BOLU ABANT IZZET BAYSAL UNIVERSITY

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

BOUNDARY OF THE GENERAL LINEAR GROUP OVER THE RATIONALS AND P-ADIC CONTINUED FRACTIONS

DOCTOR OF PHILOSOPHY

ESRA ÜNAL YILMAZ

BOLU, JUNE 2019

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APPROVAL OF THE THESIS

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DECLARATION

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Esra ÜNAL YILMAZ

ABSTRACT

BOUNDARY OF THE GENERAL LINEAR GROUP OVER THE RATIONALS AND P-ADIC CONTINUED FRACTIONS

PH.D. THESIS

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(SUPERVISOR : PROF. DR. A. MUHAMMED ULUDAĞ)

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The general linear group over the rationals $\mathrm{PGL}_2(\mathbb{Q})$ is an arithmetically infinitely presented group. Our aim in this thesis in order to explore the boundary of the general linear group, the elements of this boundary are classified in a systematic manner. Also we give a through review of p-adic continued fractions in the literature. We also get explicit series expansions for some periodic p-adic continued fractions.

KEYWORDS: Linear group, Continued fractions, P-adic continued fractions, Modular group, Boundary of Groups.

ÖZET

GENEL LİNEER GRUBUN RASYONELLER ÜZERİNDEKİ SINIRI VE P-ADİK SÜREKLİ KESİRLER
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Rasyoneller üzerindeki genel lineer grup $\mathrm{PGL}_2(\mathbb{Q})$ aritmetik olarak önemli sonsuz temsile sahip bir gruptur. Bu tezdeki amacımız rasyoneller üzerindeki genel lineer grubun sınırını keşfetmek amacıyla bu sınırın elemanlarını sistematik bir yolla sınıflandırmaktır. Aynı zamanda literatürdeki p-adik sürekli kesirler araştırılmıştır ve bazı periyodik p-adik sürekli kesirler için açılımlar elde edilmiştir.

ANAHTAR KELİMELER: Lineer Grup, Sürekli Kesirler, P-adik sürekli kesirler, Modüler grup, Grupların Sınırları.

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1. INTRODUCTION

Our aim in this thesis is to explore the "boundary" of the general linear group over the field of rationals, i.e. the group

$$\mathrm{PGL}_2(\mathbb{Q}) := \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} : ps - qr \neq 0 \right\} / \langle \mp I \rangle$$

Our central theme is that this boundary classifies in a systematic manner all iterated expressions of "arithmetic" nature, i.e. continued fractions of the form

$$n_0 + \cfrac{m_1}{n_1 + \cfrac{m_2}{n_2 + \cfrac{m_3}{n_3 + \cfrac{m_4}{n_4 + \cfrac{m_5}{\dots}}}}}$$

as well as iterated expressions of the form

$$n_0 + p^{m_1}(n_1 + p^{m_2}(n_2 + p^{m_3}(n_3 + p^{m_4}(n_4 + \dots)))),$$

and hybrids of these two forms. By "arithmetic" nature we mean that all numbers n_i, m_i are integers. It is understood that the convergence of these iterated expressions are taken in a proper sense; i.e. either in the p-adic or real topology. Sometimes these expressions do not converge at all in some topology; and in many cases two different expressions converge to the same number. Interpreting these iterated expressions as boundary elements explains why and how two different expressions converge to the same number.

By the "boundary" of $\mathrm{PGL}_2(\mathbb{Q})$ we understand the set of all words in some given presentation of $\mathrm{PGL}_2(\mathbb{Q})$. The presentation we consider is the conjectural presentation given in Uludağ (2016). Note that this group is not finitely presented. In order to view this boundary as the boundary of some topological space, we need to find a space on which $\mathrm{PGL}_2(\mathbb{Q})$ acts in some nice manner.

Since $\mathrm{PGL}_2(\mathbb{Q})$ is an infinitely presented group, this space must be infinite dimensional and it must have a very complicated structure (named "heyula" by M. Uludağ). However, the subgroup $\mathrm{PSL}_2(\mathbb{Z})$ of $\mathrm{PGL}_2(\mathbb{Q})$ and its boundary gives us some understanding on how to construct and study this space. In this case, the space acted upon is the Farey tree (i.e. the planar 2-3 regular tree) whose boundary consists of simple continued fractions, i.e. those continued fractions of the form

$$n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \cfrac{1}{\dots}}}}}$$

with $n_0 \in \mathbb{Z} \cup \{\infty\}$ and $n_i \in \mathbb{Z}_{>0} \cup \{\infty\}$ for $i > 0$. It is well known that this continued fraction always converges in the real topology and every irrational number is represented uniquely as a simple continued fraction. The boundary in this case is identified with the extended real line.

The subgroup $\mathrm{PGL}_2(\mathbb{Z})$ of $\mathrm{PGL}_2(\mathbb{Q})$ studied in Uludağ (2013) has a boundary which is also identified with the extended real line; however, in this case the unicity of representation is lost. Every real number admits an infinite number of representations of the form

$$n_0 + \cfrac{\pm 1}{n_1 + \cfrac{\pm 1}{n_2 + \cfrac{\pm 1}{n_3 + \cfrac{\pm 1}{n_4 + \cfrac{\pm 1}{\dots}}}}}$$

as a boundary element of $\mathrm{PGL}_2(\mathbb{Z})$. Different representations of the same real number are interpreted as homotopic paths of the space corresponding to $pglq$ (which is a thickening of the Farey tree).

The Baumslag-Solitar group

$$BS(p,1) = \langle T, H_p \mid H_p^{-1} T^p H_p = T \rangle$$

is another subgroup of $\mathrm{PGL}_2(\mathbb{Z})$, and the p-adic numbers

$$n_0 + p^{m_1}(n_1 + p^{m_2}(n_2 + p^{m_3}(n_3 + p^{m_4}(n_4 + \dots))))$$

with $n_i \in \mathbb{Z}_{>0}$ $(i \geq 0)$ can be viewed as elements of its boundary. The space acted upon have been constructed by Farb and Mosher (1998)

Our second aim in this thesis is to study various boundary elements of $\mathrm{PGL}_2(\mathbb{Q})$ with the perspective of preparing the ground for constructing the space on which $\mathrm{PGL}_2(\mathbb{Q})$ acts. Our approach yields a unified treatment of all p-adic continued fractions appearing in the literature. We give a through review of p-adic continued fractions in the literature and we situate them as boundary elements of $\mathrm{PGL}_2(\mathbb{Q})$. In particular we will prove that the continued fraction below converges in the p-adic topology and we express the limit as an infinite sum

$$1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-\vdots}}} = 1-\sum_{n=1}^{\infty}\frac{1}{2(2n-1)}\binom{2n}{n}x^n$$

where $|x|_p < 1 \in \mathbb{Z}_p$. In particular, this implies that

$$1-\cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{1-\vdots}}} = 1-\sum_{n=1}^{\infty}\frac{1}{2(2n-1)}\binom{2n}{n}p^n$$

We then extend this formula to continued fractions of the form

$$1-\cfrac{x}{1-\cfrac{y}{1-\cfrac{x}{1-\cfrac{y}{1-\cdots}}}}$$

and express the sum as an infinite series, where $x, y \in \mathbb{Z}_p$ satisfy $|x|_p < 1$, $|y|_p < 1$. Next we give a conjecture some special type of continued fraction with 3-period under similar assumptions. We prove some general convergence theorem for p-adic continued fractions as well.

2. CONTINUED FRACTIONS

One of the mathematical concepts with the longest history is the continued fractions. It is difficult to determine the date of origin of continued fractions. This is due to the fact that we can find examples of these continued fractions during the last 2000 years. But the real foundations are based on the end of 17th and 18th century. In fact, the origin of continued fractions is the same as the Euclidean algorithm. Euclidean algorithm applied to two integers p and q naturally gives a simple continued fraction expansion for a p/q when $p, q \in \mathbb{Z}$. We don't know whether Euclid and his contemporaries actually used this algorithm in this way. However, the emergence of the Euclidean algorithm due to its close relationship to the continued fractions indicates the first development of continued fractions.

Studies of continued fractions have been limited to papers containing specific examples for more than a thousand years. The Indian mathematician Aryabhata (d. 550 AD) solved a linear indeterminate equation with the help of continued fractions Olds. He did not develop continued fraction theory and the continued fraction which he used was limited to specific examples. Traces of continued fractions in Greek and Arabic mathematical writings only contain specific examples. The Italian mathematicians Rafael Bombelli (b. c.1530) and Pietro Cataldi (1548-1626) contributed to the continued fractions field by giving more examples. Bombelli and Cataldi gave a continued fraction expansion respectively for square root of 13 and 18. But these were some special examples for continued fractions Olds. Continued fractions has become a field of study on its own with the work of John Wallis between 1616 and 1703 Kline (1972). In Wallis's book 'Opera Mathematica (1695)' he made a significant contributions to the theory of continued fractions Kline (1972). He found how to calculate n-th convergent and investigated some properties of convergents. German mathematician Christian Huygens (1629-1695) made an application using the convergents of continued fractions. He used to find the best rational approach to gear ratios. Continued fraction theory which began with the works of Wallis and Huygens proceeded to a great extent with the work of Euler (1707-1783), Lambert (1728-1777) and Lagrange (1736-1813).

Euler's work 'De Fractionlous Continious (1738)' was a vestige containing most of modern theory of continued fractions. He proved that every rational can be written as a terminating continued fraction. Also he found the continued fraction expansion of e. Lambert generalized Euler's work and proved if x is rational then e^x and $\tan x$ are irrational. These results can be seen in Beskin (1986).

Ramanujan has made significant contributions to the theory of continued fractions. Rogers-Ramanujan continued fraction is defined by

$$R(q) := \frac{q^{1/5}}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+...} \quad |q| < 1,$$

firstly established in a paper Rogers (1894) and Andrews (1981) Rogers proved that

$$R(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

where

$$(x;q)_\infty := \prod_{k=0}^{\infty} (1 - xq^k), \quad |q| < 1.$$

Ramanujan proved some theorems about $R(q)$ in one of his notebooks Ramanujan (1957) and he also generalized Rogers-Ramanujan continued fractions.

Lagrange used continued fractions to find the value of an irrational root of a quadratic equation. He also proved that quadratic irrationals are real numbers which can be represented by periodic continued fractions. 19th century was an important period for continued fraction theory. There was considerable growth in this area. Jacobi, Perron, Hermite, Gauss, Cauchy and Stieljes made a great contributions to field of continued fractions. Jacobi and Perron generalized the classical continued fractions and introduced multidimensional continued fractions.

2.1 From the Euclidean algorithm to the real number system

In this chapter we explain boundary of the Farey tree and the correspondence with simple continued fractions.

The Euclidean algorithm has an important role for the construction of the real numbers. Since the origin of the modular group is Euclidean algorithm, that is the Euclidean algorithm can be encoded in a modular group. They should not be separated from each other. Modular groups acts on the Farey tree. This tree has a boundary. Elements of this boundary give us real numbers. However these real numbers appear as continued fractions. Note that this construction yields set of real numbers. Our aim is to adapt this construction to p-adics. Let us consider the processes of antyphairesis starting from the Euclidean algorithm.

Suppose that we are given two sticks and we are asked to compare their lenghts

A −

B −

If we cut the multiples of B from A until the remaining part is shorter than B then replace A with B, replace B with remaining part and repeating the procedure. This procedure give a simple continued fraction. We can describe this process as follows. Let us consider $A := a,\ B := b \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned}
&a = a_0 b + r_0,\ 0 < r_0 < b\\
&b = a_1 r_0 + r_1,\ 0 < r_1 < r_0\\
&r_0 = a_2 r_1 + r_2,\ 0 < r_2 < r_1\\
&\dots\\
&r_{k-2} = a_k r_{k-1}\ 0 = r_k < r_{k-1} < \cdots < r_0 < b.
\end{aligned}$$

Then these equalities give us

$$\frac{a}{b} = a_0 + \frac{r_0}{b} = a_0 + \frac{1}{\frac{b}{r_0}} = a_0 + \frac{1}{a_1 + \frac{r_1}{r_0}} = ... = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

Above fraction is simple continued fraction. If we write this fraction in operator language, we use translation and involution operator i.e.

$$T : x \to x + 1$$
$$U : x \to \frac{1}{x}$$

So we can write

$$a/b = T^{a_0} U T^{a_1} U ... U T^{a_k}(0)$$

If the sticks A and B are not commensurable i.e. there does not exist common unit of measure, then antyphairesis process does not terminate and as a result we get an infinite simple continued fraction expansion.

Example 1. Compute the continued fraction expansion of $54/16$

$54 = 3.16 + 6 \to 16 = 2.6 + 4 \to 6 = 1.4 + 2 \to 4 = 2.2 + 0.$

So we get continued fraction expansion of $\frac{54}{16}$

$$3 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}}$$

Now we will show the most primitive antyphairesis process. This antyphairesis process is a variation of Euclidean algorithm. This algorithm is processing if stick A longer than B then we can write $A = B+C$; if not $A = B-C$. After then replace A with B, B with C. We must repeat this procedure until there is no B remains. Let us consider $A := a,\ B := b \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned}
&a = b \mp r_0, \\
&b = r_0 \mp r_1, \\
&\dots \\
&r_{k-2} = r_{k-1} \mp r_k, \\
&r_{k-1} = r_k + 0 \\
&r_{k+1} = 0
\end{aligned}$$

Then these equalities give us continued fraction expansion of $\frac{a}{b}$;

$$\frac{a}{b} = 1 \mp \cfrac{1}{1 \mp \cfrac{1}{1 \mp \cfrac{1}{\ddots \mp \cfrac{1}{1}}}}$$

These continued fractions can be expressed language of operators T, S, U where $T : x \to x+1$, $U : x \to \frac{1}{x}$ and $S : x \to -\frac{1}{x}$

Example 2. Compute the continued fraction of $\frac{29}{13}$

$29 = 13 + 16 \to 13 = 16 - 3 \to 16 = 3 + 13 \to 3 = 13 - 3 \to 13 = 3 + 10 \to 3 = 10 - 7 \to 10 = 7 + 3 \to 7 = 3 + 4 \to 3 = 4 - 1 \to 4 = 1 + 3 \to 1 = 3 - 2 \to 3 = 2 + 1 \to 2 = 1 + 1 \to 1 = 1 + 0.$

So we get continued fraction expansion of $\frac{29}{13}$

$$1 + \cfrac{1}{1 - \cfrac{1}{1 + \cfrac{1}{1 - \cdots}}}$$

This type continued fraction expansion is called semiregular continued fraction expansion. This process can be encoded in terms of the operators S, U and T as

$$TUTSTUTSTUTSTUTUTSTUTSTUTUTUT(0)$$

2.2 Simple Continued Fractions

Before considering the p-adic version of continued fractions, we will give basic definitions and theorems about simple continued fractions. The theory of continued fractions are about special algorithm which is one of the most important objects in probability theory, analysis and number theory. Our aim is to apply these ideas to p-adic continued fraction as well.

A simple continued fraction is, a continued fraction by definition

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

usually with $a_i \in \mathbb{Z}^+$ $(i \geq 1)$ and $a_0 \in \mathbb{Z}$. The set of all simple continued fractions can be canonically identified with the real line. We can describe this process as follows. Let us consider $a, b \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned}
&a = a_0 b + r_0,\ 0 < r_0 < b \\
&b = a_1 r_0 + r_1,\ 0 < r_1 < r_0 \\
&r_0 = a_2 r_1 + r_2,\ 0 < r_2 < r_1 \\
&\dots \\
&r_{k-2} = a_k r_{k-1}\ 0 = r_k < r_{k-1} < \cdots < r_0 < b.
\end{aligned}$$

Then these equalities give us

$$\frac{a}{b} = a_0 + \frac{r_0}{b} = a_0 + \cfrac{1}{\cfrac{b}{r_0}} = a_0 + \cfrac{1}{a_1 + \cfrac{r_1}{r_0}} = ... = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

So we can define a representation for a real number.

Definition 2.2.1. Simple continued fraction for a real number α is defined in the following form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $a_0 \in \mathbb{Z}$ and $a_i > 0$ for $i > 0$.

For typographical reasons, this continued fraction is denoted by $[a_0; a_1, a_2, ...]$ and the following expression,

$$\frac{p_n}{q_n} = [a_0; a_1, ..., a_n]$$

is n-th convergent of α.

Theorem 2.2.2. *Let we assume that* $p_0 = a_0$, $p_{-1} = 1$, $q_0 = 1$, $q_{-1} = 0$ *where* $n \geq 2$ *then* p_n *and* q_n *be defined as following form*

$$p_n = a_n p_{n-1} + p_{n-2} \tag{2.1}$$

$$q_n = a_n q_{n-1} + q_{n-2}. \tag{2.2}$$

Proof. We will use induction method to prove. For $n = 2$

$$p_2 = a_2 p_1 + p_0$$

$$q_2 = a_2 q_1 + q_0$$

so base step of induction is verified. Let us assume for all $n < k$ 2.1 and 2.2 true.
Consider the continued fraction $[a_0; a_1, a_2, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$ and we are substituting $a'_{n-1} = a_{n-1} + \frac{1}{a_n}$ so we get

$$[a_0; a_1, a_2, ..., a_{n-2}, a'_{n-1}] = \frac{a'_{n-1} p_{n-2} + p_{n-3}}{a'_{n-1} q_{n-2} + q_{n-3}} \qquad = \frac{(a_{n-1} + \frac{1}{a_n}) p_{n-2} + p_{n-3}}{(a_{n-1} + \frac{1}{a_n}) q_{n-2} + q_{n-3}} = \frac{p_n}{q_n}$$

□

Proposition 2.2.3. *For all* $n \geq 0$

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \tag{2.3}$$

Proof. Again we can use induction. Let us assume for $k < n$ (1.3) is true. Multiplying first equation in 2.2.2 by q_{n-1} and second equation by p_{n-1} and subtracting first from second we get

$$q_n p_{n-1} - p_n q_{n-1} = -(q_{n-1} p_{n-2} - p_{n-1} q_{n-2})$$

using the induction hypothesis we have

$$-(q_{n-1}p_{n-2} - p_{n-1}q_{n-2}) = -(-1)^{n-1} = (-1)^n$$

□

Proposition 2.2.4. *Each real number α, correspond a unique continued fraction of value α. If α is rational, continued fraction is finite and if α is irrational then continued fraction is infinite.*

2.2.5 Gauss map and Gauss-Kuzmin theorem for simple continued fractions

We will introduce the Gauss map that is important for its relation with continued fractions in number theory. The Gauss map $G : [0,1] \to [0,1]$ is defined as follows,

$$G(x) = \begin{cases} 0, & if \quad x = 0 \\ \left\{ \dfrac{1}{x} \right\} & if \quad 0 < x \leq 1 \end{cases}$$

Here $\{x\}$ denotes the fractional part of x. We can write $\{x\} = x - [x]$ where $[x]$ is the integer part. Note that

$$\frac{1}{x} = n \Leftrightarrow n \leq \frac{1}{x} < n+1 \Leftrightarrow \frac{1}{n+1} < x \leq \frac{1}{n}$$

Thus one has

$$G(x) = \begin{cases} 0, & if \quad x = 0 \\ \dfrac{1}{x} - n & if \quad \frac{1}{n+1} < x \leq \frac{1}{n} \end{cases}$$

where $n \in \mathbb{N}$. The restriction of G to an interval of the form $(1/n+1, 1/n]$ is called branch. Each branch $G : (1/n+1, 1/n] \to [0,1)$ is monotone, surjective and invertible.

The Gauss map is important for its connections with continued fractions. We can take $a_0 = [x]$ but for the rest of the elements a_i we use the map G. First we define

$$a_1(x) = [\frac{1}{x}] \quad x \neq 0, \quad and \quad a_1(0) = \infty$$

It follows that the we can write

$$x = \frac{1}{a_1 + G(x)}$$

. Now for $G^{n-1}(x) \neq 0$, we define the next partial quotient of x by

$$a_n(x) = \left[\frac{1}{G^{n-1}(x)} \right]$$

Now that we have defined everything we need, we can find the partial quotients $a_1, a_2, ...$ by repeatedly applying our function $G(x)$. However, the continued fraction is finite for rational numbers. This means that there is an $n \geq 0$ so $G^n(x) = 0$. Now we write $x = \frac{p_0}{q_0}$ where x is a relatively prime and $0 < p_0 < q_0$. Now we define $G^n(x) = \dfrac{p_n}{q_n}$, so $G(\mathbb{Q} \cap [0,1]) = \mathbb{Q} \cap [0,1]$. Now

$$G\left(\frac{p_0}{q_0}\right) = \frac{1}{\frac{p_0}{q_0}} - \left[\frac{1}{\frac{p_0}{q_0}}\right] = \frac{q_0}{p_0} - a_1 = \frac{q_0 - p_0 a_1}{p_0} = \frac{p_1}{q_1}$$

Here p_1 and q_1 is relatively prime. We can now see that the following inequalities hold:

$$p_1 \leq q_0 - a_1 p_0 \leq p_0$$

Equality will occur when $p_0 = q_1$. So we find that the continued fraction in this case is finite and in other cases we find that $p_1 < p_0$. In the same way we find $p_2 < p_1$. Thus

$$0 \leq ... < p_n < p_{n-1} < ... < p_1 < p_0$$

for $p_i \in \mathbb{N}_{>0}$ with $i \geq 0$. But then there exists a_n for which $p_n = 0$. Here we stop the procedure of determining a_n and thus we find a finite continued fraction.

Measure theoretic properties are related with Gauss map. The measure of continued fraction expansions is about properties of the sequence $(a_n)_{n \in \mathbb{N}^+}$. It started is about 1800's, with a note by Gauss in his mathematical diary Brezinski (1980). We defined the Gauss map for simple continued fraction expansions. Gauss wrote the following fact in his diary

$$\lim_{n \to \infty} \lambda(G^n \leq x) = \frac{log(1+x)}{log2}$$

where $x \in [0,1]$ and λ denotes the Lebesgue measure on $[0,1]$. G^n is the n-th iterate of G. Gauss found this formula, but no one knew how he found it. Considering the fact that modern probability theory and ergodic theory started about a century later, Gauss's success is still more important. It is often difficult to find invariant measure. Twelve years later Gauss wrote a letter to Laplace and said that he did not solve satisfactorily a curious problem and that his efforts were unproductive. In modern notation, this problem is to estimate the error

$$e_n(x) := \lambda(G^{-n}[0,x]) - \frac{log(1+x)}{log2} \quad (n \geq 1,\ x \in [0,1])$$

This has been called Gauss problem. It received a first solution more than a century later, Kuzmin showed that in Kuzmin (1928)

$$e_n(x) = O(q^{\sqrt{n}})$$

as $n \to \infty$ uniformly in x with some $0 < q < 1$. This has been called the Gauss-Kuzmin theorem or the Kuzmin theorem. One year later, using a different method, Levy (1929) improved Kuzmin's result by showing that

$$|e_n(x)| \leq q^n$$

$n \in \mathbb{N}_+$, $0 \leq x < 1$, with $q = 3.5 - 2\sqrt{2} = 0.67157$....The Gauss-Kuzmin-Levy theorem is the first basic result in the rich metrical theory of continued fractions.

3. BOUNDARY OF THE GENERAL LINEAR GROUP $\mathrm{PGL}_2(\mathbb{Q})$

We may describe simple continued fractions as the boundary points of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ (or of the Farey tree on which $\mathrm{PSL}_2(\mathbb{Z})$ acts in a nice manner).
Our task in this chapter is to explain this correspondence, with the aim of building this correspondence for other subgroups of $\mathrm{PGL}_2(\mathbb{Q})$.

3.1 The Modular Group

Let $\varphi \in \mathrm{Aut}(\mathbb{Z}^2)$, the automorphism group of the abelian group $\mathbb{Z}^2$. Then φ is determined by its values $\varphi((1,0)) = (p,r)$ and $\varphi(0,1) = (q,s)$. The value $\varphi(m,n)$ is then given by

$$\varphi(m,n) = m\varphi((p,r)) + n\varphi((q,s)) = (pm + qn, rm + sn)$$

which we may write in the matrix form as

$$\varphi(m,n) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \tag{3.1}$$

A map defined by (3.1) is bijective if and only if the determinant of the matrix satisfies $ps - qr = \pm 1$ because it has the inverse

$$\varphi^{-1}(m,n) = \frac{1}{ps - qr} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \tag{3.2}$$

Since the composition of linear maps correspond to the product of their matrices, $\mathrm{Aut}(\mathbb{Z}^2)$ is isomorphic to the group of invertible integral two-by-two matrices under the matrix product:

$$\mathrm{GL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad : \quad p,q,r,s \in \mathbb{Z}, \quad ps - qr = \pm 1 \right\},$$

sometimes called the *homogeneous modular group*. The map $\det : \mathrm{GL}_2(\mathbb{Z}) \to \{\pm 1\}$ sending a matrix to its determinant is a homomorphism, whose kernel consists of unimodular integral two-by-two matrices

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad : \quad p,q,r,s \in \mathbb{Z}, \quad ps - qr = 1 \right\}$$

which is a subgroup of index 2 inside $\mathrm{GL}_2(\mathbb{Z})$. Here, "unimodular" means "of determinant 1" and "integral" means "with integer entries".

Now we introduce an equivalence relation on $\mathbb{Z}^2 \setminus \{0\}$ via

$$(m, n) \sim (m', n') \iff l(m, n) = k(m', n') \text{ for some } k, l \in \mathbb{Z}^\times \tag{3.3}$$

Denote by $[m : n]$ the equivalence class of (m, n). The set of equivalence classes is denoted $\mathbb{P}^1(\mathbb{Z})$ and is called the *projective linear over* $\mathbb{Z}$. Group automorphisms of $\mathbb{Z}^2$ are compatible with this equivalence relation. Therefore every automorphism φ as in (3.1) induces a bijection

$$f_\varphi : [m : n] \in \mathbb{P}^1(\mathbb{Z}) \longrightarrow [pm + qn : rm + sn] \tag{3.4}$$

The points of $\mathbb{P}^1(\mathbb{Z})$ are in one-to-one correspondence with the set $\mathbb{Q} \cup \{\infty\}$ via

$$[m : n] \in \mathbb{P}^1(\mathbb{Z}) \longrightarrow \frac{m}{n} \in \mathbb{Z},$$

where the equivalence class $[1 : 0]$ is sent to ∞. If we identify $\mathbb{P}^1(\mathbb{Z})$ with $\mathbb{Q} \cup \{\infty\}$ this way, then the bijection f_φ is expressed as

$$f_\varphi : x = \frac{m}{n} \in \mathbb{P}^1(\mathbb{Z}) \longrightarrow \frac{pm + qn}{rm + sn} = \frac{px + q}{rx + s} \in \mathbb{P}^1(\mathbb{Z}),$$

where $f_\varphi(\infty) = p/r$ and $f_\varphi(-s/r) := \infty$ by definition. This is an example of what is called a *linear fractional transformation* of $\mathbb{P}^1(\mathbb{Z})$. The map f_φ will be simply denoted as

$$\frac{px + q}{rx + s}.$$

The set of linear fractional transformations induced by the automorphisms of $\mathbb{Z}^2$ is denoted $\mathrm{PGL}_2(\mathbb{Z})$. In other words

$$\mathrm{PGL}_2(\mathbb{Z}) := \left\{ \frac{px + q}{rx + s} \ : \ p, q, r, s \in \mathbb{Z},\ ps - qr = \pm 1 \right\}.$$

This is a group under functional composition.

The map $p : \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Z})$ is a homomorphism with kernel $\langle \pm I \rangle$. In other words, $\mathrm{PGL}_2(\mathbb{Z}) := \mathrm{GL}_2(\mathbb{Z})/\langle \pm I \rangle$.

Finally, the subgroup $\langle \pm I \rangle$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, and the quotient

$$\mathrm{PSL}_2(\mathbb{Z}) := \left\{ \frac{px + q}{rx + s} \ : \ p, q, r, s \in \mathbb{Z},\ ps - qr = 1 \right\} = \mathrm{SL}_2(\mathbb{Z})/\langle \pm I \rangle.$$

is the **modular group**.

The submonoid of $\mathrm{PSL}_2(\mathbb{Z})$ which consists of linear fractional transformations with non-negative entries will be denoted by $\mathsf{PSL}(\mathbb{N})$:

$$\mathsf{PSL}(\mathbb{N}) := \left\{ \frac{px+q}{rx+s} \ : \ p,q,r,s \in \{0,1,2,\dots\}, \ ps-qr=1 \right\} = \mathrm{SL}_2(\mathbb{Z})/\langle \pm I \rangle.$$

The monoids $\mathrm{GL}(\mathbb{N})$, $\mathrm{PGL}(\mathbb{N})$ and $\mathrm{SL}(\mathbb{N})$ are defined in the same manner. Note that these submonoids depend on the base chosen for $\mathbb{Z}^2$.

Remark 3.1.1. Elements of $\mathrm{PGL}_2(\mathbb{Z})$ are often denoted by the corresponding matrices in $\mathrm{GL}_2(\mathbb{Z})$. In this convention, the matrices

$$M := \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \text{ and } -M := \begin{pmatrix} -p & -q \\ -r & -s \end{pmatrix},$$

both denote the same element of $\mathrm{PGL}_2(\mathbb{Z})$. Another convention is to use the notation $\pm M$ for this element.

Determinant and trace. Two characteristic properties of a matrix are its determinant and its trace. Since $\det(-M) = \det(M)$, the map $\det$ is a well-defined homomorphism on $\mathrm{PGL}_2(\mathbb{Z})$ with $\mathrm{PSL}_2(\mathbb{Z})$ as its kernel.

On the other hand, since $tr(-M) = -tr(M)$, the trace map does not descend to a well-defined map on $\mathrm{PSL}_2(\mathbb{Z})$. In order to handle this ambiguity, we will speak of the trace of an element of $\mathrm{PSL}_2(\mathbb{Z})$ up to a sign. Another solution is to use the absolute trace $|tr(M)|$ on $\mathrm{SL}_2(\mathbb{Z})$, which descends to a well-defined map on $\mathrm{PSL}_2(\mathbb{Z})$. Like the usual trace map, it is constant on conjugacy classes. Note that $\det$ is a homomorphism whereas tr is not.

Remark 3.1.2. The same discussion is valid for the trace map on $\mathrm{PGL}_2(\mathbb{Z})$, but not on $\mathrm{PGL}_2(\mathbb{R})$. Since the kernel of the map $\mathrm{GL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$ is $\mathbb{R}$ and not just ± 1, it doesn't make sense to speak of the trace of an element in the latter group.

A general picture. To resume, we have the exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \langle \pm I \rangle & & \langle \pm I \rangle & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}) & \longrightarrow & \mathrm{GL}_2(\mathbb{Z}) \simeq \mathrm{Aut}(\mathbb{Z}^2) & \xrightarrow{\det} & \langle \pm 1 \rangle & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathrm{PSL}_2(\mathbb{Z}) & \longrightarrow & \mathrm{PGL}_2(\mathbb{Z}) & \xrightarrow{\det} & \langle \pm 1 \rangle & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The groups $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{PGL}_2(\mathbb{Z})$ are degree-2 extensions of $\mathrm{PSL}_2(\mathbb{Z})$, and $\mathrm{GL}_2(\mathbb{Z})$ is an extension of degree 4. These groups are close relatives of each other, and any one of these might have been chosen as our main hero. Among these $\mathrm{PSL}_2(\mathbb{Z})$ has the simplest description as a finitely presented group, being isomorphic to the product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. $*$ is a free product.

Elements. These groups are intimately related to the following basic fact:

Lemma 3.1.3. *(Bézout) Two integers p and q are relatively prime if and only if $ps - qr = 1$ for some $r, s \in \mathbb{Z}$.*

Thus for every pair of coprime integers, we have some elements of $\mathrm{PSL}_2(\mathbb{Z})$. For example, when $p = 3$, $q = 7$ one has $5 \times 3 - 2 \times 7 = 12 \times 3 - 5 \times 7 = \cdots = 1$. This give rise to the following elements:

$$\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \cdots \in \mathrm{PSL}_2(\mathbb{Z})$$

Perhaps after the unit element, the simplest element of $\mathrm{GL}_2(\mathbb{Z})$ is the one exchanging the generators of $\mathbb{Z}^2$, i.e. the one defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding linear fractional map is

$$U : x \to \frac{1}{x} \in \mathrm{PGL}_2(\mathbb{Z}).$$

Since its determinant is -1, U is not an element of the modular group. The simplest non-identity element of $\mathrm{PSL}_2(\mathbb{Z})$ is the involution

$$S : x \to -\frac{1}{x}, \text{ with matrix form: } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Besides S, there is the following element, which is of order 3 :

$$L : x \to 1 - \frac{1}{x}, \text{ with matrix form: } \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

In fact, $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S and L. We will see that any element of finite order in $\mathrm{PSL}_2(\mathbb{Z})$ is conjugate to S, L or L^2.

3.2 Farey tree

Now let us construct Farey tree from $\mathrm{PSL}_2(\mathbb{Z})$, as follows. We can denoted by F . The set of edges of F is defined by

$$E(F) = \mathrm{PSL}_2(\mathbb{Z})$$

and the set of vertices of F is defined

$$V(F) = V_{\odot} \cup V_{\otimes}$$

where the set of degree three vertices,

$$V_{\odot} = \Big\{ \{W, WL, WL^2\} : \quad W \in \mathrm{PSL}_2(\mathbb{Z}) \Big\}$$

and the set of degree two vertices,

$$V_{\otimes} = \Big\{ \{W, WS\} : \quad W \in \mathrm{PSL}_2(\mathbb{Z}) \Big\}.$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$\mathrm{PSL}_2(\mathbb{Z})$ is generated by S, L and $S^2 = L^3 = I$

Two distinct vertices v and v' are joined by an edge, if and only if $v \cap v' \neq \emptyset$ and this edge is the single element in the intersection. Note that, if $v, v' \in V_{\odot}$ then

$$v = \{W, WL, WL^2\}$$
$$v' = \{U, UL, UL^2\}$$

If $v \cap v' \neq \emptyset$ then $WL^2 = UL$. Multiplying by L from the right side

$$WL^2.L = UL^2$$

since $L^3 = I$ then we get $W = UL^2$. Multiplying by L^2 from the right side

$$WL^2.L^2 = UL.L^2$$

then we have $WL = U$. So we get $v = v'$. Similarly, if $v, v' \in V_{\otimes}$ and $v \cap v' \neq \emptyset$ then $v = v'$. Hence, no two distinct vertices of the same type has a nonempty intersection.

This graph is connected since $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S and L, it is loop-free, since $\mathrm{PSL}_2(\mathbb{Z})$ is freely generated by S and L.

Definition 3.2.1. A loop free connected graph is called a tree.

Hence our graph is a tree. It is called the Farey tree. The action of $\mathrm{PSL}_2(\mathbb{Z})$ is on Farey tree by automorphisms. We are able to implement the antyphairesis process in p-adic algorithms. For this purpose first of of all we introduce $\mathrm{PGL}_2(\mathbb{Q})$.

3.3 The group $\mathrm{PGL}_2(\mathbb{Q})$

Parts of this section are taken from the unpublished note of Uludağ (2016). Let $\mathrm{PGL}_2(\mathbb{Q})$ be the group of projective two-by-two singular matrices with rational entries. We can define $\mathrm{PGL}_2(\mathbb{Q})$ as

$$\mathrm{PGL}_2(\mathbb{Q}) := \{[M] : \quad M \in \mathbf{M}_2(\mathbb{Z}), \quad \det(M) \neq 0\}$$

where $\mathbf{M}_2(\mathbb{Z})$ the set of all two-by-two matrices with integral entries and denote by $[M]$ the projectivization of the matrix M.

$\mathrm{PGL}_2(\mathbb{Z})$ is subgroup of $\mathrm{PGL}_2(\mathbb{Q})$ where $\mathrm{PGL}_2(\mathbb{Z})$ is the group of 2x2 integral projective matrices of determinant ± 1. The Borel subgroup $\mathbb{B}(\mathbb{Z})$ of $\mathrm{PGL}_2(\mathbb{Z})$, which by definition is the set of upper triangular elements, is generated by the translation $T : x \to 1 + x$ and the reflection $V : x \to -x$. Since $T = KV$, the Borel subgroup is also generated by the involutions V and $K : x \to 1 - x$, showing that $B(\mathbb{Z})$ is the infinite dihedral group. The group $\mathrm{PGL}_2(\mathbb{Z})$ itself is generated by its Borel subgroup $B(\mathbb{Z})$ and the involution $U : x \to 1/x$. Note that the derived subgroup of $B(\mathbb{Z})$ is $\mathbb{Z}$. Similarly, $\mathrm{PGL}_2(\mathbb{Q})$ is generated by its Borel subgroup $B(\mathbb{Q})$ and U. Here $B(\mathbb{Q})$ is infinitely generated but nevertheless is quite similar to the infinite dihedral group in that its derived subgroup is $\mathbb{Q}$. Presentation of $B(\mathbb{Q})$,

$$B(\mathbb{Q}) \simeq \langle K, H_p | H_p^{-1} T^p H_p = T, \quad [H_p H_q] = 1, \quad p, q = -1, 2, 3, 5, 7, ... \rangle$$

where the elements $H_p : x \to px$ are the homotheties. Note that $T = KV$ and $V = H^{-1}$.

Proposition 3.3.1. *The set* $\{K, S, V\} \cup \{I_p : \quad p : prime\}$ *generates* $\mathrm{PGL}_2(\mathbb{Q})$.

Proof. If we denote the translation $T_{r/s} : x \to x + r/s$ then $T_{r/s} = H_{r/s} KV H_{s/r}$ and unless $p = 0$, we have

$$\frac{px + q}{rx + s} = \frac{p}{r} \left\{ 1 - \frac{\dfrac{ps - qr}{pr}}{x + \dfrac{s}{r}} \right\} = H_{p/r} K H_{(ps-qr)/pr} T_{s/r}(x).$$

If $p = 0$ then

$$\frac{px+q}{rx+s} = \frac{q}{rx+s} = \frac{q}{r}\frac{1}{x+\frac{s}{r}} = H_{q/r}UT_{s/r}(x).$$

Now $U = KI_p(KV)^pVI_pK$ for any p. Finally, the result follows since we can express the homotheties in terms of involutions. □

Corollary 3.3.2. *The set* $\{T, U, V\} \cup \{H_p : \quad p : prime\}$ *generates* $\mathrm{PGL}_2(\mathbb{Q})$.

Proof. When $p = 2$ the relation $U = KI_p(KV)^pVI_pK$ implies $U = KI_2KVKI_2K$, i.e. either one of the generators V and U can be eliminated from a generating set. The case $p = 2$ also implies that U and V are conjugate elements in $\mathrm{PGL}_2(\mathbb{Q})$. They are not conjugates inside $\mathrm{PGL}_2(\mathbb{Z})$. □

3.4 Some submonoids of $\mathrm{PGL}_2(\mathbb{Q})$ and their boundaries

Firstly we will consider the submonoid generated by T, H_p of $\mathrm{PGL}_2(\mathbb{Q})$ where $T : x \to x + 1$ and $H_p : x \to p.x$. This monoid has the representation $\langle T, H_p \mid T^pH_p = H_pT\rangle$. We want to imitate antyphairesis procedure of the preceding section to find the boundary of the monoid $\langle T, H_p\rangle$. The way to do this we fixed a point in space and looked at the ways from the point to infinite. We denote by $\partial^+\langle T, H_p\rangle$ the set of infinite words in the monoid generated by T and H_p.

Definition 3.4.1. The boundary of the monoid $\langle T, H_p\rangle$ consists of all infinite words in T and H_p of the following kind:

$$\partial^+\langle T, H_p\rangle := \{T^{n_0}H_pT^{n_1}H_pT^{n_2}H_p... \mid n_0, n_1, n_2, ... \in \mathbb{N}\},$$

where two words are considered equivalent if they are ”homotopic”, ie. one word can be obtained from the other by applying the group relation $T^pH_p = H_pT$.

The reason for the plus sign in notation $\partial^+\langle T, H_p\rangle$ is the loss of convergence when negative powers are added. For the same number we can find infinitely many continued fraction expansion. The reason of this is homotopy. That is if two paths are homotopic then expansions converge to same number.

There is a map

$$\partial^+\langle T, H_p\rangle \longrightarrow \mathbb{Z}_p$$

$$T^{n_0}H_pT^{n_1}H_pT^{n_2}H... \longrightarrow n_0 + pn_1 + p^2n_2 + ...$$

If $n_0, n_1, n_2... \geq 0$ and p be a fixed prime then the operator sequence converges p-adic topology however it doesn't converge in the real (archimedean) topology. So the boundary of this monoid will give p-adic integers.

Let us try to describe antyphairesis process using T and H. If we have two sticks and we want to compare these length. Let us assume length of A is bigger than B then cut the B from the A until the remaining part is divisible by p or powers of p. After that we divide the remaining part into p parts then cut the B from divided part until the remaining part is divisible by p or powers of p. Applying this procedure we can get p-adic numbers.

Example 3. $T^4HT^2HT^3HTH(0) = 508$ If $p = 7$ then antyphairesis process is following $508 = 504 + 4 \rightarrow 504 = 72.7 \rightarrow 72 = 70 + 2 \rightarrow 10 = 7 + 3 \rightarrow 7 = 7.1 \rightarrow 1 = 1 + 0 \rightarrow 0 = 0.7$

Proposition 3.4.2. *The monoid* $\langle T, H_p \rangle$ *generated by* $T : x \rightarrow x + 1$ *and* $H_p : x \rightarrow p.x$ *is given*

$$< T, H_p : \ H_p^{-1}T^pH_p = T >$$

isomorphic to the Baumslag-Solitar group $BS(p, 1) \simeq BS(1, p)$

Farb and Mosher (1998) construct a space on which $\langle T, H_p \rangle$ acts in a nice manner. At the same time they study the boundary of this space which have a p-adic component.

Now we consider the monoid generated by T, H_p, U its boundary consists of infinite words in T, H_p and U. We get the monoid generated by T, H_p, U, which we denote as $\langle T, H_p, U \rangle$ where $T : x \rightarrow x + 1$, $H_p : x \rightarrow px$ and $U : x \rightarrow \frac{1}{x}$. This kind of operator sequences is more complicated than previous sequences. Also in this case we have a set consisting of previous case.

Definition 3.4.3.

$$\partial^+\langle T, H_p, U \rangle = \{T^{n_0}U^{\varepsilon_1}H_pT^{n_1}U^{\varepsilon_2}H_pT^{n_2}U^{\varepsilon_3}H_p..., n_0, n_1, n_2, ... \in \mathbb{N}, \quad \varepsilon_i \in \{+1, -1\}\}$$

where two words are considered equivalent if they have expansion giving the same number.

Some elements of boundary of this monoid one of the form

$$T^{n_0}H_pUT^{n_1}H_pUT^{n_2}H_pUT^{n_3}H_pU..., \quad (1)$$

$$T^{n_0}H_pT^{n_1}H_pT^{n_2}H_pT^{n_3}H_p... \quad (2)$$

and

$$T^{n_0}UT^{n_1}UT^{n_2}UT^{n_3}U... \quad (3)$$

These elements can be written as continued fraction expansions

$$n_0 + \cfrac{p}{n_1 + \cfrac{p}{n_2 + \cfrac{p}{n_3 + \cdots}}}, \quad (1)$$

$$n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}} \quad (2)$$

and

$$n_0 + pn_1 + p^2 n_2 + ... \quad (3)$$

.

Hence we conclude

$$\partial^+\langle T, H_p\rangle \subseteq \partial^+\langle T, H_p, U\rangle$$

and

$$\partial^+\langle T, U\rangle \subseteq \partial^+\langle T, H_p, U\rangle.$$

If we have two sticks and we want to compare these length. Let us assume length of A is bigger than B then cut the B from the A until the remaining part is divisible by p or powers of p. After that we divide the remaining part into p parts. Then replace B with divided part and repeat the procedure. Now this antyphairesis process can be demonstrated as follows;

$A : -$

$B : -$

then cut the B from A until remaining part is divisible by powers of p

$A - xB = p^m k : -$

$B : -$

then divide first stick into p^m

$A - xB/p^m = k : - - - -$

$B : -$

after that replace B with k and repeat the same procedure. Now we can give the following example.

Example 4. $T^3 H_p U T^2 H_p U T H_p U T(0) = \frac{125}{23}$. If $p = 7$ then antyphairesis process is as follows

$125/23 = 3 + 56/23 \to 56/23 = 8/23.7 \to 8/23 = 1 : 23/8 \to 23/8 = 2 + 7/8 \to 7/8 = 7.1/8 \to 1/8 = 1 : 8 \to 8 = 7 + 1 \to 7 = 7.1 \to 1 = 1 : 1 \to 1 = 1 + 0.$

We can give an example. Consider the case $n_1, n_2, n_3, ... = n$ where n is fixed number in $\mathbb{N}$ then for fixed p prime continued fraction expansion returns

$$n + \cfrac{p}{n + \cfrac{p}{n + \cfrac{p}{n + \cdots}}} \tag{3.5}$$

Theorem 3.4.4. *For the above continued fraction if $n^2 \geq 4p$ sequence converges in real topology but if $n^2 \leq 4p$ it doesn't converge in real but converges p-adic topology. Here we assume that n does not divide p.*

Proof. Since

$$n + \frac{p}{x} = x$$

we get $x^2 - nx - p = 0$. Discriminant of this quadratic equation $\Delta = n^2 + 4p$. If $n^2 < 4p$ then this contradicts x is a real number, and we conclude the continued fraction does not converge in $\mathbb{R}$. Now we will show that this continued fraction converges in p-adic topology. Let $a_n = \dfrac{P_n}{Q_n}$ is $n-$th convergent of (2.5). From the definition of Q_n and using the assumption we have $|Q_n|_p = 1$. Then

$$\begin{aligned} |a_n - a_{n-1}|_p &= |\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}}|_p \\ &= |\frac{P_n Q_{n-1} - P_{n-1} Q_n}{Q_n Q_{n-1}}|_p \\ &= |\frac{-p^{n-1}}{Q_n Q_{n-1}}|_p = |-p^{n-1}|_p \to 0 \quad (n \to \infty). \end{aligned}$$

□

The operator sequences we have formed in this way may not converge neither real nor p-adic topology. This is illustrated in the following example

Example 5. Let

$$n_0 + p(n_1 + \frac{1}{p}(n_2 + p(n_3 + \frac{1}{p} + ...))) \tag{3.6}$$

i.e.,

$$n_0 + n_2 + n_4 + ... + p(n_1 + n_3 + ...). \tag{3.7}$$

This sequence does not converge real and p-adic topology. So such sequences must excluded from the boundary.

This example shows that, unlike in the classical case of continued fractions, all infinite words in the chosen generators of $\mathrm{PGL}_2(\mathbb{Q})$ will not represent convergent elements in some topology.

4. P-ADIC CONTINUED FRACTIONS

Our aim in this chapter is to give an overview of p-adic continued fractions. We first introduce p-adic numbers.

4.1 P-adic numbers

Firstly we will introduce p-adic numbers. We will emphasize on properties of p-adic norm, Hensel lemma and its congruences. We will need properties of p-adic numbers since we are going to investigate continued fractions in the p-adic sense. Parts of this section are taken from the Gouvea (2003) and Khinchin (1964) The real numbers, denoted by $\mathbb{R}$ obtained from rationals using by a completion process. The completion procedure applied to rationals with the usual Euclidean distance yields the real numbers. This distance comes from the Euclidean norm on $\mathbb{Q}$.

There is different way to describe the closeness between rationals. Let $p \in \mathbb{N}$ be any prime number. Define a map on $\mathbb{Q}$ as follows

$$| x |_p = \begin{cases} p^{-\operatorname{ord}_p x} & if \quad x \neq 0, \\ 0 & if \quad x = 0 \end{cases}$$

where

$$\operatorname{ord}_p x = \begin{cases} highest\ power\ of\ p\ that\ divides\ x & if \quad x \in \mathbb{Z}, \\ \operatorname{ord}_p k - \operatorname{ord}_p l, & if \quad x = k/l,\ k, l \in \mathbb{Z},\ l \neq 0 \end{cases}$$

is the p-adic order of x, also called the p-adic valuation of x.

Definition 4.1.1. Let X be a field. A norm $||.||$ is a function on X satisfying the following properties.

- $||x|| \geq 0$ for all $x \in X$
- $||x|| = 0$ if and only if $x = 0$
- $||xy|| = ||x||.||y||$ for all $x, y \in X$
- $||x + y|| \leq ||x|| + ||y||\ \forall x, y \in X$

The field X together with $||.||$ is called a normed field. We can define a metric by $d(x, y) = ||x - y||$ and this metric is said to the induced by the norm.
A norm is called $Non - Archimedean$ if it satisfies the additional condition

$$||x + y|| \leq \max(||x||, ||y||); \tag{4.1}$$

otherwise, we say that norm is Archimedian.

Proposition 4.1.2. $\mid x \mid_p$ *is a non-Archimedian norm on* $\mathbb{Q}$

Proof.

1. $\mid x \mid_p = 0 \iff x = 0$
2. $\operatorname{ord}_p(xy) = \operatorname{ord}_p x + \operatorname{ord}_p y$ So $\mid xy \mid_p = \mid x \mid_p \mid y \mid_p$ is satisfied.
3. To verify $||x + y|| \leq ||x|| + ||y|| \; \forall x, y \in X$ assume $x, y \neq 0$ since $x = 0$ or $y = 0$ is trivial. Let $x = k/l$ and $y = m/n$. Then $x + y = \dfrac{kn + ml}{ln}$ and

$$\begin{aligned} &\operatorname{ord}_p(x + y) = \operatorname{ord}_p(kn + ml) - \operatorname{ord}_p(ln) \\ &\leq \min(\operatorname{ord}_p(kn), \operatorname{ord}_p(ml)) - \operatorname{ord}_p m - \operatorname{ord}_p n \\ &= \min(\operatorname{ord}_p x, \operatorname{ord}_p y) \end{aligned} \tag{4.2}$$

Therefore $\mid x + y \mid_p = \max(\mid x \mid_p, \mid y \mid_p) \leq \mid x \mid_p + \mid y \mid_p$

We have also show that p-adic norm is satisfied strong triangle inequality. So it is non-Archimedean.

Example 6.

$$\begin{aligned} &\mid 15 \mid_7 = 7^0 = 1 \mid 208 \mid_7 = 7^0 = 1 \\ &\mid 150 \mid_3 = 3^{-1} = 1/3 \\ &\mid \frac{9}{50} \mid_3 = 3^{-\operatorname{ord}_3 \frac{9}{50}} = 3^{-2} = 1/9 \end{aligned} \tag{4.3}$$

We see that the Euclidean norms of the 307 and 11 are not equal but for 7-adic norms they are equal.

Remark 4.1.3. The p-adic norm takes only discrete values $\{p^n, n \in \mathbb{Z}\} \cup \{0\}$

Remark 4.1.4. If $x, y \in \mathbb{N}$, then $x \equiv y \ (modp^n)$ if and only if $\mid x - y \mid_p \leq 1/p^n$.

Definition 4.1.5. Let p be a fixed prime. We define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ according to the p-adic norm. We call $\mathbb{Q}_p$ the field of p-adic numbers.

For any $x \in \mathbb{Q}_p$, let $\{x_n\}$ be a Cauchy sequence of rational numbers that represent x. Then by definition

$$|x|_p = \lim_{n\to\infty} |x_n|_p$$

Let us consider the series

$$\frac{a_{-m}}{p^m} + \frac{a_{-m+1}}{p^{m-1}} + ... + a_0 + a_1 p + a_2 p^2 + ...$$

where $0 < a_{-m} < p$ and $0 \leq p$ for all $i > -m$. Partial sum of series form a Cauchy sequence. For every $\epsilon > 0$ we can choose n_0 such that $p^{n_0} < \epsilon$ and for $n > k > n_0$

$$\left| \sum_{-m}^{k} a_i p^i - \sum_{-m}^{n} a_i p^i \right|_p = \left| \sum_{k+1}^{n} a_i p^i \right|_p \leq \max_{k<i\leq n} (|a_i p^i|_p) \leq p^{-n_0} < \epsilon$$

so each series as above represent an element of $\mathbb{Q}_p$

Lemma 4.1.6. *If $x \in \mathbb{Q}$ and $|x|_p \leq 1$, then for any i there exist an integer $\gamma \in \mathbb{Z}$ such that $|\gamma - x|_p \leq p^{-i}$. The integer γ can be chosen in $\{0, 1, 2, ..., p^i - 1\}$ and is unique if chosen in this range.*

Theorem 4.1.7. *Khinchin (1964) Every equivalence class x in $\mathbb{Q}_p$ satisfying $|x|_p \leq 1$ has exactly one representative Cauchy sequence x_i such that*

(1) $x_i \in \mathbb{Z}$, $0 \leq a_i < p^i$ *for* $i = 1, 2, ...$

(2) $x_i \equiv x_{i+1} \pmod{p^i}$ *for* $i = 1, 2, ...$

If $x \in \mathbb{Q}_p$ with $|x|_p \leq 1$ then we can write all the terms x_i given by the previous theorem in the following way:

$$x_i = a_0 + a_1 p + ... + a_{i-1} p^{i-1}$$

where all $a_i' s$ are in $\{0, 1, 2, ..., p-1\}$. From previous theorem condition (2) give us

$$x_{i+1} = a_0 + a_1 p + ... + a_{i-1} p^{i-1} + a_i p^i.$$

Therefore x is represented by the following convergent series in p-adic topology

$$x = \sum_{n=0}^{\infty} a_n p^n$$

which can be thought of as a number in the base p that extends infinitely far to the left. Then we can write

$$x = \cdots a_n \cdots a_2 a_1 a_0$$

If $|x|_p > 1$ then we can multiply x by a power of p, so as to get a p-adic number $x' = xp^m$ that satisfies $|x'|_p = 1$

$$x = \sum_{n=-m}^{\infty} a_n p^n$$

where $a_{-m} \neq 0$ and $a_i \in \{0, 1, 2, ..., p-1\}$. This representation is called the canonical p-adic expansion of x.

Definition 4.1.8. A p-adic number $x \in \mathbb{Q}_p$ is said to be a p-adic integer if its expansion contains only nonnegative powers of p. The set of p-adic integers is denoted $\mathbb{Z}_p$. So one has

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i \right\}.$$

Proposition 4.1.9. *A p-adic integer has a multiplicative inverse in $\mathbb{Z}_p \Leftrightarrow a_0 \neq 0$. We will denote by the set of invertible elements $\mathbb{Z}_p^{\times}$,*

$$\mathbb{Z}_p^{\times} = \left\{ \sum_{i=1}^{\infty} a_i p^i;\ a_0 \neq 0 \right\}$$

This group is called the group of p-adic units.

4.1.10 Hensel's Lemma and Congruences

Hensel's lemma is an algorithm to solve polynomial equations in $\mathbb{Q}_p$. To illustrate this, let us extract $\sqrt{8}$ in $\mathbb{Q}_7$. We will find a sequence of $a_0, a_1, a_2, ...\ 0 \leq a_i \leq 6$, such that

$$(a_0 + a_1.7 + a_2.7^2 + ...)^2 = 1 + 7$$

From above we get $a_0^2 \equiv 1\ (mod\, 7)$, which implies $a_0 = 1$ or $a_0 = 6$. If $a_0 = 1$, then

$$1 + 2a_1.7 = 1.7\ (mod\ 7^2)$$

$$2a_1 = 1\ (mod\ 7)$$

and therefore $a_1 = 4$. Continuing this way we get a series $a = 1 + 4.7 + ...$ where each a_i after a_0 is uniquely determined. The above method for solving equations for example $x^2 - 8 = 0$ in $\mathbb{Q}_7$ can be generalized as 'Hensel's Lemma'.

Theorem 4.1.11. ***Hensel's Lemma.*** *Let* $F(x) = a_0 + a_1x + ... + a_nx^n$ *be a polynomial whose coefficients are* p*-adic integers. Let*

$$F'(x) = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}$$

be the derivative of $F(x)$*. Suppose* $\tilde{c_0}$ *is a* p*-adic integer which satisfies* $F(\tilde{c_0}) \neq 0 \pmod{p}$*. Then there exist a unique* p*-adic integer* c *such that* $F(c) = 0$ *and* $c = \tilde{c_0} \pmod{p}$

4.2 P-adic continued fractions

Continued fractions expansion for real numbers give a best rational approximation. Every rational number can be written a finite continued fraction and quadratic irrational numbers have a nice structure. Since real continued fractions plays an important role for real Diaphontine approximation, for finding an answer to similar questions Mahler, Schneider, Weger, Ruban, Bundschuh and Browkin studied on theory of p-adic continued fraction.

In Mahler (1940) gave a geometric representation for p-adic integers derived from continued fractions but this representation was not natural since a continued fraction construction algorithm did not proceed simply by choosing partial quotients and constructing remainders recursively. The method for constructing continued fractions for real numbers, is quite obvious. For any real number, α, there is only one integer, b, such that $0 \leq |\alpha - b| < 1$. However if $\alpha \in \mathbb{Q}_p$, there are infinitely many integers $b \in \mathbb{Z}$ such that $0 \leq |\alpha - b|_p < 1$. This choice was handled in two different ways and two different continued fraction representations were obtained. These continued fractions, which are very small differences between them, have defined Schneider (1970) and Ruban (1970). Though there are some experimental results, the analogues of the Lagrange and Hurwitz's theorem have not been proven with the continued fraction of Browkin. Laohakosol (1985) used Ruban's continued fraction definition and he characterized rational numbers. Wang (1985) also characterized the rational numbers with infinite Ruban continued fraction expansion and he was seemingly unaware of Ruban's work. Moreover Wang (1985) gave a sufficient condition on the partial quotients for a p-adic number to be transcendental.

Weger (1988) proved that some quadratic elements do not have eventually periodic expansions using Scheneider's p-adic continued fractions.
Bundschuh (1977) described that rational numbers have finite Schneider continued fraction expansion. Bundschuh also obtain some data on quadratic irrationals which have non-

periodic Schneider continued fraction expansion. Ooto (2014) has found a similar result with Weger but Ooto used to the algorithm given by Ruban. Browkin (2001) modified Ruban's definition and gave some p-adic continued fraction algorithms; however, the periodicity has not been proved for the continued fractions obtained by applying his algorithms to quadratic elements.

Hirsh and Washington (2011) gave a combinatorial characterization of rational numbers which have terminating expansions and they proved an analogue of Khinckin's theorem. Given a real number x the Gauss map outputs a sequence of integers $[c_0; c_1, ...]$ called the convergents. Schneider's map is defined on the open ball $p\mathbb{Z}_p$, by the following procedure. For $x \in p\mathbb{Z}_p$ suppose $|x|_p = p^{\upsilon(x)}$. Then set

$$\delta(x) = \frac{p^a}{x} - b$$

with $a = \upsilon(x) \in \mathbb{N}$ and $b \in \{1, 2, ..., p-1\}$ uniquely chosen such that $\left|\frac{p^a}{x} - b\right|_p$. Applying the map δ repeatedly we see that

$$x = \cfrac{p^{a_0}}{b_1 + \cfrac{p^{a_1}}{b_2 + \cfrac{p^{a_2}}{b_3 + \ddots}}}$$

Thus in this case the algorithm outputs a sequence of pairs $\alpha(n) = (a_n, b_n) \in \mathbb{N} \times \{1, 2, ..., p-1\}$ $(n = 1, 2, ...)$.. We will consider the dynamical system $(p\mathbb{Z}_p, \mathbb{B}, \mu, T_p)$ where $\mathbb{B}$ is σ-algebra on $p\mathbb{Z}_p$ and μ is Haar measure on $p\mathbb{Z}_p$. For the Haar measure it is the case that $\mu(pa + p^m\mathbb{Z}_p) = p^{1-m}$. The following properties are due to **?**. T_p is a measure preserving with respect to μ, i.e. $\mu(T_p^{-1}(A)) = \mu(A)$ for all $A \in \mathbb{B}$.
T_p is ergodic, i.e. $\mu(B)$ or 1 for any $B \in \mathbb{B}$ with $T_p^{-1}(B) = B$.
The p-adic analogue of Khinchin's Theorem for almost all $x \in p\mathbb{Z}_p$ the p-adic continued fraction expansion satisfies

$$lim_{n\to\infty}\frac{a_1 + a_2 + ... + a_n}{n} = \frac{p}{p-1}$$

Hancl and Nair (2013) developed an analogue of Khinckin's theorem. They proved if p_n $(n = 1, 2, ...)$ denotes the sequences of rational primes then

$$lim_{n\to\infty}\frac{a_{p_1} + a_{p_2} + ... + a_{p_n}}{n} = \frac{p}{p-1}$$

almost everywhere with respect to Haar measure.

Murru and Terracini (2018) studied on multidimensional continued fractions in the field of p-adic numbers $\mathbb{Q}_p$. They gave some conditions about their convergence and they proved that convergent multidimensional continued fractions always strongly converge in $\mathbb{Q}_p$ contrarily to the real case where strong convergence is not ever guaranteed.

Dalloul (2018) give a sufficient condition on two p-adic continued fractions α, β so that the α, β, $\alpha \mp \beta$, $\alpha.\beta^{\mp 1}$ be p-adic irrationals. Moreover, he improved some results regarding the transcendentality of p-adic continued fractions.

Capuano and Zannier (2018) worked on the expansion of rationals and quadratic irrationals for the p-adic continued fractions introduced by Ruban (1970). We know that no analogue of Lagrange's theorem holds for quadratic irrational numbers and for some rational numbers may have nonterminating periodic continued fraction expansions. Capuano and Zannier (2018) gave general criterion explicitly to demonstrate the periodicity of the expansion in both the rational numbers and the quadratic irrationals.

4.2.1 Ruban's p-adic continued fractions

Ruban (1970) developed an algorithm by making p-adic analogues of integer and fractional parts that are used constituting real continued fraction. For a p-adic number γ,

$$\gamma = \sum_{n=k}^{\infty} c_n p^n,\ 0 \leq a_n < p$$

where $k \in \mathbb{Z}$, $c_n \in \{0, 1, 2, ..., p-1\}$ for $n \geq k$, and $c_n \neq 0$, the fractional part is denoted by $\{\gamma\}$

$$\{\gamma\} = \sum_{n=k}^{0} c_n p^n \quad if k \leq 0,$$

or $\{\gamma\} = 0$ otherwise. The integer part is denoted by $[\gamma]$ and

$$[\gamma] = \sum_{n=1}^{\infty} c_n p^n$$

After these notations, we can define Ruban continued fractions.

Definition 4.2.2. For $\gamma \in p\mathbb{Z}p$, a Ruban continued fraction is constructed as follows;

1. $\gamma_0 = \gamma$, $\gamma_{n+1} = d_n/\gamma_n - c_n$
2. $c_0 = 0$ and $d_0 = 0$

3. $c_n = \{\gamma_n\}$ and $d_n = 1$

for $n \geq 0$ as long as $\gamma_n \neq c_n$. If $\gamma_n = c_n$ then algorithm terminates.

From this definition $\upsilon(c_n) < 0$, $\upsilon(\gamma_n)$, and $\gamma_n = 1/[\gamma_{n-1}]$ for $n \geq 1$. Ruban showed that the continued fraction which produced in this way always converges to γ, and that any continued fraction with $c_0 = 0$, $\upsilon(c_n) < 0$, and $d_n = 1$ for $n \geq 1$ converges to a number in $p\mathbb{Z}p$.

Next we define sequences of rational numbers P_n and Q_n by

$$P_{-1} = 1, \quad Q_{-1} = 0, \quad P_0 = 0, \quad Q_0 = 1$$
$$P_{n+1} = a_n P_n + P_{n-1}, \quad Q_{n+1} = a_n Q_n + Q_{n-1} \quad (n \geq 0).$$

It is easily seen that

$$\frac{P_n}{Q_n} = \cfrac{1}{c_0 + \cfrac{1}{c_1 + \cfrac{1}{\cdots + \cfrac{1}{c_{n-1}}}}}. \tag{4.4}$$

We call $\frac{P_n}{Q_n}$ the n-th convergent of Ruban continued fraction. We must show that such a Ruban continued fraction converges to γ. Let n be a positive integer then at the n-th step of construction we have

$$\gamma_{n-1}^{-1} = c_{n-1} + \gamma_n$$

where $\gamma_{n-1}, \gamma_n \in p\mathbb{Z}_p$. If $\gamma_n = 0$ for some n then Ruban continued fraction is finite and is equal to γ. If $\gamma_n \neq 0$ for all positive integer n, then by induction

$$\gamma - \frac{P_n}{Q_n} = \frac{(-1)^n}{Q_n(\gamma_n^{-1} Q_n + Q_{n-1}}) \quad (n \geq 1)$$

since $|c_i|_p > 1$, by induction and from $Q_{n+1} = a_n Q_n + Q_{n-1}$ we have

$$|Q_n|_p = |c_0 c_1 ... c_{n-1}|_p \quad (n \geq 1).$$

Also from $\gamma_{n-1}^{-1} = c_{n-1} + \gamma_n$ we get $|\gamma_n^{-1}|_p = |c_n|_p$ $(n \geq 1)$. So

$$|\gamma - \frac{P_n}{Q_n}|_p = |c_0^2 c_1^2 ... c_{n-1}^2 c_n|_p^{-1} \to 0 \quad n \to \infty.$$

That is Ruban continued fraction converges to γ.

Given a Ruban continued fraction of the form (3.4) by induction it easily get

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}}{Q_{n-1}Q_n}$$

and then

$$|\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}}|_p = |Q_{n-1}Q_n|_p^{-1} = |c_0^2 c_1^2 ... c_{n-2}^2 c_{n-1}|_p^{-1} \to 0 \quad n \to \infty$$

implying that the Ruban continued fraction converges to some $\gamma \in p\mathbb{Z}_p$.

4.2.3 Browkin's p-adic continued fractions

Browkin's papers Browkin (2001) and Browkin (2001) on p-adic continued fractions are given several algorithms for computing continued fraction approximations to p-adic square roots. Browkin developed Ruban's continued fraction algorithm. We can see as modification of a Ruban's definition. For a p-adic number γ,

$$\gamma = \sum_{n=k}^{\infty} a_n p^n, \ 0 \leq a_n < p$$

where $k \in \mathbb{Z}$, $a_n \in \{-\frac{p-1}{2}, ..., -1, 0, 1, 2, ..., \frac{p-1}{2}\}$ for $n \geq k$, and $a_n \neq 0$, the fractional part is denoted by $\{\gamma\}$. The integer part is denoted by $[\gamma]$. The fractional and integer part defined as above Ruban's definition.

Then we can define Browkin continued fraction algorithm.

Definition 4.2.4. For $\gamma \in \mathbb{Q}p$, we define inductively c_n and d_n as follows;

1. Let $c_0 = \gamma$ and $d_0 = [\gamma]$
2. If $d_0 = c_0$, then c_1, d_1 are not defined.
3. If $d_0 \neq c_0$ then $c_1 = (c_0 - d_0)^{-1}$ and $d_1 = [c_1]$
4. If c_j, d_j are defined for $j = 0, 1, ..., k$ and $d_k = c_k$, then c_{k+1} and d_{k+1} are not defined.
5. If $d_k \neq c_k$ then let $c_{k+1} = (c_k - d_k)^{-1}$ and $d_{k+1} = [c_{k+1}]$

We call the sequence (d_n) the p-adic continued fraction of γ.

For an arbitrary sequence (d_n) and $\upsilon(d_n) < 0$ for $n \geq 0$ we define the partial quotients

$$\frac{P_n}{Q_n} = [d_0; d_1, ..., d_n] \quad (n = 0, 1, ...)$$

as usual,

$$P_0 = d_0, \quad P_1 = d_0 d_1 + 1, \quad P_n = d_n P_{n-1} + P_{n-2}$$
$$Q_0 = 1, \quad Q_1 = d_1, \quad Q_n = d_n Q_{n-1} + Q_{n-2}$$

for $n \geq 2$. If the sequence d_n is infinite, then the sequence $(\frac{P_n}{Q_n})$ is convergent and moreover if d_n was obtained by the above algorithm applied to $\gamma \in \mathbb{Q}_p$, then $\lim \dfrac{P_n}{Q_n} = \gamma$. In this case we use the notation $\gamma = [d_0; d_1, ...]$.

When the p-adic continued fraction is produced by means of Browkin's Algorithm, it has been shown to always p-adically converge. That is

$$\lim_{n \to \infty} \left| \frac{P_n}{Q_n} - \sqrt{x} \right|_p = 0 \quad \sqrt{x} \in \mathbb{Q}_p$$

in any case of whether or not it is periodic.

4.2.5 Schneider's p-adic continued fractions

P-adic continued fraction algorithm is build firstly by Mahler (1940). Schneider (1970) developed Mahler's algorithm and gave a process of p-adic continued fraction expansion. For a p-adic integer $\gamma \in \mathbb{Z}_p$

$$\gamma = \sum_{n=0}^{\infty} a_n p^n, \ 0 \leq a_n < p$$

Scheneider defined continued fraction by $c_0 = a_0$, $c_n \in \{1, 2, ..., p-1\}$, $d_n = p^{e_n}$ for some e_n and $\upsilon(\gamma_n) = 0$ for $n > 0$

Definition 4.2.6. For a p-adic integer γ a Schneider continued fraction constructing in the following way;

1. $\gamma_0 = \gamma$, $\gamma_{n+1} = d_n / \gamma_n - c_n$
2. For a chosen unique $c_n \in \{0, 1, ..., p-1\}$ such that $\upsilon(\gamma_n - c_n) > 0$
3. $d_n = p^{e_n}$ where $e_n = \upsilon(\gamma_n - c_n)$ for $n \geq 0$

as long as $\gamma_n \neq c_n$. If $\gamma_n = c_n$ then algorithm terminates.

4.2.7 Convergence of Schneider's continued fractions

We shall consider the continued fractions $SCF_p(q_n)$ with integral general terms q_n. The simplest such continued fractions are of the following type, which will be called a 'p-regular' continued fraction.

$$SCF_p(q_n) = q_1 + \cfrac{p}{q_2 + \cfrac{p}{q_3 + \cfrac{p}{q_4 + \cdots}}} \tag{4.5}$$

Our first task is to study the condition for the convergence of 4.5 in the real number field and p-adic number field. For this purpose we introduce, for each positive integer n, independent indeterminates $x_1, ..., x_n$ and put

$$x_1 + \cfrac{p}{x_2 + \cfrac{p}{\ddots + \cfrac{p}{x_n}}} \tag{4.6}$$

For $n = 1, 2, 3$ they are computed explicitly as

$$[x_1] = x_1 = \frac{x_1}{1} \tag{4.7}$$

$$[x_1, x_2] = x_1 + \frac{p}{x_2} = \frac{p + x_1 x_2}{x_2} \tag{4.8}$$

$$[x_1, x_2, x_3] = x_1 + \cfrac{p}{x_2 + \cfrac{p}{x_3}} = \frac{px_1 + px_3 + x_1 x_2 x_3}{p + x_2 x_3} \tag{4.9}$$

We shall find the sequences of pairs of coprime polynomials $P_n(x_1, ..., x_n)$, $Q_n(x_1, ..., x_n)$ is element of $\mathbb{Z}[x_1, ..., x_n]$ such that

$$[x_1, ..., x_n] = \frac{P_n(x_1, ..., x_n)}{Q_n(x_1, ..., x_n)} \tag{4.10}$$

From the above computation of $[x_1, ..., x_n]$ for $n = 1, 2, 3$ we have

$$P_1 = x_1, \quad Q_1 = 1 \tag{4.11}$$

$$P_2 = p + x_1 x_2, \quad Q_2 = x_2 \tag{4.12}$$

$$P_3 = px_1 + px_3 + x_1 x_2 x_3, \quad Q_3 = p + x_2 x_3 \tag{4.13}$$

If we set $P_0 = 1, Q_0 = 0$ then we observe that the following equalities hold for $n = 2, 3$

$$P_n = x_n P_{n-1} + pP_{n-2} \tag{4.14}$$

$$Q_n = x_n Q_{n-1} + pQ_{n-2} \tag{4.15}$$

We can show that 4.14 and 4.15 holds for all $n \geq 1$. It suffices to show that the polynomials defined by the recurring formulas satisfies 4.10. This will be proved by induction on n. Next we show that the polynomials $P_n(x_1, ..., x_n), Q_n(x_1, ..., x_n)$ determined as above satisfy the equalities

$$\begin{aligned} P_n Q_{n-1} - P_{n-1} Q_n &= (-1)^n p^{n-1} \\ P_n Q_{n-2} - P_{n-2} Q_n &= (-1)^{n-1} p^{n-2} x_n \end{aligned} \tag{4.16}$$

Kojima (2012) find some results for a special type of Schneider continued fraction. These results are shown by the following theorems.

Theorem 4.2.8. *Let q_1 be an integer and $q_2, q_3, q_4, ...$ be natural numbers. If q_n is not divisible by p for any $n \geq 2$ then $SCF_p(q_n)$ converges in $\mathbb{R}$ and in $\mathbb{Q}_p$ simultaneously.*

Proof. Suppose that q_n satisfies the same condition as 4.2.8 and let

$$a_n = [q_1, q_2, ..., q_n] \tag{4.17}$$

Then for $n \geq 2$

$$a_n - a_{n-2} = \frac{(-1)^{n-1} p^{n-2} q_n}{Q_n Q_{n-2}} \tag{4.18}$$

From the definition of Q_n we have $Q_n > 0$ for $n \geq 1$. Therefore from 4.18

$$\begin{aligned} a_1 \leq a_3 \leq a_5 \leq ..., \\ a_2 \geq a_4 \geq a_6 \geq ... \end{aligned} \tag{4.19}$$

On the other hand

$$a_n - a_{n-1} = \frac{(-1)^n p^{n-1}}{Q_n Q_{n-1}} \tag{4.20}$$

This gives $a_{2m} \geq a_{2m-1}$ for $m \geq 1$ From 4.19 the sequences a_{2m-1} are monotonically increasing and are bounded. So they are converge in $\mathbb{R}$ We can prove that

$$\lim_{m\to\infty} a_{2m-1} = \lim_{m\to\infty} a_{2m} \tag{4.21}$$

This completes the proof of convergence for a_n in $\mathbb{R}$

Note. Approximation rate for $p = 1$ is higher!

Next we prove that a_n converges in $\mathbb{Q}_p$

$$\begin{aligned} |a_n - a_{n-1}|_p &= |\frac{(-1)^n p^{n-1}}{Q_n Q_{n-1}}|_p \\ &= p^{-(n-1)} \to 0 \end{aligned} \tag{4.22}$$

This completes the proof. □

Remark 4.2.9. If we modify the condition Theorem 4.2.8 and assume that q_n are not divisible by p for all but finitely many n, then Theorem 4.2.8 becomes false. For example, let $q_1 = 0, q_2 = p$ and $q_n = p - 1$ for $n \geq 3$. Then we have

$$\begin{aligned} Q_0 &= 0 \\ Q_1 &= 1 \\ Q_2 &= p \\ Q_n &= p^{n-1} \end{aligned} \tag{4.23}$$

a_n does not converge in Q_p.

Theorem 4.2.10. *Let q_n $n \geq 1$ be a sequence of integers such that for $n \geq 2$, q_n is positive and is not divisible by p. Then the continued fraction 4.5 converges to a p-adic integer in $\mathbb{Q}_p$. Conversely any p-adic integer α is obtained as the limit of a continued fraction 4.5 satisfying the above conditions.*

Proof. As before we set the following notations

$$P_n = P_n(q_1, q_2, ..., q_n)$$

$$Q_n = Q_n(q_1, q_2, ..., q_n)$$

and $a_n = \dfrac{P_n}{Q_n}$ then as in proof of previous theorem we have $|Q_n|_p = 1$ for $n \geq 1$. Moreover from the definition of P_n, we have $|P_n|_p \leq 1$ for $n \geq 1$. This implies

$$|a_n|_p = |\frac{P_n}{Q_n}|_p = \frac{|P_n|_p}{|Q_n|_p} \leq 1.$$

So $a_n \in \mathbb{Z}_p$ for $n \geq$. Since $\mathbb{Z}_p$ is closed, the limiting value of a_n is also a p-adic integer. Next for given $\alpha \in \mathbb{Z}_p$, we construct $q_1, q_2, q_3, ...$ such that the continued fraction 4.5 converges to α in $\mathbb{Q}_p$. First we set $\alpha_1 = \alpha$ and take an integer q_1 such that $p|(\alpha_1 - q_1)$. Note that $\alpha_1 \in \mathbb{Z}_p$ so we can choose an integer q_1 with $p|(\alpha_1 - q_1)$. Next, we put $\alpha_1 - q_1 = p\alpha_2^{-1}$. Then from the condition of q_1, we see that α_2 is a p-adic unit. Now we take a natural number q_2 such that $p|(\alpha_2 - q_2)$. Since α_2 is a p-adic unit, the integer q_2 is not divisible by p. We continue in this way,

$$\alpha_1 = \alpha,$$
$$p|(\alpha_1 - q_1), \quad \alpha_1 - q_1 = p\alpha_2^{-1}$$
$$p|(\alpha_2 - q_2), \quad \alpha_2 - q_2 = p\alpha_3^{-1}$$
$$\cdots$$
$$p|(\alpha_n - q_n), \quad \alpha_n - q_n = p\alpha_{n+1}^{-1}$$

then we see that α_n are p-adic units for $n \geq 2$ and q_n are not divisible by p for $n \geq 2$. Now we show that the continued fraction

$$q_1 + \cfrac{p}{q_2 + \cfrac{p}{q_3 + \cfrac{p}{\vdots}}}$$

converges to α. From the definition of α_n, we can express by

$$\alpha = \alpha_1 = q_1 + \frac{p}{\alpha_2} = q_1 + \cfrac{p}{q_2 + \cfrac{p}{\alpha_3}} \cdots = q_1 + \cfrac{p}{q_2 + \cfrac{p}{q_3 + \cfrac{p}{\vdots + \cfrac{p}{\alpha_n}}}}.$$

Hence it follows that

$$\begin{aligned}
\alpha - a_{n-1} &= \frac{P_n(q_1, q_2, ..., q_{n-1}, \alpha_n)}{Q_n(q_1, q_2, ..., q_{n-1}, \alpha_n)} - \frac{P_{n-1}(q_1, q_2, ..., q_{n-1}}{Q_{n-1}(q_1, q_2, ..., q_{n-1}} \\
&= \frac{P_n(q_1, q_2, ..., q_{n-1}, \alpha_n) Q_{n-1}(q_1, q_2, ..., q_{n-1}}{Q_{n-1}(q_1, q_2, ..., q_{n-1} Q_n(q_1, q_2, ..., q_{n-1}, \alpha_n)} \\
&- \frac{P_{n-1}(q_1, q_2, ..., q_{n-1}) Q_n(q_1, q_2, ..., q_{n-1}, \alpha_n)}{Q_{n-1}(q_1, q_2, ..., q_{n-1} Q_n(q_1, q_2, ..., q_{n-1}, \alpha_n)} \\
&= \frac{(-1)^n p^{n-1}}{Q_{n-1}(q_1, q_2, ..., q_{n-1} Q_n(q_1, q_2, ..., q_{n-1}, \alpha_n)}.
\end{aligned}$$

on the other hand as in the proof of previous theorem we see that $|Q_n|_p = 1$ for $n \geq 1$. So we conclude that

$$|\alpha - a_{n-1}|_p = \left| \frac{(-1)^n p^{n-1}}{Q_{n-1}(q_1, q_2, ..., q_{n-1} Q_n(q_1, q_2, ..., q_{n-1}, \alpha_n)} \right|_p = \frac{1}{p^{n-1}} \to 0 \quad (n \to \infty)$$

□

Question. Assume that q_n are divisible by p for infinitely many n such that continued fraction converges in $\mathbb{R}$ but not in $\mathbb{Q}_p$. Can we find a continued fraction satisfying these?

Remark 4.2.11. Let $\alpha \in \mathbb{Z}_p$. From the proof of Theorem 4.2.10 for any sequence $q_1, q_2, q_3, ...$ which satisfy the condition in the proof, the continued fraction 4.5 converges to α. So there exists infinitely many sequences q_n such that the continued fraction 4.5 converges to α.

Question. Find an example of eventually periodic two continued fraction converging the same p-adic integer.

4.2.12 Continued Logarithm Algorithm

The continued logarithm algorithm was introduced by Gosper around 1978. His purpose of defining this concept was to use the continued fractions where it could not be used effectively. For example the Avogadro number is 23 digits long, but we just know the first six digits. Therefore even the integer part of its continued fraction is unknown. Recently Brabec, Borwein, Calkin, Lindstrom, and Mattingly has worked on continued logarithm algorithm Brabec (2006), Brabec (2007), Borwein and Lynch (2017). Continued logarithm algorithm can be seen as mutation of continued fraction algorithm Jonathan and Lynch. (2017). Every real number $x \geq 1$ is expanded continued logarithm algorithm as follows

$$x = 2^{k_0}(1 + \cfrac{1}{2^{k_1}(1 + \cfrac{1}{2^{k_2}(1 + \cfrac{1}{\ddots})})}) \tag{4.24}$$

where $k_i \in \mathbb{N}$ for $i \geq 0$. This algorithm can be described as the following way Jonathan and Lynch. (2017)

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \geq 2 \\ \dfrac{1}{x-1} & \text{if } 1 < x < 2 \end{cases}$$

The algorithm continues by iterating g until the result is 1; the division steps $x \to x/2$ are applied repeatedly until $x \to \frac{1}{x-1}$ is used, or $x = 1$.. When $x = 1$ the algorithm terminates. The number of division steps is expressed by the number of k's in (3.25).

Example 7.

$$\frac{64}{5} = 2^3(1 + \cfrac{1}{2^0(1 + \cfrac{1}{2^0(1 + \cfrac{1}{2^1})})}) \tag{4.25}$$

We can denote by $x = \langle k_0, k_1, ..., k_n \rangle$ for abbreviation. So $\dfrac{64}{5} = \langle 3, 0, 0, 1 \rangle$. Similarly $2^n = \langle n \rangle$ for $n \geq 0$. Recurrence relations for continued logarithms as follows.

Theorem 4.2.13. *Jonathan and Lynch. (2017)Let x be the real number with continued logarithm $\langle k_0, k_1, k_2, ...\rangle$, and let x_n be the n-th continued logarithm convergent. Then $x_n \frac{r_n}{s_n}$ where $r_{-1} = 1$, $s_{-1} = 0$, $r_0 = 2^{k_0}$, $s_0 = 1$ and*

$$r_{n+1} = 2^{k_{n+1}} r_n + 2^{k_n} r_{n-1}$$

and

$$s_{n+1} = 2^{k_{n+1}} s_n + 2^{k_n} s_{n-1}$$

Proof. The proof is immediately follows from classical recurrence of continued fractions.

□

4.2.14 The p-adic Division Algorithm and Euclidean Algorithm

Both the division algorithm and Euclidean algorithm are elements of classical number theory. With the goal of recreating the real aspects of classical number theory in the p-adics, we know how to re-invent these two classical algorithms. Let $\mathbb{Z}[\frac{1}{p}] = \{\frac{m}{p^k} : \quad m, k \in \mathbb{Z}\}$. Notice that $\mathbb{Z}[\frac{1}{p}]$ is closed under addition and multiplication; in fact, it is a subring of $\mathbb{Q}$. We will give some results in Lager (2009) .

Theorem 4.2.15. *Let p be an odd prime. Then given any σ and $\tau \in \mathbb{Q}_p$ where $\tau \neq 0$ there exists a unique $q \in \mathbb{Z}[\frac{1}{p}]$ with $\mid q \mid_\infty < \frac{p}{2}$ and $\eta \in \mathbb{Q}_p$ with $\mid \eta \mid_p < \mid \tau \mid_p$ such that*

$$\sigma = q\tau + \eta \tag{4.26}$$

Definition 4.2.16. (p-adic Euclidean Algorithm). Let p be an odd prime. The p-adic Euclidean algorithm applied to σ and $\tau \in \mathbb{Q}_p$ and where $\tau \neq 0$ is as follows. First apply division algorithm to σ and τ to produce

$$\sigma = q_1\tau + \eta_1 \tag{4.27}$$

In each subsequent step ”shift to the left" and apply division algorithm again,

$$\begin{aligned} \tau &= q_2\eta_1 + \eta_2 \\ \eta_1 &= q_3\eta_2 + \eta_3 \\ &\vdots \\ \eta_{i-2} &= q_i\eta_{i-1} + \eta_i \\ &\vdots \end{aligned}$$

This process either continues indefinitely or stops when $\eta i = 0$. The outputs of this algorithm are the sequences q_i and ηi. When appropriate we will consider the inputs as $\sigma = \eta_{-1}$ and $\tau = \eta_0$.

Example 8. The 7-adic Euclidean Algorithm applied to $\frac{181625}{11}$ and $\frac{10555}{2}$ yields

$$\frac{181625}{11} = (2)(\frac{10555}{2}) + (\frac{9360}{11}.7^1)$$
$$\frac{10555}{2} = (\frac{12}{7})(\frac{9360}{11}.7^1) + (\frac{-2215}{22}.7^2)$$
$$\frac{9360}{11}.7^1 = (\frac{-10}{7})(\frac{-2215}{22}.7^2) + (\frac{-5}{11}.7^4)$$
$$\frac{-2215}{22}.7^2 = (\frac{50}{7^2})(\frac{-5}{11}.7^4) + (\frac{-5}{22}.7^5)$$
$$\frac{-5}{11}.7^4 = (\frac{2}{7})(\frac{-5}{22}.7^5) + 0$$

This algorithm stops since $\eta_5 = 0$. The outputs are the sequences

$$q_i = \{2, \frac{12}{7}, \frac{-10}{7}, \frac{50}{7^2}, \frac{2}{7}\}$$

$$\eta i = \{\frac{9360}{11}.7^1, \frac{-2215}{22}.7^2, \frac{-5}{11}.7^4, \frac{-5}{22}.7^5\}$$

The classical Euclidean algorithm also computes the simple continued fraction form of $\frac{a}{b}$:

$$\frac{a}{b} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \cdots}}} = [q_1, q_2, q_3, ...]$$

where $q_i \in \mathbb{Z}$ are the quotients obtained during the Euclidean algorithm. Browkin defined a simple continued fraction form of a p-adic number, ζ, as follows:

$$\zeta = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}} = [b_1, b_2, b_3, ...]$$

where $b_i \in \mathbb{Q}$ with $\mid b_i \mid_\infty < \frac{p}{2}$. The following summarizes Browkin method. For ζ with $v(\zeta) = m$ as in $\zeta = \sum_{j=m}^{\infty} c_j p^j = c_m p^m + c_{m+1} p^{m+1} + ...$ where m is (possibly negative) integer and $c_j \in \{\frac{1-p}{2}, ..., \frac{p-1}{2}\}$ define

$$\pi(\zeta) = \begin{cases} 0, & \text{if } m > 0 \\ \sum_{j=m}^{0} c_j p^j, & \text{otherwise} \end{cases}$$

Theorem 4.2.17. *(Browkin). Let $\zeta \in \mathbb{Q}_p \setminus 0$ and let $\zeta_1 = \zeta$. For $i \geq 1$,let $b_i = \pi(\zeta_i), \zeta_{i+1} = (\zeta_i - b_i)^{-1}$ until $\zeta_i - b_i = 0$ then $\zeta = [b_1, b_2, b_3, ...]$*

Theorem 4.2.18. *Let $\zeta \in \mathbb{Q}_p \setminus 0$, and let q_i be the outputs of the p-adic Euclidean algorithm applied to ζ and 1. Then $\zeta = [q_1, q_2, q_3, ...]$. Using the theorem in our previous example one computes that*

$$\frac{\frac{181625}{11}}{\frac{10555}{2}} = 2 + \cfrac{1}{\frac{12}{7} + \cfrac{1}{\frac{-10}{7} + \cfrac{1}{\frac{-50}{7^2} + \frac{1}{\frac{2}{7}}}}}$$

Even though p-adic Euclidean algorithm and Browkin's method produce the same results, the p-adic Euclidean algorithm is computationally quicker since performing the p-adic division algorithm only requires an inversion modulo a prime power. Further last theorem tells us exactly under what conditions the p-adic Euclidean algorithm terminates.

Example 9. The 5-adic Euclidean Algorithm applied to 325 and 35 yields

$$\begin{aligned} 325 &= (0)(35) + (13.5^2) \\ 35 &= (\frac{-11}{5})(13.5^2) + (6.5^3) \\ 13.5^2 &= (\frac{-2}{5})(6.5^3) + (1.5^4) \\ 6.5^3 &= (\frac{6}{5})(5^4) + (0.5^5) \end{aligned}$$

It is important to determine coefficients in p-adic Euclidean algorithm process We take a consideration following situations

1. $q \in \mathbb{Z}[\frac{1}{p}]$
2. $\mid q \mid_\infty < \frac{p}{2}$
3. $\mid \eta \mid_p < \mid \tau \mid_p$
4. Coefficients of η_i can be rational but if coefficients of η_i is equal $\frac{a}{b}$, $(a, b) = 1$ and $b \nmid p$.
5. if $q = \frac{r}{p}$ $(r, p^2) = 1$
   In view of these circumstances, we apply p-adic Euclidean algorithm.

Corollary 4.2.19. *Lager (2009) For σ, $\tau \in \mathbb{Q}_p$ $\tau \neq 0$, the p-adic Euclidean algorithm terminates in a finite number of steps if and only if $\sigma/\sigma \in \mathbb{Q}$*

Question. In the classical setting the division algorithm leads to modular arithmetic. Is there a similar theory to come from the p-adic division algorithm?

Question. How does the sequence η_i from the computation (σ, τ) relate to the sequence $\{\eta_{i'}\}$ from computing (σ, τ)? Examples quickly show that they are not always related in the simple way the remainders are in the classical case.

5. MATERIALS AND METHODS

Traditional methods of the theory of continued fractions ergodic theory, Diophantine analysis, Pade approximant etc. In this thesis we give an overview of the methods used in the literature Beskin (1986), Olds, Moore (1964). With a similar motivation Mahler (1940) Schneider (1970) Weger (1988), Ruban (1970), Bundschuh (1977) Browkin (2001) studied on theory of p-adic continued fractions. We give their methods in this thesis.

In this thesis, in addition to these methods we develop a group theoretical approach to continued fractions. Namely p-adic continued fractions are viewed as boundary points of some groups of $\mathrm{PGL}_2(\mathbb{Q})$. This method helps us to interpret Euclid's antyphairesis process in the context of p-adic continued fractions.

One of the aims of this thesis is to find series expansions for some periodic p-adic continued fractions. In this context we develop three methods. The first method involves a kind of iterated geometric series. This method can help prove convergence. However, it is very difficult to obtain explicit expansion by using this method. The second method involves a functional equation. But we couldn't get explicit expansion. The third method involves differential function equation. Using the third method we can get explicit expansion for some periodic p-adic continued fractions. We use Mapple and Mathematica to calculate derivatives.

We have exploited these methods to find the series expansions for some simple periodic p-adic continued fractions. These methods needs to be developed further to get the series expansions of general periodic p-adic continued fractions.

6. EXPANSIONS OF SOME PERIODIC P-ADIC CONTINUED FRACTIONS

Our work in this chapter is based on the following theorem. We proved the following theorem and its generalization. We will give explicit formulas for some type of continued fractions. Expansion of the following continued fraction gives us Catalan numbers. We have achieved more general numbers for different expansions. Note that the continued fraction

$$1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-\vdots}}} \tag{6.1}$$

converges in the real topology for $|x| < 1$, $0 < x < 1$ Lorentzen and Waadeland (1992). We discovered that it was also a convergent in the p-adic case inspired by this real situation.

Theorem 6.0.1. *The continued fraction*

$$1-\cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{1-\vdots}}}$$

converges in the p-adic topology to the number $1-\sum_{n=1}^{\infty}\dfrac{1}{2(2n-1)}\binom{2n}{n}p^n$.

Before proving this theorem let us show the more general continued fractions are convergent in p-adic topology.

Theorem 6.0.2. *The following continued fraction*

$$1-\cfrac{x_0}{1-\cfrac{x_1}{1-\cfrac{x_2}{1-\ddots}}} \tag{6.2}$$

converges in p-adic topology where $|x_i|_p < 1$.

Proof. Let us assume

$$P_0 = 1,\ Q_0 = 0$$

then we have

$$P_1 = 1,\ Q_1 = 1, \tag{6.3}$$

$$P_2 = 1 - x_0,\ Q_2 = 1, \tag{6.4}$$

$$P_3 = 1 - x_1 - x_0,\ Q_3 = 1 - x_1, \tag{6.5}$$

$$... \tag{6.6}$$

$$P_n = P_{n-1} - x_{n-2}P_{n-2},\ Q_n = Q_{n-1} - x_{n-2}Q_{n-2} \tag{6.7}$$

We claim that

$$P_nQ_{n-1} - P_{n-1}Q_n = -x_{n-2}x_{n-3}x_{n-4}...x_1x_0$$

for $n = 2$ $P_2Q_1 - P_1Q_2 = -x_0$
Now let $n \geq 2$ and assume that for $n - 1$

$$P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1} = -x_{n-3}x_{n-4}x_{n-5}...x_1x_0 \tag{6.8}$$

then using the induction hypothesis

$$\begin{aligned} P_nQ_{n-1} - P_{n-1}Q_n &= x_{n-2}(P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1}) \\ &= x_{n-2}(-x_{n-3}x_{n-4}x_{n-5}...x_1x_0) = -x_{n-2}x_{n-3}x_{n-4}...x_1x_0. \end{aligned}$$

On the other hand we know that $|x_i|_p < 1$. Then

$$|Q_n|_p = |Q_{n-1} - x_{n-2}Q_{n-2}|_p = \max(1, |x_{n-2}|_p) = 1. \tag{6.9}$$

In that case, to show convergence in p-adic topology

$$\begin{aligned} |a_n - a_{n-1}|_p &= |\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}}|_p \\ &= |\frac{P_nQ_{n-1} - P_{n-1}Q_n}{Q_nQ_{n-1}}|_p \\ &= |\frac{-x_{n-2}x_{n-3}x_{n-4}...x_1x_0}{Q_nQ_{n-1}}|_p \\ &= |-x_{n-2}x_{n-3}x_{n-4}...x_1x_0|_p < \frac{1}{p^n} \to 0 \quad (n \to \infty) \end{aligned}$$

□

Now let us take a wider perspective on these continued fractions. In the most general sense we are trying find a series representation of the continued fraction.

$$f(\vec{x}) = n_0 - \cfrac{x_0}{n_1 - \cfrac{x_1}{n_2 - \cfrac{x_2}{\vdots - \cfrac{\vdots}{n_k - \cfrac{x_k}{n_0 - \cfrac{x_0}{n_1 - \cfrac{x_1}{\vdots - \cfrac{\vdots}{n_k - \cdots}}}}}}}} \tag{6.10}$$

where $|x_i| \in p\mathbb{Z}_p$ and $n_0 \in \mathbb{Z}$ $n_i \in \mathbb{N}$, $1 \leq i \leq k$. 6.10 is a continued fraction with period $k+1$. If we truncate it from $k+1-$th term then we have

$$f(\vec{x}) = n_0 - \cfrac{x_0}{n_1 - \cfrac{x_1}{\ddots - \cfrac{\vdots}{n_k - \cfrac{x_k}{f(\vec{x})}}}}. \tag{6.11}$$

By making some arrangements we get the following functional equation

$$f(\vec{x}) = \frac{A_k(\vec{n},\vec{x}).f(\vec{x}) + B_k(\vec{n},\vec{x})}{C_k(\vec{n},\vec{x}).f(\vec{x}) + D_k(\vec{n},\vec{x})}$$

where A_k, B_k, C_k, D_k is polynomial with respect to x and n_i. This is the most general form but we can solve some special cases of this equation.

If we consider for all $n_i = 1$ and $x_i = p$ with 1-period we will give an expansion as follows,

$$1 - \sum_{n=1}^{\infty} \frac{1}{2(2n-1)}\binom{2n}{n}p^n = 1 - \cfrac{p}{1 - \cfrac{p}{1 - \cfrac{p}{1 - \vdots}}} \tag{6.12}$$

This continued fraction converges in p-adic topology. We prove that the more general form in 6.0.2 $1 - \sum_{n=1}^{\infty} \frac{1}{2(2n-1)}\binom{2n}{n}p^n$ viewed as an element of $\partial^+\langle T, H\rangle$. On the other hand

$$1 - \cfrac{p}{1 - \cfrac{p}{1 - \cfrac{p}{1 - \vdots}}} \tag{6.13}$$

may be viewed as an element of $\partial^+\langle T, S\rangle$.

Now we will prove 6.12. First method we will proceed expanding right hand side of equality step by step. We will use $\frac{1}{1-p} = \sum_{n=0}^{\infty} p^n$. This process is shown below.

$$1 - p.\sum_{n=0}^{\infty}\left(\cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{\vdots}}}}\right)^n = 1 - p.\sum_{n=0}^{\infty} p^n\left(\sum_{k=0}^{\infty}\left(\cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{\vdots}}}\right)^k\right)^n \tag{6.14}$$

$$= 1 - p.\sum_{n=0}^{\infty} p^n\left(\sum_{k=0}^{\infty} p^k\left(\sum_{t=0}^{\infty}\cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{\vdots}}}\right)^t\right)^k\Bigg)^n \tag{6.15}$$

In principle, using above method we can compute the coefficients of all terms $p, p^2, p^3, ...$. However, this method is impossible to implement even on a computer. We have to deal with very complicated processes. So we proved the following theorem by using a different method.

Theorem 6.0.3. *One has the equality of p-adic numbers*

$$1 - \sum_{n=1}^{\infty}\frac{1}{2(2n-1)}\binom{2n}{n}p^n = 1 - \cfrac{p}{1-\cfrac{p}{1-\cfrac{p}{1-\vdots}}}$$

Proof. Let $f(x) = 1 - \sum_{n=1}^{\infty}\dfrac{1}{2(2n-1)}\binom{2n}{n}x^n =$. Then $1 - \dfrac{x}{f(x)} = f(x)$. We get the following equation

$$f(x) - (f(x))^2 = x \tag{6.16}$$

Taking the first derivative of this equation we have

$$\begin{aligned} f'(x) - 2f(x)f'(x) &= 1 \\ -2f(x)f'(x) &= 1 - f'(x) \end{aligned} \tag{6.17}$$

Let $f'(x) = g(x)$. Taking second derivative we get

$$-2(g'(x)f(x) + g(x)f'(x)) = -g'(x) \tag{6.18}$$

By continuing this way $n-$derivative of the equation,

$$-2\sum_{k=0}^{n}\binom{n}{k}g^{(n-k)}(x)f^k x = g^{(n)}(x). \tag{6.19}$$

Since $g(x) = f'(x)$ we can write

$$-2\sum_{k=0}^{n}\binom{n}{k}f^{(n-k+1)}(x)f^{(k)}(x) = f^{(n+1)}(x). \tag{6.20}$$

Since

$$f(x) = 1 - \sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}p^k$$

then substituting in 6.20. After some calculations equality is satisfied. □

More generally the following result is true.

Theorem 6.0.4. *Suppose that A is a p-adic integer that is $|A|_p \leq 1$. The above theorem is satisfied i.e.*

$$1 - \sum_{n=1}^{\infty}\frac{1}{2(2n-1)}\binom{2n}{n}A^n = 1 - \cfrac{A}{1 - \cfrac{A}{1 - \cfrac{A}{1 - \vdots}}} \tag{6.21}$$

Proof.

$$1 - \frac{A}{1 - \sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}A^k} = 1 - \sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}A^k \tag{6.22}$$

By making some arrangements we get

$$\left(\sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}A^k\right)\left(1 - \sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}A^k\right) = A \tag{6.23}$$

Since A is a p-adic integer we can write

$$A = \sum_{n=0}^{\infty}a_n p^n.$$

Then we are substituting in equality 6.22, we have

$$\left(\sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}\left(\sum_{n=0}^{\infty}a_n p^n\right)^k\right)\left(1 - \sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}\left(\sum_{n=0}^{\infty}a_n p^n\right)^k\right) = \sum_{n=0}^{\infty}a_n p^n$$

For $k = 1$

$$\begin{aligned}&(1 - (a_0 + a_1p + a_2p^2 + ...) - (a_0 + a_1p + a_2p^2 + ...)^2 - 2(a_0 + a_1p + a_2p^2 + ...)^3 - ...).\\ &((a_0 + a_1p + a_2p^2 + ...) + (a_0 + a_1p + a_2p^2 + ...)^2 + 2(a_0 + a_1p + a_2p^2 + ...)^3 + ...)\\ &= a_0 + a_1p + a_2p^2 + ...\end{aligned}$$

$$\left(1 - \sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}{a_0}^k\right)\left(\sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\binom{2k}{k}{a_0}^k\right) = a_0 + a_1p + a_2p^2 \tag{6.24}$$

and then generally we can write

$$(1-\sum_{n=1}^{\infty}\frac{1}{2(2n-1)}\binom{2n}{n}{a_k}^n)(\sum_{n=1}^{\infty}\frac{1}{2(2n-1)}\binom{2n}{n}{a_k}^n)=a_k \quad k=0,1,2,... \tag{6.25}$$

After some calculations we can get

$$a_0=a_0, \quad a_1=a_1,...,a_m=a_m \tag{6.26}$$

□

Now we consider a p-adic periodic continued fraction of period 2.

Theorem 6.0.5.

$$1-\cfrac{p}{1-\cfrac{p^2}{1-\cfrac{p}{1-\vdots}}}=1-p-\sum_{n=2}^{\infty}a_{n+1}p^{n+1}$$

where $a_0=1,\ a_1=-1,\ a_2=0,\ a_{n+1}=a_n+\sum_{k=1}^{n-1}a_ka_{n-1-k}$.

Corollary 6.0.6. $1-p-\sum_{n=2}^{\infty}a_{n+1}p^{n+1}$ *is a solution of the quadratic equation* $x^2+x(-p^2+p-1)+p^2$

Proof. $x=1-p-p^3-p^4-2p^5-4p^6-8p^7-17p^8+...$
Substituting x in quadratic equation we get

$$\begin{aligned}&(1-p-p^3-p^4-2p^5-4p^6-8p^7-17p^8+...)(1-p-p^3\\&-p^4-2p^5-4p^6-8p^7-17p^8+...)\\&+(1-p-p^3-p^4-2p^5-4p^6-8p^7-17p^8+...)(-p^2+p-1)+p^2\\&=0\end{aligned}$$

□

Question. Can we find an explicit formula for periodic p-adic continued fractions of length 2? The answer is yes. Now we will get this formula. The most general form is attempted to be solved in 2-period with taking $n_0=n_1=1$ and $|x|_p<1,\ |y|_p<1$.

$$\sum_{n,m=0}^{\infty}a_{n,m}x^ny^m=1-\cfrac{x}{1-\cfrac{y}{1-\cfrac{x}{1-\vdots}}}$$

$$1-\cfrac{x}{1-\cfrac{y}{1-\cfrac{x}{1-\cfrac{y}{1-\cdots}}}}=1-\cfrac{x}{1-\cfrac{y}{\alpha(x,y)}}=\alpha(x,y)=\sum_{n,m=0}^{\infty}a_{n,m}x^ny^m \tag{6.27}$$

6.27 gives us

$$\alpha^2(x,y)-(y-x+1)\alpha(x,y)+y=0 \tag{6.28}$$

Using the partial derivative operator of the multiplication of two functions $D_{n,m}(f.g) = \frac{\partial}{\partial x^n}\frac{\partial}{\partial y^n}(f.g) = \sum_{k=0}^{m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{k}D_x^iD_y^k(g)D_x^{n-i}D_y^{m-k}(f)$, we can calculate partial derivatives of the 6.28 that is we want to get

$$\begin{aligned}&\sum_{k=0}^{m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{k}D_x^i(D_y^k\alpha(x,y))D_x^{n-i}D_y^{m-k}\alpha(x,y)\\ &-\sum_{k=0}^{m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{k}D_x^i(D_y^k\alpha(x,y))D_x^{n-i}D_y^{m-k}(y-x+1)\\ &+\sum_{k=0}^{m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{k}D_x^i(D_y^k(y))D_x^{n-i}D_y^{m-k}(1)=0\end{aligned}$$

We know that $\alpha(0,0)=1$. Then we must solve the above linear recurrence equation. It appears impossible to solve this equation consisting of partial derivatives manually, so we have calculated partial derivatives of the equation using Mathematica. Let's list some derivative values of $\alpha(x,y)$ at the point of $(0,0)$;

$\alpha(0,0)=1$	$\alpha^{(0,1)}=0$	$\alpha^{(0,2)}=0$	$\alpha^{(0,3)}=0$
$\alpha^{(1,0)}=-1$	$\alpha^{(1,1)}=-1$	$\alpha^{(1,2)}=-2$	$\alpha^{(1,3)}=-6$
$\alpha^{(2,1)}=-2$	$\alpha^{(2,2)}=-12$	$\alpha^{(2,3)}=-72$	$\alpha^{(2,4)}=-480$
$\alpha^{(3,1)}=-6$	$\alpha^{(3,2)}=-72$	$\alpha^{(3,3)}=-720$	$\alpha^{(3,4)}=-7200$
$\alpha^{(4,1)}=-24$	$\alpha^{(4,2)}=-480$	$\alpha^{(4,3)}=-7200$	$\alpha^{(4,4)}=-100800$
$\alpha^{(5,1)}=-120$	$\alpha^{(5,2)}=-3600$	$\alpha^{(5,3)}=-75600$	$\alpha^{(5,4)}=-1411200$

Now we can calculate some coefficients of x^ny^m, let us assume $m=0$

$$\sum a_{n,0}x^n$$

The value of the first derivative of the above sum when we write $n=1$ is equal to $\alpha^{(1,0)}(0,0)$. So this gives coefficient $a_{1,0}$. We can calculate the coefficients $a_{1,0}$, $a_{2,0}$, $a_{3,0}, ..., a_{n,0}$ by continuing with this method. So we find

$$a_{1,0} = -1,\ a_{2,0} = 0,\ a_{3,0} = 0,\ a_{4,0} = 0,\ ..., a_{n,0} = 0$$

Let us assume $m = 1$.

$$\sum a_{n,1} x^n y$$

Taking first derivative with respect to y and then with respect to x, we get

$$a_{1,1} = -1,\ a_{2,1} = -1,\ a_{3,1} = -1, ..., a_{n,1} = -1.$$

We assume that $m = 2$ and using similar method, we have

$$a_{1,2} = -1,\ a_{2,2} = -3,\ a_{3,2} = -6, ..., a_{n,2} = \binom{n+1}{2}.$$

We can calculate a lot of coefficients using this method. But we need a general formula of these coefficients that we found experimentally. We get the general formula of coefficients. The table of coefficients are as follows;

Table 6.1

$$a_{n,m} = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \dots & a_{0,m} \\ a_{1,0} & a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,0} & a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,0} & a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$$

Theorem 6.0.7. *For $x, y \in \mathbb{Z}_p^*$ the continued fraction*

$$1 - \cfrac{x}{1 - \cfrac{y}{1 - \cfrac{x}{1 - \cfrac{y}{1 - \cdots}}}} = \sum_{n,m=0}^{\infty} a_{n,m} x^n y^m \tag{6.29}$$

converges in the $p-$adic topology and $a_{n,m}$ is given by

$$=$$

$$\begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ -1 & -1 & -1 & \dots & -1 \\ 0 & -1 & -3 & \dots & \dfrac{-C(m+1,2)C(m+1,1)}{m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & \dfrac{-C(n+1,2)C(n+1,1)}{(n+1)} & \dots & \dfrac{C(m+n-1,n)C(m+n-1,n-1)}{(m+n-1)} \end{pmatrix}$$

Proof. Firstly we can write a new equation for continued fraction

$$1-\cfrac{x}{1-\cfrac{y}{1-\cfrac{x}{1-\cfrac{y}{1-\cdots}}}}=1-\frac{x}{\alpha(y,x)}=\alpha(x,y)$$

So we get the following equation

$$\alpha(x,y)\alpha(y,x)-\alpha(y,x)+x=0 \tag{6.30}$$

since $\alpha(x,y)=\sum a_{n,m}x^ny^m$ then we can rewrite the equation 6.30

$$\sum a_{n,m}x^ny^m\sum a_{m,n}x^my^n-\sum a_{m,n}x^my^n+x=0 \tag{6.31}$$

We will use induction on the diagonal of the matrix to prove the theorem. First of all let $n+m=0$. From 6.31 we get

$$a_{0,0}^2-a_{0,0}=0$$

So $a_{0,0}=0$ or $a_{0,0}=1$. Let us continue our way considering $a_{0,0}=1$. Assuming $a_{0,0}=0$ gives a similar result. We will show this later. To show the base step of induction we assume that $n+m=1$. So this gives two equations,

$$a_{0,0}a_{0,1}+a_{0,0}a_{1,0}-a_{1,0}=0 \tag{6.32}$$

$$a_{1,0}a_{0,0}+a_{0,0}a_{0,1}-a_{0,1}+1=0 \tag{6.33}$$

then

$$\begin{bmatrix} a_{0,0} & a_{0,0}-1 \\ a_{0,0}-1 & a_{0,0} \end{bmatrix}\begin{bmatrix} a_{0,1} \\ a_{1,0} \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.34}$$

since $a_{0,0}=1$ we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_{0,1} \\ a_{1,0} \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so

$$\begin{bmatrix} a_{0,1} \\ a_{1,0}+1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $a_{0,1}=0$ and $a_{1,0}=-1$. We proved that basis step of induction. Now we assume that induction hypothesis that is suppose we know that the coefficients of $n+m\leq k-1$. We must show that in the case $n + m = k$. Using the previous hypothesis we will calculate respectively $a_{k,0},\ a_{k-1,1},\ a_{k-2,2}, ..., a_{0,k}$. To find these coefficients we write the following system of equations

$$a_{k,0}a_{0,0} + a_{k-1,0}a_{0,1} + a_{k-2,0}a_{0,2} + ... + a_{0,0}a_{0,k} - a_{0,k} = 0$$

$$a_{k-1,1}a_{0,0} + a_{k-2,1}a_{0,1} + a_{k-3,1}a_{0,2} + ... + a_{0,1}a_{0,k-1} + a_{k-1,0}a_{1,0}+$$
$$a_{k-2,0}a_{1,1} + ... + a_{0,0}a_{1,k-1} - a_{1,k-1} = 0$$

$$a_{k-2,2}a_{0,0} + a_{k-3,2}a_{0,1} + a_{k-4,2}a_{0,2} + ... + a_{0,2}a_{0,k-2}+$$
$$a_{k-2,1}a_{1,0} + a_{k-3,1}a_{1,1} + ... + a_{0,1}a_{1,k-2}+ \qquad \vdots$$
$$a_{k-2,0}a_{2,0} + a_{k-3,0}a_{2,1} + ... + a_{0,0}a_{2,k-2} - a_{2,k-2} = 0$$

$$a_{1,k-1}a_{0,0} + a_{0,k-1}a_{0,1} + a_{1,k-2}a_{1,0} + a_{0,k-2}a_{1,1}$$
$$+a_{1,k-3}a_{2,0} + a_{0,k-3}a_{2,1} + ... + a_{1,0}a_{k-1,0} + a_{0,0}a_{k-1,1} - a_{k-1,1} = 0$$

$$a_{0,k}a_{0,0} + a_{0,k-1}a_{1,0} + a_{0,k-2}a_{2,0} + ... + a_{0,0}a_{k,0} - a_{k,0} = 0$$

From these equations we get

$$a_{k,0} = 0, a_{k-1,1} = -1a_{k-2,2} = \frac{-(k-1,2)(k-1,1)}{k-1}, ..., a_{1,k-1} = -1, a_{0,k} = 0.$$

□

If we want to express 6.29 in terms of analytic functions then using the coefficients a_{nm} and generalized corresponding functions we can write the following expansion. So we get a different representation for the continued fraction 6.29 in 2-period

$$1 - 1.xy^0 - 1.xy^1 - 1.xy^2 - 1xy^3 - 1 - ... - 1.xy^m - ... \tag{6.35}$$

$$- 1.x^2y - 3x^2y^2 - 6x^2y^3 - 10x^2y^4 - ... - C(m+1,2)x^2y^m - ... \tag{6.36}$$

$$- 1x^3y - 6x^3y^2 - 20x^3y^3 - 50x^3y^4 - ... - \frac{C(m+2,3)C(m+2,2)}{n+2}x^3y^m - ... \tag{6.37}$$

$$= -\frac{x}{1-y} - \frac{x^2y}{(1-y)^3} - \frac{4x^3y}{(1-y)^5} - \frac{36x^4y(1+3y+y^2)}{(1-y)^7} - ... \tag{6.38}$$

$$- \frac{x^n(n-1)!^2y(T(n-1,1) + T(n-1,2)y + T(n-1,3)y^2 + ... + T(n-1,n-1)y^{n-2}}{(1-y)^{2n-1}} \tag{6.39}$$

where $T(n,k) = \frac{C(n-1,k-1)*C(n,k-1)}{k}$ for $1 \leq k \leq n$

Now we consider two-periodic continued fractions of the form

$$a - \cfrac{x}{b - \cfrac{y}{a - \cfrac{x}{b - \vdots}}} = \sum_{n,m=0}^{\infty} a_{nm} x^n y^m \tag{6.40}$$

we will find the a_{nm}. For this purpose,

$$a - \cfrac{x}{b - \cfrac{y}{a - \cfrac{x}{b - \vdots}}} = a - \frac{x}{b - \frac{y}{\alpha(x,y)}} = \alpha(x,y) \tag{6.41}$$

So we get $\alpha^2(x,y) + (x-y-a)\alpha(x,y) + ay = 0$. Using the partial derivative operator of the multiplication of two functions

$$D_{n,m}(f.g) = \frac{\partial}{\partial x^n}\frac{\partial}{\partial y^n}(f.g) = \sum_{k=0}^{m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{k} D_x^i D_y^k(g) D_x^{n-i} D_y^{m-k}(f)$$

, we can calculate partial derivatives of the above equation that is we want to get

$$\begin{aligned}
&\sum_{k=0}^{m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{k} D_x^i(D_y^k\alpha(x,y)) D_x^{n-i} D_y^{m-k}\alpha(x,y) \\
&-\sum_{k=0}^{m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{k} D_x^i(D_y^k\alpha(x,y)) D_x^{n-i} D_y^{m-k}(y-x+a) \\
&+\sum_{k=0}^{m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{k} D_x^i(D_y^k(ay)) D_x^{n-i} D_y^{m-k}(1) = 0
\end{aligned}$$

We know that $\alpha(0,0) = a$. Then we must solve the above linear recurrence equation. It appears to be impossible to solve this equation consisting of partial derivatives manually, so we have calculated partial derivatives of the equation using Mathematica. Let's list some derivative values of $\alpha(x,y)$ at the point of $(0,0)$;

$\alpha(0,0)=a$	$\alpha^{(0,1)}=0$	$\alpha^{(0,2)}=0$	$\alpha^{(0,3)}=0$
$\alpha^{(1,0)}=-\frac{1}{b}$	$\alpha^{(1,1)}=-\frac{1}{ab^2}$	$\alpha^{(1,2)}=-\frac{2}{a^2b^3}$	$\alpha^{(1,3)}=-\frac{6}{a^3b^4}$
$\alpha^{(2,1)}=-\frac{2}{a^2b^3}$	$\alpha^{(2,2)}=-\frac{12}{a^3b^4}$	$\alpha^{(2,3)}=-\frac{72}{a^4b^5}$	$\alpha^{(2,4)}=-\frac{480}{a^5b^6}$
$\alpha^{(3,1)}=-\frac{6}{a^3b^4}$	$\alpha^{(3,2)}=-\frac{72}{a^4b^5}$	$\alpha^{(3,3)}=-\frac{720}{a^5b^6}$	$\alpha^{(3,4)}=-\frac{7200}{a^6b^7}$
$\alpha^{(4,1)}=-\frac{24}{a^4b^5}$	$\alpha^{(4,2)}=-\frac{480}{a^5b^6}$	$\alpha^{(4,3)}=-\frac{7200}{a^6b^7}$	$\alpha^{(4,4)}=-\frac{100800}{a^7b^8}$
$\alpha^{(5,1)}=-\frac{120}{a^5b^6}$	$\alpha^{(5,2)}=-\frac{3600}{a^6b^7}$	$\alpha^{(5,3)}=-\frac{75600}{a^7b^8}$	$\alpha^{(5,4)}=-\frac{1411200}{a^11b^12}$

Now we will calculate coefficients of x^ny^m. Let us assume successively $m=1,2,...,n$. As above implementation taking derivative $\sum_{n,m}^{\infty} a_{nm}x^ny^m$ and writing the value of $\alpha^{i,j}(0,0)$. So we can get a_{nm}

$$a_{n,m}=\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \dots & a_{0,m} \\ a_{1,0} & a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,0} & a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,0} & a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$$

Theorem 6.0.8.

$$a-\cfrac{x}{b-\cfrac{y}{a-\cfrac{x}{b-\cfrac{y}{a-\cdots}}}}=\sum_{n,m=0}^{\infty} a_{n,m}x^ny^m$$

where $a_{n,m}$ is given by

Table 6.2

$$=\begin{pmatrix} a & 0 & 0 & \dots & 0 \\ -\frac{1}{b} & -\frac{1}{ab^2} & -\frac{1}{a^3b^4} & \dots & -\frac{1}{a^mb^{m+1}} \\ 0 & -\frac{1}{a^2b^3} & -\frac{3}{a^3b^4} & \dots & \frac{-C(m+1,2)C(m+1,1)}{m+1}a^{m+1}b^{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\frac{1}{a^nb^{n+1}} & \frac{-C(n+1,2)C(n+1,1)}{(n+1)a^{n+2}b^{m+3}} & \dots & \frac{C(m+n-1,n)C(m+n-1,n-1)}{(m+n-1)a^{m+n-1}b^{m+n}} \end{pmatrix}$$

These are kind of generalized Catalan numbers **?**

Proof. Firstly we can write a new equation for continued fraction

$$a - \cfrac{x}{b - \cfrac{y}{a - \cfrac{x}{b - \cfrac{y}{a - \vdots}}}} = a - \frac{x}{\frac{b}{a}\alpha(y,x)} = \alpha(x,y)$$

So we get the following equation

$$b\alpha(x,y)\alpha(y,x) - ab\alpha(y,x) + ax = 0$$

since $\alpha(x,y) = \sum a_{n,m}x^n y^m$ then we can rewrite the above equation

$$b\sum a_{n,m}x^n y^m \sum a_{m,n}x^m y^n - ab\sum a_{m,n}x^m y^n + ax = 0 \tag{6.42}$$

We will use induction on the diagonal of the matrix to prove the theorem. First of all let $n + m = 0$. From above equality we get

$$ba_{0,0}^2 - aba_{0,0} = 0$$

So $a_{0,0} = 0$ or $a_{0,0} = a$. Let us continue our way, considering $a_{0,0} = a$. If we assume $a_{0,0} = 0$, similar results can be achieved. We will show this later. To show the base step of induction we assume that $n + m = 1$. So this gives two equations

$$ba_{0,0}a_{0,1} + ba_{0,0}a_{1,0} - aba_{1,0} = 0$$
$$ba_{1,0}a_{0,0} + ba_{0,0}a_{0,1} - aba_{0,1} + a = 0$$

then

$$\begin{bmatrix} ba_{0,0} & ba_{0,0} - ab \\ ba_{0,0} - ab & ba_{0,0} \end{bmatrix} \begin{bmatrix} a_{0,1} \\ a_{1,0} \end{bmatrix} + \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since $a_{0,0} = a$ we have

$$\begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} \begin{bmatrix} a_{0,1} \\ a_{1,0} \end{bmatrix} + \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so

$$\begin{bmatrix} aba_{0,1} \\ aba_{1,0} + a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $a_{0,1} = 0$ and $a_{1,0} = -\dfrac{1}{b}$ where $a \neq 0$. We proved that base step of induction. Now we assume that induction hypothesis holds that is suppose we know that the coefficients of $n+m \leq k-1$. We must show in case of $n+m = k$. Using the previous hypothesis we will calculate respectively $a_{k,0},\ a_{k-1,1},\ a_{k-2,2}, ..., a_{0,k}$. To find these coefficients we can write the following system of equations

$$b(a_{k,0}a_{0,0} + a_{k-1,0}a_{0,1} + a_{k-2,0}a_{0,2} + ... + a_{0,0}a_{0,k}) - aba_{0,k} = 0$$

$$b(a_{k-1,1}a_{0,0} + a_{k-2,1}a_{0,1} + a_{k-3,1}a_{0,2} + ... + a_{0,1}a_{0,k-1} + a_{k-1,0}a_{1,0} + \\ a_{k-2,0}a_{1,1} + ... + a_{0,0}a_{1,k-1}) - aba_{1,k-1} = 0$$

$$b(a_{k-2,2}a_{0,0} + a_{k-3,2}a_{0,1} + a_{k-4,2}a_{0,2} + ... + a_{0,2}a_{0,k-2} + \\ a_{k-2,1}a_{1,0} + a_{k-3,1}a_{1,1} + ... + a_{0,1}a_{1,k-2} + \\ a_{k-2,0}a_{2,0} + a_{k-3,0}a_{2,1} + ... + a_{0,0}a_{2,k-2}) - aba_{2,k-2} = 0$$

$$b(a_{1,k-1}a_{0,0} + a_{0,k-1}a_{0,1} + a_{1,k-2}a_{1,0} + a_{0,k-2}a_{1,1} \\ + a_{1,k-3}a_{2,0} + a_{0,k-3}a_{2,1} + ... + a_{1,0}a_{k-1,0} + a_{0,0}a_{k-1,1}) - aba_{k-1,1} = 0$$

$\cdots$

$$b(a_{0,k}a_{0,0} + a_{0,k-1}a_{1,0} + a_{0,k-2}a_{2,0} + ... + a_{0,0}a_{k,0}) - aba_{k,0} = 0$$

From these equations we get

$$a_{k,0} = 0, a_{k-1,1} = -\frac{1}{a^{k-1}b^k}, a_{k-2,2} = -\frac{(k-1,2)(k-1,1)}{(k-1)a^{k-1}b^k}, ..., \\ a_{1,k-1} = -\frac{1}{a^{k-1}b^k}, a_{0,k} = 0$$

This completes the proof. □

Question. Can we find an explicit formula for the continued fraction of period length 3 of the following form?

$$1-\cfrac{x}{1-\cfrac{y}{1-\cfrac{z}{1-\cfrac{x}{1-\cfrac{y}{1-\cfrac{z}{1-\vdots}}}}}}=\sum_{n,k,m=0}^{\infty}x^n y^k z^m \tag{6.43}$$

We have been unable to do this. However, for the special case $y = 1$ we could get a conjecture. In this case 6.43 turns into

$$1-\cfrac{x}{1-\cfrac{1}{1-\cfrac{z}{1-\cfrac{x}{1-\cfrac{1}{1-\cfrac{z}{1-\vdots}}}}}}=\sum_{n,m=0}^{\infty}a_{nm}(x-1)^n z^m$$

We calculated the coefficients by computer and generalized them using the above method and we obtained following result.

Conjecture.

$$1-\cfrac{x}{1-\cfrac{1}{1-\cfrac{z}{1-\cfrac{x}{1-\cfrac{1}{1-\cfrac{z}{1-\vdots}}}}}}=\sum_{n,m=0}^{\infty}a_{nm}(x-1)^n z^m \tag{6.44}$$

where a_{nm} is given by;

Table 6.3

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & -1 & -2 & \dots & -m \\ 0 & 1 & 3 & \dots & C(m+1,2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (-1)^n - 1 & (-1)^{n-1}n & \dots & (-1)^{n-1}C(m+n-2,n-1) \end{pmatrix}$$

If $x = 1 + c_1p + c_2p^2 + c_3p^3 + ...$ and $z = d_1p + d_2p^2 + d_3p^3 + ...$ i.e. $|x|_p, |z|_p < 1$ then 6.44 converges in p-adic topology. So this continued fraction gives a p-adic number.

7. CONCLUSION AND RECOMMENDATION

The theory of p-adic continued fractions is a classical field of mathematics, using classical tools. In this thesis we try to interpret this theory in terms of group boundaries(namely subgroups of $\mathrm{PGL}_2(\mathbb{Q})$. In order to complete this initiative one needs to construct spaces on which these groups acts in a nice manner. Since $\mathrm{PGL}_2(\mathbb{Q})$ is an infinitely presented group, this space must be infinite dimensional and it must have a very complicated structure (named "heyula" by M. Uludağ). The subgroup $\mathrm{PSL}_2(\mathbb{Z})$ of $\mathrm{PGL}_2(\mathbb{Q})$ and its boundary gives us some understanding on how to construct and study this space. In this case, the space acted upon is the Farey tree (i.e. the planar 2-3 regular tree) whose boundary consists of simple continued fractions. The Baumslag-Solitar subgroup is another subgroup of $\mathrm{PGL}_2(\mathbb{Z})$. $P-$adic numbers can be viewed as elements of its boundary. The space acted upon have been constructed in Farb and Mosher (1998).

Also we find explicit expansion for some periodic p-adic continued fractions. This give some generalization of Catalan numbers. In addition to this n-periodic p-adic continued fractions will give us higher generalization of Catalan numbers.

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