BOLU ABANT İZZET BAYSAL UNIVERSITY
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DEPARTMENT OF MATHEMATICS

GRÖBNER-SHIRSHOV BASES AND NORMAL FORMS FOR SOME COXETER GROUPS

DOCTOR OF PHILOSOPHY

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Uğur Ustaoğlu

ABSTRACT

GRÖBNER-SHIRSHOV BASES AND NORMAL FORMS FOR SOME COXETER GROUPS
PH.D. THESIS

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The Gröbner-Shirshov bases and normal forms of the infinite Coxeter groups of type $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$ are obtained for the first time. New versions of Gröbner-Shirshov bases and normal forms of the finite Coxeter groups of type A_n, B_n and D_n are also found. Using combinatorial techniques, the product of two normal forms is attained as normal form in all Coxeter groups mentioned above and the infinite Coxeter group of type $\widetilde{A}_n$. Hence all these groups are completely revealed.

KEYWORDS: Finite Coxeter Groups, Infinite Coxeter Groups, Gröbner-Shirshov Bases, Permutation Groups, Normal Forms, Composition-Diamond Lemma .

ÖZET

BAZI COXETER GRUPLARI İÇİN GRÖBNER-SHIRSHOV TABANLARI VE NORMAL FORMLAR
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$\widetilde{B}_n$, $\widetilde{C}_n$ ve $\widetilde{D}_n$ tipi sonsuz Coxeter gruplarının Gröbner-Shirshov tabanları ve normal formları ilk defa bulundu. A_n, B_n and D_n tipi sonlu Coxeter gruplarının Gröbner-Shirshov tabanlarının ve normal formlarının yeni versiyonları bulundu. Kombinatorik teknikler kullanılarak, yukarıda bahsedilen bütün Coxeter gruplarda ve $\widetilde{A}_n$ tipi sonsuz Coxeter grubunda, iki tane normal formun çarpımının normal form olduğu elde edildi. Böylece tüm bu gruplar tamamen açığa çıkarılmıştır.

ANAHTAR KELİMELER: Sonlu Coxeter Grupları, Sonsuz Coxeter Grupları, Gröbner-Shirshov Tabanları, Permütasyon Grupları, Normal Formlar, Composition-Diamond Önsavı
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1. INTRODUCTION

Shirshov (1962) found linear bases for Lie algebras defined by generators and relations. He defined the composition of two Lie polynomials and using these compositions he gave an infinite algorithm to find the desired linear basis. Buchberger (1965) developed a similar algorithm for ideals in polynomial rings. Unlike Shirshov's algorithm his algorithm finds the basis after finitely many steps. He named these bases on behalf of his advisor as Gröbner bases. Nowadays, especially in noncommutative settings, these bases are called Gröbner-Shirshov bases. In fact Shirshov (1962) also implicitly gave an algorithm for associative algebras because he treated Lie algebras as subspace of Lie polynomials over free associative algebras. Furthermore, the cases of semigroups and groups presented by generators and defining relations are just special cases of associative algebras. Hence Gröbner-Shirshov bases for every Lie algebra, associative algebra, semigroup or group presented by generators and defining relations can be found.

The Composition-Diamond lemma for Lie algebras' (Shirshov (1962), Lemma 3) relates Gröbner-Shirshov bases for Lie algebras to their linear bases. Associative algebra version of this lemma gives a relation between Gröbner-Shirhov bases and normal forms of semigroups or groups presented by generators and defining relations. Since late nineties, finding Gröbner-Shirshov bases and normal forms of semigroups and groups are very active research area. We recommend the survey by Bokut and Chen (2014) to those who are interested in this subject.

Coxeter groups, sometimes called Weyl groups, are one of the most imported example of groups presented by generators and defining relations. Because of this, finding Gröbner-Shirshov bases of these groups attracted the attention of researchers. Gröbner-Shirshov bases for finite Coxeter groups are found in Bokut and Shiao (2001). Svechkarenko (2007) found Gröbner-Shirshov basis for finite exceptional Coxeter group of type E_8. Lee (2008) found Gröbner-Shirshov basis for finite exceptional Coxeter groups of type E_6 and E_7. Then Yılmaz et al. (2014) obtained Gröbner-Shishov bases of infinite Coxeter (Weyl) group of type $\widetilde{A}_n$. Karpuz et al. (2015) found Gröbner-Shirshov bases of some Weyl groups. For the infinite exceptional Weyl group of type F_4, Gröbner–Shirshov basis is constructed by Lee (2016).

The main purpose of this thesis is to obtain Gröbner-Shirshov bases and normal forms for infinite Coxeter groups of type $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$. We; however, observed that all articles written so far has not been interested to find multiplication of normal forms. In fact product of two normal forms is not generally a normal form. Of course, applying Shirshov's reduction process to the product one can obtain normal form but this is very time consuming. We try to find a more explicit method for finding normal form of the multiplication of two normal forms. To do this we used the combinatorial properties of Coxeter groups. In chapter 4, we reproduce Gröbner-Shirshov bases and normal forms of the finite Coxeter groups of type A_n, B_n and D_n using a different order than Bokut and Shiao (2001). Using these new normal forms we are able to find normal form of the product of two normal forms. Hence these groups are completely revealed. In chapter 5, we acquired combinatorial meaning for normal forms of the infinite Coxeter group of type $\widetilde{A}_n$ given in Yılmaz et al. (2014). The results of these two chapters are also to be published as an article (see Ustaoğlu and Yılmaz (2018)). The chapters 6, 7 and 8 are main results of this thesis. We respectively obtained Gröbner-Shirshov bases, normal forms and normal form products of the infinite Coxeter groups of type $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$.

2. BASIC CONCEPTS

2.1 Coxeter Groups

Generators and relations defined Coxeter Groups in an easy way.

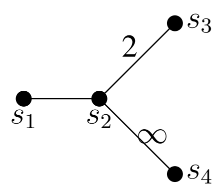
A $n \times n$ symmetric matrix m whose elements are positive integers or positive infinity is called a Coxeter matrix if it satisfies $m_{ij} = 1$ if and only if $i = j$.

Let $S = \{s_1, s_2, \ldots, s_n\}$. A Coxeter matrix can be presented by a Coxeter graph whose nodes are elements of S and whose edges are unordered pairs $\{s_i, s_j\}$ such that $m_{ij} \geq 3$. The edges with $m_{ij} \geq 4$ are weighted by $m_{ij} - 2$. We can give an example of correspondence of between the graph and the matrix.

Example 1. Let $n \times n$ symetric matrix m given by

$$m = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 4 & \infty \\ 2 & 4 & 1 & 2 \\ 2 & \infty & 2 & 1 \end{pmatrix}$$

and relation of between s_1 and s_2 is $m_{12} - 2 = 1$, then there is an edge between them; relation of between s_1 and s_3 is $m_{13} - 2 = 0$, then there is no edge between them; relation of between s_1 and s_4 is $m_{14} - 2 = 0$, then there is no edge between them; relation of between s_2 and s_3 is $m_{23} - 2 = 2$, then there are two edges between them; relation of between s_2 and s_4 is $m_{24} - 2 = \infty$, then there are infinitely many edges between them; and finally relation of between s_3 and s_4 is $m_{34} - 2 = 0$, then there is no edge between them. After then, the graph is



Also we now give the correspondence of the graph and the relations.

Definition 2.1.1. A Coxeter matrix m specifies a group W. It has a presentation Generators which are elements of S and Relations which satisfy $(s_i s_j)^{m_{ij}} = 1$ where s_i and s_j are elements of S.

Because of $m_{ii} = 1$, we have that $s_i^2 = 1$. It is simple to show that for $i \neq j$ and $m_{ij} \neq \infty$, $(s_i s_j)^{m_{ij}} = 1$ is equivalent to

$$\underbrace{s_i s_j s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij}}$$

where length of both sides is m_{ij}.

In previous example, we can give the group determined by the above Coxeter diagram has a presentation as $\{\{s_1, s_2, s_3, s_4\}, s_1^2 = s_2^2 = s_3^2 = s_4^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2, s_1 s_3 = s_3 s_1, s_1 s_4 = s_4 s_1, s_3 s_4 = s_4 s_3, s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2\}$.

(W, S) is called a Coxeter System if a group W has a presentation such as generators and relations where W and S are Coxeter group and the set of Coxeter generators, respectively. The cardinality of S is rank of (W, S). If the Coxeter graph is connected, then the system is irreducible.

In Bjorner and Brenti (2005), the following statements are equivalent and make explicit what it means for W to be determined by m via above presentation.

- If G is a group and $f : S \to G$ is a mapping such that

  $$(f(s_i)f(s_j))^{m_{ij}} = 1$$

  for all $(s_i, s_j) \in S^2$, then there is a unique extention of f to a group homomorphism $f : W \to G$.

- $W \cong F/N$, where F is the free group generated by S and N is the normal subgroup generated by $(s_i s_j)^{m_{ij}}$.

- Let S^* be the free monoid generated by S(i.e., the set of words in the alphabet S with concatenation as product). Let $\equiv$ be the equivalence relation generated by allowing insertion or deletion of any word of the form

  $$(s_i s_j)^{m_{ij}} = \underbrace{s_i s_j s_i s_j \cdots}_{2m_{ij}}.$$

  Then, $S^*/\equiv$ forms a group isomorphic to W.

Let (W, S) be a Coxeter system. Definition 2.1.1 leaves some uncertainty about the orders of pairwise products $s_i s_j$ as elements of W where $s_i, s_j \in S$. All that follows is that the order of $s_i s_j$ divides m_{ij} if m_{ij} is finite. This leaves open the possibility that distinct Coxeter graphs might determine isomorphic Coxeter systems. However, this is not the case.

Proposition 2.1.2 (Bjorner and Brenti (2005),Proposition 1.1.1)**.** (W, S) *is the Coxeter system determined by a Coxeter matrix* m*. Let* s_i *and* s_j *be distinct elements of* S*. Then, the followings hold:*

(i) s_i *and* s_j *are distinct in* W*.*

(ii) *The order of* $s_i s_j$ *in* W *is* m_{ij}*.*

Theorem 2.1.3 (Bjorner and Brenti (2005),Theorem 1.1.2)**.** *Up to isomorphism there is a one-to-one correspondence between Coxeter matrices and Coxeter systems.*

Assume that (W, S) is a Coxeter system. Then $w = s_1 s_2 \cdots s_k$ where $s_i \in S$ is a generator, $1 \leq i \leq k$, for all $w \in W$. k is called the length of w denoted by $l(w) = k$. If k is minimum for w, then word written as a product of generators is called a reduced word for w.

The all finite irreducible Coxeter systems and some infinite Coxeter systems have been classified (see Bjorner and Brenti (2005)). We give Coxeter graphs of the finite Coxeter groups of type A_n, B_n and D_n and the infinite Coxeter groups of type $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$ in the following tables.

Table 2.1. The some finite irreducible Coxeter systems

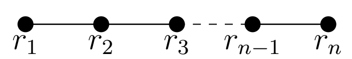
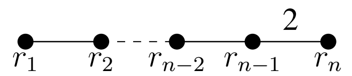
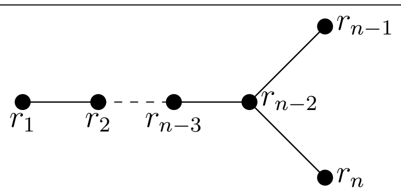
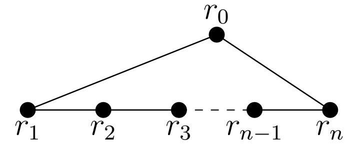
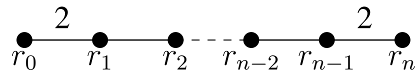
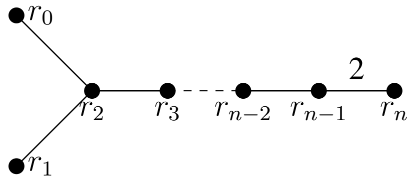
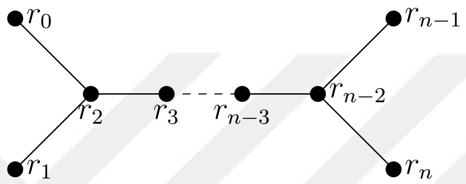
Name	Diagram
A_n $(n \geq 1)$	
B_n $(n \geq 2)$	
D_n $(n \geq 4)$	

Table 2.2. The some infinite Coxeter systems

Name	Diagram
$\widetilde{A}_n$ $(n \geq 3)$	
$\widetilde{C}_n$ $(n \geq 2)$	
$\widetilde{B}_n$ $(n \geq 3)$	
$\widetilde{D}_n$ $(n \geq 4)$	

2.2 Poincaré Series

Definition 2.2.1. Let W be a Coxeter group and $A \subseteq W$. Define

$$A(x) := \sum_{w \in A} x^{l(w)}.$$

$A(x)$ is called Poincaré series of A. (If $|A| < \infty$, then $A(x)$ is the Poincaré polynomial of A.)

Lemma 2.2.2 (Bjorner and Brenti (2005),Theorem 7.1.1)**.** *Suppose that* $W = \prod_{i=1}^{k} W_i$, *where* $W_1, W_2, \ldots, W_k$ *are irreducible Coxeter systems. Then*

$$W(x) = \prod_{i=1}^{k} W_i(x).$$

Define $[i]_x = 1 + x + x^2 + \ldots + x^{i-1}$ for $i \geq 1$.

Theorem 2.2.3 (Bjorner and Brenti (2005),Theorem 7.1.5)**.** *Let* (W, S) *be a finite irreducible Coxeter system, and* $n := |S|$. *Then, there exist positive integers* $e_1, e_2, \ldots e_n$ *such that*

$$W(x) = \prod_{i=1}^{n} [e_i + 1]_x.$$

The integers $e_1, e_2, \ldots, e_n$ are called exponents of (W, S) which is given by the following table for some finite Coxeter groups.

Table 2.3. Exponents for some finite Coxeter groups

Name	Exponents
A_n $(n \geq 1)$	$1, 2 \ldots, n$
B_n $(n \geq 2)$	$1, 3 \ldots, 2n-1$
D_n $(n \geq 4)$	$1, 3 \ldots, 2n-3, n-1$

After some easy computations, the orders of the finite Coxeter groups of type A_n, B_n and D_n can be found as $(n+1)!$, $2^n n!$ and $2^{n-1}n!$, respectively (see Bjorner and Brenti (2005) Appendix $A1$).

Theorem 2.2.4 (Bjorner and Brenti (2005),Theorem 7.1.10)**.** *Let (W, S) be infinite Coxeter system, and let $e_1, e_2, \ldots, e_n$ be the exponents of the corresponding finite group. Then,*

$$W(x) = \prod_{i=1}^{n} \frac{[e_i + 1]_x}{1 - x^{e_i}}.$$

If we remove r_0 from the graph of $\widetilde{C}_n$, then it becomes same graph of the Coxeter graph of B_n. Hence we can use the exponent of B_n for Poincaré Series of $\widetilde{C}_n$.

2.3 Some Basic Definitions of Gröbner-Shirshov Basis

First of all, we recall some concepts about the Gröbner-Shirshov basis theory. Let S be a set and S^* be the free monoid of words generated by S. We denote empty word by 1.

A well ordering $<$ on S^* is called monomial order if $x < y$ implies $axb < ayb$ for all $a, b \in S^*$. We use degree lexicographic order in this work.

Let $k\langle S\rangle$ be free associative algebra generated by S over a field k. Given $0 \neq f \in k < S >$, we denote by $\overline{f}$ the leading word in f, the biggest element of f with respect to given monomial order.

For two monic polynomials f and g, if there is a word w such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in S^*$, then the composition of f and g defined by $< f, g >_w = fb - ag$. If such a word w does not exist and $\overline{f} > \overline{g}$, then $\overline{f} = a\overline{g}b$ for some $a, b \in S^*$. In this case the composition defined as $< f, g > = f - agb$.

The transformation $f \mapsto f - agb$ is called the elimination of the leading word (ELW) of f in g. Let $R \subset k\langle S\rangle$ be set of monic polynomials and f be another monic polynomial. We say f is reduced to h modulo R if f is obtained by a sequence of ELW in elements of R and further ELW of h is not possible. A set $R \subset k\langle S\rangle$ is called a Gröbner-Shirshov Basis if any composition of polynomials from R is reduced to zero modulo R.

If $R \subset k\langle S\rangle$ is not a Gröbner-Shirshov basis, then take a composition of polynomials from R and reduce it modulo R. If reduction process produce a non zero polynomial r, then enlarge the set R by r. Repeat this process for each composition of polynomials from R until no more enlargement of R is necessary. Then the set you obtain is a Gröbner-Shirshov basis. Such a process is called Shirshov algorithm.

The following well known lemma is useful for finding normal form of a group via its Gröbner-Shirshov basis.

Lemma 2.3.1. *(Composition-Diamond lemma for associative algebras)*

Let k be a field, $A = k\langle S|R\rangle = k\langle S\rangle/Id(R)$ and $<$ a monomial ordering on S^*, where $Id(R)$ is the ideal of $k\langle S\rangle$ generated by R. Then the following statements are equivalent:

(i) *R is a Gröbner-Shirshov basis.*

(ii) *$f \in Id(R) \Rightarrow \overline{f} = a\overline{s}b$ for some $s \in R$ and $a, b \in S^*$.*

(iii) *The set of R-reduced words*

$$Red(R) = \{w \in S^* | w \neq a\overline{s}b, a, b \in S^*, s \in R\}$$

is a k-linear basis for the algebra $A = k\langle S|R\rangle$.

If a group G is defined by generators S and relations R, then we can identify each relation $a = b$ in R with a polynomial $a - b$. Thus the set of relations can be considered as a subset of $k\langle S\rangle$. Therefore one can find a Gröbner-Shirshov basis of R which we call a Gröbner-Shirshov basis of the group G. Notice that R consists of "biwords", differences of words. The Shirshov algorithm maintains this property throughout the entire computation. Hence Gröbner-Shirshov basis of a group can be thought as a special set of relations for this group. Furthermore the set $Red(R) = \{w \in S^* | w \neq a\overline{s}b, a, b \in S^*, s \in R\}$ becomes the set of all normal forms of G by the Composition-Diamond lemma.

3. MATERIALS AND METHODS

The main idea for finding Gröbner-Shirshov bases for the infinite Coxeter groups of type $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$ is to apply Shirshov algorithm for small n's. Then we generalize and prove the results for every n. This works for finite Coxeter groups. One; however, can not apply Shirshov algorithm by hand for infinite Coxeter groups. Hence we wrote some codes for Shirshov algorithm using Mathematica. But even with the help of the computer, we have not succeeded in getting the Gröbner-Shirshov bases of infinite Coxeter groups. Then we decided to partially apply Shirshov algorithm. After adding some new polynomials to the basis, we found corresponding reduced words. To do this, we also wrote some codes in Mathematica. Since number of elements in each length is known for Coxeter groups, we checked that number of reduced words of each length is equal to this known number up to certain lengths. If equality does not hold, then we continue to apply Shirshov algorithm. If equality holds for large lengths, we supposed that we found the Gröbner-Shirshov basis since Composition-Diamond lemma implies if the set of reduced words of given polynomials is equal to the set of elements of the Coxeter group, then these polynomials form a Gröbner-Shirshov basis. Then we generalize our results to every n and then try to prove them using combinatorial techniques. If we were not able to prove our generalizations, then we supposed that our generilazations were not correct. In this case we went back to the algorithm and tried to find extra polynomials on the Gröbner-Shirshov basis.

4. GRÖBNER-SHIRSHOV BASES AND NORMAL FORMS FOR SOME FINITE COXETER GROUPS

In this chapter, we will obtain Gröbner-Shirshov bases and normal forms of the finite Coxeter groups of type A_n, B_n and D_n. We also find the product of two normal forms as a normal form in all these Coxeter groups. Let us start with the finite Coxeter group of type A_n.

4.1 The Finite Coxeter Group of Type A_n

Let $S = \{r_1, r_2, \ldots, r_n\}$ and

$$r_{ij} = \begin{cases} r_i r_{i+1} \cdots r_j & \text{if } 1 \leq i < j \leq n\ ; \\ r_i r_{i+1} \cdots r_n r_{n-1} \cdots r_{2n-j} & \text{if } 1 \leq i \leq n < j \leq 2n-i; \\ r_i & \text{if } j = i; \\ 1 & \text{if } j = i-1. \end{cases}$$

Suppose that $<$ is the degree lexicographic order on S. A set R is Gröbner-Shirshov basis hereafter means it is a Gröbner-Shirshov basis with respect to $<$.

Definition 4.1.1. The finite Coxeter group of type A_n $(n \geq 1)$ is generated by $S = \{r_1, r_2, \cdots, r_n\}$ with defining relations:

(R_1) $r_i r_i = 1$ for $1 \leq i \leq n$,

(R_2) $r_i r_j = r_j r_i$ for $1 \leq i < j-1 \leq n-1$,

(R_3) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ for $1 \leq i \leq n-1$.

Hence A_n has a presentation $A_n = \langle S \mid \{R_1, R_2, R_3\} \rangle$.

Proposition 4.1.2. *A reduced Gröbner-Shirshov basis of the finite Coxeter group of type A_n contains the following polynomials:*

- $f_1^{(i)} = r_i r_i - 1$ *for* $1 \leq i \leq n$,

- $f_2^{(i,j)} = r_i r_j - r_j r_i$ *for* $1 \leq i \leq j-1 \leq n-1$,
- $f_3^{(i)} = r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}$ *for* $1 \leq i \leq n-1$,
- $g^{(i,j)} = r_{ij} r_i - r_{i+1} r_{ij}$ *for* $1 \leq i < j-1 \leq n-1$.

Proof. The first three polynomials come from defining relations of the finite Coxeter group of type A_n. We can obtain $g^{(i,j)}$ with the following compositions of inclusion.

$$
\begin{aligned}
< f_3^{(i)}, f_2^{(i,i+2)} > &= (r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}) r_{i+2} - r_i r_{i+1}(r_i r_{i+2} - r_{i+2} r_i) \\
&= r_i r_{i+1} r_{i+2} r_i - r_{i+1} r_i r_{i+1} r_{i+2} \\
&= g^{(i,i+2)}
\end{aligned}
$$

for $1 \leq i \leq n-2$.

$$
\begin{aligned}
< g^{(i,j)}, f_2^{(i,j+1)} > &= (r_{ij} r_i - r_{i+1} r_{ij}) r_{j+1} - r_{ij}(r_i r_{j+1} - r_{j+1} r_i) \\
&= r_{i,j+1} r_i - r_{i+1} r_{i,j+1} \\
&= g^{(i,j+1)}
\end{aligned}
$$

for $1 \leq i < j-1 \leq n-2$

□

Notice that we are not claiming that these polynomials form a Gröbner-Shirshov basis for the finite Coxeter group of type A_n at this point. A Gröbner-Shirshov basis may contain more polynomials.

We now give a well known combinatorial description of the finite Coxeter group of type A_n.

Proposition 4.1.3. *(Bjorner and Brenti (2005), Proposition 1.5.4) The symmetric group* S_{n+1} *with generating set* $S = \{r_1, \ldots, r_n\}$ *is the finite Coxeter group of type* A_n *where* $r_i = (i\ i+1)$ *for* $i = 1, \ldots, n$.

Definition 4.1.4. Given $u \in S_{n+1}$, let

$$I_i(u) := \{u(j) : u(j) > i \text{ for some } j < u^{-1}(i)\}$$

for $i = 1, \ldots, n+1$.

Lemma 4.1.5. *Let* $u \in S_{n+1}$ *such that* $u(l) = l$ *for* $1 \leq l \leq k-1$ *and let* $x = r_{kj_k}$ *where* $j_k = |I_k(u)| + k - 1$. *If* $v = ux^{-1}$, *then* $v(l) = l$ *for* $1 \leq l \leq k$ *and* $I_i(v) = I_i(u)$ *for* $i \neq k$.

Proof. Since $u^{-1}(l) = l$ for $1 \leq l \leq k-1$, $|I_k(u)| = u^{-1}(k) - k$ and then $j_k + 1 = |I_k(u)| + k = u^{-1}(k)$. Hence $v(l) = u(x^{-1}(l)) = u(l) = l$ for $1 \leq l \leq k-1$ and $v(k) = u(x^{-1}(k)) = u(j_k + 1) = k$.

Let $i \neq k$. Notice that $u^{-1}(i) \neq j_{k+1} = u^{-1}(k)$. Suppose $t \in I_i(u)$. Then $u(j) > i$ for some $j < u^{-1}(i)$. Since $u^{-1}(i) \neq j_k + 1$, x preserve the inequality that is $x(j) < x(u^{-1}(i)) = v^{-1}(i)$. Therefore $t = v(x(j)) > i$ for some $x(j) < v^{-1}(i)$. This implies $t \in I_i(v)$. Conversely suppose $t \in I_i(v)$. Hence $t = v(j) > i$ for some $j < v^{-1}(i)$. Since $v^{-1}(i) = x(u^{-1}(i)) \neq x(j_k + 1) = k$, x^{-1} preserve the inequality that is $x^{-1}(j) < x^{-1}(v^{-1}(i)) = u^{-1}(i)$. Therefore $t = u(x^{-1}(j)) > i$ for some $x^{-1}(j) < u^{-1}(i)$. This implies $t \in I_i(u)$. □

Theorem 4.1.6. *Any $w \in S_{n+1}$ can be represented in a form $r_{nj_n} \cdots r_{ij_i} \cdots r_{1j_1}$ where $i - 1 \leq j_i = |I_i(w)| + i - 1 \leq n$ for $i = 1, \ldots, n$.*

Proof. Let $v_0 = w$. Suppose that $v_k = v_{k-1}(r_k j_k)^{-1}$ where $j_k = |I_k(v_{k-1})| + k - 1$ for $k = 1, \ldots, n$. Lemma 4.1.5 implies $v_k(l) = l$ for $l = 1, \ldots, k$ and $I_i(v_k) = I_i(v_{k-1})$ for $i \neq k$. Hence

$$j_k = |I_k(v_{k-1})| + k - 1 = |I_k(v_{k-2})| + k - 1 = \cdots = |I_k(w)| + k - 1$$

for $k = 1, \ldots, n$. Furthermore $v_n = w(r_{1j_1})^{-1} \cdots (r_{nj_n})^{-1}$ is the identity element. Hence

$$w = r_{nj_n} \cdots r_{ij_i} \cdots r_{1j_1}.$$

□

We can now give main result of this section.

Corollary 4.1.7. *Let R be the set of polynomials given in Proposition 4.1.2. Then*

(i) $Red(R) = \{r_{nj_n} \cdots r_{ij_i} \cdots r_{1j_1} | i - 1 \leq j_i \leq n,\ i = 1, \ldots, n\}$.

(ii) R is a Gröbner-Shirshov basis for the finite Coxeter group of type A_n.

Proof. (i) One can easily check that $w \in Red(R)$ for any $w \in A$. Conversely if $w \in Red(R) \subset S^*$, then w corresponds to a permutation in S_{n+1}. Hence $w \in A$ by Theorem 4.1.6.

(ii) If R was not a Gröbner-Shirshov basis, then $R \subset \bar{R}$ where $\bar{R}$ was a Gröbner-Shirshov basis. So Red$(\bar{R})$ should contain more words than Red(R) but there are $(n+1)!$ words in Red(R) which is exactly same as number of elements in the finite Coxeter group of type A_n. Hence Red(R) is the set of normal forms and R is a Gröbner-Shirshov basis for the finite Coxeter group of type A_n by the Composition-Diamond lemma.

□

Notice that Theorem 4.1.6 gives a method for converting any element of the finite Coxeter group of type A_n into its normal form. Let us conclude this section by an example of finding the normal form of the product of two normal forms.

Example 2. Let $x = r_4 r_{23} r_{14}$ and $y = r_5 r_4 r_{35} r_{25} r_{14}$ in A_5.

$$\begin{aligned} x &= r_4 r_{23} r_{14} \\ &= (4\ 5)(2\ 3\ 4)(1\ 2\ 3\ 4\ 5) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 1 & 6 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} y &= r_5 r_4 r_{35} r_{25} r_{14} \\ &= (5\ 6)(4\ 5)(3\ 4\ 5\ 6)(2\ 3\ 4\ 5\ 6)(1\ 2\ 3\ 4\ 5) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 1 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} z &= xy \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 1 & 2 & 3 & 5 \end{pmatrix} \end{aligned}$$

Then $|I_5(z)| = 1$, $|I_4(z)| = 1$, $|I_3(z)| = 2$, $|I_2(z)| = 2$ and $|I_1(z)| = 2$. Theorem 4.1.6 implies $z = r_5 r_4 r_{34} r_{23} r_{12}$.

4.2 The Finite Coxeter Group of Type B_n

Now let us do similar computations for the finite Coxeter group of type B_n.

Definition 4.2.1. The finite Coxeter group of type B_n $(n \geq 2)$ is generated by $S = \{r_1, r_2, \cdots, r_n\}$ with defining relations:

(R_1) $r_i r_i = 1$ for $1 \leq i \leq n$,

(R_2) $r_i r_j = r_j r_i$ for $1 \leq i < j-1 < n$,

(R_3) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ for $1 \leq i < n-1$,

(R_4) $r_{n-1} r_n r_{n-1} r_n = r_n r_{n-1} r_n r_{n-1}$.

Hence B_n has a presentation $B_n =< S \mid \{R_1, R_2, R_3, R_4\} >$.

Proposition 4.2.2. *A reduced Gröbner-Shirshov basis of the finite Coxeter group of type B_n contains the following polynomials:*

- $f_1^{(i)} = r_i r_i - 1$ *for* $1 \leq i \leq n$,
- $f_2^{(i,j)} = r_i r_j - r_j r_i$ *for* $1 \leq i < j-1 < n$,
- $f_3^{(i,j)} = r_{ij} r_i - r_{i+1} r_{ij}$ *for* $1 \leq i \leq n-2$ *and* $i < j < 2n-i-1$,
- $f_4^{(i)} = r_{i,2n-i} r_{i+1} - r_{i+1} r_{i,2n-i}$.

Proof. The polynomials $f_1^{(i)}$ and $f_2^{(i,j)}$ come from the defining relations (R_1) and (R_2), respectively. Similarly the polynomials $f_3^{(i,i+1)}$ and $f_4^{(n-1)}$ comes from the defining relation (R_3) and (R_4), respectively. Now let us apply the Shirshov algorithm to these polynomials.

$< f_3^{(i,j-1)}, f_2^{(i,j)} >= f_3^{(i,j)}$ for $n \geq j$,

$< f_3^{(i,j-1)}, f_2^{(i,2n-j)} >= f_3^{(i,j)}$ for $n < j$,

$< f_3^{(i,2n-2-i)}, f_3^{(i,i+1)} >= f_4^{(i)}$.

□

Notice again that a Gröbner-Shirshov basis of the finite Coxeter group of type B_n may contain more polynomials.Using the combinatorial description of the finite Coxeter group of type B_n, we will show that the polynomials given above proposition are in fact form a Gröbner-Shirshov basis.

Let S_n^B be the group of all permutations w on $[\pm n] = \{-n, -(n-1), \dots, -1, 1, \dots, n\}$ such that

$$w(-i) = -w(i)$$

for all $i \in [\pm n]$. Clearly, such a w is uniquely determined by its values on $[n] = \{1, 2, \dots, n\}$ and we write $w = [w_1, w_2, \dots, w_n]$ where $w_i = w(i)$ for $i = 1, \dots, n$. This notation is window notation for w. If $u \in S_n$, then its extension to S_n^B is $[u] = [u_1, u_2, \dots, u_n]$ where $u_i = u(i)$ for $i = 1, 2, \dots, n$.

Proposition 4.2.3. *(Bjorner and Brenti (2005) ,Proposition 8.1.3) The group S_n^B with generating set $S = \{r_1, \dots, r_n\}$ is the finite Coxeter group of type B_n where $r_i = [(i\ i+1)]$ for $i = 1, 2, \dots, n-1$ and $r_n = (n\ \ -n)$.*

Lemma 4.2.4. *The following equality holds in B_n*

$$(i\ \ -i)r_{ij_i} = r_{i,2n-j_i-1}$$

where $i - 1 \leq j_i \leq n - 1$ for $1 \leq i \leq n - 1$.

Proof. First of all, $r_{i,2n-i} = [(i\ i+1\ \cdots\ n)](n\ \ -n)[(n\ n-1\ \cdots\ i)] = (i\ \ -i)$ for $i = 1, \dots, n$. Then $r_{i,2n-j_i-1} = r_{i,2n-i}r_{ij_i} = (i\ \ -i)r_{ij_i}$ for $i - 1 \leq j_i \leq n$. □

Theorem 4.2.5. *Given $w = [w_1, w_2, \dots, w_n] \in S_n^B$, let $u \in S_n$ such that $u(i) = |w_i|$ for $i = 1 \dots, n$. Then w can be uniquely represented in a form*

$$r_{nj_n} \cdots r_{ij_i} \cdots r_{1j_1}$$

where $j_i = |I_i(u)| + i - 1$ if $w^{-1}(i) > 0$ and $j_i = 2n - |I_i(u)| - i$ if $w^{-1}(i) < 0$.

Proof. Clearly,

$$w = (\prod_{w^{-1}(i)<0} (i\ \ -i))[u]$$

Theorem 4.1.6 implies $[u]$ has the unique representation

$$r_{n\overline{j_n}} \cdots r_{i\overline{j_i}} \cdots r_{1\overline{j_1}}$$

where $\overline{j_i} = |I_i(u)| + i - 1$ for $i = 1, \dots, n$. Here we abuse the notation. In fact $r_n \notin A_{n-1}$ but $r_{n\overline{j_n}} = 1$ since $\overline{j_n} = n - 1$. Since $(i\ \ -i)$ and r_{kj_k} commute when $k > i$, we can write

$$r_{nj_n} \cdots r_{ij_i} \cdots r_{1j_1}$$

where $r_{ij_i} = r_{i\overline{j_i}}$ if $w^{-1}(i) > 0$ and $r_{ij_i} = (i\ \ -i)r_{i\overline{j_i}}$ if $w^{-1}(i) < 0$.

Since $(i\ \ -i)r_{ij_i} = r_{i,2n-j_i-1}$ by Lemma 4.2.4,

$$(i\ \ -i)r_{i,|I_i(u)|+i-1} = r_{i,2n-(|I_i(u)|+i-1)-1} = r_{i,2n-|I_i(u)|-i}$$

when $w^{-1} < 0$. Hence w has desired representation.

□

As a consequence of this theorem, we can conclude the following.

Corollary 4.2.6. *Let R be the set of polynomials given in Proposition 4.2.2. Then*

(i) $Red(R) = \{r_{nj_n} \cdots r_{ij_i} \cdots r_{1j_1} | i-1 \leq j_i \leq 2n-i,\ i = 1, \ldots, n\}$.

(ii) R is a Gröbner-Shirshov basis for the finite Coxeter group of type B_n.

Proof. The first statement is easily follows from Theorem 4.2.5 and the fact that the words in the right hand side are in Red(R).

Notice that there are $2^n n!$ words of the form $r_{nj_n} \cdots r_{ij_i} \cdots r_{1j_1}$ where $i-1 \leq j_i \leq 2n-i$ for $i = 1, \ldots, n$. This is same as number of elements of the finite Coxeter group of type B_n. This implies Red(R) is the set of normal forms and R is a Gröbner-Shirshov basis for the finite Coxeter group of type B_n by the Composition-Diamond lemma. □

Example 3. Let us take two different words in B_4 such that $x = r_4 r_{35} r_{23} r_{14}$ and $y = r_{34} r_{25} r_{16}$.

$x = r_4 r_{35} r_{23} r_{14}$

$\quad = [1, 2, 3, -4][1, 2, -3, 4][1, 3, 4, 2][2, 3, 4, -1]$

$\quad = [-3, -4, 2, -1]$

and

$y = r_{34} r_{25} r_{16}$

$\quad = [1, 2, 4, -3][1, 3, -2, 4][2, -1, 3, 4]$

$\quad = [4, -1, -2, -3]$

$z = xy$

$\quad = [-3, -4, 2, -1][4, -1, -2, -3]$

$\quad = [-1, 3, 4, -2]$

Let $u \in S_3$ such that $[u] = [1, 3, 4, 2]$. Then $|I_4(u)| = 0$, $|I_3(u)| = 0$, $|I_2(u)| = 2$ and $|I_1(u)| = 0$. Therefore $z = r_{24} r_{17}$ by Theorem 4.2.5.

4.3 The Finite Coxeter Group of Type D_n

Finally we consider the finite Coxeter group of type D_n.

Definition 4.3.1. The finite Coxeter group of type D_n $(n \geq 4)$ is generated by $S = \{r_1, r_2, \cdots, r_n\}$ with defining relations:

(R_1) $r_i r_i = 1$ for $1 \leq i \leq n$,

(R_2) $r_i r_j = r_j r_i$ for $1 \leq i < j-1 < n$ except $(i,j) = (n-2, n)$,

(R_3) $r_{n-1} r_n = r_n r_{n-1}$,

(R_4) $r_{n-2} r_n r_{n-2} = r_n r_{n-2} r_n$,

(R_5) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ for $1 \leq i < n-1$.

Hence D_n has a presentation $D_n =< S|\{R_1, R_2, R_3, R_4, R_5\} >$.

Let

$$\widetilde{r}_{ij} = \begin{cases} r_i r_{i+1} \cdots r_{n-2} r_n r_{n-1} \cdots r_{2n-j}, & 1 \leq i < n-1 < j \leq 2n-i, \\ r_{ij}, & otherwise \end{cases}$$

Hence $\widetilde{r}_{ij}$ and r_{ij} are interchangeable if $i-1 \leq j_i < n$ for $1 \leq i \leq n$.

Proposition 4.3.2. *A reduced Gröbner-Shirshov basis of the finite Coxeter group of type D_n contains the following polynomials:*

- $f_1^{(i)} = r_i r_i - 1$ *for* $1 \leq i \leq n$,
- $f_2^{(i,j)} = r_i r_j - r_j r_i$ *for* $1 \leq i < j-1 < n$ *except* $(i,j) = (n-2, n)$,
- $f_3 = r_{n-1} r_n - r_n r_{n-1}$,
- $f_4 = \widetilde{r}_{n-2,n} r_{n-2} - r_n \widetilde{r}_{n-2,n}$,
- $f_5^{(i,j)} = \widetilde{r}_{ij} r_i - r_{i+1} \widetilde{r}_{ij}$ *for* $(1 \leq i < j-1 \leq n-1)$ *or* $(1 \leq i < n-2$ $n \leq j \leq 2n-3$ *and* $2n-j-1 > 1)$,
- $f_6^{(i)} = \widetilde{r}_{i,2n-i} r_{i+1} - r_{i+1} \widetilde{r}_{i,2n-i}$ *for* $1 \leq i \leq n-3$,
- $f_7 = \widetilde{r}_{n-2,n+2} r_n - r_{n-1} \widetilde{r}_{n-2,n+2}$,
- $f_8 = \widetilde{r}_{n-2,n+2} r_{n-1} - r_n \widetilde{r}_{n-2,n+2}$.

Proof. The polynomials $f_1^{(i)}$, $f_2^{(i,j)}$, $f_3^{(i)}$, f_4, $f_5^{(i,i+1)}$ come across the defining relations (R_1), (R_2), (R_3), (R_4) and (R_5), respectively. Now let us apply the Shirshov algorithm to these polynomials.

$< f_5^{(i,i+1)}, f_2^{(i,i+2)} >= f_5^{(i,i+2)}$ for $1 \leq i < n-2$,

$< f_5^{(i,j-1)}, f_2^{(i,j)} >= f_5^{(i,j)}$ for $i+3 \leq j \leq n-1$,

$< f_5^{(n-3,n-2)}, f_2^{(n-3,n)} >= f_5^{(n-3,n)}$,

$< f_5^{(i,n-2)}, f_2^{(i,n)} >= f_5^{(i,n)}$ for $1 \leq i < n-3$,

$< f_5^{(i,2n-j-1)}, f_2^{(i,j)} >= f_5^{(i,2n-j)}$ for $1 \leq i < n-2,\ 3 \leq j \leq n-1$ and $j-i > 1$,

$< f_5^{(i,2n-i-2)}, f_5^{(i,i+1)} >= f_6^{(i)}$ for $1 \leq i \leq n-3$,

$< f_5^{(n-2,n)}, f_4 >= r_{n-2} f_3^{(n-1)} r_{n-2} r_n - r_{n-1} r_{n-2} f_3^{(n-1)} r_{n-2} + f_7$,

$< f_4, f_5^{(n-2,n)} >= f_8$.

□

We will show that the polynomials given in the above propositions form a Gröbner-Shirshov basis using the technique used in the previous sections.

Let S_n^D be the subgroup of S_n^B consisting of all of the signed permutations having an even number of negative entries in their window notation.

Proposition 4.3.3. *(Bjorner and Brenti (2005),Proposition 8.2.3) The group S_n^D with generating set S is the finite Coxeter group of type D_n where $r_i = [(i\ i+1)]$ for $i = 1, 2, \ldots, n-1$ and $r_n = [(n-1\ \ -n)]$.*

Lemma 4.3.4. *Let $r_{ij_i} \in D_n$ where $i-1 \leq j_i \leq n-1$ for $1 \leq i \leq n-2$. Then we have the followings:*

(i) $r_{ij_i}(n\ -n) = (n\ -n) r_{ij_i}$ *for* $i-1 \leq j_i \leq n-2$ *and* $1 \leq i \leq n-2$.

(ii) $(i\ -i) r_{ij_i} = (n\ -n) \widetilde{r}_{i,2n-j_i-1}$ *for* $i-1 \leq j_i \leq n-2$ *and* $1 \leq i \leq n-2$.

(iii) $r_{i,n-1}(n\ -n) = (i\ -i) r_{i,n-1} = (n\ -n) \widetilde{r}_{in}$ *for* $1 \leq i \leq n-2$.

(iv) $(i\ -i) r_{i,n-1}(n\ -n) = r_{i,n-1} = (n\ -n)^2 r_{i,n-1}$ *for* $1 \leq i \leq n-2$.

Proof. (i) It is easily follows from definition of r_{ij_i}.

(ii) $\widetilde{r}_{i,2n-i} = [(i\ i+1\ \cdots\ n-1)][(n-1\ \ -n)][(n\ n-1\ \cdots\ i)] = (n\ \ -n)(i\ \ -i)$ for $i = 1, \dots, n-2$. So

$$\widetilde{r}_{i,2n-j_i-1} = \widetilde{r}_{i,2n-i} r_{ij_i} = (n\ \ -n)(i\ \ -i) r_{ij_i}$$

where $i-1 \leq j_i \leq n-2$ for $i = 1, 2, \dots, n-2$. Multiplying by $(n\ \ -n)$ from the left gives the desired equality.

(iii) $\widetilde{r}_{i,n-1}(n\ \ -n) = [(i \cdots n)](n\ \ -n) = (i\ \ -i)[(i\ \cdots\ n)] = (i\ \ -i)\widetilde{r}_{i,n-1} = (n\ \ -n)[(i\ \cdots\ n)][(n-1\ \ -n)] = (n\ \ -n)\widetilde{r}_{in}$ for $1 \leq i \leq n-2$.

(iv) It is a consequence of part (iii).

□

We now ready to provide a representation for each element of S_n^D but we need one more definition. Given $w = [w_1, w_2, \dots, w_n] \in S_n^D$, define

$$neg(w, i) := |\{\{w_1, w_2, \dots, w_n\} : -i < w_j < 0,\ \ 1 \leq j \leq n\}|$$

for $i = 1, \dots, n$.

Theorem 4.3.5. *Given $w = [w_1, w_2, \dots, w_n] \in S_n^D$,let $u \in S_n$ such that $u(i) = |w_i|$ for $i = 1, \dots, n$. Then w can be uniquely represented in a form*

$$\widetilde{r}_{nj_n} \cdots \widetilde{r}_{ij_i} \cdots \widetilde{r}_{1j_1}$$

where the indices are given by the following rules:

(i) If $|I_i(u)| + i - 1 \neq n-1$, then $j_i = |I_i(u)| + i - 1$ when $w^{-1}(i) > 0$ and $j_i = 2n - |I_i(u)| - i$ when $w^{-1}(i) < 0$ for $i = 1, \dots, n-2$,

(ii) If $|I_i(u)| + i - 1 = n-1$, then $j_i = n$ when $w^{-1}(i)(-1)^{neg(w,i)} < 0$ and $j_i = n-1$ when $w^{-1}(i)(-1)^{neg(w,i)} > 0$ for $i = 1, \dots, n-2$,

(iii) If $|I_{n-1}(u)| = 0$, then $(j_{n-1}, j_n) = (n-2, n-1)$ when $w^{-1}(n-1) > 0$ and $(j_{n-1}, j_n) = (n-1, n)$ when $w^{-1}(n-1) < 0$ and

(iv) If $|I_{n-1}(u)| = 1$, then $(j_{n-1}, j_n) = (n-1, n-1)$ when $w^{-1}(n) > 0$ and $(j_{n-1}, j_n) = (n-2, n)$ when $w^{-1}(n) < 0$.

Proof. Clearly,

$$w = (\prod_{w^{-1}(i)<0} (i \ -i))[u]$$

Theorem 4.1.6 implies $[u]$ has the unique representation

$$r_{n-1,\bar{j}_{n-1}} \cdots r_{i\bar{j}_i} \cdots r_{1\bar{j}_1}$$

where $\bar{j}_i = |I_i(u)| + i - 1$ for $i = 1, \dots, n-1$.

(i) Given $1 \leq i \leq n-2$, let $\bar{j}_i \neq n-1$. If $w^{-1}(i) > 0$, then

$$r_{i\bar{j}_i}(n \ -n)^{neg(w,i)} = (n \ -n)^{neg(w,i+1)} r_{i\bar{j}_i}$$

by part (i) of Lemma 4.3.4. If $w^{-1}(i) < 0$, then

$$(i \ -i) r_{i\bar{j}_i}(n \ -n)^{neg(w,i)} = (n \ -n)^{neg(w,i+1)} r_{i,2n-\bar{j}_i-1} = (n \ -n)^{neg(w,i+1)} r_{i,2n-|I_i(u)|-i}$$

by part (i) and (ii) of Lemma 4.3.4.

(ii) Given $1 \leq i \leq n-2$, let $\bar{j}_i \neq n-1$. If $w^{-1}(i) < 0$ and $neg(w,i)$ is even, then

$$(i \ -i) r_{i,n-1}(n \ -n)^{neg(w,i)} = (i \ -i) r_{i,n-1} = (n \ -n)\widetilde{r}_{in} = (n \ -n)^{neg(w,i+1)}\widetilde{r}_{in}$$

by part (iii) of Lemma 4.3.4. If $w^{-1}(i) > 0$ and $neg(w,i)$ is odd, then

$$r_{i,n-1}(n \ -n)^{neg(w,i)} = r_{i,n-1}(n \ -n) = (n \ -n)\widetilde{r}_{in} = (n \ -n)^{neg(w,i+1)}\widetilde{r}_{in}$$

by part (iii) of Lemma 4.3.4. If $w^{-1}(i) < 0$ and $neg(w,i)$ is odd, then

$$(i \ -i) r_{i,n-1}(n \ -n)^{neg(w,i)} = (i \ -i) r_{i,n-1}(n \ -n) = r_{i,n-1} = (n \ -n)^{neg(w,i+1)} r_{i,n-1}$$

by part (iv) of Lemma 4.3.4. If $w^{-1}(i) > 0$ and $neg(w,i)$ is even, then clearly

$$r_{i,n-1}(n \ -n)^{neg(w,i)} = r_{i,n-1} = (n \ -n)^{neg(w,i+1)} r_{i,n-1}.$$

(iii) If $|I_{n-1}(u)| = 0$, then $j_{n-1} = n-2$. Since number of negative entries is even in the window notation of w, $w^{-1}(n)$ and $(-1)^{neg(w,n-1)}$ have same sign when $w^{-1}(n-1) > 0$. In this case

$$(n \ -n) r_{n-1,n-2}(n \ -n) = r_{n-1,n-2} = r_{n,n-1} r_{n-1,n-2}.$$

On the other hand, clearly $w^{-1}(n)$ and $(-1)^{neg(w,n-1)}$ have opposite sign when $w^{-1}(n-1) < 0$. In this case

$$
\begin{aligned}
(n-1\ \ -(n-1))r_{n-1,n-2}(n\ \ -n) &= (n-1\ \ -(n-1))(n\ \ -n)r_{n-1,n-2} \\
&= [(n-1\ \ -n)][(n-1\ n)] \\
&= r_n r_{n-1}.
\end{aligned}
$$

(iv) If $|I_{n-1}(u) = 1|$. So $j_{n-1} = n-1$. If $w^{-1}(n) > 0$, then $w^{-1}(n-1)$ and $(-1)^{neg(w,n-1)}$ have same sign. In this case

$$(n-1\ \ -(n-1))[(n-1\ n)](n\ \ -n) = [(n-1\ n)] = r_{n-1}.$$

If $w^{-1}(n) > 0$, then $w^{-1}(n-1)$ and $(-1)^{neg(w,n-1)}$ have opposite sign. In this case

$$(n\ -n)[(n-1\ n)](n\ -n) = (n\ -n)(n-1\ -(n-1))[(n-1\ n)] = [(n-1\ -n)] = r_n.$$

□

Corollary 4.3.6. *Let R be the set of polynomials given in Proposition 4.2.2. Then*

(i) $Red(R) = \{\widetilde{r}_{nj_n}\widetilde{r}_{n-1,j_{n-1}}\cdots\widetilde{r}_{ij_i}\cdots\widetilde{r}_{1j_1} | i-1 \leq j_i \leq 2n-i,$ except $n-1 \leq j_{n-1} \leq n\}$.

(ii) R is a Gröbner-Shirshov basis for the finite Coxeter group of type D_n.

Proof. The first statement is easily follows from Theorem 4.3.5 the fact that the words in the right hand side are in Red(R). Notice that there are exactly $2^{n-1}n!$ words in Red(R). This is same as number of elements of the finite Coxeter group of type D_n. This implies $Red(R)$ is the set of normal forms and R is a Gröbner-Shirshov basis for the finite Coxeter group of type D_n by the Composition-Diamond lemma. □

Let us end this section with an application of Theorem 4.3.5.

Example 4. Let $w = [3, -5, 1, 4, -2] \in S_5^D$ and $u \in S_5$ such that $[u] = [3, 5, 1, 4, 2]$. Since $|I_1(u)| = 2$ and $w^{-1}(1) > 0$, $j_1 = 2$ by part (i) of Theorem 4.3.5. Similarly $j_3 = 2$. Since $|I_2(u)| = 3$ and $w^{-1}(2)(-1)^{neg(w,2)} < 0$, $j_2 = 5$ by part (ii) of Theorem 4.3.5. Since $|I_4(u)| = 1$ and $w^{-1}(5) < 0$, $j_4 = 3$ and $j_5 = 5$ by part (iv) of Theorem 4.3.5. Hence

$$w = \tilde{r}_5\tilde{r}_{25}\tilde{r}_{12}.$$

5. GRÖBNER-SHIRSHOV BASIS AND NORMAL FORMS FOR THE INFINITE COXETER GROUP OF TYPE $\widetilde{A}_n$

5.1 Gröbner-Shirshov Basis for The Infinite Coxeter Group of Type $\widetilde{A}_n$

Definition 5.1.1. For a positive integer $n > 2$, the infinite Coxeter group of type $\widetilde{A}_n$ has a presentation with generators $S = \{r_0, r_1, \ldots, r_n\}$ and defining relations:

(R_1) $r_i r_i = 1$ for $0 \leq i \leq n$,

(R_2) $r_i r_j = r_j r_i$ for $0 \leq i < j - 1 < n$ and $(i, j) \neq (0, n)$,

(R_3) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ for $1 \leq i \leq n-1$.

(R_4) $r_0 r_n r_0 = r_n r_0 r_n$.

A Gröbner-Shirshov basis and corresponding normal forms for affine Coxeter group of type $\widetilde{A}_n$ are founded in Yılmaz et al. (2014). Notice that they called this group as infinite Weyl group of type $\widetilde{A}_n$. The following theorem is just rewriting of their normal forms in the new notation.

5.2 Normal Forms for The Infinite Coxeter Group of Type $\widetilde{A}_n$

Theorem 5.2.1 (Yılmaz et al. (2014), Theorem 21)**.** *Let*

$$v = r_{nl_n} r_{n-1,l_{n-1}} \cdots r_{1l_1}$$

for $k - 1 \leq l_k \leq k$, $1 \leq k \leq n$; *and*

$$w = (r_0 r_{n,2n-q_t} r_{1p_t})(r_0 r_{n,2n-q_{t-1}} r_{1p_{t-1}}) \cdots (r_0 r_{n,2n-q_1} r_{1p_1})$$

for $0 \leq p_k \leq n$, $2 \leq q_k \leq n+1$ *satisfying the following conditions:*

(i) $p_k > p_{k-1}$ *and* $q_k < q_{k-1}$ *when* $2 \leq q_k - p_k < q_{k-1} - p_{k-1} \leq n+1$,

(ii) $p_k > p_{k-1}$ *and* $q_k \leq q_{k-1}$ *when* $-1 \leq p_k - q_k \leq n-2$ *and* $2 \leq q_{k-1} - p_{k-1} \leq n+1$

(iii) $p_k \geq p_{k-1}$ *and* $q_k \leq q_{k-1}$ *when* $-1 \leq p_{k-1} - q_{k-1} \leq p_k - q_k \leq n-2$.

Then any word $u \in S^*$ *has a normal form* $u = vw$.

After this point we try to give a combinatorial meaning of these normal forms. Let $N = n+1$ and $\widetilde{S}_N^A$ be the group of all bijections u of $\mathbb{Z}$ in itself such that

$$u(i+N) = u(i) + N$$

for all $i \in \mathbb{Z}$ satisfying

$$\sum_{i=1}^{N} u(i) = \frac{N(N+1)}{2}.$$

By definition such a $u \in \widetilde{S}_N^A$ is uniquely determined by its values on $\{1, \ldots, N\}$. Hence we write $u = [u_1, \ldots, u_N]$ where $u_i = u(i)$ for $i = 1, \ldots, N$ and call this window notation for u.

Note that for all $u \in \widetilde{S}_N^A$ and $i, j \in \mathbb{Z}$, $u(i) \not\equiv u(j) \mod N$ if and only if $i \not\equiv j \mod N$.

Proposition 5.2.2 (Bjorner and Brenti (2005), Proposition 8.3.3)**.** *The group* $\widetilde{S}_N^A$ *with generating set* $S = \{r_0, r_1, \ldots, r_n\}$ *is the infinite Coxeter group of type* $\widetilde{A}_n$ *where* $r_i = [(i \quad i+1)]$ *for* $i = 1, 2, \ldots, n$ *and*

$$r_0 = [0, 2, \ldots, N-1, N+1].$$

Now for each $u \in \widetilde{S}_N^A$ we find a word in S^*. This word turn out to be a normal form for u given in Theorem 5.2.1.

Lemma 5.2.3. *Let* $u = [u_1, \ldots, u_N] \in \widetilde{S}_N^A$. *For* $1 \leq i \leq N-1$, $2 \leq j \leq N$ *and* $1 \leq j - i \leq N-1$ *if*

$u(r_0 r_{n,2n-j} r_{1,i-1})^{-1} = [v_1, \ldots, v_N]$*, then*

$$v_k = \begin{cases} u_j - N, & k = 1; \\ u_{k-1}, & 2 \leq k < i+1; \\ u_k, & i+1 \leq k < j; \\ u_{k+1}, & j \leq k < N; \\ u_i + N, & k = N. \end{cases}$$

Proof. First of all, for any $w \in \widetilde{S}_N^A$, $wr_0 = [w_N - N, w_2, \ldots, w_n, w_1 + N]$ since $w(N) = w(0 + N) = w(0) + N$ and $w(N+1) = w(1) + N$. Then

$$\begin{aligned} u(r_0 r_{n,2n-j} r_{1,i-1})^{-1} &= u(r_{i-1} r_{i-1} \cdots r_1 r_j r_{j+1} \cdots r_n r_0) \\ &= u[(i\; i-1\; \cdots\; 1)(j\; j+1\; \cdots\; N)] r_0. \end{aligned}$$

Since $j - 1 \geq 1$, two cycles in above equation are disjoint and the assertion easily follows. □

Lemma 5.2.4. *Let* $u = [u_1, \ldots, u_N] \in \widetilde{S}_N^A$. *For* $2 \leq i \leq N$, $1 \leq j \leq N - 1$ *and* $1 \leq i - j \leq N - 1$ *if*

$u(r_0 r_{n,2n-j-1} r_{1,i-1})^{-1} = [v_1, \ldots, v_N]$, *then*

$$v_k = \begin{cases} u_j - N, & k = 1; \\ u_{k-1}, & 2 \leq k < j+1; \\ u_k, & j+1 \leq k < i; \\ u_{k+1}, & i \leq k < N; \\ u_i + N, & k = N. \end{cases}$$

Proof. For $1 \leq i \leq N$, $1 \leq j \leq N + 1$ and $1 \leq i - j \leq n$,

$$\begin{aligned} u(r_0 r_{n,2n-j-1} r_{1,i-1})^{-1} &= u(r_{i-1} r_{i-1} \cdots r_1 r_{j+1} \cdots r_n r_0); \\ &= u[(i\; i-1\; \cdots\; 1)(j+1\; \cdots\; N)] r_0; \\ &= u[(j\; j-1\; \cdots\; 2\; 1\; i\; i+1\; \cdots\; N]. \end{aligned}$$

The assertion easily follows. □

Definition 5.2.5. Given $u = [u_1, u_2, \ldots, u_N] \in \widetilde{S}_N^A$, define the following index set:

$$I_u = \{(i, j) | u_i < 0 \text{ and } u_j > N\}.$$

Notice that if $(i_1, j_1) \in I_u$ and $(i_2, j_2) \in I_u$ such that $i_1 - j_1 = i_2 - j_2$, then $(i, j) \in I_u$ where $i = \min\{i_1, i_2\}$ and $j = \max\{j_1, j_2\}$. Clearly $j - i < j_1 - i_1 = j_2 - i_2$. Hence there exist unique $(i, j) \in I_u$ such that

$$j - i = \max\{i_k - j_k | (i_k, j_k) \in I_u\}.$$

We denote this element by $\max I_u$.

Corollary 5.2.6. *Given $u = [u_1, u_2, \ldots, u_N] \in \widetilde{S}_N^A$, let $(i_1, j_1) = \max I_u$, .*

(i) If $i_1 < j_1$, $v = u(r_0 r_{nj_1} r_{1,i_1-1})^{-1}$ and $(i_2, j_2) = \max I_v$, then $i_2 > i_1$ and $j_2 < j_1$.

(ii) If $i_1 > j_1$, $v = u(r_0 r_{n,2n-j-1} r_{1,i-1})^{-1}$ and $(i_2, j_2) = \max I_v$, then $i_2 \geq i_1$ and $j_2 \leq j_1$.

Proof. Since $u_{i_1} < 0$ and $u_{j_1} > N$, $V_N = u_{i_1} + N < N$ and $v_1 = u_{j_1} - N > 0$. This implies $i_2 \neq 1$ and $j_1 \neq N$ in both cases.

(i) Suppose that $1 < i_2 \leq i_1$ and $j_1 \leq j_2 < N$. Then $v_{i_2} = u_{i_2-1}$ and $v_{j_2} = u_{j_1+1}$ by Lemma 6.2.2. This is however a contradiction to $(i_1, j_1) = \max I_u$.

(ii) Suppose that $1 < i_2 < i_1$ and $j_1 < j_2 < N$. Then $v_{i_2} = u_{i_2}$ and $v_{j_2} = u_{j_2}$ by Lemma 6.2.4.This is again a contradiction to $(i_1, j_1) = \max I_u$.

□

Theorem 5.2.7. *Any $u \in \widetilde{S}_N^A$ has a normal form representation as it is given in Theorem 5.2.1.*

Proof. Let $u \in \widetilde{S}_N^A$. If $I_u = \emptyset$, then $u \in S_{n+1}$ and Theorem 4.1.6 implies that u has a normal form representation

$$v = r_{nl_n} r_{n-1,l_{n-1}} \cdots r_{1l_1}$$

where $l_i = |I_i(u)| + i - 1$ for $i = 1, \ldots, n$. In this case we take $w = 1$.

Suppose that $I_u \neq \emptyset$. Let $u^{(0)} = u$ and $u^{(k)} = u^{(k-1)}(r_0 r_{n,2n-q_k} r_{1p_k})^{-1}$ where the indices p_k and q_k are determined as follows. Let $(i_k, j_k) = \max I_{u^{(k)}}$. If $i_k < j_k$, then $q_k = j_k$ and $p_k = i_k - 1$; and if $i_k > j_k$, then $q_k = j_k + 1$ and $p_k = i_k - 1$. Notice that if $1 \leq -j_k \leq n$, then $2 \leq p_k - q_k \leq n - 1$; and if $1 \leq j_k - i_k \leq n$, then $-1 \leq q_k - p_k \leq n - 1$. Furthermore Corollary 6.2.3 implies that the indices satisfy the conditions given Theorem 5.2.1.

Since $\sum_i^N |u^{(k)}(i)| = \sum_i^N |u^{(k-1)}(i)| - 2N$, after finitely many steps

$$u^t = u(r_0 r_{n,2n-q_1} r_{1p_1})^{-1} \cdots (r_0 r_{n,2n-q_t} r_{1p_t})^{-1}$$

where $I_{u^{(t)}} = \emptyset$. Hence $u^{(t)}$ has a normal form representation $v = r_{nl_n} r_{n-1,l_{n-1}} \cdots r_{1l_1}$ and then vw is a normal form representation for u where

$$w = (r_0 r_{n,2n-q_t} r_{1p_t})(r_0 r_{n,2n-q_{t-1}} r_{1p_{t-1}}) \cdots (r_0 r_{n,2n-q_1} r_{1p_1}).$$

□

Let us finish this chapter with an illustration of the previous theorem.

Example 5. Let $u = u^{(0)} = [17, -1, -5, 6, -2] \in \widetilde{S}_5^A$. So $\max I_{u^{(0)}} = (2, 4)$ and

$$u^{(1)} = [17, -1, -5, 6, -2](r_0 r_4 r_1)^{-1} = [1, 17, -5, -2, 4].$$

Then $\max I_{u^{(1)}} = (3, 2)$ implies

$$u^{(2)} = [1, 17, -5, -2, 4](r_0 r_{45} r_{12})^{-1} = [12, 1, -2, 4, 0].$$

Since $\max I_{u^{(2)}} = (3, 1)$,

$$u^{(3)} = [12, 1, -2, 4, 0](r_0 r_{46} r_{12})^{-1} = [7, 1, 4, 0, 3].$$

Applying the process one more time $\max I_{u^{(3)}} = (4, 1)$ and

$$u^{(4)} = [7, 1, 4, 0, 3](r_0 r_{46} r_{13})^{-1} = [2, 1, 4, 3, 5] = r_3 r_1.$$

Hence

$$u = (r_3 r_1)(r_0 r_{46} r_{13})(r_0 r_{46} r_{12})(r_0 r_{45} r_{12})(r_0 r_4 r_1).$$

6. GRÖBNER-SHIRSHOV BASIS AND NORMAL FORMS FOR THE INFINITE COXETER GROUP OF TYPE $\widetilde{C}_n$

6.1 Gröbner-Shirshov Basis for The Infinite Coxeter Group of Type $\widetilde{C}_n$

Definition 6.1.1. For a positive integer $n \geq 2$, the infinite Coxeter group of type $\widetilde{C}_n$ has a presentation with generators $S = \{r_0, r_1, \ldots, r_n\}$ and defining relations:

(R_1) $r_i r_i = 1$ for $0 \leq i \leq n$,

(R_2) $r_i r_j = r_j r_i$ for $0 \leq i < j-1 < n$,

(R_3) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ for $1 \leq i < n-1$,

(R_4) $r_{n-1} r_n r_{n-1} r_n = r_n r_{n-1} r_n r_{n-1}$,

(R_5) $r_0 r_1 r_0 r_1 = r_1 r_0 r_1 r_0$.

Lemma 6.1.2. *Suppose that $<$ is the degree lexicographic order on S^*. A Gröbner-Shirshov basis for the infinite Coxeter group of type $\widetilde{C}_n$ with respect to $<$ contains the following polynomials:*

- $f_1^{(i)} = r_i r_i - 1$ *for* $0 \leq i \leq n$,
- $f_2^{(i,j)} = r_i r_j - r_j r_i$ *for* $0 \leq i < j-1 < n$,
- $f_3^{(i,j)} = r_{ij} r_i - r_{i+1} r_{ij}$ *for* $1 \leq i \leq n-2$ and $i < j < 2n-i-1$,
- $f_4^{(i)} = r_{i,2n-i} r_{i+1} - r_{i+1} r_{i,2n-i}$ *for* $1 \leq i \leq n-1$,
- $f_5^{(i)} = r_0 r_{1i} r_0 r_{1i} - r_1 r_0 r_{1i} r_0 r_{1,i-1}$ *for* $1 \leq i \leq n-1$,
- $f_6^{(i)} = r_0 r_{1,2n-i} r_0 r_{1,2n+1-i} - r_1 r_0 r_{1,2n-i} r_0 r_{1,2n-i}$ *for* $2 \leq i \leq n$.

Proof. The polynomials $f_1^{(i)}$ and $f_2^{(i,j)}$ come from the defining relations (R_1) and (R_2), respectively. Similarly the polynomials $f_3^{(i,i+1)}$ and $f_4^{(n-1)}$ come from the defining relation (R_3) and (R_4), respectively. Now let us apply the Shirshov algorithm to these polynomials.

$< f_3^{(i,j-1)}, f_2^{(i,j)} >= f_3^{(i,j)}$ for $n \geq j > i$ and $1 \leq i \leq n-2$,

$< f_3^{(i,j-1)}, f_2^{(i,2n-j)} >= f_3^{(i,j)}$ for $n < j < 2n-i-1$ and $1 \leq i < n-2$ and

$< f_3^{(i,2n-2-i)}, f_3^{(i,i+1)} >= f_4^{(i)}$ for $1 \leq i < n-1$.

The polynomial $f_5^{(1)}$ comes from the defining relation (R_5). Adding this polynomial, let us continue to apply Shirshov algorithm.

$$
\begin{aligned}
< f_5^{(i-1)}, f_3^{(i-1,i)} > =& \sum_{k=1}^{i-2} r_0 r_{1,i-1} r_0 r_{1,k-1} f_2^{(k,i)} r_{k+1,i} + r_0 r_{1,i-1} f_2^{(0,i)} r_{1,i} \\
&- \sum_{k=1}^{i-2} r_1 r_0 r_{1,i-1} r_0 r_{1,k-1} f_2^{(k,i)} r_{k+1,i-1} - r_1 r_0 r_{1,i-1} f_2^{(0,i)} r_{1,i-1} + f_5^{(i)}
\end{aligned}
$$

for $1 < i \leq n-1$,

$$
\begin{aligned}
< f_5^{(n-1)}, f_4^{(n-1)} > =& \sum_{k=1}^{n-2} r_0 r_{1,n-1} r_0 r_{1,k-1} f_2^{(k,n)} r_{k+1,n+1} + r_0 r_{1,n-1} f_2^{(0,n)} r_{1,n+1} \\
&- \sum_{k=1}^{n-2} r_1 r_0 r_{1,n-1} r_0 r_{1,k-1} f_2^{(k,n)} r_{k+1,n} - r_1 r_0 r_{1,n-1} f_2^{(0,n)} r_{1,n} + f_6^{(n)}
\end{aligned}
$$

$$
\begin{aligned}
< f_6^{(i+1)}, f_4^{(i-1)} > =& \sum_{k=1}^{i-2} r_0 r_{1,2n-i-1} r_0 r_{1,k-1} f_2^{(k,i)} r_{k+1,2n-i+1} + r_0 r_{1,2n-i-1} f_2^{(0,i)} r_{1,2n-i+1} \\
&- r_1 r_0 r_{1,2n-i-1} r_0 r_{1,i-2} f_3^{(i-1,i+1)} r_i \\
&- \sum_{k=1}^{i-2} r_1 r_0 r_{1,2n-i-1} r_0 r_{1,k-1} f_2^{(k,i)} r_{k+1,2n-i} \\
&- r_1 r_0 r_{1,2n-i-1} f_2^{(0,i)} r_{1,2n-i} + f_6^{(i)}
\end{aligned}
$$

for $2 \leq i < n$. □

At this point we are not able to show that polynomials given above lemma form Gröbner-Shirshov basis for the infinite Coxeter group of type $\widetilde{C}_n$.

6.2 Normal Forms for The Infinite Coxeter Group of Type $\widetilde{C}_n$

We now give a combinatorial description of the infinite Coxeter group of type $\widetilde{C}_n$ (see Bjorner and Brenti (2005)). Let $\widetilde{S}_n^C$ be the group of all permutations u of $\mathbb{Z}$ in itself such that

$$u(i + 2n + 1) = u(i) + 2n + 1$$

and

$$u(-i) = -u(i)$$

for all $i \in \mathbb{Z}$. Notice that

$$u(k(2n+1)) = k(2n+1)$$

for all $k \in \mathbb{Z}$

Clearly, such a u is uniquely determined by its values on $[n] = \{1, 2, \dots, n\}$, and we write $u = [u_1, u_2, \dots, u_n]$ where $u_i = u(i)$ for $i = 1, \dots, n$. We call this window notation for u. Hereafter we set $N = 2n + 1$.

Proposition 6.2.1 (Bjorner and Brenti (2005), Proposition 8.4.3)**.** *The group $\widetilde{S}_n^C$ with generating set $S = \{r_0, r_1, r_2, \cdots, r_n\}$ is the infinite Coxeter group of type $\widetilde{C}_n$ where $r_0 = [2n, 2, \dots, n]$, $r_i = [(i \quad i+1)]$ for $i = 1, 2, \dots, n-1$ and $r_n = [(n \quad -n)]$.*

Lemma 6.2.2. *Let $u = [u_1, \dots, u_n] \in \widetilde{S}_n^C$. If*

$u(r_0 r_{1i})^{-1} = [v_1, \dots, v_n]$ for some $0 \leq i \leq n-1$, then

$$v_j = \begin{cases} N - u_{i+1}, & j = 1; \\ u_{j-1}, & 1 < j \leq i+1; \\ u_j, & j > i+1. \end{cases}$$

Proof. Clearly $[u_1, \dots, u_n] r_0 = [N - u_1, u_2, \dots, u_n] = u(r_0)^{-1}$ since

$$u(2n) = u(-1+N) = u(-1) + N = N - u_1.$$

Since $(r_0 r_{1i})^{-1} = r_i r_{i-1} \cdots r_1 r_0$, $u(r_0 r_{1i})^{-1} = u[(i+1 \quad i \quad \cdots \quad 1)] r_0$ for $i = 1, \dots, n-1$. The assertion easily follows. □

The following corollary is an easy consequence of the above lemma.

Corollary 6.2.3. *Given $u = [u_1, \dots, u_n] \in \widetilde{S}_n^C$ there exists $w = (r_0 r_{1i_1}) \cdots (r_0 r_{1i_s})$ where $0 \leq i_s < \cdots < i_1 \leq n-1$ satisfying $uw^{-1} = [v_1, \dots, v_n]$ with $v_i \leq n$ for $i = 1, \dots, n$.*

Proof. If $u_i \leq n$ for $i = 1, \dots, n$, then $w = 1$. Otherwise let $j_1 > \cdots > j_s$ be the all the indices satisfying $u_{j_k} > n$. By Lemma 6.2.2,

$$[v_1, \dots, v_n] = u(r_0 r_{1 j_s - 1})^{-1} \cdots (r_0 r_{1 j_1 - 1})^{-1},$$

where $v_i \leq n$ for $i = 1, \dots, n$. Hence $w = (r_0 r_{1i_1}) \cdots (r_0 r_{1i_s})$ where $i_k = j_k - 1$ is the desired element. □

Lemma 6.2.4. *Let $u = [u_1, \dots, u_n] \in \widetilde{S}_n^C$. If $u(r_0 r_{1i})^{-1} = [v_1, \dots, v_n]$ for some $n \le i \le 2n-1$, then*

$$v_j = \begin{cases} u_{t+1} + N, & j = 1; \\ u_{j-1}, & 1 < j \le t+1; \\ u_j, & j > t+1. \end{cases}$$

where $t = 2n - i - 1$.

Proof. For $n \le i \le 2n-1$, $r_{1i} = r_{1,2n-1} r_{1t} = (1 \quad -1) r_{1t}$ where $t = 2n - i - 1$. Then $(r_0 r_{1i})^{-1} = (r_t r_{t-1} \cdots r_1)(1 \quad -1) r_0$ and $u(r_{1i})^{-1} = [v_1, \dots, v_n]$ where

$$v_j = \begin{cases} u_{t+1} + N, & j = 1; \\ u_{j-1}, & 1 < j \le t+1; \\ u_j, & j > t+1. \end{cases}$$

by Lemma 6.2.2.

□

Corollary 6.2.5. *Given $u = [u_1, \dots, u_n] \in \widetilde{S}_n^C$ such that $u_i \le n$ for $i = 1, \dots, n$ there exists $w = (r_0 r_{1i_s})^{p_s} \cdots (r_0 r_{1i_1})^{p_1}$ where $n \le i_1 < \cdots < i_s \le 2n-1$ satisfying $uw^{-1} = [v_1, \dots, v_n]$ with $v_i \in [\pm n] = \{\pm 1, \dots, \pm n\}$ for $i = 1, \dots n$.*

Proof. If $u_i \in [\pm n]$ for $i = 1, \dots n$, then $w = 1$. Otherwise let $v^{(0)} = u$ and $v^{(j+1)} = v^{(j)}(r_0 r_{1i_j})^{-1}$ where $t_j = 2n - i_j$ is the largest index satisfying $v^{(i)}_{t_j} > -n$ in the windows notation of $v^{(j)}$. Lemma 6.2.4 implies that $t_j \ge t_{j+1}$ and there exists $M \in \mathbb{N}$ such that $v^{(M)}_k \in [\pm n]$ for $k = 1, \dots, n$ in the windows notation of $v^{(M)}$. Hence

$$w = (r_0 r_{1i_M})(r_0 r_{1i_{N-1}}) \cdots (r_0 r_{1i_1})$$

is the desired element. □

Definition 6.2.6. Given $u = [u_1, \dots, u_n] \in \widetilde{S}_n^C$ satisfying $0 < u_i \le n$, define

$$I_i(u) := \{u_j : u_j > i \text{ for some } j < u^{-1}(i)\}$$

for $i = 1, \dots, n$.

Lemma 6.2.7. *Suppose $u = [u_1, \dots, u_n] \in \widetilde{S}_n^C$ such that $u(l) = l$ for $1 \le l \le k-1$ and $0 < u_i \le n$ for all i's. Let $x = r_{kj_k}$ where $j_k = |I_k(u)| + k - 1$. If $v = ux^{-1}$, then $v(l) = l$ for $1 \le l \le k$ and $I_i(u) = I_i(v)$ for $i \ne k$.*

Proof. Since $u^{-1}(l) = l$ for $1 \leq l \leq k-1$, $|I_k(u)| = u^{-1}(k) - k$ and then $j_k + 1 = |I_k(u)| + k = u^{-1}(k)$. Hence $v(l) = u(x^{-1}(l)) = u(l) = l$ for $1 \leq l \leq k-1$ and $v(k) = u(x^{-1}(k)) = u(j_k + 1) = k$.

Let $i \neq k$. Notice that $u^{-1}(i) \neq j_{k+1} = u^{-1}(k)$. Suppose $t \in I_i(u)$. Then $u(j) > i$ for some $j < u^{-1}(i)$. Since $u^{-1}(i) \neq j_k + 1$, x preserves the inequality that is $x(j) < x(u^{-1}(i)) = v^{-1}(i)$. Therefore $t = v(x(j)) > i$ for some $x(j) < v^{-1}(i)$. This implies $t \in I_i(v)$. Conversely suppose $t \in I_i(v)$. Hence $t = v(j) > i$ for some $j < v^{-1}(i)$. Since $v^{-1}(i) = x(u^{-1}(i)) \neq x(j_k + 1) = k$, x^{-1} preserves the inequality that is $x^{-1}(j) < x^{-1}(v^{-1}(i)) = u^{-1}(i)$. Therefore $t = u(x^{-1}(j)) > i$ for some $x^{-1}(j) < u^{-1}(i)$. This implies $t \in I_i(u)$. □

Corollary 6.2.8. *Given $w = [w_1, w_2, \ldots, w_n] \in \widetilde{S}_n^C$ such that $w_i \in [\pm n]$ for $i = 1, \ldots, n$, let $u \in S_n$ such that $u(i) = |w_i|$ for $i = 1, \ldots, n$. Then*

$$w = r_{n-1 j_{n-1}} \cdots r_{i j_i} \cdots r_{1 j_1}$$

where $j_i = |I_i(u)| + i - 1$ if $w^{-1}(i) > 0$ and $j_i = 2n - |I_i(u)| - i$ if $w^{-1}(i) < 0$.

Proof. Let $v_0 = u$. Suppose that $v_k = v_{k-1}(r_{kj_k})^{-1}$ where $j_k = |I_k(v_{k-1})| + k - 1$ for $k = 1, \ldots, n$. Lemma 6.2.7 implies $v_k(l) = l$ for $l = 1, \ldots, k$ and $I_i(v_k) = I_i(v_{k-1})$ for $i \neq k$. Hence

$$j_k = |I_k(v_{k-1})| + k - 1 = |I_k(v_{k-2})| + k - 1 = \cdots = |I_k(u)| + k - 1$$

for $k = 1, \ldots, n$. Furthermore $v_n = w(r_{1j_1})^{-1} \cdots (r_{nj_n})^{-1}$ is the identity element. Hence

$$u = r_{nj_n} \cdots r_{ij_i} \cdots r_{1j_1}.$$

Notice that $r_{i,2n-i} = [(i \quad i+1 \quad \cdots \quad n)](n \quad -n)[(n \quad n-1 \quad \cdots \quad i)] = (i \quad -i)$ for $i = 1, \ldots, n$. Then $(i \quad -i) r_{ij_i} = r_{i,2n-i} r_{ij_i} = r_{i,2n-j_i-1}$. Clearly,

$$w = [w_1, \ldots, w_n] = \Big(\prod_{w^{-1}(i) < 0} (i \quad -i) \Big) [u_1, \ldots, u_n].$$

Since $(i \quad i)$ commutes with r_{kj_k} for $k > i$ and $(i \quad -i) r_{ij_i} = r_{i,2n-j_i-1}$,

$$w = r_{n-1 j_{n-1}} \cdots r_{ij_i} \cdots r_{1j_1}$$

where $j_i = |I_i(u)| + i - 1$ if $w^{-1}(i) > 0$ and $j_i = 2n - (|I_i(u)| + i - 1) - 1 = 2n - |I_i(u)| - i$ if $w^{-1}(i) < 0$. □

Definition 6.2.9. Let

$$j \leqslant i = \begin{cases} j < i, & \text{if } j < n; \\ j \leq i, & \text{if } j \leq n. \end{cases}$$

Definition 6.2.10. Let

$$W_C = \{(r_{nl_n} \cdots r_{il_i} \cdots r_{1l_1}) \prod_{k=1}^{t} (r_0 r_{1j_k})\}$$

where $i-1 \leq l_i \leq 2n-i$, $0 \leq j_k \leqslant j_{k-1} \leq 2n-1$ and $t \in \mathbb{N}$.

Theorem 6.2.11. *Any $u \in \widetilde{S}_n^C$ can be represented with a word in W_C.*

Proof. Let Let $u = [u_1, \ldots, u_n] \subset \widetilde{S}_n^C$. If $u = 1$, then clearly $u \in W_C$ with $j_i = i-1$ f and $\alpha_i =$ for all i's. Given $u = [u_1, \ldots, u_n] \subset \widetilde{S}_n^C$ we can find a word p satisfying $v = [u_1, \ldots, u_n] = up^{-1}$ such that $v_i \leq n$ for $i = 1, \ldots, n$ and at least one $v_j < -n$ by Corollary 6.2.3. Then we can find a word q satisfying $w = [w_1, \ldots, w_n] = vq^{-1}$ such that $w_i \in [\pm n]$ by Corollary 6.2.5. Furthermore $w \in W_C$ by Corollary 6.2.8 and clearly $u = wqp \in W_C$. □

Before giving the main result, we need the following lemma.

Lemma 6.2.12. *For $n \in \mathbb{N}$,*

$$\prod_{i=1}^{n} \frac{1+x^i}{1-x^{i+n}} = \prod_{i=1}^{n} \frac{1}{1-x^{2i-1}}.$$

Proof. The equation is valid when $n = 1$. Assume that

$$\prod_{i=1}^{n} \frac{1+x^i}{1-x^{i+n}} = \prod_{i=1}^{n} \frac{1}{1-x^{2i-1}}$$

Then

$$\begin{aligned} \prod_{i=1}^{n+1} \frac{1+x^i}{1-x^{i+n+1}} &= \frac{(1+x^{n+1})(1+x^n)\cdots(1+x)}{(1-x^{2n+2})(1-x^{2n+1})(1-x^{2n})\cdots(1-x^{n+2})} \\ &= \frac{(1+x^{n+1})(1-x^{n+1})}{(1-x^{2n+2})(1-x^{2n+1})} \prod_{i=1}^{n} \frac{1+q^i}{1-x^{i+n}} \\ &= \frac{1}{(1-x^{2n+1})} \prod_{i=1}^{n} \frac{1}{1-x^{2i-1}} \\ &= \prod_{i=1}^{n+1} \frac{1}{1-x^{2i+1}}. \end{aligned}$$

□

Lemma 6.2.13. *The generating function for words in W_C is*

$$\prod_{i=1}^{n}(1+x+\ldots+x^{2i-1})\frac{1+x^i}{1-x^{n+i}}.$$

Proof. The generating function for the words of the form $r_{nl_n}\cdots r_{il_i}\cdots r_{1l_1}$ is

$$\prod_{i=1}^{n}(1+x+\ldots+x^{2i-1}).$$

The generating function for the word of the form $(r_0r_{1j_k})^t$ where $n \leq j_k \leq 2n-1$ and $t \in \mathbb{N}$ is $\frac{1}{1-x^{j_k+1}}$ and for $(r_0r_{1j_k})^t$ where $0 \leq j_k < n$ and $t \in \{0,1\}$ is $1+x^{j_k+1}$. Hence the generating function for the words $\prod_{k=1}^{t}(r_0^C r_{1j_k}^C)$ where $t \geq 0$ and $0 \leq j_k \leqslant j_{k-1} \leq 2n-1$ is

$$\prod_{i=1}^{n}\frac{1+x^i}{1-x^{n+i}}.$$

□

Notice that the generating function for the infinite Coxeter group of type $\widetilde{C}_n$ is

$$\prod_{i=1}^{n}\frac{1+x+\ldots+x^{2i-1}}{1-x^{2i-1}}.$$

By Lemma 6.2.12

$$\prod_{i=1}^{n}\frac{1+x+\ldots+x^{2i-1}}{1-x^{2i-1}}=\prod_{i=1}^{n}(1+x+\ldots+x^{2i-1})\frac{1+x^i}{1-x^{n+i}}$$

which is equal to generating function of words in W_C.

Now we are ready to find the main result.

Theorem 6.2.14. *Let R^C be the set of all polynomials given in Lemma 6.1.2. Then*

(i) $W_C =$ *Red*(R^C).

(ii) R^C *is a Gröbner-Shirshov basis for the infinite Coxeter group of type* $\widetilde{C}_n$.

Proof. (i) It is easy to see that any word in W_C is R^C-reduced. Hence $W_C \subseteq \text{Red}(R^C)$. Conversely if $w \in \text{Red}(R^C)$, then w can be written as a permutation in $\widetilde{S}_n^C$ and this permutation has a corresponding word in W_C by Theorem 6.2.11. Hence $\text{Red}(R^C) \subseteq W_C$.

(ii) We know that any polynomial in R^C is a part of a Gröbner-Shirshov basis of the infinite Coxeter group of type $\widetilde{C}_n$. If R^C were not a Gröbner-Shirshov basis, then $\text{Red}(R^C) = W_C$ should be a proper subset of the set of normal forms of the infinite Coxeter group of type $\widetilde{C}_n$ by Composition-Diamond lemma. This contradicts to the fact that W_C and normal forms of the infinite Coxeter group of type $\widetilde{C}_n$ have same generating functions.

□

Notice that Theorem 6.2.11 gives a method converting any $u \in \widetilde{S}_n^C$ into its normal form. Let us finish this chapter with an example of this conversion.

Example 6. Let $u = [24, 38, 17, -5] \in \widetilde{S}_4^C$. Corollary 6.2.3 implies $uw^{-1} = [-8, -29, -15, -5]$ where $w = r_0r_{12}r_0r_1r_0$. Applying the process explained in the proof of the Corollary 6.2.5, we get

$[-8, -29, -15, -5](r_0r_{14})^{-1} = [4, -8, -29, -15]$,

$[4, -8, -29, -15](r_0r_{14})^{-1} = [-6, 4, -8, -29]$,

$[-6, 4, -8, -29](r_0r_{14})^{-1} = [-20, -6, 4, -8]$,

$[-20, -6, 4, -8](r_0r_{14})^{-1} = [1, -20, -6, 4]$,

$[1, -20, -6, 4](r_0r_{15})^{-1} = [3, 1, -20, 4]$,

$[3, 1, -20, 4](r_0r_{15})^{-1} = [-11, 3, 1, 4]$ and

$[-11, 3, 1, 4](r_0r_{17})^{-1} = [-2, 3, 1, 4]$.

Thus

$$[-8, -29, -15, -5]((r_0r_{17})(r_0r_{15})^2(r_0r_{14})^4)^{-1} = [-2, 3, 1, 4].$$

Corollary 6.2.8 implies $[-2, 3, 1, 4] = r_{26}r_{12}$.

Hence $u = (r_{26}r_{12})(r_0r_{17})(r_0r_{15})^2(r_0r_{14})^4(r_0r_{12}r_0r_1r_0)$.

7. GRÖBNER-SHIRSHOV BASIS AND NORMAL FORMS FOR THE INFINITE COXETER GROUP OF TYPE $\widetilde{B}_n$

7.1 Gröbner-Shirshov Basis for The Infinite Coxeter Group of Type $\widetilde{B}_n$

Definition 7.1.1. For a positive integer $n \geq 2$, the infinite Coxeter group of type $\widetilde{B}_n$ has a presentation with generators $S = \{r_0, r_1, \ldots, r_n\}$ and defining relations:

(R_1) $r_i r_i = 1$ for $0 \leq i \leq n$,

(R_2) $r_i r_j = r_j r_i$ for $0 \leq i < j - 1 < n$ but $(i, j) \neq (0, 2)$,

(R_3) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ for $1 \leq i < n - 1$,

(R_4) $r_0 r_1 = r_1 r_0$,

(R_5) $r_{n-1} r_n r_{n-1} r_n = r_n r_{n-1} r_n r_{n-1}$,

(R_6) $r_0 r_2 r_0 = r_2 r_0 r_2$.

After this point we will not use superscripts unless we need to distinguish between groups $\widetilde{B}_n$ and $\widetilde{C}_n$.

Lemma 7.1.2. *Suppose that $<$ is the degree lexicographic order on S^*. A Gröbner-Shirshov basis for the infinite Coxeter group of type $\widetilde{B}_n$ with respect to $<$ contains the following polynomials:*

- $g_1^{(i)} = r_i r_i - 1$ *for* $0 \leq i \leq n$,
- $g_2^{(i,j)} = r_i r_j - r_j r_i$ *for* $0 \leq i < j - 1 < n$ *but* $(i, j) \neq (0, 2)$,
- $g_3^{(i,j)} = r_{ij} r_i - r_{i+1} r_{ij}$ *for* $1 \leq i \leq n - 2$ *and* $i < j < 2n - i - 1$,
- $g_4^{(i)} = r_{i,2n-i} r_{i+1} - r_{i+1} r_{i,2n-i}$ *for* $1 \leq i \leq n - 1$,
- $g_5^{(i)} = r_0 r_{2i} r_{1i} - r_1 r_0 r_{2i} r_{1,i-1}$ *for* $1 \leq i \leq n - 1$,
- $g_6^{(i)} = r_0 r_{2,2n-i} r_{1,2n-i+1} - r_1 r_0 r_{2,2n-i} r_{1,2n-i}$ *for* $2 \leq i \leq n$,

- $g_7^{(i,j)} = r_0 r_{2i} r_{1j} r_0 - r_2 r_0 r_{2i} r_{1j}$ *for* $2 \leq i \leq 2n-3$ *and* $0 \leq j \leq 1$,

- $g_8^{(i,j)} = r_0 r_{2i} r_{1j} r_0 r_{2j} - r_2 r_0 r_{2i} r_{1j} r_0 r_{2,j-1}$ *for* $2 \leq j < i \leq n$,

- $g_9^{(i,j)} = r_0 r_{2,2n-i} r_{1j} r_0 r_{2j} - r_2 r_0 r_{2,2n-i} r_{1j} r_0 r_{2,j-1}$,
  for $3 \leq i \leq n-1$ *and* $2 \leq j \leq n-1$,

- $g_{10}^{(i,j)} = r_0 r_{2,2n-2} r_{1i} r_0 r_{2j} r_1 - r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2j}$,
  for $1 \leq i \leq 2$ *and* $2 \leq j \leq 2n-3$,

- $g_{11}^{(i,j)} = r_0 r_{2,2n-2} r_{1i} r_0 r_{2j} r_{1,i-1} - r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2j} r_{1,i-2}$,
  for $3 \leq i \leq n-1$, $3 \leq j \leq n$ *and* $i \leq j$,

- $g_{12} = r_0 r_{2,2n-2} r_0 r_2 - r_2 r_0 r_{2,2n-2} r_0$,

- $g_{13}^{(i,j)} = r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-j} r_{1i} - r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-j} r_{1,i-1}$,
  for $2 \leq j \leq i \leq n-1$,

- $g_{14}^{(i,j)} = r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,2n-j} r_{1,2n-i} - r_2 r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,2n-j} r_{1,2n-i-1}$,
  for $2 \leq j \leq i \leq n-1$,

- $g_{15}^{(i,j)} = r_0 r_{2,2n-i} r_{1,2n-j-1} r_0 r_{2,2n-j} - r_2 r_0 r_{2,2n-i} r_{1,2n-j-1} r_0 r_{2,2n-j-1}$,
  for $2 \leq i-1 \leq j \leq n-1$,

- $g_{16} = r_0 r_{2,2n-2} r_1 r_0 r_{2,2n-2} r_1 r_2 - r_2 r_0 r_{2,2n-2} r_1 r_0 r_0 r_{2,2n-2} r_1$,
  for $2 \leq i-1 \leq j \leq n-1$,

- $g_{17}^{(i,j)} = r_0 r_{2,2n-2} r_{1j} r_0 r_{2,2n-i} r_{1,j-1} - r_2 r_0 r_{2,2n-2} r_{1j} r_0 r_{2,2n-i} r_{1,j-2}$,
  for $3 \leq j < i \leq n-1$.

Proof. $g_1^{(i)}, g_2^{(i,j)}, g_3^{(i,i+1)}, g_4^{(n-1)}, g_5^{(1)}$ and $g_7^{(2,0)}$ are defining relations for the infinite Coxeter group of type $\widetilde{B}_n$.

$< g_3^{(i,i+1)}, g_2^{(i,i+2)} >= g_3^{(i,i+2)}$,

$< g_3^{(i,j-1)}, g_2^{(i,j)} >= g_3^{(i,j)}$ for $i+2 < j \leq n$,

$< g_3^{(i,j)}, g_2^{(i,2n-j-1)} >= g_3^{(i,j+1)}$ for $n < j < 2n-i-1$,

$< g_3^{(i,2n-i-2)}, g_3^{(i,i+1)} >= g_4^{(i)}$ for $1 \leq i < n-1$,

$$< g_5^{(i-1)}, g_3^{(i-1,i)} > = \sum_{k=1}^{i-2} r_0 r_{2,i-1} r_{1,k-1} g_2^{(k,i)} r_{k+1,i} - \sum_{k=1}^{i-2} r_1 r_0 r_{2,i-1} r_{1,k-1} g_2^{(k,i)} r_{k+1,i-1} + g_5^{(i)}$$

for $1 < i \leq n-1$,

$$< g_5^{(n-1)}, g_4^{(n-1)} > = \sum_{k=1}^{n-2} r_0 r_{2,n-1} r_{1,k-1} g_2^{(k,n)} r_{k+1,n+1} - \sum_{k=1}^{n-2} r_1 r_0 r_{2,n-1} r_{1,k-1} g_2^{(k,n)} r_{k+1,n} + g_6^{(n)},$$

$$< g_6^{(i+1)}, g_4^{(i-1)} > = \sum_{k=1}^{i-2} r_0 r_{2,2n-i-1} r_{1,k-1} g_2^{(k,i)} r_{k+1,2n-i+1} - r_1 r_0 r_{2,2n-i-1} r_{1,i-2} g_3^{(i-1,2n-i-1)} r_i - \sum_{k=1}^{i-2} r_1 r_0 r_{2,2n-i-1} r_{1,k-1} g_2^{(k,i)} r_{k+1,2n-i} + g_6^{(i)},$$

$< g_7^{(i-1,0)}, g_2^{(0,i)} >= g_7^{(i,0)}$ for $3 \leq i \leq 2n-3$,

$< g_7^{(2,0)}, g_5^{(1)} >= g_7^{(2,1)}$,

$< g_7^{(i-1,1)}, g_2^{(0,i)} >= r_0 r_{2,i-1} g_{(1,i)} r_0 - r_2 r_0 r_{2,i-1} g_{(1,i)} + g_7^{(i,1)}$ for $3 \leq i \leq n$,

$< g_7^{(i-1,1)}, g_2^{(0,2n-i)} >= r_0 r_{2,2n-i+1} g_{(1,i)} r_0 - r_2 r_0 r_{2,i-1} g_{(1,i)} + g_7^{(i,1)}$ for $n < i \leq 2n-3$,

$< g_7^{(i,1)}, g_7(2,0) >= g_8^{(i,2)}$ for $3 \leq i \leq n$,

$$< g_8^{(i,j-1)}, g_3^{(j-1,j)} > = \sum_{k=1}^{j-2} r_0 r_{2i} r_{1,j-1} r_0 r_{2,k-1} g_2^{(k,j)} r_{k+1,j} + r_0 r_{2i} r_{1,j-1} g_2^{(0,j)} r_{2,j} - \sum_{k=1}^{j-2} r_2 r_0 r_{2i} r_{1,j-1} r_0 r_{2,k-1} g_2^{(k,j)} r_{k+1,j-1} - r_2 r_0 r_{2i} r_{1,j-1} g_2^{(0,j)} r_{2,j-1} + g_8^{(i,j)}$$

for $3 \leq j < i \leq n$,

$< g_7^{(2n-i,1)}, g_7^{(2,0)} >= g_9^{(i,2)}$ for $3 \leq i \leq n-1$,

$$
\begin{aligned}
< g_9^{(i,j-1)}, g_3^{(j-1,j)} > = & \sum_{k=1}^{j-2} r_0 r_{2,2n-i} r_{1,j-1} r_0 r_{2,k-1} g_2^{(k,j)} r_{k+1,j} + r_0 r_{2,2n-i} r_{1,j-1} g_2^{(0,j)} r_{2,j} \\
& - \sum_{k=1}^{j-2} r_2 r_0 r_{2,2n-i} r_{1,j-1} r_0 r_{2,k-1} g_2^{(k,j)} r_{k+1,j-1} \\
& - r_2 r_0 r_{2,2n-i} r_{1,j-1} g_2^{(0,j)} r_{2,j-1} + g_9^{(i,j)}
\end{aligned}
$$

for $3 \leq i \leq n-1$ and $3 \leq j \leq n-1$,

$< g_7^{(2n-3,0)}, g_7^{(2,0)} >= g_{12}$,

$< g_{12}, g_5^{(2)} >= g_{10}^{(1,2)} - r_2 r_0 r_{2,2n-2} g_5^{(1)} r_2$,

$< g_{10}^{(1,j-1)}, g_2^{(1,j)} >= g_{10}^{(1,j)}$ for $2 < j \leq n$,

$< g_{10}^{(1,j)}, g_7^{(j,1)} >= g_{10}^{(2,j)} - r_2 r_0 r_{2,2n-2} r_1 g_7^{(j,0)}$ for $2 \leq j \leq n$,

$< g_{10}^{(i,n)}, g_2^{(1,n-1)} >= g_{10}^{(i,n+1)}$ for $1 \leq i \leq 2$,

$< g_{10}^{(i,2n-j-1)}, g_2^{(1,j)} >= g_{10}^{(i,2n-j)}$ for $1 \leq i \leq 2$ and $3 \leq j < n-1$,

$$
\begin{aligned}
< g_{10}^{(2,j)}, g_3^{(1,2)} > = & \; r_0 r_{2,2n-2} r_{12} r_0 g_3^{(2,j)} r_{12} + r_0 r_{2,2n-2} r_{12} g_2^{(0,3)} r_{2,j} r_{12} \\
& - r_2 r_0 r_{2,2n-2} r_{12} r_0 g_3^{(2,j)} r_1 - r_2 r_0 r_{2,2n-2} r_{12} g_2^{(0,3)} + g_{11}^{(3,j)}
\end{aligned}
$$

for $3 \leq j \leq n$,

$$
\begin{aligned}
< g_{11}^{(i-1,j)}, g_3^{(i-2,i-1)} > = & \sum_{k=1}^{i-3} r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2j} r_{1,k-1} g_2^{(k,i-1)} r_{k+1,i-1} \\
& + r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,i-2} g_3^{(i-1,j)} r_{1,i-1} \\
& - \sum_{k=1}^{i-3} r_2 r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2j} r_{1,k-1} g_2^{(k,i-1)} r_{k+1,i-2} \\
& - r_2 r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,i-2} g_3^{(i-1,j)} r_{1,i-2} + g_{11}^{(i,j)}
\end{aligned}
$$

for $4 \leq i \leq n-1$, $4 \leq j \leq n$ and $i \leq j$,

$< g_{10}^{(2,2n-3)}, g_3^{(1,2)} >= g_{13}^{(2,2n-2)}$,

$$< g_{13}^{(i-1,j)}, g_3^{(i-1,i)} > = \sum_{k=1}^{i-2} r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,2n-j} r_{1,k-1} g_2^{(k,i)} r_{k+1,i}$$
$$+ r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,i-2} g_4^{(i-1)} r_{2n-i+2,2n-j} r_{1i}$$
$$- \sum_{k=1}^{i-2} r_2 r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,2n-j} r_{1,k-1} g_2^{(k,i)} r_{k+1,i-1}$$
$$- r_2 r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,i-2} g_4^{(i-1)} r_{2n-i+2,2n-j} r_{1,i-1} + g_{13}^{(i,j)}$$

for $2 \le j < i \le n-1$,

$$< g_{13}^{(i,i-1)}, g_3^{(i-2,i)} > = \sum_{k=1}^{i-3} r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-i+1} r_{1,k-1} g_2^{(k,i-1)} r_{k+1,i}$$
$$+ r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-i} g_1^{(i-1)} r_{1i}$$
$$- r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-i+1} r_{1,i-3} g_3^{(i-2,i-1)}$$
$$- \sum_{k=1}^{i-3} r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-i+1} r_{1,k-1} g_2^{(k,i-1)} r_{k+1,i-1}$$
$$- r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-i} g_1^{(i-1)} r_{1,i-1} + g_{13}^{(i,i)}$$

for $3 \le i \le n-1$,

$$< g_{13}^{(n-1,j)}, g_4^{(n-1)} > = \sum_{k=1}^{n-2} r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,2n-j} r_{1,k-1} g_2^{(k,n)} r_{k+1,n+1}$$
$$+ \sum_{k=n+2}^{j} r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,k-1} g_2^{(2n-k,n)} r_{k+1,2n-j} r_{1,n+1}$$
$$+ r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,n-2} g_4^{(n-1)} r_{n+2,2n-j} r_{1,n+1}$$
$$+ \sum_{k=2}^{n-2} r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,k-1} g_2^{(k,n)} r_{k+1,2n-j} r_{1,n+1}$$
$$+ r_0 r_{2,2n-2} r_{1,n-1} g_2^{(0,n)} r_{2,2n-j} r_{1,n+1}$$
$$- \sum_{k=1}^{n-2} r_2 r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,2n-j} r_{1,k-1} g_2^{(k,n)} r_{k+1,n}$$
$$- \sum_{k=n+2}^{j} r_2 r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,k-1} g_2^{(2n-k,n)} r_{k+1,2n-j} r_{1,n}$$
$$- r_2 r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,n-2} g_4^{(n-1)} r_{n+2,2n-j} r_{1n}$$
$$- \sum_{k=2}^{n-2} r_2 r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,k-1} g_2^{(k,n)} r_{k+1,2n-j} r_{1,n}$$
$$- r_2 r_0 r_{2,2n-2} r_{1,n-1} g_2^{(0,n)} r_{2,2n-j} r_{1,n}$$
$$+ g_{14}^{(n-1,j)}$$

for $2 \leq j \leq n-1$,

$$
\begin{aligned}
< g_{14}^{(i+1,j)}, g_4^{(i)} > = & \sum_{k=1}^{i-1} r_0 r_{2,2n-2} r_{1,2n-i-2} r_0 r_{2,2n-j} r_{1,k-1} g_2^{(k,i+1)} r_{k+1,i-1} r_{i,2n-i} \\
& + \sum_{k=j}^{i-1} r_0 r_{2,2n-2} r_{1,2n-i-2} r_0 r_{2,2n-k-1} g_2^{(k,i+1)} r_{2n-k+1,2n-j} r_{1,2n-i} \\
& + r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,i-1} g_4^{(i)} r_{2n-i+1,2n-j} r_{1,2n-i} \\
& + \sum_{k=2}^{i-1} r_0 r_{2,2n-2} r_{1,2n-i-2} r_0 r_{2,k-1} g_2^{(k,i+1)} r_{k+1,2n-j} r_{1,2n-i} \\
& + r_0 r_{2,2n-2} r_{1,2n-i-2} g_2^{(0,i+1)} r_{2,2n-j} r_{1,2n-i} \\
& - r_2 r_0 r_{2,2n-2} r_{1,2n-i-2} r_0 r_{2,2n-j} r_{1,i-1} g_3^{(i,2n-i-2)} r_{i+1} \\
& - \sum_{k=1}^{i-1} r_2 r_0 r_{2,2n-2} r_{1,2n-i-2} r_0 r_{2,2n-j} r_{1,k-1} g_2^{(k,i+1)} r_{k+1,i-1} r_{i,2n-i} \\
& - \sum_{k=j}^{i-1} r_2 r_0 r_{2,2n-2} r_{1,2n-i-2} r_0 r_{2,2n-k-1} g_2^{(k,i+1)} r_{2n-k+1,2n-j} r_{1,2n-i-1} \\
& - r_2 r_0 r_{2,2n-2} r_{1,2n-i-2} r_0 r_{2,i-1} g_4^{(i)} r_{2n-i+1,2n-j} r_{i,2n-i-1} \\
& - \sum_{k=2}^{i-1} r_2 r_0 r_{2,2n-2} r_{1,2n-i-2} r_0 r_{2,k-1} g_2^{(k,i+1)} r_{k+1,2n-j} r_{i,2n-i-1} \\
& - r_2 r_0 r_{2,2n-2} r_{1,2n-i-2} g_2^{(0,i+1)} r_{2,2n-j} r_{i,2n-i-1} + g_{14}^{(i,j)}
\end{aligned}
$$

for $2 \leq j \leq i < n-1$,

$$
\begin{aligned}
< g_9^{(i,n-1)}, g_4^{(n-1)} > = & \sum_{k=2}^{n-2} r_0 r_{2,2n-i} r_{1,n-1} r_0 r_{2,k-1} g_2^{(k,n)} r_{k+1,n+1} \\
& + r_0 r_{2,2n-i} r_{1,n-1} g_2^{(0,n)} r_{2,n+1} \\
& - \sum_{k=2}^{n-2} r_2 r_0 r_{2,2n-i} r_{1,n-1} r_0 r_{2,k-1} g_2^{(k,n)} r_{k+1,n} \\
& - r_2 r_0 r_{2,2n-i} r_{1,n-1} g_2^{(0,n)} r_{2,n} + g_{15}^{(i,n-1)}
\end{aligned}
$$

for $3 \leq i \leq n-1$,

$$
\begin{aligned}
< g_8^{(n,n-1)}, g_4^{n-1} > = & \sum_{k=2}^{n-2} r_0 r_{2,n} r_{1,n-1} r_0 r_{2,k-1} g_2^{(k,n)} r_{k+1,n+1} \\
& + r_0 r_{2,n} r_{1,n-1} g_2^{(0,n)} r_{2,n+1} \\
& - \sum_{k=2}^{n-2} r_2 r_0 r_{2,n} r_{1,n-1} r_0 r_{2,k-1} g_2^{(k,n)} r_{k+1,n} \\
& - r_2 r_0 r_{2,n} r_{1,n-1} g_2^{(0,n)} r_{2,n} + g_{15}^{(n,n-1)},
\end{aligned}
$$

$$
\begin{aligned}
< g_{15}^{(i,j+1)}, g_4^{(j)} > = & \sum_{k=2}^{j-1} r_0 r_{2,2n-i} r_{1,2n-j-2} r_0 r_{2,k-1} g_2^{(k,j+1)} r_{k+1,2n-j} \\
& + r_0 r_{2,2n-i} r_{1,2n-j-2} g_2^{(0,j+1)} r_{2,2n-j} \\
& - r_2 r_0 r_{2,2n-i} r_{1,2n-j-2} r_0 r_{2,j-1} g_3^{(j,2n-j-2)} r_{j+1} \\
& - \sum_{k=2}^{j-1} r_2 r_0 r_{2,2n-i} r_{1,2n-j-2} r_0 r_{2,k-1} g_2^{(k,j+1)} r_{k+1,2n-j-1} \\
& - r_2 r_0 r_{2,2n-i} r_{1,2n-j-2} g_2^{(0,j+1)} r_{2,2n-j-1} + g_{15}^{(i,j)}
\end{aligned}
$$

for $2 \leq i-1 \leq j < n-1$,

$< g_{10}^{(1,2n-3)}, g_3^{(1,2)} >= g_{16}$,

$< g_{10}^{(2,i)}, g_3^{(1,2)} >= r_0 r_{2,2n-2} r_{12} r_0 g_3^{(2,2n-i)} r_{12} - r_2 r_0 r_{2,2n-2} r_{12} r_0 g_3^{(2,2n-i)} r_1 + g_{17}^{(i,3)}$ for $3 < i \leq n-1$,

$$
\begin{aligned}
< g_{17}^{(i,j-1)}, g_3^{(j-2,j-1)} > = & \sum_{k=1}^{j-3} r_0 r_{2,2n-2} r_{1,j-1} r_0 r_{2,2n-i} r_{1,k-1} g_2^{(k,j-1)} r_{k+1,j-1} \\
& + r_0 r_{2,2n-2} r_{1,j-1} r_0 r_{2,j-2} g_3^{(j-1,2n-i)} r_{1,j-1} \\
& + \sum_{k=2}^{j-2} r_0 r_{2,2n-2} r_{1,j-1} r_0 r_{2,k-1} g_2^{(k,j)} r_{k+1,2n-i} r_{1,j-1} \\
& + r_0 r_{2,2n-2} r_{1,j-1} g_2^{(0,j)} r_{2,2n-i} r_{1,j-1} \\
& - \sum_{k=1}^{j-3} r_2 r_0 r_{2,2n-2} r_{1,j-1} r_0 r_{2,2n-i} r_{1,k-1} g_2^{(k,j-1)} r_{k+1,j-2} \\
& - r_2 r_0 r_{2,2n-2} r_{1,j-1} r_0 r_{2,j-2} g_3^{(j-1,2n-i)} r_{1,j-2} \\
& - \sum_{k=2}^{j-2} r_2 r_0 r_{2,2n-2} r_{1,j-1} r_0 r_{2,k-1} g_2^{(k,j)} r_{k+1,2n-i} r_{1,j-2} \\
& - r_2 r_0 r_{2,2n-2} r_{1,j-1} g_2^{(0,j)} r_{2,2n-i} r_{1,j-1} + g_{17}^{(i,j)}
\end{aligned}
$$

for $4 \leq j < i \leq n-1$. □

At this point we are not able to show that polynomials given above lemma form Gröbner-Shirshov basis for the infinite Coxeter group of type $\widetilde{B}_n$.

7.2 Normal Forms for The Infinite Coxeter Group of Type $\widetilde{B}_n$

For $v \in \widetilde{S_n}^C$ define

$$v[i,j] = |\{k \in \mathbb{Z} : k \leq i, v(k) \geq j\}|$$

for all $i, j \in \mathbb{Z}$.

Let $\widetilde{S}_n^B = \{u \in \widetilde{S}_n^C : u[n, n+1] \equiv 0 \bmod 2\}$ It is clear that $\widetilde{S}_n^B$ is a subgroup of $\widetilde{S}_n^C$ of index 2. Furthermore for any $u \in \widetilde{S}_n^B$

$$u = (r^C_{nj_n} r^C_{n-1,j_{n-1}} \cdots r^C_{1j_1})(r^C_0 r^C_{1,2n-1})^{\alpha_{2n-1}} \cdots (r^C_0 r^C_1)^{\alpha_1} (r^C_0)^{\alpha_0}$$

where $\sum_{k=0}^{2n-1} \alpha_k$ is an even number.

The following proposition says that $\widetilde{S}_n^B$ is the infinite Coxeter group of type $\widetilde{B}_n$.

Proposition 7.2.1. *(Bjorner and Brenti (2005), Proposition 8.5.3)*

The group $\widetilde{S}_n^B$ with generating set $\{r^B_0, r^B_1, \ldots, r^B_n\}$ is the infinite Coxeter group of type $\widetilde{B}_n$ where $r^B_i = r^C_i$ for $i = 1, 2, \ldots, n$ and $r^B_0 = [2n-1, 2n, 3, \ldots, n]$.

We now try to find normal form representations of elements of $\widetilde{B}_n$ with respect to these generators. First of all, we give some relations between words in $\widetilde{B}_n$ and words in $\widetilde{C}_n$.

Lemma 7.2.2. *The followings are equivalent.*

(i) $r^C_0 r^C_1 r^C_0 = r^B_0$,

(ii) $(r^C_0 r^C_{1i})(r^C_0 r^C_{1j}) = r^B_0 r^B_{2i} r^B_{1j}$ *for* $0 \leq j \lessdot i \leq 2n-2$.

Proof. (i)

$$\begin{aligned} r^C_0 r^C_1 r^C_0 &= [2n, 2, \ldots, n][2, 1, 3, \ldots, n][2n, 2, \ldots, n], \\ &= [2n-1, 2n, 3, \ldots, n], \\ &= r^B_0. \end{aligned}$$

(ii)

$$\begin{aligned} r^B_0 r^B_{2i} r^B_{1j} &= r^C_0 r^C_1 r^C_0 r^C_{2i} r^C_{1j}, \\ &= r^C_0 r^C_{1i} r^C_0 r^C_{1j} \quad \text{by a series of ELW in} f_2^{(0,k)}. \end{aligned}$$

□

Notice that length of word in $\widetilde{C}_n$ is two more than the length of corresponding word in $\widetilde{B}_n$.

Lemma 7.2.3. *In the infinite Coxeter group of type $\widetilde{C}_n$, we have*

$$(r_0^C r_{1,2n-2}^C)(r_0^C r_{1j}^C)(r_0^C r_{1i}^C) = \begin{cases} (r_0^C r_{1,2n-1}^C)(r_0^C r_{1i}^C)(r_0^C r_{1,j-1}^C), & \text{if } i+j < 2n, \\ (r_0^C r_{1,2n-1}^C)(r_0^C r_{1,i-1}^C)(r_0^C r_{1j}^C), & \text{if } i+j \geq 2n. \end{cases}$$

for $1 \leq i, j \leq 2n-1$ satisfying $j \leq i$ when $i < n$ or $i < j$ when $i \geq n$.

Proof. If $i+j < 2n$, there are two possibilities: either $1 \leq j \leq i < n$ or $1 \leq j < n \leq i < 2n-j$. In both cases we have

$$\begin{aligned}(r_0^C r_{1,2n-2}^C)(r_0^C r_{1j}^C)(r_0^C r_{1i}^C) &= (r_0^C r_{1,2n-1}^C)(r_0^C r_{1j}^C)(r_0^C r_{1j-1}^C)(r_0^C r_{j+1,i}^C) \quad \text{by an ELW in } f_5^{(j)}, \\ &= (r_0^C r_{1,2n-1}^C)(r_0^C r_{1i}^C)(r_0^C r_{1,j-1}^C) \quad \text{by a series of ELW in } f_2.\end{aligned}$$

If $2n \leq i+j$, then $n \leq j < i \leq 2n-2$. Let $i = 2n-k$ and $j = 2n-l$. Therefore

$$(r_0^C r_{1,2n-2}^C)(r_0^C r_{1j}^C)(r_0^C r_{1i}^C) = (r_0^C r_{1,2n-1}^C)(r_0^C r_{1j}^C)(r_0^C r_{1j}^C) r_{l-2} r_{l-3} \cdots r_k$$

by ELW in $f_6^{(j)}$. Furthermore

$$\begin{aligned}(r_0^C r_{1j}^C) r_t &= (r_0^c r_{1,t-1}^C) r_{t+1}^C r_{tj}^C \quad \text{by an ELW in } f_3^{(t,j)}, \\ &= r_{t+1}^C r_{1j}^C \quad \text{by a series of ELW in } f_2.\end{aligned}$$

for $l-2 \leq t \leq k$. Then desired equality easily follows.

□

Corollary 7.2.4.

$$(r_0^C r_{1,2n-1}^C)(r_0^C r_{1i}^C)(r_0^C r_{1j}^C) = \begin{cases} (r_0^B r_{2,2n-2}^B r_{1,j+1}^B)(r_0^C r_{1i}^C), & i+j < 2n-1, \\ (r_0^B r_{2,2n-2}^B r_{1j}^B)(r_0^C r_{1,i+1}^C), & i+j \geq 2n-1. \end{cases}$$

for $1 \leq j \leqslant i \leq 2n-2$.

Lemma 7.2.5. *Let $m \geq 1$.*

(i) $(r_0^C r_{1,2n-1}^C)^{2m} = (r_0^B r_{2,2n-2}^B r_1^B)^{2m}$,

(ii) $(r_0^C r_{1,2n-1}^C)^{2m-1}(r_0^C r_{1j}^C) = (r_0^B r_{2,2n-2}^B r_1^B)^{2m-1}(r_0^B r_{2j}^B)$ *for* $2 \leq j \leq 2n-2$,

(iii) $(r_0^C r_{1,2n-1}^C)^{2m-1} r_0^C = (r_0^B r_{2,2n-2}^B r_1^B)^{2(m-1)}(r_0^B r_{2,2n-2}^B)(r_0^B)$,

Proof.

(i) We use induction on m.

$$\begin{aligned}(r_0^B r_{2,2n-2}^B r_1^B)(r_0^B r_{2,2n-2}^B r_1^B) &= (r_0^C r_{1,2n-2}^C r_0^C r_1^C)(r_0^C r_{1,2n-2}^C r_0^C r_1^C),\\ &= (r_0^C r_{1,2n-2}^C)(r_1^C r_0^C r_1^C r_0^C)(r_{2,2n-2}^C r_0^C r_1^C),\\ &= (r_0^C r_{1,2n-1}^C)(r_0^C r_{1,2n-2}^C)(r_0^C r_0^C r_1^C),\\ &= (r_0^C r_{1,2n-1}^C)^2.\end{aligned}$$

First equality comes from Lemma 7.2.2, second and third equalities come from ELW in $f_5^{(1)}$ and $f_2^{(0,k)}$, respectively.

Suppose that $(r_0^B r_{2,2n-2}^B r_1^B)^{2k} = (r_0^C r_{1,2n-1}^C)^{2k}$ for a positive integer k. Then

$$\begin{aligned}(r_0^B r_{2,2n-2}^B r_1^B)^{2(k+1)} &= (r_0^C r_{1,2n-1}^C)^{2k}(r_0^B r_{2,2n-2}^B r_1^B)^2\\ &= (r_0^C r_{1,2n-1}^C)^{2(k+1)}\end{aligned}$$

(ii)

$$\begin{aligned}(r_0^B r_{2,2n-2}^B r_1^B)^{2m+1}(r_0^B r_{2j}^B) &= (r_0^C r_{1,2n-1}^C)^{2m}(r_0^C r_{1,2n-2}^C r_0^C r_1^C)(r_0^C r_{1j}^C r_0^C) \text{ by Lemma 7.2.2},\\ &= (r_0^C r_{1,2n-1}^C)^{2m} r_0^C r_{1,2n-2}^C r_1^C r_0^C r_1^C r_0^C r_{2j}^C r_0^C \text{ by ELW in} f_5^{(1)},\\ &= (r_0^C r_{1,2n-1}^C)^{2m+1} r_0^C r_{1j}^C r_0^C r_0^C \text{ by a series of ELW in} f_2^{(0,k)},\\ &= (r_0^C r_{1,2n-1}^C)^{2m+1} r_0^C r_{1j}^C \text{ by ELW in} f_1^{(0)}.\end{aligned}$$

(iii)

$$\begin{aligned}(r_0^B r_{2,2n-1}^B)(r_0^B) &= (r_0^C r_{1,2n-2}^C r_0^C)(r_0^C r_1^C r_0^C) \text{ by Lemma 7.2.2}\\ &= r_0^C r_{1,2n-1}^C r_0^C\end{aligned}$$

The rest is an easy consequence of part (i). $\square$

Notice that length of word in $\widetilde{C}_n$ is $2m$ more than the length of the corresponding word in $\widetilde{B}_n$.

Definition 7.2.6. We define the following words in $\widetilde{B}_n$:

(i) $w_0 = r^B_{nl_n} \cdots r^B_{il_i} \cdots r^B_{1l_1}$ for $i-1 \leq l_i \leq 2n-i$ and $i = 1, \ldots, n$.

(ii) $w_1 = \prod_{k=1}^{t}(r^B_0 r^B_{2,2n-2} r^B_{1i_k})$ for $t \geq 0$ and $1 \leq i_k \leqslant i_{k-1} \leq 2n-2$.

(iii) $w_2 = \prod_{k=1}^{s}(r^B_0 r^B_{2,j_{2k-1}} r^B_{1j_{2k}})$ for $s \geq 0$ and $0 \leq j_k \leqslant j_{k-1} \leq 2n-3$.

(iv) $w_3 = \begin{cases} (r^B_0 r^B_{2n-2} r^B_1)^{2m}, & \\ (r^B_0 r^B_{2n-2} r^B_1)^{2m-1}(r^B_0 r^B_{2j}), & \text{for } m \geq 0 \text{ and } 1 \leq j \leq 2n-2, \\ (r^B_0 r^B_{2n-2} r^B_1)^{2(m-1)}(r^B_0 r^B_{2,2n-2}) r^B_0. & \end{cases}$

(v) $w_4 = w_0 w_1 w_2$ where $i_t \geq 2$ and either $j_1 \leqslant i_t$ or $j_1 \nleqslant i_t$ but
$\begin{cases} j_2 \leqslant i_t, & i_t + j_1 \geq 2n; \\ j_2 + 1 < i_t, & i_t + j_1 < 2n. \end{cases}$,

(vi) $w_5 = w_0 w_1 w_3$.

Let $W_B = \{w_4, w_5\}$.

Theorem 7.2.7. *Any word $w \in W_C$ in which number of appearance of r_0 is even can be transformed a word in W_B.*

Proof. Since $r^B_i = r^C_i$ for $i = 1, \ldots, n$, we only consider the word of the form

$$w = (r^C_0 r^C_{1,2n-1})^m \prod_{k=1}^{t}(r^C_0 r^C_{1j_k})$$

where $m + t$ is even $0 \leq j_k \leqslant j_{k-1} \leq 2n-2$.

If $m = 0$, then Lemma 7.2.2 implies that

$$w = \prod_{k=1}^{\frac{t}{2}} (r_0^B r_{2,j_{2k-1}}^B r_{j_{2k}}^B).$$

Therefore $w \in W_B$.

Suppose that $m \geq 1$ and $2n-2 = j_1 = j_2 = \cdots = j_l > j_{l+1}$. Then

$$w = \Big(\prod_{k=1}^{\lfloor \frac{l+1}{2} \rfloor} (r_0^B r_{2,j_{2k-1}}^B r_{1,j_{2k}}^B) \Big) w'$$

where

$$w' = (r_0^C r_{1,2n-1}^C)^m \prod_{k=2\lfloor \frac{l+1}{2} \rfloor + 1}^{t} (r_0^C r_{1j_k}^C)$$

by several application of Corollary 7.2.4 and Lemma 7.2.2.

Let us rewrite w' as follows

$$w' = (r_0^C r_{1,2n-1}^C)^m (r_0^C r_{1i}^C) \prod_{k=0}^{p} (r_0^C r_{1i_k}^C).$$

Suppose that $i + k + i_k \geq 2n - 1$ for $0 \leq k \leq q \leq p$ and $i + q + 1 + i_k < 2n - 1$ for $q + 1 \leq k \leq p$.

Let $a = (2n-2) - i$. Now we investigate case by case.

Case (i): $q \geq a - 1$ and $m > a$.

Corollary 7.2.4 and Lemma 7.2.2 imply that

$$w' = \prod_{k=0}^{a} (r_0^B r_{2,2n-2}^B r_{1i_k}^B) w''$$

$$\begin{aligned} w' &= \prod_{k=0}^{a-1} (r_0^B r_{2,2n-2}^B r_{1i_k}^B)(r_0^C r_{1,2n-1}^C)^{m-a}(r_0^C r_{1,2n-2}^C) \prod_{k=a}^{p} (r_0^C r_{1,i_k}^C) \\ &= \prod_{k=0}^{a} (r_0^B r_{2,2n-2}^B r_{1i_k}^B) w'' \end{aligned}$$

where

$$w'' = (r_0^C r_{1,2n-1}^C)^{m-a} \prod_{k=a+1}^{p} (r_0^C r_{1,i_k}^C).$$

Now same process can be applied to $w^{''}$. This should be repeated until one of the conditions is not met. Hence we can assume that $w^{'}$ does not satisfy one of the conditions without loss of generality.

Case (ii): $q \geq a-1$ and $m = a$ Corollary 7.2.4 and Lemma 7.2.2 imply that

$$w^{'} = \prod_{k=0}^{m}(r_0^B r_{2,2n-2}^B r_{1i_k}^B) \prod_{k=\frac{m+2}{2}}^{\frac{p}{2}} (r_0^B r_{2,i_{2k-1}}^B r_{1,i_{2k}})$$

Since $i_a \leqslant i_{a+1}$, $w^{'} \in W_B$ and so is w.

Case (iii): $q \geq a-1$ and $m < a$

Corollary 7.2.4 and Lemma 7.2.2 implies that

$$\begin{aligned} w^{'} &= \Big(\prod_{k=0}^{m-1}(r_0^B r_{2,2n-2}^B r_{1i_k}^B)\Big)(r_0^C r_{1,i+m}^C) \prod_{k=m}^{p}(r_0^C r_{1,i_k}^C) \\ &= \Big(\prod_{k=0}^{m-1}(r_0^B r_{2,2n-2}^B r_{1i_k}^B)\Big)(r_0^B r_{2,i+m}^B r_{1i_m}^B) \prod_{k=\frac{m+2}{2}}^{\frac{p}{2}} (r_0^B r_{2,i_{2k-1}}^B r_{1,i_{2k}}). \end{aligned}$$

If $i+m \leqslant i_{m-1}$, then clearly $w^{'} \in W_B$ which implies $w \in W_B$. Suppose $i+m \nleqslant i_{m-1}$. Since $i_{m-1}+m+i \geq 2n$ and $i_m \leqslant i_{m-1}$, $w^{'} \in W_B$ and so is w.

Case (iv): $q < a-1$ and $m \leq q$

Same as case (iii).

Case (v): $q < a-1$ and $q < m \leq p$

$$\begin{aligned} w^{'} &= \Big(\prod_{k=0}^{q}(r_0^B r_{2,2n-2}^B r_{1i_k}^B)\Big)(r_0^C r_{1,2n-1}^C)^{m-q-1}(r_0^C r_{1,i+q+1}^C) \prod_{k=q+1}^{p} (r_0^C r_{1i_k}^C) \\ &= \Big(\prod_{k=0}^{q}(r_0^B r_{2,2n-2} r_{1i_k}^B)\Big)\Big(\prod_{k=q+1}^{m-1} (r_0^B r_{2,2n-2}^B r_{1i_k+1}^B)\Big)(r_0^B r_{2,i+q+1}^B r_{1i_m}^B) \prod_{k=\frac{m+2}{2}}^{\frac{p}{2}} (r_0^B r_{2,i_{2k-1}}^B r_{1,i_{2k}}) \end{aligned}$$

by Corollary 7.2.4 and Lemma 7.2.2. Notice that $i_q > i_{q+1}$. If $i+q+1 \leqslant i_{m-1}+1$, then clearly $w^{'} \in W_B$ which implies $w \in W_B$. Suppose $i+q+1 \nleqslant i_{m-1}+1$. Then

$i_{m-1}+1 \leq i+q+1$ and $i_m+i+q+1 < 2n-1$. This implies $i_m < n$ and then $i_m+1 < i_{m-1}+1$. Hence $w' \in W_B$ and so is w.

Case (vi): Applying Corollary 7.2.4 and Lemma 7.2.2 several times gives

$$\begin{aligned} w' &= \Big(\prod_{k=0}^{q}(r_0^B r_{2,2n-2} r_{1i_k}^B)\Big)\Big(\prod_{k=q+1}^{p}(r_0^B r_{2,2n-2}^B r_{1i_k+1}^B)\Big)(r_0^C r_{1,2n-1}^C)^{m-p-1}(r_0^C r_{1,i+q+1}^C) \\ &= \Big(\prod_{k=0}^{q}(r_0^B r_{2,2n-2} r_{1i_k}^B)\Big)\Big(\prod_{k=q+1}^{p}(r_0^B r_{2,2n-2}^B r_{1i_k+1}^B)\Big)w'' \end{aligned}$$

where

$$w'' = \begin{cases} (r_0^B r_{2,2n-2}^B r_1^B)^{m-p}, & i+q+1 = 2n-2 \\ (r_0^B r_{2,2n-2}^B r_1^B)^{m-p-1}(r_0^B r_{2,i+q+1}^B), & 1 \leq i+q+1 \leq 2n-3 \\ (r_0^B r_{2,2n-2}^B r_1^B)^{m-p-2}(r_0^B r_{2,2n-2}^B)(r_0^B), & i+q+1 = 0 \end{cases}$$

by Lemma 7.2.5. Then clearly $w' \in W_B$ and so is w. □

Lemma 7.2.8. *The generating function for words in W_B is*

$$\prod_{i=1}^{n}(1+x+\cdots+x^{2i-1})\frac{1+x^i}{1-x^{n+i}}.$$

Proof. We found one to one corresponding between words in W_B and words in W_C with number of occurrence of r_0 is even. Let $w = (r_{nl_n}^C r_{n-1,l_{n-1}}^C \cdots r_{1l_1}^C)\prod_{k=1}^{t}(r_0^C r_{1j_k}^C)$ where t is even and $0 \leq j_k \leqslant j_{k-1} \leq 2n-1$. Since $r_i^C = r_i^B$ for $i = 1, \ldots, n$, $r_{nl_n}^C r_{n-1,l_{n-1}}^C \cdots r_{1l_1}^C = r_{nl_n}^B r_{n-1,l_{n-1}}^B \cdots r_{1l_1}^B$. Clearly its generating function is

$$\prod_{i=1}^{n}(1+x+x^2+\cdots+x^{2i-1}).$$

When converting the $\prod_{k=1}^{t}(r_0^C r_{1j_k}^C)$ into a word in W_B, the corresponding word losses its length by number of occurrence of r_0. The generating function for the words in the form $\prod_{k=1}^{t}(r_0^C r_{1j_k}^C)$ where $t \geq 0$ in W_C is

$$\prod_{i=1}^{n}\frac{1+x^i}{1-x^{n+i}}.$$

Hence generating function for the corresponding words in W_B is

$$\frac{1}{1-x^n}\prod_{i=1}^{n-1}\frac{1+x^i}{1+x^{n+i}} = \prod_{i=1}^{n-1}\frac{1+x^i}{1-x^{n+i}}$$

since $\frac{1}{1-x^n} = \frac{1+x^n}{1-x^{2n}}$. Notice that we consider all words of the form $\prod_{k=1}^{t}(r_0^C r_{1j_k}^C)$ where $t \geq 0$. We can add or remove r_0^C to end of the word if number of occurrence r_0^C is odd and this will not change the result. □

Notice that the generating function for the infinite Coxeter group of type $\widetilde{B}_n$ is

$$\prod_{i=1}^{n} \frac{1 + x + \cdots + x^{2i-1}}{1 - x^{2i-1}}.$$

By Lemma 6.2.12

$$\prod_{i=1}^{n}(1 + x + \cdots + x^{2i-1})(\frac{1 + x^i}{1 - x^{n+i}}) = \prod_{i=1}^{n} \frac{1 + x + cdots + x^{2i-1}}{1 - x^{2i-1}}$$

which is equal to generating function of words in W_B.

Now we are ready to find the main result.

Theorem 7.2.9. *Let R^B be the set of all polynomials given in Lemma 7.1.2. Then*

(i) $W_B = \textit{Red}(R^B)$.

(ii) R^B *is a Gröbner-Shirshov basis for the infinite Coxeter group of type* $\widetilde{B}_n$.

Proof. (i) It is easy to see that any word in W_B is R^B-reduced. Hence $W_B \subseteq \text{Red}(R^B)$. Conversely if $w \in \text{Red}(R^B)$, then w can be written as a permutation in $\widetilde{S}_n^B$ and this permutation has a corresponding word in W_B by Theorem 7.2.7. Hence $\text{Red}(R^B) \subseteq W_B$.

(ii) We know that any polynomial in R^B is a part of a Gröbner-Shirshov basis of the infinite Coxeter group of type $\widetilde{B}_n$. If R^B were not a Gröbner-Shirshov basis, then $\text{Red}(R^B) = W_B$ should be a proper subset of the set of normal forms of the infinite Coxeter group of type $\widetilde{B}_n$ by Composition-Diamond lemma. This contradicts to the fact that W_B and normal forms of the infinite Coxeter group of type $\widetilde{B}_n$ have same generating functions.

□

Let us finish the chapter by an example.

Any word $w \in W_C$ with the number of appearance of r_0 is even can be transformed a word in W_B by Theorem 7.2.7. This transformation can be made by using Corollary 7.2.4 and

Lemma 7.2.5 repeatedly. Now we will take a word $w \in \widetilde{S}_n^B$ and we find corresponding word in the infinite Coxeter group of type $\widetilde{C}_n$. Using the techniques given in Chapter 6, then we convert it into a word in the infinite Coxeter group of type $\widetilde{B}_n$.

Example 7. Let $u = [42, 5, 17, -20] \in \widetilde{S}_4^B$. Since $\widetilde{S}_4^B \subseteq \widetilde{S}_4^C$, $u \in \widetilde{S}_4^C$. Now we will deal with in $\widetilde{S}_4^C$. Corollary 6.2.3 implies $uw^{-1} = [-8, 4, -33, -20]$ where $w = r_0r_{12}r_0r_1r_0$. Applying the process explained in the proof of the Corollary 6.2.5, we get

$[-8, 4, -33, -20](r_0r_{14})^{-1} = [-11, -8, 4, -33]$,

$[-11, -8, 4, -33](r_0r_{14})^{-1} = [-24, -11, -8, 4]$,

$[-24, -11, -8, 4](r_0r_{15})^{-1} = [1, -24, -11, 4]$,

$[1, -24, -11, 4](r_0r_{15})^{-1} = [-2, 1, -24, 4]$,

$[-2, 1, -24, 4](r_0r_{15})^{-1} = [-15, -2, 1, 4]$,

$[-15, -2, 1, 4](r_0r_{17})^{-1} = [-6, -2, 1, 4]$ and

$[-6, -2, 1, 4](r_0r_{17})^{-1} = [3, -2, 1, 4]$.

Thus $[-8, 4, -33, -20]((r_0r_{17})^2(r_0r_{15})^3(r_0r_{14})^2)^{-1} = [3, -2, 1, 4]$.

Corollary 6.2.8 implies $[3, -2, 1, 4] = r_{25}r_{12}$. Hence

$u = (r_{25}r_{12})(r_0r_{17})^2(r_0r_{15})^3(r_0r_{14})^2(r_0r_{12}r_0r_1r_0) \in \widetilde{S}_4^C$.

Applying Corollary 7.2.4 repeatedly the word $(r_0r_{17})^2(r_0r_{15})^3(r_0r_{14})^2(r_0r_{12}r_0r_1r_0)$ can be converted the following words in each step.

$(r_0^Cr_{17}^C)(r_0^Cr_{17}^C)(r_0^Cr_{15}^C)(r_0^Cr_{15}^C)(r_0^Cr_{15}^C)(r_0^Cr_{14}^C)(r_0^Cr_{14}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C)$

$= (r_0^Cr_{17}^C)(r_0^Br_{26}^Br_{15}^B)(r_0^Cr_{16}^C)(r_0^Cr_{15}^C)(r_0^Cr_{14}^C)(r_0^Cr_{14}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C)$

$= (r_0^Br_{26}^Br_{15}^B)(r_0^Cr_{17}^C)(r_0^Cr_{16}^C)(r_0^Cr_{15}^C)(r_0^Cr_{14}^C)(r_0^Cr_{14}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C)$

$= (r_0^Br_{26}^Br_{15}^B)(r_0^Br_{26}^Br_{15}^B)(r_0^Cr_{17}^C)(r_0^Cr_{14}^C)(r_0^Cr_{14}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C)$

$= (r_0^Br_{26}^Br_{15}^B)(r_0^Br_{26}^Br_{15}^B)(r_0^Br_{26}^Br_{14}^B)(r_0^Cr_{15}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C)$

Applying Lemma 7.2.2,

$(r_0^Cr_{15}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C) = (r_0^Br_{15}^Br_{12}^B)(r_0^B)$.

Since $r_i^B = r_i^C$ for $i \neq 0$, then $r_{25}^C r_{12}^C = r_{25}^B r_{12}^B$.

Therefore

$u = (r_{25}^B r_{12}^B)(r_0^B r_{26}^B r_{15}^B)^2 (r_0^B r_{26}^B r_{14}^B)(r_0^B r_{15}^B r_{12}^B)(r_0^B) \in \widetilde{S}_4^B.$

8. GRÖBNER-SHIRSHOV BASIS AND NORMAL FORMS FOR THE INFINITE COXETER GROUPS OF TYPE $\widetilde{D}_n$

8.1 Gröbner-Shirshov Basis for The Infinite Coxeter Group of Type $\widetilde{D}_n$

Definition 8.1.1. For a positive integer $n \geq 4$, the infinite Coxeter group of type $\widetilde{D}_n$ has a presentation with generators $S = \{r_0, r_1, \ldots, r_n\}$ and defining relations:

(R_1) $r_i r_i = 1$ for $0 \leq i \leq n$,

(R_2) $r_i r_j = r_j r_i$ for $0 < i < j - 1 < n$ but $(i,j) \neq (0,2)$ and $(i,j) \neq (n-2,n)$,

(R_3) $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ for $1 \leq i < n-1$,

(R_4) $r_{n-2} r_n r_{n-2} = r_n r_{n-2} r_n$,

(R_5) $r_0 r_2 r_0 = r_2 r_0 r_2$.

For convenience let us define

$$r_{ij} = \begin{cases} r_i r_{i+1} \cdots r_j, & \text{if } 1 \leq i < j < n\,; \\ r_i r_{i+1} \cdots r_{n-2} r_n r_{n-1} \cdots r_{2n-j}, & \text{if } 1 \leq i \leq n-1 < j \leq 2n-i; \\ r_i, & \text{if } j = i; \\ 1, & \text{if } j = i-1. \end{cases}$$

After this point we will not use superscripts unless we need to distinguish between groups $\widetilde{B}_n$ and $\widetilde{D}_n$.

Lemma 8.1.2. *Suppose that $<$ is the degree lexicographic order on S^*. A Gröbner-Shirshov basis for the infinite Coxeter group of type $\widetilde{D}_n$ with respect to $<$ contains the following polynomials:*

- $h_1^{(i)} = r_i r_i - 1$ *for* $0 \leq i \leq n$,
- $h_2^{(i,j)} = r_i r_j - r_j r_i$ *for* $1 < j - i$ *but* $(i,j) \neq (0,2)$ *and* $(i,j) \neq (n-2,n)$,
- $h_3^{(i)} = r_{i,i+1} - r_{i+1} r_i$ *for* $i = 0, n-1$,

- $h_4 = r_{n-2,n}r_{n-2} - r_nr_{n-2,n}$,
- $h_5^{(i,j)} = r_{ij}r_i - r_{i+1}r_{ij}$ *for* $(1 \leq i < j \leq n-1)$ *or* $(1 \leq i < n-2$ *and* $n \leq j \leq 2n-3$ *and* $2n-j-1 > 1)$,
- $h_6^{(i)} = r_{i,2n-i}r_{i+1} - r_{i+1}r_{i,2n-i}$ *for* $1 \leq i \leq n-3$,
- $h_7 = r_{n-2,n+2}r_n - r_{n-1}r_{n-2,n+2}$,
- $h_8 = r_{n-2,n+2}r_{n-1} - r_nr_{n-2,n+2}$,
- $h_9^{(i,j)} = r_0r_{2i}r_{1j}r_0 - r_2r_0r_{2i}r_{1j}$ *for* $0 \leq j \leq 1$ *and* $2 \leq i \leq 2n-3$,
- $h_{10}^{(i)} = r_0r_{2i}r_{1i} - r_1r_0r_{2i}r_{1,i-1}$ *for* $2 \leq i \leq n-1$,
- $h_{11} = r_0r_{2n}r_{1n} - r_1r_0r_{2n}r_{1,n-2}$,
- $h_{12} = r_0r_{2,n-1}r_{1,n+1} - r_1r_0r_{2,n-1}r_{1n}$,
- $h_{13}^{(i)} = r_0r_{2,2n-i}r_{1,2n-i+1} - r_1r_0r_{2,2n-i}r_{1,2n-i}$ *for* $2 \leq i < n$,
- $h_{14}^{(i,j)} = r_0r_{2i}r_{1j}r_0r_{2j} - r_2r_0r_{2i}r_{1j}r_0r_{2,j-1}$ *for* $(2 \leq j \leq n-1$ *and* $n \leq i \leq 2n-3)$ *or* $(2 \leq j < n-1$ *and* $3 \leq i \leq n-1$ *and* $j < i)$,
- $h_{15} = r_0r_{2,n-1}r_{1n}r_0r_{2n} - r_2r_0r_{2,n-1}r_{1n}r_0r_{2,n-2}$,
- $h_{16}^{(i,j)} = r_0r_{2,2n-2}r_{1i}r_0r_{2j}r_{1,i-1} - r_2r_0r_{2,2n-2}r_{1i}r_0r_{2j}r_{1,i-2}$ *for* $2 \leq i \leq j \leq n-1$,
- $h_{17} = r_0r_{2,2n-2}r_0r_2 - r_2r_0r_{2,2n-2}r_0$,
- $h_{18}^{(i,j)} = r_0r_{2,2n-2}r_{1i}r_0r_{2j}r_1 - r_2r_0r_{2,2n-2}r_{1i}r_0r_{2j}$ *for* $(i = 1$ *and* $2 \leq j \leq n-1)$ *or* $(1 \leq i \leq 2$ *and* $n \leq j \leq 2n-3)$,
- $h_{19}^{(i)} = r_0r_{2,2n-i}r_{1,n-1}r_0r_{2,n+1} - r_2r_0r_{2,2n-i}r_{1,n-1}r_0r_{2n}$ *for* $3 \leq i \leq n$,
- $h_{20}^{(i)} = r_0r_{2,2n-i}r_{1n}r_0r_{2n} - r_2r_0r_{2,2n-i}r_{1n}r_0r_{2,n-2}$ *for* $3 \leq i \leq n-1$,
- $h_{21}^{(i)} = r_0r_{2,2n-2}r_{1i}r_0r_{2,2n-2}r_{12} - r_2r_0r_{2,2n-2}r_{1i}r_0r_{2,2n-2}r_1$ *for* $1 \leq i \leq 2$,
- $h_{22} = r_0r_{2,2n-2}r_{1n}r_0r_{2n}r_{1,n-2} - r_2r_0r_{2,2n-2}r_{1n}r_0r_{2n}r_{1,n-3}$,
- $h_{23}^{(i)} = r_0r_{2,2n-2}r_{1,n-1}r_0r_{2,2n-i}r_{1n} - r_2r_0r_{2,2n-2}r_{1,n-1}r_0r_{2,2n-i}r_{1,n-2}$ *for* $2 \leq i \leq n-1$,
- $h_{24}^{(i)} = r_0r_{2,2n-2}r_{1n}r_0r_{2,2n-i}r_{1,n-1} - r_2r_0r_{2,2n-2}r_{1n}r_0r_{2,2n-i}r_{1,n-2}$ *for* $2 \leq i \leq n-1$,

- $h_{25}^{(i,j)} = r_0 r_{2,2n-2} r_{1,2n-i} r_0 r_{2,2n-j} r_{1,2n-i+1} - r_2 r_0 r_{2,2n-2} r_{1,2n-i} r_0 r_{2,2n-j} r_{1,2n-i}$
  for $2 \leq j < i \leq n$,

- $h_{26}^{(i,j)} = r_0 r_{2,2n-i} r_{1,2n-j} r_0 r_{2,2n-j+1} - r_2 r_0 r_{2,2n-i} r_{1,2n-j} r_0 r_{2,2n-j}$
  for $3 \leq i \leq j \leq n-1$,

- $h_{27}^{(i,j,k)} = r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-j} r_{1k} - r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-j} r_{1,k-1}$
  for $(k = i$ *and* $2 \leq j \leq n-2$ *and* $3 \leq i \leq n-2)$
  or $(k = i-1$ *and* $3 \leq i \leq n-2$ *and* $i < j \leq n)$,

Proof. $h_1^{(i)}, h_2^{(i,j)}, h_3^{(i)}, h_4, h_5^{(i,i+1)}$ and $h_9^{(2,0)}$ are defining relations for the infinite Coxeter group of type $\widetilde{D}_n$.

$< h_5^{(i,i+1)}, h_2^{(i,i+2)} >= h_5^{(i,i+2)}$ for $1 \leq i < n-2$,

$< h_5^{(i,j-1)}, h_2^{(i,j)} >= h_5^{(i,j)}$ for $i+3 \leq j \leq n-1$,

$< h_5^{(n-3,n-2)}, h_2^{(n-3,n)} >= h_5^{(n-3,n)}$,

$< h_5^{(i,n-2)}, f_2^{(i,n)} >= h_5^{(i,n)}$ for $1 \leq i < n-3$,

$< h_5^{(i,2n-j-1)}, h_2^{(i,j)} >= h_5^{(i,2n-j)}$ for $1 \leq i < n-2$, $3 \leq j \leq n-1$ and $j - i > 1$,

$< h_5^{(i,2n-i-2)}, h_5^{(i,i+1)} >= h_6^{(i)}$ for $1 \leq i \leq n-3$,

$< h_5^{(n-2,n)}, h_4 >= r_{n-2} h_3^{(n-1)} r_{n-2} r_n - r_{n-1} r_{n-2} h_3^{(n-1)} r_{n-2} + h_7$,

$< h_4, h_5^{(n-2,n)} >= h_8$,

$< h_9^{(i-1,0)}, h_2^{(0,i)} >= h_9^{(i,0)}$ for $3 \leq i < n$,

$< h_9^{(n-2,0)}, h_2^{(0,n)} >= h_9^{(n,0)}$,

$< h_9^{(2n-i-1,0)}, h_2^{(0,i)} >= h_9^{(2n-i,0)}$ for $3 \leq i \leq n-1$,

$< h_9^{(2,0)}, h_3^{(0)} >= h_9^{(2,1)}$,

$< h_9^{(i-1,1)}, h_2^{(0,i)} >= r_0 r_{2,i-1} h_2^{(1,i)} r_0 - r_2 r_0 r_{2,i-1} h_{(1,i)} + h_9^{(i,1)}$ for $3 \leq i < n$,

$< h_9^{(2n-i,0)}, h_3^{(0)} >= h_9^{(2n-i,1)}$ for $3 \leq i \leq n$,

$< h_3^{(0)}, h_5^{(1,2)} >= h_{10}^{(2)}$,

$$< h_{10}^{(i-1)}, h_5^{(i-1,i)} > = \sum_{k=1}^{i-2} r_0 r_{2,i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,i} - \sum_{k=1}^{i-2} r_1 r_0 r_{2,i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,i-1} + h_{10}^{(i)}$$

for $3 \le i \le n-1$,

$$< h_{10}^{(n-2)}, h_4 > = \sum_{k=1}^{n-3} r_0 r_{2,n-2} r_{1,k-1} h_2^{(k,n)} r_{k+1,n} - \sum_{k=1}^{n-3} r_1 r_0 r_{2,n-2} r_{1,k-1} h_2^{(k,n)} r_{k+1,n-2} + h_{11},$$

$$< h_{10}^{(n-1)}, h_3^{(n-1)} >= h_{12};$$

$$< h_{12}, h_8 > = \sum_{k=1}^{n-3} r_0 r_{2,n-1} r_{1,k-1} h_2^{(k,n)} r_{k+1,n+2} + r_0 r_{2,n-2} h_3^{(n-1)} r_{1,n+2} - r_1 r_0 r_{2,n-1} r_{1,n-3} h_4 r_{n-1} - \sum_{k=1}^{n-3} r_1 r_0 r_{2,n-1} r_{1,k-1} h_2^{(k,n)} r_{k+1,n+1} - r_1 r_0 r_{2,n-2} h_3^{(n-1)} r_{1,n+1} + h_{13}^{(n-1)},$$

$$< h_{13}^{(i+1)}, h_6^{(i-1)} > = \sum_{k=1}^{i-2} r_0 r_{2,2n-i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,2n-i+1} - r_1 r_0 r_{2,2n-i-1} r_{1,i-2} h_5^{(i-1,2n-i-1)} r_i - \sum_{k=1}^{i-2} r_1 r_0 r_{2,2n-i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,2n-i} + h_{13}^{(i)}$$

for $2 \le i \le n-2$,

$< h_9^{(i,1)}, h_9^{(2,0)} >= h_{14}^{(i,2)}$ for $3 \le i \le 2n-3$,

$$< h_{14}^{(i,j-1)}, h_5^{(j-1,j)} > = \sum_{k=2}^{j-2} r_0 r_{2i} r_{1,j-1} r_0 r_{2,k-1} h_2^{(k,j)} r_{k+1,j} - \sum_{k=2}^{j-2} r_2 r_0 r_{2i} r_{1,j-1} r_0 r_{2,k-1} h_2^{(k,j)} r_{k+1,j-1} + h_{14}^{(i,j)}$$

for $3 \le i \le 2n-3$,

$$
\begin{aligned}
< h_{14}^{(n-1,n-2)}, h_4 > = & \sum_{k=1}^{n-3} r_0 r_{2,n-1} r_{1,n-2} r_0 r_{2,k-1} h_2^{(k,n)} r_{k+1,n} + r_0 r_{2,n-1} r_{1,n-2} h_2^{(0,n)} r_{2n} \\
& - \sum_{k=1}^{n-3} r_2 r_0 r_{2,n-1} r_{1,n-2} r_0 r_{2,k-1} h_2^{(k,n)} r_{k+1,n-2} \\
& - r_2 r_0 r_{2,n-1} r_{1,n-2} h_2^{(0,n)} r_{2,n-2} + h_{15},
\end{aligned}
$$

$$
\begin{aligned}
< h_{14}^{(2n-3,i)}, h_{10}^{(i)} > = & r_0 r_{2,2n-3} h_5^{(1,i)} r_0 r_{2i} r_{1,i-1} - r_2 r_0 r_{2,2n-3} r_{1i} h_{10}^{(i-1)} r_i \\
& - r_2 r_0 r_{2,2n-3} h_5^{(1,i)} r_0 r_{2,i-1} r_{1,i-2} r_i \\
& - \sum_{k=1}^{i-2} r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,i-1} + h_{16}^{(i,i)}
\end{aligned}
$$

for $2 \leq i < n$,

$$
\begin{aligned}
< h_{16}^{(i,j-1)}, h_2^{(i-1,j)} > = & \sum_{k=1}^{i-2} r_0 r_{2,2n-2} r_{1i} r_0 r_{2,j-1} r_{1,k-1} h_2^{(k,j)} r_{k+1,i-1} \\
& - \sum_{k=1}^{i-2} r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,j-1} r_{1,k-1} h_2^{(k,j)} r_{k+1,i-2} \\
& + h_{16}^{(i,j)}
\end{aligned}
$$

for $2 \leq i < j \leq n-1$,

$< h_9^{(2n-3,0)}, h_9^{(2,0)} >= h_{17}$,

$< h_{17}, h_{10}^{(2)} >= h_{18}^{(1,2)}$,

$< h_{18}^{(1,i-1)}, h_2^{(1,i)} >= h_{18}^{(1,i)}$ for $2 < i < n$,

$< h_{18}^{(1,n-2)}, h_2^{(1,n)} >= h_{18}^{(1,n)}$,

$< h_{18}^{(1,2n-j-1)}, h_2^{(1,j)} >= h_{18}^{(1,2n-j)}$ for $3 \leq j < n$,

$< h_{16}^{(2,n-2)}, h_2^{(1,n)} >= h_{18}^{(2,n)}$,

$< h_{18}^{(2,2n-j-1)}, h_2^{(1,j)} >= h_{18}^{(2,j)}$,

$< h_{14}^{(2n-i,n-1)}, h_3^{(n-1)} >= h_{19}^{(i)}$ for $3 \leq i \leq n$,

$$
\begin{aligned}
< h_{14}^{(2n-i,n-2)}, h_4 > = &\sum_{k=2}^{n-3} r_0 r_{2,2n-i} r_{1,n-2} r_0 r_{2,k-1} h_2^{(k,n)} r_{k+1,n-2} r_{k+1,n} \\
&+ r_0 r_{2,2n-i} r_{1,n-2} h_2^{(0,n)} r_{2n} \\
&- \sum_{k=2}^{n-3} r_2 r_0 r_{2,2n-i} r_{1,n-2} r_0 r_{2,k-1} h_2^{(k,n)} r_{k+1,n-2} \\
&- r_2 r_0 r_{2,2n-i} r_{1,n-2} h_2^{(0,n)} r_{2,n-2} + h_{20}^{(i)}
\end{aligned}
$$

for $3 \leq i < n$,

$< h_{18}^{(i,2n-3)}, h_5^{(1,2)} >= h_{21}^{(i)}$ for $1 \leq i \leq 2$,

$$
\begin{aligned}
< h_{20}^{(3)}, h_{11} > = & r_0 r_{2,2n-3} h_5^{(1,n)} r_0 r_{2n} r_{1,n-2} - \sum_{k=1}^{n-3} r_2 r_0 r_{2,2n-3} r_{1n} r_0 r_{2,2n-2} r_{1,k-1} h_2^{(k,n)} r_{k+1.n-3} \\
&+ h_{22},
\end{aligned}
$$

$$
\begin{aligned}
< h_{16}^{(n-1,n-1)}, h_4 > = &\sum_{k=1}^{n-3} r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,n-1} r_{1,k-1} h_2^{(k,n)} r_{k+1,n} \\
&+ r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,n-2} h_3^{(n-1)} r_{1n} \\
&- \sum_{k=1}^{n-3} r_2 r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,n-1} r_{1,k-1} h_2^{(k,n)} r_{k+1,n-2} \\
&- r_2 r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,n-2} h_3^{(n-1)} r_{1,n-2} + h_{23}^{(n-1)},
\end{aligned}
$$

$$
\begin{aligned}
< h_{23}^{(i+1)}, h_5^{(i-1,n)} > = &\sum_{k=1}^{i-2} r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,2n-i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,n} \\
&- r_2 r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,2n-i-1} r_{1,i-2} h_5^{(i-1,n-2)} \\
&- \sum_{k=1}^{i-2} r_2 r_0 r_{2,2n-2} r_{1,n-1} r_0 r_{2,2n-i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,n-2} \\
&+ h_{23}^{(i)}, \quad 2 \leq i < n-1
\end{aligned}
$$

for $2 \leq i < n-1$,

$$
\begin{aligned}
< h_{22}, h_5^{(n-2,n-1)} > = &\sum_{k=1}^{n-3} r_0 r_{2,2n-2} r_{1n} r_{2n} r_{1,k-1} h_2^{(k,n-1)} r_{k+1,n-1} \\
&- \sum_{k=1}^{n-3} r_2 r_0 r_{2,2n-2} r_{1n} r_{2n} r_{1,k-1} h_2^{(k,n-1)} r_{k+1,n-2} \\
&+ h_{24}^{(n-1)},
\end{aligned}
$$

$$
\begin{aligned}
< h_{24}^{(i+1)}, h_5^{(i-1,n-1)} > = &\sum_{k=1}^{i-2} r_0 r_{2,2n-2} r_{1n} r_0 r_{2,2n-i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,n-1} \\
&- r_2 r_0 r_{2,2n-2} r_{1n} r_0 r_{2,2n-i-1} r_{1,i-2} h_5^{(i-1,n-2)} \\
&- \sum_{k=1}^{i-2} r_2 r_0 r_{2,2n-2} r_{1n} r_0 r_{2,2n-i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,n-2} \\
&+ h_{24}^{(i)}
\end{aligned}
$$

for $2 \leq i < n-1$,

$< h_{24}^{(j)}, h_3^{(n-1)} >= h_{25}^{(n,j)}$ for $2 \leq j < n$,

$$
\begin{aligned}
< h_{23}^{(j)}, h_7 > = &\sum_{k=1}^{n-3} r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,2n-j} r_{1,k-1} h_2^{(k,n-1)} r_{k+1,n+2} \\
&+ \sum_{k=j}^{n-3} r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,2n-k-1} h_2^{(k,n-1)} r_{2n-k+1,2n-j} r_{1,n+2} \\
&+ r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,2n-3} h_8 r_{n+3,2n-j} r_{1,n+2} \\
&+ \sum_{k=2}^{n-3} r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,k-1} h_2^{(k,n)} r_{k+1,2n-j} r_{1,n+2} \\
&+ r_0 r_{2,2n-j} r_{1,n-1} h_2^{(0,n)} r_{2,2n-j} r_{1,n+2} \\
&+ r_0 r_{2,2n-j} r_{1,n-2} h_3^{(n-1)} r_0 r_{2,2n-j} r_{1,n+2} \\
&- r_2 r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,2n-j} r_{1,n-3} h_5^{(n-2)} r_n \\
&- r_2 r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,2n-j} r_{1,n-3} r_{n-1} r_{n-2} h_3^{(n-1)} \\
&- \sum_{k=1}^{n-3} r_2 r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,2n-j} r_{1,k-1} h_2^{(k,n-1)} r_{k+1,n+1} \\
&- \sum_{k=j}^{n-3} r_2 r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,2n-k-1} h_2^{(k,n-1)} r_{2n-k+1,2n-j} r_{1,n+1} \\
&- r_2 r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,n-3} h_8 r_{n+3,2n-j} r_{1,n+1} \\
&- \sum_{k=2}^{n-3} r_2 r_0 r_{2,2n-j} r_{1,n-1} r_0 r_{2,k-1} h_2^{(k,n)} r_{k+1,2n-j} r_{1,n+1} \\
&- r_2 r_0 r_{2,2n-j} r_{1,n-1} h_2^{(0,n)} r_{2,2n-j} r_{1,n+1} \\
&- r_2 r_0 r_{2,2n-j} r_{1,n-2} h_3^{(n-1)} r_0 r_{2,2n-j} r_{1,n+1} + h_{25}^{(n-1,j)}
\end{aligned}
$$

for $2 \leq j < n-1$,

$$
\begin{aligned}
< h_{25}^{(i+1,j)}, h_6^{(i-1)} > &= \sum_{k=1}^{i-2} r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,2n-j} r_{1,k-1} h_2^{(k,i)} r_{k+1,2n-i+1} \\
&+ \sum_{k=j}^{i-2} r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,2n-k-1} h_2^{(k,i)} r_{2n-k+1,2n-j} r_{1,2n-i+1} \\
&+ r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,i-2} h_6^{(i-1)} r_{2n-i+2,2n-j} r_{1,2n-i+1} \\
&+ \sum_{k=2}^{i-2} r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,k-1} h_2^{(k,i)} r_{k+1,2n-j} r_{1,2n-i+1} \\
&+ r_0 r_{2,2n-2} r_{1,2n-i-1} h_2^{(0,i)} r_{2,2n-j} r_{1,2n-i+1} \\
&- r_2 r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,2n-j} r_{1,i-2} h_5^{(i-1,2n-i-1)} r_i \\
&- \sum_{k=1}^{i-2} r_2 r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,2n-j} r_{1,k-1} h_2^{(k,i)} r_{k+1,2n-i} \\
&- \sum_{k=j}^{i-2} r_2 r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,2n-k-1} h_2^{(k,i)} r_{2n-k+1,2n-j} r_{1,2n-i} \\
&- r_2 r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,i-2} h_6^{(i-1)} r_{2n-i+2,2n-j} r_{1,2n-i} \\
&- \sum_{k=2}^{i-2} r_2 r_0 r_{2,2n-2} r_{1,2n-i-1} r_0 r_{2,k-1} h_2^{(k,i)} r_{k+1,2n-j} r_{1,2n-i} \\
&- r_2 r_0 r_{2,2n-2} r_{1,2n-i-1} h_2^{(0,i)} r_{2,2n-j} r_{1,2n-i} + h_{25}^{(i,j)}
\end{aligned}
$$

for $3 \leq i \leq n-2$ and $2 \leq j < i$,

$$
\begin{aligned}
< h_{19}^{(i)}, h_8 > &= \sum_{k=2}^{n-3} r_0 r_{2,2n-i} r_{1,n-1} r_0 r_{2,k-1} h_2^{(k,n)} r_{k+1,n+2} \\
&+ r_0 r_{2,2n-i} r_{1,n-1} h_2^{(0,n)} r_{2,n+2} \\
&- r_2 r_0 r_{2,2n-i} r_{1,n-1} r_0 r_{2,n-3} h_4 r_{n-1} \\
&- \sum_{k=2}^{n-3} r_2 r_0 r_{2,2n-i} r_{1,n-1} r_0 r_{2,k-1} h_2^{(k,n)} r_{k+1,n+1} \\
&- r_2 r_0 r_{2,2n-i} r_{1,n-1} h_2^{(0,n)} r_{2,n+1} + h_{26}^{(i,n-1)}
\end{aligned}
$$

for $3 \leq i < n$,

$$< h_{26}^{(i,j+1)}, h_6^{(j-1)} > = \sum_{k=2}^{j-2} r_0 r_{2,2n-i} r_{1,2n-j-1} r_0 r_{2,k-1} h_2^{(k,j)} r_{k+1,2n-j+1}$$
$$+ r_0 r_{2,2n-i} r_{1,2n-j-1} h_2^{(0,j)} r_{2,2n-j+1}$$
$$- r_2 r_0 r_{2,2n-i} r_{1,2n-j-1} r_0 r_{2,j-2} h_5^{(j-1,2n-j-1)} r_j$$
$$- \sum_{k=2}^{j-2} r_2 r_0 r_{2,2n-i} r_{1,2n-j-1} r_0 r_{2,k-1} h_2^{(k,j)} r_{k+1,2n-j}$$
$$- r_2 r_0 r_{2,2n-i} r_{1,2n-j-1} h_2^{(0,j)} r_{2,2n-j} + h_{26}^{(i,j)}$$

for $3 \le i < n,\ \ 3 \le j < n$ and $i \le j$

$$< h_{18}^{(2,2n-j)}, h_5^{(1,2)} > = r_0 r_{2,2n-2} r_{12} r_0 h_5^{(2,2n-j)} r_{12} + r_0 r_{2,2n-2} r_{12} h_2^{(0,3)} r_{2,2n-j} r_{12}$$
$$- r_2 r_0 r_{2,2n-2} r_{12} r_0 h_5^{(2,2n-j)} r_1 - r_0 r_{2,2n-2} r_{12} h_2^{(0,3)} r_{2,2n-j} r_1$$
$$+ h_{27}^{(3,j,2)}$$

for $3 < j \le n$,

$$< h_{27}^{(i-1,j,i-2)}, h_5^{(i-2,i-1)} > = \sum_{k=1}^{i-3} r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,2n-j} r_{1,k-1} h_2^{(k,i-1)} r_{k+1,i-1}$$
$$+ r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,i-2} h_5^{(i-1,2n-j)} r_{1,i-1}$$
$$+ \sum_{k=2}^{i-2} r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,k-1} h_2^{(k,i)} r_{k+1,2n-j} r_{1,i-1}$$
$$+ r_0 r_{2,2n-2} r_{1,i-1} h_2^{(0,i)} r_{2,2n-j} r_{1,i-1}$$
$$- \sum_{k=1}^{i-3} r_2 r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,2n-j} r_{1,k-1} h_2^{(k,i-1)} r_{k+1,i-2}$$
$$- r_2 r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,i-2} h_5^{(i-1,2n-j)} r_{1,i-2}$$
$$- \sum_{k=2}^{i-2} r_2 r_0 r_{2,2n-2} r_{1,i-1} r_0 r_{2,k-1} h_2^{(k,i)} r_{k+1,2n-j} r_{1,i-2}$$
$$- r_2 r_0 r_{2,2n-2} r_{1,i-1} h_2^{(0,i)} r_{2,2n-j} r_{1,i-2} + h_{27}^{(i,j,i-1)}$$

for $3 < i \le n-2,\ i < j \le n$ and $k = i-1$,

$$< h_{27}^{(i,i+1,i-1)}, h_5^{(i-1,i)} > = \sum_{k=1}^{i-2} r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,i}$$
$$- \sum_{k=1}^{i-2} r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-i-1} r_{1,k-1} h_2^{(k,i)} r_{k+1,i-1}$$
$$+ h_{27}^{(i,i,i)}$$

for $3 \leq i \leq n-2$,

$$< h_{27}^{(i,j+1,i)}, h_5^{(j-1,i)} > = \sum_{k=1}^{j-2} r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-j-1} r_{1,k-1} h_2^{(k,j)} r_{k+1,i}$$
$$- r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-j-1} r_{1,j-2} h_5^{(j-1,i-1)}$$
$$- \sum_{k=1}^{j-2} r_2 r_0 r_{2,2n-2} r_{1i} r_0 r_{2,2n-j-1} r_{1,k-1} h_2^{(k,j)} r_{k+1,i-1}$$
$$h_{27}^{(i,j,i)}$$

for $3 \leq i \leq n-2$ and $2 \leq j < i$ □

At this point we are not able to show that polynomials given above lemma form Gröbner-Shirshov basis for the infinite Coxeter group of type $\widetilde{D}_n$.

8.2 Normal Forms for The Infinite Coxeter Group of Type $\widetilde{D}_n$

For $v \in \widetilde{S_n}^C$ define

$$v[i,j] = |\{k \in \mathbb{Z} : k \leq i, v(k) \geq j\}|$$

for all $i, j \in \mathbb{Z}$.

$\widetilde{S}_n^B$ is a subgroup of $\widetilde{S}_n^C$ of index 2.

Let $\widetilde{S}_n^D$ be the subgroup of $\widetilde{S}_n^B$ consisting of all the elements of $\widetilde{S}_n^B$ that have, in their complete notation, an even number of negative entries to the right of 0.

$$\widetilde{S}_n^D = \{u \in \widetilde{S}_n^B \ : \ u[0,1] \equiv 0 \ (mod 2)\}$$

Thus, $\widetilde{S}_n^D$ is a subgroup of $\widetilde{S}_n^B$ of index 2.

Proposition 8.2.1. *(Bjorner and Brenti (2005), Proposition 8.6.3)*

The group $\widetilde{S}_n^D$ with generating set $\{r_0^D, r_1^D, \ldots, r_n^D\}$ is the infinite Coxeter group of type $\widetilde{D}_n$ where $r_i^D = r_i^B$ for $i = 0, 1, 2, \ldots, n-1$ and $r_n^D = [(n-1 \ \ -n)]$.

We now try to find normal form representations of elements of $\widetilde{D}_n$ with respect to these generators. First of all, we give some relations between words in $\widetilde{D}_n$ and words in $\widetilde{B}_n$.

Lemma 8.2.2. *(i)* $r_n^B r_{n-1}^B = r_n^D r_n^B$,

(ii) $r_n^B r_{n-1}^B r_n^B = r_n^D$,

(iii) $r_{n-1}^B r_n^B r_{n-1}^B = r_n^D r_{n-1}^D r_n^B$,

(iv) $r_n^B r_{n-1}^B r_n^B r_{n-1}^B = r_n^D r_{n-1}^D$.

Proof. (i)

$$\begin{aligned} r_n^B r_{n-1}^B &= [(n \ \ {-n})][(n-1 \ n)] \\ &= [(n-1 \ \ {-n})][(n \ \ {-n})] \\ &= r_n^D r_n^B \end{aligned}$$

(ii)

$$\begin{aligned} r_n^B r_{n-1}^B r_n^B &= r_n^D r_n^B r_n^B \quad \text{by part (i)} \\ &= r_n^D \quad \text{since } r_n^B r_n^B = 1 \end{aligned}$$

(iii)

$$\begin{aligned} r_{n-1}^B r_n^B r_{n-1}^B &= r_{n-1}^B r_n^D r_n^B \quad \text{by part (i)} \\ &= r_{n-1}^D r_n^D r_n^B \quad \text{since } r_{n-1}^B = r_{n-1}^D \\ &= r_n^D r_{n-1}^D r_n^B \quad \text{by ELW in } h_3^{(n-1)} \end{aligned}$$

(iv)

$$\begin{aligned} r_n^B r_{n-1}^B r_n^B r_{n-1}^B &= r_n^D r_{n-1}^B \quad \text{by part (ii)} \\ &= r_n^D r_{n-1}^D \quad \text{since } r_{n-1}^B = r_{n-1}^D \end{aligned}$$

□

Lemma 8.2.3. *For* $1 \leq i \leq n-2$

$$r_{ij_i}^B = \begin{cases} r_{ij_i}^D, & j_i < n; \\ r_{i,n-1}^D r_n^B, & j_i = n; \\ r_{ij_i}^D r_n^B, & j_i > n. \end{cases}$$

Proof. Since $r_i^B = r_i^D$ for $1 \le i \le n-1$, $r_{ij_i}^B = r_{ij_i}^D$ for $j_i < n$. Similarly $r_{in}^B = r_{i,n-1}^D r_n^B$. Suppose that $j_i > n$ and $i \le n-2$. Then

$$\begin{aligned}
r_{ij_i}^D r_n^B &= r_{i,n-2}^B r_n^B r_{n-1}^B r_n^B r_{n-1}^B \cdots r_{2n-j_i}^B r_n^B \text{ by part (ii) of Lemma 8.2.2,} \\
&= r_{i,n-2}^B r_n^B r_{n-1}^B r_n^B r_{n-1}^B r_n^B \cdots r_{2n-j_i}^B \text{ by ELW's in } g_2^{(k,n)}\ k = 2n-j_i, \ldots, n-2, \\
&= r_{i,n-2}^B r_n^B r_n^B r_{n-1}^B r_n^B r_{n-1}^B \cdots r_{2n-j_i}^B \text{ by ELW's in } g_4^{(n-1)}, \\
&= r_{ij_i}^B \text{ since } r_n^B r_n^B = 1.
\end{aligned}$$

□

Lemma 8.2.4. *For* $1 \le i \le n-2$

$$r_n^B r_{ij_i}^B = \begin{cases} r_{ij_i}^D r_n^B, & j_i \le n-2; \\ r_{in}^D r_n^B, & j_i = n-1; \\ r_{in}^D, & j_i = n; \\ r_{ij_i}^D, & j_i > n. \end{cases}$$

Proof. (i)

$$\begin{aligned}
r_n^B r_{ij_i}^B &= [(n\ \ -n)][(i\ i+1\ \cdots\ j_i+1)] \\
&= [(i\ i+1\ \cdots\ j_i+1)][(n\ \ -n)] \\
&= r_{ij_i}^B r_n^B \text{ since } j_i + 1 < n \\
&= r_{ij_i}^D r_n^B \text{ since } r_{ij_i}^B = r_{ij_i}^D
\end{aligned}$$

(ii)

$$\begin{aligned}
r_{in}^D r_n^B &= r_{i,n-2}^B r_n^B r_{n-1}^B r_n^B r_n^B \\
&= r_{i,n-2}^B r_n^B r_{n-1}^B \text{ by ELW's in } g_1^{(n)} \\
&= r_n^B r_{i,n-1}^B \text{ by ELW's in } g_2^{(k,n)} \text{ for } k = n-2, \ldots, i
\end{aligned}$$

(iii)

$$\begin{aligned}
r_n^B r_{in}^B &= r_n^B r_{i,n-1}^B r_n^B \\
&= r_{i,n}^D r_n^B r_n^B \text{ by part (ii)} \\
&= r_{i,n}^D \text{ since } r_n^B r_n^B = 1.
\end{aligned}$$

(iv)

$$\begin{aligned}
r_n^B r_{ij_i}^B &= r_n^B r_{i,n}^B r_{n-1}^B \cdots r_{2n-j_i}^B \\
&= r_{in}^D r_{n-1}^B \cdots r_{2n-j_i}^B \text{ by part (iii)} \\
&= r_{ij_i}^D \text{ since } r_k^B = r_k^D \text{ for} k \ne n.
\end{aligned}$$

□

Definition 8.2.5. Let $w = r^B_{nj_n} r^B_{n-1,j_{n-1}} \cdots r^B_{ij_i} \cdots r^B_{1j_1}$ where $i-1 \leq j_i \leq 2n-i$ for $1 \leq i \leq n$. Define $n(w)$ to be number of appearance of r_n in w.

The following corollary is a consequence of the equalities $r^B_n r^B_0 = r^B_0 r^B_n$, $r^B_0 = r^D_0$ and the above lemmas.

Corollary 8.2.6. *Let* $1 \leq j \lessdot i \leq 2n-2$.

$$r^B_0 r^B_{2i} r^B_{1j} = \begin{cases} r^D_0 r^D_{2i} r^D_{1j}, & i \leq n-1 \textit{ or } j > n \\ r^D_0 r^D_{2,n-1} r^D_{1j} r^B_n, & i = n \textit{ and } j < n-1 \\ r^D_0 r^D_{2,n-1} r^D_{1n} r^B_n, & i = n \textit{ and } j = n-1 \\ r^D_0 r^D_{2,n-1} r^D_{1n}, & i = n \textit{ and } j = n \\ r^D_0 r^D_{2i} r^D_{1j} r^B_n, & i > n \textit{ and } j < n-1 \\ r^D_0 r^D_{2i} r^D_{1n} r^B_n, & i > n \textit{ and } j = n-1 \\ r^D_0 r^D_{2i} r^D_{1n}, & i > n \textit{ and } j = n \end{cases}$$

Corollary 8.2.7. *Let* $1 \leq j \lessdot i \leq 2n-2$.

$$r^B_n r^B_0 r^B_{2i} r^B_{1j} = \begin{cases} r^D_0 r^D_{2i} r^D_{1j} r^B_n, & i \leq n-1 \textit{ or } j > n \\ r^D_0 r^D_{2n} r^D_{1j} r^B_n, & i = n-1 \\ r^D_0 r^D_{2i} r^D_{1j}, & i \geq n \textit{ and } j < n \\ r^D_0 r^D_{2i} r^D_{1,n-1} r^B_n, & i \geq n \textit{ and } j = n \end{cases}$$

Definition 8.2.8.

$$i \lessdot j = \begin{cases} i \leq j, & \text{if } i \geq n+1; \\ j = n-1 \text{ or } j \geq n+1, & \text{if } i = n; \\ i < j, & \text{if } i \leq n-1. \end{cases}$$

Notice that n and $n-1$ are not comparable by themselves but $n \lessdot n-1$ and $n-1 \lessdot n$.

Definition 8.2.9.

$$i \lesssim j = \begin{cases} i \leq j, & \text{if } i \geq n; \\ j = n-1 \text{ or } j \geq n+1, & \text{if } i = n-1; \\ i < j, & \text{if } i < n-1. \end{cases}$$

Notice that n and $n-1$ are not comparable by each other.

Definition 8.2.10. We define the following words in $\widetilde{D}_n$,

(i) $w_0 = r^D_{nl_n} \cdots r^D_{il_i} \cdots r^D_{1l_1}$ where $i-1 \leq l_i \leq 2n-i$ for $i = 1, \ldots, n$ except $n-2 \leq l_{n-1} \leq n-1$.

(ii) $w_1 = \prod_{k=1}^{t}(r^D_0 r^D_{2,2n-2} r^D_{1,i_k})$ for $t \geq 0$, $1 \leq i_k \lesssim i_{k-1} \leq 2n-2$.

(iii) $w_2 = \prod_{k=1}^{s}(r^D_0 r^D_{2,j_{2k-1}} r^D_{1,j_{2k}})$ for $s \geq 0$, $1 \leq j_k \lessdot j_{k-1} \leq 2n-3$.

(iv) $$w_3 = \begin{cases} (r^D_0 r^D_{2,2n-2} r^D_{1,j_{2k}})^{2m}, \\ (r^D_0 r^D_{2,2n-2} r^D_{1,j_{2k}})^{2m-1}(r^D_0 r^D_{2j}), \\ (r^D_0 r^D_{2,2n-2} r^D_{1,j_{2k}})^{2(m-1)}(r^D_0 r^D_{2,2n-2}) r^D_0, \end{cases} \quad \text{for } m \geq 0 \text{ and } 1 \leq j \leq 2n-2.$$

(v) $w_4 = w_0 w_1 w_2$ where $i_t \geq 2$ and either $j_1 \lessdot i_t$ or $j_1 \not\lessdot i_t$ but $\begin{cases} j_2 \lesssim i_t, & i_t + j_1 \geq 2n; \\ j_2 + 1 < i_t, & i_t + j_1 < 2n. \end{cases}$

(vi) $w_5 = w_0 w_1 w_3$

Let $W_D = \{w_4, w_5\}$.

Theorem 8.2.11. *Any word $w \in W_B$ where $n(w)$ is even can be transformed a word in W_D.*

Proof. Let $w_0 = r^B_{nj_n} r^B_{n-1,j_{n-1}} \cdots r^B_{ij_i} \cdots r^B_{1j_1}$ where $i-1 \leq j_i \leq 2n-i$ for $1 \leq i \leq n$. Let $t_i = n(r^B_{nj_n} \cdots r^B_{i+1,j_{i+1}})$. Then

$$w_0 = \begin{cases} (r^D_{n,l_n} \cdots r^D_{il_i} \cdots r^D_{1,l_1}), & n(w) \text{ is even;} \\ (r^D_{n,l_n} \cdots r^D_{il_i} \cdots r^D_{1,l_1}) r^B_n, & n(w) \text{ is odd.} \end{cases}$$

where

$$l_n = \begin{cases} n, & j_n = n \text{ or } j_{n-1} = n+1; \\ n-1, & \text{otherwise.} \end{cases},$$

$$l_{n-1} = \begin{cases} n-1, & j_{n-1} = n-1 \text{ or } j_{n-1} = n; \text{ and } j_n = n-1; \\ n-1, & j_{n-1} = n+1; \\ n-2, & \text{otherwise.} \end{cases}$$

and

$$l_i = \begin{cases} j_i, & j_i \neq n-1, n; \\ n-1, & j_i = n-1 \text{ or } j_i = n; \text{ and } t_i \text{ is even;} \\ n, & j_i = n-1 \text{ or } j_i = n; \text{ and } t_i \text{ is odd.} \end{cases}$$

for $i = n-2, n-3, \ldots, 1$.

Here the values of l_n and l_{n-1} are easily follows from Lemma 8.2.2. The values of other l_i's can be obtained by recursively applying either Lemma 8.2.3 or Lemma 8.2.4 for $i = n-2, n-3, \ldots, 1$ and using the fact $r_n^B r_n^B = 1$.

Let $w_1 = \prod_{k=1}^{t}(r_0^B r_{2,2n-2}^B r_{i_k}^B)$ for $t \geq 0$ and $1 \leq i_k \lessdot i_{k-1} \leq 2n-2$ and let ζ be the number of i_k's which is less than or equal to $n-1$ in w_1. Then several applications of Corollary 8.2.6 and Corollary 8.2.7 imply that

$$w_1 = \begin{cases} \prod_{k=1}^{t}(r_0^D r_{2,2n-2}^D r_{i_k}^D), & \zeta \text{ is even;} \\ (\prod_{k=1}^{t}(r_0^D r_{2,2n-2}^D r_{i_k}^D))r_0^B, & \zeta \text{ is odd.} \end{cases}$$

where $j_k = i_k$ if $i_k \neq n-1$ and $j_k = n$ if $i_k = n-1$.

Now consider $\bar{w}_1 = r_n^B w_1$. Similarly

$$\bar{w}_1 = \begin{cases} \prod_{k=1}^{t}(r_0^D r_{2,2n-2}^D r_{i_k}^D), & \zeta \text{ is odd;} \\ (\prod_{k=1}^{t}(r_0^D r_{2,2n-2}^D r_{i_k}^D))r_0^B, & \zeta \text{ is even.} \end{cases}$$

where $j_k = i_k$ if $i_k \neq n$ and $j_k = n-1$ if $i_k = n$.

Hence both w_1 and $\bar{w}_1$ can be transformed one of the following

$$\begin{cases} \prod_{k=1}^{t}(r_0^D r_{2,2n-2}^D r_{i_k}^D), \\ (\prod_{k=1}^{t}(r_0^D r_{2,2n-2}^D r_{i_k}^D))r_0^B, \end{cases}$$

where for $t \geq 0$, $1 \leq i_k \lesssim i_{k-1} \leq 2n-2$.

□

Lemma 8.2.12.

$$\prod_{i=1}^{n-1}[(1+x+x^2+\ldots+x^i)(1+x^i)] = (1+x+x^2+\ldots+x^{n-1})\prod_{i=1}^{n-1}(1+x+x^2+\ldots+x^{2i-1})$$

Proof. We use case analysis. If n is odd, then we must show that

$$\prod_{k=1}^{\frac{n-3}{2}}(1+x+x^2+\cdots+x^{2k})\prod_{t=1}^{n-1}(1+x^t) = \prod_{m=0}^{\frac{n-3}{2}}(1+x+x^2+\cdots+x^{n+2m})$$

Let us prove that

$$\begin{aligned}\prod_{k=1}^{\frac{n-3}{2}}(1+x+x^2+\cdots+x^{2k})\prod_{t=1}^{n-1}(1+x^t) &= \prod_{k=1}^{\frac{n-3}{2}}(\frac{1-x^{2k+1}}{1-x})\prod_{t=1}^{n-1}(\frac{1-x^{2t}}{1-x^t}) \\ &= \frac{(1-x^{n+1})(1-x^{n+3})\cdots(1-x^{2n-2})}{(1-x)^{\frac{n-1}{2}}} \\ &= \frac{1-x^{n+1}}{1-x}\frac{1-x^{n+3}}{1-x}\cdots\frac{1-x^{2n-2}}{1-x} \\ &= \prod_{m=0}^{\frac{n-3}{2}}(1+x+x^2+\cdots+x^{n+2m}).\end{aligned}$$

If n is even, then we must show that

$$\prod_{k=1}^{\frac{n-2}{2}}(1+x+x^2+\cdots+x^{2k})\prod_{t=1}^{n-1}(1+x^t) = \prod_{m=0}^{\frac{n-1}{2}}(1+x+x^2+\cdots+x^{n+2m-2}).$$

Let us prove that

$$\begin{aligned}\prod_{k=1}^{\frac{n-2}{2}}(1+x+x^2+\cdots+x^{2k})\prod_{t=1}^{n-1}(1+x^t) &= \prod_{k=1}^{\frac{n-2}{2}}(\frac{1-x^{2k+1}}{1-x^k})\prod_{t=1}^{n-1}(\frac{1-x^{2t}}{1-x^t}) \\ &= \frac{(1-x^{n})(1-x^{n+2})\cdots(1-x^{2n-2})}{(1-x)^{\frac{n}{2}}} \\ &= \frac{1-x^{n}}{1-x}\frac{1-x^{n+2}}{1-x}\cdots\frac{1-x^{2n-2}}{1-x} \\ &= \prod_{m=0}^{\frac{n-1}{2}}(1+x+x^2+\cdots+x^{n+2m-2}).\end{aligned}$$

□

Lemma 8.2.13. *The generating function for word in W_D is*

$$\frac{1+x+x^2+\cdots+x^{n-1}}{1-x^{n-1}}\prod_{i=1}^{n-1}\frac{1+x^i}{1-x^{n-1+i}}.$$

Proof. We found one to one corresponding between words in W_D and the words in W_C with number of occurrences of r_0 and r_n are even. Let

$$w = (r^C_{nl_n}r^C_{n-1,l_{n-1}}\cdots r^C_{1,l_1})\prod_{k=1}^{t}(r^C_0 r^C_{1j_k})$$

where t is even, $n(w)$ is even and $0 \leq j_k \leqslant j_{k-1} \leq 2n-1$.

First consider the part of $r^C_{nl_n}r^C_{n-1,l_{n-1}}\cdots r^C_{1,l_1} = r^B_{nl_n}r^B_{n-1,l_{n-1}}\cdots r^B_{1,l_1}$. Theorem 8.2.11 implies that corresponding word in W_D is of the form

$$r^D_n r^D_{n-1} r^D_{n-2,j_{n-2}}\cdots r^D_{1j_1}$$

where $i-1 \leq j_i \leq 2n-i$. The generating function for these words in this form is

$(1+x)^2 \prod_{i=2}^{n-1}(1+x+x^2+\cdots+x^{i-1}+2x^i+x^{i+1}+\cdots+x^{2i})$

$= \prod_{i=1}^{n-1}(1+x+\cdots+x^i)(1+x^i)$

$= (1+x+x^2+\cdots+x^{n-1}) \prod_{i=1}^{n-1}(1+x+x^2+\cdots+x^{2i-1})$ by Lemma 8.2.12.

Consider the word $\overline{w} = \prod_{k=1}^{t}(r_0^C r_{1j_k}^C)$ where t is even and $n(\overline{w})$ is even. Notice that we assume $n(r_{nj_n}^B r_{n-1,j_{n-1}}^B \cdots r_{1,j_1}^B)$ is even; otherwise, we have to consider the word $r_n^B \overline{w}$. When converting the word $\prod_{k=1}^{t}(r_0^C r_{1j_k}^C)$ into a word in W_D, the corresponding word losses its length by number of occurrence of r_0 and r_n. The generating function for the words in the form $\prod_{k=1}^{t}(r_0^C r_{1j_k}^C)$ in W_C is

$$\prod_{i=1}^{n} \frac{1+x^i}{1-x^{n+i}}.$$

Hence generating function for the corresponding word in W_D is

$$\frac{(1+x)(1+x^2)\cdots(1+x^{n-1})}{(1-x^{n-1})(1-x^n)\cdots(1-x^{2n-2})} = \frac{1}{x^{n-1}} \prod_{i=1}^{n-1} \frac{1+x^i}{1-x^{n-1+i}}.$$

□

Notice that the generating function for the infinite Coxeter group of type $\widetilde{D}_n$ is

$$\frac{1+x+x^2+\cdots+x^{n-1}}{1-x^{n-1}} \prod_{i=1}^{n-1} \frac{1+x+\ldots+x^{2i-1}}{1-x^{2i-1}}.$$

By Lemma 6.2.12

$$\frac{1+x+x^2+\cdots+x^{n-1}}{1-x^{n-1}} \prod_{i=1}^{n-1} \frac{1+x+\ldots+x^{2i-1}}{1-x^{2i-1}} = (\prod_{i=1}^{n-1}(1+x+\ldots+x^{2i-1})(\frac{1+x^i}{1-x^{n-1+i}}))$$

which is equal to generating function of words in W_D.

Now we are ready to find the main result.

Theorem 8.2.14. *Let R^D be the set of all polynomials given in Lemma 8.1.2. Then*

(i) $W_D =$ *Red*(R^D).

(ii) R^D *is a Gröbner-Shirshov basis for the infinite Coxeter group of type* $\widetilde{D}_n$.

Proof. (i) It is easy to see that any word in W_D is R^D-reduced. Hence $W_D \subseteq \text{Red}(R^D)$. Conversely if $w \in \text{Red}(R^D)$, then w can be written as a permutation in $\widetilde{S}_n^D$ and this permutation has a corresponding word in W_D by Theorem 8.2.11. Hence $\text{Red}(R^D) \subseteq W_D$.

(ii) We know that any polynomial in R^D is a part of a Gröbner-Shirshov basis of the infinite Coxeter group of type $\widetilde{D}_n$. If R^D were not a Gröbner-Shirshov basis, then $\text{Red}(R^D) = W_B$ should be a proper subset of the set of normal forms of the infinite Coxeter group of type $\widetilde{D}_n$ by Composition-Diamond lemma. This contradicts to the fact that W_D and normal forms of the infinite Coxeter group of type $\widetilde{D}_n$ have same generating functions.

□

Let us finish this chapter with an example. We will take $u \in \widetilde{S}_n^D \subseteq \widetilde{S}_n^C$. First we find corresponding word in W_C. Then we convert it a word in W_B and find a word in W_D.

Example 8. Let $u = [44, 7, 31, -6] \in \widetilde{S}_4^D$. Since $\widetilde{S}_4^D \subseteq \widetilde{S}_4^C$, $u \in \widetilde{S}_4^C$. Now we will deal with in $\widetilde{S}_4^C$. Corollary 6.2.3 implies $uw^{-1} = [-22, 2, -35, -6]$ where $w = r_0r_{12}r_0r_1r_0$. Applying the process explained in the proof of the Corollary 6.2.5, we get

$[-22, 2, -35, -6](r_0r_{14})^{-1} = [3, -22, 2, -35]$,

$[3, -22, 2, -35](r_0r_{14})^{-1} = [-26, 3, -22, 2]$,

$[-26, 3, -22, 2](r_0r_{15})^{-1} = [-13, -26, 3, 2]$,

$[-13, -26, 3, 2](r_0r_{16})^{-1} = [-17, -13, 3, 2]$,

$[-17, -13, 3, 2](r_0r_{16})^{-1} = [-4, -17, 3, 2]$,

$[-4, -17, 3, 2](r_0r_{16})^{-1} = [-8, -4, 3, 2]$ and

$[-8, -4, 3, 2](r_0r_{17})^{-1} = [1, -4, 3, 2]$.

Thus $[-22, 2, -35, -6]((r_0r_{17})(r_0r_{16})^3(r_0r_{15})(r_0r_{14})^2)^{-1} = [1, -4, 3, 2]$.

Corollary 6.2.8 implies $[1, -4, 3, 2] = r_4r_3r_{23}$. Hence

$u = (r_4r_3r_{23})(r_0r_{17})(r_0r_{16})^3(r_0r_{15})(r_0r_{14})^2(r_0r_{12}r_0r_1r_0) \in \widetilde{S}_4^C$.

Applying the process of the Corollary 7.2.4,

$(r_0^Cr_{17}^C)(r_0^Cr_{16}^C)(r_0^Cr_{16}^C)(r_0^Cr_{16}^C)(r_0^Cr_{15}^C)(r_0^Cr_{14}^C)(r_0^Cr_{14}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C)$

$= (r_0^Br_{26}^Br_{16}^B)(r_0^Cr_{17}^C)(r_0^Cr_{16}^C)(r_0^Cr_{15}^C)(r_0^Cr_{14}^C)(r_0^Cr_{14}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C)$

$= (r_0^Br_{26}^Br_{16}^B)(r_0^Br_{26}^Br_{15}^B)(r_0^Cr_{17}^C)(r_0^Cr_{14}^C)(r_0^Cr_{14}^C)(r_0^Cr_{12}^C)(r_0^Cr_1^C)(r_0^C)$

$= (r_0^B r_{26}^B r_{16}^B)(r_0^B r_{26}^B r_{15}^B)(r_0^B r_{26}^B r_{14}^B)(r_0^C r_{15}^C)(r_0^C r_{12}^C)(r_0^C r_1^C)(r_0^C)$.

Applying the process of the Lemma 7.2.2,

$(r_0^B r_{26}^B r_{16}^B)(r_0^B r_{26}^B r_{15}^B)(r_0^B r_{26}^B r_{14}^B)(r_0^C r_{15}^C)(r_0^C r_{12}^C)(r_0^C r_1^C)(r_0^C)$

$= (r_0^B r_{26}^B r_{16}^B)(r_0^B r_{26}^B r_{15}^B)(r_0^B r_{26}^B r_{14}^B)(r_0^B r_{25}^B r_{12}^B)(r_0^B)$.

Since $r_i^C = r_i^B$ for $i \neq 0$, then $(r_4^C r_3^C r_{23}^C) = (r_4^B r_3^B r_{23}^B)$. Hence

$u = (r_4^B r_3^B r_{23}^B)(r_0^B r_{26}^B r_{16}^B)(r_0^B r_{26}^B r_{15}^B)(r_0^B r_{26}^B r_{14}^B)(r_0^B r_{25}^B r_{12}^B)(r_0^B) \in \widetilde{S}_4^B$.

Now we convert u from $\widetilde{B}_4$ to $\widetilde{D}_4$ using the techniques in this chapter.

$r_4^B r_3^B = r_4^D r_4^B$ by Lemma 8.2.2, then

$u = (r_4^D r_4^B r_{23}^B)(r_0^B r_{26}^B r_{16}^B)(r_0^B r_{26}^B r_{15}^B)(r_0^B r_{26}^B r_{14}^B)(r_0^B r_{25}^B r_{12}^B)(r_0^B)$,

$r_4^B r_{23}^B = r_{24}^D r_4^B$ by Lemma 8.2.4, then

$u = (r_4^D r_{24}^D r_4^B)(r_0^B r_{26}^B r_{16}^B)(r_0^B r_{26}^B r_{15}^B)(r_0^B r_{26}^B r_{14}^B)(r_0^B r_{25}^B r_{12}^B)(r_0^B)$,

$r_4^B r_0^B r_{26}^B r_{16}^B = r_0^D r_{26}^D r_{16}^D r_4^B$ by Corollary 8.2.7, then

$u = (r_4^D r_{24}^D)(r_0^D r_{26}^D r_{16}^D r_4^B)(r_0^B r_{26}^B r_{15}^B)(r_0^B r_{26}^B r_{14}^B)(r_0^B r_{25}^B r_{12}^B)(r_0^B)$,

$r_4^B r_0^B r_{26}^B r_{15}^B = r_0^D r_{26}^D r_{15}^D r_4^B$ by Corollary 8.2.7, then

$u = (r_4^D r_{24}^D)(r_0^D r_{26}^D r_{16}^D)(r_0^D r_{26}^D r_{15}^D r_4^B)(r_0^B r_{26}^B r_{14}^B)(r_0^B r_{25}^B r_{12}^B)(r_0^B)$,

$r_4^B r_0^B r_{26}^B r_{14}^B = r_0^D r_{26}^D r_{13}^D r_4^B$ by Corollary 8.2.7, then

$u = (r_4^D r_{24}^D)(r_0^D r_{26}^D r_{16}^D)(r_0^D r_{26}^D r_{15}^D)(r_0^D r_{26}^D r_{13}^D r_4^B)(r_0^B r_{25}^B r_{12}^B)(r_0^B)$,

$r_4^B r_0^B r_{25}^B r_{12}^B = r_0^D r_{25}^D r_{12}^D$ by Corollary 8.2.7, then

$u = (r_4^D r_{24}^D)(r_0^D r_{26}^D r_{16}^D)(r_0^D r_{26}^D r_{15}^D)(r_0^D r_{26}^D r_{13}^D)(r_0^D r_{25}^D r_{12}^D)(r_0^B)$,

Since $r_0^B = r_0^D$, then

$u = (r_4^D r_{24}^D)(r_0^D r_{26}^D r_{16}^D)(r_0^D r_{26}^D r_{15}^D)^2(r_0^D r_{26}^D r_{13}^D)(r_0^D r_{25}^D r_{12}^D) \in \widetilde{S}_4^D$.

9. CONCLUSION AND RECOMMENDATION

In this thesis we found Gröbner-Shirshov basis and normal forms for the infinite Coxeter groups of type $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$ with respect to degree lexicographic order and we order the generators as $r_0 > r_1 > \cdots > r_n$. We also made some experiments by changing the order of the generators. The resulting bases were more complex than our original basis. Experiments can be carried out using orders other than degree lexicographic order.

As a further study, Gröbner-Shirshov basis and normal form calculations can be made for the infinite Coxeter groups of type $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$. But even Gröbner-Shirshov bases of the finite Coxeter groups of type E_6 ,E_7 and E_8 are quite complicated. Calculations for infinite groups of these types require serious programming knowledge and patience.

Using the combinatorial meanings that we found some new combinatorial properties of the Coxeter groups can be obtained for normal forms. For example deciding two elements are comparable with respect to Bruhat order is open problem for the infinite Coxeter groups of type $\widetilde{B}_n$ and $\widetilde{D}_n$. We will try to use our normal forms and their combinatorial meanings to solve this problem.

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