = a \underbrace{<\boldsymbol{t}(s), \boldsymbol{t}(s)>_L}_{time-like} + b \underbrace{<\boldsymbol{t}(s), \boldsymbol{n}(s)>_L}_{\underbrace{0}_{since\ orthogonal}}$$

$$<v, \boldsymbol{t}(s)>_L = a.(-1) + b.0$$

$$<v, \boldsymbol{t}(s)>_L = -a$$

$$<v, \boldsymbol{n}(s)>_L = a \underbrace{<\boldsymbol{t}(s), \boldsymbol{n}(s)>_L}_{\underbrace{0}_{since\ orthogonal}} + b \underbrace{<\boldsymbol{n}(s), \boldsymbol{n}(s)>_L}_{space-like}$$

$$<v, \boldsymbol{n}(s)>_L = a.0 + b.1$$

$$<v, \boldsymbol{n}(s)>_L = b$$

$$v = -<v, \boldsymbol{t}(s)>_L.\boldsymbol{t}(s) + <v, \boldsymbol{n}(s)>_L.\boldsymbol{n}(s)$$

$$\underbrace{<v, v>_L}_{time-like} = <v, \boldsymbol{t}(s)>_L^2.(-1) + <v, \boldsymbol{n}(s)>_L^2.(1)$$

$$-1 = -<v, \boldsymbol{t}(s)>_L^2 + <v, \boldsymbol{n}(s)>_L^2$$

We know that $-\cosh\big(\theta(s)\big) = <\boldsymbol{t}(s), v>_L$

$$-1 = -\cosh^2\big(\theta(s)\big) + <v, \boldsymbol{n}(s)>_L^2$$

$$<v, \boldsymbol{n}(s)>_L^2 = -1 + \cosh^2\big(\theta(s)\big)$$

Since we know that $\cosh^2(\theta) - \sinh^2(\theta) = 1$

$$<v, \boldsymbol{n}(s)>_L^2 = \sinh^2\big(\theta(s)\big)$$

$$<v, \boldsymbol{n}(s)>_L = \pm\sinh\big(\theta(s)\big) \qquad (**)$$

$$-\theta'(s)\sinh\big(\theta(s)\big) = <\boldsymbol{\kappa}(s).\boldsymbol{n}(s), v>_L = \boldsymbol{\kappa}(s) <\boldsymbol{n}(s), v>_L \qquad (*)$$

$\kappa(s) = \pm\theta'(s)$ since $(*)$ and $(**)$.

Finallly, for the curves in E_1^2 of constant curvature. Suppose that the curvature $\boldsymbol{\kappa}$ is a constant $a \neq 0$. Then

$$\theta(s) = \int_{s_0}^{s} a \, dt = as + b \,, \;\; b \in IR.$$

From

$$\theta(s) = \int_{s_0}^{s} \kappa(t) \, dt \;\;,$$

Curves have curvature a:

(1) The space-like curve

$$\alpha(s) = \frac{1}{a} (\sinh(as+b), \, \cosh(as+b)).$$

$$< \alpha'(s), \alpha'(s) >_L = \, 1.$$

(2) The time-like curve

$$\beta(s) = \frac{1}{a} (\cosh(as+b), \sinh(as+b)).$$

$$< \beta'(s), \beta'(s) >_L = \, -1.$$ (Lopez, 2014)

According to the Euclidean space , α and β curves are Euclidean hyperbolas.

3.4.3 Theorem

Let $\alpha : I \rightarrow E_1^3$ be a Frenet curve included in a plane of E_1^3. α is a circle $\Leftrightarrow$ $\kappa = c, c \neq 0$ ($where\ c\ is\ constant$) and the $\boldsymbol{\tau}$ =0. (Lopez, 2014)

3.4.4 Theorem

Let D be the light-like plane $y = z$. The only space-like curves in D with constant pseudo torsion $\lambda \neq 0$ are,

$$\alpha(s) = \left(s + d \,, \frac{a}{\lambda^2} e^{\lambda s} + bs + c \,, \frac{a}{\lambda^2} e^{\lambda s} + bs + c \;\right), \;\; a, b, c, d \;\in IR.$$ (Lopez, 2014)

Proof:

Let $\alpha(s) = \big(x(s), y(s), y(s)\big)$. α is parametrized by arclength parameter then

$$x'(s) = \pm 1 \ \text{ when } \ x(s) = s$$

$$\alpha'(s) = \boldsymbol{t}(s) = (1\,, y'(s), y'(s)\,)$$

$$< \alpha'(s), \alpha'(s) >_L = \ 1 \ \ \text{since} \ \ \alpha \text{ and } \boldsymbol{t} \text{ is a space} - \text{like.}$$

$$\boldsymbol{n}(s) = \boldsymbol{t}(s) \ = \big(0, y''(s), y''(s)\big)$$

$< \boldsymbol{n}(s)\,, \boldsymbol{n}(s) >_L = \ 0$ since $\boldsymbol{n}$ $is\ a$ light $-$ like.

$\boldsymbol{b}$ is unit null vector satisfiying $< \boldsymbol{n}\,, \boldsymbol{b} >_L = \ -1.$

$$\boldsymbol{b}(s) = \left(\frac{y'}{y''}, \frac{-1 + (y')^2}{2y''}, \frac{1 + (y')^2}{2y''}\right).$$

$y'' \neq 0$ because conversly, $y(s) = as + b,\ a, b \in IR$, showing that α is straight-line. $\alpha(s) = (\pm s, as + b, as + b\,)$. The computation of the pseudo torsion $\lambda = -< \boldsymbol{n}'\,, \boldsymbol{b} >_L$.

$$\boldsymbol{n}'(s) = \big(0, y'''(s), y'''(s)\big)$$

$$\lambda = -\left(0 + \frac{y'''(-1 + (y')^2)}{2y''} - \frac{y'''(1 + (y')^2)}{2y''}\right) = \frac{y'''}{y''} \ \text{ with } y'' \neq 0.$$

Because $\frac{y'''}{y''} = \lambda$ by solving $y(s) = \frac{a}{\lambda^2} e^{\lambda s} + bs + c.$

3.5 Helices In E_1^3

A curve is called a general helix or cylindrical helix if its tangent makes a constant angle with a fixed line in Euclidean space.. A curve is a general helix $\Leftrightarrow$ τ/κ is a constant function. For instance, plane curves are helices. We expand this concept to the Lorentz-Minkowski space. The problem is two defined the angle two vectors. Difficulty is caused causal characters of vectors.

3.5.1 Definition

A helix $\alpha : I \to E_1^3$ is a unit velocity curve (or parametrized by the pseudo-arclength if α is light-like) such that there exists a vector $w \in E_1^3$ with $< \boldsymbol{t}(s), w >_L$ is constant. Any line parallel to this direction w is named the axis of the helix.

Especially, a plane curve and a straight-line are helices. As a result $\boldsymbol{\tau}/_{\kappa}$ is constant. (Lopez, 2014)

3.5.2 Theorem

Let $\beta : I \to E_1^3$ be aa Frenet curve. β is a helix $\Leftrightarrow \boldsymbol{\tau}/_{\kappa}$ is constant. (O'Neill, 1983)

4. SURFACES IN LORENTZ SPACE

First we will define the notion of space-like and time-like surface. We will define the $\boldsymbol{H}$ and K for them. We will calculate these curvatures by using parametrizations. We will define umbilical and minimal surfaces of E_1^3. Nevertheless, we will see the effect of causal characters, for instance , the surfaces can not be closed and the Weingarten map for time-like surfaces might not be diagonalizable.

4.1 Spacelike and Timelike Surfaces In $\boldsymbol{E_1^3}$

Let M be a smooth and connected surface for non-empty boundary ∂M. Let $x: M \to E_1^3$ be an immersion, that is, a differentiable map such that its differentiable map $dx_p: T_pM \to IR^3$ is injective. We identify the tangent plane T_pM with $(dx)_p(T_pM)$. $x^*(<,>_L)_p$ is the pullback metric ,

$$x^*(<,>_L)_p(u,v) = < dx_p(u), dx_p(v) >_L \text{ where } u,v \in T_pM.$$

$x: (M, x^*(<,>_L)) \to (E_1^3, <,>_L)$ is an isometric immersion. The metric $x^* <,>_L$ can be of 3-types,

a) T_pM is a space-like plane when $x^* <,>_L$ is positive definite.

b) T_pM is a time-like plane when $x^* <,>_L$ is a metric with index 1.

c) T_pM is a light-like plane when $x^* <,>_L$ is a degenerate metric. (Lopez, 2014)

4.1.1 Definition

Let M be a surface. An immersion $x: M \to E^3$ is called space-like (respectively time-like, light-like) if all tangent planes $\left(T_pM, x^*(<,>_L)\right)$ are space-like (respectively time-like, light-like).

A space-like or time-like surface are a non-degenerate surface. As the curves of E_1^3, given an immersed surface in E_1^3, the causal character might change in different points of the same surface. A surface is not necessarily classified in one of the above types. For instance , in the sphere

$$S^2 = \{ (x , y, z) \in IR^3 \mid x^2 + y^2 + z^2 = 1 \}$$

$$v = (x, y, z) \in S^2 \Rightarrow < v, v >_L = x^2 + y^2 - z^2$$

We know that $x^2 + y^2 + z^2 = 1 \Rightarrow x^2 + y^2 = 1 - z^2$. So

$$v = (x, y, z) \in S^2 \Rightarrow < v, v >_L = x^2 + y^2 - z^2 = 1 - 2z^2$$

(1) The region $A = \{ (x , y, z) \in S^2 \mid \ | z | < \frac{1}{\sqrt{2}} \}$ is time-like.

(2) The region $B = \{ (x , y, z) \in S^2 \mid \ | z | > \frac{1}{\sqrt{2}} \}$ is space-like.

(3) The region $C = \{ (x , y, z) \in S^2 \mid \ | z | = \frac{1}{\sqrt{2}} \}$ is light-like.

For a space-like (resp. time-like) surface M and $p \in M$ we have the decomposition $E_1^3 = \left(T_pM \right) \oplus (T_pM)^{\perp}$, where $(T_pM)^{\perp}$ is a time-like (resp. space-like) subspace of dimension 1. A Gauss map is a differentiable map $n\colon M \to E_1^3$ such that $| n(p)| = 1$ and $n(p) \in (T_pM)^{\perp} \ \forall\, p \in M$. For a non-degenerate surface this is equivalent to existence of a Gauss map, also called an orientation of M. (Lopez, 2014)

4.1.2 Proposition

Let S be a compact surface and let $x\colon S \to E_1^3$ be a space-like, time-like or light-like immersion. Then $\partial S \neq \emptyset$. (Lopez, 2014)

Proof:

Let $\partial S = \emptyset$. Consider that the immersion is space-like (respectively time-like or light-like). Let $\alpha \in E^3$ be a space-like (respectively time-like) vector. Since S is compact, let $p_0 \in S$ be the minimum point of the function
$f(p) = < x\,(p), \alpha >_L$. As $\partial S = \emptyset$, then p_0 is critical point of the function f so $< (dx)_{p_0}(w), \alpha >_L, \forall w \in T_{p_0} S$. Then $a \in (T_{p_0} S)^{\perp}$, a contradiction because $(T_{p_0} S)^{\perp}$ is time-like (respectively space-like or light-like).

4.1.3 Proposition

Let $x: S \rightarrow E_1^3$ be a space-like immersion of a surface S. Consider the projection map $\pi: S \rightarrow IR^2,\ \pi(x, y, z) = (x, y)$.

a) The projection π is local diffeomorphism.

b) Assume that S is compact and that $x\,|_{\partial S}$ is a diffeomorphism between ∂S and a plane, closed, simple curve. Then $x\,(S)$ is a graph on the planer domain determined by $x(\partial S)$. **(Lopez, 2014)**

4.1.4 Example

A plane $D = d_0 + span\,\{w\}^{\perp}$ the causal character of D coincides with the one of. w is a unit time-like or space-like vector $\Rightarrow$ a Gauss map is a $n(d) = w$. (Lopez, 2014)

4.1.5 Example

A hyperbolic plane of center $p_0 \in E_1^3$ and radius $r > 0$ is

$$\mathbb{H}^2(r; p_0) = \{\, p \in E_1^3 \mid \, < p - p_0, p - p_0 >_L = -r^2, < p - p_0, e_3 >_L < 0 \,\}$$

here $e_3 = (0, 0, 1)$. The set $\{\, p \in E_1^3 \mid \, < p - p_0, p - p_0 >_L = -r^2\}$ has two connected components and that the condition $< p - p_0, e_3 >_L < 0$ chooses from them. Let p_0 be origin in IR^3 and $r = 1$ is denoted by $\mathbb{H}^2(1; O(0,0,0)) = \mathbb{H}^2$, that is

$$\mathbb{H}^2 = \{\, p \in E_1^3 \mid \, < p, p >_L = -1, < p, e_3 >_L < 0 \,\}$$

$$\mathbb{H}^2 = \{\, (x, y, z) \in E_1^3 \mid x^2 + y^2 - z^2 = -1, z > 0 \,\}.$$

This surface is one part of a hyperboloid of two sheets. A hyperbolic plane is a space-like surface. Actually, if $w \in T_p\mathbb{H}^2(r\,;\,p_0)$ and $\alpha = \alpha(t) \subset \mathbb{H}^2(r\,;\,p_0)$ is the curve that represent w, then $< \alpha(t) - p_0, \alpha(t) - p_0 >_L = -r^2$. By differentiating with respect to t

$2 < \alpha'(t), \alpha(t) - p_0 >_L = 0$ let $t = 0 \Rightarrow < \alpha'(0), \alpha(0) - p_0 >_L = 0 \Rightarrow < w,\ p - p_0 >_L = 0.$

This means that $T_pS = span\,\{p - p_0\}^{\perp}$. As $p - p_0$ is a time-like vector then S is a space-like surface. Morever, $n(p) = \frac{(p - p_0)}{r}$ is a Gauss map. Since $< n,\ e_3 >_L < 0, n$ is future directed. (Lopez, 2014)

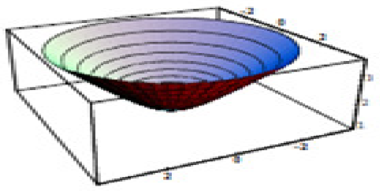


Figure 4.1. Hyperbolic Plane of $\boldsymbol{E_1^3}$

4.1.6 Example

The pseudo-sphere of center p_0 and radius r is

$$\mathbb{S}_1^2(\,r;p_0) = \{\,p \in E_1^3 \mid \,<p-p_0, p-p_0\ >_L = \ r^2\}.$$

The tangent plane at p is $T_pM = span\,\{\,p-p_0\}^\perp$ and $(p) = \frac{(p-p_0)}{r}, n(p)$ is a space-like since $r^2 > 0. n(p)$ is a space-like vector, so the surface is time-like. p_0 is the origin and $r = 1$ the surface is named The De Sitter Space and we denote by $\mathbb{S}_1^2$. Then

$$\mathbb{S}_1^2 = (1; O(0,0,0)\,) = \mathbb{S}_1^2 = \{\,(x\,,y,z)\ \in E_1^3 |\, x^2+y^2-z^2 = 1\,\}.$$

According to Euclidean geometry, this surface is a ruled hyperboloid.

Additionally this surface called is a hyperboloid of one sheet. (Lopez, 2014)

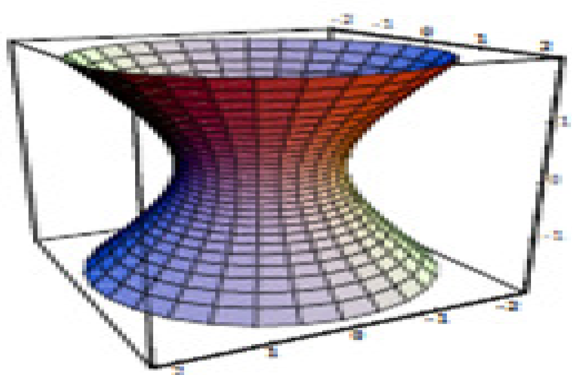


Figure 4.2. Pseudo-sphere of $\boldsymbol{E_1^3}$

4.1.7 Example

The light-like cone of center p_0 is

$$C(p_0) = \{ p \in E_1^3 \mid < p - p_0, p - p_0 >_L = 0\} - \{p_0\}$$

$$T_p\, C\,(p_0) = span\,\{ p - p_0\}^{\perp}.$$

The surface is light-like. If p_0 is origin of IR^3, then $C(p_0)$ is the light-like cone C of E_1^3. (Lopez, 2014)

$$C(0\,; O(0,0,0)\,) = C = \{ (x\,, y, z) \in E_1^3 \mid x^2 + y^2 - z^2 = 0 \} - \{(0\,,0\,,0)\}.$$

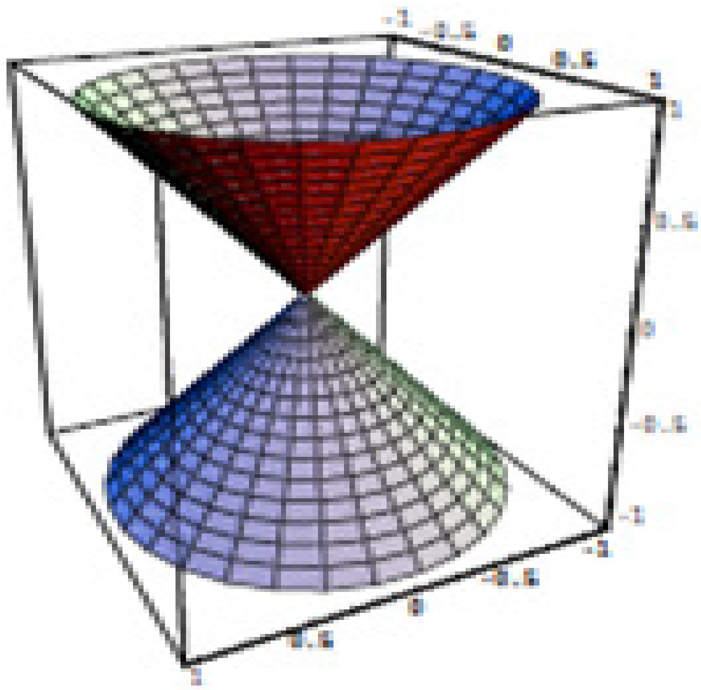


Figure 4.3. Light-like Cone Of E_1^3

4.1.8 Example

$h: \Omega \subset \mathrm{IR}^2 \rightarrow IR$ be a smooth function defined on a domain $\Omega \subset IR^2$. The graph of h defined by

$$h_1 = graph\,(h) = \{(x, y, h(x, y)) \mid (\,x, y) \in \Omega\,\}.$$

Consider h_1 as the image of immersion $\phi_1 : \Omega \rightarrow E_1^3$, given by

$$\phi_1\,(\,x, y) = (x, y, h(x, y)).$$

$$\text{As } \phi_{1x}\,(x,y) = (1,0,h_x),\ \ \phi_{1y}\,(x,y) = \big(0,1,h_y\big).$$

$$E = <\ \phi_{1x},\, \phi_{1x}\ >_L = 1 - h_x^2$$

$$F = <\ \phi_{1x},\, \phi_{1y}\ >_L = -h_x h_y$$

$$G = <\ \phi_{1y},\, \phi_{1y}\ >_L = 1 - h_y^2$$

$$\begin{pmatrix} 1-h_x^2 & -h_x h_y \\ -h_x h_y & 1-h_y^2 \end{pmatrix}$$

and determinant is $1 - h_x^2 - h_y^2 = 1 - ||\ \nabla_L h\ ||^2$

(1) ϕ_1 is space-like if $||\ \nabla_L h\ ||^2 < 1$.

(2) ϕ_1 is time-like if $||\ \nabla_L h\ ||^2 > 1$.

(3) ϕ_1 is light-like if $||\ \nabla_L h\ ||^2 = 1$. (Lopez, 2014)

$$h_1 = graph\,(h) = \big\{\big(x,y,h(x,y)\big) \mid (x,y) \in \Omega\ \big\}$$

$$h_2 = graph\,(h) = \{(x,h(x,z),z) \mid (x,z) \in \Omega\ \}$$

$$h_3 = graph\,(h) = \{(h(y,z),y\,,z) \mid (y,z) \in \Omega\ \}$$

Consider h_3 as the image of immersion $\phi_3 \colon \Omega\ \to E_1^3$, given by

$$\phi_3\,(y,z) = (f(y,z),y,z).$$

$$\text{As } \phi_{3y}\,(y,z) = \big(h_y,1,0\big),\ \ \phi_{3z}\,(y,z) = (h_z,0,1).$$

$$E = <\ \phi_{1x},\, \phi_{1x}\ >_L = 1 + h_y^2$$

$$F = <\ \phi_{1x},\, \phi_{1y}\ >_L = h_y h_z$$

$$G = <\ \phi_{1y},\, \phi_{1y}\ >_L = h_z^2 - 1$$

$$\begin{pmatrix} 1+h_y^2 & h_y h_z \\ h_y h_z & h_z^2 - 1 \end{pmatrix}.$$

The determinat is $-h_y^2 + h_z^2 - 1$ which is different from $1 - ||\ \nabla_L h\ ||^2$and the mark gives the causal character of the surface. Hence the same function h might give a surface with a different causal character. For instance $\Omega = IR^2$ and $h(x,y) = 0 \Rightarrow$ h_1 is a space-like plane but h_3 is a time-like plane.

4.1.9 Example

Let $f(x,y,z) = x^2 + y^2 - z^2$. Then $p = (x,y,z)$ and $v = (v_1, v_2, v_3)$

$$(df)_p(v) = 2xv_1 + 2yv_2 - 2zv_3 \Rightarrow (df)_p(v) = 2(xv_1 + yv_2 - zv_3).$$

p is critical point only if $p = (0,0,0)$. $f(0,0,0) = 0$ and $\forall\, a \neq 0$,

$S_a = f^{-1}(\{a\})$ is a surface.

$$\nabla_L f = (2x, 2y, 2z) = 2(x,y,z)$$

$$\nabla_e f = (2x, 2y, -2z) = 2(x,y,-z)$$

$$< \nabla_L f, \nabla_L f >_L = 4(x^2 + y^2 - z^2) = 4\, f(x,y,z) = 4a.$$

Consider $a \in \{-1,1\}$.

i. If $a < 0$ then $< \nabla_L f, \nabla_L f >_L = -4$ the surface is space-like.
ii. If $a > 0$ then $< \nabla_L f, \nabla_L f >_L = 4$ the surface is time-like. (Lopez, 2014)

4.1.10 Proposition

A space-like (respectively time-like) surface is locally the graph of a function defined in the plane $z = 0$ (respectively $x = 0$ or $y = 0$). (Lopez, 2014)

4.1.11 Theorem

Let M be a surface and let $x: M \to E_1^3$ be space-like immersion. Then M is orientable. (Lopez, 2014)

4.2 Mean Curvature Of Space-like And Time-like Surfaces

Let $x: M \to E_1^3$ be a space-like or time-like immersion of a surface M and let n be its Gauss map. $\mathfrak{I}(M)$ refers to be space of tangent vector fields to M it is denote by ∇^0 the Levi-Civita connection of E_1^3. If $Y \in \mathfrak{I}(M)$, we obtain the decomposition

$$\nabla_X^0 Y = (\nabla_X^0 Y)^{\mathrm{T}} + (\nabla_X^0 Y)^{\perp},$$

where T and $\perp$ indicate the tangential part and the normal part according to M of $\nabla_X^0 Y$, respectively. ∇ refers the induced connection on M by the immersion x,

$$\nabla_X Y = (\nabla_X^0 Y)^{\mathrm{T}}.$$

We define the second fundamental form of x as the tensorial, symmetric map

$$\sigma: \mathfrak{J}(M)\, x\, \mathfrak{J}(M) \rightarrow \mathfrak{J}(M)^{\perp},\ \sigma(X,Y) = (\nabla^0_X Y)^{\perp}.$$

The expression of the Gauss formula is

$$\nabla^0_X Y = \nabla_X Y + \sigma(X,Y),\ X, Y \in \mathfrak{J}(M)\ \mathbf{(4.1)}$$

Consider Z as a normal vector field to x and let $A_Z(X)$ be tangential component of $-\nabla^0_X Z$,

$$A_Z(X) = -(\nabla^0_X Z)^{\mathrm{T}}.$$

We have from $(\mathbf{4.1})$

$$< A_Z(X), Y >_L = < \sigma(X,Y), Z >_L.\ \ (\mathbf{4.2})$$

Because σ is symmetric, $(\mathbf{4.2})$ implies

$$< A_Z(X), Y >_L = < X, A_Z(Y) >_L .(\mathbf{4.3})$$

A_Z is self-adjoint according to the metric of. Let N be a unit normal vector field on M.The immersion is space-like ⇒ the surface is always orientable by 4.1.1 theorem.

Denote

$$< n, n >_L = \epsilon \begin{cases} -1 & if\ M\ is\ space-like \\ 1 & if\ M\ is\ time-like. \end{cases}$$

Take in the above formula $Z = N$. Since $< N, N >_L$ is constant, we have $< \nabla^0_X n, n >_L = 0$. Then $\nabla^0_X N$ is tangent to M. Denote

$$-\nabla^0_X n = A_N(X)\ (Weingarten\ formula)\ (\mathbf{4.4})$$ (Lopez, 2014)

4.2.1 Definition

The Weingarten endomorphism at $p \in M$ is described by

$$A_p : T_pM \rightarrow T_pM,\ A_p = (A_N\ (n))_p.$$

Morever $(\mathbf{4.4})$ gives

$$A_p\ (v) = -\nabla_v^0 n = \ -(dN)_P(v), v \in T_pM.$$

We will write AX instead of $A_N(X)$.

Since $\sigma(X, Y\ )$ is commensurate to n from $(\mathbf{4.1})$ and $(\mathbf{4.2})$

$$\sigma(X, Y\ ) = \epsilon < \ \sigma(X, Y\ ), n\ \ >_L n = \epsilon < AX, Y >_L n. (\mathbf{4.5})$$

Now $(*)$ writes as

$$\nabla_X^0 Y = \ \nabla_X Y + \epsilon < AX, Y >_L n. \text{(Lopez, 2014)}$$

4.2.2 Definition

Let M be a surface and let $x: M \to E_1^3$ be a space-like or time-like immersion. $\vec{\boldsymbol{H}}$ is the mean curvature vector field.

$$\vec{\boldsymbol{H}} = \frac{1}{2}\ trace\ (\sigma).$$

The mean curvatures function $\vec{\boldsymbol{H}}$ is defined by the relation $\vec{\boldsymbol{H}} = \boldsymbol{H}n.$ For this reason

$$\vec{\boldsymbol{H}} = \epsilon < \vec{H}, n >_L.$$

$\vec{\boldsymbol{H}}$ is a vector field perpendicular to M, $\vec{\boldsymbol{H}}\ \in\ \Im(M)^{\perp}$. We can write $\vec{\boldsymbol{H}}$ and $\boldsymbol{H}$ in terms of a local tangent basis. Let $\{e_1, e_2\}$ be an orthonormal local tangent vector fields on M where e_1 is space-like and $< e_2, e_2\ \ >_L = -\epsilon$. Then $(\mathbf{4.5})$ gives

$$\vec{\boldsymbol{H}} = \frac{1}{2}\ trace\ (\sigma) = \ \frac{1}{2}\ \big(\sigma(e_1, e_1) - \epsilon\sigma(\ e_2\ , e_2)\big)$$

$$\vec{\boldsymbol{H}} = \frac{1}{2}(\epsilon < Ae_1, e_1 >_L - < Ae_2\ , e_2\ \ >_L)n$$

$$\vec{\boldsymbol{H}} = \frac{1}{2}(< Ae_1, e_1 >_L - \epsilon\ \ < Ae_2\ , e_2\ \ >_L)n = \left(\frac{\epsilon}{2}\ \text{trace}\ A\right)n.$$

On the other hand,

$\boldsymbol{H} = \epsilon < \vec{\boldsymbol{H}}, n >_L = \frac{\epsilon}{2}(< Ae_1, e_1 >_L - \epsilon \ < Ae_2 , e_2 \ >_L) = \frac{\epsilon}{2}$ trace (A). (Lopez, 2014)

4.2.3 Corollary

The mean curvature of a space-like or time-like surface is

$$\boldsymbol{H} = \frac{\epsilon}{2} \text{ trace } (A). \quad (\mathbf{4.6})$$

We define the Gauss curvature K of the surface. For a surface, $\rho = 2\mathrm{K}$ where ρ is the scalar curvature. We calculate the curvature tensor of the surface. (O'Neill, 1983)

Denote by R^0 and R the curvature tensors of E_1^3 and M, respectively. Because $R^0 = 0$, we can calculate. Let $X, Y, Z \in \mathfrak{J}(M)$. We know that

$$R^0(X,Y)Z = \nabla^0_X \nabla^0_Y Z - \nabla^0_Y \nabla^0_X Z - \nabla^0_{[X,Y]} Z.$$

Also , $\nabla^0_Y Z = \nabla_Y Z + \sigma(Y,Z)$. Since $\sigma(Y,Z) = \epsilon < AY, Z >_L n$, and using **(4.1)** we have

$$\nabla^0_X \nabla^0_Y Z = \nabla^0_X(\nabla_Y Z) + \nabla^0_X \sigma(Y,Z)$$

$$\nabla^0_X \nabla^0_Y Z = \nabla_X \nabla_Y Z + \sigma(X, \nabla_Y Z) - \epsilon < AY, Z >_L AX + \epsilon < AY, Z >_L n.$$

The tangential part on M is $\nabla_X \nabla_Y Z - \epsilon < AY, Z >_L AX$. Likewise, we compute $\nabla^0_Y \nabla^0_X Z$ and $\nabla^0_{[X,Y]} Z$ and considering the tangential parts. Using that $R^0 = 0$ and that $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we conclude

$$R(X,Y)Z = -\epsilon < AY, Z >_L AX + \epsilon < AX, Z >_L AY$$

$$R(X,Y)Z = \epsilon(-< AY, Z >_L AX + < AX, Z >_L AY). \quad (\mathbf{4.7})$$

Thus we calculate the Ricci tensor the scalar curvature ρ. For Ricci tensor, we get

$$\text{Ric } (X,Y) = \text{trace } (v \to R(X,v)Y) = < R(X,e_1)Y, e_1 >_L - \epsilon < R(X,e_2)Y, e_2 >_L$$

$$\text{Ric } (X,Y) = \epsilon\big(< AX, Y >_L (< Ae_1, e_1 >_L - \epsilon \ < Ae_2 , e_2 \ >_L)\big) - \epsilon < AX, AY >_L$$

$$\text{Ric}(X,Y) = \epsilon(\text{trace}(A) < AX, Y >_L - < AX, AY >_L)$$

$$\text{Ric}(X,Y) = 2H < AX, Y >_L - \epsilon < AX, AY >_L.$$ Hence

$$\rho = \text{trace}(\text{Ric}) = R(e_1, e_1) - \epsilon R(e_2, e_2)$$

$$\rho = 2\boldsymbol{H}(< Ae_1, e_1 >_L - \epsilon < Ae_2, e_2 >_L) - \epsilon(< Ae_1, e_1 >_L - \epsilon < Ae_2, e_2 >_L)$$

$$\rho = \epsilon\big(\text{trace}(A)^2 - \text{trace}(A^2)\big) = 4\epsilon\boldsymbol{H^2} - \epsilon\,\text{trace}(A^2)$$

$$\rho = 2\epsilon \det(A). \qquad \textbf{(4.8)}$$

This matrix A in the basis $\{e_1, e_2\}$ is

$$A = \begin{pmatrix} < Ae_1, e_1 >_L & < Ae_2, e_1 >_L \\ -\epsilon < Ae_1, e_2 >_L & -\epsilon < Ae_2, e_2 >_L \end{pmatrix}.$$

As $= 2\text{K}$, the Gauss curvature K is

$$\text{K} = \epsilon\det(A) = \frac{\epsilon}{2}\left(4\boldsymbol{H^2} - \text{trace}\left(A^2\right)\right). \qquad (\mathbf{4.9})$$ (Lopez, 2014)

4.2.4 Corollary

The Weingarten map A of a space-like or time-like surface of E_1^3.

$$\text{K} = \epsilon\det(A). \qquad (\mathbf{4.10})$$

We can calculate K in 2 - dimensional manifold, the Gauss curvature coincides with the local curvature of the 2 - dimensional plane generated by $\{e_1, e_2\}$ of the tangent plane. As a result of $(\mathbf{4.7})$, we get

$$\text{K} = \frac{< R(e_1, e_2)e_2, e_1 >_L}{< e_1, e_1 >_L < e_2, e_2 >_L - < e_1, e_2 >_L^2}$$

$$\text{K} = \frac{\epsilon(< Ae_1, e_1 >_L < Ae_2, e_2 >_L - < Ae_1, e_2 >_L < Ae_2, e_1 >_L)}{-\epsilon}$$

$$\text{K} = -(< Ae_1, e_1 >_L < Ae_2, e_2 >_L - < Ae_1, e_2 >_L^2).$$

This expression coincides with $(\mathbf{4.9})$. (Lopez, 2014)

4.2.5 Definition

Let $x: M \to E_1^3$ a space-like or time-like immersion and $p \in M$. If the Weingarten map A_p is diagonalizable, the eigenvalues of A_p are called the principal curvature at p. Denote by $\lambda_1(p)$ and $\lambda_2(p)$. From $(\mathbf{4.6})$ and $(\mathbf{4.10})$ (Lopez, 2014)

4.2.6 Corollary

Assume that A_p is diagonalizable in a space-like or time-like surface of E_1^3.

$$\boldsymbol{H}(p) = \epsilon \frac{\lambda_1(p)+\lambda_2(p)}{2}, \quad \mathrm{K}(p) = \epsilon\, \lambda_1(p)\lambda_2(p). \text{ (Carmo, 1976)}$$

4.2.7 Definition

Let $x: M \to E_1^3$ be a space-like or time-like immersion. A point $p \in M$ is named if $\exists\ \lambda(p) \in IR$ such that

$$< \sigma(u,v), n(p) >_L = \lambda(p) < u,v >_L, \qquad u,v \in T_pM.$$

A surface is named completely umbilical if all points are umbilic.

Hence, an umbilic is a point where the first and the second fundamental forms are proportional. Besides, it is equivalent to say that

$$< A_p u\,, v >_L = \lambda(p) < u,v >_L .$$

Especially, and from **(4.2)**, A_p must be diagonalizable since

$$< Ae_1, e_2 >_L = 0.$$

Hence we can say that p is umbilical $\Leftrightarrow \lambda_1(p) = \lambda_2(p)$. In Euclidean space, it is well know the inequality $\boldsymbol{H}^2 - \mathrm{K} \geq 0$ and hold only in a umbilic. (Lopez, 2014)

4.2.8 Proposition

Suppose that M is a space-like or time-like surface of E_1^3, $p \in M$ and A_p is diagonalizable.

$$\boldsymbol{H}(p)^2 - \epsilon\mathrm{K}(p) \geq 0$$

and the equality $\Leftrightarrow p$ is umbilic. Especially, in a time-like surface.

$$H(p)^2 - \mathrm{K}(p) < 0 \Rightarrow p \text{ is not umbilic. (Lopez, 2014)}$$

Proof:

From the definition of $\boldsymbol{H}$ and K, we have

$$0 \leq \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 = \left(\frac{\lambda_1 + \lambda_2}{2}\right)^2 - \lambda_1\lambda_2 = \boldsymbol{H}^2 - \epsilon \mathrm{K}.$$

Furthermore the equality holds at a point $p \Leftrightarrow \lambda_1(p) = \lambda_2(p)$, that is, p is an umbilic. The diagonalization of the Weingarten map depends on the existence of real roots of its characteristic polynomial $P(\lambda)$. A simple calculation leads to $P(\lambda) = \lambda^2 - 2\boldsymbol{H}\epsilon\lambda + \epsilon\mathrm{K}$ and its discriminat is $\Delta = 4(\boldsymbol{H}^2 - \epsilon\mathrm{K})$.

(1) $\boldsymbol{H}^2 - \epsilon\mathrm{K} > 0 \Rightarrow$ there are two different real roots of $P(\lambda)$ and the Weingarten map is diagonalizable.

(2) $\boldsymbol{H}^2 - \epsilon\mathrm{K} < 0 \Rightarrow A$ is not diagonalizable.

(3) $\boldsymbol{H}^2 - \epsilon\mathrm{K} = 0 \Rightarrow$ there is a double root of $P(\lambda)$.

a) $\epsilon = -1 \Rightarrow$ the root $\lambda = -\boldsymbol{H}$ is the eigenvalue of A and the point is umbilic.

b) $\epsilon = 1 \Rightarrow$ the matrix could be or not be diagonalizable.

$$|\sigma|^2 = \sum_{i,j=1}^{2} < Ae_i, e_j >_L^2 = 4\boldsymbol{H}^2 - 2\epsilon\mathrm{K},$$

and if A_p is diagonalizable, $|\sigma|^2 = \lambda_1^2 + \lambda_2^2$. There exist non-umbilical time-like surfaces such that $\boldsymbol{H}^2 - \mathrm{K} = 0$ on the surface.

4.2.9 Example

Plane

Consider a non-degenerate plane $D = d_0 + < x, x >_L^{\perp}$, with $|x|_L = 1. n = x$ and dn =0. Here $\lambda_1 = \lambda_2 = \boldsymbol{H} = \mathrm{K} = 0$.

4.2.10 Example

Hyperbolic plane

$$\mathbb{H}^2(r; p_0) = \{ p \in E_1^3 \mid < p - p_0, p - p_0 >_L = -r^2, < p - p_0, e_3 >_L < 0 \}.$$

The unit normal vector pointing to the future of $\mathbb{H}^2(r; p_0)$ is

$n(p) = \frac{(p-p_0)}{r}$. Then $A = {}^{I}/_{r}$ and

$$\lambda_1 = \lambda_2 = {}^{-1}/_{r}\,, \boldsymbol{H} = {}^{1}/_{r}\,, \mathrm{K} = {}^{1}/_{r^2}.$$

Hence a hyperbolic plane has constant negative curvature. $\mathbb{H}^2(r;p_0)$ is a 2 dimensional space form of negative curvature and called the hyperbolic plane.

4.2.11 Example

Pseudo-sphere

$$\mathbb{S}_1^2(\,r;p_0) = \{\,p \in E_1^3 \mid \,< p - p_0, p - p_0 \;>_L = \; r^2\}.$$

For $\mathbb{S}_1^2(\,r;p_0)$, the Gauss map is $n(p) = \frac{(p-p_0)}{r}$. Then $A = {}^{-I}/_{r}$. In this way

$$\lambda_1 = \lambda_2 = {}^{-1}/_{r}\,, \boldsymbol{H} = {}^{-1}/_{r}\,, \mathrm{K} = {}^{1}/_{r^2}\ .$$

Hence a pseudo-sphere has constant positive curvature.

4.3 Local Calculation of the Curvature and Examples

We calculate the curvatures of a space-like or time-like surface by using local parametrization. (Carmo, 1976)

Consider a local parametrization

$$X: U \subset IR^2 \rightarrow E_1^3,\;\; X = X(u,v),$$

of a (space-like or time-like) immersion x. Let $B = \{X_u, X_v\,\}$ be a local basis of the tangent plane at each point of $X(U)$. The Lorentz - Minkowski 1st fundamental form is the metric on T_pM,

$$I_p =<,>_p : T_pM \;\times T_pM \;\rightarrow\; IR$$

$$I_p(u\,,v) =< u\,,v\; >_p$$

$$I =< dX, dX\; >_L = Edu^2 + 2Fdudv + Gdv^2$$

According to $\boldsymbol{b}$, let $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ be the matricial phrase of the first fundamental form, where

$$E =< X_u, X_u\; >_L\,, F =< X_u, X_v\; >_L\,, G =< X_v, X_v\; >_L\ .$$

Denote $\det(I) = EG - F^2$. The surface is space-like if $\det(I) = EG - F^2 > 0$ and it is time-like if $\det(I) = EG - F^2 < 0$. We take the normal vector field

$$n = \frac{X_u \times_L X_v}{|X_u \times_L X_v|_L}.$$

We use the notation $< n, n >_L = \epsilon$ again. Here

$$|X_u \times_L X_v|_L = \sqrt{-\epsilon(EG - F^2)} = \sqrt{-\epsilon \det(I)}.$$

The Minkowski second fundamental form of p

$$\sigma_p : T_pM \times T_pM \rightarrow IR$$

$$\sigma_p(u, v) = -< (dN)_p(u), v >_L = < A_p(u), v >_L$$

$$II = < -dX, dn >_L = Ldu^2 + 2Mdudv + Ndv^2.$$

Let $\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ be a matricial phrase of σ with respect to

$$L = < X_u, -n_u >_L = < n, X_{uu} >_L$$

$$M = < X_u, -n_v >_L = < X_v, -n_u >_L = < n, X_{uv} >_L$$

$$L = < X_v, -n_v >_L = < n, X_{vv} >_L$$

A is the Weingarten map. Then

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

For this reason the Minkowski mean curvature $\boldsymbol{H}$ and Minkowski Gauss curvature K are defined as expected by

$$\boldsymbol{H} = \epsilon \frac{EN + GL - 2MF}{2(EG - F^2)}$$

$$\mathrm{K} = \epsilon \frac{\det II}{\det I} = \epsilon \frac{LN - M^2}{EG - F^2}$$. (Weinstein, 1996)

4.3.1 Example

Let $f \in C^2(\Omega)$ be a smooth function and consider the surface M given by $z = f(x, y)$. Let $\psi : \Omega \rightarrow E_1^3$ denote the usual parametrization

$$\psi(x, y) = (x, y, f(x, y)).$$

$$\psi_x\,(x, y) = (1, 0, f_x),\;\; \psi_y\,(x, y) = \left(0, 1, f_y\right).$$

First fundamental form coeffients are,

$$E = <\ \psi_x, \psi_x\ >_L = 1 - f_x^2$$

$$F = <\ \psi_x, \psi_y\ >_L = -f_x f_y$$

$$G = <\ \psi_y, \psi_y\ >_L = 1 - f_y^2$$

$$\begin{pmatrix} 1 - f_x^2 & -f_x f_y \\ -f_x f_y & 1 - f_y^2 \end{pmatrix}.$$

Thus the $\det I = EG - F^2 = 1 - f_x^2 - f_y^2 = 1 - ||\ \nabla_L f\ ||^2$. On Ω

(1) Immersion is space-like if $||\ \nabla_L f\ ||^2 < 1$

(2) Immersion is time-like if $||\ \nabla_L f\ ||^2 > 1$.

The mean curvature $\boldsymbol{H}$ satisfies

$$\left(1 - f_y^2\right) f_{xx} + 2 f_x f_y + \left(1 - f_x^2\right) f_{yy} = -2H\left(-\epsilon\left(1 - |\nabla_L f\ |^2\right)\right)^{3/2}.$$

Likewise, the Gauss curvature K is

$$\mathrm{K} = -\frac{f_{xx} f_{yy} - f_{xy}^2}{(1 - f_x^2 - f_y^2)^2}.$$

4.3.2 Example

Let $\alpha: I \to E_1^3$ be a null curve and we denote by $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ the Frenet trihedron.

Let

$$X: I\ x\ IR \to E_1^3\ ,\ \ X(s, t) = \alpha(s) + t\boldsymbol{t}(\boldsymbol{s}).$$

This surface is named a *B*-scroll. (Graves, 1979)

We calculate the matrix of the Weingarten map the basis on $\{X_s, X_t\}$.

$$X_s = \alpha' + t\,\boldsymbol{b}' = \boldsymbol{t} + t\boldsymbol{\tau}\boldsymbol{n} \text{ and } X_t = \boldsymbol{b}, \text{ then}$$

$$E = <\ X_s, X_s\ >_L = t^2\,\boldsymbol{\tau}^2$$

$$F = <\ X_s, X_t\ >_L = -1$$

$$G = <\ X_t, X_t\ >_L = 0$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} t^2\,\boldsymbol{\tau}^2 & -1 \\ -1 & 0 \end{pmatrix}.$$

Thus the $\det I = EG - F^2 = -1.$ So the determinat is negative, the surface is time-like. Since

$$X_{ss} = t\boldsymbol{\tau}^2\boldsymbol{t} + (1 + t\boldsymbol{\tau}'\,)\boldsymbol{n} + \; t\boldsymbol{\tau}\boldsymbol{b}$$

$$X_{st} = \boldsymbol{\tau}\boldsymbol{n}$$

$$X_{tt} = \; 0$$

The second fundamental form is

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} -1 - t(\boldsymbol{\tau}' - t\boldsymbol{\tau}^3) & -\boldsymbol{\tau} \\ -\boldsymbol{\tau} & 0 \end{pmatrix}.$$

Thus the $\det II = LN - M^2 = -\boldsymbol{\tau}^2.$

$$\mathrm{K} = \frac{\det II}{\det I} = \frac{-\boldsymbol{\tau}^2}{-1} = \boldsymbol{\tau}^2$$

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = \frac{-2(-\tau\,).(-1)}{-2} \; = \; \tau.$$

Weingarten endomorphism

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$A = \begin{pmatrix} \tau & 0 \\ 1 + t\boldsymbol{\tau}' & \boldsymbol{\tau} \end{pmatrix}.$$

This matrix is not diagonalizable. Since $\boldsymbol{H^2} - \epsilon\mathrm{K} = 0, \epsilon = 1$ so it is not umbical.

4.3.3 Example

The surfaces we will now examine are all minimal surfaces ($\boldsymbol{H} = 0$). (Dillen, 1999), (Kobayashi, 1983), (Woestijne, 1990)

(1) Helicoid of the 1st kind is

$$X(s,t\,) = (\,s\cos(t)\,, s\sin(t)\,, ht\,), \qquad s > h > 0.$$

$$X_s = (\,\cos(t)\,, \sin(t)\,, 0\,)$$

$$X_t = (-s\sin(t)\,, s\cos(t)\,, h\,)$$

$$E = < X_s, X_s >_L = 1$$

$$F = < X_s, X_t >_L = 0$$

$$G = < X_t, X_t >_L = s^2 - h^2$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s^2 - h^2 \end{pmatrix}.$$

Thus the $\det I = EG - F^2 = s^2 - h^2$, the surface $X(s,t)$ is space-like since $s > h > 0$.

$$n = \frac{X_s \times_L X_t}{|X_s \times_L X_t|_L}$$

$$n = \frac{1}{\sqrt{s^2 - h^2}} \left(h\sin(t), -h\cos(t), 0\right)$$

$$X_{ss} = (0, 0, 0)$$

$$X_{st} = (-\sin(t), \cos(t), 0)$$

$$X_{tt} = (-s\cos(t), -s\sin(t), 0)$$

$$L = < n, X_{ss} >_L = 0$$

$$M = < n, X_{st} >_L = \frac{-h}{\sqrt{s^2 - h^2}}$$

$$N = < n, X_{tt} >_L = 0$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 0 & \frac{-h}{\sqrt{s^2 - h^2}} \\ \frac{-h}{\sqrt{s^2 - h^2}} & 0 \end{pmatrix}$$

$$\det II = LN - M^2 = -\frac{h^2}{s^2 - h^2}$$

$$\mathrm{K} = \frac{\det II}{\det I} = -\frac{h^2}{(s^2 - h^2)^2}$$

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = 0.$$

The Weingarten map is

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & \dfrac{-h}{\sqrt{s^2 - h^2}} \\ \dfrac{-h}{\sqrt{(s^2 - h^2)^3}} & 0 \end{pmatrix}.$$

(2) Helicoid of the 2^{nd} kind is

$$X(s,t) = (ht, s\cos h(t), s\sin h(t)), \qquad h > 0, \qquad s \in (h, \infty).$$

$$X_s = (0, \cosh(t), \sinh(t))$$

$$X_t = (h, s\sin h(t), s\cos h(t))$$

$$E = < X_s, X_s >_L = 1$$

$$F = < X_s, X_t >_L = 0$$

$$G = < X_t, X_t >_L = h^2 - s^2$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h^2 - s^2 \end{pmatrix}.$$

Thus the $\det I = EG - F^2 = h^2 - s^2 < 0$, the surface $X(s,t)$ is time-like.

$$n = \frac{X_s \times_L X_t}{|X_s \times_L X_t|_L}$$

$$n = \frac{1}{\sqrt{s^2 - h^2}} (s, h\sin h(t), h\cos h(t))$$

$$X_{ss} = (0, 0, 0)$$

$$X_{st} = (0, \sin h(t), \cos h(t))$$

$$X_{tt} = (0, s\cos h(t), s\sin h(t))$$

$$L = < n, X_{ss} >_L = 0$$

$$M = < n, X_{st} >_L = \frac{-h}{\sqrt{s^2 - h^2}}$$

$$N = < n, X_{tt} >_L = 0$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{-h}{\sqrt{s^2 - h^2}} \\ \dfrac{-h}{\sqrt{s^2 - h^2}} & 0 \end{pmatrix}$$

$$\det II = LN - M^2 = -\frac{h^2}{s^2 - h^2}$$

$$\mathrm{K} = \frac{\det II}{\det I} = \frac{h^2}{(s^2 - h^2)^2}$$

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = 0.$$

The Weingarten map is

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & \dfrac{-h}{\sqrt{s^2 - h^2}} \\ \dfrac{-h}{\sqrt{(s^2 - h^2)^{3/2}}} & 0 \end{pmatrix}.$$

This matrix is not diagonalizable. Since $\boldsymbol{H^2} - \mathrm{K} < 0$, so it is not umbical.

(3) Helicoid of the 3rd is parametrization

$$X(s,t) = (ht, s\sin h(t), s\cos h(t)), \qquad h > 0, \qquad s \in IR.$$

$$X_s = (0, \sinh(t), \cosh(t))$$

$$X_t = (h, s\cos h(t), s\sin h(t))$$

$$E = < X_s, X_s >_L = -1$$

$$F = < X_s, X_t >_L = 0$$

$$G = < X_t, X_t >_L = h^2 + s^2$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & h^2 + s^2 \end{pmatrix}.$$

Thus the $\det I = EG - F^2 = -h^2 - s^2 < 0$, the surface $X(s,t)$ is time-like.

$$n = \frac{X_s \times_L X_t}{|X_s \times_L X_t|_L}$$

$$n = \frac{1}{\sqrt{s^2 + h^2}} (-s, h\cos h(t), h\sin h(t))$$

$$X_{ss} = (0, 0, 0)$$

$$X_{st} = (0, \cos h(t), \sinh(t))$$

$$X_{tt} = (0, s\sinh(t), s\cosh(t))$$

$$L = < n, X_{ss} >_L = 0$$

$$M = < n, X_{st} >_L = \frac{h}{\sqrt{s^2 + h^2}}$$

$$N = < n, X_{tt} >_L = 0$$

$$II=\begin{pmatrix}L & M\\ M & N\end{pmatrix}=\begin{pmatrix}0 & \dfrac{h}{\sqrt{s^2+h^2}}\\ \dfrac{h}{\sqrt{s^2+h^2}} & 0\end{pmatrix}$$

$$\det II = LN-M^2 = -\frac{h^2}{s^2+h^2}$$

$$\mathrm{K}=\frac{\det II}{\det I}=\frac{h^2}{(s^2+h^2)^2}$$

$$\boldsymbol{H}=\frac{EN+GL-2MF}{2(EG-F^2)}=0.$$

The Weingarten map is

$$A=\begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1}\begin{pmatrix}L & M\\ M & N\end{pmatrix}$$

$$A=\begin{pmatrix}0 & \dfrac{-h}{\sqrt{s^2+h^2}}\\ \dfrac{h}{\sqrt{(s^2+h^2)^{3/2}}} & 0\end{pmatrix}.$$

This matrix is not diagonalizable. Since $\boldsymbol{H^2}-\mathrm{K}<0$, so it is not umbical.

(4) The Cayley's surface is

$$X(s,t)=\left(st-ht+h\frac{t^3}{3}, s+ht^2, st+ht+h\frac{t^3}{3}\right), \qquad h,s>0.$$

$$X_s=(t,1,\mathrm{t})$$

$$X_t=(s-h+ht^2, 2th, s+h+ht^2)$$

$$E=<X_s, X_s>_L=1$$

$$F=<X_s, X_t>_L=0$$

$$G=<X_t, X_t>_L=-4sh$$

$$I=\begin{pmatrix}E & F\\ F & G\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -4sh\end{pmatrix}.$$

Thus the $\det I = EG-F^2=-4sh<0$, the surface $X(s,t)$ is time-like.

$$n=\frac{X_s\times_L X_t}{|X_s\times_L X_t|_L}$$

$$n = \frac{1}{2\sqrt{sh}} \left(s + h - ht^2, -2ht, s - h - ht^2 \right)$$

$$X_{ss} = (0, 0, 0)$$

$$X_{st} = (1, 0, 1)$$

$$X_{tt} = (-2ht, 2h, 2ht)$$

$$L = < n, X_{ss} >_L = 0$$

$$M = < n, X_{st} >_L = \sqrt{\frac{h}{s}}$$

$$N = < n, X_{tt} >_L = 2\sqrt{\frac{h}{s}}\, t(-s - h + ht^2)$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{h}{s}} \\ \sqrt{\frac{h}{s}} & 2\sqrt{\frac{h}{s}}\, t(-s - h + ht^2) \end{pmatrix}.$$

$$\det II = LN - M^2 = -\frac{h}{s}$$

$$\mathrm{K} = \frac{\det II}{\det I} = \frac{1}{4s^2}$$

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = 0.$$

The Weingarten map is

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & \sqrt{\frac{h}{s}} \\ \frac{-1}{4\sqrt{s^3 h}} & \frac{-1}{2\sqrt{s^3 h}}\, t(-s - h + ht^2) \end{pmatrix}.$$

This matrix is not diagonalizable. Since $\boldsymbol{H}^2 - \mathrm{K} < 0$, so it is not umbical.

4.3.4 Example

Ruled surfaces is a class of surfaces of interest in Minkowski space E_1^3. (Dillen, 1999). We will use examples of ruled surface to calculate $\boldsymbol{H}$ and K. The samples we will give. The surface is time-like and the Weingarten endomorphism map is not diagonalizable and $\boldsymbol{H^2} - \mathrm{K} = 0.$

(1) Consider the immersion

$$X(s,t) = (s\cos(t), s\sin(t), s + ht), \qquad h > 0.$$

$$X_s = (\cos(t), \sin(t), 1)$$

$$X_t = (-s\sin(t), s\cos(t), h)$$

$$E = < X_s, X_s >_L = 0$$

$$F = < X_s, X_t >_L = -h$$

$$G = < X_t, X_t >_L = s^2 - h^2$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 0 & -h \\ -h & s^2 - h^2 \end{pmatrix}.$$

Thus the $\det I = EG - F^2 = -h^2$, the surface $X(s,t)$ is time-like since $h > 0$.

$$n = \frac{X_s \times_L X_t}{|X_s \times_L X_t|_L}$$

$$n = \frac{1}{h}(h\sin(t) - s\cos(t), -h\cos(t) - s\sin(t), s)$$

$$X_{ss} = (0, 0, 0)$$

$$X_{st} = (-\sin(t), \cos(t), 0)$$

$$X_{tt} = (-s\cos(t), -s\sin(t), 0)$$

$$L = < n, X_{ss} >_L = 0$$

$$M = < n, X_{st} >_L = -1$$

$$N = < n, X_{tt} >_L = \frac{s^2}{h}$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & \frac{s^2}{h} \end{pmatrix}$$

$$\det II = LN - M^2 = -1$$

$$\mathrm{K} = \frac{\det II}{\det I} = \frac{1}{h^2}$$

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = \frac{1}{h}.$$

The Weingarten map is

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{h} & -1 \\ 0 & \frac{1}{h} \end{pmatrix}.$$

This matrix is not diagonalizable. Since $\boldsymbol{H}^2 - \mathrm{K} = 0$ and $\epsilon = 1$ so it is not umbical.

(2) Let $a \neq 0$. The surface

$$X(s,t) = (ht, (s+a)\cosh(t) + s\sinh(t), (s+a)\sinh(t) + s\cosh(t)).$$

$$X_s = (0, \cosh(t) + \sinh(t), \sinh(t) + \cosh(t))$$

$$X_t = (h, (s+a)\sinh(t) + s\cosh(t), (s+a)\cosh(t) + s\sinh(t))$$

$$E = < X_s, X_s >_L = 0$$

$$F = < X_s, X_t >_L = -a$$

$$G = < X_t, X_t >_L = h^2 - 2sa - a^2$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 0 & -a \\ -a & h^2 - 2sa - a^2 \end{pmatrix}.$$

Thus the $\det I = EG - F^2 = -a^2 < 0$, the surface $X(s,t)$ is time-like.

$$n = \frac{X_s \times_L X_t}{|X_s \times_L X_t|_L}$$

$$n = \frac{1}{|a|} (a, h\sinh(t) + h\cosh(t), h\cosh(t) + h\sinh(t))$$

$$X_{ss} = (0,0,0)$$

$$X_{st} = (0, \cosh(t) + \sinh(t), \sinh(t) + \cosh(t))$$

$$X_{tt} = (0, (s+a)\cosh(t) + s\sinh(t), (s+a)\sinh(t) + s\cosh(t)).$$

$$L = < n, X_{ss} >_L = 0$$

$$M = < n, X_{st} >_L = 0$$

$$N = < n, X_{tt} >_L = \frac{1}{|a|} ha$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{|a|} ha \end{pmatrix}$$

$$\det II = LN - M^2 = 0$$

$$\mathrm{K} = \frac{\det II}{\det I} = 0$$

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = 0.$$

The Weingarten map is

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -\dfrac{h}{|a|} \\ 0 & 0 \end{pmatrix}.$$

This matrix is not diagonalizable. Since $\boldsymbol{H} = \mathrm{K} = 0$ and $\epsilon = 1$ so it is not umbical.

(3) The parabolic null cylinder is

$$X(s,t) = \left(s + h\left(-t + \frac{t^3}{3}\right), ht^2, s + h\left(t + \frac{t^3}{3}\right)\right), \qquad h > 0.$$

$$X_s = (1,0,1)$$

$$X_t = (-h + ht^2, 2th, h + ht^2)$$

$$E = < X_s, X_s >_L = 0$$

$$F = < X_s, X_t >_L = -2h$$

$$G = < X_t, X_t >_L = 0$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 0 & -2h \\ -2h & 0 \end{pmatrix}.$$

Thus the $\det I = EG - F^2 = -4h^2 < 0$, the surface $X(s,t)$ is a time-like.

$$n = \frac{X_s \times_L X_t}{|X_s \times_L X_t|_L}$$

$$n = \frac{1}{2h}(-2ht, -2ht, -2ht)$$

$$X_{ss} = (0,0,0)$$

$$X_{st} = (0,0,0)$$

$$X_{tt} = (2ht, 2h, 2ht)$$

$$L = < n, X_{ss} >_L = 0$$

$$M = < n, X_{st} >_L = 0$$

$$N = < n, X_{tt} >_L = -2h$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2h \end{pmatrix}$$

$$\det II = LN - M^2 = 0$$

$$\mathrm{K} = \frac{\det II}{\det I} = 0$$

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = 0.$$

The Weingarten map is

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This matrix is not diagonalizable. Since $\boldsymbol{H} = \mathrm{K} = 0$ and $\epsilon = 1$ so it is not umbical.

4.3.5 Theorem

The only totally umbilical surfaces in Lorentz - Minkowski space are a plane, these are the pseudosphere or hyperbolic plane. (Lopez, 2014)

5. MINIMAL AND MAXIMAL SURFACES

The study of minimal surfaces started with the more intuitive meaning of minimal surfaces, namely surfaces of least area among a family of surfaces having the same boundary. Lagrange defined in 1760 the minimal surfaces as surfaces whose mean curvature vanishes. A surface M in E_1^n is called minimal if and only if the mean curvature vector field is equal to zero, so $\boldsymbol{H} = 0$.

The minimal surfaces in the Lorentz-Minkowski space E_1^3 with metric $g = dx_1^2 + dx_2^2 - dx_3^2$ were studied by Kobayashi in 1983. He classified all the spacelike minimal – he called them 'maximal' because the second variation of volume is always negative definite for spacelike surfaces in E_1^3 – rotation surfaces and ruled surfaces.

4.4 Minimal Surfaces In $\boldsymbol{E^3}$

A minimal surface is a surface M with mean curvature $\boldsymbol{H} = 0$ at all points $p \in M$. The mean curvature is the average of the principal curvatures. Denote by the principal curvatures k_1 and k_2, then

$$\boldsymbol{H} = \frac{k_1 + k_2}{2}.$$

A linear transformation from the tangent space of the surface at that point to itself $S_p\colon T_pM \to T_pM$. We use the shape operator to find the mean curvature.

Let a surface $M \subseteq E^3$ be parametrized by $\vec{x}(u,v)\colon \Omega \subseteq IR^2 \to M$. Then the unit normal vector to the surface is

$$\vec{n} = \frac{X_s \times X_t}{|X_s \times X_t|}.$$

Define E, F, G, L, M and N as

$$E = <\vec{x}_u, \vec{x}_u>$$
$$F = <\vec{x}_u, \vec{x}_v>$$
$$G = <\vec{x}_v, \vec{x}_v>$$

$$L = < \vec{n}, \vec{x}_{uu} >$$

$$M = < \vec{n}, \vec{x}_{uv} >$$

The shape operator is

$$S = \frac{1}{EG - F^2} \begin{pmatrix} GL - FM & GM - FN \\ EM - FL & EN - FM \end{pmatrix}.$$

The mean curvature is

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = \frac{1}{2}\, trace(S).$$

5.1.1 Example (Minimal Surface)

The helicoid of parametrizations

$$X(s, t) = (s\cos(t), s\sin(t), ht), \qquad s > h > 0$$

$$X_s = (\cos(t), \sin(t), 0)$$

$$X_t = (-s\sin(t), s\cos(t), h)$$

$$E = < X_s, X_s > = 1$$

$$F = < X_s, X_t > = 0$$

$$G = < X_t, X_t > = s^2 - h^2$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s^2 - h^2 \end{pmatrix}.$$

Thus the det $I = EG - F^2 = s^2 - h^2 > 0$, since $s > h > 0$.

$$n = \frac{X_s \times X_t}{|X_s \times X_t|}$$

$$n = \frac{1}{\sqrt{s^2 - h^2}} (h\sin(t), -h\cos(t), 0)$$

$$X_{ss} = (0, 0, 0)$$

$$X_{st} = (-\sin(t), \cos(t), 0)$$

$$X_{tt} = (-s\cos(t), -s\sin(t), 0)$$

$$L = < n, X_{ss} > = 0$$

$$M = < n, X_{st} > = \frac{-h}{\sqrt{s^2 - h^2}}$$

$$N = < n, X_{tt} > = 0$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 0 & \frac{-h}{\sqrt{s^2 - h^2}} \\ \frac{-h}{\sqrt{s^2 - h^2}} & 0 \end{pmatrix}$$

$$\det II = -\frac{h^2}{s^2 - h^2}$$

$$\mathrm{K} = \frac{\det II}{\det I} = -\frac{h^2}{(s^2 - h^2)^2}$$

$$\boldsymbol{H} = \frac{EN + GL - 2MF}{2(EG - F^2)} = 0.$$

$\boldsymbol{H} = 0 \Rightarrow X(s, t\ )$ is minimal surface.

In conclusion, all of the helicoids variety are minimal surfaces.

5.1.2 Isothermal Patch

If $E = G$ and $F = 0$ then $\vec{x}(u, v)\colon \Omega \to M$ is a patch such that it is called an isothermal patch. We can say that geometrically means $\vec{x}_u$ and $\vec{x}_v$ are orthogonal, therefore angles are preserved $\vec{x}$ stretches the patch the same amount in the u and v directions.

$$\vec{x} \text{ is an isothermal patch } \Rightarrow \boldsymbol{H} = \frac{EN + EL}{2(E^2)} = \frac{N + L}{2E}.$$

Any surface can be parametrized using an isothermal patch. Every minimal surface IR^3 has locally isothermal parametrization. (Oprea, 2007)

5.1.2.1 Theorem

If $\vec{x}(u, v)$ is isothermal, then $\Delta\vec{x} = (2E\boldsymbol{H}).\vec{n}.$ (Oprea, 2007)

Proof:

We know that $\Delta x = x_{uu} + x_{vv}$

$$x_{uu} = \frac{E_u}{2E}\, x_u - \frac{E_v}{2G}\, x_v + L\vec{n}$$

$$x_{uv} = \frac{E_v}{2E}\, x_u - \frac{G_u}{2G}\, x_v + M\vec{n}$$

$$x_{vv} = -\frac{G_u}{2E}\, x_u + \frac{G_v}{2G}\, x_v + N\vec{n}$$

Since $\vec{x}$ is isothermal $E = G$ and $F = 0$

$$\Delta x = x_{uu} + x_{vv}$$

$$\Delta x = \left(\frac{E_u}{2E}\, x_u - \frac{E_v}{2G}\, x_v + L\vec{n}\right) + \left(-\frac{G_u}{2E}\, x_u + \frac{G_v}{2G}\, x_v + N\vec{n}\right)$$

$$\Delta x = (L + N).\vec{n}$$

We know that

$$\boldsymbol{H} = \frac{N + L}{2E} \Rightarrow L + N = 2E\boldsymbol{H}.$$

$\Delta\vec{x} = (\boldsymbol{2EH}).\vec{n}$. (Oprea, 2007)

5.1.2.2 Corollary

A surface $M: \vec{x}(u, v) = \big(x^1(u, v), x^2(u, v), x^3(u, v)\big)$, with isothermal coordinates is minimal if and only if x^1, x^2 and x^3 are all harmonic. (Oprea, 2007)

Proof:

($\Rightarrow$) If M is minimal, then $\boldsymbol{H} = 0 \Rightarrow \Delta\vec{x} = (2E\boldsymbol{H}).\vec{n} = 0 \Rightarrow x^1, x^2, x^3$ are harmonic.

($\Leftarrow$) x^1, x^2, x^3 are harmonic $\Rightarrow \Delta\vec{x} = 0 \Rightarrow (2E\boldsymbol{H}).\vec{n} = 0.\ \vec{n}$ is unit normal vector, So $\vec{n} \neq 0$ and $E = \langle \vec{x}_u, \vec{x}_u \rangle = |\vec{x}_u|^2 \neq 0$. Therefore $\boldsymbol{H} = 0 \Rightarrow M$ is minimal.

For a curve (α) parametrized by arc length,

$$\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{d^2\alpha}{ds^2}\right|.$$

Because $\vec{x}(u,v)$ is not parametrized by arc length, the principle curvatures are not exactly the magnitude of the second derivatives $|\vec{x}_{uu}|$ and $|\vec{x}_{vv}|$, but they are certainly related.

$$\vec{x}^{j}_{uu} + \vec{x}^{j}_{vv} = 0 \text{ for } j \in \{1,2,3\} \Rightarrow k_1 + k_2 = 0 \text{ and vice versa.}$$

5.1.3 Transition to Holomorphic Functions from Isothermal Patches

Let M be a minimal surface described by isothermal patch $\vec{x}(u,v)$. Let $z = u + iv$ and $\bar{z} = u - iv$, so then

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Notice that $z + \bar{z} = 2u$ and $z - \bar{z} = 2vi$, so

$$u = \frac{z+\bar{z}}{2} \text{ and } v = \frac{z-\bar{z}}{2i}.$$

This means that $\vec{x}(u,v)$ may be written as

$$\vec{x}(z,\bar{z}) = \left(x^1(z,\bar{z}), x^2(z,\bar{z}), x^3(z,\bar{z})\right),$$

and the derivative of j^{th} components is

$$\frac{\partial x^j}{\partial z} = \frac{1}{2}\left(x^j_u - i x^j_v\right).$$

Define

$$\phi = \frac{\partial \vec{x}}{\partial z} = (x^1_z, x^2_z, x^3_z)$$

$$(\phi)^2 = (x^1_z)^2 + (x^2_z)^2 + (x^3_z)^2.$$

Then

$$(\phi^j)^2 = (x^j_z)^2 = \left(\tfrac{1}{2}\left(x^j_u - i x^j_v\right)\right)^2 = \tfrac{1}{4}\left((x^j_u)^2 - (x^j_v)^2 - 2i x^j_u x^j_v\right), \text{ so}$$

$$(\phi)^2 = \frac{1}{4}\sum_{j=1}^{3}\left((x^j_u)^2 - (x^j_v)^2 - 2i x^j_u x^j_v\right)$$

$$(\phi)^2 = \frac{1}{4}\left(|\vec{x_u}|^2 - |\vec{x_v}|^2 - 2i\,\vec{x_u}.\,\vec{x_v}\right)$$

$$(\phi)^2 = \frac{1}{4}(E - G - 2iF).$$

Since $\vec{x}$ is isothermal,

$$(\phi)^2 = \frac{1}{4}(E - E) = 0. \text{ (Oprea 2007)}$$

5.1.3.1 Lemma

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = \frac{1}{4}\Delta\vec{x}. \text{ (Oprea 2007), (Stein and Shakarchi 2003)}$$

Proof:

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{1}{2}\left(\frac{\partial \vec{x}}{\partial u} - i\frac{\partial \vec{x}}{\partial v}\right)\right)$$

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = \frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial}{\partial u}\left(\frac{\partial \vec{x}}{\partial u} - i\frac{\partial \vec{x}}{\partial v}\right) + i\frac{\partial}{\partial v}\left(\frac{\partial \vec{x}}{\partial u} - i\frac{\partial \vec{x}}{\partial v}\right)\right)\right)$$

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = \frac{1}{4}\left(\frac{\partial^2 \vec{x}}{\partial u^2} - i\frac{\partial^2 \vec{x}}{\partial u \partial v} + i\frac{\partial^2 \vec{x}}{\partial u \partial v} + \frac{\partial^2 \vec{x}}{\partial v^2}\right)$$

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = \frac{1}{4}\left(\frac{\partial^2 \vec{x}}{\partial u^2} + \frac{\partial^2 \vec{x}}{\partial v^2}\right)$$

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = \frac{1}{4}\Delta\vec{x}.$$

5.1.3.2 Theorem

Assume M is a surface with patch $\vec{x}$. Let $\vec{\phi} = \frac{\partial \vec{x}}{\partial z}$ and suppose $(\phi)^2 = 0$ (i.e., $\vec{x}$ is isothermal). M is minimal $\Leftrightarrow$ each ϕ^j is holomorphic. (Oprea, 2007)

Proof:

$$f \text{ is holomorphic} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \text{ (Stein and Shakarchi, 2003)}$$

($\Rightarrow$) M is minimal $\Rightarrow$ x^j is harmonic for $j \in \{1, 2, 3\}$.

x^j harmonic $\Rightarrow$ $\Delta\vec{x} = 0 \Rightarrow \frac{1}{4}\Delta\vec{x} = 0 \Rightarrow \frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = 0$ by the 5.1.2.1 lemma

Because $\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = 0, \phi^j$ is holomorphic.

($\Leftarrow$) ϕ^j is holomorphic $\Rightarrow$ $\frac{\partial \vec{\phi}}{\partial \bar{z}} = 0.$

$$\frac{\partial \vec{\phi}}{\partial \bar{z}} = 0 \Rightarrow \frac{\partial}{\partial \bar{z}}\left(\frac{\partial \vec{x}}{\partial z}\right) = \frac{1}{4}\Delta \vec{x} = 0 \Rightarrow \Delta \vec{x} = 0 \Rightarrow \forall\, x^j \text{ harmonic } \Rightarrow M \text{ is minimal.}$$

5.1.3.3 Corollary

$$x^j(z,\bar{z}) = c_j + 2Re(\int \phi^j dz\,).\text{(Oprea 2007)}$$

Proof:

$$z = u + iv \Rightarrow dz = du + idv$$

$$\phi^j dz = \frac{1}{2}\left(x_u^j - ix_v^j\right)(du + idv) = \frac{1}{2}\left(x_u^j du + x_v^j dv\right) + \frac{1}{2}i(x_u^j dv - x_v^j du)$$

$$\bar{\phi}^j d\bar{z} = \frac{1}{2}\left(x_u^j + ix_v^j\right)(du - idv) = \frac{1}{2}\left(x_u^j du + x_v^j dv\right) - \frac{1}{2}i(x_u^j dv - x_v^j du)$$

Then we have

$$dx^j = \frac{\partial x^j}{\partial z}dz + \frac{\partial x^j}{\partial \bar{z}}d\bar{z}$$

$$dx^j = \phi^j dz + \bar{\phi}^j d\bar{z}$$

$$dx^j = \frac{1}{2}\left(x_u^j du + x_v^j dv\right) + \frac{1}{2}\left(x_u^j du + x_v^j dv\right)$$

$$dx^j = x_u^j du + x_v^j dv$$

$$dx^j = 2Re\left(\phi^j dz\right) \Rightarrow x^j = 2Re\left(\int \phi^j dz\right) + c_j \quad, c_j \text{ is constant.}$$

5.1.4 Weierstrass-Enneper Representations of Minimal Surfaces

We will give the Weierstrass-Enneper Representation for minimal surfaces might be established.

5.1.4.1 Theorem (The Weierstrass-Enneper Representation I)

If f is holomorphic on a domain D, g is meromorphic on , and fg^2 is holomorphic on D, then a minimal surface is defined by

$$\vec{x}(z,\bar{z}) = \left(x^1(z,\bar{z}), x^2(z,\bar{z}), x^3(z,\bar{z})\right),$$

where

$$x^1(z,\bar{z}) = Re\left(\int \frac{1}{2} f(1-g^2)\, dz \right)$$

$$x^2(z,\bar{z}) = Re\left(\int \frac{i}{2} f(1+g^2)\, dz \right)$$

$$x^3(z,\bar{z}) = Re\left(\int fg\, dz \right).$$

(Oprea, 2007), (Weinstein, 1996)

Proof:

We know that M is a minimal surface defined by isothermal parametrizations $\vec{x}(z,\bar{z})$. Since M is minimal we know that ϕ^js are complex analytic functions. Since $\vec{x}$ is isothermal we have

$$(\phi)^2 = \frac{1}{4}\sum_{j=1}^{3}((x_u^j)^2 - (x_v^j)^2 - 2ix_u^j x_v^j) = 0.$$

Since M is minimal we have

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0$$

$$(\phi^1)^2 + (\phi^2)^2 = -(\phi^3)^2$$

$$(\phi^1 + i\phi^2).(\phi^1 - i\phi^2) = -(\phi^3)^2.$$

For non-zero $\phi^1 - i\phi^2$. Let $f = \phi^1 - i\phi^2$ and $g = \frac{\phi^3}{f}$.

$$\frac{(\phi^1 + i\phi^2).(\phi^1 - i\phi^2)}{(\phi^1 - i\phi^2)} = \frac{-(\phi^3)^2}{(\phi^1 - i\phi^2)}$$

$$(\phi^1 + i\phi^2) = \frac{-(\phi^3)^2}{(\phi^1 - i\phi^2)}$$

$$(\phi^1 + i\phi^2) = -\underbrace{\phi^3}_{fg}.\underbrace{\frac{\phi^3}{(\phi^1 - i\phi^2)}}_{f}$$

$$(\phi^1 + i\phi^2) = -fg.g = -fg^2$$

We have $\phi^1 - i\phi^2 = f$ and $\phi^1 + i\phi^2 = -fg^2$

$$(\phi^1 + i\phi^2) + (\phi^1 - i\phi^2) = -fg^2 + f \Rightarrow 2\phi^1 = f(1 - g^2)$$

$$\phi^1 = \frac{1}{2}f(1-g^2) \Rightarrow x^1(z,\bar{z}) = Re\left(\int \frac{1}{2}f(1-g^2)\,dz\right)$$

$$(\phi^1 + i\phi^2) - (\phi^1 - i\phi^2) = (-fg^2) - f \Rightarrow 2i\phi^2 = -f(1+g^2)$$

$$\phi^2 = \frac{1}{2}if(1+g^2) \Rightarrow x^2(z,\bar{z}) = Re\left(\int \frac{i}{2}f(1+g^2)\,dz\right)$$

$$g = \frac{\phi^3}{f} \Rightarrow \phi^3 = fg \Rightarrow x^3(z,\bar{z}) = Re\left(\int fg\,dz\right).$$

5.1.4.2 Theorem (The Weierstrass-Enneper Representation II)

For any holomorphic function $F(\tau)$, a minimal surface is defined by

$$\vec{x}(z,\bar{z}) = \left(x^1(z,\bar{z}), x^2(z,\bar{z}), x^3(z,\bar{z})\right)$$

where

$$x^1(z,\bar{z}) = Re\left(\int (1-\tau^2)\,F(\tau)d\tau\right)$$

$$x^2(z,\bar{z}) = Re\left(\int i(1+\tau^2)\,F(\tau)d\tau\right)$$

$$x^3(z,\bar{z}) = Re\left(\int 2\tau\,F(\tau)d\tau\right).$$

(Oprea, 2007), (Weinstein, 1996)

Proof:

Suppose in The Weierstrass-Enneper Representation I using only one holomorphic function that is a composition of functions. g is holomorphic with g^{-1} is also holomorphic $\Rightarrow$ we consider g as a new complex variable $\tau = g$ with $d\tau = g'dz$ $\left(\text{which means } \frac{d\tau}{dz} = \frac{dg}{dz}\right)$. Define

$$F(\tau) = \frac{f}{g'} \quad \text{and obtain } F(\tau)d\tau = fdz.$$

Substitute τ for g and $F(\tau)d\tau$ for fdz in the Weierstrass-Enneper Representation I , we get

$$x^1(z,\bar{z}) = Re\left(\int (1-\tau^2)\, F(\tau) d\tau\right)$$

$$x^2(z,\bar{z}) = Re\left(\int i(1+\tau^2)\ F(\tau) d\tau\right)$$

$$x^3(z,\bar{z}) = Re\left(\int 2\tau\, F(\tau) d\tau\right).$$

5.1.4.2.1 Example

The most common parametrization for Enneper's surface is

$$\overrightarrow{x}(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, -v - u^2v + \frac{1}{3}v^3, u^2 - v^2\right).$$

First show that this is an isothermal patch.

$$\vec{x}_u = (1 - u^2 + v^2, -2uv, 2u\,)$$

$$\vec{x}_v = (2uv, -1 - u^2 + v^2, -2v\,)$$

$$E = <\vec{x}_u\,, \vec{x}_u> = 1 + 2u^2 + 2v^2 + 2u^2v^2 + u^4 + v^4$$

$$G = <\vec{x}_v\,, \vec{x}_v> = 1 + 2u^2 + 2v^2 + 2u^2v^2 + u^4 + v^4$$

$$F =< \vec{x}_v\,, \vec{x}_v> = 2uv(1 - u^2 + v^2) - 2uv(-1 - u^2 + v^2) - 4uv = 0.$$

Because $E = G$ and $= 0, \overrightarrow{x}(u,v)$ is isothermal. Let $z = u + iv$ and $\vec{\phi} = \vec{x}_u - i\vec{x}_v$. Then

$$\vec{\phi} = (1 - u^2 + v^2, -2uv, 2u\,) - i(2uv, -1 - u^2 + v^2, -2v\,)$$

$$\vec{\phi} = \big((1 - u^2 + v^2 - i2uv), -2uv - i(-1 - u^2 + v^2), (2u + i2v)\big)$$

$$\vec{\phi} = \big(1 - (u^2 + i2uv - +v^2), -2uv + i + iu^2 - iv^2, 2(u + iv)\big)$$

$$\vec{\phi} = (1 - (u + iv)^2, i(1 + u^2 + i2uv - v^2), 2(u + iv)\,)$$

$$\vec{\phi} = (1 - (u + iv)^2, i(1 + (u + iv)^2\,), 2(u + iv)\,)$$

$$\vec{\phi} = (1 - z^2, i(1 + z^2), 2z\ \,)$$

Note that $\phi^1(z) = 1 - z^2$, $\phi^2(z) = 1 + z^2$, and $\phi^3(z) = 2z$ are all holomorphic. (Koreavar, 2002)

Now we will examine the reversal. We know $\vec{\phi} = \vec{x}_u - i\vec{x}_v$ and $\phi^1(z) = 1 - z^2$, $\phi^2(z) = 1 + z^2$, and $\phi^3(z) = 2z$, and we want $\vec{x}(u, v)$ to be real-valued. Let

(1) $x^1 = Re(\int (1 - z^2)\, dz\,)$

$$x^1 = Re\left(z - \frac{1}{3}\, z^3\right)$$

$$x^1 = Re(u + iv - \frac{1}{3}(u + iv)^3)$$

$$x^1 = Re\,\left(u + iv - \frac{1}{3}(u^3 + 3u^2vi - 3uv^2 + iv^3)\right)$$

$$x^1 = u - \frac{1}{3}u^3 + uv^2$$

(2) $x^2 = Re(\int i(1 + z^2)\, dz\,)$

$$x^2 = Re\left(\, i\left(z + \frac{1}{3}\, z^3\right)\right)$$

$$x^2 = Re\,(i\,(u + iv + \frac{1}{3}(u + iv)^3))$$

$$x^2 = Re\,\left(\,ui - v + \frac{1}{3}(u^3 + 3u^2vi - 3uv^2 + iv^3)\right)$$

$$x^2 = -\,v + \frac{1}{3}v^3 - uv^2$$

(3) $x^3 = Re\,(\,\int 2zdz\,)$

$$x^3 = Re\,(z^2)$$

$$x^3 = Re\,((u + iv\,)^2)$$

$$x^3 = Re\,(\,u^2 + 2uvi - v^2)$$

$$x^3 = u^2 - v^2$$

We get,

$$\vec{x}(u, v) = \left(\,u - \frac{1}{3}u^3 + uv^2, -v - u^2v + \frac{1}{3}v^3, u^2 - v^2\right)$$

which is Enneper's surface. (Koreavar, 2002)

The enneper's surface may be obtained from $f = 1$ and $g = z$.

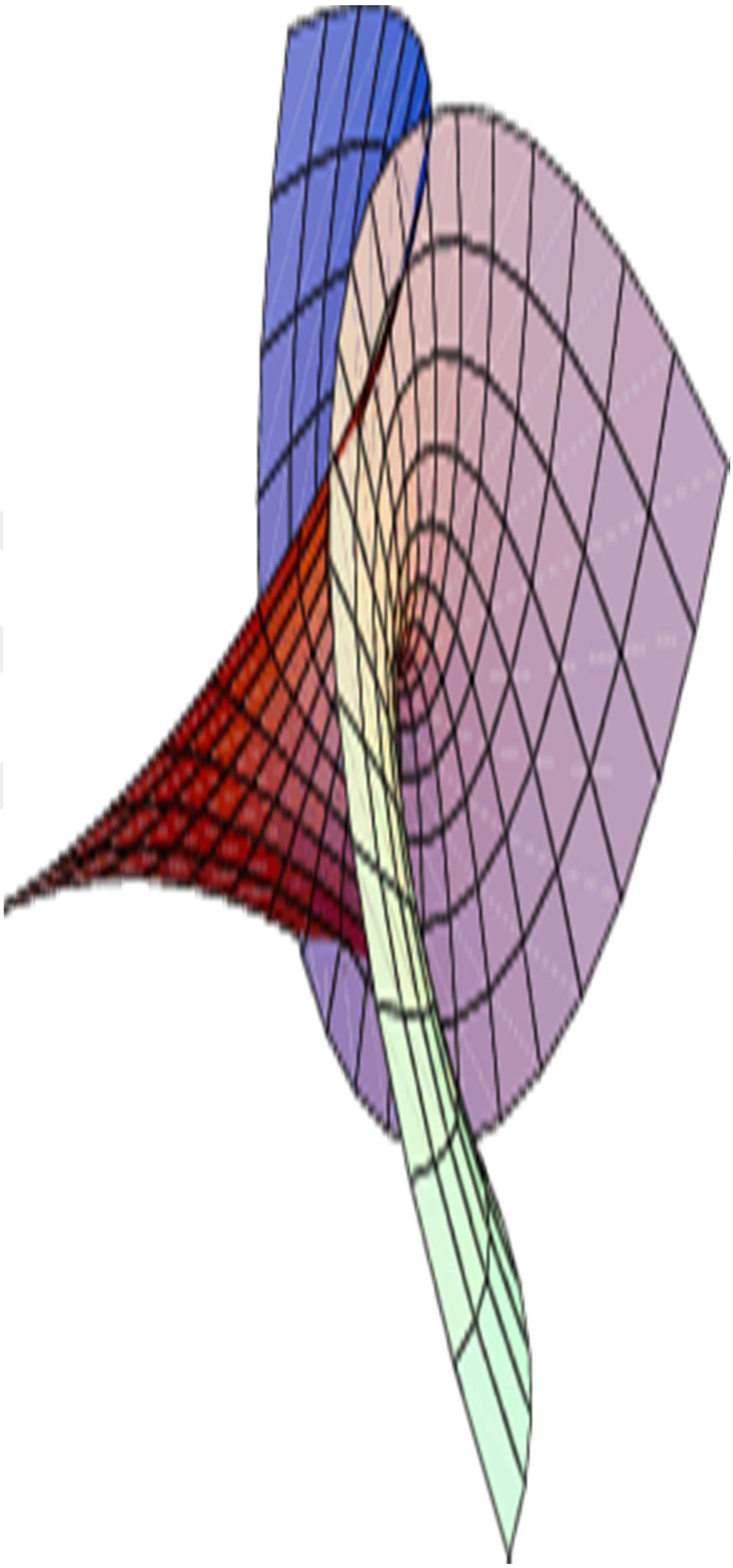


Figure 5.1. Enneper's Minimal Surface

5.1.4.2.2 Example

A helicoid may be obtained from $F(\tau) = \frac{i}{2\tau^2}$ where $\tau = e^z$. (Oprea, 2007)

Notice that $\tau = e^z, \tau^{-1} = Log\,(\mathrm{z})$, and $F(e^z) = \frac{i}{2e^{2z}}$ are all holomorphic on the domain of Log (z). I have used *Log* (z) instead of $log(z)$ because *Log* (z) is the principal branch of the *log* and branches of the *log* are holomorphic, but *log* itself is not.

(1) $x^1 = Re\left(\int (1-\tau^2)\frac{i}{2\tau^2}d\tau \right)$

$$x^1 = Re\left(\frac{-i}{2\tau} - \frac{i}{2}\tau\right)$$

$$x^1 = Re\left(-\frac{i}{2}(e^{-z} + e^z)\right)$$

$$x^1 = Re\left(-\frac{i}{2}\left(e^{-(u+iv)} + e^{(u+iv)}\right)\right)$$

$$x^1 = Re\left(-\frac{i}{2}\left(e^{-u}\big(\cos(-v) + isin(-v)\big) + e^u\big(\cos(v) + isin(v)\big)\right)\right)$$

$$x^1 = Re\left(-\frac{i}{2}e^{-u}\cos(-v) + \frac{1}{2}e^{-u}\sin(-v) - \frac{i}{2}e^u\cos(v) + \frac{1}{2}e^u\sin(v)\right)$$

$$x^1 = \frac{1}{2}e^{-u}\sin(-v) + \frac{1}{2}e^u\sin(v)$$

(2) $x^2 = Re\left(\int i(1+\tau^2)\frac{i}{2\tau^2}d\tau \right)$

$$x^2 = Re\left(\frac{1}{2\tau} - \frac{1}{2}\tau\right)$$

$$x^2 = Re\left(\frac{1}{2}(e^{-z} - e^z)\right)$$

$$x^2 = Re\left(\frac{1}{2}\left(e^{-(u+iv)} + e^{(u+iv)}\right)\right)$$

$$x^2 = Re\left(\frac{1}{2}\left(e^{-u}\big(\cos(-v) + isin(-v)\big) - e^u\big(\cos(v) + isin(v)\big)\right)\right)$$

$$x^2 = Re\left(\frac{1}{2}e^{-u}\cos(-v) + \frac{i}{2}e^{-u}\sin(-v) - \frac{1}{2}e^u\cos(v) + \frac{i}{2}e^u\sin(v)\right)$$

$$x^2 = \frac{1}{2}e^{-u}\cos(-v) - \frac{1}{2}e^u\cos(v)$$

(3) $x^3 = Re\left(\int 2\tau \left(\frac{i}{2\tau^2}\right) d\tau \right)$

$$x^3 = Re(i\, Log\, |\, \tau\, |\,)$$

$$x^3 = Re(i\, Log\, |\, e^z\, |\,)$$

$$x^3 = Re(iz)$$

$$x^3 = Re\big(i(u + iv\,)\big)$$

$$x^3 = Re(iu - v)$$

$$x^3 = -v$$

So $\vec{x}(u, v) = \left(\frac{1}{2} e^{-u} \sin(-v) + \frac{1}{2} e^{u} \sin(v)\,, \frac{1}{2} e^{-u} \cos(-v) - \frac{1}{2} e^{u} \cos\,(v)\,, -v \right)$

is an isothermal patch for the helicoid.

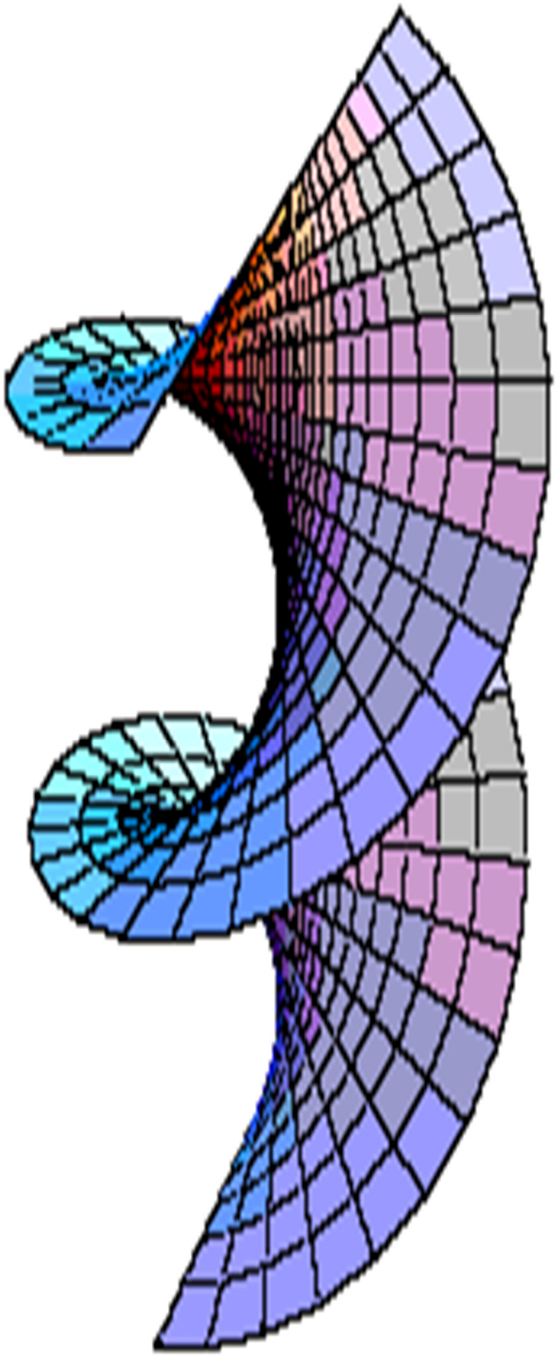


Figure 5.2. Helicoid of Minimal Surface

5.2 Maximal Surface In E_1^3

A spacelike surface in $\boldsymbol{H} = 0$ is called a maximal surfaces.

We will give Weierstrass-Enneper representation for these surfaces and also explain with examples.

5.2.1 Weierstrass-Enneper Formulas for Maximal Surfaces E_1^3

For a space-like surface in E_1^3, the Gauss map is described to be a mapping which appoints to each point of the surface the unit normal vector at the point.

Therefore

$$\mathbb{H}^2 = \{ (x, y, z) \in E_1^3 | \, x^2 + y^2 - z^2 = -1, z > 0 \},$$

which has constant negative curvature -1 according to the induced metric. We describe a stereographic mapping σ for $\mathbb{H}^2$

$$\sigma: C \backslash \{|z| = 1\} \to \mathbb{H}^2 ; \;\; z \to \left(\frac{-2\, Re\, z}{|z|^2 - 1}, \frac{-2\, Im\, z}{|z|^2 - 1}, \frac{|z|^2 + 1}{|z|^2 - 1} \right) \text{ and } \sigma(\infty) = (0,0,1).$$

$\sigma(z)$ is the intersection of $\mathbb{H}^2$ and the line joining $(Re\, z, Im\, z, 0)$ and the "north pole" $(0,0,1)$ of $\mathbb{H}^2$. (Kobayashi 1983)

5.2.1.1 Theorem (Weierstrass-Enneper Formula of 1st Kind)

Any maximal spacelike surface in E_1^3 is represented as

$$\phi(z) = Re \int \left(\frac{1}{2} f(1 + g^2), \frac{i}{2} f(1 - g^2), -fg \right) dz, \qquad z \in D, \quad (*)$$

where D is a domain in C, and f (respectively g) is a holomorphic (respectively meromorphic) function on D such that $|g(z)| \neq 1$ for $z \in D$. Morever,

(1) The Gauss map n is given by $n(z) = \sigma\big(g(z)\big)$, where σ is a map defined

$$\sigma: C \backslash \{|z| = 1\} \to \mathbb{H}^2 ;$$

$$z \to \left(\frac{-2\, Re\, z}{|z|^2 - 1}, \frac{-2\, Im\, z}{|z|^2 - 1}, \frac{|z|^2 + 1}{|z|^2 - 1} \right) \text{ and } \sigma(\infty) = (0,0,1) ;$$

(2) The induced metric is

$$ds = \left(\frac{|f|.|1-|g|^2}{2}\right)|dz|$$

(3) The Gauss curvature is

$$\boldsymbol{K} = \left\{\frac{4|g'|}{|f|(1-|g|^2)^2}\right\}^2.$$

(Kobayashi, 1983)

Proof:

Suppose that $\phi: D \to E_1^3$ is a maximal space-like surface. From the maximality, $\Delta\phi = 0$ where Δ is the Laplacian defined by the induced metric on D, which is a positive definite Riemannian metric. In particular, D can not be a closed surface. Thus, taking the universal covering of D.We might suppose D is domain in ₵ and that ϕ is a conformal mapping. Set

$$2\partial_z\phi = (\phi^1, \phi^2, \phi^3) \text{ where } \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad z = u + iv.$$

Then, the conformality of ϕ implies that $(\phi^1)^2 + (\phi^2)^2 - (\phi^3)^2 = 0$, and $\Delta\phi = 0$ implies that $\bar{\partial}_z\partial_z\phi = 0$, i.e., $\phi^i$are holomorphic.

$$(\phi^1)^2 + (\phi^2)^2 - (\phi^3)^2 = 0$$

$$(\phi^1)^2 + (\phi^2)^2 = (\phi^3)^2$$

$$(\phi^1 + i\phi^2).(\phi^1 - i\phi^2) = (\phi^3)^2$$

Let $f = \phi^1 - i\phi^2$ and $g = -\frac{\phi^3}{f}$.

$$\frac{(\phi^1 + i\phi^2).(\phi^1 - i\phi^2)}{(\phi^1 - i\phi^2)} = \frac{(\phi^3)^2}{(\phi^1 - i\phi^2)}$$

$$(\phi^1 + i\phi^2) = \frac{(\phi^3)^2}{(\phi^1 - i\phi^2)}$$

$$(\phi^1 + i\phi^2) = -(\phi^3).\left(\frac{-\phi^3}{\phi^1 - i\phi^2}\right)$$

$$(\phi^1 + i\phi^2) = fg.g = fg^2$$

We have $\phi^1 + i\phi^2 = fg^2$ and $\phi^1 - i\phi^2 = f$

$$(\phi^1 + i\phi^2) + (\phi^1 - i\phi^2) = fg^2 + f \Rightarrow 2\phi^1 = f(1 + g^2)$$

$$2\partial_z \phi = (\phi^1, \phi^2, \phi^3)$$

$$\phi^1 = \frac{1}{2} f(1+g^2) \Rightarrow \phi^1(z) = Re\left(\int \frac{1}{2} f(1+g^2)\, dz \right)$$

$$(\phi^1 + i\phi^2) - (\phi^1 - i\phi^2) = (fg^2) - f = fg^2 - f \Rightarrow 2i\phi^2 = f(1-g^2)$$

$$\phi^2 = \frac{1}{2} if(1-g^2) \Rightarrow \phi^2(z) = Re\left(\int \frac{i}{2} f(1-g^2)\, dz \right)$$

$$g = \frac{-\phi^3}{f} \Rightarrow \phi^3 = -fg \Rightarrow \phi^3(z) = Re\left(\int -fg\, dz \right).$$

Holomorphic function $F(\tau)$, a maximal surface is

$$\phi^1(z) = Re\left(\int (1+\tau^2)\, F(\tau) d\tau \right)$$

$$\phi^2(z) = Re\left(\int i(1-\tau^2)\ F(\tau) d\tau \right)$$

$$\phi^3(z) = Re\left(\int -2\tau\, F(\tau) d\tau \right).$$

Suppose in the Weierstrass-Enneper Representation of the 1^{st} kind using only one holomorphic function that is a composition of functions. g is holomorphic and with g^{-1} also holomorphic $\Rightarrow$ we consider g as a new complex variable $\tau = g$ with $d\tau = g' dz$ $\left(\text{which means } \frac{d\tau}{dz} = \frac{dg}{dz}\right)$. Define

$$F(\tau) = \frac{f}{2g'} \quad \text{and obtain } F(\tau) d\tau = \frac{f}{2} dz.$$

Substitute τ for g and $F(\tau)d\tau$ for fdz in the Weierstrass-Enneper Representation of the 1^{st} kind then we get

$$\phi^1(z) = Re\left(\int (1+\tau^2)\, F(\tau) d\tau \right)$$

$$\phi^2(z) = Re\left(\int i(1-\tau^2)\ F(\tau) d\tau \right)$$

$$\phi^3(z) = Re\left(\int -2\tau\, F(\tau) d\tau \right).$$

(1) Unit normal vector is defined as

$$n = \frac{\phi_u \times_L \phi_v}{|\phi_u \times_L \phi_v|_L}.$$

$$\phi_u \times_L \phi_v = \begin{vmatrix} i & j & -k \\ 2\,Re\,\phi^1 & 2\,Re\,\phi^2 & 2\,Re\,\phi^3 \\ 2\,Im\,\phi^1 & 2\,Im\,\phi^2 & 2\,Im\,\phi^3 \end{vmatrix}$$

$$\phi_u \times_L \phi_v = -4(Re\,\phi^2 Im\,\phi^3 - Re\,\phi^3 Im\,\phi^2, Re\,\phi^3 Im\,\phi^1 - Re\,\phi^1 Im\,\phi^3, Re\,\phi^2 Im\,\phi^1 - Re\,\phi^1 Im\,\phi^2)$$

$$\phi_u \times_L \phi_v = -4(\,Im(\phi^2.\bar{\phi}^3), Im(\phi^3\bar{\phi}^1), Im\,(\phi^2\bar{\phi}^1)\,)$$

We know that $\phi^1 = \frac{1}{2}f(1+g^2), \phi^2 = \frac{1}{2}if(1-g^2)$, and $\phi^3 = -fg$

$$\phi^2\bar{\phi}^3 = \frac{1}{2}if(1-g^2).\left(-\overline{fg}\,\right) = -\frac{1}{2}if(1-g^2).\left(\bar{f}.\bar{g}\right)$$

$$\phi^2\bar{\phi}^3 = -\frac{1}{2}if.\bar{f}(1-g^2).\bar{g} = -\frac{i}{2}|f|^2(\bar{g}-\bar{g}g^2)$$

$$\phi^2\bar{\phi}^3 = \frac{1}{2}|f|^2(-i.\bar{g}+i.\bar{g}.g.g) = \frac{1}{2}|f|^2(\,-i.\bar{g}+i\bar{g}|g|^2)$$

$$Im(\phi^2.\bar{\phi}^3) = Im\left[\frac{1}{2}|f|^2(\,-i.\bar{g}+i\bar{g}|g|^2)\right]$$

$$Im(\phi^2.\bar{\phi}^3) = \frac{1}{2}|f|^2 Im[(\,-i.\bar{g}+i\bar{g}|g|^2)]$$

$$Im(\phi^2.\bar{\phi}^3) = \frac{1}{2}|f|^2[Im(-i.\bar{g}) + |g|^2.Im(i.\bar{g})]$$

$$Im(\phi^2.\bar{\phi}^3) = \frac{1}{2}|f|^2[-Re\,(g) + |g|^2.Re(g)]$$

$$Im(\phi^2.\bar{\phi}^3) = \frac{1}{2}|f|^2.Re(g)[|g|^2-1]$$

$$\phi^3\bar{\phi}^1 = -fg\left[\frac{1}{2}\,\overline{f(1+g^2)}\right] = -\frac{1}{2}fg[\,\bar{f}.\overline{(1+g^2)}\,]$$

$$\phi^3\bar{\phi}^1 = -\frac{1}{2}fg[\,\bar{f}.(1+\bar{g}^2)] = -\frac{1}{2}f.\bar{f}\,[\,g.(1+\bar{g}^2)\,]$$

$$\phi^3\bar{\phi}^1 = -\frac{1}{2}|f|^2[\,g+g.\bar{g}^2\,] = -\frac{1}{2}|f|^2[\,g+g.\bar{g}.\bar{g}]$$

$$\phi^3\bar{\phi}^1 = -\frac{1}{2}|f|^2(g+\bar{g}.|g|^2)$$

$$Im(\,\phi^3\bar{\phi}^1) = Im\left[-\frac{1}{2}|f|^2(g+\bar{g}.|g|^2)\right]$$

$$Im(\,\phi^3\bar{\phi}^1) = -\frac{1}{2}|f|^2 Im\,[(g+\bar{g}.|g|^2)]$$

$$Im(\,\phi^3\bar{\phi}^1) = -\frac{1}{2}|f|^2[\,Im\,(g) + |g|^2.Im\,(\bar{g}\,)]$$

$$Im(\,\phi^3\bar{\phi}^1) = -\frac{1}{2}|f|^2[\,Im\,(g) - |g|^2.Im\,(g)]$$

$$Im(\phi^3\bar{\phi}^1) = \frac{1}{2}|f|^2[-Im\,(g) + |g|^2.Im\,(g)]$$

$$Im(\phi^3\bar{\phi}^1) = \frac{1}{2}|f|^2 Im\,(g)[\,|g|^2 - 1\,]$$

$$\phi^2\bar{\phi}^1 = \frac{i}{2}\,f(1-g^2).\left[\frac{1}{2}\,\overline{f(1+g^2)}\right] = \frac{i}{4}\,f(1-g^2).[\,\bar{f}.\overline{(1+g^2)}\,]$$

$$\phi^2\bar{\phi}^1 = \frac{i}{4}\,f.\bar{f}.(1-g^2)(1+\bar{g}^2) = \frac{i}{4}|f|^2(1+\bar{g}^2-g^2-g^2.\bar{g}^2)$$

$$\phi^2\bar{\phi}^1 = \frac{i}{4}|f|^2(1+\bar{g}^2-g^2-|g|^4\,)$$

$$\phi^2\bar{\phi}^1 = \frac{1}{4}|f|^2(i+i.\bar{g}^2-i.g^2-i.|g|^4\,)$$

$$Im(\phi^2\bar{\phi}^1) = Im\left[\frac{1}{4}|f|^2(i+i.\bar{g}^2-i.g^2-i.|g|^4\,)\right]$$

$$Im(\phi^2\bar{\phi}^1) = \frac{1}{4}|f|^2[\,Im\,(i+i.\bar{g}^2-i.g^2-i.|g|^4\,)]$$

$$Im(\phi^2\bar{\phi}^1) = \frac{1}{4}|f|^2[\,Im\,(\,i) + Im(i.\bar{g}^2) - Im(i.g^2) - |g|^4.Im(i)\,]$$

$$Im(\phi^2\bar{\phi}^1) = \frac{1}{4}|f|^2[\,1 + Im(g) - Im(g) - |g|^4.1\,]$$

$$Im(\phi^2\bar{\phi}^1) = \frac{1}{4}|f|^2(1-|g|^4\,)$$

$$Im(\phi^2\bar{\phi}^1) = -\frac{1}{4}|f|^2(|g|^4-1\,)$$

$$Im(\phi^2\bar{\phi}^1) = -\frac{1}{4}|f|^2.[\,|g|^2-1\,].[\,|g|^2+1\,]$$

$$\phi_u \times_L \phi_v = -4(\,Im(\phi^2.\bar{\phi}^3), Im(\phi^1\bar{\phi}^3), Im\,(\phi^2\bar{\phi}^1)\,)$$

$$\phi_u \times_L \phi_v = |f|^2.[\,|g|^2-1\,].(-2\,Re(g), -2\,Im\,(g),\ |g|^2+1\ )$$

We use the notation $< n, n\ >_L = \epsilon = -1$ since the surface is space-like. Here

$$|\phi_u \times_L \phi_v|_L = \sqrt{-\epsilon(EG-F^2)} = \sqrt{-\epsilon\det(I)} = \sqrt{\det(I)} = \sqrt{(E.E)} = E.$$

Since $\phi(u,v)$ is isothermal. We define $|\phi_u \times_L \phi_v|_L = E = \lambda^2$ and we shall find $E = \lambda^2$.

$$\phi = \frac{\partial x}{\partial z} = \left(\frac{\partial x_1}{\partial z}, \frac{\partial x_2}{\partial z}, \frac{\partial x_3}{\partial z}\right) = (\phi^1, \phi^2, \phi^3)$$

$$|\phi|^2 = |\phi^1|^2 + |\phi^2|^2 - |\phi^3|^2$$

Since *x* is isothermal parametrization

$$E = |x_u|^2 = |x_v|^2 = G \text{ and } F = < x_u, x_v >_L = 0.$$

Let $\lambda^2 = |x_u|^2 = |x_v|^2$

$$|\phi|^2 = |\phi^1|^2 + |\phi^2|^2 - |\phi^3|^2 = \left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 - \left|\frac{\partial x_3}{\partial z}\right|^2$$

$$|\phi|^2 = \left|\frac{1}{2}(\frac{\partial x_1}{\partial u} - i\frac{\partial x_1}{\partial v})\right|^2 + \left|\frac{1}{2}(\frac{\partial x_2}{\partial u} - i\frac{\partial x_2}{\partial v})\right|^2 - \left|\frac{1}{2}(\frac{\partial x_3}{\partial u} - i\frac{\partial x_3}{\partial v})\right|^2$$

We note that

$$\left|\frac{1}{2}(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v})\right|^2 = \left[\frac{1}{2}(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v})\right].\left[\frac{1}{2}(\frac{\partial x_k}{\partial u} + i\frac{\partial x_k}{\partial v})\right].$$

$$\left|\frac{1}{2}(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v})\right|^2 = \frac{1}{4}\left[(\frac{\partial x_k}{\partial u})^2 + (\frac{\partial x_k}{\partial v})^2\right] \text{ for } k = 1, 2, 3$$

$$|\phi|^2 = \sum_{k=1}^{3} \frac{1}{4}\left[(\frac{\partial x_k}{\partial u})^2 + (\frac{\partial x_k}{\partial v})^2\right] = \frac{1}{4}|x_u|^2 + \frac{1}{4}|x_v|^2$$

$$|\phi|^2 = \frac{1}{4}E + \frac{1}{4}E = \frac{1}{2}\lambda^2$$

$$\lambda^2 = 2|\phi|^2 = 2(|\phi^1|^2 + |\phi^2|^2 - |\phi^3|^2)$$

We will use

$$\phi^1 = \tfrac{1}{2}(1+\tau^2)F(\tau), \phi^2 = \tfrac{1}{2}i(1-\tau^2)F(\tau), \phi^3 = -\tau F(\tau).$$

Then letting $F = (F(\tau))$

$$\lambda^2 = 2\left(|\frac{1}{2}(1+\tau^2)F(\tau)|^2 + |\frac{1}{2}i(1-\tau^2)F(\tau)|^2 - |-\tau F(\tau)|^2\right)$$

$$\lambda^2 = 2.\frac{1}{4}|F|^2(|(1+\tau^2)|^2 + |i(1-\tau^2)|^2 - 4|-\tau|^2)$$

$$\lambda^2 = \frac{1}{2}|F|^2((1+\tau^2)(1+\bar{\tau}^2) + (1-\tau^2)(1-\bar{\tau}^2) - 4|\tau|^2)$$

$$\lambda^2 = \frac{1}{2}|F|^2(1 + (\tau^2+\bar{\tau}^2) + |\tau|^4 + 1 - (\tau^2+\bar{\tau}^2) + |\tau|^4 - 4|\tau|^2)$$

$$\lambda^2 = \frac{1}{2}|F|^2(2 - 4|\tau|^2 + 2|\tau|^4)$$

$$\lambda^2 = |F|^2(1 - 2|\tau|^2 + |\tau|^4)$$

$$\lambda^2 = |F|^2(1-|\tau|^2)^2 = E$$

$$n = \frac{\phi_u \times_L \phi_v}{|\phi_u \times_L \phi_v|_L}$$

$$\phi_u \times_L \phi_v = |f|^2.[\,|g|^2 - 1\,].(-2\,Re(g), -2\,Im\,(g),\, |g|^2 + 1\,)$$

$$|\phi_u \times_L \phi_v|_L = E = |F|^2(1 - |\tau|^2)^2 = |F(\tau)|^2(1 - |\tau|^2)^2$$

$$|\phi_u \times_L \phi_v|_L = E = |f|^2[\,|g|^2 - 1\,][\,|g|^2 - 1\,]$$

$$|\phi_u \times_L \phi_v|_L = |f|^2[\,|g|^2 - 1\,][\,|g|^2 - 1\,]$$

$$n = \frac{|f|^2.[\,|g|^2 - 1\,]}{|f|^2[\,|g|^2 - 1\,][\,|g|^2 - 1\,]}(-2\,Re(g), -2\,Im\,(g),\, |g|^2 + 1\,)$$

$$n = \left(\frac{-2\,Re(g)}{|g|^2 - 1}, \frac{-2\,Im\,(g)}{|g|^2 - 1}, \frac{|g|^2 + 1}{|g|^2 - 1}\right)$$

So $n(z) = \sigma\big(g(z)\big)$

(2) The induced metric is given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$ds^2 = E(du^2 + dv^2) \text{ since } E = G \text{ and } F = 0$$

$$ds^2 = \lambda^2|d\tau|^2$$

We know that $\lambda^2 = |F|^2(1 - |\tau|^2)^2$

$$ds^2 = |F|^2(1 - |\tau|^2)^2\,|d\tau|^2$$

We defined $\tau = g$ with $d\tau = g'dz$ $\left(\text{which means } \frac{d\tau}{dz} = \frac{dg}{dz}\right)$ and

$$F(\tau) = \frac{f}{2g'} \quad \text{and obtain } F(\tau)d\tau = \frac{f}{2}dz.$$

$$ds^2 = \frac{|f|^2}{4}(1 - |g|^2)^2\,|dz|^2$$

So

$$ds = \left(\frac{|f|.|1 - |g|^2|}{2}\right)|dz|.$$

(3) We know that Gauss Theorem Egregium. The Gauss curvature $\boldsymbol{K}$ depends on the metric $E, G, F = 0$:

$$\boldsymbol{K} = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right).$$

We know that is isothermal $E = G,\ F = 0$ and $\lambda^2 = E$.

$$\boldsymbol{K} = -\frac{1}{2\sqrt{\ \lambda^2\lambda^2}}\left(\frac{\partial}{\partial v}\left(\frac{(\lambda^2)_v}{\sqrt{\lambda^2\lambda^2}}\right)+\frac{\partial}{\partial u}\left(\frac{(\lambda^2\,)_u}{\sqrt{\lambda^2\lambda^2}}\right)\right)$$

$$\boldsymbol{K} = -\frac{1}{2\lambda^2}\left(\frac{\partial}{\partial v}\left(\frac{(\lambda^2)_v}{\sqrt{\lambda^2\lambda^2}}\right)+\frac{\partial}{\partial u}\left(\frac{(\lambda^2\,)_u}{\sqrt{\lambda^2\lambda^2}}\right)\right)$$

$$\boldsymbol{K} = -\frac{1}{2\lambda^2}\left(\frac{\partial^2}{\partial v^2}(In\,\lambda^2\,)+\frac{\partial^2}{\partial u^2}(In\,\lambda^2\,)\right)$$

$$\boldsymbol{K} = -\frac{1}{2\lambda^2}(\Delta\, In\,\lambda^2\ )$$

$$\boldsymbol{K} = -\frac{1}{\lambda^2}(\Delta\, In\,\lambda\ )$$

$$\lambda^2 = |F|^2(1-|\tau|^2)^2 \ \Rightarrow\ \lambda = |F|.\,|1-|\tau|^2|$$

$$\boldsymbol{K} = -\frac{1}{|F|^2(1-|\tau|^2)^2}(\,\Delta\, In(|F|.\,|1-|\tau|^2|)\,)$$

$$\boldsymbol{K} = -\frac{\Delta\, In|F| + \Delta\, In(|1-|\tau|^2|)}{|F|^2(1-|\tau|^2)^2}$$

We know that

$$\Delta h = 4\left(\frac{\partial}{\partial\bar{z}}\left(\frac{\partial h}{\partial z}\right)\right)$$

$$\Delta\, In|F| = 4\left(\frac{\partial}{\partial\bar{\tau}}\left(\frac{\partial(In|F|)}{\partial\tau}\right)\right) = 4\,\frac{\partial}{\partial\bar{\tau}}\left(\frac{\partial(In(F.\bar{F})^{1/2})}{\partial\tau}\right)$$

$$\Delta\, In|F| = 2\frac{\partial}{\partial\bar{\tau}}\left(\frac{\partial\big(In(F.\bar{F})\big)}{\partial\tau}\right) = 2\frac{\partial}{\partial\bar{\tau}}\left(\frac{\partial(In\,F + In\,\bar{F})}{\partial\tau}\right)$$

$$\Delta\, In|F| = 2\frac{\partial}{\partial\bar{\tau}}\left(\frac{\partial(In\,F)}{\partial\tau}+\frac{\partial(In\,\bar{F})}{\partial\tau}\right)$$

$$\Delta\, In|F| = 2\frac{\partial}{\partial\bar{\tau}}\left(\frac{F_\tau}{F}+\frac{\bar{F}_\tau}{\bar{F}}\right)$$

Since F is holomorphic, then $\bar{F}$ can not be holomorphic. (Oprea, 2007)

Thus, $(\bar{F}_\tau) = 0$. this implies that

$$\Delta\, In|F| = 2\frac{\partial}{\partial\bar{\tau}}\left(\frac{F_\tau}{F}\right) = 0$$

Since, F_τ, and, hence, $\frac{F_\tau}{F}$ are holomorphic, we also have that

$$\Delta\, In(|1-|\tau|^2|) = 4\left(\frac{\partial}{\partial\bar{\tau}}\left(\frac{\partial(In(|1-|\tau|^2|))}{\partial\tau}\right)\right)$$

$$\Delta\, In(|1-|\tau|^2|) = 4\frac{\partial}{\partial\bar{\tau}}\left(\frac{\partial(In(1-\tau.\bar{\tau}))}{\partial\tau}\right)$$

$$\Delta\, In(|1-|\tau|^2|) = 4\frac{\partial}{\partial\bar{\tau}}\left(-\frac{\bar{\tau}}{1-|\tau|^2}\right)$$

$$\Delta\, In(|1-|\tau|^2|) = -4\left(\frac{1-|\tau|^2+\tau.\bar{\tau}}{(1-|\tau|^2)^2}\right)$$

$$\Delta\, In(|1-|\tau|^2|) = -4\left(\frac{1-|\tau|^2+|\tau|^2}{(1-|\tau|^2)^2}\right)$$

$$\Delta\, In(|1-|\tau|^2|) = \left(\frac{-4}{(1-|\tau|^2)^2}\right)$$

$$\boldsymbol{K} = -\frac{\Delta\, In|F| + \Delta\, In(|1-|\tau|^2|)}{|F|^2(1-|\tau|^2)^2}$$

$$\boldsymbol{K} = -\frac{0+\left(\frac{-4}{(1-|\tau|^2)^2}\right)}{|F|^2(1-|\tau|^2)^2}$$

$$\boldsymbol{K} = \frac{4}{|F|^2(1-|\tau|^2)^4} = \frac{4}{|F(\tau)|^2(1-|\tau|^2)^4}$$

We defined $\tau = g$ with $d\tau = g'dz$ $\left(\text{which means } \frac{d\tau}{dz} = \frac{dg}{dz}\right)$ and

$$F(\tau) = \frac{f(z)}{2g'(z)} \quad \text{and obtain } F(\tau)d\tau = \frac{f}{2}dz$$

$$\boldsymbol{K} = \frac{4}{\left|\frac{f(z)}{2g'(z)}\right|^2(1-|g(z)|^2)^4} = \frac{4.4\,|g'(z)|^2}{|f(z)|^2(1-|g(z)|^2)^4}$$

$$\boldsymbol{K} = \left\{\frac{4|g'(z)|}{|f(z)|(1-|g(z)|^2)^2}\right\}^2.$$

As an immediate consequence.

5.2.1.2 Theorem (Weierstrass-Enneper Formula of 2^{nd} Kind)

Any maximal space-like surface in E_1^3 is represented as

$$\phi(z) = Re\int\left(\frac{1}{2}f(g^2+1), ifg, \frac{1}{2}f(g^2-1)\right)dz, \qquad Re\, g \neq 0, \ (**)$$

The Gauss map n is

$$n(z) = \sigma\left(\frac{(1-g)}{(1+g)}\right).$$ (Kobayashi, 1983)

Proof:

Replace f and g in $(*)$ by

$$\frac{f(1+g)^2}{2} \text{ and } \frac{(1-g)}{(1+g)}, \qquad \text{respectively}.$$

$$\phi^1 = \frac{1}{2}f(1+g^2) = \frac{1}{2}\left(\frac{f(1+g)^2}{2}\right).\left(1+\left(\frac{(1-g)}{(1+g)}\right)^2\right)$$

$$\phi^1 = \frac{1}{4}(f\,(1+g)^2).\left(\frac{2(1+g^2)}{(1+g)^2}\right) = \frac{1}{2}f(g^2+1).$$

$$\phi^2 = \frac{1}{2}if(1-g^2) = \frac{1}{2}i\left(\frac{f(1+g)^2}{2}\right).\left(1-\left(\frac{(1-g)}{(1+g)}\right)^2\right)$$

$$\phi^2 = \frac{1}{2}i\left(\frac{f(1+g)^2}{2}\right).\left(\frac{4g}{(1+g)^2}\right) = ifg$$

$$\phi^3 = -fg = -\left(\frac{f(1+g)^2}{2}\right).\left(\frac{(1-g)}{(1+g)}\right) = \frac{1}{2}f(g^2-1).$$

5.2.2 Examples Of Maximal Surfaces

5.2.2.1 Example

The first example of maximal surfaces in E_1^3 is a space-like plane. The only complete maximal surface in E_1^3 is plane with g is $constant\ in$ $(*)$ or $(**)$. (Calabi 1970, Cheng and Yau 1976)

5.2.2.2 Example (Enneper's Surface of 1st Kind)

Set $f = 2,\ g = z$ and $D = C \setminus \{\ |z| < 1\ \}$ in

$$\phi(z) = Re\ \int (\tfrac{1}{2}f(1+g^2), \tfrac{i}{2}f(1-g^2), -fg\,)dz,\ z \in D \text{ where } z = u + iv.$$

$$\phi(z) = Re\ \int (\frac{1}{2}2(1+z^2), \frac{i}{2}2(1-z^2), -2z\,)dz$$

$$\phi(z) = Re\ \int ((1+z^2), i(1-z^2), -2z\,)dz.$$

$$\phi^1 = Re\left(\int (z^2+1)\,dz \right)$$

$$\phi^1 = Re\left(z + \frac{1}{3}\ z^3\right) = Re\left(u + iv + \frac{1}{3}(u+iv)^3\right)$$

$$\phi^1 = Re\left(\ u + iv + \frac{1}{3}(u^3 + 3u^2vi - 3uv^2 - iv^3)\right)$$

$$\phi^1 = \left(\ u - uv^2 + \frac{1}{3}u^3\ \right)$$

$$\phi^2 = Re\left(\int i(1-z^2)\,dz \right)$$

$$\phi^2 = Re\left(i\left(z - \frac{1}{3}\ z^3\right)\right) = Re\left(i\left(u + iv - \frac{1}{3}(u+iv)^3\right)\right)$$

$$\phi^2 = Re\left(i\left(u + iv - \frac{1}{3}(u^3 + 3u^2vi - 3uv^2 - iv^3)\right)\right)$$

$$\phi^2 = Re\left(ui - v - \frac{1}{3}(u^3i - 3u^2v - 3uv^2i + v^3)\right)$$

$$\phi^2 = \left(-v + u^2v - \frac{1}{3}v^3\right)$$

$$\phi^3 = Re\left(\int -2z\,dz \right) = Re\ (-z^2) = Re\ (\,-(u+iv)^2)$$

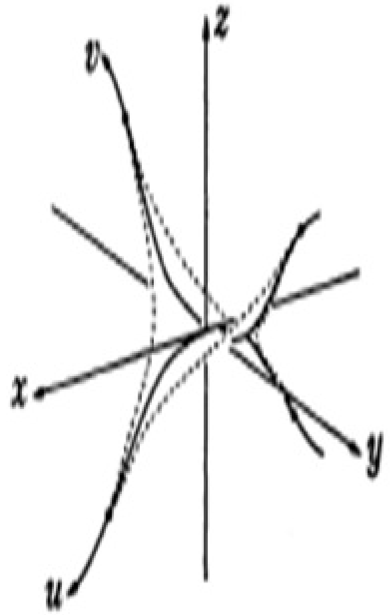
$$\phi^3 = Re\ (\,-u^2 - 2uiv + v^2\,)$$

$$\phi^3 = (v^2 - u^2\,)$$

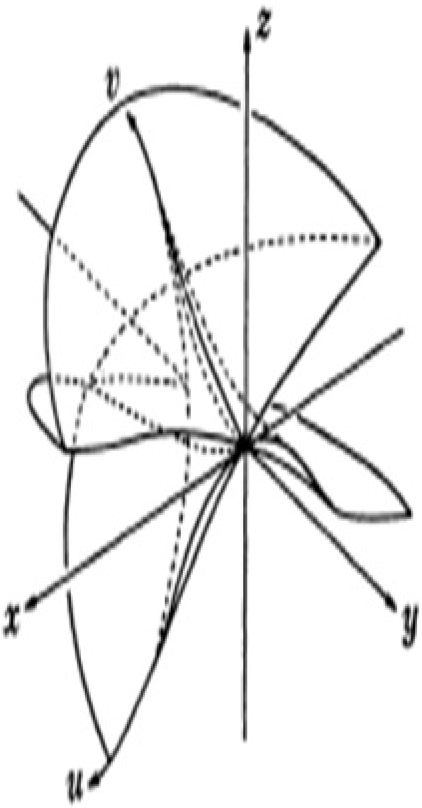
We have a maximal surface given by

$$\phi\ (z) = \left(\ u - uv^2 + \frac{1}{3}u^3\ , -v + u^2v - \frac{1}{3}v^3, v^2 - u^2\ \right) \text{ where } z = u + iv$$

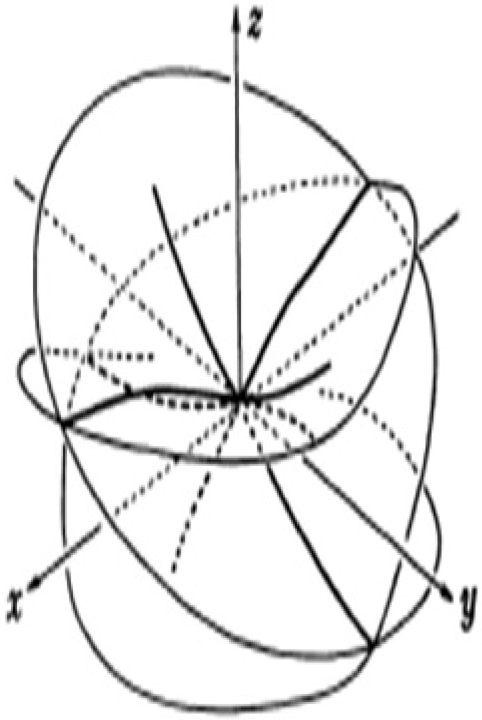
(Kobayashi, 1983)



$\{u^2 + v^2 \leq 1\}$



$\{y \leq 0\}$



$\left\{v^2 - \frac{1}{3}u^2 \geq 1\right\} \cup \left\{u^2 - \frac{1}{3}v^2 \geq 1\right\}$

Figure 5.3. Enneper's Surface of the $\mathbf{1^{st}}$ Kind

5.2.2.3 Example (Enneper's Surface of 2^{nd} Kind)

This given by putting $f = 2\alpha,\ g = z$ in

$$\phi(z) = Re \int \left(\frac{1}{2}f(g^2+1), ifg, \frac{1}{2}f(g^2-1)\right) dz, Re\ g \neq 0,$$

where α is a non-zero real constant.

$$\phi(z) = Re \int \left(\frac{1}{2}.2\alpha.(z^2+1), i2\alpha z, \frac{1}{2}.2\alpha.(z^2-1)\right) dz$$

$$\phi(z) = Re \int \left(\alpha.(z^2+1), i2\alpha z, \alpha.(z^2-1)\right) dz$$

$$\phi^1 = Re\left(\int \alpha(z^2+1)\, dz\right)$$

$$\phi^1 = \alpha.Re\left(z + \frac{1}{3}z^3\right) = \alpha.Re\left(u + iv + \frac{1}{3}(u+iv)^3\right)$$

$$\phi^1 = \alpha.Re\left(u + iv + \frac{1}{3}(u^3 + 3u^2vi - 3uv^2 - iv^3)\right)$$

$$\phi^1 = \alpha\left(u - uv^2 + \frac{1}{3}u^3\right)$$

$$\phi^2 = Re\left(\int 2z\alpha i\, dz\right) = \alpha.Re\,(z^2 i) = \alpha.Re\,(\,i\,(u+iv)^2)$$

$$\phi^2 = \alpha.Re\,(\,u^2 i + 2uv - iv^2\,)$$

$$\phi^2 = \alpha(-2uv\,)$$

$$\phi^3 = Re\left(\int \alpha(z^2-1)\, dz\right)$$

$$\phi^3 = \alpha.Re\left(\frac{1}{3}z^3 - z\right) = \alpha.Re\left(\frac{1}{3}(u+iv)^3 - u + iv\right)$$

$$\phi^3 = \alpha.Re\left(\frac{1}{3}(u^3 + 3u^2vi - 3uv^2 - iv^3) - (u+iv)\right)$$

$$\phi^3 = \alpha\left(-u - uv^2 + \frac{1}{3}u^3\right)$$

Then we have a maximal surface an explicit formula is given as follows:

$$\phi\,(z) = \alpha\,\left(u - uv^2 + \frac{1}{3}u^3, -2uv, -u - u^2v + \frac{1}{3}u^3\right)$$

where $z = u + iv\,, u \neq 0.$ This surface is a rotation surface with a light-like axis $(1\,,0,1)$, which can be seen from the following expression (5.2.3):

$$\phi(z) = \begin{pmatrix} 1-\frac{1}{2}v^2 & v & \frac{1}{2}v^2 \\ -v & 1 & v \\ -\frac{1}{2}v^2 & v & 1+\frac{1}{2}v^2 \end{pmatrix} \begin{pmatrix} \alpha u + \frac{\alpha}{3}u^3 \\ 0 \\ -\alpha u + \frac{\alpha}{3}u^3 \end{pmatrix}, u \neq 0.$$

As shown in below. (Kobayashi, 1983)

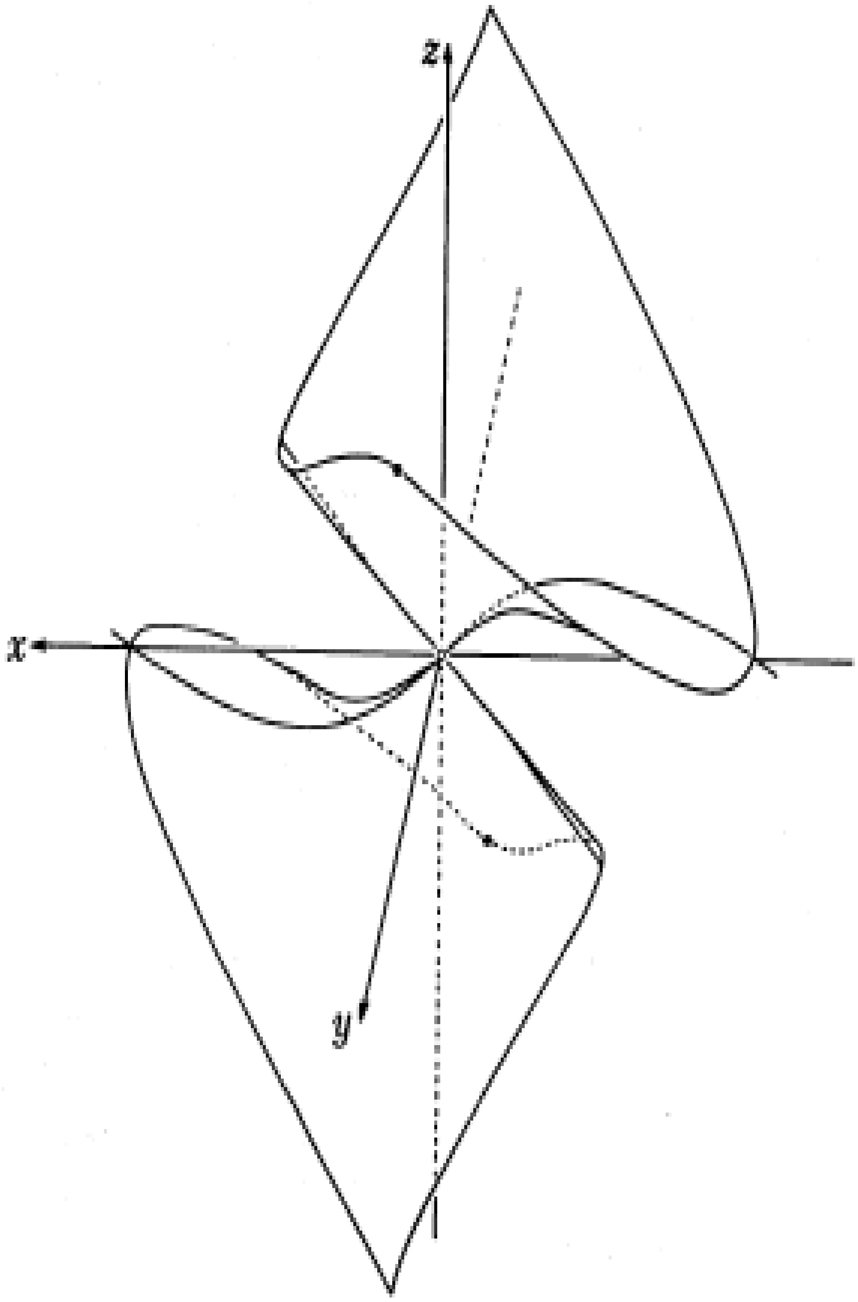


Figure 5.4. Enneper's Surface of the $\mathbf{2^{nd}}$ Kind

5.2.2.4 Example (Conjugate of Ennper's Surface of the 2^{nd} Kind)

We define the conjugate surface of Enneper's surface of the 2^{nd} kind by putting $f = 2\alpha i,\ g = z$ in

$$\phi(z) = Re \int \left(\frac{1}{2}f(g^2+1), ifg, \frac{1}{2}f(g^2-1)\right) dz, Re\ g \neq 0.$$

$$\phi(z) = Re \int \left(\frac{1}{2}.2\alpha i.(z^2+1), i.2\alpha.i.z, \frac{1}{2}.2\alpha i.(z^2-1)\right) dz$$

$$\phi(z) = Re \int \left(\alpha i.(z^2+1), -2\alpha z, \alpha i.(z^2-1)\right) dz$$

$$\phi^1 = Re\left(\int \alpha i(z^2+1)\, dz\right)$$

$$\phi^1 = \alpha.Re\left(zi + \frac{i}{3}z^3\right) = \alpha.Re\left(ui - v + \frac{i}{3}(u+iv)^3\right)$$

$$\phi^1 = \alpha.Re\left(ui - v + \frac{i}{3}(u^3 + 3u^2vi - 3uv^2 - iv^3)\right)$$

$$\phi^1 = \alpha\left(-v - u^2v + \frac{1}{3}v^3\right)$$

$$\phi^2 = Re\left(\int -2z\alpha\, dz\right) = \alpha.Re\,(-z^2) = \alpha.Re\,(-(u+iv)^2)$$

$$\phi^2 = \alpha.Re\,(-u^2 - 2uvi + v^2)$$

$$\phi^2 = \alpha(v^2 - u^2)$$

$$\phi^3 = Re\left(\int \alpha i(z^2-1)\, dz\right)$$

$$\phi^3 = \alpha.Re\left(\frac{i}{3}z^3 - zi\right) = \alpha.Re\left(\frac{i}{3}(u+iv)^3 - ui + v\right)$$

$$\phi^3 = \alpha.Re\left(\frac{i}{3}(u^3 + 3u^2vi - 3uv^2 - iv^3) - ui + v\right)$$

$$\phi^3 = \alpha\left(v - u^2v + \frac{1}{3}v^3\right)$$

Then we have a maximal surface

$$\phi\,(z) = \alpha\left(-v - u^2v + \frac{1}{3}v^3, v^2 - u^2, v - u^2v + \frac{1}{3}v^3\right)$$

$$\phi\,(z) = \alpha\left(-v + \frac{1}{3}v^3, v^2, v + \frac{1}{3}v^3\right) - \alpha u^2(v, 1, v), \qquad u \neq 0.$$

As a consequence, this is a ruled surface.

As shown below. (Kobayashi, 1983)

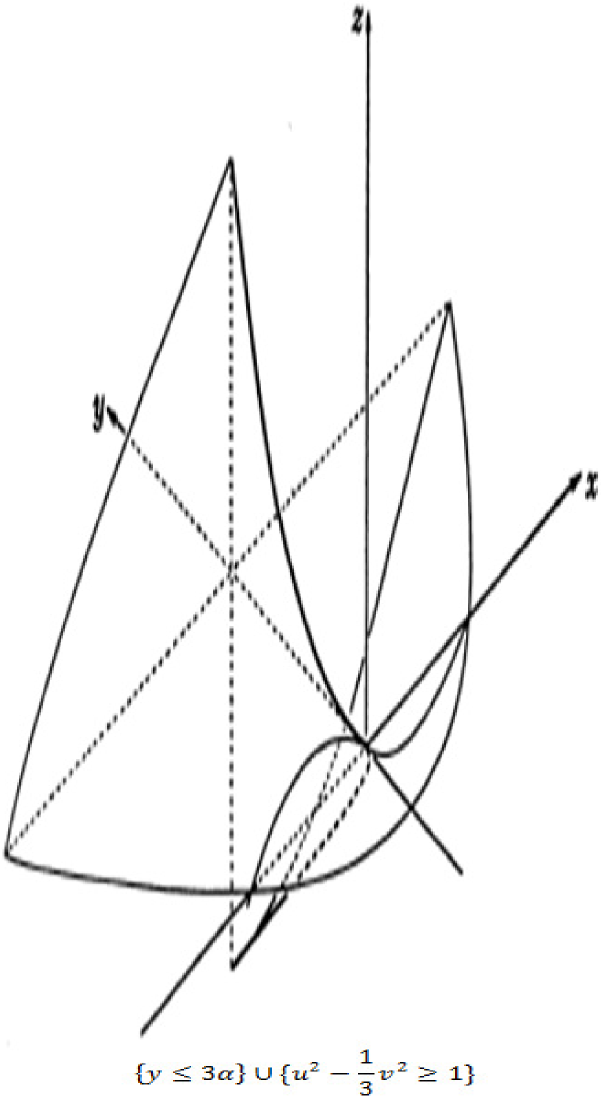


Figure 5.5. Conjugate of Enneper's surface of the **2^{nd}** Kind

5.2.2.5 Example (Catenoid of the 1^{st} Kind)

This rotattion surface is defined by

$$x^2 + y^2 - \alpha^2 \sinh^2\left(\frac{z}{\alpha}\right) = 0\,, \qquad (z \neq 0),$$

where α is a non-zero real. In view of the Weierstrass-Enneper formula, it is given by putting $f = \alpha z^{-2},\ g = z$ in

$$\phi(z) = Re\ \int (\tfrac{1}{2} f(1 + g^2), \tfrac{i}{2} f(1 - g^2), -fg\ )dz,\ z \in D \text{ where } z = u + iv.$$

As shown below. (Kobayashi, 1983)

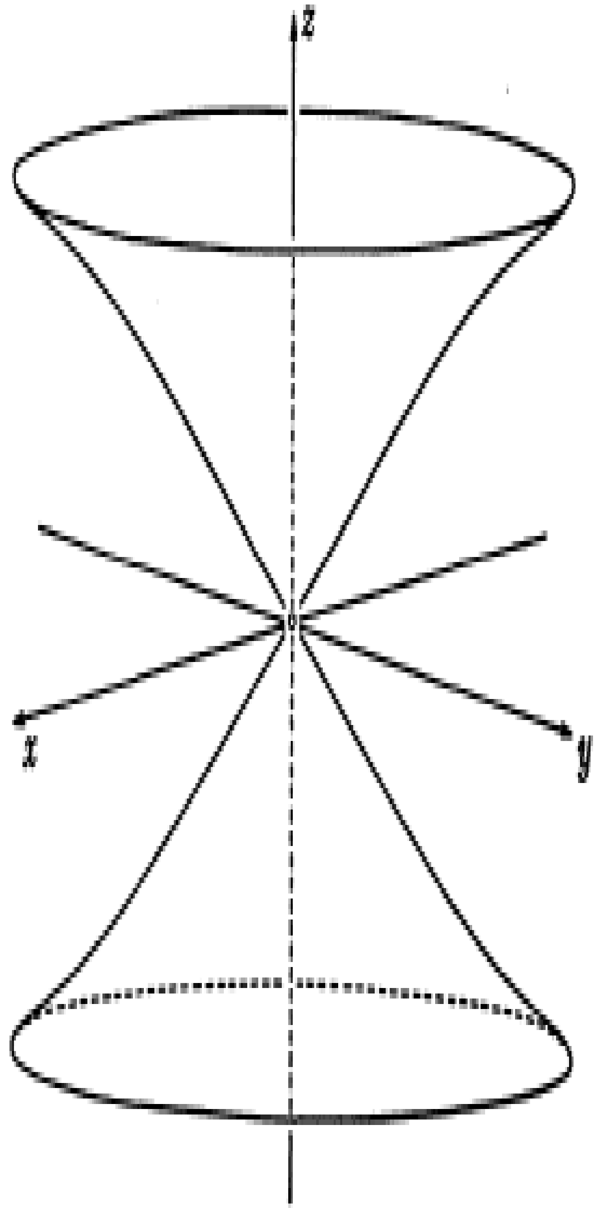


Figure 5.6. Catenoid of the $\mathbf{1^{st}}$ Kind

5.2.2.6 Example (Helicoid)

The conjugate surface of a catenoid of the 1st kind, that is, the surface defined by setting $f = i\alpha z^{-2},\ g = z$ in

$$\phi(z) = Re\ \int(\tfrac{1}{2}f(1+g^2), \tfrac{i}{2}f(1-g^2), -fg\,)dz,\ z \in D \text{ where } z = u + iv,$$

is given by

$$\phi(z) = (\,0, 0, \alpha\theta) + \alpha\cosh(\log r\,(-\sin\theta\,, \cos\theta, 0\,)), \qquad z = re^{i\theta}, \qquad (r \neq 1).$$

Note that this is an open subset of the usual helicoid;

$$x\cos\left(\frac{z}{\alpha}\right) + y\sin\left(\frac{z}{\alpha}\right) = 0.$$

Hence, it is also a minimal surface with respect to the metric $dx^2 + dy^2 + dz^2$. Conversely, this property characterizes the helicoid. (Kobayashi, 1983)

5.2.2.7 Example (Catenoid of the 2^{nd} Kind)

This is a rotation surface defined by setting $f = \alpha z^{-2},\ g = z$ in

$$\phi(z) = Re\int\left(\frac{1}{2}f(g^2+1), ifg, \frac{1}{2}f(g^2-1)\right)dz, Re\ g\ \neq 0.$$

$$\phi(z) = \begin{pmatrix} \cosh\log r & 0 & \sinh\log r \\ 0 & 1 & 0 \\ \sinh\log r & 0 & \cosh\log r \end{pmatrix}\begin{pmatrix} 0 \\ -\alpha\theta \\ \alpha\cos\theta \end{pmatrix},\ z = re^{i\theta}, (\cos\theta \neq 0\ ).$$

As shown in below. (Kobayashi, 1983)

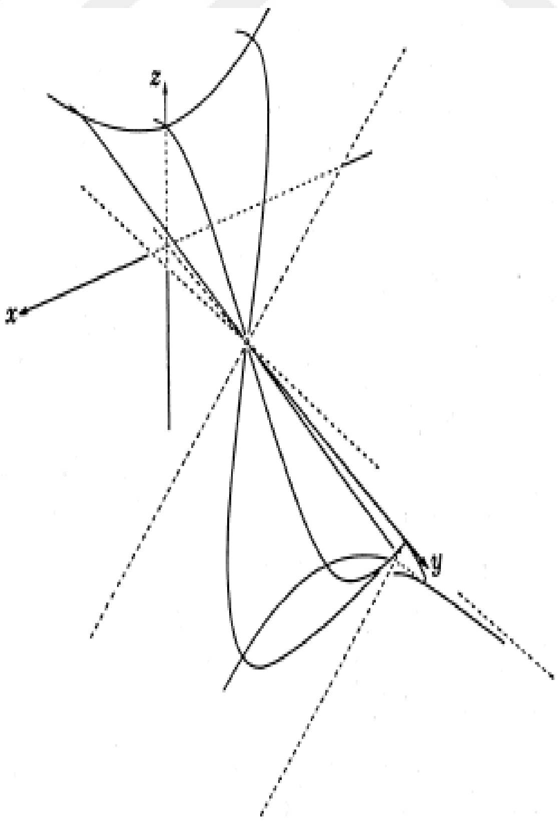


Figure 5.7. Catenoid of the 2^{nd} Kind

5.2.2.8 Example (Helicoid of 2^{nd} Kind)

This is a ruled surface defined by

$$z + x\tanh\left(\frac{y}{\alpha}\right) = 0 \ \left(x^2 \leq \alpha^2 \cosh\left(\frac{y}{a}\right)\right),$$

which corresponds to $f = i\alpha z^{-2},\ g = z$ in

$$\phi(z) = Re \int \left(\frac{1}{2}f(g^2+1), ifg, \frac{1}{2}f(g^2-1)\right) dz, Re\ g \neq 0.$$

(Kobayashi, 1983)

5.2.2.9 Example (Scherk's Surface of the 1^{st} Kind)

This is maximal surface defined by

$$\mathrm{z} = \log\cosh y - \log\cosh x,\quad (\cosh^{-2} x + \cosh^{-2} y > 1)\,,$$

which is obtained by setting $f = 4(1-z^4)^{-1},\ g = z$ in

$$\phi(z) = Re \int (\tfrac{1}{2}f(1+g^2), \tfrac{i}{2}f(1-g^2), -fg\,)dz,\ z \in D \text{ where } z = u + iv.$$

As shown below. (Kobayashi, 1983)

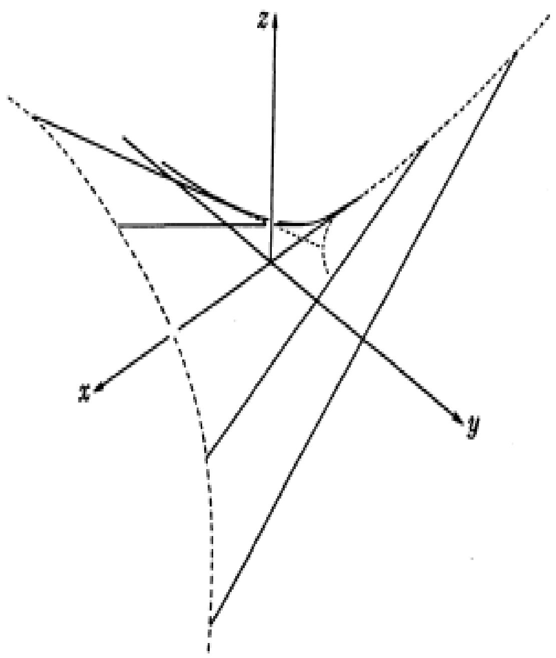


Figure 5.8. Helicoid of the $\mathbf{2^{nd}}$ Kind

5.2.3 Rotation Surfaces

The purpose of this section is to determine the maximal rotation surfaces in E_1^3. A surface is called a rotation surface with axis l if it is invariant under the action of the group of motions in E_1^3 which fix each point of the line l.

5.2.3.1 Theorem

Every maximal rotation surface in E_1^3 is congruent to a part of one of the following:

i. (x, y) - plane ;
ii. Catenoid of the 1^{st} kind ;
iii. Catenoid of the 2^{nd} kind ;
iv. Enneper's surface of the 2^{nd} kind. (Kobayashi, 1983)

Proof:

The (x, y) - plane is obviously a rotation surface with time-like axis, and every space-like plane is congruent to it. So, we suppose that the given maximal rotation surface is not a plane.

If the axis is time-like (respectively space-like), we might assume that the axis is the $z -$ axis (respectively $y -$ axis), because every time-like (respectively space-like) unit vector is transformed to $(0, 0, 1)$ (respectively $(0, 1, 0)$) by a Lorentz transformation. Then the surface is expressed as follows:

Case 1:

$X(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$ if the axis is timelike. The maximal surface equation is then given by $f.f'' = (f')^2 - 1$ for $(f')^2 - 1 > 0$.

$$X_u = (-f(v)\sin u, f(v)\cos u, 0)$$

$$X_v = (f'(v)\sin u, f'(v)\cos u, g'(v))$$

$$E = < X_u, X_u >_L = \left(f(v)\right)^2$$

$$F = < X_u, X_v >_L = 0$$

$G = < X_v, X_v >_L = \left(f'(v)\right)^2 - \left(g'(v)\right)^2 = 1$ since profile curve is unit velocity.

$$I = Edu^2 + 2Fdudv + Gdv^2$$

$$I = \left(f(v)\right)^2 du^2 + 2.0dudv + 1dv^2$$

$$I = \left(f(v)\right)^2 du^2 + dv^2$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \left(f(v)\right)^2 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det I = \left(f(v)\right)^2$$

$$II = Ldu^2 + 2Mdudv + Ndv^2$$

$$L = < n, X_{uu} >_L, \qquad M = < n, X_{uv} >_L \ \ and \ \ N = < n, X_{vv} >_L$$

$$X_{uu} = (-f(v)\cos u, -f(v)\sin u, 0)$$

$$X_{uv} = (-f'(v)\sin u, f'(v)\cos u, 0)$$

$$X_{vv} = (f''(v)\cos u, f''(v)\sin u, g''(v))$$

$$n = (f(v)g'(v)\cos u, f(v)g'(v)\sin u, f(v)f'(v))$$

$$L = f(v)g'(v)$$

$$M = 0$$

$$N = f''(v)g'(v) - f'(v)g''(v)$$

$$II = \left(f(v)g'(v)\right)du^2 + 2.0\ dudv + \left(f''(v)g'(v) - f'(v)g''(v)\right)dv^2$$

$$II = \left(f(v)g'(v)\right)du^2 + \left(f''(v)g'(v) - f'(v)g''(v)\right)dv^2$$

$$sp = \begin{pmatrix} f^2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -fg' & 0 \\ 0 & f''g' - f'g'' \end{pmatrix}.$$

$$sp = \begin{pmatrix} \frac{g'}{f} & 0 \\ 0 & f''g' - f'g'' \end{pmatrix}.$$

Since profile curve is unit velocity curve $(f')^2 - (g')^2 = 1.$ Differentiate both sides $2.f'.f'' - 2.g'.g'' = 0 \Rightarrow f'.f'' = g'.g''$

$$(f''g' - f'g'').g' = f''.\ g'.\ g' - f'.\ \underbrace{g''.\ g'}$$

$$(f''g' - f'g'').g' = f''.\ (g')^2 - f'.(f'.f'')$$

$$(f''g' - f'g'').g' = f''.\ (g')^2 - (f')^2.f''$$

$$(f''g' - f'g'').g' = f''((g')^2 - (f')^2). We\ know\ that\ (f')^2 - (g')^2 = 1.$$

$$(f''g' - f'g'').g' = -f'' \underbrace{((f')^2 - (g')^2)}_{1}$$

$$(f''g' - f'g'').g' = -f''$$

$$f''g' - f'g'' = \frac{-f''}{g'}$$

$$\boldsymbol{H} = \frac{1}{2}\left(f''g' - f'g'' + \frac{g'}{f}\right)$$

$$\boldsymbol{H} = \frac{1}{2}\left(\frac{-f''}{g'} + \frac{g'}{f}\right)$$

X is maximal if and only if $\frac{g'}{f} = \frac{f''}{g'} \Rightarrow (g')^2 = f.f''$. We know that

$$(f')^2 - (g')^2 = 1 \Rightarrow (g')^2 = (f')^2 - 1.$$

$$(f')^2 - 1 = f.f''$$

Solving the differentiable equation let $h = f'$, $f' = \frac{df}{dv}$

$$f'' = \frac{dh}{dv} = \frac{dh}{df}.\frac{df}{dv} = h.\frac{dh}{df}$$

$$f.f'' = (f')^2 - 1 \Rightarrow fh\frac{dh}{df} = h^2 - 1, \qquad g' \neq 0, \qquad h^2 \neq 1.$$

$$\int \frac{hdh}{h^2 - 1} = \int \frac{df}{f}$$

$$h = \sqrt{a^2 f^2 + 1}$$

$$\frac{df}{dv} = h = \sqrt{a^2 f^2 + 1} \Rightarrow \frac{df}{\sqrt{a^2 f^2 + 1}} = dv$$

$$\int \frac{df}{\sqrt{a^2 f^2 + 1}} = \int dv$$

Let $af = \theta \Rightarrow adf = d\theta \Rightarrow df = \frac{1}{a}d\theta$

$$\frac{1}{a}\int \frac{d\theta}{\sqrt{\theta^2 + 1}} = v$$

$$\frac{1}{a}\sinh^{-1}\theta = v$$

$$\frac{1}{a}\sinh^{-1}(af) = v + b$$

$$\sinh^{-1}(af) = a(v + b)$$

$$af = \sinh a(v + b)$$

$$f = a^{-1}\sinh(av + b)$$

Case 2:

$X(u,v) = (\,g(v)\sinh u,\ f(t),\ g(v)\cosh u)$ if the axis is spacelike. The maximal surface equation is then given by $g.g'' = (g')^2 - 1$ for $(g')^2 - 1 < 0$.

$$X_u = (g(v)\cosh u\,, 0, g(v)\sinh u\,)$$

$$X_v = (g'(v)\sinh u, f'(v)\,, g'(v)\cosh u\,)$$

$$E = < X_u\,, X_u >_L = \left(g(v)\right)^2$$

$$F = < X_u, X_v >_L = 0$$

$$G = < X_v, X_v >_L = \left(f'(v)\right)^2 - \left(g'(v)\right)^2 = 1 \text{ since profile curve is unit velocity.}$$

$$I = Edu^2 + 2Fdudv + Gdv^2$$

$$I = \left(g(v)\right)^2 du^2 + 2.0dudv + 1dv^2$$

$$I = \left(g(v)\right)^2 du^2 + dv^2$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \left(g(v)\right)^2 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det I = \left(g(v)\right)^2$$

$$II = Ldu^2 + 2Mdudv + Ndv^2$$

$$L = < n, X_{uu} >_L, \qquad M = < n, X_{uv} >_L \ and\ N = < n, X_{vv} >_L$$

$$X_{uu} = (\,g(v)\sinh u, 0, g(v)\cosh u)$$

$$X_{uv} = (g'(v)\cosh u, 0, g'(v)\sinh u\,)$$

$$X_{vv} = (g''(v)\sinh u\,, f''(v), g''(v)\cosh u)$$

$$n = (-f'(v)\cosh u, -g'(v), -f'(v)\cosh u\ )$$

$$L = f'(v).\,g(v)$$

$$M = 0$$

$$N = f'(v)g''(v) - f''(v)g'(v)$$

$$II = \left(\,f'(v).\,g(v)\right)du^2 + 2.0\ dudv + \left(f'(v)g''(v) - f''(v)g'(v)\right)dv^2$$

$$sp = \begin{pmatrix} g^2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} f'g & 0 \\ 0 & f'g'' - f''g' \end{pmatrix}.$$

$$sp = \begin{pmatrix} \frac{f'}{g} & 0 \\ 0 & f'g'' - f''g' \end{pmatrix}.$$

Since profile curve is unit velocity curve $(f')^2 - (g')^2 = 1.$ Differentiate both sides $2.f'.f'' - 2.g'.g'' = 0 \Rightarrow f'.f'' = g'.g''$

$$(f'g'' - f''g').f' = f'.\,f'.\,g'' - g'.\,\underbrace{f''.\,f'}$$

$$(f'g'' - f''g').f' = g''.\,(f')^2 - g'.(g'.g'')$$

$$(f'g''-f''g').f' = g''.\,(f')^2-(g')^2.g''$$

$$(f'g''-f''g').f' = g''(\,(f')^2-(\,g'\,)^2).\,We\ know\ that\ (f')^2-(g')^2=1.$$

$$(f'g''-f''g').f' = g''\underbrace{((f')^2-(g')^2)}_{1}$$

$$(f'g''-f''g').f' = g''$$

$$f'g''-f''g' = \frac{g''}{f'}$$

$$\boldsymbol{H}=\frac{1}{2}\left(f'g''-f''g'+\frac{f'}{g}\right)$$

$$\boldsymbol{H}=\frac{1}{2}\left(\frac{g''}{f'}+\frac{f'}{g}\right)$$

X is maximal if and only if $-\frac{f'}{g}=\frac{g''}{f'}\Rightarrow(f')^2=-g.g''$. We know that

$$(f')^2-(g')^2=1\ \Rightarrow(f')^2=(g')^2+1.$$

$$(g')^2+1=-g.g''$$

Solving the differentiable equation let $h=g'$, $g'=\frac{df}{dv}$

$$g''=\frac{dh}{dv}=\frac{dh}{dg}.\frac{dg}{dv}=h.\frac{dh}{dg}$$

$$-g.g''=(g')^2+1\ \Rightarrow -gh\,\frac{dh}{dg}=\ h^2+1,\qquad f'\neq 0,\qquad h^2\neq -1.$$

$$-\int\frac{hdh}{h^2+1}=\int\frac{dg}{g}$$

$$h=\frac{\sqrt{1-a^2g^2}}{ag}$$

$$\frac{dg}{dv}=\ h=\frac{\sqrt{1-a^2g^2}}{ag}\Rightarrow\frac{ag\,dg}{\sqrt{1-(ag)^2}}=dv$$

$$\int\frac{ag\,dg}{\sqrt{1-(ag)^2}}=\int dv$$

Let $ag=\theta\ \Rightarrow adg=d\theta\Rightarrow dg=\frac{1}{a}\,d\theta$

$$\frac{1}{a}\int\frac{d\theta}{\sqrt{1-\theta^2}}=v$$

$$\frac{1}{a}\cos^{-1}\theta = v$$

$$\frac{1}{a}\cos^{-1}(ag) = v + b$$

$$\cos^{-1}(ag) = a(v + b)$$

$$ag = \cos a(v + b)$$

$$g = a^{-1}\cos(av + c)$$

Thus, we have $f(v) = a^{-1}\sinh(av + b)$ and $g(v) = a^{-1}\cos(av + c)$, where a and b are integral constants.

Hence, the surface is locally congruent to a catenoid of the 1st kind or a catenoid of the 2nd kind according to that the axis is time-like or space-like.

If the axis is light-like, we might suppose that it is $\boldsymbol{IR}.(1,0,1)$. Note that the subgroup of the Lorentz group which fixes $(1,0,1)$ is

$$\left\{\begin{pmatrix} 1-\frac{u^2}{2} & u & \frac{u^2}{2} \\ -u & 1 & u \\ -\frac{u^2}{2} & u & 1+\frac{u}{2} \end{pmatrix}; u \in IR\right\}.$$

Thus, the surface can be written as

$$X(u,v) = \begin{pmatrix} 1-\frac{u^2}{2} & u & \frac{u^2}{2} \\ -u & 1 & u \\ -\frac{u^2}{2} & u & 1+\frac{u}{2} \end{pmatrix}\begin{pmatrix} h(v)+v \\ 0 \\ h(v)-v \end{pmatrix}. \ (*)$$

The maximal surface equation for $(*)$ is given by

$$vh'' - 2h' = 0, \qquad v \neq 0, \ h' > 0.$$

Hence, we have the solution $h(v) = at^3 + b$, $a > 0$, which shows that the surface is Enneper's surface of the 2nd kind.

5.2.4 Ruled Surfaces

As for maximal ruled surfaces, we have following:

5.2.4.1 Theorem

Every maximal ruled surfaces in E_1^3 is congruent to a part of one of the following:

i. (x, y) - plane;
ii. Helicoid;
iii. Helicoid of the 2^{nd} kind;
iv. Conjugate of Enneper's surface of the 2^{nd} kind. (Kobayashi, 1983)

Proof:

Every space-like ruled surface can be written

$$X(t,s) = \alpha(s) + t.\boldsymbol{n}(s)$$

$$\alpha'(s) = \boldsymbol{t}(s)$$

$$< \alpha'(s), \alpha'(s) >_L = 1$$

$$span\{\ \boldsymbol{t}(s) = \alpha'(s)\ \}^{\perp} \text{ is a time-like subspace } E_1^2.$$

$\boldsymbol{t}'(s)$ may be space-like.

$\boldsymbol{t}'(s)$ may be time-like.

$\boldsymbol{t}'(s)$ may be light-like.

We will examine $\boldsymbol{t}'(s)$ may be space-like.

$\boldsymbol{t}'(s)$ is a space-like vector.

$$\kappa(s) = < \boldsymbol{t}'(\boldsymbol{s}), \boldsymbol{n}(\boldsymbol{s}) >_L \Rightarrow \underbrace{\boldsymbol{t}'(\boldsymbol{s})}_{space-like} = \kappa(s).\ \underbrace{\boldsymbol{n}(\boldsymbol{s})}_{space-like}$$

$$< \boldsymbol{n}(s), \boldsymbol{n}(s) >_L = 1$$

$$< \boldsymbol{t}(s), \boldsymbol{t}(s) >_L = 1 \Rightarrow < \boldsymbol{t}(s), \boldsymbol{t}'(s) >_L = 0 \Rightarrow < \boldsymbol{t}(s), \boldsymbol{n}(s) >_L = 0$$

where $\alpha(s)$ is a space-like curve in E_1^3 with arclength parameter and $\boldsymbol{n}(s)$ is a unit normal vector field along $\alpha(s)$. Note that $\boldsymbol{n}(s)$ is an asymptotic vector field on the surface and that

$$\alpha'(s) + t.\boldsymbol{n}'(s) \;\Rightarrow \boldsymbol{t}(s) + t.\boldsymbol{n}'(s) \text{ is perpendicular to } \boldsymbol{n}(s).$$

It follows from the maximality that $\alpha'(s) + t.\boldsymbol{n}'(s)$ is an asymptotic direction. Especially, putting $t = 0$, we can see that $\boldsymbol{n}(s)$ is the principal normal vector of $\alpha(s)$. Thus, we need only to determine the curve $\alpha(s)$ to get the surface

$$X(t,s) = \alpha(s) + t.\boldsymbol{n}(s),$$
$$< \alpha'(s), \alpha'(s) >_L = < \boldsymbol{n}(s), \boldsymbol{n}(s) >_L = 1, \qquad \mathbf{(5.1)}$$
$$< \boldsymbol{t}(s), \boldsymbol{n}(s) >_L = 0.$$

Denoting by $\boldsymbol{b}(s)$ the binormal vector of (s), we have the Frenet-Serret formula:

$$\boldsymbol{t}'(\boldsymbol{s}) = \kappa(s).\boldsymbol{n}(\boldsymbol{s})$$
$$\boldsymbol{n}'(\boldsymbol{s}) = -(\kappa s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s}), \qquad \mathbf{(5.2)}$$
$$\boldsymbol{b}'(\boldsymbol{s}) = \boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s})$$

where κand τ are curvature and torsion of $\alpha(s)$, respectively. Hence,

$$\boldsymbol{t}(s) + t.\boldsymbol{n}'(s) = \boldsymbol{t}(s) + t\big(-\kappa(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})\big)$$
$$\boldsymbol{t}(s) + t.\boldsymbol{n}'(s) = \boldsymbol{t}(s) - t.\kappa(s).\boldsymbol{t}(\boldsymbol{s}) + t.\boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s})$$
$$\boldsymbol{t}(s) + t.\boldsymbol{n}'(s) = \boldsymbol{t}(s)\big(1 - t.(\kappa s)\big) + t.\boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s}) \qquad \mathbf{(5.3)}$$

$$\underbrace{\boldsymbol{t}'(s) + t.\boldsymbol{n}''(s)}$$
$$\kappa(s).\boldsymbol{n}(\boldsymbol{s}) + t\left(-\kappa(s).\underbrace{\boldsymbol{n}'(\boldsymbol{s})}_{\kappa(s).\boldsymbol{n}(\boldsymbol{s})} + \boldsymbol{\tau}(s).\underbrace{\boldsymbol{b}'(\boldsymbol{s})}_{\tau(s).\boldsymbol{n}(\boldsymbol{s})} - \kappa'(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}'(s).\boldsymbol{b}(\boldsymbol{s})\right)$$
$$\kappa(s).\boldsymbol{n}(\boldsymbol{s}) + t\,\big(-\kappa(s).(\kappa s).\boldsymbol{n}(\boldsymbol{s}) + \boldsymbol{\tau}(s).\boldsymbol{\tau}(s).\boldsymbol{n}(\boldsymbol{s}) - \kappa'(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}'(s).\boldsymbol{b}(\boldsymbol{s})\big)$$
$$\underbrace{\boldsymbol{t}'(s) + t.\boldsymbol{n}''(s)}$$
$$\boldsymbol{n}(\boldsymbol{s})\left(\kappa(s) - t\big(\kappa(s)\big)^2 + t\big(\boldsymbol{\tau}(s)\big)^2\right) + t\big(-\kappa'(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}'(s).\boldsymbol{b}(\boldsymbol{s})\big). \qquad \mathbf{(5.4)}$$

Since $\alpha' + t\boldsymbol{n}'$ is an asymptotic direction, $\alpha'' + t\boldsymbol{n}''$ is tangent to the surface. That is $t\big(-\kappa'(s).\boldsymbol{t}(\boldsymbol{s}) + \boldsymbol{\tau}'(s).\boldsymbol{b}(\boldsymbol{s})\big)$ must be parallel to

$$\boldsymbol{t}(s)\big(1 - t.\kappa(s)\big) + t.\boldsymbol{\tau}(s).\boldsymbol{b}(\boldsymbol{s}) \text{ for any } t \text{ and } s.$$

Thus ,κ and $\boldsymbol{\tau}$ are constant.

Then, if $|\kappa| > |\boldsymbol{\tau}| > 0$ ($respectively\ \ |\boldsymbol{\tau}| > |\kappa| > 0$),

$$\tilde{\alpha}(s) = \alpha(s) + \left(\frac{\kappa}{\kappa^2 - \boldsymbol{\tau}^2}\right)\boldsymbol{n}(\boldsymbol{s})$$

is a time-like (respectively space-like) line by $(\mathbf{5.3})$ and $(\mathbf{5.4})$**.** Therefore, from $(\mathbf{5.2})$, we can see that $\alpha(s)$ is congruent to

$$\left(\frac{\kappa}{\kappa^2 - \boldsymbol{\tau}^2}\cos\left(\sqrt{\kappa^2 - \boldsymbol{\tau}^2}\right)s, \frac{\kappa}{\kappa^2 - \boldsymbol{\tau}^2}\sin\left(\sqrt{\kappa^2 - \boldsymbol{\tau}^2}\right)s, \frac{\boldsymbol{\tau}}{\kappa^2 - \boldsymbol{\tau}^2}\ s\right) \qquad (\mathbf{5.5})$$

if $|\kappa| > |\boldsymbol{\tau}| > 0$;

Or

$$\left(\frac{\kappa}{\boldsymbol{\tau}^2 - \kappa^2}\cosh\left(\sqrt{\boldsymbol{\tau}^2 - \kappa^2}\right)s, \frac{\boldsymbol{\tau}}{\boldsymbol{\tau}^2 - \kappa^2}s, \frac{\kappa}{\boldsymbol{\tau}^2 - \kappa^2}\sinh\left(\sqrt{\boldsymbol{\tau}^2 - \kappa^2}\right)\ s\right) \qquad (\mathbf{5.6})$$

if $|\boldsymbol{\tau}| > |\kappa| > 0$.

The surface defined by $(\mathbf{5.1})$ is compatible to a part of helicoid or 2^{nd} kind.

If $|\boldsymbol{\tau}| = |\kappa| \neq 0$, we have $\boldsymbol{n}'' = 0$ by $(\mathbf{5.2})$, hence $\alpha'''' = 0. \alpha(s)$ is a polynomial of degree 3. We have the conjugate of Enneper's surface of the 2^{nd} kind.

6. CONCLUSION

Curves and surfaces in E_1^3 have some similar properties with these in E^3. We have seen that curves and surfaces in E_1^3 differs by their causal character.We investigate Weierstrass-Enneper representation the surfaces by comparing minimal surfaces in E^3 with maximal surfaces in E_1^3. The Weierstrass-Enneper representation of minimal and maximal surfaces gives linkage between differential geometry and complex analysis. This representation is used for the classification of these surfaces. These methods might be used for the surfaces in E_1^n and classification of these opens a way to investigate new examples.

7. REFERENCES

Bonnor WB (1969) "Null Curves in a Minkowski Space-Time", Tensor (N. S) 20, 229-242.

Carmo Mdo (1976) Differential Geometry of Curves and Surfaces, Prentice-Hall, Saddle River.

Clelland JN (2012) "Totally Quasi-Umbilical Timelike surfaces in $R^{1,2}$", Asian J. Math. 16, No. 2, 189-208.

Dillen F, Kühnel W (1999) "Ruled Weingarten Surfaces in Minkowski 3-Space", Manuscripta Math. 98, No. 3, 307-320.

Ferrández A, Giménez A, Lucas P (2001) "Null Helices in Lorentzian Space Forms", Internat. J. Modern Phys. A 16, 4845-4863.

Graves LK (1979) "Codimension One Isometric Immersions Between Lorentz Space", Trans. Amer. Math. Soc. 252, 367-392.

Inoguchi J, Lee S (2008) "Null Curves in Minkowski 3-Space", International Elec. J. Geom. 1, 40-83.

Kobayashi O (1983) "Maximal Surfaces in The 3-dimensional Minkowski Space L^3", Tokyo J. Math. 6, 297-309.

Korevaar N (March 26 2002) "Making Minimal Surfaces with Complex Analysis", University of Utah.

Kühnel W (2002) Differential Geometry Curves-Surfaces-Manifolds, Second Edition, Volume 16.

López R (1999) "Constant Mean Curvature Surfaces with Boundary in Euclidean Three-Space", Tsukuba J. Math. 23, 27-36.

López R (1999) "Constant Mean Curvature Hypersurfaces Foliated by Spheres", Differential Geom. Appl. 11, 245-256.

López R (2002) "Cyclic Hypersurfaces of Constant Curvature", Advanced Studies in Pure Mathematics, 34, Minimal Surfaces Geometric Analysis and Symplectic Geometry, 185-199.

López R (2008) "Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space", arXiv: 0810.3351.

López R (2013) Constant Mean Curvature Surfaces with Boundary, Springer-Verlag, Berlin.

López R (2014) "Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space", International Elec. J. Geom. Volume 7, No. 1, 44-107.

Mira P, Pastor JA (2003) "Helicoidal Maximal Surfaces in Lorentz-Minkowski Space", Monatsh. Math. 140, 315-334.

O'Neill B (1966) Elementary Differential Geometry, Academic Press, New York.

O'Neill B (1983) Semi-Riemannian Geometry with Applications to Relativity Pure and Applied Mathematics, 103. Academic Press, Inc. New York.

Oprea J (2007) Differential Geometry and Its Applications, Mathematical Association of America, Inc. USA.

Osserman R (1969) A Survey of Minimal Surfaces, Van Nostrand, New York.

Pressley (2001) Elementary Differential Geometry, Springer, London.

Stein E, Shakarchi R (2003) Princeton Lectures in Analysis II Complex Analysis, Princeton University Press, PrincetonNew Jersey.

Walrave J (1995) Curves and Surfaces in Minkowski Space, Thesis (Ph. D), Katholieke Universiteit Leuven (Belgium).

Weinstein T (1996) an Introduction to Lorentz Surfaces, de Gruyter Expositions in Mathematics, 22. Walter de Gruyter & Co. Berlin.

Woestijne IW (1990) "Minimal Surfaces in The 3-dimensional Minkowski Space", Geometry and Topology of Submanifolds: II , World Scientific Press, 344-369, Singapore.

8. CURRICULUM VITAE

Name SURNAME	**:** Safiye Dilan CEYLAN
Place and Date of Birth	**:** Fatih, 03.03.1992
Universities	
Bachelor's Degree	**:** Bolu Abant İzzet Baysal University, Deparment of Mathematics
e-mail	**:** ceylan_1614@hotmail.com