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THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS

MONOTONE AND OPEN WHITNEY MAPS DEFINED ON HYPERSPACES

MASTER OF SCIENCE

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APPROVAL OF THE THESIS

MONOTONE AND OPEN WHITNEY MAPS DEFINED ON HYPERSPACES submitted by **Gamze Bakanakoğlu** in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics Department, the Graduate School of Natural and Applied Sciences of BOLU ABANT IZZET BAYSAL UNIVERSITY in 02/01/19** by

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Gamze Bakanakoğlu

ABSTRACT

MONOTONE AND OPEN WHITNEY MAPS DEFINED ON HYPERSPACES
M.S. THESIS
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Let X be a nonempty continuum. It is known that there exists a map, which is called Whitney map, satisfying some special properties. Whitney map for 2^X need not be monotone and open. However, Whitney maps for defined on $C(X)$ i.e the closed subset of 2^X, have these properties. One of the aims of this dissertation is to investigate whether Whitney maps defined 2^X are monotone and open. In addition, if the Whitney map is denoted by ω, the structure of Whitney level denoted by $\omega^{-1}(t)$ (in some articles it is called Whitney contunia too) is investigated whenever X is Peano continuum. Because, locally connectedness of Whitney level for an arbitrary $t \in [0, \omega(X)]$ is not known yet.

This dissertation is organized as follows, we give, first, a comprehensive introduction part to explain a motivation of this dissertation after that general definitions and theorem(s) used in this thesis are given, and then the notion of a hyperspace is mentioned briefly. The existence and extension of a Whitney map is the crucial parts of this dissertation. Since our study depends on its existence, we work on it and its point inverses whenever X is, especially, a locally connected continuum. Chapter six and seven are related to these properties. In the last chapter, we give some observations which is not in literature and we mention what we are working on and we also pose some questions for interested reader.

KEYWORDS: Hyperspace, Whitney Maps, Hausdorff Metric, Monotone and Open maps, Whitney Level,Peano Continuum .

ÖZET

HİPER UZAYLAR ÜZERİNDE TANIMLI MONOTON VE AÇIK WHITNEY DÖNÜŞÜMLER
YÜKSEK LİSANS TEZİ
GAMZE BAKANAKOĞLU,
BOLU ABANT İZZET BAYSAL UNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ
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X boştan farklı bir continuum (tıkız , bağlantılı metrik uzay) olmak üzere, 2^X üzerinde tanımlı Whitney adı verilen belirli özellikleri sağlayan bir dönüşümün varlığı bilinmektedir, bu dönüşümün açık ya da monoton olma özelliklerine sahip olması gerekmemesine rağmen 2^X'in kapalı altkümesi olan $C(X)$ üzerinde tanımlı her Whitney dönüşümün monoton ve açık olduğu bilinmektedir. Eğer X continuum üzerine lokal bağlantılılığı eklersek ki, literatürde buna Peano continuum denir, 2^X üzerindeki her Whitney dönüşümün açık ve monoton özelliğine sahip olduğunun incelenmesi bu tezin amacının bir tanesidir ve diğeri eğer Whitney dönüşümü ω ile gösterirsek X'in Peano contiuum olması durumunda $w^{-1}(t)$ ile gösterilen Whitney seviye (bazı makalelerde buna Whitney continua da denir) yapısı da ayrıca incelenecektir. Çünkü bildiğimiz kadarı ile $[0, \omega(X)]$ aralığında aldığımız her t için bu Whitney seviye yapısının lokal bağlantılı olup olmadığı henüz bilinmemektedir.

Bu tez şu şekilde düzenlenmiştir. Biz öncelikle tezin motivasyonunu açıklamak için ayrintılı bir giriş bölümününü verdik ardından bu tezde kullandığımız tanımları ve teoremler verildi ve sonra kısaca Hyperspace kavramından bahsettik. Whitney dönüşümlerin varlığı ve genişletilebilirliği önemli bir bölümdür. Çünkü bizim çalışmamız onun varlığına bağlıdır. Biz özellikle X'in lokal bağlantılı olması durumunda onun ve onun noktasal tersinin özellikleri üzerinde çalışacağız. Söylediğimiz gibi Whitney dönüşümler bölümünün arkasından onun özellikleri ve onun noktasal tersinin özelliklerinin olduğu iki bölüm gelmektedir. Son olarak, sonuç bölümünde, literatürde olmayan bir gözlemimizi vereceğiz ve ne üzerinde çalışıyor olduğumuzdan bahsedeceğiz ve üstelik ilgilenen okuyucular için açık problem bırakacağız.

ANAHTAR KELİMELER: Hiper Uzaylar, Whitney Dönüşümler, Husdorff uzaklık, Monoton ve Açık Dönüşümler, Whitney Seviye, Peano Continuum.

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LIST OF ABBREVIATIONS AND SYMBOLS

$\mathbf{2^X}$	: The hyperspace 2^X of all nonempty closed subset of X with Vietories topology.
$\overline{\mathbf{A}}$	: The closure of A.
A map	: A continuous function.
$\mathbf{B_V}$	: Base for Vietories topology.
$\mathbf{C(X)}$	: The connected members of 2^X.
$\mathbf{C_n(X)}$	: The subsets of 2^X consisting at most n components sets.
$\mathbf{H_d}$	: Hausdorff metric.
$\mathbf{f \simeq g}$	: f is homotopic to g.
$\mathbf{F_n(X)}$	: The subset of 2^X consisting of at most n point sets.
$\mathbb{N}$	: Natural numbers.
$\mathcal{P}$	: Topological property.
$\mathbb{R}$	: Real numbers.
$\mathbb{R}_{\geq \mathbf{0}}$	: Positive Real Numbers and zero.
$\mathbf{T_V}$	: Vietories topology.
ω	: Whitney map.
$\omega^{\mathbf{-1}}$	: Whitney level.
$\mathbb{Z}_+$	: Positive integers.
PoSP	: Partially ordered space.
iff	: If and only If.

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Thanks them all who contribute this thesis.

1. INTRODUCTION

Let (X, T) be a T_1 topological space and define the following set

$$CL(X) = \{A \subset X | A \text{ is nonempty and closed in } X\}$$

we will put a topology on $CL(X)$, that is called Hyperspace of a topological space X or hypertopology, such that the function $i : (X, T) \to (CL(X), \tau)$ defined as $i(x) = \{x\}$ is a homomorphism onto its image. The two famous hyperspaces, Vietoris topology and the Hausdorff metric topology denoted by τ_{H_d}, have been introduced and investigated by Leopold Vietoris(1891-2002) and Hausdorff in 1914. It goes without saying that in case of X is compact space with metric d, then the topology produced by the Hausdorff metric coincide with the Vietoris topology.

As we can see above, the fundamental motivation of hyperspace is to obtain information on the structure of a topological space X by studying properties of the hyperspace $CL(X)$. By the way, I just wanted to notice that $CL(X)$ and 2^X are the same when X is compact. But especially geometric model of hyperspaces is not only quite complicated but also unknown in many cases, so the hyperspace theory created one of its own way to tackle this difficulty by means of Whitney map, defined by J.L. Kelly in 1942.

By a Whitney map we mean a real-valued continuous function ω defined on a hyperspace $\mathscr{H}$ of X that satisfies the following two conditions: $\omega(\{x\}) = 0$ for every $x \in X$ and if $A \subsetneq B$ then $\omega(A) < \omega(B)$. The subject of openness, monotonicity and confluence of Whitney maps has been investigated by M.M. Awartani, W.J. Charatonik, S.B.Nadler Jr., and A. Illanes, see (Charatonik, 1984; Awartani, 1993; Charatonik and Samuelewicz, 2002; Illanes, 1993). In here, we should note that J.J.Charatonik observed in (Charatonik, 1964) that monotone surjective maps and open surjective maps between continua are confluent. One can ask whether the Whitney map ω is also monotone, answer of this question has been given by Janusz R. Prajs affirmatively by showing more general result that is each confluent Whitney map is monotone. By the way, in (Charatonik, 1988) W.J. Charatonik gave short form of Prajs's proof.

It has been proved by Eberthart and Nadler in (Eberhart and Nadler, 1971) that Whitney maps defined on $C(X)$ when X is continuum are monotone and open. Whitney maps, however, for 2^X need not have either of these two nice properties. For example Nadler showed in (Eberhart and Nadler, 1971) that there is a Whitney map for 2^X that is not open (and not monotone) when X is a closed interval. In 1992, we encountered necessary and sufficient condition, given by Illanes, on a Whitney map for 2^X for being open (or monotone). In addition to that in case of being either locally connected (Illanes, 1986) or arc-smooth (Illanes, 1990) of X, then we can talk about then existence of an open (and monotone) Whitney map for 2^X that has been showed by Illanes.

Basic knowledge about Whitney level, denoted by $\omega^{-1}(t)$, has been given in (Nadler, 1978). We also investigate the structure of Whitney level whenever X is a locally connected continuum (Peano continuum) by using (Goodykoobtz and Nadler, 1982). In this paper, they look for conditions under which positive Whitney level, defined on Peano continua, is a Hilbert cube. By a Hilbert cube, we mean that a space which is a homomorphic to $I^\infty = \prod_{i=1}^\infty [0,1]_i$.

2. AIM AND SCOPE OF THE STUDY

2.1 Fundamental Definitions

As indicated in abstract and introduction, one of the main goal of this dissertation is to investigate the structure of point inverse of the Whitney map defined on $C(X)$ and 2^X, but we focus on especially for 2^X. To be more specific, we wonder the structure of a point inverse of the Whitney map defined on 2^X when X is a Peano continuum. We give some definitions and theorems used in this thesis and we follow (Illanes and Nadler, 1999; Willard, 1970; Macias, 2018). But note that we will omit some well known definitions such as topological space, metric space, partially ordered set, and so on.

Definition 2.1.1. *Let (X, d) be a metric space. The metric H_d given by the following formula*

$$H_d(A, B) = inf\{\epsilon > 0 : A \subset B_\epsilon(B) \text{ and } B \subset B_\epsilon(A)\}$$

where $B_\epsilon(A) = \{x \in X :$ there is point $a \in A$ such that $d(a, x) < \epsilon\}$ is called the Hausdorff metric on 2^X

Definition 2.1.2. *A function $f : (X, d) \to (Y, p)$ between two metric spaces is said to be a **map** if it is a continuous function.*

Definition 2.1.3. *A map $f : X \to Y$ between contunia is said to be*

1. ***open** if it maps open sets to open sets.*
2. ***monotone** if $f^{-1}(t)$ is connected for each $t \in Y$ or equivalently if $f^{-1}(B)$ is connected for each subcontinuum B of $f(X)$.*
3. ***confluent** if for each subcontinuum Q of Y, each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q.*

Definition 2.1.4. *A continuum X is said to be **unicoherent** provided that $A \cap B$ is connected whenever A and B are subcontinua of X such that $A \cup B = X$.*

Definition 2.1.5. *Let X and Y be topological spaces. The maps $f : X \to Y$ and $g : X \to Y$ are called a **homotopic** if there is a continuous function $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ for all $x \in X$ and $F(x, 1) = g(x)$ for all $x \in X$. In this case F is called a **homotopy** between f and g.*

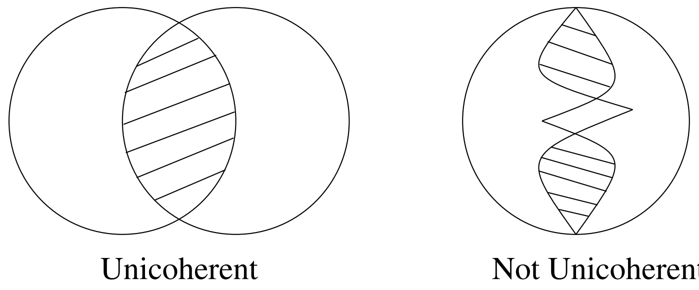

Figure 2.1: Unicoherent

Definition 2.1.6. *If a continuum is the union of two proper subcontinua, then it is called **decomposable**, and a continuum that is not decomposable is said to be **indecomposable**. A continuum is said to be **hereditarily indecomposable** if all of its subcontinua are indecomposable.*

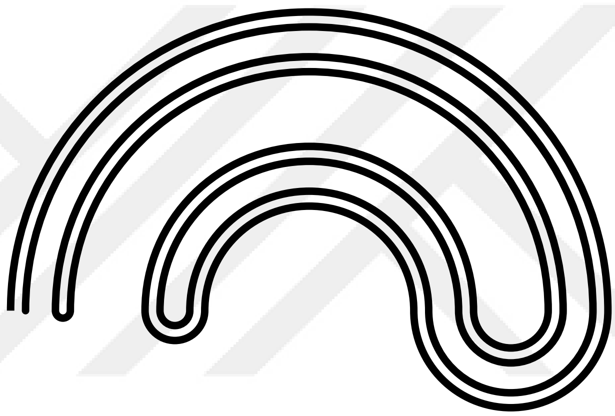

Figure 2.2: Buckethandle Continuum (Indecomposable continuum)

Definition 2.1.7. *A **chain** means a finite collection of open sets* $\mathcal{U} = \{U_1, \ldots, U_n\}$ *such that* $U_i \cap U_j \neq \emptyset$ *iff* $|i - j| \leq 1$. *A member of* $\mathcal{U}$ *is called a **link** of* $\mathcal{U}$.

Observe that the links do not need to be connected

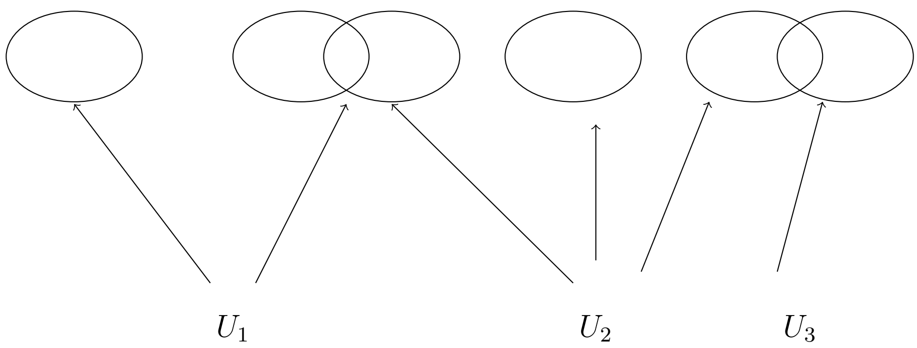

Figure 2.3: Links of chain

Definition 2.1.8. *A continuum X is said to be* ***chainable*** *if for each $\epsilon > 0$, there is a chain in X, covering X such that each link has diameter less than ϵ. See the following pictures.*

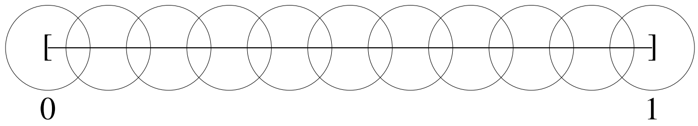

Figure 2.4: Unit interval

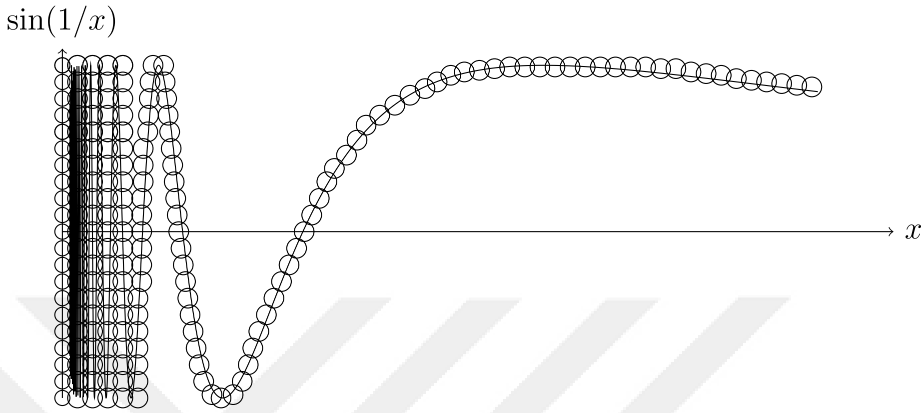

Figure 2.5: Topologist sine curve

The notion of **circle-like** is very similar to the notion of chainable except a little difference, see (Illanes and Nadler, 1999)[p. 259] and for the example see the picture 2.6

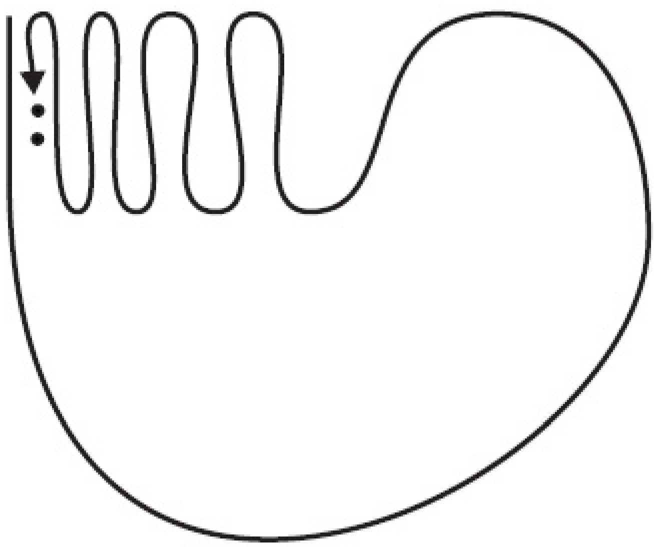

Figure 2.6: Warsaw Circle

Definition 2.1.9. *A metric space X said to be* ***contractible*** *provided that the identity map, 1_X, of X is homotopic to a constant map g.*

Definition 2.1.10. *A space X is **path connected** iff for any two points x and y in X there exists a map $f : [0, 1] \to X$ such that $f(0) = x$, $f(1) = y$.*

Theorem 2.1.11. *If X is a contractible metric space, then X is path connected.*

Definition 2.1.12. *A **retraction** is a continuous function, r, from space Y into Y such that r is identity on its range. A subset Z of Y is said to be **retract** of Y provided that there is a retraction of Y onto Z. A compactum K is called an **absolute retract** (written AR) provided that whenever K is embedded in a metric space, Y, the embedded copy of K is a retract of Y.*

Definition 2.1.13. *A topological property P such as connectedness or being continuum is called **Whitney property** with the purpose that if X has property P, then $\omega^{-1}(t)$ for each Whitney map ω on $C(X)$ and for each t whenever $0 \leq t < \omega(X)$ have that property P.*

3. THE NOTION OF HYPERSPACE

3.1 Definition of Hyperspace and its structure

This section is devoted mainly to the structure of a Hyperspace, we present the Vietoris topology and define it by giving its base and subbase.

The basic motivation of hyperspaces is to obtain new spaces from old ones by using the underlying topology. For a given topological space X, The hyperspace of X can be defined as the set of its nonempty closed sets, i.e.

$$2^X = \{K \subset X | K \text{ is closed and nonempty } \}.$$

there are other Hyperspaces can be given but we will use that Hyperspace to construct a special map, so as called Whitney map, on it to attain our main problem.

It is well known that X is always a closed subspace of X, so it is a point in 2^X, sometimes it is called the *fat* point or *top* point of that hyperspace. If X is T_1, then every point is closed, so the singleton set $\{x\} \subset X$, with $x \in X$, is a point of 2^X and hence

$$F_1(X) = \{\{x\} | x \in X\} \subset 2^X,$$

is the canonical copy of X in 2^X. It is the image of the inclusion map

$$\begin{aligned} i : X &\to 2^X \\ x &\mapsto \{x\}. \end{aligned}$$

One can endow hyperspaces with a number of topologies. If X is compactum, the most common topology for hyperspaces is the one induced by Hausdorff metric (distance) which is given in Definition 2.1.1. We know that in this case, i.e. If X is a compactum, the topology induced by Hausdorff distance and the Vietoris topology coincide. By the way, we note that $C(X)$ denotes the connected subsets of 2^X. In an other saying that it is all subcontinuum of X.

As in the Figure 3.1, the top element of *hyperspace* is X, while $F_1(X) = \{\{x\} : x \in X\}$ is the bottom element of it. By $C_n(X)$, we denote the set of elements of 2^X which has at most n components, whilst the set of elements of 2^X which has at most n points is denoted by $F_n(X)$. These are the other *hyperspaces* of X.

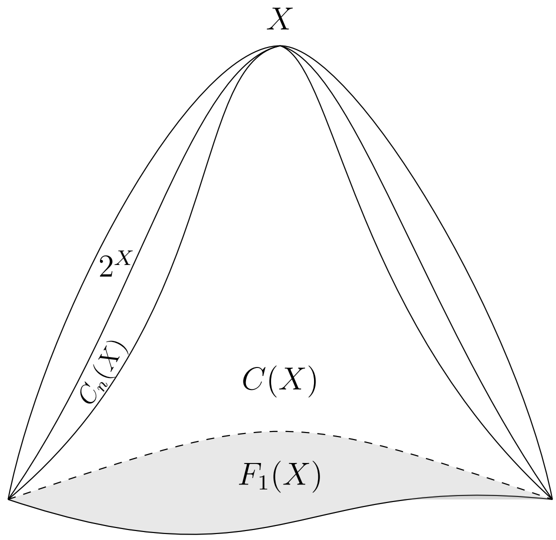

Figure 3.1: Hyperspaces

Let look closer to the structure of the Vietoris topology.

For open subsets $U_1, U_2, \ldots, U_n$ of X, the symbol $\langle U_1, U_2, \ldots, U_n \rangle$ denotes the basic open subset of 2^X in the Vietoris topology defined by $\langle U_1, U_2, \ldots, U_n \rangle = \{A \in 2^X : A \subseteq U_1 \cup U_2 \cup \ldots \cup U_n$ and $A \cap U_i \neq \emptyset$ for $i \in \{1, 2, \ldots, n\}\}$.

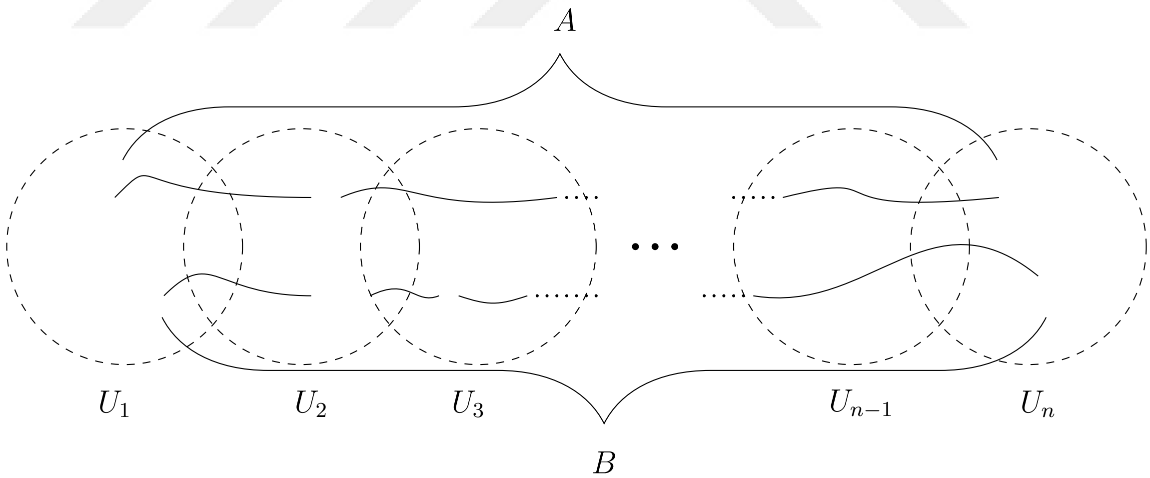

Figure 3.2: Base for Vietoris Topology

Its subbasic open subsets are the sets of the form $\langle U \rangle$ and $\langle X, U \rangle$ whenever U is an open sets i.e $\mathcal{S} = \{\langle U \rangle : U \text{ is an open set }\} \cup \{\langle X, U \rangle : U \text{ is an open set }\}$ is a subbase for a Vietoris topology.

A different description for the Vietoris topology is as follows;

Let X be a topological space and for each nonempty subset U of X, define:

$$U^+ = \{A \in 2^X : A \subseteq U\} \text{ and } U^- = \{A \in 2^X : A \cap U \neq \emptyset\}$$

Sets in U^- hit U, whereas sets in U^+ miss the complement U^c of U.

The Vietoris topology is the topology generated by U^+ and U^-, and a basic open set of the Vietoris topology is of the form:

$$\langle U_1, U_2, \ldots, U_n \rangle = \left(\bigcup_{i \leq n} U_i \right)^+ \cap \left(\bigcap_{i \leq n} U_i^- \right)$$

where $U_1, U_2, \ldots, U_n$ are nonempty open subsets of X. The set-theoretic relationship between the above sets and the subbasic open subsets of the Vietoris topology can be given as follows:

$$U^+ = \langle U \rangle \text{ and } U^- = \langle X, U \rangle.$$

We refer the reader to (Dogan, 2017)[Page 3-4, Example 3.1.3-3.14] for some examples of the Vietoris topology.

It is a well known fact that $C(X)$ is a metric continuum which is a closed subset of 2^X. We shall give some of the previous known results on $C(X)$ as follows (Curtis, 1974).

1. $C(X) \times I^\infty \approx I^\infty$ iff X is a Peano space, and $C(X) \approx I^\infty$ iff X is a non-degenerate Peano space containing no free arc. By a free arc, we mean that it is an arc α such that $X = \alpha \cup Y$, where $\alpha \cap Y \subset \alpha$ and Y is closed,

2. $C(X)$ is continuous image of the Cantor fan (Figure 3.4). By Cantor fan, we mean cone over Cantor set, more technically, it is quotient space $(X \times I)/\mathcal{G}$ where $\mathcal{G} = \{\{(x,t)\} | x \in C \text{ and } t \in [0,1)\} \cup \{X \times \{1\}\}$ (Figure 3.3), denoted by F_C or $Cone(C)$ where C is a Cantor set. Indeed, every compactum is a continuous image of the Cantor Fan.

3. Since Cantor Fan is an arcwise connected continuum and arcwise connectedness is preserved by continuous functions. It follows that $C(X)$ is arcwise connected. This fact by means of order arc can also be proved, see (Illanes and Nadler, 1999)[p.113, 14.9 Theorem].

4. $C(X)$ is Peanian[1] if and only if X is Peanian,

[1] A continuous image of a linear interval = a compact, connected, and locally connected continuum.

5. $C(X)$ is contractible,

6. $C(X)$ is an absolute retract(briefly AR).

All the results given above are true for 2^X excluding the second part of (1). In this case, there is no need to contain a free arc.

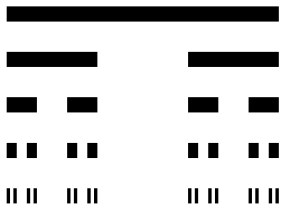

Figure 3.3: Cantor Set

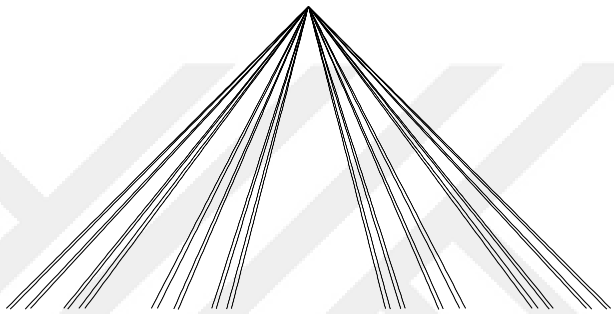

Figure 3.4: Cantor Fan

We want to point out here that if X is a continuum (without being locally connectednes) then there is a retraction $r : 2^X \to C(X)$, it was shown by Nadler (Nadler, 1974). In 1939, Wojdyslawski (Wojdyslawski, 1939) proved that $C(X)$ is an AR iff X is a locally connected continuum. Hence when X is a locally connected, $C(X)$ is a retract of 2^X, we just want to notice the importance of being locally connected for a continuum X, we will generally use this property in the following sections. Please see the following picture to figure out the geometric model of some hyperspaces.

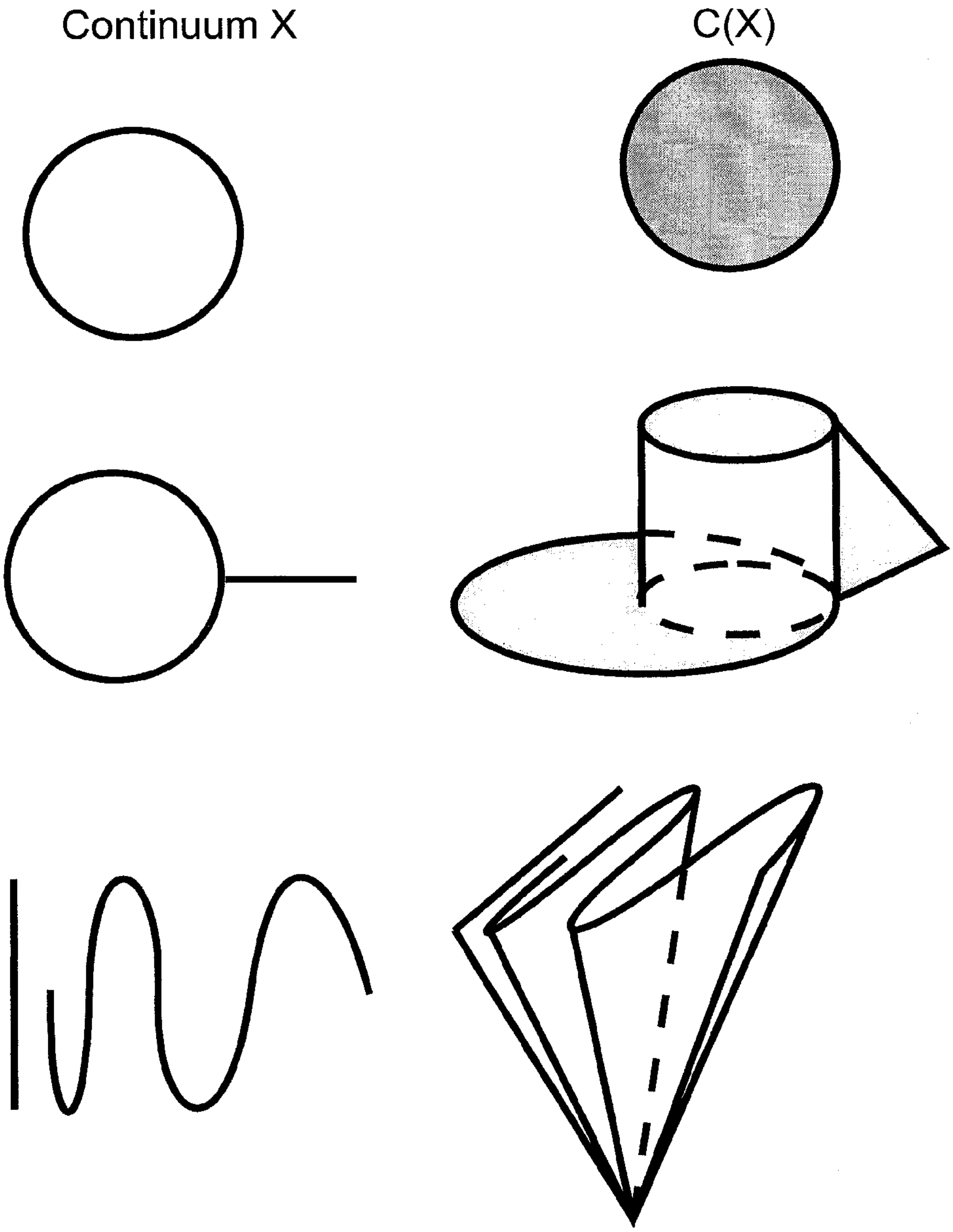

Figure 3.5: Geometric model of $C(X)$

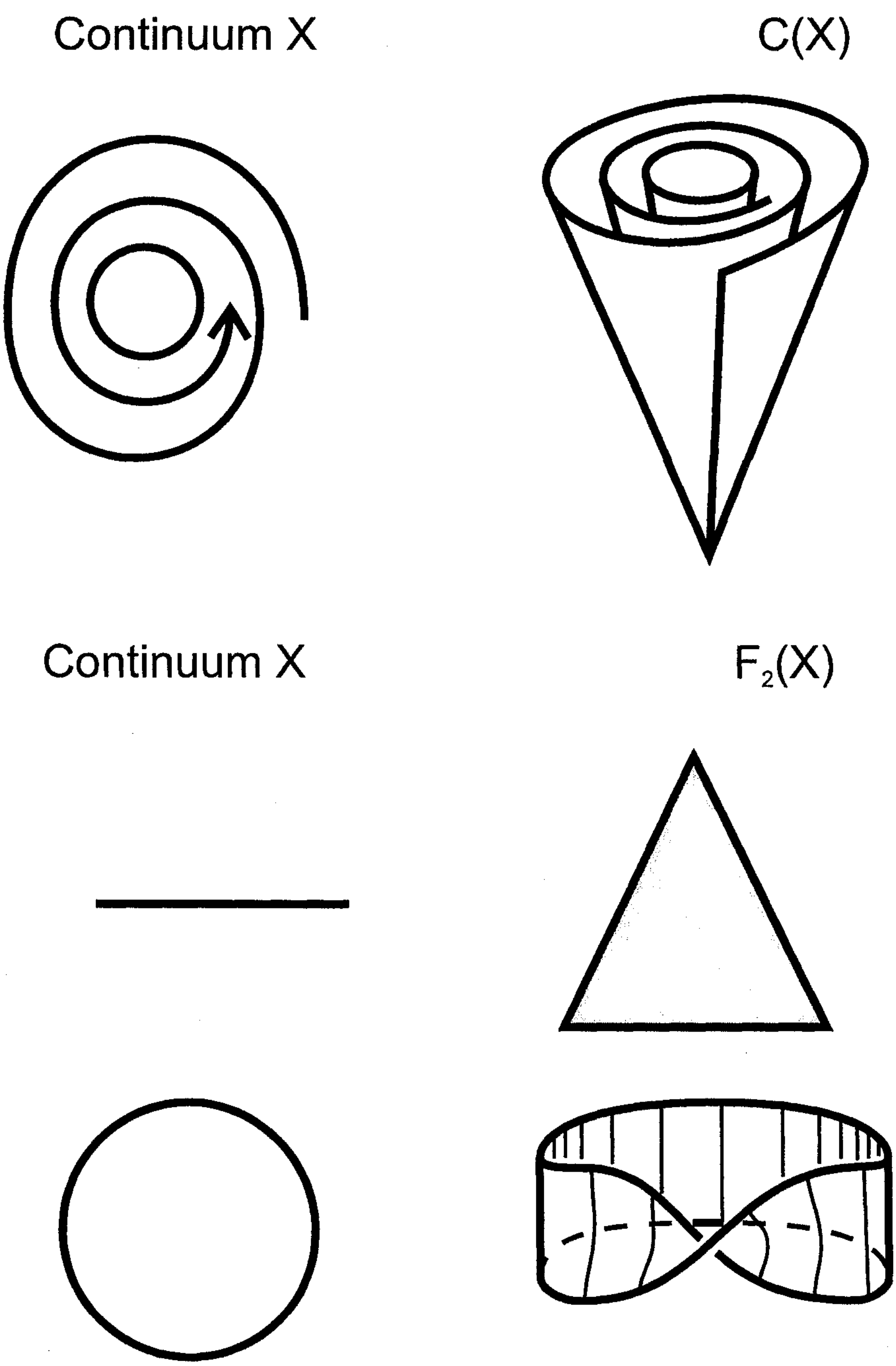

Figure 3.6: Geometric model of $C(X)$ and $F_2(X)$

4. WHITNEY MAPS

4.1 Existence of Whitney Maps

In 1978 , Nadler recognized that the property of admitting a Whitney map is playing crucial role in studying, especially, the arc structure of hyperspaces (Nadler, 1978). The Whitney maps are so important for us as well to tell something about our aim, we will investigate the structure of Whitney level to be able to do this the only tool we have Whitney maps. That's why we need to be ensure that they exist. There is a two approaches to existence of it. The first one is taken from Nadler's excellent book, "Hyperspaces of sets" that we do not write it here to not stay away from our focus and the second one Ward's approach by means of PoSP we give this approaches more detail but not much because it is different than Nadler's book. So three distinct construction of Whitney maps on 2^X were given by Nadler in the book (Nadler, 1978, pp. 25-27). Actually, we do not need the connectedness property of X, so from this point we will take X as a compactum (compact metric space). A reader who wishes to accept the existence of a Whitney map for any hyperspace of X can easily skip to the next section or the interested reader can take a look (Illanes and Nadler, 1999, p. 107, 13.4 Theorem).

In 1980, Ward used a different technique by means of a partially order space (briefly PoSP) to show the existence of whitney maps on 2^X (Ward, 1980). He thought that if the hyperspace is taken into consideration as a spacial type of PoSP (without doubt that it is much larger than the hyperspaces of compactum), then some problems concerning in hyperspaces would be more tractable.

By a *partially ordered space* X, we mean that it is a topological space endowed with a partially order "$\leq$" for which its graph is a closed subset of $X \times X$. Obviously it's known that if X is a regular Hausdorff space then 2^X is a PoSP with respect to inclusion (Kuratowski, 1966, p. 167 Theorem 1). In order to show the existence of Whitney map on 2^X, we shall give the definition of a Whitney map on a PoSP.

A *Whitney map* for a PoSP X is a map, $\omega : X \to [0, 1]$, satisfying following conditions:

(i) if $x \in Min\ (X)$ then $\omega(x) = 0$,

(ii) if $x \in Max\ (X)$ then $\omega(x) = 1$,

(iii) if $x < y$ in X then $\omega(x) < \omega(y)$.

where $Min\ (X)$ $(Max\ (X))$ is the set of minimal (maximal) element of X. By a minimal element m of X we mean whenever $x \in X$ and $x \leq m$ $(m \leq x)$ it follows that $x = m$.

obviously, in the settings of hyperspaces a Whitney map is considered as an ordinary Whitney map defined on 2^Y for a continuum Y.

The following fundamental theorem was given by Ward in (Ward, 1980, p. 373).

Theorem 4.1.1. *If X is a compact metric PoSP such that $Min\ (X)$ and $Max\ (X)$ are disjoint closed sets, then X admits a Whitney map.*

In this theorem, the condition on $Min\ (X)$ and $Max\ (X)$ is essential, for example let $X = [0, 1]$ and define the partial order on X by $x \leq y$ iff $x = y$ or $y = 1$. Then X is a compact metric PoSP, $Min\ X = [0, 1)$ and $Max\ (X) = \{1\}$. Whence, if $\omega : X \to [0, 1]$ satisfies (i) and (ii), then ω is not continuous.

We can easily deduce the following corollary.

Corollary 4.1.2. *If X is a non-degenerate compactum, then 2^X admits a Whitney map.*

Proof. We know from (Nadler, 1978, p. 7) that 2^X is a compact metric space and we have noted that 2^X is a PoSP, it follows that $Min\ X = \{\{x\} : x \in (X)\}$ and $Max\ (X) = \{X\}$ are disjoint closed subsets of 2^X and hence it admits a Whitney map. □

4.2 Extension of Whitney Maps

We ensure from previous section that the Whitney map exists on 2^X when X is a continuum (actually compactum). So we can deal with the extension of it and there are two questions coming out asked by Nadler (Nadler, 1978, 14.71.5) and Bruce Hughes. The first question is that can a Whitney map defined on $C(X)$ be extended to 2^X? And the other one, let take X as a subcontinuum of continuum Y can a Whitney map defined on $C(X)$ (resp.,2^X) be extended to $C(Y)$ (resp., 2^Y)? Again Ward in (Ward, 1981) gave an answer for both affirmatively. Before giving Ward's approach. Let's recall that the theorem given in Illanes and Nadler (1999)[16.10 Theorem p. 132], i.e; if X is a nonempty and compact metric space then any Whitney map defined on any closed subset of 2^X can be extended to 2^X.

In very recently, Ivan Lončar gives some generalizations of this theorem using generalized Whitney maps. We refer the reader to the article (Lončar, 2017) for details.

In the sequel we will give Ward's approach to these two questions omitting proof of theorems in (Ward, 1981). Firstly, we start by defining the following two sets. $L(x) = \{p \in P : p \leq x\}$ and $M(x) = \{p \in P : x \leq p\}$ where P is a PoSP and $x \in P$ and then using these two sets we will give the followings;

$$L(A) = \bigcup\{L(x) : x \in A\}, M(A) = \bigcup\{M(x) : x \in A\}.$$

where $A \subset P$.

Before mentioning one of the fundamental theorem in (Ward, 1981), we just wanted to bear in mind the definition of an order-preserving function. Let A and B be partially ordered sets, for a function $f : A \to B$, if $f(x) \leq f(y)$ in B holds, whenever $x \leq y$ in A, then f is called *order-preserving*. Ward proved that if A be closed subset of compact PoSP then the continuous function which is order preserving defined on A has a continuous extension.

Since $C(X)$ is a closed subset of 2^X (O'Neill, 2009, p. 8), we can obtain answer for the first question directly from Ward's result. For the answer of the second question, We will refer to the main theorem of a Ward given in (Ward, 1981, p. 467).

So by using Ward's theorem, we are ready to give the answer of the second question asked by Bruce Hughes as a corollary.

Corollary 4.2.1. *Let X be a subcontinuum of continuum Y, then each Whitney map defined on $C(X)$ (resp. 2^X) can be extended to $C(Y)$ (resp. 2^Y).*

Proof. It can easily be seen from the proof of corollary 3.4 in (Ward, 1981, p. 468). □

5. PROPERTIES OF WHITNEY MAPS FOR 2^X

5.1 Open and monotone Whitney map for 2^X and $C(X)$

In this section, we draw our attention to Whitney maps for 2^X whenever X is a continuum. It's a well know the fact that Whitney maps for $C(X)$ are always monotone and open (Eberhart and Nadler, 1971, p. 1032). However Whitney maps defined on 2^X need not to be monotone (or open) even if X is an arc (Nadler, 1978)[p. 466 14.61].

Illanes showed in the following theorem that if X is a continuum there exists a Whitney map on 2^X which is not open and monotone.

Theorem 5.1.1. *(Illanes and Nadler, 1999)[24.2. Theorem p.208] Let X be a continuum. Then there exists a Whitney map for 2^X which is neither open nor monotone.*

Proof. Let d denote a metric on X, choose two fix points p and q in X such that $p \neq q$. For $\epsilon = \frac{d(p,q)}{3}$ define two disjoint sets $A = cl_X(B(\epsilon, p))$ and $B = cl_X(B(\epsilon, q))$. Now we will define a set; $\mathcal{H} = \{\{a, b\} \in 2^X : a \in A$ and $b \in B\}$ and a function on $\mathcal{H}$ as;

$$\nu\ (a, b) = \frac{1}{2} + d(a, p) + d(b, q).$$

Since the function from $A \times B$ to $\mathcal{H}$ given by $(a, b) \mapsto \{a, b\}$ is a homeomorphism, $\mathcal{H}$ is compact and ν is continuous. Hence ν is a Whitney map for $\mathcal{H}$.

There is no doubt that $\nu\ (a, b) = \dfrac{1}{2}$ iff $\{a, b\} = \{p, q\}$. We, however, know from the previous section that we have extension of the Whitney map ν on 2^X, namely; we have Whitney map $\mu : 2^X \longmapsto R$ that extends ν.

Let $\mathcal{U}$ be the ϵ-ball with center $\{p, q\}$. If $C \in \mathcal{U} \setminus \{p, q\}$, then it follows $C \subset A \cup B$ moreover $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$. So C contains an element $\{a, b\}$ of $\mathcal{H} \setminus \{p, q\}$, and hence $\mu(C) \geq \mu(\{a, b\}) > \frac{1}{2}$.

Then we deduce that $\frac{1}{2} \in \mu(\mathcal{U}) \in [\frac{1}{2}, \mu(X)]$ and $\mathcal{U} \cap \mu^{-1}(\frac{1}{2}) = \{p, q\}$,i.e. $\{p, q\}$ is an isolated point of $\mu^{-1}(\frac{1}{2})$, so $\mu(\mathcal{U})$ is not open $[0, \mu(X)]$. Therefore, μ is neither monotone nor open. $\square$

Also the following examples shows that fact.

Example 5.1.2. *Let α be the polygonal arc (with the usual metric) in the plane given by $\alpha = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : max\{|x|, |y|\} = 1$ such that if $x = 1$ then $y \leq 0$ or $y \geq 1/2\}$ and a Whitney map on 2^α as done in formulas (1) and (2) of (Whitney, 1932)[p.275] is neither monotone and open.*

Example 5.1.3. *(Charatonik, 1984) Let S be the unit circle in $\mathbb{R}^2$ and define functions $f, g : [1, \infty) \to \mathbb{R}^2$ as follows*

$$f(t) = \left(1 + \frac{1}{t}\right) exp(it) \text{ and } g(t) = \left(1 - \frac{1}{t}\right) exp(-it)$$

then define

$$X = S \cup f([1, \infty)) \cup g([1, \infty)).$$

W.J. Charatonik showed that there are no confluent Whitney maps for 2^X. Therefore there are neither open nor monotone Whitney maps for 2^X.

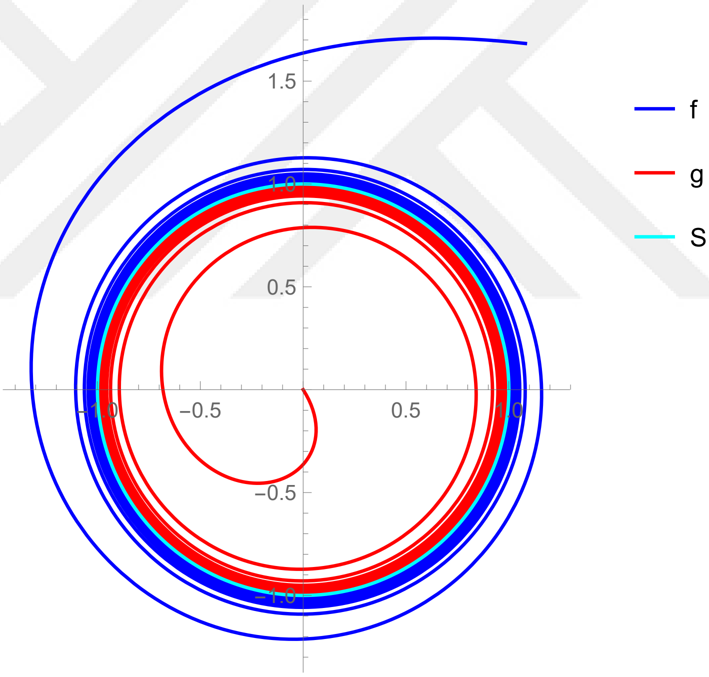

Figure 5.1: The Space X

We sure that a Whitney map for 2^X is not always monotone and open. But, of course, it may be true in some cases, for example, when X is the convex arc $[0, 1]$.

Although one can construct confluent maps which are not monotone, however both concepts coincide for Whitney maps defined on 2^X as it is shown in the following theorem by W.J. Charatonik.

Theorem 5.1.4. *Let X be a continuum. For a Whitney map $\omega : 2^X \to [0, \omega(X)]$, the following conditions are equivalent:*

(i) *ω is monotone,*

(ii) *ω is confluent,*

(iii) *$\omega^{-1}[0, t]$ is connected for each $t \in [0, \omega(X)]$.*

Corollary 5.1.5. *(Illanes and Nadler, 1999)[p. 209] Let X be a continuum. Then every open Whitney map for 2^X is monotone*

With respect to open Whitney maps for 2^X, we have the following equivalences.

Theorem 5.1.6. *Let X be a continuum. Let $\omega : 2^X \to J = [0, \omega(X)]$ be a Whitney map. Then the following statements are equivalent*

(i) *ω is open,*

(ii) *the function $t \to \omega^{-1}(t)$ from J into 2^{2^X} is continuous,*

(iii) *if $t \in J$ and $\{t_n\}_{n=1}^{\infty}$ is a sequence in $[0, t)$ such that $t_n \to t$, then $\omega^{-1}(t_n) \to \omega^{-1}(t)$ in (2^{2^X}),*

(iv) *the local minima of ω occur only on $F_1(X)$, and*

(v) *$\omega|_{F(X)} : F(X) \to J$ is open.*

We have talked about existence of a monotone and open Whitney map for $C(X)$ and especially for 2^X when X is a continuum and if it exists for 2^X, we gave equivalent condition. However, existence of a monotone and open Whitney map for 2^X is shown by Illanes see Illanes (1986). Before giving this fundamental result, we are planning to give some necessary background information briefly.

A metric ρ for X is said to be *convex* if for a given $x, y \in X$, there exists $z \in X$ such that $\rho(x, z) = \rho(x, y)/2 = \rho(z, y)$. It is well know that the existence of a convex metric, ρ, for X, implies the existence of an isometry $\sigma : [0, \rho(x, y)] \to X$ such that $\sigma(0) = x$ and $\sigma(\rho(x, y)) = y$.

X is said to admit a convex metric ρ if the original topology on X coincide with the induced topology by convex metric ρ for X. We know the following fact from K. Menger that admitting a convex metric, ρ, for a space X means X is a Peano continuum, namely; locally connected continuum. The converse is also true, it was shown by R.H. Bing and E.E. Moise independently.

Now we ready to construct a Whitney map as follows:

Construction. *(Illanes, 1986)[p. 516] Let ρ be a (convex)metric on X. For $A \in 2^X$ and define $\omega_n(A) = \{\epsilon > 0 : \exists\, x_1, \ldots, x_n \in X$ such that $A \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)\}$ for a given positive integer n. It is clear that $\omega_n : 2^X \to [0, \infty)$ is continuous. Moreover, it satisfies $\omega_n(\{x\}) = 0$ for $x \in X$ and $\omega_n(A) \leq \mathrm{diam}(X)$. So now, we can construct Whitney map is as follows;*

$$\omega(A) = \sum w_n(A)/2^n.$$

Now we ready to mention about the existence of an open and monotone Whitney map in case of the metric, d, defined on X is convex. But we sketch the proof skipping details we refer to (Illanes, 1986)[p. 517] for the reader who interested in the details of the proof.

Theorem 5.1.7. *If d is a convex metric for X, then the Whitney map,ω, constructed above is monotone and open.*

Proof. We start the proof by showing $\omega(F)$ is an interior point of $\omega(U)$ where F a finite nondegenerate subset and U is an open subset of 2^X such that $F \in U$.

To prove ω is monotone. we will show that $\omega^{-1}([0, t])$ is connected for every $t \in \mathbb{R}^+$ to be able to do this we first show $\omega^{-1}([0, t))$ is connected for any positive $t \in \mathbb{R}^+$. Let $\xi = \{F \in 2^X : \omega(F) < t$ and F is finite $\}$ be a dense subset of $\omega^{-1}([0, t))$, so one can show that ξ is connected because it is easily shown that $F_1(X) \subset \epsilon$ and $F_1(X)$ is connected and it follows that $w^{-1}([0, t])$ is connected for any $t \in \mathbb{R}^+$. After that, we only need to prove that $w^{-1}[t, \infty)$ is connected. Take $t \in \mathbb{R}^+$ and $A \in \omega^{-1}([t, \infty))$ the set $\{A \cup \{x\} \in 2^X : x \in X\}$ is a connected subset of $\omega^{-1}([t, \infty))$ and so $\omega^{-1}([t, \infty))$ is connected. Since 2^X is unicoherent (see Definition 2.1.4), $w^{-1}(t) = w^{-1}([0, t]) \bigcap w^{-1}([t, \infty))$ is connected. □

One can easily deduce the following corollary.

Corollary 5.1.8. *(Illanes, 1986)[p. 517] If X is a Peano continuum, then there exists a monotone and open Whitney map for 2^X.*

We will dedicate the following section to the point inverses of Whitney maps, and we will see that admissible Whitney map is a backbone of the next section, so now we will mention about the notion of arc-smoothness because the arc-smoothness is not only a sufficient condition to have admissible Whitney maps for $C(X)$ and 2^X but also give existence of monotone and open Whitney map for 2^X and if there exists a admissible Whitney map, then we have several results on the point inverse of this Whitney map. For the concept of the arc-smoothness, or special type of contractibility, we refer to (Goodykoontz, 1983). In (Illanes, 1990)[Theorem 1.3], Illanes has showed that if X is an arc-smooth continuum, then there exists a Whitney map $\omega : 2^X \to \mathbb{R}^+$, defined as in the same article page 1071, such that $\omega^{-1}(t)$ and $(\omega \mid_{C(X)})^{-1}(t)$ are arc-smooth continua for every $t \in (0, \omega(X))$ and ω and $\omega \mid_{C(X)}$ are admissible, so arc-smoothness is a Whitney property. We refer to (Illanes, 1990) and (Goodykoontz, 1983) for interested readers. Also Jack T. Goodykoontz in (Goodykoontz, 1974) showed that being arc-smooth of X implies being arc-smooth each of hyperspacess we interested in, namely 2^X or $C(X)$. He also determined some other conditions to say that 2^X or $C(X)$ is arc-smooth. For example, the most interesting one for us that if X is locally connected then $C(X)$ is arc-smooth at each of its points, and 2^X is arc-smooth when X is hereditarily indecomposable.

As we mentioned the importance of being arc-smooth of X just above, we will see in Theorem 6.1.9 at the following section that being arc-smooth of X implies the existence an open and monotone Whitney map defined on 2^X. Why is the importance of existence of monotone and open Whitney map? Just because if it is monotone, then we know at least that all the Whitney levels are connected. So these are continua as well. What about the openness of it? It is directly related homeomorphism that can be construct between the levels and X. So if know the structure of the levels we can say the same things for X.

6. WHITNEY LEVELS IN PEANO CONTINUA

6.1 On Some Properties of Whitney levels

In this section we will give some results on Whitney levels in hyperspace when X is a Peano continuum by using Whitney map defined in Section 3. In addition to that we investigate the structure of Whitney level of admissible Whitney maps.

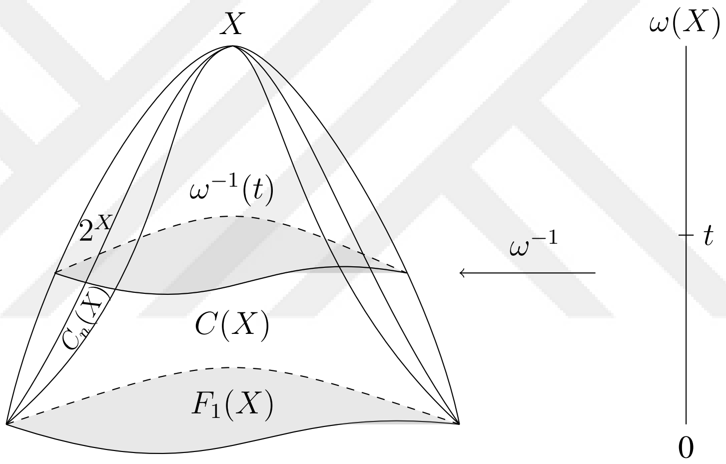

Figure 6.1: Whitney level

If ω is a Whitney map for $\mathscr{H}$ and $0 \leq t < \omega(X)$ then $\omega^{-1}(t)$ is called a *Whitney level*; if $t \in (0, \omega(X))$, then it is called a positive *Whitney level*. Whitney levels are covering X and converging to $\omega^{-1}(0)$ as t approaches to zero. By using definition of Whitney map we have $w^{-1}(0) = F_1(X)$ and hence $w^{-1}(0)$ is homeomorphic to X. Thus, Whitney levels approximate X as t approaches to zero. Working on the structure of positive Whitney levels and figure out the properties preserved under the convergence to the zero level of it will be interesting. You will see some geometric model of Whitney level in the following picture

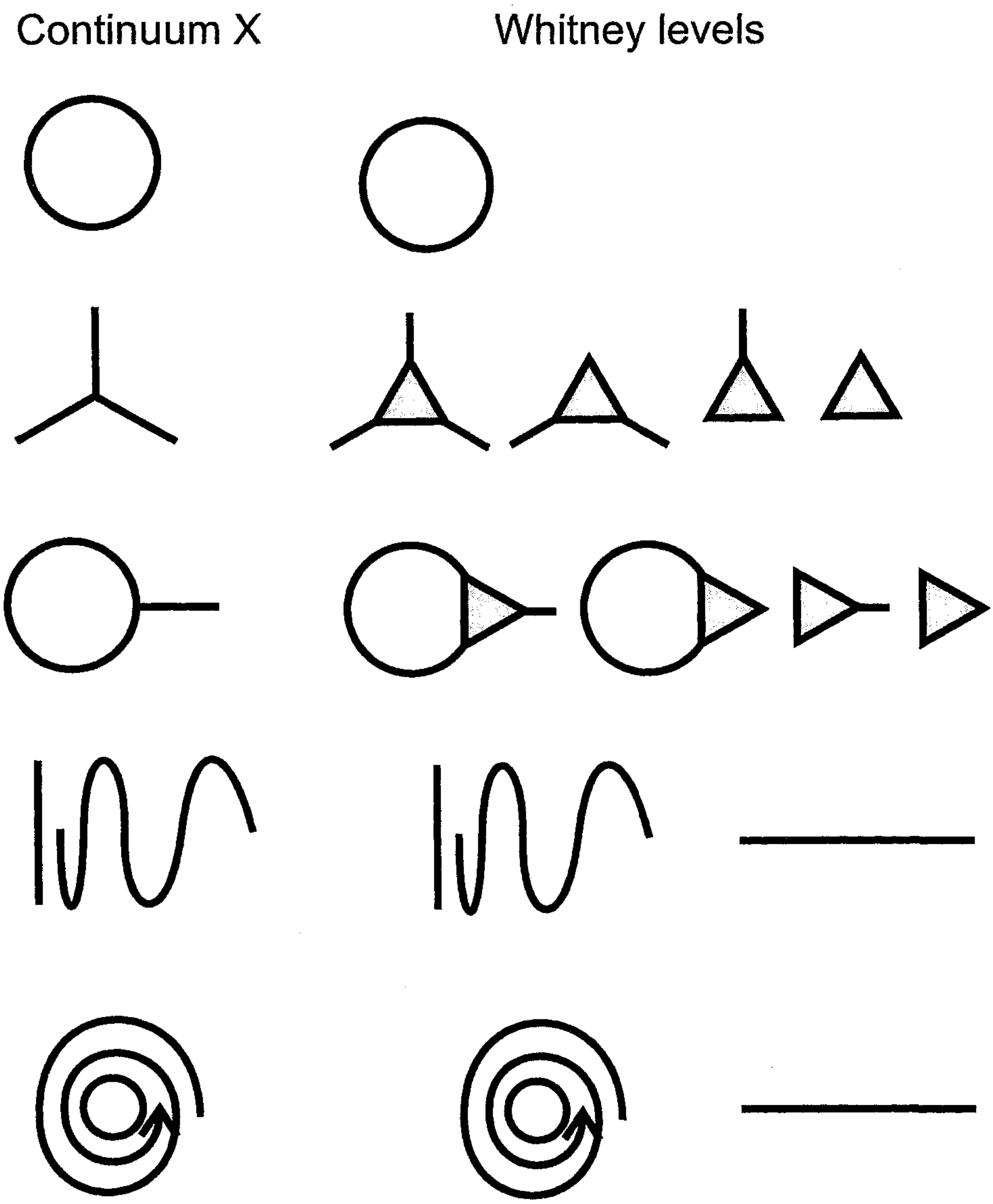

Figure 6.2: Geometric model of Whitney level

Investigation of the structure of Whitney level will be under the condition of existence of admissible Whitney maps. Before doing this we will look at what we have achieved with the existence of the admissible Whitney map. But first, let us give the definition of admissible Whitney maps.

Definition 6.1.1. *(Goodykoobtz and Nadler, 1982)[p. 674 Definition 2.1] Let $\mathscr{H}$ denote either one of the hyperspaces 2^X or $C(X)$. A whitney map ω for $\mathscr{H}$ is called an* admissible Whitney map *for $\mathscr{H}$ provided that there is a (continuous) homotopy $h : \mathscr{H} \times [0,1] \to \mathscr{H}$ satisfying the following two conditions:*

(i) for all $A \in \mathscr{H}$, $h(A,1) = A$ and $h(A,0) \in F_1(X)$;

(ii) if $\omega(h(A,t)) > 0$ for some $A \in \mathscr{H}$ and $t \in [0,1]$, then $w(h(A,s)) < w(h(A,t))$ whenever $0 \leq s < t$.

A homotopy $h : \mathscr{H} \times [0,1] \to \mathscr{H}$ satisfying (i) and (ii) is called an ω- admissible deformation for $\mathscr{H}$. For brief information about homotopy we refer to (Willard, 1970)[pp. 222-226]. We know from (Goodykoobtz and Nadler, 1982)[Proposition 2.2] that the existence of an admissible Whitney map is a topological invariant and also under the condition of the existence of an admissible Whitney map for 2^X or $C(X)$, X is arcwise connected. For the details we refer to (Goodykoobtz and Nadler, 1982)[Theorem 2.3].

Theorem 6.1.2. *(Goodykoobtz and Nadler, 1982, (2.4) Theorem) Let ω be an admissible Whitney map for $\mathscr{H} = 2^X$ or $C(X)$. Then, X is contractible if and only is $\mathscr{H}$ is contractible.*

We should note here that we do not need to have an admissible Whitney map to show that $\mathscr{H}$ is contractible whenever X is contractible, for the interested readers we refer to (Kelley, 1942). Now, we ready to give the following result which will be useful to obtain general result about Peano continuum with admissible Whitney map.

Corollary 6.1.3. *(Goodykoobtz and Nadler, 1982)[(2.5) Corollary] If there is an admissible Whitney map for 2^X or $C(X)$ and if X is a Peano continuum, then X is contractible.*

Proof. It is well known that If X is a Peano continuum, then $\mathscr{H} = 2^X$ or $C(X)$ are contractible, and hence by Theorem 6.1.2 we obtain that X is contractible. $\square$

Theorem 6.1.4. *(Goodykoobtz and Nadler, 1982)[(2.7) Theorem] If ω is an admissible Whitney map for $\mathscr{H} = 2^X$ or $C(X)$, then, for any t_0 such that $0 < t_0 < w(X)$, $\omega^{-1}(t_0)$ is a retract of $w^{-1}([t_0, \omega(X)])$.*

At this point let us note the following theorem given by (Nadler, 1978)[Theorem (14.73.32) p. 503].

Theorem 6.1.5. *Let X be a continuum and let ω be a Whitney map for $\mathscr{H} = C(X)$ (which need not to be an admissible). If $\omega^{-1}(t_0)$ is locally connected for some $t_0 \in [0, \omega(X)]$, then $\omega^{-1}([t_0, \omega(X)])$ is locally connected and, hence, an absolute retract.*

Proof. If we choose t_0 as $\omega(X)$, we have $\omega^{-1}(t_0) = \{X\}$, so let assume that $t_0 \in [0, \omega(X))$. Then $\omega^{-1}(t_0)$ is a continuum (if we choose $\mathscr{H} = 2^X$, then the Whitney map need not be monotone) by (Nadler, 1978)[(14.2) p.400] and, by assumption $\omega^{-1}(t_0)$ is L.C. Hence, by (Nadler, 1978)[(1.92) p.134], $C(\omega^{-1}(t_0))$ is a locally connected continuum. Therefore $\omega^{-1}([t_0, \omega(X)])$ is locally connected by (Nadler, 1978)[(14.73.8) p.487] because the property of being a local connected continuum is invariant under continuous mappings (Kuratowski, 1968)[Theorem 5, p. 257]. Hence by (Nadler, 1978)[(0.74.1) p.52], $\omega^{-1}([t_0, \omega(X)])$ is an absolute retract. $\square$

Now, we have the following theorem related to the positive Whitney level's structure under the condition of contractibility of $\mathscr{H}$ and also we have a more exhaustive version of half of Theorem 6.1.2.

Theorem 6.1.6. *(Goodykoobtz and Nadler, 1982)[(2.8) Theorem] Let ω be an admissible Whitney map for $\mathscr{H} = 2^X$ or $C(X)$. If $\mathscr{H}$ is contractible, then $w^{-1}(t_0)$ is contractible for each $t_0 \in [0, \omega(X)]$.*

In (Curtis and Schori, 1978) Curtis ans Schori showed that 2^X is a Hilbert Cube whenever X is any Peano continua, they also showed the similar result for $C(X)$ in case of X contains no free arc. In (Goodykoobtz and Nadler, 1982) Goodykoontz and Nadler obtained similar result with the Curtis-Schori's one under the condition of existence of an admissible Whitney map. That's why the following theorem is important, because it was used at "Main Result" section in (Goodykoobtz and Nadler, 1982)[proof of (4.1) Theorem].

Theorem 6.1.7. *(Goodykoobtz and Nadler, 1982)[(2.9) Theorem] If there is an admissible Whitney map ω for 2^X or $C(X)$ and if X is Peano continuum, then $w^{-1}(t_0)$ is an absolute retract for each t_0 such that $0 < t_0 < \omega(X)$.*

By using Theorem 6.1.2, Corollary 6.1.3, and Theorem 6.1.6 we have the following trivial result.

Corollary 6.1.8. *If there is an admissible Whitney map* ω *for* $\mathscr{H} = 2^X$ *or* $C(X)$ *and if* X *is Peano continuum, then* $w^{-1}(t_0)$ *is contractible.*

Proof. By using Corollary 6.1.3 we obtain that X is contractible and hence by Theorem 6.1.2 $\mathscr{H}$ is contractible, then, by Theorem 6.1.6 $w^{-1}(t_0)$ is contractible for any t_0 such that $0 < t_0 < w(X)$, so contractibility is a Whitney property in the light of hypothesis of corollary. □

Another important result is given with the following theorem about Whitney levels of admissible Whitney maps.

Theorem 6.1.9. *(Goodykoobtz and Nadler, 1982)[Theorem 2.12] Let* X *be a continuum. If* ω *is an admissible Whitney map for* 2^X*, then* ω *is open, and* $\omega^{-1}(t)$ *is a continuum for each* $t \in [0, \omega(X)]$.

Proof. Definition 6.1.1 (b) implies that there is no local minima for ω at any element $A \in 2^X - F_1(X)$. Applying Theorem 5.1.6 we deduce that ω is open. The last part of the theorem follows from Corollary 5.1.5. □

We have already used an admissible Whitney map to obtain some properties of Whitney levels. A useful sufficient condition for the existence of an admissible Whitney map has been given by Goodykoontz and Nadler. The reader may wish to see (Goodykoobtz and Nadler, 1982)[pp. 677-680].

As we mentioned before, Goodykoontz and Nadler have obtained parallel result of the Curtis-Schori theorem when the Whitney map is admissible. To obtain to that result, they have used Toruńczyk's characterization of the Hilbert cube, this characterization consist a new notions named Z-set and Z-maps, so will give them firstly.

A closed subset A of a continuum Y with metric d is said to be a *Z-set* in Y provided that for each $\epsilon > 0$ there is a continuous function $f_\epsilon : Y \to Y \setminus A$ for which $d(f_\epsilon(y), y) < \epsilon$ for all $y \in Y$ (Chapman, 1976)[p.2]. A *Z-map* is a continuous function $f : Y \to Y$ such that $f(Y)$ is a Z-set in Y (Toruńczyk, 1980)[p.33]. Now, we ready to give Toruńczyk's characterization.

If the identity map defined on Y, where Y is a compact metric absolute retract, is uniform limit of Z-maps, then Y is a Hilbert cube.

Now, we will give the following most general result pointing out that the positive Whitney levels are Hilbert cube. We omit the proof of this result because it is technical and depends on the section 3 in (Goodykoobtz and Nadler, 1982)[p. 680], the interested reader may look at (Goodykoobtz and Nadler, 1982)[pp. 688-691] for more details.

Theorem 6.1.10. *(Goodykoobtz and Nadler, 1982)[(4.1) Theorem] Let X be a Peano continuum. If there is an admissible Whitney map ω for 2^X, then $\omega^{-1}(t_0)$ is a Hilbert cube whenever $0 < t_0 < \omega(X)$. If there is an admissible Whitney map ω for $C(X)$ and if X contains no free arc, then $\omega^{-1}(t_0)$ is a Hilbert cube whenever $0 < t_0 < \omega(X)$.*

We interested in structure of point inverses of a Whitney map under some suitable assumptions. As we mentioned in Section 2, Whitney maps on $C(X)$ are better behaved than Whitney map on 2^X. So from now on, we will give brief information on the structure of point inverses of a Whitney map for $C(X)$. For example, let me mention about Krasinkiewicz result, that is, if X is chainable (respectively, proper circle-like), then $\omega^{-1}(t)$ is chainable (respectively, proper circle-like) for all $t < 1$; and more if X is $[0, 1]$ (respectively, S^1), then $\omega^{-1}(t)$ is an arc (respectively, simple closed curve) for all $t < 1$. In (Nadler, 1975), Nadler has obtained two important results on point inverses of a Whitney map for $C(X)$. The first one, arcwise connectivity of X implies arcwise connectivity of $\omega^{-1}(t)$ for all $t \in [0, 1]$ (Nadler, 1975)[Theorem 2, p.399], and the second, local connectivity of X implies local connectivity of $\omega^{-1}(t)$ for all $t \in [0, 1]$ (Nadler, 1975)[Theorem 3, p.401]. Let us remind you the letter X in here is a nondegenerate metric continuum. In addition to that we follow the corollary from (Nadler, 1975)[Corollary 3, p. 403].

Corollary 6.1.11. *The following four statements are equivalent for a Whitney map ω defined on $C(X)$*

1. *The continuum X is locally connected,*
2. *For each $t \in [0, 1]$, $\omega^{-1}(t)$ is locally connected,*
3. *For each $t \in [0, 1]$, $\omega^{-1}([0, t])$ is locally connected,*
4. *For each $t \in [0, 1]$, $\omega^{-1}([t, 1])$ is locally connected.*

Of course, here one question arises, under which condition we can say something for local connectivity of point inverses of a Whitney map for 2^X, whenever X is locally connected? We know from Remark (Nadler, 1975)[p.403] that the Theorem 3 in (Nadler, 1975) fails for Whitney maps on 2^X even when X is an arc in plane.

Actually, this question is a backbone of this dissertation. We wish to solve this question, or at least we wish to say some words on it. To the best of our knowledge, this question is still unsolved. To see that this is so, let follow the conclusion section.

7. CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusion

As indicated from the last part of the previous section, we wish to say something for local connectivity of point inverses of a Whitney map for 2^X, whenever X is locally connected. So, actually, Nadler gave the answer negatively when the range of Whitney map ω is I. However, under the assumption of existence of admissible Whitney map for $\mathscr{H} = 2^X$ and taking X as Peano continuum, we have the most fundamental observation about this question that is $\omega^{-1}(t_0)$ is an absolute retract continuum for $t_0 \in (0, \omega(X))$ by Theorem 6.1.7, so now by using(Charatonik and Prajs, 2001) it is locally connected. This observation is not appeared in the literature.

By the way, let us note that the relation of being dendrite of X and existence of admissible Whitney maps for 2^X and $C(X)$; that is, there exist admissible Whitney maps for 2^X and $C(X)$ when X is any dendrite, we refer to (Goodykoobtz and Nadler, 1982)[(2.16) Theorem p.679]. By dendrite, we mean a Peano continuum which contains no simple closed curve. So by Theorem 6.1.9 we have an open and monotone map defined on 2^X, this is supporting Illanes's result see Corollary 5.1.7 and now the question rises that what can we say about local connectivity of point inverses of Whitney map for 2^X in case of X is dendrite?

Another way to show the local connectedness of point inverses of a Whitney map for 2^X whenever X is locally connected would be showing that $\omega^{-1}(t)$ for $t \in [0, \omega(X)]$ is a retract of 2^X without the assumption of the existence of an admissible Whitney map. Because if X is locally connected, then 2^X is locally connected (Nadler, 1978)[(1.92) Theorem, p. 134], and every retract of a locally connected space is locally connected (Hu, 1965)[Proposition 10.1, p.27]. Again this way is not appeared in literature, we are working on it nowadays. Actually, we do not know whether it is enough to assume X being locally connected to say something about the local connectedness of the point inverses of a Whitney map for 2^X, actually it seems to be not. In addition, what if X is dendroid or arc-smooth , then what can we say about local connectedness of point inverses of a Whitney map for 2^X?

7.2 Recommendations

One of the most important and widely used properties of hyperspaces is that they behave nicely with to respect to inverse limit, the hyperspace of inverse limit is homeomorphic to inverse limit of the hyperspaces. So the readers specifically interested in this topic may prefer to look at the structure of hyperspace in terms of inverse limit. We suggest that readers are referred to (Nadler, 1978)[p. 159] and (Macias, 2018)[p.53 Chapter 2].

As a recent advance, Whitney blocks for $C(X)$ and its topological properties have been worked by María Elena Aguilera and Alejendro Illanes(Aguilera and Illanes, 2016; Aguilera, 2017, 2018), interested readers may prefer take a look at this subject.

In addition to that in this section we will pose the following questions. Interested reader will be work on it.

Let X be dendrite (or smooth dendroid). By smooth dendroid, we mean for some point p which belongs to an arcwise connected hereditarily unicoherent continuum X. Simplest example of dendrite is arc. Another simple example is the locally connected fan, F_ω see Figure 7.1. F_ω is the dendrite that has only one ramification point with order ω.

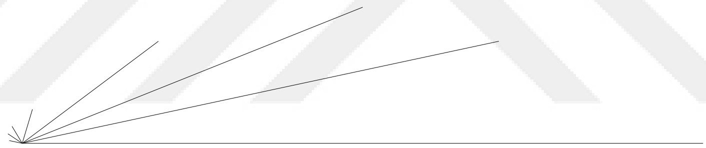

Figure 7.1: locally connected Fan F_w

What can we say about the structure of point inverse of Whitney map defined on 2^X?

We will pose some other problems with respect to Z-sets which are known if X is a Peano continuum.

When X is a continuum which hyperspaces have the property that every countable, closed subset of the hyperspace is a Z-set?

For any continua, is $F_1(X)$ a Z-set in $C(X)$ or 2^X? If not, for which continua can we say this?

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