BOLU ABANT IZZET BAYSAL UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON WEAKLY PRIME RADICAL

MASTER OF SCIENCE

ZENNURE TUBA LAÇIN

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DECLARATION

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Zennure Tuba LAÇİN

ABSTRACT

ON WEAKLY PRIME RADICAL
MSC THESIS
ZENNURE TUBA LAÇIN
BOLU ABANT IZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
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If T is an $S-$ module and Q is a submodule of T, which is proper, then Q is called prime if $xn \in Q$ implies $n \in Q$ or $xT \subseteq Q$ for some $x \in S$, $n \in T$. Also, if $xym \in Q$ implies $xm \in Q$ or $ym \in Q$ for some $x, y \in S$, $m \in T$, then Q is called weakly prime submodule. One can easily show that prime submodules are weakly prime.

We get some properties of weakly prime radical which are always true for prime radical. $(Q : T)$ is always a prime ideal when Q is a prime submodule. We have shown that if Q is a weakly prime submodule, $(Q : m)$ is a prime ideal for every $m \in T - Q$.

In this thesis, we give the definition of a weakly quasi-primary submodule which generalizes the concept of a weakly primary submodule. Also we show that every weakly quasi-primary submodule Q is weakly prime if and only if $\langle E_T(Q) \rangle = Q$. If S is a commutative ring with identity whose prime ideals are totally ordered, then it is shown that a weakly prime radical is a weakly prime submodule, and the weakly radical formula holds for S. Finally, we prove that divided domains satisfy weakly radical formula.

KEYWORDS: Weakly prime submodule, Weakly prime radical, Weakly radical formula, Divided domain, Pseudo-valuation domain.

ÖZET

ZAYIF ASAL RADIKAL ÜZERINE
YÜKSEK LISANS TEZI
ZENNURE TUBA LAÇIN
BOLU ABANT İZZET BAYSAL ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ
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T bir S modül ve Q da T nin bir alt modülü ise, eğer $x \in S$, $n \in T$ için $xn \in Q$ varken, $n \in Q$ ve ya $xT \subseteq Q$ ise Q ya asal alt modül denir. Diğer taraftan $x, y \in S$ ve $m \in T$ için $xym \in Q$ varken $xm \in Q$ ya da $ym \in Q$ elde ediliyor ise Q ya zayıf asal alt modül denir. Her asal alt modülün zayıf asal alt modül olduğu açıktır.

Asal radikal için her zaman doğru olan bazı özelliklerin zayıf asal radikal için de sağlandığını gösterdik. Q 'nun S $-$modül, T'nin asal bir alt modülü olması durumunda, $(Q : T)$ nin asal ideal olduğu iyi bilinmektedir. Q 'nun zayıf asal alt modül olması durumunda $(Q : m)$ nin her $m \in T - Q$ için asal ideal olduğunu gösterdik.

Bu tezde zayıf asal alt modül kavramını genelleştiren zayıf yarı-asal alt modüller kavramını tanıtıyoruz. Aynı zamanda, her zayıf yarı-asal alt modül Q 'nun zayıf asal alt modül olabilmesi için gerek ve yeter şartın $\langle E_T(Q) \rangle = Q$ olduğunu gösterdik. Eğer S, asal idealleri tam sıralı olan değişmeli ve birimli bir halka ise, zayıf asal radikalin zayıf asal alt modül olduğu ve aynı zamanda zayıf radikal formülün S için sağlandığı gösterildi. Son olarak, bölünmüş bölgelerin zayıf radikal formülünü sağladığını kanıtladık.

ANAHTAR KELİMELER: Zayıf asal alt modül, Zayıf asal radikal, Zayıf radikal formül, Bölünmüş bölge, Sözde değerleme bölgesi.

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LIST OF ABBREVIATIONS AND SYMBOLS

$\boldsymbol{rad_T(Q)}$	**:** Intersection of all prime submodules of T containing Q
$\boldsymbol{wrad_T(Q)}$	**:** Intersection of all weakly prime submodules of T containing Q
$\boldsymbol{E_T(Q)}$	**:** Envelope of Q in T
$\boldsymbol{UE_T(Q)}$	**:** Union of envelopes of Q in T
PVD	**:** Pseudo-Valuation Domain
ACC	**:** Ascending Chain Condition
$\boldsymbol{Spec(S)}$	**:** The set of all prime ideals of S

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1. INTRODUCTION

If S is a commutative ring and A is an ideal of S, then radical of the ideal A is defined as intersection of all prime ideals of S containing A. It is also defined by

$$\sqrt{A} = \{s \in S: \ s^t \in A \ for\ some\ positive\ integer\ t\}.$$

This characterization had been generalized to modules over a commutative ring (McCasland and Moore, 1991). This generalization corresponds to two different concepts in the modules. One of them is the radical of a submodule and the other is the envelope of a submodule. A proper submodule Q of T, is $q-$prime ($q-$primary) if $an \in Q$ for $a \in S$, $n \in T$ gives $n \in Q$ or $a \in q = (Q : T)$ ($n \in Q$ or $a \in q = \sqrt{Q : T}$) where $(Q : T)$ is the set of all elements of S which takes every element of T into Q. According to this definition, $rad_T(Q)$, prime radical of Q in T is just intersection of all prime submodules of T containing Q. If there is no prime submodule of T or no prime submodule of T contains Q, then $rad_T(Q) = T$. The envelope of Q in T, $E_T(Q)$, is the set of all elements $y \in T$ where there exists $a \in S$, $n \in T$ such that $a^k n \in Q$ and $y = an$ for some positive integer k. The submodule generated by envelope is denoted by $\langle E_T(Q) \rangle$. Q satisfies the radical formula (Q s.t.r.f.) if $rad_T(Q) = \langle E_T(Q) \rangle$. If every submodule Q of T s.t.r.f., then T satisfies the radical formula (T s.t.r.f.). Also if every $S-$ module satisfies the radical formula, then a ring S satisfies the radical formula. Since a ring S considered as an $S-$ module, every submodule of S satisfies the radical formula by the definition of the radical of an ideal.

The notion of weakly prime submodule was introduced in (Behboodi at all, 2011). If for $x, y \in S$ and $n \in T$, $xyn \in Q$ implies $xn \in Q$ or $yn \in Q$, then proper submodule Q of an $S-$ module T is weakly prime. A proper submodule Q of T is weakly primary if $xyn \in Q$ where $x, y \in S$ and $n \in T$, then $yn \in Q$ or $x^t n \in Q$ for some $t \geq 1$. Also, Q is weakly quasi-primary, if $xyn \in Q$ implies either $x^t n \in Q$ or $y^t n \in Q$ for some $t \in \mathbb{N}$ where $x, y \in S$ and $n \in T$ and Q is proper. The weakly prime radical of Q in T, $wrad_T(Q)$, is the intersection of all weakly prime submodules of T containing Q. If there is no weakly prime submodule containing Q, then $wrad_T(Q) = T$. So, $wrad_T(T) = T$. Since every prime submodule is weakly prime, $wrad_T(Q) \subseteq rad_T(Q)$. If for a submodule Q of the module T, $wrad_T(Q) = \langle E_T(Q) \rangle$, then it is said that Q satisfies the weakly radical formula. T satisfies the weakly radical formula, if for every submodule Q of T, Q satisfies the weakly radical formula. A ring S satisfies the weakly radical formula, if every $S-$ module satisfies the weakly radical formula.

Azizi (2007) gave the definition of the nth envelope of a submodule. For a submodule Q of T, $E_0(Q) = Q$, $E_1(Q) = E_T(Q)$, $E_2(Q) = E_T(\langle E_1(Q) \rangle)$, and for any positive integer k, $E_{k+1}(Q) = E_T(\langle E_k(Q) \rangle)$ inductively. Here, $E_k(Q)$ is called as the kth envelope of Q. Azizi defined the weakly radical formula of degree k (Azizi, 2009). If $\langle E_k(Q) \rangle = wrad_T(Q)$ for every submodule Q of T, then T satisfies the weakly radical formula of degree k (s.t.w.r.f. of degree k). If every $S-$ module satisfies weakly radical formula of degree k, then S satisfies weakly radical formula of degree k (S s.t.w.r.f. of degree k).

For any submodule Q of M, we consider $UE_T(Q) = \bigcup_{k \in \mathbb{N}} \langle E_k(Q) \rangle$. $UE_T(Q)$ is the union of envelopes of Q (Azizi, 2009). It is obvious

$$Q \subseteq \langle E_k(Q) \rangle \subseteq UE_T(Q) \subseteq wrad_T(Q) \subseteq rad_T(Q), \text{ for any } k \in \mathbb{N}.$$

The weakly radical formula holds for T if $wrad_T(Q) = UE_T(Q)$ for every submodule Q of T and if for every $S-$ module weakly radical formula holds, then it is said that the weakly radical formula holds for a ring S.

An integral domain S is called a valuation domain if all its ideals form a chain under inclusion and S is a divided domain if each prime ideal is comparable to every principal ideal of S. Badawi (1995) showed that prime ideals of S are linearly ordered iff for each $x, y \in S$, $x|y^t$ or $y|x^t$ for some $t \geq 1$ when S is a commutative ring with identity. Also he gave some equivalent conditions for a ring S with linearly ordered prime ideals. Also, Badawi showed that S is a divided domain iff for every $x, y \in S$, either $x|y$ or $y|x^n$ for some n ≥ 1 which implies that prime ideals of divided domains are totally ordered (Badawi, 1995). An integral domain S is pseudo-valuation domain (PVD), if every prime ideal of S is strongly prime. If S is a

PVD, for non-unit elements x, y of S, either $x|y$ or $y|x^2$ (Badawi, 1995). Thus every PVD is a divided domain.

This introduction forms Chapter 1.

The definitions of weakly prime submodule and weakly prime radical are given in Chapter 2. Some characterizations of weakly prime submodules and the properties of weakly prime radical are stated in the second section of Chapter 2. In the third section, we discussed under what conditions weakly prime radical of a module is distributive over intersection.

In Chapter 3, the envelope of a module is introduced and modules which satisfy the weakly radical formula are investigated. If S is a commutative ring with identity satisfying conditions of Theorem 1 of Badawi (1995) , then we showed that every proper submodule Q of an $S-$ module T is weakly quasi-primary and hence $wrad_T(Q)$ is weakly prime submodule for any submodule Q of an $S-$ module T. Also, we showed that divided domains satisfy the weakly radical formula.

Chapter 4 is the conclusion chapter which contains some suggestions for future study.

Throughout S is a commutative ring with identity and T is a unitary $S-$ module.

2. WEAKLY PRIME RADICAL OF SUBMODULES

2.1 Prime Submodules

Definition 2.1 *Let S be a ring and T be an $S-$ module. A submodule Q of T which is proper, is called **prime** if for some $a \in S$, $n \in T$, $an \in Q$ implies $n \in Q$ or $aT \subseteq Q$.*

Definition 2.2 *Let Q be any submodule of an $S-$ module T. $(Q : T)$ denotes the set of elements $x \in S$ such that $xT \subseteq Q$. This is usually called **residual by Q**. $(Q : T)$ is an ideal of S and $(Q : T)$ is the annihilator of the module T/Q.*

For any ideal I of S $(\mathrm{Q} :_{\mathrm{T}} \mathrm{I})$ denotes the set of elements $\mathrm{n} \in \mathrm{T}$ such that $\mathrm{In} \subseteq \mathrm{Q}$.

If Q is prime submodule of T with $\mathrm{p} = (Q : T)$, then Q is called $p-$ prime submodule.

Example 2.3 *Let K be a field, U be a vector space over the field K. Then every proper subspace V of U is a $\langle 0 \rangle -$prime submodule of U.*

Proof. Let V be a proper subspace of U and $f \in K$, $u \in U$ such that $fu \in V$. If $f = 0$, then $fU \subseteq V$. If $f \neq 0$, then f^{-1} exists and $u \in V$. Thus, V is a prime submodule. Since V is proper subspace of U, $(V : U) = \langle 0 \rangle$. □

Lemma 2.4 *(Jenkins, 1991, Lemma 1) Let S be a ring and T be an $S-$module. Then a submodule Q of T is prime iff $\mathcal{P} = (Q : T)$ is a prime ideal of S and T/Q is torsion-free $S/\mathcal{P}-$module.*

Lemma 2.5 *Let $\{N_j : j \in J\}$ be a family of submodules of T. Then $\left(\bigcap_{j\in J} N_j : T\right) = \bigcap_{j\in J}(N_j : T)$.*

Proof. Let $x \in (\bigcap_{j\in J} N_j : T)$. Then $xT \subseteq \bigcap_{j\in J} N_j \subseteq N_j$ for all $j \in J$. Therefore $x \in \left(N_j : T\right)$ for all $j \in J$, so that $x \in \bigcap_{j\in J}(N_j : T)$. Conversely, let $y \in \bigcap_{j\in J}(N_j : T)$. Then $y \in \left(N_j : T\right)$ for all $j \in J$, and $yT \subseteq \bigcap_{j\in J} N_j$. Thus $y \in (\bigcap_{j\in J} N_j : T)$. □

Lemma 2.6 *Let S be a finitely generated module, $\{ N_j : j \in J\}$ be a totally ordered family of prime submodules. Then $(\bigcup_{j\in J} N_j : T) = \bigcup_{j\in J}(N_j : T)$.*

Proof. Let $x \in (\bigcup_{j\in J} N_j : T)$. Then $xT \subseteq \bigcup_{j\in J} N_j$. Since T is finitely generated, $T = Sm_1 + Sm_2 + \cdots + Sm_n$ for some $m_1, m_2, \dots, m_n \in T$. Hence for all $m_i \in T$, xm_i is an element of N_{j_i} for some $j_i \in J$. Since $\{ N_j : j \in J\}$ is totally ordered, there exist $k \in I$ such that $N_{j_1}, N_{j_2}, \dots, N_{j_n} \subseteq N_k$. Thus $xT \subseteq N_k$ and $x \in \bigcup_{j\in J}(N_j : T)$ Now let $x' \in \bigcup_{j\in J}(N_j : T)$. Then $x' \in (N_j : T)$ for some $j \in J$ and $x'T \subseteq N_j \subseteq \bigcup_{j\in J} N_j$. Hence $x' \in (\bigcup_{j\in J} N_j : T)$.□

Definition 2.7 *Let A be an ideal of a ring S. The set $\sqrt{A} = \{s \in S : \; s^t \in A \; for \; some \; t \in \mathbb{Z}^+\}$ is called the* ***radical*** *of A.*

Definition 2.8 *Let T be an $S-$ module. A submodule Q of T is* ***primary submodule*** *if for $x \in S$ and $n \in T$, $xn \in Q$ implies $n \in Q$ or $x^kT \subseteq Q$ for some positive integer k.*

If $\sqrt{(Q : T)} = p$, then Q is a $p-$ primary submodule. Clearly every prime submodule of a module T is primary.

Definition 2.9 *Let T be an $S-$ module and Q be a submodule of T with $Q \neq T$. The* ***prime radical*** *of Q in T, $rad_T(Q)$, is defined as the intersection of all prime submodules of T containing Q. If no prime submodule contains Q, then $rad_T(Q) = T$, and thus $rad_T(T) = T$.*

2.2 Weakly Prime Submodules and Weakly Prime Radical

In this section we will deal with the weakly prime radical of submodules and its properties.

Definition 2.10 *Let T be an $S-$ module. A submodule W of T which is proper is **weakly prime** if $abn \in W$, where $a, b \in S$ and $n \in T$, then either $an \in W$ or $bn \in W$.*

Every prime submodule is weakly prime. Let us show by example that the converse is not true.

Let $S = \mathbb{Q}[x, y]$, $P = \langle x \rangle$ be a non-zero prime ideal of S, T be a free $S-$ module $S \oplus S$. Then $Q = 0 \oplus P$ is weakly prime submodule of T which is not prime. It is clear that $(Q : T) = 0$. Then if we take $(0, y) \notin Q$ and $x \notin (Q : T)$, we have $x(0, y) = (0, xy) \in Q$ which means that Q is not prime. To show that Q is weakly prime, let $a, b \in S$, $m = (m_1, m_2) \in T$ such that $abm \in Q$. Then $abm = (abm_1, abm_2) \in 0 \oplus P$ implies that $abm_1 = 0$ and $abm_2 \in P$. Since S is integral domain, $a = 0$ or $bm_1 = 0$. If $a = 0$, then $am \in 0 \oplus P$. If $bm_1 = 0$, then $b = 0$ or $m_1 = 0$. $b = 0$ implies that $bm \in 0 \oplus P$ similar to the case $a = 0$. Now suppose that $m_1 = 0$. $abm_2 \in P$ implies that $a \in P$ or $bm_2 \in P$. If $a \in P$, then $am = a(0, m_2) \in 0 \oplus P$. If $bm_2 \in P$, then $bm = b(0, m_2) = (0, bm_2) \in 0 \oplus P$. Thus in each case $am \in Q$ or $bm \in Q$.

Proposition 2.11 *Let W be any weakly prime submodule of T and I be an ideal of* S. *Then $(W : I) = \{n \in T : In \subseteq W\}$ is a weakly prime submodule of T.*

Proof. Let $a, b \in S$ and $n \in T$ such that $abn \in (W : I)$. We know that $I(abn) \subseteq W$, for every $r \in I$, $r(abn) \in W$ implies that $rn \in W$ or $abn \in W$ since W is weakly prime. If $rn \in W$, then $n \in (W : I)$. If $abn \in W$, then $an \in W$ or $bn \in W$. Thus for all cases, $an \in (W : I)$ or $bn \in (W : I)$. Therefore $(W : I)$ is weakly prime submodule of T. □

Lemma 2.12 *W is a weakly prime submodule of a module T iff $(W : m)$ is a prime ideal for every $m \in T - W$.*

Proof. Let $ab \in (W : m)$ and W be a submodule of T which is weakly prime. Hence $abm \in W$ where $a, b \in S$, $m \in T - W$. So that $am \in W$ or $bm \in W$ implies that $a \in (W : m)$ or $b \in (W : m)$. Conversely, suppose for every $m \in T - W$, $(W : m)$ is prime ideal. Assume that $abm \in W$ for some $m \in T - W$. Then $ab \in (W : m)$. Since $(W : m)$ is prime, $a \in (W : m)$ or $b \in (W : m)$. Then $am \in W$ or $bm \in W$. Therefore W is weakly prime submodule. □

Azizi (2008) showed that if Q is a weakly prime submodule, then

$$\{(Q : m) : m \in T - Q\}$$

forms a chain of prime ideals. Hence $(Q : T)$ is a prime ideal.

We can write the following definition, since every prime submodule is weakly prime.

Definition 2.13 *Let T be an $S-$ module. and Q be a submodule of T with $Q \neq T$. The **weakly prime radical** of Q in T, $wrad_T(Q)$, is defined as the intersection of all weakly prime submodule of T containing Q. If there is no weakly prime submodule containing Q, then $wrad_T(Q) = T$, and thus $wrad_T(T) = T$.*

Proposition 2.14 *Let $\{W_i : i \in I\}$ be a non-empty family of weakly prime submodules of an $S-$ module T. Suppose that the family is totally ordered by inclusion. Then $\bigcap_{i\in I} W_i$ is a weakly prime submodule of T.*

Proof. Let $xym \in \bigcap_{i\in I} W_i$ for $x, y \in S$ and $m \in T$. Then $xym \in W_i$ for all $i \in I$. Since W_i is weakly prime, $xm \in W_i$ or $ym \in W_i$. Since the family is totally ordered, $xm \in \bigcap_{i\in I} W_i$ or $ym \in \bigcap_{i\in I} W_i$. Thus, $\bigcap_{i\in I} W_i$ is a weakly prime submodule. □

Proposition 2.15 *Let T be an $S-$ module. If a submodule N of T is contained in a weakly prime submodule K, then K contains a minimal weakly prime submodule of N.*

Proof. Let $A = \{L : L \text{ } is \text{ } weakly \text{ } prime \text{ } submodule \text{ } of \text{ } T \text{ } and \text{ } N \subseteq L \subseteq K\}$. $A \neq \emptyset$ since $K \in A$. If $L_1, L_2 \in A$, then let us define a relation $\leq$ such that $L_1 \leq L_2$ if $L_2 \subseteq L_1$. This gives a partial order on A. Let A' be a non-empty totally ordered subset of A. Consider the intersection of all the elements of A', say $\bar{L} = \bigcap_{i\in I} L_i$ where $L_i \in A'$. Since A' is totally ordered, $\bar{L}$ is a weakly prime submodule. Then $N \subseteq \bar{L} \subseteq K$. Hence $\bar{L} \in$ A and $\bar{L} \subseteq L_i$ for all $L_i \in A'$. We have $L_i \leq \bar{L}$. Thus $\bar{L}$ is an upper bound for A'. By Zorn's lemma, A contains a maximal element, say Y. Since $Y \in A$, we have $N \subseteq Y \subseteq K$ and Y is a weakly prime submodule of T. To complete the proof we will show that Y is a minimal weakly prime submodule of N. Now suppose $\bar{Y}$ is a weakly prime submodule of T such that $N \subseteq \bar{Y} \subseteq K$ and $\bar{Y} \subseteq Y$. Then $\bar{Y} \in A$ and $Y \leq \bar{Y}$. Thus, $Y = \bar{Y}$, since Y is a maximal element of A. Therefore Y is a minimal weakly prime submodule of N.□

Corollary 2.16 *Every proper submodule of finitely generated module possesses at least one minimal weakly prime submodule.*

Proof. Let T be a finitely generated module, Q be a proper submodule of T. Then there exists submodule N' of T such that $Q \subseteq N'$ and N' is maximal. Since N' is maximal, it is a prime submodule and hence N' is weakly prime (Lu, 1984). Then by Proposition 2.15, N' contains a minimal weakly prime submodule of Q. Thus Q has at least one minimal weakly prime submodule. □

By using above corollary, we can give the characterization of the weakly prime radical of submodules of finitely generated modules.

Theorem 2.17 *Let T be finitely generated $S-$ module. Then the weakly prime radical of a proper submodule N of T is just the intersection of its minimal weakly prime submodules.*

Proof. Let N be submodule of T with $N \neq T$. By Corollary 2.16, N has at least one minimal weakly prime submodule, say W_i. Let L be the intersection of all minimal weakly prime submodules of T containing N. By the definition of $wrad_T(N)$, $wrad_T(N) \subseteq \bigcap_{i\in I} W_i = L$. On the other hand, if W is any weakly prime submodule containing N, then W contains some minimal weakly prime submodule Q_i of N by Proposition 2.16. Hence $L = \bigcap_{i\in I} W_i \subseteq wrad_T(N)$. □

In the following proposition we will give some basic properties of the weakly prime radical.

Proposition 2.18 *Let T be an $S-$ module, J be an index set and let N, N_j be submodules of T for $j \in J$ and I be an ideal of S. Then*

(i) $N \subseteq wrad_T(N)$,

(ii) $wrad_T(wrad_T(N)) = wrad_T(N)$,

(iii) $wrad_T(\bigcap_{j\in J} N_j) \subseteq \bigcap_{j\in J} wrad_T(N_j) = wrad_T(\bigcap_{j\in J} wrad_T(N_j))$

(iv) $\sum_{j\in J} wrad_T(N_j) \subseteq wrad_T(\sum_{j\in J} N_j) = wrad_T(\sum_{j\in J} wrad_T(N_j))$

(v) $wrad_T(IT) = wrad_T(\sqrt{I}T) = wrad_T(I^nT)$ *for every ideal I and for every positive integer n.*

(vi) $\sqrt{(N:T)} \subseteq (wrad_T(N) : T)$,

(vii) *If T is finitely generated, then $wrad_T(N) = T$ iff $N = T$*

Proof. (i) Let $x \in N$. Then $x \in W$ for every weakly prime submodule W of T containing N. Hence $x \in wrad_T(N)$.

(ii) $wrad_T(N) \subseteq wrad_T(wrad_T(N))$ is clear by (i). Since $N \subseteq wrad_T(N) \subseteq W_i$ for all weakly prime submodules W_i containing $wrad_T(N)$, we have $\bigcap_{i\in I} W_i \subseteq P_j$ for all weakly prime submodules P_j containing N. Hence $wrad_T(wrad_T(N)) = \bigcap_{i\in I} W_i \subseteq wrad_T(N)$.

(iii) Let $wrad_T(\bigcap_{j\in J} N_j) = \bigcap_{i\in I} W_i$ for all weakly prime submodules W_i containing $\bigcap_{j\in J} N_j$ and let $\{Q_{ij}\}$ be the set of weakly prime submodules containing N_j. Since $\bigcap_{j\in J} N_j \subseteq N_j \subseteq Q_{ij}$ for all i and j, $wrad_T(\bigcap_{j\in J} N_j) \subseteq wrad_T(N_j)$. Hence $wrad_T(\bigcap_{j\in J} N_j) \subseteq \bigcap_{j\in J} wrad_T(N_j)$.

Since $\bigcap_{j\in J} wrad_T(N_j) \subseteq wrad_T(N_j)$ for all $j \in J$, $wrad_T\left(\bigcap_{j\in J} wrad_T(N_j)\right) \subseteq wrad_T\left(wrad_T(N_j)\right) = wrad_T(N_j)$ for all $j \in J$. Hence $wrad_T\left(\bigcap_{j\in J} wrad_T(N_j)\right) \subseteq \bigcap_{j\in J} wrad_T(N_j)$. By (i), $\bigcap_{j\in J} wrad_T(N_j) = wrad_T\left(\bigcap_{j\in J} wrad_T(N_j)\right)$ is clear.

(iv) Since $N_j \subseteq \sum_{j\in J} N_j$ for all j, $wrad_T(N_j) \subseteq wrad_T(\sum_{j\in J} N_j)$. Then $\sum_{j\in J} wrad_T(N_j) \subseteq wrad_T(\sum_{j\in J} N_j)$. Let us show that $wrad_T(\sum_{j\in J} N_j) =$

$wrad_T(\sum_{j\in J} wrad_T(N_j))$. It is clear that $\sum_{j\in J} N_j \subseteq \sum_{j\in J} wrad_T\,(N_j)$. Then $wrad_T\left(\sum_{j\in J} N_j\right) \subseteq wrad_T\left(\sum_{j\in J} wrad_T\,(N_j)\right)$.

Let $\{Q_i\}$ be a set of weakly prime submodules of T containing $\sum_{j\in J} wrad_T\,(N_j)$ and let W_k be any weakly prime submodule of T such that $\sum_{j\in J} N_j \subseteq W_k$. Since $N_j \subseteq \sum_{j\in J} N_j$, $wrad_T(N_j) \subseteq W_k$ for all j . Hence $\sum_{j\in J} wrad_T(N_j) \subseteq W_k$ for all weakly prime submodules W_k of T containing $\sum_{j\in J} N_j$. Then $\bigcap_{i\in I} Q_i \subseteq W_k$ for all W_k . Therefore $wrad_T(\sum_{j\in J} wrad_T(N_j)) \subseteq wrad_T(\sum_{j\in J} N_j)$.

(v) This is trivially true if $wrad_T(IT) = T$. If $wrad_T(IT) \neq T$, then there exists a weakly prime submodule Q such that $IT \subseteq Q$. We know $I \subseteq (IT : T) \subseteq (Q : T)$. Then $\sqrt{I} \subseteq \sqrt{(Q:T)} = (Q:T)$. If $x \in \sqrt{I}$, then there exists $k \in \mathbb{Z}^+$ such that $x^k \in I \subseteq (Q:T)$. Hence $x \in (Q:T)$. So that $\sqrt{I}T \subseteq (Q:T)T = Q$. Thus $wrad_T\left(\sqrt{I}T\right) \subseteq wrad_T(IT)$. Since $I \subseteq \sqrt{I}$, $IT \subseteq \sqrt{I}T$ and $wrad_T(IT) \subseteq wrad_T(\sqrt{I}T)$.

Since $\sqrt{I^n} = \sqrt{I}$ for every positive integer n , $wrad_T(I^nT) = wrad_T\left(\sqrt{I^n}T\right) = wrad_T\left(\sqrt{I}T\right) = wrad_T(IT)$.

(vi) Let $0 \neq z \in \sqrt{(N:T)}$. Then there exists $k \in \mathbb{Z}^+$ such that $z^k \in (N:T)$, which implies that $z^kT \subseteq N \subseteq W$ for all weakly prime submodules W containing N. Therefore $z^k \in (W:T)$ and $z \in (W:T)$ since $(W:T)$ is a radical ideal. So $zT \subseteq wrad_T(N)$ and thus $z \in (wrad_T(N):T)$. Therefore $\sqrt{(N:T)} \subseteq (wrad_T(N):T)$.

(vii) Let $N = T$, then $wrad_T(N) = wrad_T(T) = T$. Conversely assume that $wrad_T(N) = T$ and $N \neq T$. By Corollary 2.16, N has at least one minimal weakly prime submodule W such that $N \subseteq W$ since T is finitely generated. Hence $T = wrad_T(N) \subseteq W$ and $T \subseteq W$. This contradicts with the fact that W is weakly prime. Therefore $N = T$.

Corollary 2.19 *Let T be finitely generated module, Y and K be submodules of T. Then $wrad_T(Y) + wrad_T(K) = T$ if and only if $Y + K = T$.*

Proof. Assume that $wrad_T(Y) + wrad_T(K) = T$. Then by above proposition (vii), $wrad_T\big(wrad_T(Y) + wrad_T(K)\big) = T$. By the same proposition $wrad_T(Y + K) = T$ and $Y + K = T$.

Conversely, suppose that $Y + K = T$. Then by using above proposition, $T = wrad_T(Y + K) = wrad_T\big(wrad_T(Y) + wrad_T(K)\big)$. Hence $wrad_T(Y) + wrad_T(K) = T$. □

Proposition 2.20 *Let Q be a submodule of an $S-$ module $T$and I be an ideal of S. Then $wrad_T(wrad_T(Q){:}\, I) = (wrad_T(Q){:}\, I)$.*

Proof. Let $x \in (wrad_T(Q){:}\, I)$. Then $x \in P$ for all weakly prime submodule P of T containing $wrad_T(Q){:}\, I$. Hence $x \in wrad_T(wrad_T(Q){:}\, I)$. Now assume that $wrad_T(Q) = \bigcap_{j\in J} W_j$, for all weakly prime submodule W_j of T containing Q. Then $wrad_T(wrad_T(Q) : I) = wrad_T(\bigcap_{j\in J} W_j : I) = wrad_T(\bigcap_{j\in J}(W_j : I))$ and $wrad_T(\bigcap_{j\in J}\big(W_j : I\big)) \subseteq \bigcap_{j\in J} wrad_T(W_j : I)$ by Proposition 2.18. Proposition 2.11 implies that $\bigcap_{j\in J} wrad_T\big(W_j : I\big) = \bigcap_{j\in J}\big(W_j : I\big) \subseteq (W_j : I)$, for all weakly prime submodule W_j of T containing Q . $I\big(wrad_T(wrad_T(Q){:}\, I)\big) \subseteq I\big(W_j : I\big) \subseteq W_j$ and then $I\big(wrad_T(wrad_T(Q) : I)\big) \subseteq wrad_T(N)$. Hence $wrad_T(wrad_T(Q) : I) \subseteq (wrad_T(Q) : I)$. □

Corollary 2.21 *Let Q be a submodule of an $S-$ module $T$and J be an ideal of S. Then $wrad_T(Q : J) \subseteq (wrad_T(Q) : J)$.*

Proof. We know that $Q \subseteq wrad_T(Q)$. Then $(Q : J) \subseteq (wrad_T(Q) : J)$. We have $wrad_T(Q : J) \subseteq wrad_T(wrad_T(Q) : J)$ and then by Proposition 2.20, $wrad_T(Q : J) \subseteq (wrad_T(Q) : J)$. □

Proposition 2.22 *Let Q be a submodule of $S-$ module T, I and J be ideals of S. Then $wrad_T(IJQ) = wrad_T(IQ) \cap wrad_T(JQ)$.*

Proof. Let $A_1 = \{\mathrm{W} :\ \mathrm{W}$ is a weakly prime submodule of T such that $\mathrm{IJQ} \subseteq \mathrm{W}\}$, $A_2 = \{\mathrm{W}' : \mathrm{W}'$ is a weakly prime submodule of T such that $\mathrm{IQ} \subseteq \mathrm{W}'\}$, and $A_3 = \{\bar{\mathrm{W}}: \bar{\mathrm{W}}$ is a weakly prime submodule of T such that $\mathrm{JQ} \subseteq \bar{\mathrm{W}}\}$. Since each

$W \in A_1$ is weakly prime, $IJQ \subseteq W$ implies that $IQ \subseteq W$ or $JQ \subseteq W$. Therefore $A_1 = A_2 \cup A_3$ and $wrad_T(IJQ) = \bigcap_{W \in A_1} W = \left(\bigcap_{W' \in A_2} W'\right) \cap \left(\bigcap_{\bar{W} \in A_3} \bar{W}\right)$. Then $wrad_T(IJQ) = wrad_T(IQ) \cap wrad_T(JQ)$. □

Definition 2.23 *Let T be an $S-$ module, Q be a submodule of T with $Q \neq T$. Q is* ***weakly primary*** *if $xym \in Q$, where $x, y \in S$, $m \in T$, implies $xm \in Q$ or $y^n m \in Q$ for some integer n.*

Proposition 2.24 *Let T be a finitely generated $S-$ module. $\mathcal{M}$ be a maximal ideal of S, and Q be a weakly primary submodule of T such that $\sqrt{(Q : T)} = \mathcal{M}$. Then $wrad_T(Q)$ is a weakly prime submodule with $(wrad_T(Q){:}T) = \mathcal{M}$ and $wrad_T(Q) = wrad_T(Q + \mathcal{M}T) = Q + \mathcal{M}T$.*

Proof. By Proposition 2.18 (vi), $\mathcal{M} = \sqrt{(Q : T)} \subseteq (wrad_T(Q){:}T)$. Since $\mathcal{M}$ is a maximal ideal, $(wrad_T(Q) : T) = S$ or $\mathcal{M}$. If $(wrad_T(Q){:}T) = S$, then $T = ST \subseteq wrad_T(Q)$. By Proposition 2.18 (vii), $Q = T$ which is a contradiction. Hence $(wrad_T(Q){:}T) = \mathcal{M}$. Then $wrad_T(Q)$ is a prime submodule (Lu, 1984). Thus $wrad_T(Q)$ is a weakly prime submodule.

Every weakly prime submodule containing $Q + \mathcal{M}$T also contains Q. Hence $wrad_T(Q) \subseteq wrad_T(Q + \mathcal{M}T)$. Now assume that $wrad_T(Q) = \bigcap_{i \in I} P_i$ for every weakly prime submodule P_i of T containing Q. Then $\sqrt{(Q : T)} \subseteq (wrad_T(Q){:}T) = (\bigcap_{i \in I} P_i{:}T) = \bigcap_{i \in I}(P_i{:}T) \subseteq (P_i{:}T)$ for every $i \in I$, by Proposition 2.18. Hence $\mathcal{M}T \subseteq (P_i{:}T)T \subseteq P_i$, so that $Q + \mathcal{M}T \subseteq Q + P_i = P_i$ for every $i \in I$. Thus $wrad_T(Q + \mathcal{M}T) \subseteq \bigcap_{i \in I} P_i = wrad_T(Q)$. In order to complete the proof, it is enough to show that $wrad_T(Q + \mathcal{M}T) = Q + \mathcal{M}$T. It is clear that $Q + \mathcal{M}T \subseteq wrad_T(Q + \mathcal{M}T) = wrad_T(Q)$. Since $\mathcal{M}T \subseteq Q + \mathcal{M}T$, we have $(\mathcal{M}T{:}T) \subseteq \left((Q + \mathcal{M}T){:}T\right) \subseteq (wrad_T(Q){:}T)$. Thus $\left((Q + \mathcal{M}T){:}T\right) = \mathcal{M}$, so that $Q + \mathcal{M}$T is a weakly prime submodule of T. Hence $wrad_T(Q) \subseteq Q + \mathcal{M}T$ since $Q \subseteq Q + \mathcal{M}$T. Therefore $wrad_T(Q + \mathcal{M}T) = Q + \mathcal{M}T$. □

2.3 The Weakly Prime Radical of an Intersection

The equality $wrad_T(N \cap L) = wrad_T(N) \cap wrad_T(L)$ is not always true for submodules N and L. If U is a vector space, and W_1, W_2 are subspaces of U, then

$$wrad_U(W_1 \cap W_2) = wrad_U(W_1) \cap wrad_U(W_2).$$

In this section, we will investigate the under which conditions this equality is true.

Lemma 2.25 *Let Q and L be submodule of an $S-$ module T, and K be a weakly prime submodule of T with $Q \cap L \subseteq K$. If $(Q : T) \nsubseteq (K : T)$, then $L \subseteq K$.*

Proof. Suppose $Q \cap L \subseteq K$, $(Q : T) \nsubseteq (K : T)$ and $L \nsubseteq$ K . Since $Q \cap L \subseteq K$, $\big((Q \cap L): T\big) \subseteq (K : T)$ so that $(Q : T) \cap (L : T) \subseteq (K : T)$. Then $(Q : T) \subseteq (K : T)$ or $(L : T) \subseteq (K : T)$. This gives us that $(Q : T) \subseteq (K : T)$ since $L \nsubseteq$ K. This is a contradiction. Therefore $L \subseteq$ K. □

Proposition 2.26 *Let M be a submodule of an $S-$ module T, I be an ideal of S. If Q is a weakly prime submodule of T such that $IT \cap M \subseteq Q$, then $IT \subseteq Q$ or $M \subseteq Q$.*

Proof. Since $IT \cap M \subseteq Q$, $\big((IT \cap M): T\big) \subseteq (Q : T)$. We have $(IT : T) \cap (M : T) \subseteq (Q : T)$. Then $(IT : T) \subseteq (Q : T)$ or $(M : T) \subseteq (Q : T)$. If $(IT : T) \subseteq (Q : T)$, then $I \subseteq (IT : T) \subseteq (Q : T)$. This gives that $IT \subseteq (Q : T)T \subseteq Q$. If $(IT : T) \nsubseteq (Q : T)$, then $M \subseteq Q$ by Lemma 2.25. □

Proposition 2.27 *Let K and M be submodules of an $S-$ module T such that whenever $K \cap M \subseteq W$, we have $K \subseteq W$ or $M \subseteq W$ for any weakly prime submodule W of T. Then $wrad_T(K \cap M) = wrad_T(K) \cap wrad_T(M)$.*

Proof. If $wrad_T(K \cap M) = T$, then $wrad_T(K) = wrad_T(M) = T$ and $wrad_T(K \cap M) = wrad_T(K) \cap wrad_T(M)$. If $wrad_T(K \cap M) \neq T$, then there exists a weakly prime submodule W of T such that $K \cap M \subseteq W$. By hypothesis, $K \subseteq W$ or $\subseteq W$. So that $wrad_T(K) \subseteq W$ or $wrad_T(M) \subseteq W$. Since this is true for every weakly prime submodule T containing $K \cap M$, $wrad_T(K) \cap wrad_T(M) \subseteq wrad_T(K \cap M)$. Therefore $wrad_T(K \cap M) = wrad_T(K) \cap wrad_T(M)$. □

Proposition 2.28 *Let K and M be submodules of an $S-$module T where $\sqrt{(K:T)} + \sqrt{(M:T)} = S$. Then $wrad_T(K \cap M) = wrad_T(K) \cap wrad_T(M)$.*

Proof. Suppose W is a weakly prime submodule containing $K \cap M$ with $P = (W:T)$. Then $(K:T) \cap (M:T) \subseteq P$ so that $(K:T) \subseteq P$ or $(M:T) \subseteq P$. If $(K:T) \subseteq P$, then $(M:T) \nsubseteq P$. Hence $K \subseteq W$ by Lemma 2.25. Therefore $K \subseteq W$ or $M \subseteq W$. By Proposition 2.27, $wrad_T(K \cap M) = wrad_T(K) \cap wrad_T(M)$. □

3. PSEUDO-VALUATION DOMAIN

McCasland and Moore (1991), generalized the concept of radical of an ideal to modules over commutative rings. The definitions of the radical and envelope of a submodule are the result of this generalization. McCasland and Smith (2008), gave on algorithm for the computation of radical of a submodule over a Noetherian ring. There is no known algorithm for the computation of weakly prime radical.

In this thesis, we tried to find an algorithm for the computation of weakly prime radical. At the same time, we tried the find conditions on modules and rings which satisfy the weakly radical formula.

In the first section of this chapter, the information about the envelope of a submodule is given. The second section concerns the modules satisfying weakly radical formula.

3.1 The Envelope of a Submodule

Let T be an $S-$ module, A be an ideal of S. It is well-known, that the radical of A is $\sqrt{A} = \{s \in S : s^n \in A, for\ some\ n \in \mathbb{Z}^+\}$. A similar definition is possible for modules which is called the envelope of a submodule.

Definition 3.1 *Let T be an $S-$ module and Q be a submodule of T. The envelope of Q in T, $E_T(Q)$, is defined as the set*

$$\{sn : s \in S, n \in T\ such\ that\ s^k n \in Q\ for\ some\ k \in \mathbb{Z}^+\}.$$

In general, $E_T(Q)$ is not a submodule. For example, if we take $T = \mathbb{Z} \oplus \mathbb{Z}$, $Q = (3,0)\mathbb{Z}$ and $S = \mathbb{Z}$, then $(10,0) = 2(5,0) \in E_T(Q)$ since $2^5(5,0) = (150,0) = 50(3,0) \in Q$. Also it is clear that $(3,0) \in E_T(Q)$. But $(10,0) - (3,0) = (7,0) \notin E_T(Q)$ since there does not exist any $k \in \mathbb{Z}^+$ such that $7^k(1,0) \in (3,0)\mathbb{Z}$. So we are considering the submodule generated by envelope, $\langle E_T(Q) \rangle$.

It is clear that $\langle E_T(Q)\rangle$ is always contained in $rad_T(Q)$ for any submodule Q.

Theorem 3.2 *(Yılmaz and Cansu, 2014, Theorem 2.5) Let M be a submodule of module T over S, where $M = Q_1 \cap Q_2 \cap \dots \cap Q_l$ is a minimal primary decomposition of M such that $\sqrt{(Q_i : T)} = p_i$ for all $i = 1,2,\dots,l$. If $A = \{1,2,\dots,l\}$ and $\emptyset \neq T \subsetneq A$, then*

$$\langle E_T(M)\rangle = M + \left(\bigcap_{i=1}^{l} p_i\right)T + \sum_{\emptyset \neq T \subsetneq A}\left(\bigcap_{i\in T} p_i\right)\left(\bigcap_{i\in A-T} Q_i\right)$$

Corollary 3.3 *(Yılmaz and Cansu, (2014), Lemma 3.1) If Q is a weakly prime submodule of an S −module T, then $\langle E_T(Q)\rangle = Q$.*

3.2 Pseudo-valuation Domain

Main results obtained in the thesis are given in this section. First we will give some necessary definitions and then we will give the results we have found.

Definition 3.4 *An integral domain S is* ***valuation domain*** *if all its ideals form a chain under inclusion.*

Definition 3.5 *An integral domain S is* ***divided domain*** *if for every prime ideal P of S, either $P \subseteq \langle x\rangle$ or $\langle x\rangle \subseteq P$ for all $x \in S$.*

Definition 3.6 *Let S be an integral domain and K be the set of all non-zero elements in S. Then the ring of quotients of S by K, $K^{-1}S$, will be a field and it is called as* ***the quotient field*** *of an integral domain S.*

Proposition 3.7 *(Larsen and McCarthy, 1971) For an integral domain S, the following statements are equivalent.*

(i) S is a valuation domain.
(ii) If $a, b \in S$, then either $\langle a\rangle \subseteq \langle b\rangle$ or $\langle b\rangle \subseteq \langle a\rangle$.
(iii) If x belongs to the quotient field D of S, then either $x \in D$ or $x^{-1} \in D$

Definition 3.8 *Let S be an integral domain with quotient field D. A prime ideal Q of S is* ***strongly prime*** *if $u, v \in D$ and $uv \in Q$ implies $u \in Q$ or $v \in Q$.*

Definition 3.9 *An integral domain S is called a* ***pseudo-valuation domain*** *(PVD) if every prime ideal of S is strongly prime.*

Proposition 3.10 *Every valuation domain is a PVD.*

Proof. Let D be a valuation domain and U be the quotient field of D. Let Q be a prime ideal of D. Assume $ab \in Q$ where $a, b \in U$. If both a and b are in D, it is done. If $a \notin D$, then $a^{-1} \in D$. Thus $b = aba^{-1} \in Q$, as desired. □

Theorem 3.11 (Badawi, 1995, Theorem 1) If S is *a commutative ring with identity, then the following statements are equivalent.*

a) The prime ideals of S are linearly ordered.
b) The radical ideals of S are linearly ordered.
c) Each proper radical ideal of S is prime.
d) The radicals of principal ideals of S are linearly ordered.
e) For each $x, y \in S$, there is an $n \geq 1$ such that either $x|y^n$ or $y|x^n$.

Proposition 3.12 *If S is a divided domain, then prime ideals of S are totally ordered.*

Proof. Let P, Q be two prime ideals of S and $P \nsubseteq Q$. Then $\langle x \rangle \subseteq P$ or $P \subseteq \langle x \rangle$ for all $x \in Q$. Since $P \nsubseteq Q$, $\langle x \rangle \subseteq P$, which means that $x \in P$ for all $x \in Q$. Thus $Q \subseteq P$. □

By using the following proposition, we can give the characterization of divided domains.

Proposition 3.13 *(Badawi, 1995, Proposition 2) If S is an integral domain, then the following statements are equivalent.*

a) S is a divided domain.
b) The ideals I and $\sqrt{J}$ are comparable for every pair of proper ideals I and J of S.

c) *The ideals* $\langle x\rangle$ *and* $\sqrt{\langle y\rangle}$ *are comparable, for every* $x, y \in S$.
d) *For every* $x, y \in S$*, either* $x|y$ *or* $y|x^k$ *for some* $k \geq 1$.

If S is a PVD, Badawi (1995) showed that for any non-unit elements x, y of S, either $x|y$ or $y|x^2$. In the same paper, he gave the following characterization of PVDs.

Proposition 3.14 *(Badawi, 1995, Proposition 3) Let* S *be an integral domain and* K *be the set of all non-unit elements of* S*. Then the followings are equivalent.*

a) S *is a PVD with the maximal ideal* K*,*
b) *For every proper ideal* A *of* S*, either* $J \subset I$ *or* $IA \subset J$ *for each pair* I *and* J *of ideals of* S.
c) *For every* $p, q \in S$*, either* $qS \subset pS$ *or* $pzS \subset qS$ *for every non-unit* $z \in S$.
d) *For every* $p, q \in S$*, either* $p|q$ *or* $q|pz$ *for every non-unit* $z \in S$.
e) *For every* $p, q \in S$*, either* $qS \subset pS$ *or* $pK \subset qS$.
f) *For every* $p, q \in S$*, either* $qK \subset pS$ *or* $pS \subset qK$.

Definition 3.15 *Let* T *be an* $S-$ *module. A submodule* Q *of* T *with* $Q \neq T$*, is* ***weakly quasi-primary*** *if* $abm \in Q$*, then either* $a^k m \in Q$ *or* $b^k m \in Q$ *for some* $k \in \mathbb{N}$ *where* $a, b \in S$*,* $m \in T$.

If Q is a weakly quasi-primary submodule, then by the definition $\sqrt{(Q{:}T)}$ is a prime ideal.

We know that the envelope of every weakly prime submodule equals itself. In the following lemma, we proved that every weakly quasi- primary submodule is weakly prime iff $\langle E_T(Q)\rangle = Q$.

Lemma 3.16 *Let* T *be an* $S-$ *module,* Q *be a proper submodule of* T.

(i) *Suppose that* Q *is a weakly primary submodule. Then* Q *is weakly prime iff* $\langle E_T(Q)\rangle = Q$.

(ii) Suppose that Q is a weakly quasi-primary submodule. Then Q is weakly prime if and only if $\langle E_T(Q)\rangle = Q$.

Proof. If Q is weakly prime, then $\langle E_T(Q)\rangle = Q$, so we have to show only one side of the assertion. Hence suppose that $\langle E_T(Q)\rangle = Q$.

(i) Let $xym \in Q$ where $x, y \in S$, $m \in T$. If $xm \notin Q$ then $y^k m \in Q$ for $k \geq 1$ since Q is weakly primary. Then $ym \in Q$ since $\langle E_T(Q)\rangle = Q$. Therefore Q is a weakly prime submodule of T.

(ii) Let $xym \in Q$ where $x, y \in S$, $m \in T$. Assume that $xm \notin Q$. Then $x^k m \notin Q$ for any k. Otherwise $\langle E_T(Q)\rangle = Q$ implies that $xm \in Q$. Therefore $y^k m \in Q$ for some $k \geq 1$. Then $ym \in Q$ since $\langle E_T(Q)\rangle = Q$. Thus Q is a weakly prime submodule of T.

Theorem 3.17 *Let S be a commutative ring with identity whose prime ideals are totally ordered and T be an $S-$ module. Then each proper submodule Q of T is weakly quasi-primary. Furthermore, every proper submodule Q of T is weakly prime if and only if $Q = \langle E_T(Q)\rangle$.*

Proof. Let $x, y \in S$ and $m \in T\backslash Q$, so $(Q : m)$ is a proper ideal of S. We suppose that $xym \in Q$. Therefore $xy \in (Q : m) \subseteq \sqrt{(Q : m)}$. Since $\sqrt{(Q : m)}$ is a proper radical ideal of S, $\sqrt{(Q : m)}$ is a prime ideal. So either $x \in \sqrt{(Q : m)}$ or $y \in \sqrt{(Q : m)}$. This implies either $x^t m \in Q$ or $y^t m \in Q$ for some positive integer t. Since every proper submodule of T is weakly quasi-primary by above theorem, the proof is clear by Lemma 3.16. □

Chin-Pi Lu (1990) prove that if T is finitely generated $S-$module, $\mathcal{M}$ is maximal ideal of S and Q is primary submodule of T with $\sqrt{(Q : T)} = \mathcal{M}$, then $rad_T(Q) = rad_T(Q + \mathcal{M}T) = Q + \mathcal{M}T$. In the following lemma we showed that $wrad_T(N) = wrad_T(N + pT)$ for every submodule N of a module T, if S is a ring with totally ordered prime ideals and $\sqrt{(N : T)} = p$ is a prime ideal.

Lemma 3.18 *Let S be a ring with totally ordered prime ideals. Then $wrad_T(N) = wrad_T(N + pT)$ for every submodule N of an $S-$module T where $p = \sqrt{(N : T)}$ is a prime ideal.*

Proof. $wrad_T(N) \subseteq wrad_T(N + pT)$ is clear. We say, $wrad_T(N) = \bigcap_{i \in I} W_i$ where W_i is weakly prime submodule containing N with $q_i = (W_i : T)$. Therefore $(N : T) \subseteq (W_i : T)$ implies that $p \subseteq q_i$. So that $pT \subseteq q_i T$. Hence $N + pT \subseteq W_i + p\,T \subseteq W_i + q_i T = W_i$ for every i. Hence $wrad_T(N + pT) \subseteq wrad_T(N)$ and thus $wrad_T(N + pT) = wrad_T(N)$. □

Lemma 3.19 *Let S be a ring with totally ordered prime ideals, T be an $S-$ module and $\wp$ be a maximal ideal of S. If $(N : T) = \wp$, then $wrad_T(N) = N + \wp T$ for any submodule N of T.*

Proof. $N + \wp T \subseteq wrad_T(N + \wp T) = wrad_T(N)$, by above lemma. Therefore $\wp T \subseteq N + \wp T$ implies that $\wp \subseteq \big((N + \wp T) : T\big) \subseteq (wrad_T(N) : T)$. Since $wrad_T(N)$ is a weakly prime submodule, $(wrad_T(N) : T)$ is a prime ideal. So $(wrad_T(N) : T) = \wp$ and hence $\big((N + \wp T) : T\big) = \wp$. Then $N + \wp T$ is a prime submodule; thus a weakly prime submodule of T containing N. So that $wrad_T(N) \subseteq N + \wp$T. Therefore $wrad_T(N) = N + \wp T$. □

Azizi (2009) gave the definition of kth envelope of a submodule and union of envelopes.

Definition 3.20 *For a submodule Q of T, define $E_0(Q) = Q$, $E_1(Q) = E_T(Q)$, $E_2(Q) = E_T(\langle E_1(Q)\rangle)$, and for any $k \in \mathbb{Z}^+$, $E_{k+1}(Q) = E_T(\langle E_k(Q)\rangle)$.*

For any submodule Q of the module T,

$$\mathrm{UE_T(Q)} = \bigcup_{\mathrm{k} \in \mathbb{N}} \langle \mathrm{E_k(Q)} \rangle;$$

is called the union of envelopes of Q. It is clear that

$Q \subseteq \langle E_k(Q)\rangle \subseteq UE_T(Q) \subseteq wrad_T(Q) \subseteq rad_T(Q)$, for any k ∈ ℕ (Azizi, 2009).

Definition 3.21 *Let T be an S −module. If $UE_T(Q) = wrad_T(Q)$, for every submodule Q of T, it is said that* ***weakly radical formula holds*** *for T. If the weakly radical formula holds for every S −module, then weakly radical formula holds for a ring S.*

Definition 3.22 *Let $z \in \mathbb{Z}^+$. If $\langle E_z(N)\rangle = wrad_T(N)$ for every submodule N of T, then we will say T* ***satisfies the weakly radical formula*** *(s.t.w.r.f.)* ***of degree z.*** *If every $S -$ module s.t.w.r.f. of degree z, then S is called s.t.w.r.f. of degree z.*

In any module, intersection of weakly prime submodules is not weakly prime. In the following theorem we gave some conditions, to guarantee that this intersection is weakly prime.

Definition 3.23 *Let S be a ring and T be an $S -$ module. An increasing sequence $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots$ of submodules of T is called as* ***an ascending chain****. If every ascending chain of submodules of T is finite, it is said that T satisfies* ***ascending chain condition*** *(ACC).*

Theorem 3.24 *Let S be a valuation domain. If T is an $S -$ module satisfying ACC, then $wrad_T(M)$ is weakly prime for any submodule M of T.*

Proof. Let M be a submodule of T. Since T satisfies ACC, M will be written as an intersection of finite number of primary submodules. In particular M has a minimal primary decomposition where $M = Q_1 \cap Q_2 \cap \ldots \cap Q_s$ and each Q_s is $P_s -$ primary. Since S is a valuation ring, prime ideals form a chain. So we may assume, after renumbering of Q_s's if necessary $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_s$. Yılmaz and Cansu (2014, Theorem 3.4), implies that if $\langle E_T(M)\rangle = M$, then M is a weakly prime submodule. So $M = wrad_T(M)$. If $\langle E_T(M)\rangle \neq M$, we can apply the same argument to $\langle E_T(M)\rangle$. Hence we obtain a chain of submodules $M \subseteq \langle E_T(M)\rangle \subseteq \langle E_2(M)\rangle \subseteq \cdots$ where $E_k(M) =< E_{k-1}(M) >$ for $k = 2,3, \ldots$ and $E_1(M) = \langle E_T(M)\rangle$. Since T satisfies ACC, this chain must be terminates, that is there exists a positive integer k such that $\langle E_k(M)\rangle = \langle E_l(M)\rangle$ for $l \geq k$. Hence $\langle E_k(M)\rangle$ is a weakly prime submodule and furthermore $\langle E_k(M)\rangle = wrad_T(M)$. □

Also we can show that if S is a commutative ring with identity whose prime ideals are totally ordered, then $wrad_T(Q)$ is weakly prime for any submodule Q of T.

Theorem 3.25 *Let S be a commutative ring with identity whose prime ideals are totally ordered. Let T be an $S-$ module.*

a) *If $Q = \bigcap_{i\in I} P_i$ where P_i's are weakly prime submodules of T, then Q is also a weakly prime submodule.*
b) *For any submodule L of T, $wrad_T(L)$ is a weakly prime submodule.*

Proof. (a) By Theorem 3.17, Q is weakly quasi-primary submodule. Let $xym \in Q$ where $x, y \in S$, $m \in T$, then either $x^k m \in Q$ or $y^k m \in Q$ for some positive integer k. Hence either $x^k m \in P_i$ or $y^k m \in P_i$ for every $i \in I$. Since each P_i is weakly prime submodule, either $xm \in P_i$ or $ym \in P_i$ for every $i \in I$. This implies either $xm \in Q$ or $ym \in Q$. Thus Q is a weakly prime submodule.

(b) It is clear by (a).

Theorem 3.26 *Let S be a commutative ring with identity whose prime ideals are totally ordered. If an $S-$ module T satisfies ACC, then T satisfies weakly radical formula of degree k for some positive integer k.*

Proof. Let L be a submodule of T. We can obtain the following chain of submodules $\subseteq \langle E_1(L)\rangle \subseteq \langle E_2(L)\rangle \subseteq \cdots$. Notice that each submodule on the chain is weakly quasi-primary. Since T satisfies ACC, this chain terminates. Hence there exists an integer l such that $\langle E_l(L)\rangle = \langle E_{l+1}(L)\rangle = \cdots$. This implies that $\langle E_l(L)\rangle$ is weakly prime submodule of T. So $wrad_T(L) \subseteq \langle E_l(L)\rangle$. On the other hand $\langle E_l(L)\rangle \subseteq wrad_T(L)$ is always true. Thus $\langle E_l(L)\rangle = wrad_T(L)$. □

Yılmaz and Cansu (2014) showed that if $N = Q_1 \cap Q_2 \cap \ldots \cap Q_s$ is the reduced primary decomposition of N with chain of prime ideals

$$p_1 = \sqrt{Q_1 : T} \subseteq p_2 = \sqrt{Q_2 : T} \subseteq \cdots\cdots \subseteq p_s = \sqrt{Q_s : T},$$

then

$$\langle E_T(N)\rangle = N + p_1T + \sum_{i=2}^{s} p_i(\bigcap_{j=1}^{i-1} Q_j).$$

Theorem 3.26 implies that if S is a commutative ring with identity whose prime ideals are totally ordered and T satisfies ACC, then for any submodule N of T

$$\langle E_k(N)\rangle = wrad_T(N)$$

for some positive integer k. We think that we can find a method for computing $wrad_T(N)$ by applying the envelope formula of Yılmaz and Cansu for the submodules $N, \langle E_1(N)\rangle, \langle E_2(N)\rangle, \dots, \langle E_k(N)\rangle$ for some positive integer k.

Also, we can give the following theorem.

Theorem 3.27 *Let S be a commutative ring with identity whose prime ideals are totally ordered. Then weakly radical formula holds for S.*

Proof. Let N be a submodule of an $S-$ module T. It suffices to show that $\bigcup_{i\in I} E_i(N)$ is a weakly prime submodule of T. Let $abx \in \bigcup_{i\in I} E_i(N)$ for some $a, b \in S$ and $x \in T$. Then either $a^n = bt$ or $b^n = as$ for some $s, t \in S$. We may assume $a^n = bt$. It is clear that $(ab)^n x = b^{n+1}tx \in \bigcup_{i\in I} E_i(N)$. Therefore $b^{n+1}tx \in \langle E_k(N)\rangle$ for some $k \in \mathbb{N}$. This implies that $btx \in \langle E_T\langle E_k(N)\rangle\rangle = \langle E_{k+1}(N)\rangle$. Therefore $a^n x = btx \in \langle E_{k+1}(N)\rangle$ implies that $ax \in \langle E_{k+2}(N)\rangle \subseteq \bigcup_{i\in I} E_i(N)$. □

Lemma 3.28 *Let S be a divided domain, N be a submodule of an $S-$ module T. If $s^n x \in \langle E_T(N)\rangle$, then $sx \in \langle E_T(N)\rangle$ for some $s \in S$ and $x \in N$.*

Proof. Suppose that $s^n x \in \langle E_T(N)\rangle$ and $sx \notin \langle E_T(N)\rangle$. Then $s^n x = a_1m_1 + \cdots + a_km_k$, $a_i \in S$, $m_i \in T$, $1 \le i \le k$ such that ${a_i}^{t_i} m_i \in N$. Let $t = \max\{t_1, t_2, \dots, t_k\}$, then ${a_i}^t m_i \in N$. Since S is a divided domain, $a_i | {a_1}^d$, that is, ${a_1}^d = a_iu_i$ for some $u_i \in S$. Thus ${a_1}^{td} m_i = (a_iu_i)^t m_i = {a_i}^t {u_i}^t m_i \in N$. Since S is a divided domain, $s^n | a_1$ or $a_1 | (s^n)^l$. If $a_1 | s^{nl}$, then $s^{nl} = a_1u$ for some $u \in S$. So $s^{n(dtl+1)}x = (s^{nl})^{td} s^n x = {a_1}^{td} u^{td} (\sum a_im_i) \in N$, implies that $sx \in \langle E_T(N)\rangle$, which is a contradiction. Hence, $s^n | a_1$, which means that $a_1 = s^n l_1$ for some $l_1 \in S$. Therefore $s^n(x - l_1m_1) = \sum_{i=2}^{k} a_im_i$ and $(sl_1)^{ntd} m_1 = (s^n {l_1}^n)^{td} m_1$, $(s^n {l_1}^n)^{td} m_1 = (s^n)^{td} {l_1}^{td} {l_1}^{td(n-1)} m_1 \in N$ implies that $sl_1m_1 \in \langle E_T(N)\rangle$. If we say $y = x - l_1m_1$, then $sy = sx - sl_1m_1$. Since $sx \notin \langle E_T(N)\rangle$, $sy \notin \langle E_T(N)\rangle$. Now we can apply the same argument to s^ny. After k steps, we get $sx = \sum_{i=1}^{k} sl_im_i$, where $sl_im_i \in \langle E_T(N)\rangle$. So $sx \in \langle E_T(N)\rangle$. □

Theorem 3.29 *Divided domains satisfy the weakly radical formula*.

Proof. Let S be a divided domain, T be an $S-$ module and Q be a submodule of T. Suppose that $xym \in \langle E_T(Q)\rangle$ for some $x, y \in S$ and $m \in T$. Since S is a divided domain, either $x|y$ or $y|x^n$ for some $n \in \mathbb{Z}^+$. If $a|b$, then $b = au$ for some $u \in S$. Then $b^2m = uabm \in \langle E_T(N)\rangle$. By Lemma 3.28, $bm \in \langle E_T(N)\rangle$. If $a^n = ub$ for some $u \in S$, then $a^{n+1}m = ubam \in \langle E_T(N)\rangle$ and $am \in \langle E_T(N)\rangle$ by Lemma 3.28. Hence $\langle E_T(N)\rangle$ is weakly prime. Thus $wrad_T(N) = \langle E_T(N)\rangle$. □

4. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, we show that for any weakly prime submodule of an $S-$ module T and for any ideal I, $(W:I)$ is a weakly prime submodule of T. Also we prove that weakly prime radical of any submodule of a finitely generated module is just the intersection of its minimal weakly prime submodules. In the second section of the second chapter of the thesis, we state some properties of weakly prime radical. Also, we tried to find some equalities about the weakly prime radical of the intersection of two submodules.

We show that weakly quasi-primary submodule Q is weakly prime if and only if $\langle E_T(Q)\rangle = Q$. If S is a ring with identity where its prime ideals are totally ordered, then it is shown that every proper submodule Q of a module T is weakly quasi-primary. The intersection of weakly prime submodules is not weakly prime in general, we prove that if S is a valuation domain and T is an $S-$ module satisfying ACC, then $wrad_T(M)$ is weakly prime for any submodule M of T.

In the second section of third chapter, we deal with the commutative ring S with identity whose prime ideals are totally ordered. We prove that for any submodule L of an $S-$ module T, $wrad_T(L)$ is weakly prime and $wrad_T(L) = wrad_T(L + pT)$ if $p = \sqrt{(L:T)}$ is a prime ideal. We show that $wrad_T(L) = L + pT$ for any submodule L of T if $p = (N:T)$ is a maximal ideal.

Theorem 3.27 showed that weakly radical formula holds for S. The main result of the thesis gives that if T is an $S-$ module where S is divided domain, then T satisfies the weakly radical formula. A commutative ring S is called treed ring if $Spec(S)$ as a poset under inclusion is a tree, that is, no maximal ideal of S contains incomparable prime ideals. Since every divided domain is a treed ring, the question that we want to answer is that; does every module T over a treed ring satisfy the weakly radical formula.

Another related question for our future study is to find a method for computing the weakly prime radical of a submodule.

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