BOLU ABANT IZZET BAYSAL UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

SOLVING INVERSE KINEMATICS PROBLEM BY GRÖBNER BASES

MASTER OF SCIENCE

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APPROVAL OF THE THESIS

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ÖZLEM ALTUNBEZEL

ABSTRACT

SOLVING INVERSE KINEMATICS PROBLEM BY GRÖBNER BASES
MSC THESIS
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BOLU ABANT İZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
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The inverse kinematics problem is one of the most important problems in robotics because the solutions provide control over the position and orientation of the robot hand. It is shown that Gröbner Basis Theory is an alternative method for solving the inverse kinematics problem. The aim of the thesis is to compare two different methods for specialization issue of parameters in the solution set of the problem. While the first method finds specializations by extra colon ideal computations, the second method computes a comprehensive Gröbner system of the problem. The advantages and disadvantages of both methods are explained with examples.

KEYWORDS: Gröbner Basis, Inverse Kinematics Problem, Comprehensive Gröbner System.

ÖZET

TERS KINEMATIK ROBOTIK PROBLEMİNİN GRÖBNER TABANLARIYLA ÇÖZÜLMESİ
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Ters kinematik problemi robotikteki en önemli problemlerden biridir çünkü çözümler robot elinin pozisyonu ve yönü üzerinde kontrol sağlar. Gröbner Taban Teorisinin ters kinematik problemini çözmek için alternatif bir yöntem olduğu gösterilmiştir. Tezin amacı, problemin çözüm setinde parametrelere değer verilmesi sorunu için iki farklı yöntemi karşılaştırmaktır. İlk yöntem, ekstra kolon ideal hesaplamaları ile parametrelere verilmesi gereken değerleri bulurken, ikinci yöntem problemin kapsamlı bir Gröbner sistemini hesaplar. Her iki yöntemin avantajları ve dezavantajları örneklerle açıklanmaktadır.

ANAHTAR KELİMELER: Gröbner Tabanları, Ters Kinematik Problemi, Kapsamlı Gröbner Sistemi

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LIST OF ABBREVIATIONS AND SYMBOLS

$>_{lex}$	**:** Lexicographic Order
$>_{grevlex}$	**:** Graded (Degree) Reverse Lexicographic Order
$>_{grlex}$	**:** Graded (Degree) Lexicographic Order
$\boldsymbol{LC}(\boldsymbol{f})$	**:** Leading Coefficient of f
$\boldsymbol{LM}(\boldsymbol{f})$	**:** Leading Monomial of f
$\boldsymbol{LT}(\boldsymbol{f})$	**:** Leading Term of f
$\boldsymbol{LCM}(\boldsymbol{f}, \boldsymbol{g})$	**:** Least Common Multiple of f and g
$\boldsymbol{S}(\boldsymbol{f}, \boldsymbol{g})$	**:** S-polynomial of f and g

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1. INTRODUCTION

Gröbner basis theory was introduced in (Buchberger, 1965). The theory allows calculations in multivariate polynomial systems. Gröbner basis can also be considered as a generalization of Gaussian elimination of the linear systems to the polynomial systems. Hence they can be used anywhere where some polynomial system of equations appear. The basic theory can be found in Cox et al (2007) along with a lot of applications.

One of the area that Gröbner bases can be used is the robotics. Because the kinematics of robots with prismatic and revolute joints can be described by multivariate polynomial equations. There are two basic kinematics for robots. While forward kinematics decides the position and orientation of the robot arm for given lengths of prismatic joints and angles of revolute joints, the inverse kinematics determines possible lengths and angles from a predetermined goal position of the robot arm. Solving the inverse kinematic problem can be a difficult task. The set of possible configurations can be described as the set of solutions of a multivariable polynomial system. This is where the Gröbner basis theory comes into play. An excellent exposition of the subject can be found in (Cox et al, 2007, Chapter 6).

Kendricks (2007) compare two methods for solving the inverse kinematic robotics problem. The classical method, uses the Denavit Hartenberg Matrix, and Gröbner basis method. As a result, he found that Gröbner Basis Theory is more advantageous, and furthermore, more beneficial to the field of mathematics and robotics engineering. Recently, Gröbner basis theory has been used frequently in the analysis of the movements of various robots, for example see (Abłamowicz, 2010; Naderi et al, 2016; Kumar et al, 2017; Husty et al 2019).

In this thesis, we consider a shortcoming of Gröbner basis theory for solving the inverse kinematic robotics problem. The trouble is the original Gröbner basis for the problem may not be a Gröbner basis under certain configurations of the robot. There are two solutions to this problem. The first way is to find all possible specializations of system of polynomial equation of the given robot that cause the problem. Another way is to find a comprenhensive Gröbner system of problem. The comprehensive Gröbner system is a set of specializations and Gröbner basis under this specializations. The detailed information about comprehensive Gröbner system can be found in (Kapur et al, 2013).

We begin by introducing the basics of Gröbner basis theory in Chapter 2. The inverse kinematic robotics problem is explained in details with an example in Chapter 3. The main goal of the Chapter 4 is to determine which specializations does not preserve the Gröbner basis. A method for finding such specializations is described. The outlines of this method is given in by Cox et al (2007) without any proof. We fully explain the method with proofs of necessary theorems. Chapter 5 is devoted to comprehensive Gröbner bases. We try to explain how comprehensive Gröbner basis can be used for solving specialization problem. A brief conclusion follows comparing two methods given in Chapter 4 and Chapter 5.

2. GRÖBNER BASIS THEORY

In this chapter, the basic concepts of Gröbner basis theory which are needed for solving the inverse kinematic problems are given.

2.1 Monomial Ordering

Let k be a field. The division algorithm in $k[x]$ is a well-known process. In order to generalize this division algorithm into multivariable polynomial ring $k[x_1, x_2, \ldots, x_n]$, we have to define an ordering on monomials.

Definition 2.1.1.

A monomial ordering is a total order relation $>$ on the set of monomials $\boldsymbol{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}$ in $\boldsymbol{k}[\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}]$ such that

1) $>$ is compatible with multiplication.
2) $>$ is a well-ordering. That means every non-empty set of monomials has smallest element under $>$.

Definition 2.1.2 (Lexicographic Order)

Let $\boldsymbol{n_1 = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}$ and $\boldsymbol{n_2 = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}}$ be two monomials. We say that $\boldsymbol{n_1 >_{lex} n_2}$ if the first non-zero $\boldsymbol{\alpha_i - \beta_i}$ is positive $\boldsymbol{i = 1, 2, \ldots, n}$.

Definition 2.1.3 (Graded (Degree) Reverse Lexicographic Order)

Let $n_1 = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ and $n_2 = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$ be two monomials. We say that $n_1 >_{grevlex} n_2$

1) If $\alpha_1 + \alpha_2 + \cdots + \alpha_n > \beta_1 + \beta_2 + \cdots + \beta_n$ or

2) If $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_n$ then the last non-zero $\alpha_i - \beta_i$ is negative for $i = 1,2, \dots, n$.

Definition 2.1.4 (Graded (Degree) Lexicographic Order)

Let $n_1 = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ and $n_2 = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$ be two monomials. We say that $n_1 >_{grlex} n_2$

1) If $\alpha_1 + \alpha_2 + \cdots + \alpha_n > \beta_1 + \beta_2 + \cdots + \beta_n$ or
2) If $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_n$ and $n_1 >_{lex} n_2$.

Definition 2.1.5.

Let $\boldsymbol{f} = \sum_{\boldsymbol{\alpha}} \boldsymbol{a_\alpha} \boldsymbol{x^\alpha}$ be a non-zero polynomial and let $>$ be a monomial order

- The multidegree of f is
  Multidegree $(f) = max\{\alpha | a_\alpha \neq 0\}$
- The leading coefficient of f is
  $LC(f) = a_{multidegree(f)}$
- The leading monomial of f is
  $LM(f) = x^{multidegree(f)}$
- The leading term of f is
  $LT(f) = LC(f).LM(f)$

Example 2.1.6.

Let $\boldsymbol{f(x,y,z) = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4}$ with respect to lex order, we would have,

$f(x,y,z) = -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3$,

- $LM(f) = x^5yz^4$,
- $LC(f) = -3$,
- $LT(f) = -3x^5yz^4$,

- $multidegree(f) = (5,1,4)$,

and with respect to grlex order, we would have,

$$f(x,y,z) = -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3,$$

- $LM(f) = x^5yz^4$,
- $LC(f) = -3$,
- $LT(f) = -3x^5yz^4$,
- $multidegree(f) = (5,1,4)$

And finally, with respect to grevlex order, we would have,

$$f(x,y,z) = 2x^2y^8 - 3x^5yz^4 - xy^4 + xyz^3,$$

- $LM(f) = x^2y^8$,
- $LC(f) = 2$,
- $LT(f) = 2x^2y^8$,
- $multidegree(f) = (2,8,0)$.

2.2 Division Algorithm

Fix a monomial order $>$ on $k[x_1, \dots, x_n]$ and let $F = (f_1, f_2, \dots, f_s)$ be an ordered s-tuple of polynomials. Then every $f \in k[x_1, \dots, x_n]$ can be written as $f = a_1f_1 + a_2f_2 + \cdots + a_sf_s + r$ where $a_i, r \in k[x_1, \dots, x_n]$ and either $r = 0$ or no term of r is divisible by $LT(f_1), LT(f_2), \dots, LT(f_s)$.

Furthermore $multidegree(f) \geq multidegree(a_if_i)$. The polynomials a_i, $r \in k[x_1, \dots, x_n]$ can be found by using the following algorithm.

Input $f_1, f_2, \dots, f_s, f$

Output $a_1, a_2, \dots, a_s, r$

$a_1 \coloneqq 0, a_2 \coloneqq 0, \dots, a_s \coloneqq 0$

$p \coloneqq f$

While

$p \neq 0$ Do

$i{:}\, 1$

divisionoccured ≔ false

While $i \leq s \;\; and$ divisionoccured ≔ false

If $LT(f_i)$ divides $LT(p)$ then

$$a_i := a_i + \frac{LT(p)}{LT(f_i)}$$

$$p := p - \frac{LT(p)}{LT(f_i)} \cdot f_i$$

divisionoccured ≔ true

Else

$$i = i + 1.$$

If divisionoccured ≔ false then

$$r \coloneqq r + LT(p)$$

$$p \coloneqq p - LT(p).$$

Example 2.2.1.

Let $f = xy^2 + xy + y^3 + 1$, $f_1 = y^2 - 1$ and $f_2 = xy + 1$. By using lexicographic order we will apply division algorithm. Hence $LT(f) = xy^2$, $LT(f_1) = y^2$ and $LT(f_2) = xy$. Since $\frac{LT(f)}{LT(f_1)} = x$ we redefine f as

$$f \coloneqq f - xf_1 = xy + x + y^3 + 1.$$

Now $LT(f) = xy$ is not divisible by $LT(f_1)$, but divisible by $LT(f_2)$ so we continue with $\frac{LT(f)}{LT(f_2)} = 1$, we redefine f as $f \coloneqq f - 1. f_2 = x + y^3$.

Now $LT(f) = x$ is divisible neither $LT(f_1)$ nor $LT(f_2)$. Therefore, we move x to the remainder and continue by $f \coloneqq f - x = y^3$.

Then $LT(f)$ is divisible by $LT(f_1)$ so we continue with $\frac{LT(f)}{LT(f_1)}$ and

$$f \coloneqq f - y. f_1 = y.$$

Since y is not divisible by $LT(f_1)$ and $LT(f_2)$, the remainder is $x + y$. Hence

$$f = (x + y).f_1 + f_2 + x + y.$$

Even if we use same monomial order the remainder depends on order of division. In other words remainder is not unique. In our example if we divide f first by f_2 remainder is 0.

We have to find a special generating set for the ideal $< g_1, g_2, \dots, g_s >$ so that the remainder with respect to this set is unique. This generating set is called Gröbner Basis which is defined in the next section.

2.3 Gröbner Basis

Definition 2.3.1.

Let $\boldsymbol{I} \subseteq \boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$ be an ideal. Fix a monomial ordering in $\{\boldsymbol{g_1}, \boldsymbol{g_2}, \dots, \boldsymbol{g_n}\}$ is called Gröbner basis for $\boldsymbol{I}$ if

$$< \boldsymbol{LM}(\boldsymbol{g_1}), \dots, \boldsymbol{LM}(\boldsymbol{g_s}) > = < \boldsymbol{LM}(\boldsymbol{f}) : \boldsymbol{f} \in \boldsymbol{I} >.$$

Now, we give two important properties of Gröbner basis.

Proposition 2.3.2. (Adams W W and Loustaunau P (1994))

If $\{g_1, g_2, \dots, g_n\}$ *is a Gröbner basis for an ideal* $I \subseteq k[x_1, \dots, x_n]$, *then* $I =< g_1, \dots, g_n >$.

Theorem 2.3.3. (Adams W W and Loustaunau P (1994))

Let $\boldsymbol{G} = \{\boldsymbol{g_1}, \boldsymbol{g_2}, \dots, \boldsymbol{g_n}\}$ *be a Gröbner basis for an ideal* $\boldsymbol{I} \subseteq \boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$. *If* $\boldsymbol{f} \in \boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$, *then the remainder of* $\boldsymbol{f}$ *upon division by* $\boldsymbol{G}$ *is unique.*

The next step is to give a criterion for a Gröbner basis. First, we have to define a special polynomial which is called as $S - polynomial$.

Definition 2.3.4 ($\boldsymbol{S}$-polynomial).

Let $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$.

- If $multidegree(f) = \alpha$ and $multidegree(g) = \beta$, then $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ where $\gamma_i = \max\{\alpha_i, \beta_i\}$. We call x^γ the least common multiple of $LM(f)$ and $LM(g)$ written

$$x^\gamma = LCM\big(LM(f), LM(g)\big).$$

- The S-polynomial of f and g is

$$S(f, g) = \frac{x^\gamma}{LT(f)} f - \frac{x^\gamma}{LT(g)} g$$

Theorem 2.3.5 (Buchberger's Criterion). (Adams W W and Loustaunau P (1994),Theorem 1.6.7)

Suppose $\boldsymbol{G} = \{\boldsymbol{g_1}, \boldsymbol{g_2}, \dots, \boldsymbol{g_s}\}$ *is a generating set for an ideal* $\boldsymbol{I}$. *Then* $\boldsymbol{G}$ *is a Gröbner basis of I if and only if for all* $\boldsymbol{1} \le \boldsymbol{i} < \boldsymbol{j} \le \boldsymbol{s}$ *the remainder of* $\boldsymbol{S}(\boldsymbol{g_i}, \boldsymbol{g_j})$ *upon division by* $\boldsymbol{G}$ *is zero.*

2.4 Buchberger Algorithm

Buchberger's criterion suggests the following algorithm for finding a Gröbner basis of an ideal from a given generating set of this ideal.

Theorem 2.4.1 (Buchberger's Algorithm).

Let $\boldsymbol{G} = \{\boldsymbol{f_1}, \boldsymbol{f_2}, \dots, \boldsymbol{f_s}\} \subseteq \boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$ *and* $\boldsymbol{I}$ *be the ideal generated by* $\boldsymbol{G}$*. Compute an* $\boldsymbol{S - polynomial}\ \boldsymbol{S}(\boldsymbol{f_i}, \boldsymbol{f_j})$ *for* $\boldsymbol{i \neq j}$ *and divide it by* $\boldsymbol{G}$*. If remainder is not zero, then enlarge* $\boldsymbol{G}$ *by the remainder. Repeat this process until all of the remainders of the* $\boldsymbol{S - polynomials}$ *zero. The result is a Gröbner basis for* $\boldsymbol{I}$*.*

Using the ascending chain condition of ideals in $k[x_1, \dots, x_n]$, one can show that this algorithm terminates after finitely many steps.

Definition 2.4.2.

A Gröbner basis $\boldsymbol{G}$ of an ideal is called minimal if for each

$$f \in G, LM(f) \neq LM(g)\ for\ all\ g \in G \backslash \{f\}.$$

A minimal Gröbner basis G is called reduced if for each $f \in G$, no term of f is divisible by $LM(g)$ for all $g \in G\backslash\{f\}$.

A minimal Gröbner basis can be obtained from a Gröbner basis by simply keeping only one of the polynomials with same leading monomial and dropping other from the bases.

A reduced Gröbner basis can be obtained from a minimal Gröbner basis by dividing each polynomial in the basis by other polynomials of the basis and replacing original polynomial by the remainder.

Example 2.4.3.

We find a Gröbner basis for $< x^2y + z, xz + y > \subseteq Q[x, y, z]$ with respect to deglex with $x > y > z$.

Let $f_1 = x^2y + z$ and $f_2 = xz + y \in Q[x, y, z]$.

Compute S − polynomials.

$$S(f_1, f_2) = z.f_1 - xyf_2 = -xy^2 + z^2 = f_3$$
$$S(f_1, f_3) = yf_1 + xf_3 = -xz^2 + yz = -zf_2 + 0.$$

Remainder is zero so $S(f_1, f_3)$ do not produce a new element.

$$S(f_2, f_3) = y^2f_2 + zf_3 = y^3 + z^3 = f_4$$
$$S(f_1, f_4) = y^2f_1 - x^2f_4 = -x^2z^3 + y^2z = (-xz^2 + yz)f_2 + 0.$$

Remainder is zero so $S(f_1, f_4)$ do not produce a new element.

$$S(f_2, f_4) = y^3f_2 - xzf_4 = -xz^4 + y^4 = -z^3f_2 + yf_4 + 0\,.$$

Remainder is zero so $S(f_2, f_4)$ do not produce a new element.

$$S(f_3, f_4) = -yf_3 - xf_4 = -xz^3 - yz^2 = -z^2f_2 + 0.$$

Remainder is zero so $S(f_3, f_4)$ do not produce a new element.

Hence, $G = \{f_1, f_2, f_3, f_4\} = \{x^2y + z, xz + y, -xy^2 + z^2, y^3 + z^3\}$ is a Gröbner basis.

2.5 Applications of Gröbner Basis

Definition 2.5.1.

Let $\boldsymbol{k}$ be a field , and let $\boldsymbol{f_1}, \dots, \boldsymbol{f_s}$ be polynomials in $\boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$. Then we set

$$\mathbf{V}(\boldsymbol{f_1}, \dots, \boldsymbol{f_s}) = \{(\boldsymbol{a_1}, \dots, \boldsymbol{a_n}) \in \boldsymbol{k^n} : \boldsymbol{f_i}(\boldsymbol{a_1}, \dots, \boldsymbol{a_n}) = \mathbf{0} \ \boldsymbol{for\ all}\ \mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{s}\}.$$

We call $\mathbf{V}(\boldsymbol{f_1}, \dots, \boldsymbol{f_s})$ as the affine variety defined by $\boldsymbol{f_1}, \dots, \boldsymbol{f_s}$.

In other words the variety of a set of polynomials is in fact the solution set of the corresponding system of polynomial equations. A Gröbner basis for an ideal of a system of polynomial equations lex order simplifies system considerably.

Example 2.5.2.

Let solve the system of polynomial equations;

$$x^2 + y + z = 1$$
$$x + y^2 + z = 1$$
$$x + y + z^2 = 1$$

Consider the ideal;

$$I =< x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1 >$$

We compute the Gröbner basis G for I with respect to the lexicographic ordering $x > y > z$. We get;

$$G = \{x + y + z^2 - 1, y^2 - y - z^2 + z, 2yz^2 + z^4 - z^2, z^6 - 4z^4 + 4z^3 - z^2\}$$

Notice that the last polynomial g_4 involves only variable z, so we start from g_4:

$$z^6 - 4z^4 + 4z^3 - z^2 = z^2(z-1)^2(z^2 + 2z - 1) = 0$$

this equation gives z values :

$$z_1 = 0$$

$$z_2 = 1$$

$$z_3 = -1 + \sqrt{2}$$

$$z_4 = -1 - \sqrt{2}$$

By replacing each z values in equations, we get all possible solutions of the system as below:

$$z = 0 \;\rightarrow\; y = 0 \;\rightarrow\; x = 1$$

$$z = 0 \;\rightarrow\; y = 1 \;\rightarrow\; x = 0$$

$$z = 1 \;\rightarrow\; y = 0 \;\rightarrow\; x = 0$$

$$z = -1 + \sqrt{2} \;\rightarrow\; y = -1 + \sqrt{2} \;\rightarrow\; x = -1 + \sqrt{2}$$

$$z = -1 - \sqrt{2} \;\rightarrow\; y = -1 - \sqrt{2} \;\rightarrow\; x = -1 - \sqrt{2}$$

Definition 2.5.3.

Given $\boldsymbol{I} =< \boldsymbol{f_1}, \dots, \boldsymbol{f_s} > \subset \boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$, the $\boldsymbol{l^{th}}$ elimination ideal $\boldsymbol{I_l}$ is the ideal of $\boldsymbol{k}[\boldsymbol{x_{l+1}}, \dots, \boldsymbol{x_n}]$ defined by

$$I_l = I \cap \; k[x_{l+1}, \dots, x_n].$$

Theorem 2.5.4. (The Elimination Theorem) (Cox D, Little J, O'Shea D (2007), Theorem 3.1.2)

Let $\boldsymbol{I} \subset \boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$ be an ideal and let $\boldsymbol{G}$ be a Gröbner basis of $\boldsymbol{I}$ with respect to lex order where $\boldsymbol{x_1} > \boldsymbol{x_2} > \cdots > \boldsymbol{x_n}$. Then, for every $\boldsymbol{0} \le \boldsymbol{l} \le \boldsymbol{n}$, the set $\boldsymbol{G_l} = \boldsymbol{G} \cap \boldsymbol{k}[\boldsymbol{x_{l+1}}, \dots, \boldsymbol{x_n}]$ is a Gröbner basis of the $\boldsymbol{l^{th}}$ elimination ideal $\boldsymbol{I_l}$.

Theorem 2.5.5. (Cox D, Little J, O'Shea D (2007), Theorem4. 3.11.)

Let $\boldsymbol{I}, \boldsymbol{J}$ *be ideals in* $\boldsymbol{k}[\boldsymbol{x_1}, \dots, \boldsymbol{x_n}]$. *Then*

$$I \cap J = (tI + (1-t)J) \cap k[x_1, \dots, x_n].$$

The above two theorems give to the following method for computing intersections of ideals: If $I = \langle f_1, \dots, f_s \rangle$ and $J = \langle g_1, \dots, g_l \rangle$ are two ideals in $k[x_1 \cdots x_n]$, then consider the ideal

$$\langle tf_1, \dots, tf_s, (1-t)g_1, \dots, (1-t)g_l \rangle$$

and compute the Gröbner basis relative to lex order where $t > x_i$. Then polynomials not involving the variable t form basis for the intersection.

Definition 2.5.6.

If $\boldsymbol{I}, \boldsymbol{J}$ be ideals in $\mathbf{k}[\mathbf{x_1}, \dots, \mathbf{x_n}]$, then $\boldsymbol{I}:\boldsymbol{J}$ is the set

$$\{f \in k[x_1, \dots, x_n]: fg \in I \; for \; all \; g \in J\}$$

and is called the ideal quotient (or colon ideal) of I by J.

Proposition 2.5.7. (Cox D, Little J, O'Shea D (2007), Proposition 4.4.10)

Let I, I_l, J, J_l *and* K *be ideals in* $k[x_1, \dots, x_n]$ *for* $1 \le l \le r$. *Then*

$$I:\left(\sum_{l=1}^{r} J_l\right) = \bigcap_{l=1}^{r} (I:J_l).$$

Theorem 2.5.8. (Cox D, Little J, O'Shea D (2007), Theorem 4.4.11)

Let $\boldsymbol{I}$ be an ideal and $\boldsymbol{g}$ an element of $\boldsymbol{k}[\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}]$. If $\{\boldsymbol{h_1}, \ldots, \boldsymbol{h_p}\}$ is a basis of the ideal $\boldsymbol{I} \cap \langle \boldsymbol{g} \rangle$, then $\{\boldsymbol{h_1}/\boldsymbol{g}, \ldots, \boldsymbol{h_p}/\boldsymbol{g}\}$ is a basis of $\boldsymbol{I}: \langle \boldsymbol{g} \rangle$.

Hence a method for computing ideal quotient can be given as follows: Given $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_l \rangle$ are two ideals in $k[x_1 \cdots x_n]$, first for each g_i compute a generating set for $I: \langle g_i \rangle$ and then compute a basis for the intersection of these generating sets as explained above.

3. ROBOTICS

3.1 Geometric Description of Robots

We restrict ourselves the robots constructed by rigid segments which are connected by joints in series. One end of our robot will be fixed and the other end will have a hand. In general, this hand has a mechanism for grasping objects or for performing some task. Hence, the main goal for a robotic problem is to obtain the position and orientation of the hand.

Many actual robots are constructed using

- Planar revolute joints, and
- Prismatic joints.

A planar revolute joint permits a rotation of one segment relative to another. We will assume that both of the segments in question lie in one plane and all motions of the joint will leave the two segments in that plane.

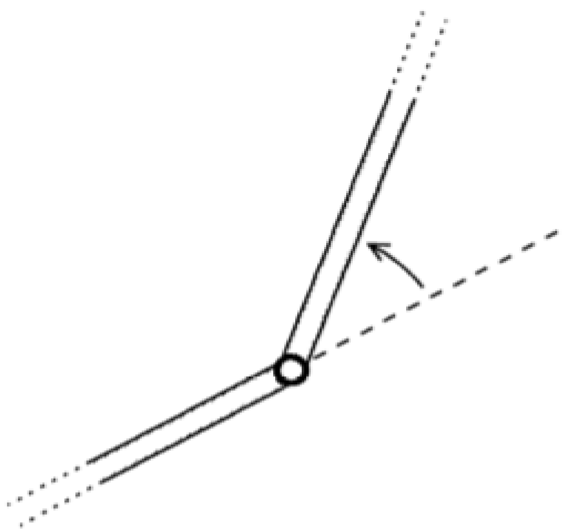

Figure 3.1. Revolute Joint

A prismatic joint permits one segment of a robot to move by sliding or translation along an axis. The following sketch shows a schematic view of a prismatic joint between two segments of a robot lying in a plane. Such a joint permits translational motion along a line in the plane.

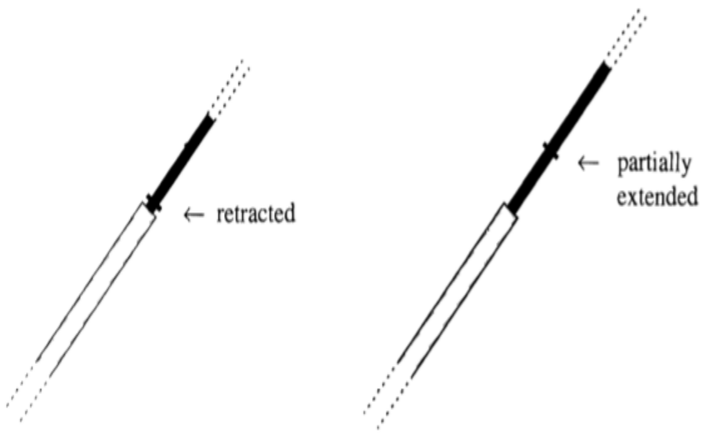

Figure 3.2. Prismatic Joint

Example 3.1.1.

Consider the following planar robot "arm" with three revolute joints and one prismatic joint. All motions of the robot take place in the plane of the paper.

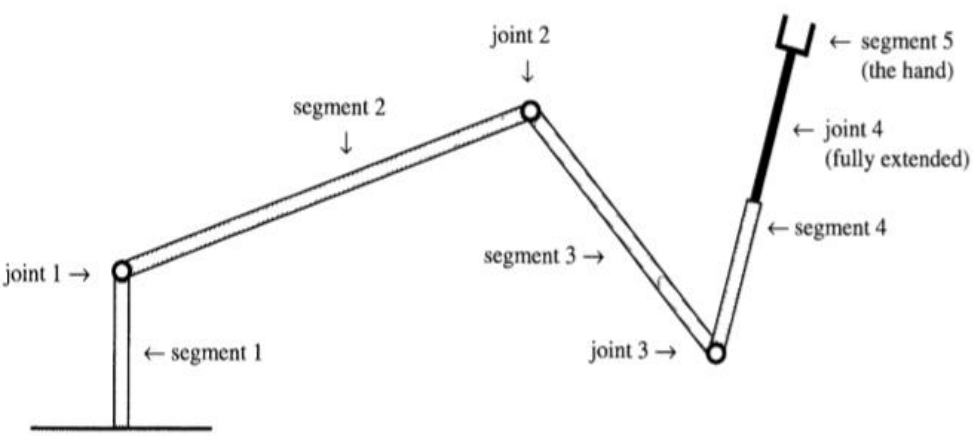

Figure 3.3. Planar Robot "Arm" with Three Revolute Joints and One Prismatic Joint

Suppose that a robot has joint $J_1, J_2, \dots, J_p$. The cartesian product

$$\mathcal{J} = J_1 \times J_2 \times \dots \times J_p$$

is called joint space of the robot.

We can represent patch position of the hand by the point (a, b, c) in $\mathbb{R}^3$ and the orientation of the hand by a unit vector $\vec{u}$ in $\mathbb{R}^3$.

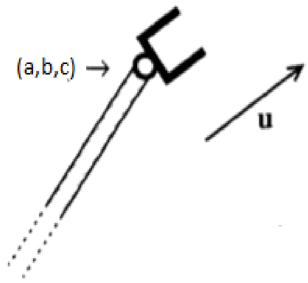

Figure 3.4. Hand Configuration

Let $U \subseteq \mathbb{R}^3$ be the set of all possible points where the hand of the robot can be placed and let V be the set of the all possible unit vectors of orientation of the hand of the robot. The cartesian product $C = U \times V$ is called the configuration space of the hand of the robot.

Hence we can define the following function

$$f: \mathcal{J} \to C \;.$$

In terms of this function there are two types of problem in robotics. First one is forward kinematic problem. In this problem we have explicit description of in terms of joint settings. In other words, coordinates on $\mathcal{J}$ are given. We try to find coordinates in C. The second problem is inverse kinematic problem. In this problem we know the position and the orientation of the hand and try to find positions of joints. More precisely, given $c \in C$, we must find $f^{-1}(c) \subseteq J$. In real world, we generally need solutions of inverse kinematic problem.

3.2 The Inverse Kinematic Problem

We try to explain this problem with an example. Consider the following robot in $\mathbb{R}^3$

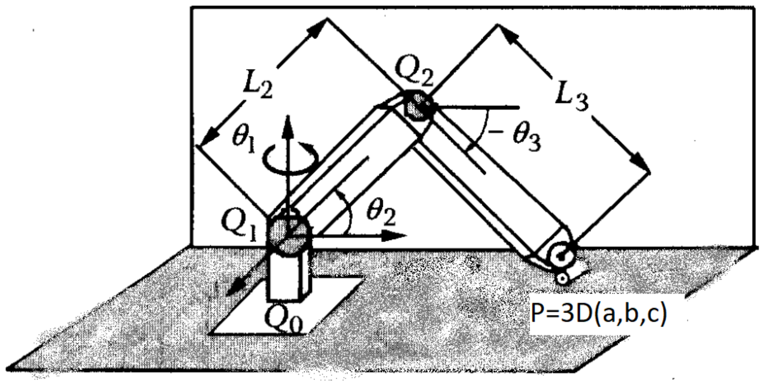

Figure 3.5. Three Arms and Three Degrees Robot Manipulator

There are three segments which are always located in the same vertical plane of $\mathbb{R}^3$. As usual the first segment is anchored. Hence we replace origin of the coordinate system at the end of the first segment. For simplicity, we do not care about the orientation of the hand. We only consider the position of the hand. The angle θ_1 rotates with respect to the axis perpendicular to the ground and determines a vertical plane where the rest of the robot is going to move. Furthermore, θ_2 (respectively, θ_3) is the counter clockwise angle between the first two segments (respectively, the last two segments).

Suppose that $P(a, b, c)$ is the position of the hand. It is easy to verify that

$$a = -\sin(\theta_1)(L_2\cos(\theta_2) + L_3\cos(\theta_3)),$$
$$b = \cos(\theta_1)\,(L_2\cos(\theta_2) + L_3\cos(\theta_3)),$$
$$c = L_2\sin(\theta_2) + L_3\sin(\theta_3).$$

First of all, we need to find an equivalent system of polynomial equations. Let $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ for $i = 1,2,3$. Hence the above system of equation becomes:

$$\text{(I)} \qquad \begin{aligned} a &= -s_1(L_2c_2 + L_3c_3), \\ b &= c_1(L_2c_2 + L_3c_3), \\ c &= L_2s_2 + L_3s_3\ . \end{aligned}$$

We also need to add

$$\text{(II)} \qquad \begin{aligned} c_1^2 + s_1^2 &= 1, \\ c_2^2 + s_2^2 &= 1 \text{ and} \\ c_3^2 + s_3^2 &= 1 \end{aligned}$$

to the system.

To solve this system, we compute the reduced Gröbner basis of the ideal

$$< a + s_1(L_2c_2 + L_3c_3),\ b - c_1\,(L_2c_2 + L_3c_3),\ c - L_2s_2 + L_3s_3, c_1^2 + s_1^2 - 1,$$
$$c_2^2 + s_2^2 - 1, c_3^2 + s_3^2 - 1 > \subseteq \mathbb{R}(a, b, c, L_2, L_3)[s_1, c_1, s_2, c_3]$$

using lexicographic order with respect to $s_3 > c_3 > s_2 > c_2 > s_1 > c_1$.

The solution with Buchberger's Algorithm is as follows:

Let us order the terms of the f_i's ;

$$f_1 = L_3c_3s_1 + L_2c_2s_1 + a$$
$$f_2 = -L_3c_3c_1 - L_2c_2c_1 + b$$
$$f_3 = -L_3s_3 - L_2s_2 + c$$
$$f_4 = {s_1}^2 + {c_1}^2 - 1$$
$$f_5 = s_2^2 + c_2^2 - 1$$
$$f_6 = s_3^2 + c_3^2 - 1$$

Let $F = \{f_1, f_2, f_3, f_4, f_5, f_6\}$. Now we can apply Buchberger's algorithm to F.

$$S(f_1, f_2) = c_1f_1 + s_1f_2 = bs_1 + ac_1.$$

Both term of $bs_1 + ac_1$ are not divisible by *F.* Hence we let $f_7 = bs_1 + ac_1$ and

$$F = F \cup \{f_7\}$$

Clearly $S(f_1, f_3)$ has zero remainder upon division by F. Let us continue

$$S(f_1, f_4) = s_1 f_1 - L_3 c_3 f_4 = -L_3 c_3 c_1^2 - L_3 c_3 + L_2 c_2 s_1^2 + a s_1$$

Applying division algorithm

$$-L_3 c_3 c_1^2 - L_3 c_3 + L_2 c_2 s_1^2 + a s_1 = c_1 f_2 + L_2 c_2 f_4 + \frac{a}{b} f_1 + L_2 c_2 + L_3 c_3 - \frac{a^2 + b^2}{b} + c_1.$$

Let $f_8 = -bL_3 c_3 - bL_2 c_2 + (a^2 + b^2) c_1$ and $F = F \cup \{f_8\}$.

Then $S(f_1, f_5), S(f_1, f_6), S(f_1, f_7), S(f_2, f_3), S(f_2, f_4), S(f_2, f_5), S(f_2, f_6)$ and $S(f_2, f_7)$ produce zero remainders upon division by *F.*

$$S(f_2, f_8) = -bf_2 + c_1 f_8 = -b^2 + a^2 c_1^2 + b^2 c_1^2$$

Let $f_9 = (a^2 + b^2) c_1^2 - b^2$ and $F = F \cup \{f_9\}$

$$S(f_1, f_8) = bf_1 + s_1 f_8 = ab + a^2 c_1 s_1 + b^2 c_1 s_1$$

Applying division algorithm,

$$\frac{a^2 + b^2}{b} c_1 f_7 - \frac{a}{b} f_9$$

Remainder is zero so $S(f_1, f_8)$ do not produce a new element.

$$\begin{aligned} S(f_2, f_9) &= (a^2 + b^2) c_1 f_2 + L_3 c_3 f_9 \\ &= a^2 b c_1 + b^3 c_1 - a^2 c_1^2 c_2 L_2 - b^2 c_1^2 c_2 L_2 - b^2 c_3 L_3 \\ &= b f_8 - c_2 L_2 f_9 + 0 \end{aligned}$$

The last equality follows from the division algorithm.

$$S(f_3, f_6) = -s_3 f_3 - L_3 f_6 = L_3 - c_3^2 L_3 - c s_3 + L_2 s_2 s_3$$

Applying division algorithm,

$$\frac{a^2+b^2}{bL_3}f_2+\left(\frac{c}{L_3}-\frac{L_2s_2}{L_3}\right)f_3-\frac{L_2^2}{L_3}f_5+\left(\frac{c_3}{b}-\frac{c_2L_2}{bL_3}\right)f_8+\frac{2c_1c_2(a^2L_2+b^2L_2)}{bL_3}$$
$$+\frac{-a^2-b^2-c^2-L_2^2+L_3^2}{L_3}+\frac{2cL_2s_2}{L_3}$$

Let $f_{10}=2bcL_2s_2+2c_1c_2(a^2L_2+b^2L_2)+b(-a^2-b^2-c^2-L_2^2+L_3^2)$ and

$$F=F\cup\{f_{10}\}$$

$$S(f_4,f_7)=bf_4-s_1f_7=-b+bc_1^2-ac_1s_1$$

Applying division algorithm,

$$-b+bc_1^2-ac_1s_1=-\frac{ac_1}{b}f_7+\frac{1}{b}f_9+0$$

Remainder is zero, so $S(f_7,f_9)$ do not produce a new element.

$$\begin{aligned}S(f_5,f_{10})&=2cL_2f_5b-s_2f_{10}\\&=-2bcL_2+2bcc_2^2L_2+a^2bs_2+b^3s_2+bc^2s_2-2a^2c_1c_2L_2s_2\\&-2b^2c_1c_2L_2s_2+bL_2^2s_2-bL_3^2s_2\end{aligned}$$

Applying division algorithm,

$$\frac{c_2^2(4a^2L_2^2+4b^2L_2^2)}{2bcL_2}f_9+\left(\frac{(-a^2-b^2)c_1c_2}{bc}+\frac{a^2b+b^3+bc^2+bL_2^2-bL_3^2}{2bcL_2}\right)f_{10}$$
$$+\frac{2c_2^2(a^2bL_2+b^3L_2+bc^2L_2)}{c}$$
$$-\frac{2c_1c_2(a^4+2a^2b^2+b^4+a^2c^2+b^2c^2+a^2L_2^2+b^2L_2^2-a^2L_3^2-b^2L_3^2)}{c}$$
$$+\frac{1}{2cL_2}(a^4b+2a^2b^3+b^5+2a^2bc^2+2b^3c^2+bc^4+2a^2bL_2^2+2b^3L_2^2-2bc^2L_2^2$$
$$+bL_2^4-2a^2bL_3^2-2b^3L_3^2-2bc^2L_3^2-2bL_2^2L_3^2+bL_3^4)$$

Let $f_{11} = c_2^2(4a^2bL_2^2 + 4b^3L_2^2 + 4bc^2L_2^2) + 4c_1c_2L_2(a^4 + 2a^2b^2 + b^4 + a^2c^2 + b^2c^2 + a^2L_2^2 + b^2L_2^2 - a^2L_3^2 - b^2L_3^2) + a^4b + 2a^2b^3 + b^5 + 2a^2bc^2 + 2b^3c^2 + bc^4 + 2a^2bL_2^2 + 2b^3L_2^2 - 2bc^2L_2^2 + bL_2^4 - 2a^2bL_3^2 - 2b^3L_3^2 - 2bc^2L_3^2 - 2bL_2^2L_3^2 + bL_3^4$

One can show that the remaining S-polynomials has zero reaminders upon division by F. Hence $F = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}\}$ is a Gröbner basis. After applying minimalization process as explained in Sectin 2.4,

$$G = \{f_3, f_7, f_8, f_9, f_{10}, f_{11}\}$$

is a minimal Gröbner basis. Then applying reduction process as explained in Section 2.4, the following polynomials form a reduced Gröbner basis:

$$g_1 = c_1^2 - \frac{b^2}{a^2 + b^2},$$

$$g_2 = s_1 + \frac{a}{b}\, c_1,$$

(III)
$$g_3 = c_2^2 - c_1c_2\left(\frac{a^2 + b^2}{L_2b} + \frac{(a^2 + b^2)(L_2^2 - L_3^2)}{L_2b(a^2 + b^2 + c^2)}\right) + \frac{(a^2 + b^2)^2}{4L_2^2(a^2 + b^2 + c^2)} + \frac{c^2(2a^2 + 2b^2 + c^2)}{4L_2^2(a^2 + b^2 + c^2)} + \frac{1}{2} + \frac{L_2^2}{4(a^2 + b^2 + c^2)} - \frac{L_3^2}{2L_2^2} - \frac{L_3^2(2L_2^2 - L_3^2)}{4L_2^2(a^2 + b^2 + c^2)},$$

$$g_4 = s_2 + \frac{(a^2 + b^2)}{b.c} c_1c_2 - \frac{(a^2 + b^2 + c^2)}{2.c.L_2} - \frac{(L_2^2 - L_3^2)}{2.c.L_2},$$

$$g_5 = c_3 + \frac{L_2}{L_3}\, c_2 - \frac{(a^2 + b^2)}{b.L_3} c_1,$$

$$g_6 = s_3 - \frac{L_2(a^2 + b^2)}{b.c.L_3}\, c_1c_2 + \frac{a^2 + b^2 - c^2 + L_2^2 - L_3^2}{2.c.L_3}.$$

Notice that from $g_1 = 0$, c_1 can be solved as a function of a and b. Then replacing this result into $g_2 = 0$, s_1 can be obtained as a function of a and b and so on. Hence from the above reduced Gröbner basis, c_1, s_1, c_2, s_2, c_3 and s_3 can easily solved in terms of free variables a, b, c, L_2 and L_3.

On the other hand, in practice, we give some certain values to these variables. The replacement of variables is called specialization. Next, we will examine how a

Gröbner basis change under specialization. In other words, we take a specialization of the variables a, b, c, L_2, L_3 and write corresponding ideal. Then we compute the reduced Gröbner basis of this ideal using same order as above. Finally, we check whether this new Gröbner basis is same as Gröbner basis given in (III).

Let $L_2 = L_3 = 1$. The ideal becomes;

$< f_1 = a + s_1(l_2c_2 + l_3c_3), f_2 = b - c_1(l_2c_2 + l_3c_3), f_3 = c - l_2s_2 - l_3s_3, f_4 = {c_1}^2 + {s_1}^2 - 1, f_5 = {c_2}^2 + {s_2}^2 - 1, f_6 = {c_3}^2 + {s_3}^2 - 1 >$

If we compute the Gröebner Basis of the ideal, then we get :

$$\{g_1 = -b^2 + (a^2 + b^2)c_1^2, g_2 = ac_1 + bs_1, g_3 = -a^4b - 2a^2b^3 - b^5 - 2a^2bc^2 - 2b^3c^2 - bc^4 + 4bc^2l^2 + (4a^4l + 8a^2b^2l + 4b^4l + 4a^2c^2l + 4b^2c^2l)c_1c_2 + (-4a^2bl^2 - 4b^3l^2 - 4bc^2l^2)c_2^2, g_4 = a^2b + b^3 + bc^2 + (-2a^2l - 2b^2l)c_1c_2 - 2bcls_2, g_5 = (a^2 + b^2)c_1 - blc_2 - blc_3, g_6 = a^2b + b^3 - bc^2 + (-2a^2l - 2b^2l)c_1c_2 + 2bcls_3\}$$

If we replace $L_2 = L_3 = 1$ in (III) then we obtain same result. (III) remains Gröbner basis under this specialization . Using the Gröbner basis, the set of solutions for this specialization is the followings:

Solution (1)

$$c_1 = \frac{b}{\sqrt{a^2 + b^2}}$$

$$s_1 = -\frac{a}{\sqrt{a^2 + b^2}}$$

$$c_2 = \frac{a^4 + b^4 + b^2c^2 + a^2(2b^2 + c^2) - \sqrt{-(a^2 + b^2)c^2(a^4 + b^4 + c^2(-4 + c^2) + 2b^2(-2 + c^2) + 2a^2(-2 + b^2 + c^2))}}{2\sqrt{a^2 + b^2}(a^2 + b^2 + c^2)}$$

$$s_2 = \frac{a^2c^2+b^2c^2+c^4+\sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2c(a^2+b^2+c^2)}$$

$$c_3 = \frac{a^4+b^4+b^2c^2+a^2(2b^2+c^2)+\sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))})}{2\sqrt{a^2+b^2}(a^2+b^2+c^2)}$$

$$s_3 = \frac{a^2c^2+b^2c^2+c^4-\sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2c(a^2+b^2+c^2)}$$

Solution (2)

$$c_1 = -\frac{b}{\sqrt{a^2+b^2}}$$

$$s_1 = \frac{a}{\sqrt{a^2+b^2}}$$

$$c_2 = \frac{-a^4-b^4-b^2c^2-a^2(2b^2+c^2)-\sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))})}{2\sqrt{a^2+b^2}(a^2+b^2+c^2)}$$

$$s_2 = \frac{a^2c^2+b^2c^2+c^4-\sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2c(a^2+b^2+c^2)}$$

$$c_3 = \frac{-a^4-b^4-b^2c^2-a^2(2b^2+c^2)+\sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2\sqrt{a^2+b^2}(a^2+b^2+c^2)}$$

$$s_3 = \frac{a^2c^2+b^2c^2+c^4+\sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2c(a^2+b^2+c^2)}$$

Solution (3)

$$c_1 = -\frac{b}{\sqrt{a^2+b^2}}$$

$$s_1 = \frac{a}{\sqrt{a^2 + b^2}}$$

$$c_2 = \frac{-a^4 - b^4 - b^2c^2 - a^2(2b^2 + c^2) - \sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))})}{2\sqrt{a^2+b^2}(a^2+b^2+c^2)}$$

$$s_2 = \frac{a^2c^2 + b^2c^2 + c^4 + \sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2c(a^2+b^2+c^2)}$$

$$c_3 = \frac{-a^4 - b^4 - b^2c^2 - a^2(2b^2 + c^2) + \sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2\sqrt{a^2+b^2}(a^2+b^2+c^2)}$$

$$s_3 = \frac{a^2c^2 + b^2c^2 + c^4 - \sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2c(a^2+b^2+c^2)}$$

Solution (4)

$$c_1 = \frac{b}{\sqrt{a^2 + b^2}}$$

$$s_1 = -\frac{a}{\sqrt{a^2 + b^2}}$$

$$c_2 = \frac{a^4 + b^4 + b^2c^2 + a^2(2b^2 + c^2) + \sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2\sqrt{a^2+b^2}(a^2+b^2+c^2)}$$

$$s_2 = \frac{a^2c^2 + b^2c^2 + c^4 - \sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2c(a^2+b^2+c^2)}$$

$$c_3 = \frac{a^4 + b^4 + b^2c^2 + a^2(2b^2 + c^2) - \sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))})}{2\sqrt{a^2+b^2}(a^2+b^2+c^2)}$$

$$s_3 = \frac{a^2c^2 + b^2c^2 + c^4 + \sqrt{-(a^2+b^2)c^2(a^4+b^4+c^2(-4+c^2)+2b^2(-2+c^2)+2a^2(-2+b^2+c^2))}}{2c(a^2+b^2+c^2)}$$

Let $L_2 = L_3 = 1$ and $b = 0$, but there is an algebraic problem since some denominators in (III) vanish at $b = 0$. So (III) can not be a Gröbner basis for this specialization. In such a situation we must substitute $b = 0$ and $L_2 = L_3 = 1$ into our ideal and then recompute the Gröbner basis. Under this specialization the new basis is

$\{g_1 = c_1,\ g_2 = -1 + s_1^2,\ g_3 = a^4 - 4c^2 + 2a^2c^2 + c^4 + (4a^2 + 4c^2)c_2^2 + (4a^3 + 4ac^2)c_2s_1,\ g_4 = -a^2 - c^2 - 2ac_2s_1 + 2cs_2,\ g_5 = -c_2 - c_3 - as_1,\ g_6 = -a^2 + c^2 - 2ac_2s_1 - 2cs_3\}$

By using this Gröbner basis, we find the possible solutions.

Solution (1)

$$c_1 = 0$$

$$s_1 = -1$$

$$c_2 = \frac{a^3 + ac^2 + \sqrt{4a^2c^2 - a^4c^2 + 4c^4 - 2a^2c^4 - c^6}}{2(a^2 + c^2)}$$

$$s_2 = \frac{a^2c^2 + c^4 - a\sqrt{-c^2(-4a^2 + a^4 - 4c^2 + 2a^2c^2 + c^4)}}{2a^2c + 2c^3}$$

$$c_3 = \frac{a^3 + ac^2 - \sqrt{-c^2(a^4 + c^2(-4 + c^2) + 2a^2(-2 + c^2))}}{2(a^2 + c^2)}$$

$$s_3 = \frac{a^2c^2 + c^4 + a\sqrt{-c^2(-4a^2 + a^4 - 4c^2 + 2a^2c^2 + c^4)}}{2a^2c + 2c^3}$$

Solution (2)

$$c_1 = 0$$

$$s_1 = -1$$

$$c_2 = \frac{a^3 + ac^2 - \sqrt{4a^2c^2 - a^4c^2 + 4c^4 - 2a^2c^4 - c^6}}{2(a^2 + c^2)}$$

$$s_2 = \frac{a^2c^2 + c^4 + a\sqrt{-c^2(-4a^2 + a^4 - 4c^2 + 2a^2c^2 + c^4)}}{2a^2c + 2c^3}$$

$$c_3 = \frac{a^3 + ac^2 + \sqrt{-c^2(a^4 + c^2(-4 + c^2) + 2a^2(-2 + c^2))}}{2(a^2 + c^2)}$$

$$s_3 = \frac{a^2c^2 + c^4 - a\sqrt{-c^2(-4a^2 + a^4 - 4c^2 + 2a^2c^2 + c^4)}}{2a^2c + 2c^3}$$

Solution (3)

$$c_1 = 0$$

$$s_1 = 1$$

$$c_2 = \frac{-a^3 - ac^2 - \sqrt{4a^2c^2 - a^4c^2 + 4c^4 - 2a^2c^4 - c^6}}{2(a^2 + c^2)}$$

$$s_2 = \frac{a^2c^2 + c^4 - a\sqrt{-c^2(-4a^2 + a^4 - 4c^2 + 2a^2c^2 + c^4)}}{2a^2c + 2c^3}$$

$$c_3 = -\frac{a^3 + ac^2 - \sqrt{-c^2(a^4 + c^2(-4 + c^2) + 2a^2(-2 + c^2))}}{2(a^2 + c^2)}$$

$$s_3 = \frac{a^2c^2 + c^4 + a\sqrt{-c^2(-4a^2 + a^4 - 4c^2 + 2a^2c^2 + c^4)}}{2a^2c + 2c^3}$$

Solution (4)

$$c_1 = 0$$

$$s_1 = 1$$

$$c_2 = \frac{-a^3 - ac^2 + \sqrt{4a^2c^2 - a^4c^2 + 4c^4 - 2a^2c^4 - c^6}}{2(a^2 + c^2)}$$

$$s_2 = \frac{a^2c^2 + c^4 + a\sqrt{-c^2(-4a^2 + a^4 - 4c^2 + 2a^2c^2 + c^4)}}{2a^2c + 2c^3}$$

$$c_3 = -\frac{a^3 + ac^2 + \sqrt{-c^2(a^4 + c^2(-4 + c^2) + 2a^2(-2 + c^2))}}{2(a^2 + c^2)}$$

$$s_3 = \frac{a^2c^2 + c^4 - a\sqrt{-c^2(-4a^2 + a^4 - 4c^2 + 2a^2c^2 + c^4)}}{2a^2c + 2c^3}$$

As we can see from above example, for a specialization that makes some denominators zero the original Gröbner basis is no longer a Gröbner basis. In the next chapter we try to find all specializations which causes this phenomenon.

4. GRÖBNER BASIS UNDER SPECIALIZATION

As the example from previous chapter shows that one can expect problems when a specialization causes any of the denominators in Gröbner basis to vanish. However, vanishings of denominators are not the only specializations that create the problem. In this chapter, the question that we try to answer is how to determine which specializations apart from vanishings of denominators are the bad ones.

The outlines of ideas given in this section suggested by (Cox et al, 2007, Chapter 6). However, they did not give any proof. We detailed their suggestions and prove the claims.

Lemma 4.1.

Let $I = < f_1, f_2, \dots, f_s > \subseteq k(t_1, \dots, t_m)[x_1, \dots, x_n]$ *such that each* f_i *is monic polynomial. Furthermore, suppose that* $\{g_1, g_2, \dots, g_t\}$ *is the reduced Gröbner basis under a choosen order. Finally, let* $(t_1, t_2, \dots, t_m) \to (a_1, a_2, \dots, a_m) \in k^m$ *be a specialization of the parameters for which the denominators of neither the* f_i*'s nor* g_s*'s vanish at* $(a_1, a_2, \dots, a_m)$.

(i) There exist polynomials $A_{ij} \in k(t_1, \dots, t_m)[x_1, \dots, x_n]$ *such that* $f_i = \sum_{j=1}^{t} A_{ij}\, g_j$, $(1 \leq i \leq s)$. *Furthermore, none of the denominators of* A_{ij} *vanish at* $(a_1, a_2, \dots, a_m)$.

(ii) If there are polynomials $B_{ji} \in k(t_1, \dots, t_m)[x_1, \dots, x_n]$ *such that* $g_j = \sum_{i=1}^{s} B_{ji}\, f_i$, $(1 \leq j \leq t)$ *and none of the denominators of* B_{ji} *vanish at* $(a_1, a_2, \dots, a_m)$, *then* $\{g_1(a_1, \dots, a_n), g_2(a_1, \dots, a_n), \dots, g_t(a_1, \dots, a_n)\}$ *is a Gröbner basis for* $\langle f_1(a_1, \dots, a_n), f_2(a_1, \dots, a_n), \dots, f_s(a_1, \dots, a_n)\rangle$.

Proof.

(i) Since $\{g_1, g_2, \dots, g_t\}$ is a Gröbner basis for I and $f_i \in I$, clearly there are polynomials $A_{ij} \in k(t_1, \dots, t_m)[x_1, \dots, x_n]$ such that

$$f_i = \sum_{j=1}^{t} A_{ij}\, g_j, \ (1 \leq i \leq s).$$

These polynomials occur during the division process of f_i by $\{g_1, \dots, g_t\}$. Since $\{g_1, g_2, \dots, g_t\}$ is the reduced Gröbner basis, $g_j's$ are monic polynomials. Hence the set of denominators of $A_{ij}'s$ are equal to the set of denominators of $f_i's$ and this implies denominators of $A_{ij}'s$ do not vanish at $(a_1, a_2, \dots, a_m)$.

(ii) Since none of the denominators of $A_{ij}'s$ vanish at $(a_1, a_2, \dots, a_m)$. $f_i(a_1, \dots, a_m) = \sum_{j=1}^{t} A_{ij}\,(a_1, \dots, a_m) g_j(a_1, \dots, a_m);\ 1 \leq i \leq s$. So

$$\langle f_1(a_1, \dots, a_m), \dots, f_s(a_1, \dots, a_m)\rangle \subseteq \langle g_1(a_1, \dots, a_m), \dots, g_t(a_1, \dots, a_m)\rangle.$$

Similarly $g_j(a_1, \dots, a_m) = \sum_{i=1}^{s} B_{ji}\,(a_1, \dots, a_m) f_i(a_1, \dots, a_m);\ 1 \leq j \leq t$ and $\langle f_1(a_1, \dots, a_m), \dots, f_s(a_1, \dots, a_m)\rangle \subseteq \langle g_1(a_1, \dots, a_m), \dots, g_t(a_1, \dots, a_m)\rangle$. Now we have to show that $\bar{G} = \{(a_1, \dots, a_m), \dots, g_t(a_1, \dots, a_m)\}$ is a Gröbner basis for $\bar{I} = \langle f_1(a_1, \dots, a_m), \dots, f_s(a_1, \dots, a_m)\rangle \subseteq k[x_1, \dots, x_n]$.

Consider $S(g_i, g_j)$ in $k(t_1, \dots, t_m)[x_1, \dots, x_n]$. Since $G = \{g_1, \dots, g_t\}$ is a Gröbner basis, the division of $S(g_i, g_j)$ by G produce a zero remainder. By construction of $S(g_i, g_j)$, none of the denominators of $S(g_i, g_j)$ vanish at $(a_1, \dots, a_m)$. Since all $g_j's$ are monic polynomials we will not get any denominator vanishing at $(a_1, \dots, a_m)$. Hence the division of $S(g_i(a_1, \dots, a_m), g_j(a_1, \dots, a_m))$ by $\bar{G}$ produce a zero divisor. By Buchberger's algorithm, $\bar{G}$ is a Gröbner basis. ■

The following is an easy consequence of the above lemma.

Corollary 4.2.

Let $\boldsymbol{d_1, \dots, d_m} \in \boldsymbol{k[t_1, \dots, t_m]}$ *be all denominators which appear among* $\boldsymbol{f_i}$, $\boldsymbol{g_j}$ *and* $\boldsymbol{B_{ji}}$, *and let* $\boldsymbol{W = V(d_1.d_2 \cdots d_m) \subset k^m}$. *Then* $\{\boldsymbol{g_1, \dots, g_t}\}$*is a Gröbner basis for* $< \boldsymbol{f_1, \dots, f_s} >$ *under all specializations*

$$(t_1, \dots, t_m) \to (a_1, \dots, a_m) \in k^m - W.$$

The next goal is to find the variety W. The problem is that the polynomial's $B'_{ji}s$ can not be obtained by division algorithm since $\{f_1, \dots, f_s\}$ is not a Gröbner basis. Because of this, we try to find a method to obtain W without computing $B'_{ji}s$.

Lemma 4.3.

Multiplying each f_i *and* g_j *by appropriate polynomials in* $k[t_1, \dots, t_m]$, *we obtain* $\tilde{f}_i$, $\tilde{g}_j \in k[t_1, \dots, t_m, x_1, \dots, x_n]$.

Let $\tilde{I} = \langle \tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_s \rangle \subseteq k[t_1, \dots, t_m, x_1, \dots, x_n]$.

If $d \in k[t_1, \dots, t_m]$ *is polynomial which clears all denominators for the* f_i, *the* g_j, *and the* $B_{ji}{}'s$, *then*

$$d \in \left(\tilde{I} \colon \tilde{g}_j\right) \cap k[t_1, \dots, t_m].$$

Proof .

Clearly

$$d\tilde{g}_j = d \sum_{i=1}^{s} B_{ji}\, \tilde{f}_i = \sum_{i=1}^{s} d\, B_{ji} f_i.$$

Since d clears all denominators for the f_i, the g_j, and the $B_{ji}{}'s$,

$$d\tilde{g}_j \in \langle \tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_s \rangle.$$

Hence $d \in \tilde{I}: \tilde{g}_j$.

The assertation easily follows since $d \in k[t_1, \dots, t_m]$. ■

Now we came the main result of this section.

Theorem 4.4.

$$W = V\left[\bigcap_{i=1}^{t}\left(\tilde{I}: \langle\tilde{g}_j\rangle\right) \cap k[t_1, \dots, t_m]\right].$$

Proof .

Let $e = d_1 d_2 \dots d_m$ be the product of all denominator of the f_i, the g_j and the $B'_{ji}s$. Clearly $e \in \left[\bigcap_{j=1}^{t}\left(\tilde{I}: \tilde{g}_j\right)\right] \cap k[t_1, \dots, t_m]$ by the above lemma.

On the other hand if $d \in \left[\bigcap_{j=1}^{t}\left(\tilde{I}: \langle\tilde{g}_j\rangle\right)\right] \cap k[t_1, \dots, t_m]$, then $d \in k[t_1, \dots, t_m]$ and $d\tilde{g}_j = \sum_{i=1}^{s} d\, B_{ji}\tilde{f}_i \in \tilde{I} \subseteq k[t_1, \dots, t_m, x_1, \dots, x_n]$.

This implies d clear all denominators $d_1, d_2, \dots, d_m$. That means for each d_i there is $h_i \in k[t_1, \dots, t_m, x_1, \dots, x_n]$ such that $d = h_i d_i$. Therefore $d = h_1 h_2 \dots h_m e$ and so $d \in < e >$.

Hence $< e > = \left[\bigcap_{j=1}^{t}\left(\tilde{I}: \langle\tilde{g}_j\rangle\right)\right] \cap k[t_1, \dots, t_m]$ which implies

$$V(e) = W = V\left(\left[\bigcap_{i=1}^{t}\left(\tilde{I}: \langle g_j\rangle\right)\right] \cap k[t_1, \dots, t_n]\right). ■$$

Using Gröbner basis theory, the ideal $\left[\bigcap_{i=1}^{t}\left(\tilde{I}: g_j\right)\right] \cap k[t_1, \dots, t_n]$ can be computed. Hence specializations other than which makes denominators zero are obtained unless the ideal is whole polynomial ring. In this case, we say that specializations which makes some of the denominators zero are only specializations for which the new Gröbner basis is not same as original Gröbner basis.

Let us apply this process to our example. Hence

$$\tilde{I} = \langle h_1, h_2, h_3, h_4, h_5, h_6 \rangle$$

where

$$h_1 = (a^2 + b^2)c_1^2 - b^2$$
$$h_2 = ac_1 + bs_1$$
$$\begin{aligned} h_3 = (&4a^4L_2 + 8a^2b^2L_2 + 4b^4L_2 + 4a^2c^2L_2 + 4b^2c^2L_2 + 4a^2L_2^3 + 4b^2L_2^3 \\ &- 4a^2L_2L_3^2 - 4b^2L_2L_3^2)c_1c_2 + (-4a^2bL_2^2 - 4b^3L_2^2 - 4bc^2L_2^2)c_2^2 \\ &- a^4b - 2a^2b^3 - b^5 - 2a^2bc^2 - 2b^3c^2 - bc^4 - 2a^2bL_2^2 - 2b^3L_2^2 \\ &+ 2bc^2L_2^2 - bL_2^4 \end{aligned}$$
$$h_4 = a^2b + b^3 + bc^2 + bL_2^2 + c_1c_2(-2a^2L_2 - 2b^2L_2) - bL_3^2 - 2bcL_2s_2$$
$$h_5 = (a^2 + b^2)c_1 - bc_2L_2 - bc_3L_3$$
$$h_6 = a^2b + b^3 - bc^2 + bL_2^2 + c_1c_2(-2a^2L_2 - 2b^2L_2) - bL_3^2 + 2bcL_3s_3$$

The computations of the colon ideals and the intersection of ideals explained in Chapter 2. Because of this here we only give the results of the computations without details.

Therefore $\tilde{I}: \langle h_i \rangle = h_i$ for $1 \leq i \leq 5$ and $\tilde{I}: \langle h_6 \rangle = k[s_1, c_1, \dots, l_2, l_3]$. Hence $\left(\bigcap_{i=1}^{6} \left(\tilde{I}: \langle h_j \rangle\right)\right) \cap k[s_1, c_1, s_2, c_2, s_3, c_3, l_2, l_3] = k[s_i c_i, \dots, l_2, l_3]$ which implies

$$W = V\left(\bigcap_{i=1}^{6} \left(\tilde{I}: g_j\right) \cap k[s_1, c_1, s_2, c_2, s_3, c_3, l_2, l_3]\right) = \emptyset.$$

This means that for this example there is no need to consider other specializations except for which makes denominators zero.

Let us finish this chapter with an example to show that a specialization which does not cause any of the denominators in Gröbner basis to vanish can be still a bad one.

Consider the ideal $I = \langle x + t\, y, x + y \rangle \subseteq k(t)[x, y]$. It is easy to show that $G = \{x, y\}$ is a Gröbner basis with respect to lex order. Notice that there is no denominators in either of the bases. On the other hand,

$$I: \langle x \rangle = I: \langle y \rangle = \langle x + y, t - 1 \rangle.$$

Since $(I: \langle x \rangle \cap I: \langle y \rangle) \cap k[t] = t - 1$, the specialization $t = 1$ cause the problem. In fact, if we put $t = 1$ in the original ideal, it becomes $I = \langle x + y \rangle$ which clearly have a different solution set.

5. COMPREHENSIVE GRÖBNER SYSTEM

The concept of comprehensive Gröbner basis was developed for finding solutions of parameterized polynomial systems. For a parametric polynomial ideal, a basis is called comprehensive Gröbner basis if for every specialization of its parameters, the specialization of the basis is a Gröbner basis of the associated specialized polynomial ideal. In many engineering problems such as inverse kinematics problem only set of specialization of parameters and corresponding Gröbner bases are needed. The basis of the original parametric ideal is unnecessary in most cases. This leads us to the concept of comprehensive Gröbner system. The difficulty of computing a comprehensive Gröbner basis of a parametric ideal is that all the polynomials in this comprehensive Gröbner basis should be belong to the ideal, while the polynomials in a comprehensive Gröbner system does not necessarily have to be belong to ideal. Hence we only study the comprehensive Gröbner systems in this chapter. Kapur et al (2013) gave an efficient method for computing comprehensive Gröbner bases and systems. We have just examined in details the algorithm for computing comprehensive Gröbner system from this article.

Let k be an algebraically closed field and $R = k[t_1, \dots, t_m]$. Consider the polynomial ring $R[x_1, x_2, \dots, x_n]$. Here we assume that x_i's and $t_j's$ are distinct variables. While working on $k[t_1, \dots, t_m, x_1, \dots, x_n]$ we use a block monomial order $\ll$ that $t_j \ll x_i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. On the other hand, if we consider the monomials on $R[x_1, \dots, x_n]$, then we restrict the monomial order $\ll$ to $\{x_1, \dots, x_n\}$.

A specialization of R is a homomorphism $\sigma: R \to k$. In this section, we only consider the specializations induced by elements in k^m. More precisely for any $a = (a_1, \dots, a_m) \in k^m$ we define $\sigma_a: R \to k$ by $f \to f(a)$. Notice that we can extend σ_a to $\sigma_{\bar{a}} : R[x_1, \dots, x_n] \to k[x_1, \dots, x_n]$ by applying σ_a coefficientwise.

Definition 5.1

Let F be a subset of $R[x_1, x_2, \ldots, x_n]$ and $A_1, A_2, \ldots, A_l$ be a subset of k^m such that $k^m \in A_1 \cup A_2 \cup \ldots \cup A_l$. A finite set $\mathcal{G} = \{(A_1, G_1), (A_2, G_2), \ldots, (A_l, G_l)\}$ is called a comprehensive Gröbner system for $< F >$ if $\sigma_{\bar{a}}(G_i)$ is a Gröbner basis for the ideal $< \sigma_{\bar{a}}(F) > \subseteq k[x_1, \ldots, x_n]$ for any $a \in A_i$ and $1 \leq i \leq l$. Each (A_i, G_i) is called a branch of $\mathcal{G}$.

Since we will work on $R[x_1, x_2, \ldots, x_n]$ where R is not a field but a ring, we have to redefine S- polynomial and a reduction step in division algorithm. The detailed information about Gröbner bases over a ring can be found in (Adams and Loustaunau, 1994, Chapter 4).

Definition 5.2

Let $p, q \in R[x_1, x_2, \ldots, x_n]$.

(i) The S-polynomial of p and q is defined as

$$S(p,q) = \frac{LC(q)x^\gamma}{LM(p)}p - \frac{LC(p)x^\gamma}{LM(q)}q \ \ where \ x^\gamma = LCM\big(LM(p), LM(q)\big).$$

(ii) If $LM(p)$ divides $LM(q)$, then the reduction of q with respect to p is defined as

$$LC(p)q - LC(q)\frac{LM(q)}{LM(p)}p.$$

Example 5.3

Let $p = 2t_1t_2x_1 + t_1^2x_2,\ q = t_1^2x_1x_2 + t_2^2x_1 \in k[t_1, t_2][x_1, x_2]$. Suppose that we use a block order $<$ with $\{t_1, t_2\} < \{x_1, x_2\}$ and within block use lexicographic order. Hence

$LC(p) = 2t_1t_2,\ LM(p) = x_1,\ LC(q) = t_1^2,\ LM(q) = x_1x_2$. Hence

$$S(p,q) = \frac{t_1^2 x_1 x_2}{x_1}(2t_1t_2 - t_1^2 x_2) - \frac{2t_1t_2x_1}{x_1x_2}(t_1^2x_1x_2 + t_2^2x_1)$$

$$S(p,q) = t_1^4x_2^2 - 2t_1t_2^3x_1.$$

Furthermore, the reduction of q with respect to p is

$$2t_1t_2(t_1^2x_1x_2 + t_2^2x_1) - t_1^2\frac{x_1x_2}{x_1}(2t_1t_2x_1 + t_1^2x_2) = 2t_1t_2^3 - t_1^4x_2^2.$$

Now, we try to give an algorithm for computing a comprehensive Gröbner system, but first we need some results. The following theorem and its corollaries are modified versions of the corresponding theorem and corollaries in Kapur et al (2013) accordance with our purposes.

Theorem 5.4. (Kapur et al (2013), Theorem 4.1)

Given a Gröbner basis $\boldsymbol{G}$ for an ideal $<\boldsymbol{F}>\subseteq \boldsymbol{R}[\boldsymbol{x_1},\dots,\boldsymbol{x_n}]$ with respect to a monomial order $<$ and a specialization $\boldsymbol{\sigma}:\boldsymbol{R}\to\boldsymbol{k}$, let

$$G_m = \{g \in G \mid \sigma(LC(g)) \neq 0\}.$$

Then $\sigma(G_m) = \{\sigma(g) \mid g \in G_m\}$ is a Gröbner basis for $<\sigma(F)>$ in $k[x_1,\dots,x_n]$.

The following corollaries are usefull to define the algorithm for a comprehensive Gröbner system.

Corollary 5.5. (Kapur et al (2013), Corollary 4.4)

Let G be a Gröbner basis for the ideal $<F>\subset R[x_1,\dots,x_n]$, $G_0 \subseteq G$, and $G_r = \{LC(g) \mid g \in G_0\}$. Furthermore, suppose that $G_m \subseteq G \setminus G_0$ such that

$< LM(G_m) >=< LM(G \setminus G_0) >$ *and* $LM(G_m)$ *is minimal set of monomials. If* $\sigma: R \to k$ *is specialization such that*

(i) $\sigma(g) = 0$ *for* $g \in G_r$ *and*

(ii) $\sigma(h) \neq 0$, *where* $h = \prod_{g \in G_m} LC(g)$

then $\sigma(G_m)$ *is Gröbner basis for* $< \sigma(F) >$.

If we select $G_0 = G \cap k[t_1, \dots, t_n]$, then $G_r = G_0$ we get.

Corollary 5.6. (Kapur et al (2013), Corollary 4.5) *Let G be a Gröbner basis for the ideal* $< F > \subseteq R[x_1, \dots, x_n]$, $G_r = G \cap k[t_1, \dots, t_n]$. *Furthermore, suppose that* $G_m \subseteq G/_{G_0}$ *such that* $< LM(G_m) >=< LM(G \setminus G_r) >$ *and* $LM(G_m)$ *is minimal set of monomials. If* $\sigma: R \to k$ *is specialization such that*

(i) $\sigma(g) = 0$ *for* $g \in G_r$ *and*

(ii) $\sigma(h) \neq 0$, *where* $h = \prod_{g \in G_m} LC(g)$

then $\sigma(G_m)$ *is Gröbner basis for* $< \sigma(F) >$. *That means* G_m *in fact a Gröbner basis for* $< F >$ *on the set* $V(G_r) \setminus V(h)$.

Now, we will give an algorithm for computing a comprehensive Gröbner system. Kapur et al (2013) gave several algorithms for computing comprehensive Gröbner basis and comprehensive Gröbner systems in different settings. The following algorithm is extracted from these algorithms, to compute a comprehensive Gröbner system accordance with our purposes.

Algorithm 5.6

Input: $F \subseteq k[t_1, \dots, t_m][x_1, \dots, x_n]$
Output: A comprehensive Gröbner system for $< F >$
$E = \emptyset, i = 1, G_r = \emptyset,$
While $h \neq 1$

$G :=$ Gröbner Basis of $< F >$

$G_r := G \cap (G \cap k[t_1, \dots, t_m]) \cup h$

IF $V(E) \setminus V(G_r) \neq \emptyset$ THEN

$G_i := G_r$, $A_i := V(E) \setminus V(G_r)$, $i := i + 1$

END IF

$G_m :=$ Minimal Basis $(G \setminus G_r)$

$h := LCM\big(LC(G_m)\big)$

$G_i := G_m$, $A_i := V(G_r) \setminus V(h)$, $i := i + 1$

$E = G_r \cup \{h\}$

$F = F|_{h=0}$

END WHILE

Let us apply the algorithm to an example.

Example 5.7

Let $F = \{ax - b, by - a, cx^2 - y, cy^2 - x\} \subseteq \mathbb{R}[a, b, c][x, y]$. We define block order $\{a, b, c\} < \{x, y\}$; within each block graded reverse lexicographic order is used.

Step 1 :

$E = \emptyset, h = 0, i = 1, G_r = \emptyset$

$G = \{x^3 - y^3, cx^2 - y, cy^2 - x, ax - b, bx - acy, \quad a^2y - b^2c,$

$by - a, a^6 - b^6, a^3c - b^3, b^3c - a^3, ac^2 - a, bc^2 - b\}$

$G_r = \{a^6 - b^2, a^3c - b^3, b^3c - a^3ac^2 - a, bc^2 - b\}$

$V(E) \setminus V(G_r) \neq \emptyset \Rightarrow G_1 = G_r, A_1 = \mathbb{R}^3 \setminus V(G_r), i = 2$

$G \setminus G_r = \{x^3 - y^3, cx^2 - y, cy^2 - x, ax - b, bx - acy, a^2y - b^2c, by - a\}$

$G_m = \{bx - acy, by - a\}$

$h = LCM\{b, b\} = b$

$G_2 = G_m$, $A_2 = V(G_r) \setminus V(b), i = 3$

$E = \big\{a^6 - b^6, a^3c - b^3, b^3c - a^3, ac^2 - a, bc^2 - b, b\big\} = \{a^3, ac^2 - a, b\}$

$F = \{ax, -a, cx^2 - y, cy^2 - x,\}$

Step 2 :

$G = \{x^3 - y^3, cx^2 - y, cy^2 - x, a\}$

$G_r = \{a, b\}$

$V(E)\backslash V(G_r) \neq 0 \Rightarrow$ Now new branch for Gröbner system.

$G\backslash G_r = \{x^3 - y^3, cx^2 - y, cy^2 - x\}$

$G_m = \{cx^2 - y, cy^2 - x\}$

$h = LCM\{c, c\} = c$

$G_3 = G_m$, $A_3 = V(a, b)\backslash V(c), i = 4$

$E = \{a^3, ac^2 - a, b, c\} = \{a, b, c\}$

$F = \{ax, -a, -y, x\}$

Step 3:

$G = \{a, x, y\}$

$G_r = \{a, b, c\}$

$V(E)\backslash V(G_r) \neq 0 \Rightarrow$ Now new branch

$G_m = \{x, y\}$

$h = 1$

$G_4 = G_m$, $A_4 = V(a, b, c)$

Since $h = 1$, algorithm determinates. Hence, we obtain a comprehensive Gröbner system for $\langle F \rangle$ as follows:

$$\begin{cases} \{a^6 - b^6,\ a^3c - b^3,\ b^3c - a^3, ac^2 - a, bc^2 - b\} & \mathbb{R}^3\backslash V(a^6 - b^6,\ a^3c - b^3,\ b^3c - a^3, ac^2 - a, bc^2 - b); \\ \{bx - acy, by - a\} & V(a^6 - b^6,\ a^3c - b^3,\ b^3c - a^3, ac^2 - a, bc^2 - b)\backslash V(b); \\ \{cx^2 - y, cy^2 - x\} & V(a, b)\backslash V(c); \\ \{x, y\} & V(a, b, c). \end{cases}$$

When we apply this algorithm to our inverse kinematics problem, we get

$$
\begin{aligned}
G = \{&(a^2+b^2)c_1^2 - b^2, bs_1 + ac_1, b - bc_1^2 + ac_1s_1, -1 + c_1^2 + s_1^2, a^4 + 2a^2b^2 \\
&+ b^4 + 2a^2c^2 + 2b^2c^2 + c^4 + 2a^2L_2^2 + 2b^2L_2^2 - 2c^2L_2^2 + L_2^4 \\
&+ c_2^2(4a^2L_2^2 + 4b^2L_2^2 + 4c^2L_2^2) - 2a^2L_3^2 - 2b^2L_3^2 - 2c^2L_3^2 - 2L_2^2L_3^2 \\
&+ L_3^4 + c_1c_2(-8a^2bL_2 - 4b^3L_2 - 4bc^2L_2 - 4bL_2^3 + 4bL_2L_3^2) \\
&+ c_2(4a^3L_2 + 4ac^2L_2 + 4aL_2^3 - 4aL_2L_3^2)s_1, -a^2 - b^2 - c^2 \\
&+ 2bc_1c_2L_2 - L_2^2 + L_3^2 - 2ac_2L_2s_1 + 2cL_2s_2, a^2L_2 + b^2L_2 - 3c^2L_2 \\
&+ L_2^3 + c_2^2(4a^2L_2 + 4b^2L_2 + 4c^2L_2) - L_2L_3^2 + c_1c_2(-4a^2b - 2b^3 \\
&- 2bc^2 - 4bL_2^2 + 2bL_3^2) + c_2(2a^3 + 2ac^2 + 4aL_2^2 - 2aL_3^2)s_1 \\
&+ (2a^2c + 2b^2c + 2c^3 - 2cL_3^2)s_2, -2cL_2s_1 + 2cc_2^2L_2s_1 - abc_1s_2 \\
&+ 2ac_2L_2s_2 + (a^2 + c^2 + L_2^2 - L_3^2)s_1s_2, 2cc_1L_2 - 2cc_1c_2^2L_2 \\
&+ 2bc_2L_2s_2 + c_1(-a^2 - b^2 - c^2 - L_2^2 + L_3^2)s_2, -1 + c_2^2 + s_2^2, -bc_1 \\
&+ c_2L_2 + c_3L_3 + as_1, -c + L_2s_2 + L_3s_3, -2cc_2c_3L_2 - cL_3 \\
&+ 2bc_1c_3L_2s_2 - L_2L_3s_2 - 2ac_3L_2s_1s_2 + (a^2 + b^2 + c^2 - L_2^2)s_3, cc_3 \\
&- c_3L_2s_2 - bc_1s_3 + c_2L_2s_3 + as_1s_3, -bc_1c_3 + c_2c_3L_2 + L_3 + ac_3s_1 \\
&- cs_3 + L_2s_2s_3, -ac_2c_3 - c_3L_2s_1 - c_2L_3s_1 + cc_3s_1s_2 + cc_2s_1s_3 \\
&+ as_2s_3, -bc_2c_3 + c_1c_3L_2 + c_1c_2L_3 - cc_1c_3s_2 - cc_1c_2s_3 \\
&+ bs_2s_3, -cc_2c_3 + bc_1c_3s_2 - L_3s_2 - ac_3s_1s_2 + bc_1c_2s_3 - L_2s_3 \\
&- ac_2s_1s_3 + cs_2s_3, -1 + c_3^2 + s_3^2\}
\end{aligned}
$$

The Gröbner basis contains 19 polynomials which is much more than our original Gröbner basis of 6 polynomials. There is no polynomial involving only parameters. So $G_r = \emptyset$. Therefore

$$G_m = \{(a^2+b^2)c_1^2 - b^2, bs_1 + ac_1, (4a^2L_2^2 + 4b^2L_2^2 + 4c^2L_2^2)c_2^2 + (-8a^2bL_2 - 4b^3L_2 - 4bc^2L_2 - 4bL_2^3 + 4bL_2L_3^2)c_1c_2 + (4a^3L_2 + 4ac^2L_2 + 4aL_2^3 - 4aL_2L_3^2)c_2s_1 + a^4 + 2a^2b^2 + b^4 + 2a^2c^2 + 2b^2c^2 + c^4 + 2a^2L_2^2 + 2b^2L_2^2 - 2c^2L_2^2 + L_2^4 - 2a^2L_3^2 - 2b^2L_3^2 - 2c^2L_3^2 - 2L_2^2L_3^2 + L_3^4, 2cL_2s_2 - 2ac_2L_2s_1 + 2bc_1c_2L_2 - a^2 - b^2 - c^2 - L_2^2 + L_3^2, c_3L_3 + c_2L_2 + as_1 - bc_1, L_3s_3 + L_2s_2 - c\}$$

Then we found $h = bc(a^2+b^2)(a^2+b^2+c^2)L_2^2L_3$ which is the product of denominators of the our original Gröbner basis as we expected. It seems that comprehensive Gröbner system is not useful for the problems in which only bad specializations are the ones that some denominators in Gröbner basis to vanish.

6. CONCLUSION

We have demonstrated that Gröbner Basis Theory is a good alternative method for solving the inverse kinematics problems. We examine two methods for specializations of parameters in Gröbner basis of the inverse kinematics problems. In the first method some extra colon ideal computations have to be done for finding specializations. After finding these specializations a new Gröbner basis computations is needed for them. Then, we have to do some colon ideal computations for these new Gröbner basis again. The process continues in this way. On the other hand, specializations and associated Gröbner bases are automatically founded during the computation of the comprehensive Gröbner system. There is no need for extra colon ideal computations. However, the calculations in comprehensive Gröbner system are made in a ring not in a field. It means that we have to compute Gröbner bases in more variables. This may cause problems because the Gröbner basis computation is very sensitive to the number of variables. The size of the Gröbner basis may increase dramatically when the number of variables increases. Therefore, if the number of specializations are relatively less, the first method is recommended. If, on the other hand, the number of specializations is excessive, the computation of a comprehensive Gröbner system should be preferred.

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