BOLU ABANT IZZET BAYSAL UNIVERSITY

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

LIFTING PROBLEM FOR HOMOGENEOUS IDEALS

MASTER OF SCIENCE

OYA AYDOĞAN

BOLU, AUGUST 2019

BOLU ABANT IZZET BAYSAL UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

LIFTING PROBLEM FOR HOMOGENEOUS IDEALS

MASTER OF THESIS

OYA AYDOĞAN

BOLU, AUGUST 2019

APPROVAL OF THE THESIS

LIFTING PROBLEM FOR HOMOGENEOUS IDEALS submitted by **Oya AYDOĞAN** and defended before the below named jury in partial fulfillment of the requirements for the degree of **Master of Science** in **Department of Mathematics, The Graduate School of Natural and Applied Sciences of Bolu Abant Izzet Baysal University in 22.08.2019** by

Examining Committee Members	**Signature**
Supervisor Assoc. Prof. Dr. Erol Yılmaz Bolu Abant İzzet Baysal University	....
Member Assoc. Prof. Dr. Ergün Yaraneri İstanbul Technical University	....
Member Asist. Prof. Dr. Ali Öztürk Bolu Abant İzzet Baysal University	....

Prof. Dr. Ömer ÖZYURT ....

Director of Graduate School of Natural and Applied Sciences

To my family

DECLARATION

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

OYA AYDOĞAN

ABSTRACT

LIFTING PROBLEM FOR HOMOGENEOUS IDEALS
MSC THESIS
OYA AYDOĞAN
BOLU ABANT IZZET BAYSAL UNIVERSITY GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS
(SUPERVISOR: ASSOC. PROF. DR. EROL YILMAZ)

BOLU, AUGUST 2019

The lifting problem for homogeneous ideals is studied. A new method for finding liftings of a homogeneous ideal is developed. This method was compared with the current methods. The results are demonstrated with examples.

KEYWORDS: Lifting problem, Syzygy modules, Gröbner bases, H-bases.

ÖZET

HOMOJEN İDEALLER İÇİN KALDIRAÇ PROBLEMİ
YÜKSEK LİSANS TEZİ
OYA AYDOĞAN
BOLU ABANT İZZET BAYSAL ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ
MATEMATİK ANABİLİM DALI
(TEZ DANIŞMANI: DOÇ. DR. EROL YILMAZ)

BOLU, AĞUSTOS - 2019

Homojen idealler için kaldırma problemi incelenmiştir. Homojen bir idealin kaldırılması için yeni bir yöntem geliştirilmiştir. Bu yöntem mevcut yöntemlerle karşılaştırıldı. Sonuçlar örneklerle gösterilmiştir.

ANAHTAR KELİMELER: Kaldıraç problemi, Syzygy modüller, Gröbner tabanları, H-tabanları

TABLE OF CONTENTS

Page

LIST OF FIGURES

LIST OF ABBREVIATIONS AND SYMBOLS

$>_{lex}$	**:** Lexicographic Order
$>_{degrevlex}$	**:** Degree Reverse Lexicographic Order
$>_{deglex}$	**:** Degree Lexicographic Order
$\boldsymbol{LC}(\boldsymbol{f})$	**:** Leading Coefficient of f
$\boldsymbol{LM}(\boldsymbol{f})$	**:** Leading Monomial of f
$\boldsymbol{LT}(\boldsymbol{f})$	**:** Leading Term of f
$\boldsymbol{H}(\boldsymbol{f})$	**:** Leading form of f
$\boldsymbol{LCM}(\boldsymbol{f}, \boldsymbol{g})$	**:** Least Common Multiple of f and g
$\boldsymbol{S}(\boldsymbol{f}, \boldsymbol{g})$	**:** S -polynomialof f and g

ACKNOWLEDGEMENTS

The author wishes to express her deepest gratitude to her supervisor Assoc. Prof. Dr. Erol Yılmaz for his guidance, advice, criticism, encouragements and insight throughout the research.

1. INTRODUCTION

In this thesis we consider the lifting problem for homogeneous ideals proposed first in (Roitman, 1988). Later, Robert (1989) studied the lifting problem over an algebraically closed field. Carra Ferro and Robbiano (1990) gave a connection between tge lifting problem and H-bases. Then using this connection and super G-bases (super Gröbner bases) they were able to found the liftings of monomial ideals. Migliore and Nagel (2000) also studied the liftings of monomial ideals on more geometric point of view. Luo and Yılmaz (2001) gave a relation between syzygy modules and H-bases. Using this relation, they found some liftings of some homogeneous ideals. More recently, Bertone et al (2016) use Gröbner bases to obtain a method for finding liftings.

We will concentrate to the last two papers in this thesis. In Chapter 2, we give some basic concepts about Gröbner bases and the syzygy modules. In Chapter 3, we formally define the lifting problem and investigate the method suggested by Bertone et al (2016) for the solution of the lifting problem. It seems their method contains many unnecessary computations and finds relatively less liftings. In Chapter 4, we study linear algebraic characterization of homogeneous ideals. Then using the idea of Luo and Yılmaz (2001) with this characterization we are able to develope a new method for finding of liftings of homogeneous ideals. Our method looks more efficient that method of Bertone et al (2016). It involves less computation and finds more liftimgs than the other method. The results are demostrated by examples. We use symbolic computation software Mathematica for calculations.

2. GRÖBNER BASES AND SYZYGY MODULES

In this chapter, some basic concepts about Gröbner basis theory and the syzygy modules which are required to solve the lifting problem are given.

2.1. Monomial Ordering

Definition 2.1.1.

A product of the form $x_1^{\alpha_1} x_2^{\alpha_2} \ldots\ldots x_n^{\alpha_n}$ is called a monomial in $x_1, \ldots\ldots, x_n$ where $\alpha_1, \ldots\ldots, \alpha_n$ are positive integers. The total degree of the monomial is the sum of $\alpha_1 + \alpha_2 + \cdots. + \alpha_n$. We simplify the notation for monomials by writing x^α where $x = (x_1, \ldots\ldots, x_n)$ and $\alpha = (\alpha_1, \ldots\ldots, \alpha_n)$ and we let $|\alpha|$ be indicate the total degree of x^α.

Then a finite K-linear combination of monomials is called a polynomial in $x_1, \ldots\ldots, x_n$ where K is a field. The set of such polynomials form a commutative ring which is called the ring of polynomials over K and denoted by $K[x_1, \ldots, x_n]$. In order to define a division algorithm on $K[x_1, \ldots, x_n]$, we need to define an order on monomials and this order must satisfy some properties.

Definition 2.1.1.

A monomial order is a total order relation $>$ on the set of monomials that is satisfy the following properties

(i) If $x^\alpha < x^\beta$, then $x^\gamma x^\alpha < x^\gamma x^\beta$ for every monomial x^γ.

(ii) Every non-empty set of monomials has smallest element under $>$.

The most known monomial order is the lexicographic order.

Definition 2.1.2 (Lexicographic Order)

Let $\boldsymbol{p_1} = \boldsymbol{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}}$ and $\boldsymbol{p_2} = \boldsymbol{x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}}$ be two monomials. We say $\boldsymbol{p_1} >_{lex} \boldsymbol{p_2}$ if the left most entry non-zero $\boldsymbol{\alpha_j - \beta_j}$ is positive $\boldsymbol{j = 1, 2, \dots, n}$.

If a monomial order $<$ is called degree compatible if $|\alpha| > |\beta|$ implies $x^\alpha < x^\beta$. Next we will define the two most used degree compatible monomial order.

Definition 2.1.3 (Degree Reverse Lexicographic Order)

Let $\boldsymbol{p_1} = \boldsymbol{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}}$ and $\boldsymbol{p_2} = \boldsymbol{x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}}$ be two monomials. We say that $\boldsymbol{p_1} >_{degrevlex} \boldsymbol{p_2}$

(i) If $|\alpha| > |\beta|$ or

(ii) If $|\alpha| = |\beta|$ then the last entry non-zero $\boldsymbol{\alpha_j - \beta_j}$ is negative for $\boldsymbol{j = 1, 2, \dots, n}$.

Definition 2.1.4 (Degree Lexicographic Order)

Let $\boldsymbol{p_1} = \boldsymbol{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}}$ and $\boldsymbol{p_2} = \boldsymbol{x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}}$ be two monomials. We say $\boldsymbol{p_1} >_{deglex} \boldsymbol{p_2}$

1) If $|\alpha| > |\beta|$ or
2) If $|\alpha| = |\beta|$ and $\boldsymbol{p_1} >_{lex} \boldsymbol{p_2}$.

Definition 2.1.5.

Let $h = \sum_\alpha m_\alpha x^\alpha$ be a nonzero polynomial in $K[x_1, \dots\dots, x_n]$ and let $>$ be a monomial order.

i)The multidegree of h is

$multideg(h) = \max\{\alpha \epsilon \mathbb{Z}_{\geq 0}^n : m_\alpha \neq 0\}$

the maximum is taken with respect to $>$.

ii)The leading coefficient of h is

$LC(h)= m_{multideg(h)}$

iii)The leading monomial of h is

$LM(h)=x^{multideg(h)}$

iv) The leading term of h is

$LT\ (h)= LC(h)LM(h)$

Example 2.1.6.

i) Let $h = 11x^6y^2z^3 + 13x^5y^6 - 9xy^2 + 3y^2z$ with respect to lex order, then we have

$multideg(h) = (6,2,3)$

$LC(h) = 11$

$LM(h) = x^6y^2z^3$

$LT(h) = 11x^6y^2z^3$

ii) Let $h = 11x^6y^2z^3 + 13x^5y^6 - 9xy^2 + 3y^2z$ and with respect to degree lexicographic order, then we have

$multideg(h) = (6,2,3)$

$LC(h) = 11$

$LM(h) = x^6y^2z^3$

$LT(h) = 11x^6y^2z^3$

iii) Let $h = 11x^6y^2z^3 + 13x^5y^6 - 9xy^2 + 3y^2z$ and with respect to degree reverse lexicographic order, then we have

$h = 13x^5y^6 + 11x^6y^2z^3 - 9xy^2 + 3y^2z$

$multideg(h) = (5,6,0)$

$LC(h) = 13$

$LM(h) = x^5y^6$

$LT(h) = 13x^5y^6$

2.2 Division Algorithm

Theorem 2.2.1. (Cox et al (2007), Theorem 2.3.3.)

Let > be a monomial order on the set of monomials of $K[x_1, \ldots\ldots, x_n]$ *and* $h = (h_1, \ldots. h_s)$ *be an ordered s-tuple of polynomials in* $K[x_1, \ldots\ldots, x_n]$*. Then every* $h \epsilon K[x_1, \ldots\ldots, x_n]$ *can be written as*

$$h = m_1h_1 + m_2h_2 + \cdots .. + m_sh_s + r,$$

where $m_{i\,,} r \in K[x_1, \ldots\ldots, x_n]$ *and either* $r = 0$ *or r is a k- linear combination of a monomials, none of which is divisible by any* $LT(h_1) \ldots. LT(h_s)$*. The polynomial r is called remainder. Furthermore,* $multideg(h) \geq multideg(m_ih_i)$ *if* $m_ih_i \neq 0$.

The polynomials $m_i, r \in K[x_1, \ldots, x_n]$ can be found as follows. Let $r = 0$ for the begging. If $LT(h) = x^{\gamma}LT(h_i)$ for some $1 \leq i \leq s$ and some monomial x^{γ}, then define new h as

$$h \coloneqq h - x^{\gamma}h_i.$$

Otherwise $h \coloneqq h - LT(h)$ and $r \coloneqq r + LT(h)$. Repeat this until $h = 0$.

Let us illustrate this algorithm with an example.

Example 2.2.2.

Let $h = x^2y + xy + y^2 - 1$, $h_1 = x^2 - 1$ and $h_2 = xy + 1$. We will apply the division algorithm with respect to lexicographic order.

Hence $LT(h) = x^2y$, $LT(h_1) = x^2$ and $LT(h_2) = xy$. Since $\frac{LT(h)}{LT(h_1)} = y$ we redefine h as

$$h \coloneqq h - yh_1 = xy + y^2 + y - 1.$$

Now $LT(h) = xy$ is not divisible by $LT(h_1)$, but divisible by $LT(h_2)$ so we continue with $\frac{LT(h)}{LT(h_2)} = 1$, we redefine h as $h \coloneqq h - 1.h_2 = y^2 + y - 2$.

Now $LT(h) = y^2$ is divisible neither $LT(h_1)$ nor $LT(h_2)$. Therefore we move y^2 to the remainder and continue by $h \coloneqq h - y^2 = y - 2$.

Since both y and -2 are not divisible by $LT(h_1)$ and $LT(h_2)$, the remainder is $y^2 + y - 2$. Hence

$$h = y.h_1 + h_2 + y^2 + y - 2.$$

This division algorithm is an imperfect generalization of its one-variable counterpart. Because if we change order of divisors for the the algorithm, then the remainder may differ. Since we can not fix this problem, we need a special generating set for the ideal $< g_1, g_2, \ldots, g_s >$ that makes the remainder unique. This generating set is called Gröbner Basis. We will define Gröbner basis and explain its properties in this section.

2.3. Gröbner Basis

Definition 2.3.1.

Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal**.** Fix a monomial ordering $>$. Define the set

$$LT(I) = \{LT(f) : f \in I\}.$$

If $I = \langle f_1, \dots, f_s \rangle$, then clearly $\langle LT(f_1), \dots, LT(f_s) \rangle \subseteq \langle LT(I) \rangle$. However, $\langle LT(I) \rangle$ can a strictly larger ideal. This bring us to the definition of Gröbner basis.

Definition 2.3.2.

Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal**.** Fix a monomial ordering $>$. A finite subset $\mathcal{G} = \{g_1, \dots, g_t\}$ of an ideal I is said to be a Gröbner Basis if

$$\langle LT(g_1), \dots \dots .. LT(g_t) \rangle = \langle LT(I) \rangle.$$

The most significant property of the Gröbner basis is the following.

Theorem 2.3.3. (Adams and Loustaunau (1994), Theorem 1.6.7)

Let $\mathcal{G} = \{g_1, g_2, \dots, g_n\}$ *be a Gröbner basis for an ideal* $I \subseteq K[x_1, \dots, x_n]$. *If* $f \in K[x_1, \dots, x_n]$, *then the remainder of* f *upon division by* $\mathcal{G}$ *is unique.*

Before giving a criterion for the Gröbner basis, we need the following definition.

Definition 2.3.4.

Let $h, g \in K[x_1, \dots, x_n]$ be nonzero polynomials.

(i) If $multideg(h) = \alpha$ and $multideg(g) = \beta$, then let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ where $\gamma_i = \max\{\alpha_i, \beta_i\}$ for all i x^γ is called the least common multiple of $LM(h)$ and $LM(g)$, written $x^\gamma = LCM\big(LM(h), LM(g)\big)$

(ii) The S-polynomial of $h\ and\ g$ is combination,

$$S(h,g) = \frac{x^\gamma}{LT(h)}h - \frac{x^\gamma}{LT(g)}g$$

Theorem 2.3.5. (Adams and Loustaunau (1994), Theorem 1.6.7.)

Suppose $\mathcal{G} = \{g_1, g_2, \dots, g_s\}$ is a generating set for an ideal I. Then $\mathcal{G}$ is a Gröbner basis of I if and only if for all $\mathbf{1} \leq \boldsymbol{i} < \boldsymbol{j} \leq \boldsymbol{s}$ the remainder of $S(g_i, g_j)$ upon division by $\mathcal{G}$ is zero.

This theorem is known Buchberger's criterion. Buchberger is the inventor of Gröbner basis. He also give an algorithm for finding Gröbner basis of an ideal from any generating set of this ideal. This algorithm can be described as follows: Starting from a set of polynomails $\mathcal{G} = \{f_1, \dots, f_s\}$. Compute the remainders of S-polynomials $S(f_i, f_j)$ upon division by $\mathcal{G}$ for $1 \leq i < j \leq s$. If the remainder $r \neq 0$, then enlarge $\mathcal{G}$ by r. Otherwise do nothing and pass to next S-polynomial. Continue this process until computing all S-polynomials. At the end, the result is a Gröbner basis.

Definition 2.3.6.

(i) A Gröbner basis $\boldsymbol{\mathcal{G}}$ of an ideal is called minimal if for each $f \in \mathcal{G}, LM(f) \neq LM(g)\ for\ all\ g \in \mathcal{G}\backslash\{f\}$.

(ii) A minimal Gröbner basis $\mathcal{G}$ is called reduced if for each $f \in \mathcal{G}$, no term of f is divisible by $LM(g)$ for all $g \in \mathcal{G}\backslash\{f\}$.

Example 2.3.7.

We will find a Gröbner basis for $\langle x^2y - y + x, xy^2 - x \rangle \subseteq \mathbb{Q}[x, y]$ with respect to the graded lex order with $x > y$.

Let $f_1 = x^2y - y + x$, $f_2 = xy^2 - x$.

Compute S-polynomial

$$S(f_1, f_2) = yf_1 - xf_2 = x^2 + xy - y^2 \neq 0$$

Remainder is not zero, so we have to add $f_3 = x^2 + xy - y^2$ generating set.

$$S(f_1, f_3) = 1. f_1 - yf_3 = -xy^2 + x + y^3 - y = -1. f_2 + y^3 - y \neq 0$$

Remainder is not zero, so we have to add $f_4 = y^3 - y$ generating set.

$$S(f_2, f_3) = xf_2 - y^2f_3 = yf_2 + yf_4 + f_3 + 0$$

$$S(f_1, f_4) = y^2f_1 - x^2f_4 = x^2y + xy^2 - y^3 = yf_3 + 0$$

$$S(f_2, f_4) = yf_2 - xf_4 = 0$$

$$S(f_3, f_4) = y^3f_3 - x^2f_4 = x^2y + xy^4 - y^5 = 1. f_1 + (y^2 + 1)f_2 - (y^2 + 1)f_4 + 0$$

So Gröbner basis of $\langle x^2y - y + x, xy^2 - x \rangle$ is a

$$\mathcal{G} = \{f_1, f_2, f_3, f_4\} = \{x^2y - y + x, xy^2 - x, x^2 + xy - y^2, y^3 - y\}$$

Also,

$$f_1 = x^2y - y + x = -1. f_2 + yf_3 + 1. f_4 + 0$$

Hence, the reduced Gröbner basis of $\langle x^2y - y + x, xy^2 - x \rangle$ is

$$\mathcal{G} = \{xy^2 - x, x^2 + xy - y^2, y^3 - y\}$$

2.4. Syzygy Modules

Definition 2.4.1.

Let $R = K[x_1, \dots\dots, x_n]$, and let $[f_1 \quad \cdots \quad f_s] \in R^s$. The set of all s-tuples $[a_1 \quad \cdots \quad a_s] \in R^s$ such that $a_1 f_1 + \cdots + a_s f_s = 0$ is an R -submodule of R^s called the syzygy module of $[f_1 \quad \cdots \quad f_s]$ and denoted by $Syz(f_1, \dots . f_s)$.

The syzygy modules are finitely generated $R-$submodules, and set of generators for them can be computed. We briefly explain the process of finding a generating set for a syzygy module. For details see (Adams and Loustaunau,1994).

If $\mathcal{G} = \{g_1, \dots, g_s\}$ is a Gröbner basis with respect to some monomial order. It is easy to compute a generating set for $Syz(g_1, \dots . g_s)$.

Let $S(g_i, g_j)$ be the S-polynomial of g_i *and* g_j

$$S(g_i, g_j) = \frac{x^{\gamma_{ij}}}{LT(g_i)} . g_i - \frac{x^{\gamma_{ij}}}{LT(g_j)} . g_j$$

where $x^{\gamma_{ij}}$ is the least common multiple of $LM(g_i)$ and $LM(g_j)$. Since $\mathcal{G}$ is a Gröbner basis, the remainder of $S(g_i, g_j)$ on division by $\mathcal{G}$ is zero. Hence

$$S(g_i, g_j) = \sum_{v=1}^{s} h_{ijv}\, g_v$$

for some $h_{ijv} \in R$ by division algorithm. The s-tuple

$$\boldsymbol{s_{ij}} = \frac{x^{\gamma_{ij}}}{LT(g_i)} . \boldsymbol{e_i} - \frac{x^{\gamma_{ij}}}{LT(g_j)} . \boldsymbol{e_j} - (h_{ij1}, \dots, h_{ijs}) \in Syz(g_1, \dots . g_s).$$

In fact, the set $\{\boldsymbol{s_{ij}}, 1 \leq i, j \leq s\}$ generates $Syz(g_1, \dots, g_s)$.

If $\{f_1, \dots . f_s\}$ does not form a Gröbner basis. A reduced Gröbner basis $\{g_1, \dots . g_t\}$ for $\langle f_1, \dots . f_s \rangle$ should be computed. Let $F = [f_1, \dots . f_s\ ]$ and $\mathcal{G} = [\ g_1, \dots . g_t]$. Since the elements of F and $\mathcal{G}$ generates the same ideal, there are a $t\ x\ s$

matrix S and a $t \ x \ s$ matrix T with entries in R, such that $F = \mathcal{G}S$ and $\mathcal{G} = F\ T$. Then the matrix S can be obtained by the Division Algorithm and the matrix T is obtained by keeping track of the reductions in Buchberger's algorithm. A generating set $\{\boldsymbol{s_{ij}}, 1 \leq i,j \leq t\}$ for $Syz(g_1, \dots, g_t)$ can be computed. Then the set of s-tuples $\{T\boldsymbol{s_{ij}}, 1 \leq i,j \leq t\}$ together with columns of the matrix $I - TS$ generates the syzygy module $Syz(f_1, \dots . f_s)$.

Example 2.4.2.

Consider $F = [f_1 \ \ f_2 \ \ f_3]$ where $f_1 = x + yz$, $f_2 = y^2 + z$, $f_3 = xz - y$. We will find a generating set for $Syz(f_1, f_2, f_3)$.

Firstly, we compute a reduced Gröbner basis $\mathcal{G}$ with respect to lexicographic order with $x > y > z$ for $< f_1, f_2, f_3 >$. Applying Buchberger's algorithm we found $\mathcal{G} = [g_1, g_2, g_3, g_4]$ is the reduced Gröbner basis where

$g_1 = x + yz$, $g_2 = y^2 + z, g_3 = yz^2 + y, g_4 = z^3 + z$. Furthermore,

$$[g_1 \ \ g_2 \ \ g_3 \ \ g_4] = [f_1 \ \ f_2 \ \ f_3]\begin{bmatrix} 1 & 0 & z & -\mathrm{yz} \\ 0 & 1 & 0 & z^2+1 \\ 0 & 0 & -1 & y \end{bmatrix}$$

and

$$[f_1 \ \ f_2 \ \ f_3] = [g_1 \ \ g_2 \ \ g_3 \ \ g_4]\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We need to reduce all $S(g_i, g_j)$'s for $1 \leq i < j \leq 4$ in order to compute generators for $Syz(g_1, g_2, g_3, g_4)$. Since $S(g_1, g_2) = y^2 g_1 - x g_2 = -xz + y^3 z = -z g_1 + yz g_2, \boldsymbol{s_{12}} = (y^2 + z, -x - yz, 0{,}0)^T$. Similarly $S(g_1, g_3) = yz^2 g_1 - x g_3 = -xy + y^2 z^3 = -y g_1 + yz g_3$ implies $\boldsymbol{s_{13}} = (yz^2 + y, 0, -x - yz, 0)^T$. Continuing same way one can get other syzygies, $\boldsymbol{s_{14}} = (z^3 + z, 0, -z^2, -x)^T$, $\boldsymbol{s_{23}} = (0, z^2 + 1, -y, -1)^T$, $\boldsymbol{s_{24}} = (0, z^3 + z, 0, -y^2 - z)^T$ and $\mathbf{s_{34}} = (0{,}0, z, -y)^T$.

Then

$$\begin{bmatrix} 1 & 0 & z & -\mathrm{yz} \\ 0 & 1 & 0 & z^2+1 \\ 0 & 0 & -1 & y \end{bmatrix} \boldsymbol{s_{12}} = \begin{bmatrix} y^2+z \\ -x-yz \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & z & -\mathrm{yz} \\ 0 & 1 & 0 & z^2+1 \\ 0 & 0 & -1 & y \end{bmatrix} \boldsymbol{s_{13}} = \begin{bmatrix} -\mathrm{xz}+y \\ 0 \\ x+yz \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & z & -\mathrm{yz} \\ 0 & 1 & 0 & z^2+1 \\ 0 & 0 & -1 & y \end{bmatrix} \boldsymbol{s_{14}} = \begin{bmatrix} \mathrm{xyz}+z \\ -xz^2-x \\ -xy+z^2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & z & -\mathrm{yz} \\ 0 & 1 & 0 & z^2+1 \\ 0 & 0 & -1 & y \end{bmatrix} \boldsymbol{s_{23}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & z & -\mathrm{yz} \\ 0 & 1 & 0 & z^2+1 \\ 0 & 0 & -1 & y \end{bmatrix} \boldsymbol{s_{24}} = \begin{bmatrix} y^3z+yz^2 \\ -y^2z^2-y^2 \\ -y^3-yz \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & z & -\mathrm{yz} \\ 0 & 1 & 0 & z^2+1 \\ 0 & 0 & -1 & y \end{bmatrix} \boldsymbol{s_{34}} = \begin{bmatrix} y^2z+z^2 \\ -\mathrm{yz}^2-y \\ -y^2-z \end{bmatrix}.$$

Note that

$$(y^3z+yz^2, -y^2z^2-y^2, -y^3-yz)^T = y\,(y^2z+z^2, -y-yz^2, -y^2-z)^T,$$

$$(z+xyz, -x-xz^2, -xy+z^2)^T = -\mathrm{y}(-\mathrm{xz}+y, 0, x+yz)^T - z\,(y^2z+z^2, -y-yz^2, -y^2-z)^T + (z+1)(y^2+z, -x-yz, 0)^T.$$

Hence

$$Syz(f_1, f_2, f_3) = \langle (y^2+z, -x-y\,z, 0)^T, (-\mathrm{x}\,z+y, 0, x+yz)^T, (y^2z+z^2, -y-yz^2, -y^2-z)^T \rangle.$$

3. GRÖBNER BASIS APPROACH TO THE LIFTING PROBLEM

In this chapter we investigate a method for the solution of the lifting problem of homogeneous ideals using the Gröbner bases. This method suggested by (Bertone et al, 2016).

Let K be a field and let $J \subseteq K[x_1, \dots, x_{n-1}]$ be an ideal generated by homogeneous polynomials $f_1, \dots, f_s$. This kind of ideals are called homogeneous ideals. It is well known that if a polynomial f is in a homogeneous ideal if and only if each homogeneous component of f is in this ideal (see Cox et al, 2007, Section 8.3).

Definition 3.1.

Let J be a homogeneous ideal in $K[x_1, \dots, x_{n-1}]$. A homogeneous ideal $I \subseteq K[x_1, \dots, x_{n-1}, x_n]$ is called a lifting of J with respect to x_n if x_n is not a zero divisor in $K[x_1, \dots, x_{n-1}, x_n]/I$ and $J = \langle f(x_1, \dots, x_{n-1}, 0) : f \in I \rangle$.

Definition 3.2.

For a given monomial order $\prec$ on $K[x_1, \dots, x_{n-1}]$, a corresponding degree reverse monomial order $\prec_n$ on $K[x_1, \dots, x_{n-1}, x_n]$ is defined as follows for the same degree monomials $\boldsymbol{x^\alpha}$ and $\boldsymbol{x^\beta}$, $\boldsymbol{x^\alpha} \prec_n \boldsymbol{x^\beta}$ if $\alpha_n > \beta_n$ or $\alpha_n = \beta_n$ and $\boldsymbol{x^\alpha}/{x_n}^{\alpha_n} \prec \boldsymbol{x^\beta}/{x_n}^{\alpha_n}$.

For every degree reverse monomial order, we have that if the leading term of a homogeneous polynomial f is divisible by ${x_n}^r$ then the polynomial f is divisible by ${x_n}^r$.

The next theorem given by Bertone et al (2016) is reformulation of (Carra Ferro and Robbiano, 1990, Theorem 2.5.) in terms of Gröbner bases. Notice that in the next chapter we will use the original version of this theorem and obtain a more effective method to find the liftings of a homogeneous ideal.

Theorem 3.3. (Bertone et al (2016), Theorem 3.2.)

Let $J \subseteq K[x_1, \dots, x_{n-1}]$ *and* $I \subseteq K[x_1, \dots, x_{n-1}, x_n]$ *be homogeneous ideals. Then* I *is a lifting of* J *if and only if the reduced Gröbner basis of* I *with respect to* $\prec_n$ *is* $\{f_\alpha + g_\alpha\}_\alpha$*, where* $\{f_\alpha\}_\alpha$ *is the reduced Gröbner basis of* J *with respect to* $\prec$ *and* $g_\alpha \in \langle x_n \rangle$*.*

A monomial ordering < is called sequential if for every monomial x^α there exists only finitely many monomials x^β with $x^\alpha < x^\beta$. Clearly every degree compatible monomial ordering is sequential. Ferro (1988) explain a method for finding all ideals that have reduced Gröbner basis with respect to a given sequential monomial ordering with the same associated monomial ideal. Given the homogeneous ideal $J \subseteq K[x_1, \dots, x_{n-1}]$ and its reduced Gröbner basis $\mathcal{G} = \{f_\alpha\}_\alpha$ with respect to a sequential monomial order $\prec$, let $\mathcal{N}(J)$ be denoted the set monomials not belonging to $\langle LM(f_\alpha)\rangle_\alpha$. Define

$$g_\alpha = \sum C_{\alpha\gamma} \boldsymbol{x}^{\gamma} {x_n}^{\gamma_n} \ , \qquad \mathcal{G} = \{f_\alpha + g_\alpha\}_\alpha$$

where the summation runs over $\{\boldsymbol{x}^{\gamma} \in N(J) \colon \deg(\boldsymbol{x}^{\gamma} {x_n}^{\gamma_n}) = \deg(f_\alpha)\}$. Then let $C = \{C_{\alpha\gamma}\}_{\alpha,\gamma}$.

Definition 3.5.

The ideal $\mathfrak{h}_0 \subseteq K[C]$ is generated by the coefficients of the monomials in a complete reduction with respect to $\mathcal{G}$ of $S(f_\alpha + g_\alpha, f_\beta + g_\beta)$ for every α and β.

Bertone et al (2016) shows that the ideal $\mathfrak{h}_0$ is independent of the reduction process of $S-polynomials$ by the division algorithm.

Definition 3.6.

Define the family of ideals

$$S = \{I: LT_{\prec}(I) = \langle LM(f_\alpha)\rangle_\alpha\}.$$

Theorem 3.7. (Carre Ferro (1988), Lemma 4.)

There exists a bijection between S and the affine scheme $V(\mathfrak{h}_0)$. That means $I = \langle g_\alpha \rangle_\alpha \in S$ if and only if the parameters $C_{\alpha\gamma}$ are replaced by constants $c_{\alpha\gamma} \in K$ that satisfy the conditions in $\mathfrak{h}_0$.

Bertone et al (2016) use this idea together with Theorem 3.3 to obtain the family of all liftings of a homogeneous ideal. They start with a homogeneous ideal $J \subseteq K[x_1, \dots, x_{n-1}]$ and then they find its reduced Gröbner basis $G = \{f_\alpha\}_\alpha$ *with* respect to a monomial order $\prec$. After that using corresponding degree reverse monomial order $\prec_n$ on $K[x_1, \dots, x_{n-1}, x_n]$ they found the family of ideals

$$S = \left\{I \subseteq K[x_1, \dots, x_{n-1}, x_n]: LT_{\prec_n}(I) = \langle LM(f_\alpha)\rangle_\alpha\right\}$$

which corresponding the family of some liftings of J.

Bertone et al (2016) also shows that the scheme is independent of the selected monomial order. If the ideal $\mathfrak{h}_1 \subseteq K[D]$ is another ideal in Definition 3.5 obtained by using a different monomial order and parameters, they explicitly construct an isomorphism between $K[C]/\mathfrak{h}_0$ and $K[D]/\mathfrak{h}_1$.

Let us illustrate this method with an example.

Example 3.8

Take the ideal $J = \langle f_1 = {x_1}^2, f_2 = x_1x_2, f_3 = {x_2}^4 + x_1{x_3}^3 \rangle \subseteq \mathbb{Q}[x_1, x_2, x_3]$. The given generating set is already a Gröbner basis for J with respect to degree reverse lexicographic order. We set

$$\mathcal{G} = \{f_1 + g_1, f_2 + g_2, f_3 + g_3\} \subseteq \mathbb{Q}[x_1, x_2, x_3, x_4] \text{ where}$$

$$g_1 = C_1x_1x_4 + C_2x_2x_4 + C_3x_3x_4 + C_4{x_4}^2,$$

$$g_2 = C_5x_1x_4 + C_6x_2x_4 + C_7x_3x_4 + C_8{x_4}^2 \text{ and}$$

$$g_3 = C_9{x_2}^3x_4 + C_{10}{x_2}^2x_3x_4 + C_{11}x_1{x_3}^2x_4 + C_{12}x_2{x_3}^2x_4 + C_{13}{x_3}^3x_4 + C_{14}{x_2}^2{x_4}^2 + C_{15}x_1x_3{x_4}^2 + C_{16}x_2x_3{x_4}^2 + C_{17}{x_3}^2{x_4}^2 + C_{18}x_1{x_4}^3 + C_{19}x_2{x_4}^3 + C_{20}x_3{x_4}^3 + C_{21}{x_4}^4.$$

We will obtain the equations that the parameters must satisfy if $\mathcal{G}$ is the reduced Gröbner basis with respect to degree reverse lexicographic order. Hence, we apply the division algorithm to the S-polynomial

$S(f_1 + g_1, f_2 + g_2) = x_2(f_1 + g_1) - x_1(f_2 + g_2) = -C_5x_1^2x_4 + C_1x_1x_2x_4 - C_6x_1x_2x_4 + C_2x_2^2x_4 - C_7x_1x_3x_4 + C_3x_2x_3x_4 - C_8x_1x_4^2 + C_4x_2x_4^2$.

Applying the division algorithm

$S(f_1 + g_1, f_2 + g_2) = -C_5x_4(f_1 + g_1) + (C_1 - C_6)x_4(f_2 + g_2) + C_2x_2^2x_4 - C_7x_1x_3x_4 + C_3x_2x_3x_4 + (C_5C_6 - C_8)x_1x_4^2 + (C_4 + C_2C_5 - C_1C_6 + C_6^2)x_2x_4^2 + (C_3C_5 - C_1C_7 + C_6C_7)x_3x_4^2 + (C_4C_5 - C_1C_8 + C_6C_8)x_4^3$.

$S(f_1 + g_1, f_3 + g_3) = {x_2}^4(f_1 + g_1) - {x_1}^2(f_3 + g_3) = -x_1^3x_3^3 - C_9x_1^2x_2^3x_4 + C_1x_1x_2^4x_4 + C_2x_2^5x_4 - C_{10}x_1^2x_2^2x_3x_4 + C_3x_2^4x_3x_4 - C_{11}x_1^3x_3^2x_4 - C_{12}x_1^2x_2x_3^2x_4 - C_{13}x_1^2x_3^3x_4 - C_{14}x_1^2x_2^2x_4^2 + C_4x_2^4x_4^2 - C_{15}x_1^3x_3x_4^2 - C_{16}x_1^2x_2x_3x_4^2 - C_{17}x_1^2x_3^2x_4^2 - C_{18}x_1^3x_4^3 - C_{19}x_1^2x_2x_4^3 - C_{20}x_1^2x_3x_4^3 - C_{21}x_1^2x_4^4$.

Applying the division algorithm

$S(f_1+g_1, f_3+g_3) = (-x_1x_3^3 - C_9x_2^3x_4 - C_{10}x_2^2x_3x_4 - C_{11}x_1x_3^2x_4 - C_{12}x_2x_3^2x_4 + (C_1 - C_{13})x_3^3x_4 - C_{14}x_2^2x_4^2 - C_{15}x_1x_3x_4^2 - C_{16}x_2x_3x_4^2 + (C_1C_{11} - C_{17})x_3^2x_4^2 - C_{18}x_1x_4^3 - C_{19}x_2x_4^3 + (C_1C_{15} - C_{20})x_3x_4^3 + (C_1C_{18} - C_{21})x_4^4)(f_1 + g_1) + (C_1x_2^3x_4 + (-C_1C_5 + C_1C_9)x_2^2x_4^2 + C_1C_{10}x_2x_3x_4^2 + C_1C_{12}x_3^2x_4^2 + (C_1C_5^2 - C_1C_5C_9 + C_1C_{14})x_2x_4^3 + (-C_1C_5C_{10} + C_1C_{16})x_3x_4^3 + (-C_1C_5^3 + C_1C_5^2C_9 - C_1C_5C_{14} + C_1C_{19})x_4^4)(f_2 + g_2) + (C_2x_2x_4 + C_3x_3x_4 + (C_4 - C_1C_6)x_4^2)(f_3 + g_3) - C_1C_7x_2^3x_3x_4^2 + (-C_1^2 + C_1C_6 + C_1C_{13})x_1x_3^3x_4^2 - C_1C_2x_2x_3^3x_4^2 - C_1C_3x_3^4x_4^2 + (C_1C_5C_6 - C_1C_8)x_2^3x_4^3 + (C_1C_5C_7 - C_1C_7C_9)x_2^2x_3x_4^3 + (-C_1^2C_{11} + C_1C_6C_{11} - C_1C_5C_{12} + C_1C_{17})x_1x_3^2x_4^3 + (-C_1C_7C_{10} - C_1C_2C_{11})x_2x_3^2x_4^3 + (-C_1C_4 - C_1C_3C_{11} - C_1C_7C_{12} + C_1C_6C_{13})x_3^3x_4^3 + (-C_1C_5^2C_6 + C_1C_5C_8 + C_1C_5C_6C_9 - C_1C_8C_9)x_2^2x_4^4 + (C_1C_5^2C_{10} - C_1^2C_{15} + C_1C_6C_{15} - C_1C_5C_{16} + C_1C_{20})x_1x_3x_4^4 + (-C_1C_5^2C_7 + C_1C_5C_7C_9 + C_1C_5C_6C_{10} - C_1C_8C_{10} - C_1C_7C_{14} - C_1C_2C_{15})x_2x_3x_4^4 + (C_1C_5C_7C_{10} - C_1C_4C_{11} - C_1C_8C_{12} - C_1C_3C_{15} - C_1C_7C_{16} + C_1C_6C_{17})x_3^2x_4^4 + (C_1C_5^4 - C_1C_5^3C_9 + C_1C_5^2C_{14} - C_1^2C_{18} + C_1C_6C_{18} - C_1C_5C_{19} + C_1C_{21})x_1x_4^5 + (C_1C_5^3C_6 - C_1C_5^2C_8 - C_1C_5^2C_6C_9 + C_1C_5C_8C_9 + C_1C_5C_6C_{14} - C_1C_8C_{14} - C_1C_2C_{18})x_2x_4^5 + (C_1C_5^3C_7 - C_1C_5^2C_7C_9 + C_1C_5C_8C_{10} + C_1C_5C_7C_{14} - C_1C_4C_{15} - C_1C_8C_{16} - C_1C_3C_{18} - C_1C_7C_{19} + C_1C_6C_{20})x_3x_4^5 + (C_1C_5^3C_8 - C_1C_5^2C_8C_9 + C_1C_5C_8C_{14} - C_1C_4C_{18} - C_1C_8C_{19} + C_1C_6C_{21})x_4^6$.

$S(f_2+g_2, f_3+g_3) = {x_2}^3(f_2+g_2) - x_1(f_3+g_3) = x_1^2x_3^3 + C_5x_1x_2^3x_4 - C_9x_1x_2^3x_4 + C_6x_2^4x_4 - C_{10}x_1x_2^2x_3x_4 + C_7x_2^3x_3x_4 - C_{11}x_1^2x_3^2x_4 - C_{12}x_1x_2x_3^2x_4 - C_{13}x_1x_3^3x_4 - C_{14}x_1x_2^2x_4^2 + C_8x_2^3x_4^2 - C_{15}x_1^2x_3x_4^2 - C_{16}x_1x_2x_3x_4^2 - C_{17}x_1x_3^2x_4^2 - C_{18}x_1^2x_4^3 - C_{19}x_1x_2x_4^3 - C_{20}x_1x_3x_4^3 - C_{21}x_1x_4^4.$

Applying the division algorithm

$S(f_2+g_2, f_3+g_3) = C_6x_4(f_1+g_1) + ((C_5 - C_9)x_2^2x_4 - C_{10}x_2x_3x_4 - C_{12}x_3^2x_4 + (-C_5^2 + C_5C_9 - C_{14})x_2x_4^2 + (C_5C_{10} - C_{16})x_3x_4^2 + (C_5^3 - C_5^2C_9 + C_5C_{14} - C_{19})x_4^3)(f_2 + g_2) + (-x_3^3 - C_{11}x_3^2x_4 - C_{15}x_3x_4^2 - C_{18}x_4^3)(f_3 + g_3) + C_7x_2^3x_3x_4 + (C_1 - C_6 - C_{13})x_1x_3^3x_4 + C_2x_2x_3^3x_4 + C_3x_3^4x_4 + (-C_5C_6 + C_8)x_2^3x_4^2 + (-C_5C_7 + C_7C_9)x_2^2x_3x_4^2 + (C_1C_{11} - C_6C_{11} + C_5C_{12} - C_{17})x_1x_3^2x_4^2 + (C_7C_{10} + C_2C_{11})x_2x_3^2x_4^2 + (C_4 + C_3C_{11} + C_7C_{12} - C_6C_{13})x_3^3x_4^2 + (C_5^2C_6 - C_5C_8 - C_5C_6C_9 +$

$C_8C_9)x_2^2x_4^3 + (-C_5^2C_{10} + C_1C_{15} - C_6C_{15} + C_5C_{16} - C_{20})x_1x_3x_4^3 + (C_5^2C_7 - C_5C_7C_9 - C_5C_6C_{10} + C_8C_{10} + C_7C_{14} + C_2C_{15})x_2x_3x_4^3 + (-C_5C_7C_{10} + C_4C_{11} + C_8C_{12} + C_3C_{15} + C_7C_{16} - C_6C_{17})x_3^2x_4^3 + (-C_5^4 + C_5^3C_9 - C_5^2C_{14} + C_1C_{18} - C_6C_{18} + C_5C_{19} - C_{21})x_1x_4^4 + (-C_5^3C_6 + C_5^2C_8 + C_5^2C_6C_9 - C_5C_8C_9 - C_5C_6C_{14} + C_8C_{14} + C_2C_{18})x_2x_4^4 + (-C_5^3C_7 + C_5^2C_7C_9 - C_5C_8C_{10} - C_5C_7C_{14} + C_4C_{15} + C_8C_{16} + C_3C_{18} + C_7C_{19} - C_6C_{20})x_3x_4^4 + (-C_5^3C_8 + C_5^2C_8C_9 - C_5C_8C_{14} + C_4C_{18} + C_8C_{19} - C_6C_{21})x_4^5.$

The coefficient of variables in the remainders generates the ideal $\mathfrak{h}_0$. After elimination of variables,

$$\mathfrak{h}_0 = \langle C_2, C_3, C_4 - C_1C_6 + C_6^2, C_7, C_8 - C_5C_6, C_{13} - C_1 + C_6, C_{17} - C_1C_{11} + C_6C_{11} - C_5C_{12}, C_{20} + C_5^2C_{10} - C_1C_{15} + C_6C_{15} - C_5C_{16}, C_{21} + C_5^4 - C_5^3C_9 + C_5^2C_{14} - C_1C_{18} + C_6C_{18} - C_5C_{19}\rangle.$$

Hence the members of family of the liftings of J are given by

$$f_1 + g_1 = {x_1}^2 + C_1x_1x_4 + (C_1C_6 - C_6^2){x_4}^2,$$

$$f_2 + g_2 = x_1x_2 + C_5x_1x_4 + C_6x_2x_4 + C_5C_6{x_4}^2$$

and

$$f_3 + g_3 = {x_2}^4 + x_1{x_3}^3 + C_9{x_2}^3x_4 + C_{10}{x_2}^2x_3x_4 + C_{11}x_1{x_3}^2x_4 + C_{12}x_2{x_3}^2x_4 + (C_1 - C_6){x_3}^3x_4 + C_{14}{x_2}^2{x_4}^2 + C_{15}x_1x_3{x_4}^2 + C_{16}x_2x_3{x_4}^2 + (C_1C_{11} - C_6C_{11} + C_5C_{12}){x_3}^2{x_4}^2 + C_{18}x_1{x_4}^3 + C_{19}x_2{x_4}^3 + (-C_5^2C_{10} + C_1C_{15} - C_6C_{15} + C_5C_{16})x_3{x_4}^3 + (-C_5^4 + C_5^3C_9 - C_5^2C_{14} + C_1C_{18} - C_6C_{18} + C_5C_{19}){x_4}^4$$

where $C_1, C_5, C_6, C_9, C_{10}, C_{11}, C_{12}, C_{14}, C_{15}, C_{16}, C_{18}, C_{19}$ are free parameters.

In the next example we will show that this method is not appropriate to the nature of the lifting problem.

Example 3.9.

Let us consider the same ideal $J = \langle f_1 = {x_1}^2, f_2 = x_1x_2, f_3 = {x_2}^4 + x_1{x_3}^3 \rangle \subseteq \mathbb{Q}[x_1, x_2, x_3]$. The Gröbner basis of J with respect to degree lexicographic order is $H = \{f_1 = {x_1}^2, f_2 = x_1x_2, f_3 = {x_2}^4 + x_1{x_3}^3, f_4 = x_2^5\}$. In fact,

$$S(f_2, f_3) = {x_3}^3 f_2 - x_2 f_3 = -x_2^5.$$

It is well known that the number of polynomials in each minimal generating set of a homogeneous ideal is fixed. This number is three for our case. Hence H is not a minimal generating set for J and f_4 is redundant. Now we define the set $\mathcal{H} = \{f_1 + g_1, f_2 + g_2, f_3 + g_3, f_4 + g_4\} \subseteq \mathbb{Q}[x_1, x_2, x_3, x_4]$ where

$$g_1 = D_1x_1x_4 + D_2x_2x_4 + D_3x_3x_4 + D_4x_4^2,$$

$$g_2 = D_5x_1x_4 + D_6x_2x_4 + D_7x_3x_4 + D_8x_4^2,$$

$$g_3 = D_{12}x_2^3x_4 + D_{13}x_2^2x_3x_4 + D_9x_1x_3^2x_4 + D_{15}x_2x_3^2x_4 + D_{18}x_3^3x_4 + D_{14}x_2^2x_4^2 + D_{10}x_1x_3x_4^2 + D_{16}x_2x_3x_4^2 + D_{19}x_3^2x_4^2 + D_{11}x_1x_4^3 + D_{17}x_2x_4^3 + D_{20}x_3x_4^3 + D_{21}x_4^4,$$

$$g_4 = D_{25}x_2^4x_4 + D_{26}x_2^3x_3x_4 + D_{28}x_2^2x_3^2x_4 + D_{31}x_2x_3^3x_4 + D_{35}x_3^4x_4 + D_{27}x_2^3x_4^2 + D_{29}x_2^2x_3x_4^2 + D_{22}x_1x_3^2x_4^2 + D_{32}x_2x_3^2x_4^2 + D_{36}x_3^3x_4^2 + D_{30}x_2^2x_4^3 + D_{23}x_1x_3x_4^3 + D_{33}x_2x_3x_4^3 + D_{37}x_3^2x_4^3 + D_{24}x_1x_4^4 + D_{34}x_2x_4^4 + D_{38}x_3x_4^4 + D_{39}x_4^5.$$

We will show that $f_4 + g_4$ is also redundant in family of liftings of J. Since f_4 is the monic form of the remainder of $S(f_2, f_3)$ upon division by $\{f_1, f_2, f_3\}$, a step of the division of $S(f_2 + g_2, f_3 + g_3)$ by $\mathcal{H}$ will be

$$S(f_2 + g_2, D_5x_4) - c(f_4 + g_4)$$

where $c \in \mathbb{Q}$. Therefore a constant multiple of g_4 will go to the remainder. Hence the parameter D_i 's in g_4 can be obtained as polynomial functions of other parameters. In this particular example,

$$S(f_2 + g_2, f_3 + g_3) = (-D_9x_3^2x_4 - D_{10}x_3x_4^2 - D_{11}x_4^3)(f_2 + g_2) + D_5x_4(f_3 + g_3) - (f_4 + g_4) + (-D_5 - D_{12} + D_{25})x_2^4x_4 + (-D_{13} + D_{26})x_2^3x_3x_4 + (-D_{15} +$$

$D_{28})\, x_2^2x_3^2x_4 + (D_6 - D_{18} + D_{31})x_2x_3^3x_4 + (D_7 + D_{35})x_3^4x_4 + (-D_5D_{12} - D_{14} + D_{27})x_2^3x_4^2 + (-D_5D_{13} - D_{16} + D_{29})x_2^2x_3x_4^2 + D_{22}x_1x_3^2x_4^2 + (D_6D_9 - D_5D_{15} - D_{19} + D_{32})x_2x_3^2x_4^2 + (D_8 + D_7D_9 - D_5D_{18} + D_{36})x_3^3x_4^2 + (-D_5D_{14} - D_{17} + D_{30})x_2^2x_4^3 + D_{23}x_1x_3x_4^3 + (D_6D_{10} - D_5D_{16} - D_{20} + D_{33})x_2x_3x_4^3 + (D_8D_9 + D_7D_{10} - D_5D_{19} + D_{37})x_3^2x_4^3 + D_{24}x_1x_4^4 + (D_6D_{11} - D_5D_{17} - D_{21} + D_{34})x_2x_4^4 + (D_8D_{10} + D_7D_{11} - D_5D_{20} + D_{38})x_3x_4^4 + (D_8D_{11} - D_5D_{21} + D_{39})x_4^5.$

After replacing parameters of g_4 with appropriate polynomial functions of other parameters, it is clear that $f_4 + g_4$ can be written as a combination of $\{f_1 + g_1, f_2 + g_2, f_3 + g_3\}$. In this particular example

$$f_4 + g_4 = (-D_9x_3^2x_4 - D_{10}x_3x_4^2 - D_{11}x_4^3)(f_2 + g_2) + D_5x_4(f_3 + g_3).$$

Therefore $f_4 + g_4$ is redundant. That means after doing huge number of divisions during the process of finding family of liftings with this method, one can eliminate $f_4 + g_4$ from the generating set of liftings.

This applies to all polynomials that are subsequently added to the generating set during the Gröbner basis computation. Hence, if the original generating set is not a Gröbner basis with respect to some monomial order, this method causes a lot of unnecessary computations.

4. A NEW METHOD FOR FINDING LIFTINGS

4.1 H-Bases

Definition 4.1.1.

Let $f \in K[x_1, \dots, x_{n-1}]$ is a nonzero polynomial. If $f = f_d + f_{d-1} + \cdots + f_0$ with $\deg(f_i) = i$, then $H(f) = f_d$ is called leading form of f.

Definition 4.1.2.

Let $I \subseteq K[x_1, \dots, x_{n-1}]$ be an ideal and let $H(I)$ be the ideal generated by leading forms of elements of I. A generating set $\{h_1, \dots, h_s\}$ of I is called H-basis or Macaulay basis of I provided that $H(I) = \langle H(h_1), \dots, H(h_s) \rangle$.

It is well known that a Gröbner basis with respect to a degree compatible monomial order is also a H-basis.

Definition 4.1.3.

(a) Let $f \in K[x_1, \dots, x_{n-1}]$ is a nonzero polynomial such $f = f_d + f_{d-1} + \cdots + f_0$ with $\deg(f_i) = i$. Then $f^h = f_d + x_n f_{d-1} + \cdots + {x_n}^d f_0$ is called homogenization of f with respect to x_n.

(b) Let $F \in K[x_1, \dots, x_{n-1}, x_n]$ is a nonzero homogeneous polynomial. Then $F_a = F(x_1, \dots, x_{n-1}, 1)$ is called dehomogenization of F with respect to x_n.

(c) For an ideal $J \subseteq K[x_1, \dots, x_{n-1}]$, the ideal $J^h = \langle f^h : f \in J \rangle$ is called the homogenization of J.

(d) For a homogeneous ideal $I \subseteq K[x_1, \dots, x_n]$, the ideal $I_a = \langle F_a : F \in I \rangle$ is called dehomogenization of I.

It is clear that if $I = \langle F_1, \dots, F_r \rangle \subseteq K[x_1, \dots, x_n]$ where $F_i's$ are homogeneous polynomials, then $I_a = \langle (F_1)_a, \dots, (F_r)_a \rangle$. On the other hand, the corresponding property is not valid for the homogenization ideals.

Lemma 4.1.4. (Carre Ferro and Robbiano (1990), Lemma 2.3.)

The followings are equivalent:

(a) The set $\{f_1, \dots, f_s\}$ *is an H-basis of an ideal* $I \subseteq K[x_1, \dots, x_{n-1}]$.

(b) $I^h = \langle {f_1}^h, \dots, {f_s}^h \rangle$.

(c) x_n *is not zero divisor on* $K[x_1, \dots, x_n]/\langle {f_1}^h, \dots, {f_s}^h \rangle$.

The next lemma gives a relation between the lifting problem and H-basis.

Lemma 4.1.5. (Carre Ferro and Robbiano (1990), Lemma 2.4.)

Let $I \subseteq K[x_1, \dots, x_n]$ *be a homogeneous ideal. The following conditions are equivalent.*

(a) x_n *is not a zero divisor in* $K[x_1, \dots, x_{n-1}, x_n]/I$.

(b) $I = (I_a)^h$.

(c) $H(I_a) = I(x_1, \dots, x_{n-1}, 0)$.

Now, we are ready to give the main result of this section. Recall that Bertone et al (2016) used the modified version of the following lemma when they finding the family of liftings via Gröbner bases.

Theorem 4.1.6. (Carre Ferro and Robbiano (1990), Theorem 2.5.)

Let $J = \langle f_1, \ldots, f_r \rangle \subseteq K[x_1, \ldots, x_{n-1}]$ *where* $f_i's$ *are homogeneous polynomials.*

(a) Let $h_i = f_i + g_i$ *where* $deg(g_i) < deg(f_i)$ *for* $i = 1, \ldots, r$. *If* $\{h_1, \ldots, h_r\}$ *is a H-basis for the ideal* $\mathcal{U} = \langle h_1, \ldots, h_r \rangle$, *then* $I = \mathcal{U}^h$ *is a lifting of* J.

(b) If I *is a lifting of* J, *then there exist polynomials* $g_1, \ldots, g_r$ *where* $deg(g_i) < deg(f_i)$ *for* $i = 1, \ldots, r$ *such that* $\{f_1 + g_1, \ldots, f_r + g_r\}$ *is a H-basis and* $I = \langle (f_1 + g_1)^h, \ldots, (f_r + g_r)^h \rangle$.

This theorem suggests the following. Given a homogeneous $J = \langle f_1, \ldots, f_r \rangle \subseteq K[x_1, \ldots, x_{n-1}]$, define

$$g_\alpha = \sum_{deg(\boldsymbol{x}^\gamma) < deg(f_\alpha)} C_{\alpha\gamma} \boldsymbol{x}^\gamma, \qquad \mathcal{H} = \{f_\alpha + g_\alpha\}_\alpha.$$

Then find the conditions the parameters $C_{\alpha\gamma}$ must satisfies for $\mathcal{H}$ to be a H-basis. We study this in the next section.

4.2 Family of Liftings and Syzygies

The key point here is the following theorem given by (Luo and Yılmaz, 2001).

Theorem 4.2.1. (Luo and Yılmaz, 2001, Theorem 2.4.)

Let $I = \langle h_1, \ldots, h_t \rangle \subseteq K[x_1, \ldots, x_{n-1}]$. *Let the columns of the* $t \times l$ *matrix* $S = (s_{ij})$ *be a generating set of* $Syz(H(h_1), \ldots, H(h_t))$. *We may assume further that each* $s_{ji} f_j$ *is a homogeneous polynomial of same degree for* $j = 1, \ldots, t$. *Then* $\mathcal{H} = \{h_1, \ldots, h_t\}$ *is a H-basis for* I *if and only if*

$$q_i = \sum_{j=1}^{t} s_{ji} h_j = \sum_{j}^{t} a_{ji} h_j \,, \qquad 1 \le i \le l$$

for some $a_{ji} \in K[x_1, \dots, x_{n-1}]$ *such that* $deg(q_i) = max\{deg(a_{ji}h_j), j = 1, \dots, t\}$.

Based on Theorem 4.1.6 and Theorem 4.2.1, Luo and Yılmaz (2001) propose the following method for finding family of lifting of a homogeneous ideal. Given a homogeneous $J = \langle f_1, \dots, f_r \rangle \subseteq K[x_1, \dots, x_{n-1}]$, define

$$g_i = \sum_{deg(\boldsymbol{x}^{\gamma}) < deg(f_i)} C_{i\gamma} \boldsymbol{x}^{\gamma}, \quad h_{\boldsymbol{i}} = f_{\boldsymbol{i}} + g_i.$$

Furthermore, for each q_i in Theorem 4.2.1 define

$$a_{ji} = \sum_{deg(\boldsymbol{x}^{\gamma}) < deg(q_i) - \deg(h_j)} D_{ij\gamma} \boldsymbol{x}^{\gamma}.$$

Then compare the coefficient of monomials of equation given Theorem 4.2.1 to find relations among the parameters $C_{i\gamma}'s$ and $D_{ij\gamma}'s$. This is not a convenient method because there are extra parameters $D_{ij\gamma}^{'}s$. Even though in their example they are able to solve these extra parameters in terms of $C_{i\gamma}'s$, there is no guarantee that this will always occur.

4.3. Vector Spaces and Homogeneous Ideals

Let $I \subseteq K[x_1, \dots, x_{n-1}]$ be an ideal generated by homogeneous polynomials $f_1, \dots, f_s$. For a nonzero integer d,

$$V_d(I) = \{f \in I \colon \deg(f) = d \text{ or } f = 0\}$$

can be considered a subspace of the vector space $\mathcal{P}_d$ of degree d polynomials over K. Furthermore all monomial multiples of the form ${x_1}^{\alpha_1} \cdots {x_{n-1}}^{\alpha_{n-1}} f_i$ with $\alpha_1 + \cdots + \alpha_{n-1} + \deg(f_i) = d$ is a spanning set for $V_d(I)$. Hence the problem whether a polynomial f is in I or not can be solved by linear algebra techniques. A minimal generating set for a homogeneous ideal can be obtained by dropping redundant elements. Number of elements in any minimal generating will be same number of polynomials.

Now let us consider a problem that we will often encounter when finding lifting family of a homogeneous ideal by a new method. Given a polynomial

$$f = \sum_{deg(\boldsymbol{x}^{\gamma})=d} C_{\gamma}\boldsymbol{x}^{\gamma},$$

we want to obtain the equations that the parameters $C_{\gamma}'s$ must satisfy for the polynomial f to be in I. This problem can be solved by linear algebraic techniques as follows. Suppose that $\mathcal{B}$ be an ordered set of monomials of degree d. Consider a matrix M whose columns are the coordinate vectors of the polynomials of the form ${x_1}^{\alpha_1} \cdots {x_{n-1}}^{\alpha_{n-1}} f_i$ with $\alpha_1 + \cdots + \alpha_{n-1} + \deg(f_i) = d$ relative to $\mathcal{B}$ and the last column is the coordinate vector of f with respect to $\mathcal{B}$. Obtain a row reduced echelon form matrix $\widetilde{M}$ which is equivalent to M. The last entries of the pivot rows of $\widetilde{M}$ gives the coefficients, as a linear combinations of $C_{\gamma}'s$, of the homogeneous polynomials $h_1, \dots, h_s$ satisfying $f = h_1 f_1 + \cdots + h_s f_s$ where either $h_i f_i = 0$ or $\deg(h_i f_i) = d$. Then consider the rows of $\widetilde{M}$ with only the last entry is zero. These non-zero elements give linear equations that $C_{\gamma}'s$ must satisfy for the polynomial f to be in I. Let us illustrate this process with an example.

Example 4.3.1.

Consider the homogeneous ideal

$$I = \langle f_1, f_2, f_3 \rangle = \langle {x_1}^2 + x_2 x_3, {x_2}^2 + {x_3}^2, {x_1}^3 + x_1 {x_2}^2 + {x_2}^2 x_3 \rangle \, .$$

We try to answer the following question. Under what conditions on parameters is the general polynomial of degree 3

$$g = a_1 x_1^3 + a_2 x_1^2 x_2 + a_4 x_1 x_2^2 + a_7 x_2^3 + a_3 x_1^2 x_3 + a_5 x_1 x_2 x_3 + a_8 x_2^2 x_3 + a_6 x_1 x_3^2 + a_9 x_2 x_3^2 + a_{10} x_3^3$$

in I?

Clearly $\{x_1f_1, x_2f_1, x_3f_1, x_1f_2, x_2f_2, x_3f_2, f_3\}$ is a spanning set for the vector space $V_3(I)$. Hence consider the augmented matrix

	x_1f_1	x_2f_1	x_3f_1	x_1f_2	x_2f_2	x_3f_3	f_3	g
x_1^3	1	0	0	0	0	0	1	a_1
$x_1^2x_2$	0	1	0	0	0	0	0	a_2
$x_1^2x_3$	0	0	1	0	0	0	0	a_3
$x_1x_2^2$	0	0	0	1	0	0	1	a_4
$x_1x_2x_3$	1	0	0	0	0	0	0	a_5
$x_1x_3^2$	0	0	0	1	0	0	0	a_6
x_2^3	0	0	0	0	1	0	0	a_7
$x_2^2x_3$	0	1	0	0	0	1	1	a_8
$x_2x_3^2$	0	0	1	0	1	0	0	a_9
x_2^3	0	0	0	0	0	1	0	a_{10}

After applying a series of elementary row operations, the row reduced echelon form of this matrix is the following

	x_1f_1	x_2f_1	x_3f_1	x_1f_2	x_2f_2	x_3f_3	f_3	g
x_1^3	1	0	0	0	0	0	0	a_5
$x_1^2x_2$	0	1	0	0	0	0	0	$-a_4 + a_6 + a_8 - a_{10}$
$x_1^2x_3$	0	0	1	0	0	0	0	$-a_7 + a_9$
$x_1x_2^2$	0	0	0	1	0	0	0	a_6
$x_1x_2x_3$	0	0	0	0	1	0	0	a_7
$x_1x_3^2$	0	0	0	0	0	1	0	a_{10}
x_2^3	0	0	0	0	0	0	1	$a_4 - a_6$
$x_2^2x_3$	0	0	0	0	0	0	0	$a_1 - a_4 - a_5 + a_6$
$x_2x_3^2$	0	0	0	0	0	0	0	$a_2 + a_4 - a_6 - a_8 + a_{10}$
x_2^3	0	0	0	0	0	0	0	$a_3 + a_7 - a_9$

Hence $g \in I$ if and only if $a_1 - a_4 - a_5 + a_6 = 0, a_2 + a_4 - a_6 - a_8 + a_{10}$ and $a_3 + a_7 - a_9 = 0$. Under these conditions

$$g = (a_5x_1 + (-a_4 + a_6 + a_8 - a_{10})x_2 + (-a_7 + a_9)x_3)f_1 + (a_6x_1 + a_7x_2 + a_{10}x_3)f_2 + (a_4 - a_6)f_3.$$

4.4. New Method

We start the following version of the division algorithm.

Theorem 4.4.1.

Suppose that $\{f_1, \ldots, f_s\}$ *is an H-basis for an ideal* $I \subseteq K[x_1, \ldots, x_n]$. *Then for any* $f \in I$, *there exist polynomials* $a_1, \ldots, a_s$ *such that*

$$f = a_1f_1 + \cdots + a_sf_s$$

where $deg(a_if_i) \leq deg(f)$ *when* $a_i \neq 0$.

Proof.

Since $\{\mathrm{f}_1, \ldots, \mathrm{f}_s\}$ is an H-basis and $f \in I$, $H(f) = b_1H(f_1) + \cdots + b_sH(f_s)$ where the polynomials $b_1, \ldots, b_s$ can be found by linear algebraic techniques as explained in the previous section. Now redefine f as

$$f \coloneqq f - b_1f_1 - \cdots - b_sf_s.$$

Then apply the same process repeatedly until f become zero.

Now we are ready to give a new method for solving the lifting problem. Given a homogeneous $J = \langle f_1, \ldots, f_r \rangle \subseteq K[x_1, \ldots, x_{n-1}]$, define

$$g_i = \sum_{deg(\boldsymbol{x}^{\gamma}) < deg(f_i)} C_{i\gamma}\boldsymbol{x}^{\gamma}, \quad h_{\boldsymbol{i}} = f_{\boldsymbol{i}} + g_i.$$

Theorem 4.1.6 implies that $\{f_1 + g_1, \ldots, f_r + g_r\}$ is a H-basis if and only if $I = \langle (f_1 + g_1)^h, \ldots, (f_r + g_r)^h \rangle$ is a lifting of J. Hence we have to decide under what conditions $\{f_1 + g_1, \ldots, f_r + g_r\}$ is an H-basis. For this we use Theorem 4.2.1. First we find a minimal generating set for $Syz(f_1, \ldots, f_r)$. Using Theorem 4.4.1 for each syzygy $(s_1, \ldots, s_r)^T$ in this generating set, find the polynomials $a_1, \ldots, a_s$ such that

$$s_1 g_1 + \cdots + s_r g_r = a_1(f_1 + g_1) + \cdots + a_r(f_r + g_r)$$

where $\deg(a_i(f_i + g_i)) \leq \deg(s_1 g_1 + \cdots + s_r g_r)$ when $a_i \neq 0$. During the division process the relations that parameters $C_{i\gamma}$ should satisfy can be obtained as in Example 4.3.1. Let $\mathfrak{h}_2$ be the ideal generated by collection of these relations. That means I is a lifting if and only if the parameters $\mathrm{C}_{\alpha\gamma}$ are replaced by constants $\mathrm{c}_{\alpha\gamma} \in \mathrm{K}$ that satisfy the conditions in $\mathfrak{h}_2$. This is a Gröbner basis free method. Even though the best method for finding a generating set for a syzygy module involves a Gröbner basis computation, there is no need for Gröbner basis computation for $\{f_1 + g_1, \ldots, f_r + g_r\}$. In some instances the generation set of the syzygy module may be known in advance. No Gröbner basis computation is required in this case. Let us demostrate this method with an example.

Example 4.4.2.

Let us take the same ideal $J = \langle f_1 = x_1{}^2, f_2 = x_1 x_2, f_3 = x_2{}^4 + x_1 x_3{}^3 \rangle \subseteq \mathbb{Q}[x_1, x_2, x_3]$. We set

$$\mathcal{H} = \{f_1 + g_1, f_2 + g_2, f_3 + g_3\} \subseteq \mathbb{Q}[x_1, x_2, x_3]$$

where

$$g_1 = F_1 x_1 + F_2 x_2 + F_3 x_3 + F_4,$$

$$g_2 = F_5 x_1 + F_6 x_2 + F_7 x_3 + F_8$$

and

$$\begin{aligned} g_3 = {} & F_9 x_1{}^3 + F_{10} x_1{}^2 x_2 + F_{11} x_1{}^2 x_3 + F_{12} x_1 x_2{}^2 + F_{13} x_1 x_2 x_3 + F_{14} x_1 x_3{}^2 + F_{15} x_2{}^3 \\ & + F_{16} x_2{}^2 x_3 + F_{17} x_2 x_3{}^2 + F_{18} x_3{}^3 + F_{19} x_1{}^2 + F_{20} x_1 x_2 + F_{21} x_1 x_3 \\ & + F_{22} x_2{}^2 + F_{23} x_2 x_3 + F_{24} x_3{}^2 + F_{25} x_1 + F_{26} x_2 + F_{27} x_3 + F_{28}. \end{aligned}$$

Using the technique expalined in the introduction chapter,

$$Syz(f_1, f_2, f_3) = \langle (x_2, -x_1, 0)^T, (0, x_2^4 + x_1 x_3^3, -x_1 x_2)^T, (x_3^3, x_2^3, -x_1)^T \rangle.$$

The next step is to write the general polynomial of degree $3, 4$ and 5 as a polynomial combinations of f_1, f_2 and f_3. We do not look the polynomials of degree 2 which are obvious and the polynomials of degree 6 or higher because the syzygies in $Syz(f_1, f_2, f_3)$ can produce at most a polynomial of degree 5. So

$$u_1 x_1^3 + u_2 x_1^2 x_2 + u_4 x_1 x_2^2 + u_7 x_2^3 + u_3 x_1^2 x_3 + u_5 x_1 x_2 x_3 + u_8 x_2^2 x_3 + u_6 x_1 x_3^2 + u_9 x_2 x_3^2 + u_{10} x_3^3 = (u_1 x_1 + u_2 x_2 + u_3 x_3) f_1 + (u_4 x_2 + u_5 x_3) f_2$$

under conditions $u_6 = u_7 = u_8 = u_9 = u_{10} = 0$;

$$v_1 x_1^4 + v_2 x_1^3 x_2 + v_4 x_1^2 x_2^2 + v_7 x_1 x_2^3 + v_{11} x_2^4 + v_3 x_1^3 x_3 + v_5 x_1^2 x_2 x_3 + v_8 x_1 x_2^2 x_3 + v_{12} x_2^3 x_3 + v_6 x_1^2 x_3^2 + v_9 x_1 x_2 x_3^2 + v_{13} x_2^2 x_3^2 + v_{10} x_1 x_3^3 + v_{14} x_2 x_3^3 + v_{15} x_3^4 = (v_1 x_1^2 + v_2 x_1 x_2 + v_4 x_2^2 + v_3 x_1 x_3 + v_5 x_2 x_3 + v_6 x_3^2) f_1 + (v_7 x_1 x_3 + v_8 x_2 x_3 + v_9 {x_3}^2) f_2 + v_{11} f_3$$

under conditions $v_{10} = v_{11}, v_{12} = v_{13} = v_{14} = v_{15} = 0$ and

$$\begin{aligned} w_1 x_1^5 &+ w_2 x_1^4 x_2 + w_4 x_1^3 x_2^2 + w_7 x_1^2 x_2^3 + w_{11} x_1 x_2^4 + w_{16} x_2^5 + w_3 x_1^4 x_3 + w_5 x_1^3 x_2 x_3 \\ &+ w_8 x_1^2 x_2^2 x_3 + w_{12} x_1 x_2^3 x_3 + w_{17} x_2^4 x_3 + w_6 x_1^3 x_3^2 + w_9 x_1^2 x_2 x_3^2 \\ &+ w_{13} x_1 x_2^2 x_3^2 + w_{18} x_2^3 x_3^2 + w_{10} x_1^2 x_3^3 + w_{14} x_1 x_2 x_3^3 + w_{19} x_2^2 x_3^3 \\ &+ w_{15} x_1 x_3^4 + w_{20} x_2 x_3^4 + w_{21} x_3^5 \\ &= (w_1 x_1^3 + w_2 x_1^2 x_2 + w_4 x_1 x_2^2 + w_7 x_2^3 + w_3 x_1^2 x_3 + w_5 x_1 x_2 x_3 \\ &+ w_8 x_2^2 x_3 + w_6 x_1 x_3^2 + w_9 x_2 x_3^2 + w_{10} x_3^3) f_1 \\ &+ (w_{11} x_2^3 + w_{12} x_2^2 x_3 + w_{13} x_2 x_3^2 + (w_{14} - w_{16}) x_3^3) f_2 + (w_{16} x_2 \\ &+ w_{17} x_3) f_3 \end{aligned}$$

under conditions $w_{15} = w_{17}, w_{18} = w_{19} = w_{20} = w_{21} = 0$.

Now start from the first syzygy.

$$q := x_2 g_1 - x_1 g_2 - F_5 x_1^2 + F_1 x_1 x_2 - F_6 x_1 x_2 + F_2 x_2^2 - F_7 x_1 x_3 + F_3 x_2 x_3 - F_8 x_1 + F_4 x_2$$

.

Then,

$H(q) = (-F_5 f_1 + (F_1 - F_6) f_2)$ and we also have $F_2 = 0, F_3 = 0, F_7 = 0$. Using the division

$$q \coloneqq q - (-F_5(f_1 + g_1) + (F_1 - F_6)(f_2 + g_2)) \\ = (F_5F_6 - F_8)x_1 + (F_4 - F_1F_6 + F_6^2)x_2 + F_4F_5 - F_1F_8 + F_6F_8$$

and we should also have $F_8 = F_5F_6$ and $F_4 = F_1F_6 - F_6^2$.

Next consider the second syzygies.

$$q := {x_3}^3 g_1 + {x_2}^3 g_2 - x_1 g_3.$$

Then,

$$H(q) = (-F_9x_1^2 - F_{10}x_1x_2 - F_{12}x_2^2 - F_{11}x_1x_3 - F_{13}x_2x_3 - F_{14}x_3^2)f_1 + ((F_5 \\ - F_{15})x_2^2 - F_{16}x_2x_3 - F_{17}x_3^2)f_2 + F_6f_3$$

and we should have $F_6 = F_1 - F_{18}$. Updating above relations we also get $F_4 = F_1F_{18} - F_{18}^2, F_8 = F_1F_5 - F_5F_{18}$. Let

$$q \coloneqq q - (-F_9x_1^2 - F_{10}x_1x_2 - F_{12}x_2^2 - F_{11}x_1x_3 - F_{13}x_2x_3 - F_{14}x_3^2)(f_1 + g_1) \\ + ((F_5 - F_{15})x_2^2 - F_{16}x_2x_3 - F_{17}x_3^2)(f_2 + g_2) + F_6(f_3 + g_3).$$

$$H(q) = ((F_9F_{18} - F_{19})x_1 + (F_{10}F_{18} - F_{20})x_2 + (F_{11}F_{18} - F_{21}))f_1 + ((-F_5^2 \\ + F_2F_{10} + F_5F_{15} + F_{12}F_{18} - F_{22})x_2 + (F_5F_{16} + F_{13}F_{18} - F_{23})x_3)f_2$$

under the condition $F_{24} = F_5F_{17} + F_{14}F_{18}$.

Then,

$$q \coloneqq q - ((F_9F_{18} - F_{19})x_1 + (F_{10}F_{18} - F_{20})x_2 + (F_{11}F_{18} - F_{21}))(f_1 + g_1) \\ + ((-F_5^2 + F_2F_{10} + F_5F_{15} + F_{12}F_{18} - F_{22})x_2 + (F_5F_{16} + F_{13}F_{18} \\ - F_{23})x_3)(f_2 + g_2).$$

$$H(q) = ((-F_9F_{18}^2 + F_{18}F_{19} - F_{25})f_1 + (F_5^3 - F_5^2F_{15} - F_5F_{12}F_{18} - F_{10}F_{18}^2 + F_{18}F_{20} \\ + F_5F_{22} - F_{26})f_2)$$

with $F_{27} = -F_5^2F_{16} - F_5F_{13}F_{18} - F_{11}F_{18}^2 + F_{18}F_{21} + F_5F_{23}$.

$$q \coloneqq q - ((-F_9F_{18}^2 + F_{18}F_{19} - F_{25})(f_1 + g_1) + (F_5^3 - F_5^2F_{15} - F_5F_{12}F_{18} - F_{10}F_{18}^2 + F_{18}F_{20} + F_5F_{22} - F_{26})(f_2 + g_2)).$$

Then $q = 0$ under the condition $F_{28} = -F_5^4 + F_5^3F_{15} + F_5^2F_{12}F_{18} + F_5F_{10}F_{18}^2 + F_9F_{18}^3 - F_{18}^2F_{19} - F_5F_{18}F_{20} - F_5^2F_{22} + F_{18}F_{25} + F_5F_{26}$.

The last syzygy do not produce any new realtion between parameters. Therefore the ideal of parameters is

$$\mathfrak{h}_0 = \langle F_2, F_3, F_7, F_6 - F_1 + F_{18}, F_{24} - F_5F_{17} - F_{14}F_{18}, F_4 - F_1F_{18} + F_{18}^2, F_8 - F_1F_5 + F_5F_{18}, F_{27} - F_5^2F_{16} + F_5F_{13}F_{18} + F_{11}F_{18}^2 - F_{18}F_{21} - F_5F_{23},\ F_{28} - F_5^4 - F_5^3F_{15} - F_5^2F_{12}F_{18} - F_5F_{10}F_{18}^2 - F_9F_{18}^3 + F_{18}^2F_{19} + F_5F_{18}F_{20} + F_5^2F_{22} - F_{18}F_{25} - F_5F_{26}\rangle.$$

Hence $I = \langle (f_1 + g_1)^h,\ (f_2 + g_2)^h, (f_3 + g_3)^h \rangle \subseteq K[x_1, x_2, x_3, x_4]$ is a lifting of J if parameters replaced by the constants in the affine scheme $V(\mathfrak{h}_2)$.

One can show that $\mathfrak{h}_2 \subseteq \mathfrak{h}_0$ and $\mathfrak{h}_2 \subseteq \mathfrak{h}_0$. Hence $V(\mathfrak{h}_0) \subseteq V(\mathfrak{h}_2)$ and $V(\mathfrak{h}_1) \subseteq V(\mathfrak{h}_2)$. For example, if we take $F_1 = 1, F_{10} = 0$ and the other free variable equal to zero we obtain a lifting $I = \langle {x_1}^2 - x_1x_4, x_1x_2 + x_2x_4, {x_2}^4 + x_1{x_3}^3 + {x_1}^3x_4 \rangle$. This lifting was not obtained in either example of the previous chapter. The reason beyond this result is the fact that a Gröbner basis with respect to a degree compatible monomial order is always a H-basis but converse is not true.

5. CONCLUSIONS AND RECOMMENDATIONS

A new method for the solution of the lifting problem for homogeneus ideals is developed. This method is more suited to the spirit of the problem than the method using Gröbner bases. Because the lifting is more related to the H-bases than Gröbner bases. Hence it finds more lifting than the method using Gröbner basis. However, we are not able to show the liftings we found are all liftings of the given homogeneous ideal. This will be an interesting problem for the future studies.

Bertone et al (2016) also embeded the zero locus of their liftings into the Hilbert scheme and found some intererting topological properties of this zero locus. Similar results can be obtained for the zero locus of our liftings.

6. REFERENCES

.

Adams W W and Loustaunau P (1994) An Introduction to Gröbner Bases, Graduate Studies in Mathematics (3), American Mathematical Society, Rhode Island, USA.

Bertone C, Cioffi F, Guida M and Roggero M (2016) “The scheme of liftings and applications”, J. Pure and Appl. Algebra, 220: 34-54.

Carra Ferro C (1998), “Gröbner bases and Hilbert schemes. I”, J. Symb. Comput. 6(2–3) :219–230.

Carra Ferro G and Robbiano L (1990), “On super G-bases”, J. Pure Appl. Algebra 68(3): 279–292.

Cox D, Little J and O'Shea D (2007) Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geomety and Commutative Algebra, Third Edition, Springer, New York, USA.

Luo T and Yılmaz E (2001), “On the lifting problem for homogeneous ideals”, J. Pure Appl. Algebra 162(2–3): 327–335.

Migliore J and Nagel U (2000), “Lifting monomial ideals”, Commun. Algebra 28(12): 5679–5701.

Roitman M (1988) “On the lifting problem for homogeneous ideals in polynomial rings”, J. Pure Appl. Algebra 51(1–2): 205–215.

7. CURRICULUM VITAE

Name SURNAME : Oya AYDOĞAN

Place and Date of Birth : Gerede 06/04/1991

Universities

Bachelor's Degree : Bolu Abant İzzet Baysal University
Department of Mathematics

e-mail : oya.aydogan7@gmail.com