BOLU ABANT İZZET BAYSAL UNIVERSITY
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

ESTIMATION OF THE REGION OF ATTRACTION FOR NONLINEAR AUTONOMOUS SYSTEMS

DOCTOR OF PHILOSOPHY

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APPROVAL OF THE THESIS

ESTIMATION OF THE REGION OF ATTRACTION FOR NONLINEAR AUTONOMOUS SYSTEMS submitted by **SERPİL YAŞAR** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in **Mathematics Department, the Graduate School of Natural and Applied Sciences of BOLU ABANT İZZET BAYSAL UNIVERSITY in 16/01/2020** by

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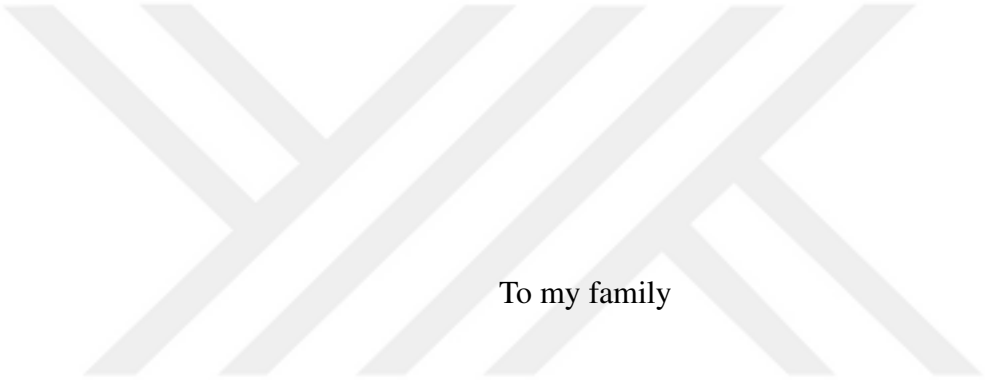

To my family

DECLARATION

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SERPİL YAŞAR

ABSTRACT

ESTIMATION OF THE REGION OF ATTRACTION FOR NONLINEAR AUTONOMOUS SYSTEMS
PH.D. THESIS
SERPİL YAŞAR,
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We study on the determination of the estimations for the regions of attraction of the nonlinear autonomous systems with polynomial vector fields. The aim is to obtain estimations for the regions of attraction of specific examples of nonlinear autonomous systems by using different methods . The study begins with a review of the Lyapunov stability theory and LaSalle's extension principle of that theory. We introduce dynamical properties of the specific examples of nonlinear systems under consideration. We present the conditions for the characterization of the boundary of region of attraction, and state and prove theorems on the unboundedness of the regions of attraction of the examples. The estimations for the regions of attraction are determined as subsets of those regions by applying Lyapunov methods while the estimations for regions of attraction and also for their boundaries are computed by using non-Lyapunov methods. When implementing these method,s we either modify some of the given algorithms in the literature or state new algorithms instead of using them.

KEYWORDS: Region of attraction, nonlinear autonomous system, polynomial vector field, Lyapunov method, non-Lyapunov method, Lyapunov stability theory, LaSalle's extension principle .

ÖZET

DOĞRUSAL OLMAYAN OTONOM SİSTEMLER İÇİN ÇEKİM BÖLGESİ KESTİRİMİ
DOKTORA TEZİ
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Polinom vektör alanlı, lineer olmayan otonom sistemlerin çekim bölgeleri için kestirimlerin belirlenmesi üzerine çalıştık. Amaç, farklı yöntemler kullanarak, lineer olmayan otonom sistemlerin spesifik örneklerinin çekim bölgelerinin kestirimlerinin elde edilmesidir. Çalışma, Lyapunov kararlılık teorisi ve o teorinin LaSalle genişletme prensipiyle başladı. İncelenen lineer olmayan sistemlerin dinamik özellikleri tanıtıldı. Çekim bölgesinin sınırının karakterize edilmesi için koşullar verildi ve çalışılan örneklerin çekim bölgelerinin sınırsız olması üzerine teoremler ifade edilip, ispatlandı. Lyapunov yöntemleri uygulanarak çekim bölgelerinin kestirimleri, o bölgelerin alt kümeleri olarak belirlenirken, çekim bölgeleri ve de onların sınırlarının kestirimleri, Lyapunov olmayan yöntemleri kullanarak hesaplandı. Bu yöntemleri uygulamaya koyarken, literatürde verilmiş olan algoritmaların bir kısmında ya değişiklikler yaptık ya da onların yerine yeni algoritmalar verdik.

ANAHTAR KELİMELER: Çekim bölgesi, lineer olmayan otonom sistem, polinom vektör alanı, Lyapunov yöntem, Lyapunov olmayan yöntem, Lyapunov kararlılık teorisi, LaSalle genişletme prensibi .

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1. INTRODUCTION

The determination of the region of attraction (or stability region, or domain of attraction) for nonlinear systems is an important problem from theoretical and practical views. The first results concerning the exact determination of the region of attraction, (ROA) are probably the results of Barbashin (1951), Krasovskii and Barbashin (1954) and of Zubov (1961). Since the computation of the exact stability region might be difficult or even impossible, the estimation of ROA becomes important in the study of the nonlinear systems analysis. Numerous approaches have been proposed for the estimation of the stability region in the literature. The methods, providing a guaranteed subset of the ROA can be divided in two main classes, i.e., Lyapunov methods, (LaSalle (1967), (Davison and Cowan (1969)) Davidson and Kurak (1971), Rodden (1964), Vannelli and Vidyasagar (1985), Kirin et al. (1981), Margolis and Vogt (1963)), and non-Lyapunov methods (Genesio et al. (1985), Sanchez et al. (2009), Chiang et al. (1988), Chiang and Thorp (1989), Noldus and Loccufier (1995)). The methods in both classes are mainly based on Lyapunov stability theory and LaSalle's extension principle of that theory.

Lyapunov methods compute a Lyapunov function, (LF) as a local stability certificate. The sublevel sets of this LF, in which the function decreases along the own trajectories of the system, provide invariant subsets of the ROA. The Zubov method, (Zubov (1962), Kirin et al. (1981), Margolis and Vogt (1963)), Groebner bases technique, (Forsman and Glad (1990), Forsman (1991)), Lie series method, (Kormanik and Li (1972)), rational solution, (Vannelli and Vidyasagar (1985), Hachicho and Tibken (2007)), the Sum of Squares programming, (Wloszek (2003), Parrilo (2003)) and other numerical techniques (Kaslik et al. (2005), Rezaiee-Pajand and Moghaddasie (2012), Giesl (2008), Chesi et al. (2005), Johansen (2000)) are the examples for the Lyapunov methods. The estimated ROA near a stable equilibrium point are calculated from Lyapunov methods are usually conservative subsets of the true ROA. This is the main disadvantage of these methods. On the other hand, non-Lyapunov methods are based on a topological characterizations of the system and use the backward integration algorithms. The actual boundaries of ROA can also be estimated by the non-Lyapunov methods.

The common feature of the non-Lyapunov methods is the integration of the system

backward in time. In this process, integration starts from a given set of initial conditions lying in a surface enclosing an equilibrium point. The computed trajectories converge towards the boundary of the ROA of the system as time decreases. So, such methods generally yield better estimations than Lyapunov methods. Backward integration, (Sanchez et al. (2009)), trajectory reversing method, (Genesio et al. (1985)), and its modifications, (Chiang et al. (1988), Chiang and Thorp (1989), Loccufier and Noldus (1995, 2000)) are the most popular non-Lyapunov methods in the literature.

In this thesis, Lyapunov and non-Lyapunov methods, are applied to determine estimations for the ROA of the specific examples of the nonlinear autonomous dynamical systems such as, Genesio system, Lorenz system, Rossler system, Belousov-Zhabotinskii reaction, and another nonlinear system, (Tibken and Hachicho (2000)). As the examples of the Lyapunov methods, the Zubov method, Groebner bases technique and sum of squares programming method, are used to compute the estimations for the ROA. From the class of non-Lyapunov methods, backward integration method, trajectory reversing method and eigenfunction estimation method are applied for the same purpose.

Chap.2 represents theoretical backgrounds on the Lyapunov stability theory and the LaSalle's extension principle of the Lyapunov stability theory. Dynamical properties and the theorems on the unboundedness of ROAs of the worked systems together with the characterization of the boundary of the ROA are stated in Chap.3. The Lyapunov and non-Lyapunov methods and their applications are studied in Chap.4 and 5, respectively. Results and discussions are given in Chap.6, and conclusions are drawn together with recommendations in Chap.7.

All computations throughout the thesis are performed by using symbolic mathematics packages, Mathematica and Matlab.

2. PRELIMINARIES

This chapter provides the necessary theoretical background for the development of our study. Various methods, proposed in the literature for the estimation of ROA belong to LF approach and mainly based on La Salle's extension of Lyapunov stability theory. So, we present Lyapunov indirect and direct methods, and LaSalle's extension principle of Lyapunov stability theory.

Throughout the thesis, we consider the autonomous systems:

$$\dot{x}(t) = f(x(t)) \tag{2.1}$$

where $t \in \mathbb{R}^+, x \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$. We suppose that f is smooth enough so that system (2.1) has a unique solution $\Phi(t; x_0)$ of (2.1) for all $t \geq t_0$, where $x_0 = \Phi(t_0; x_0) \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}^+$. Without loss of generality, it is assumed that $t_0 = 0$. We also assume that $x = 0$ is an isolated equilibrium point of (2.1), that is, there is a neighborhood of $x = 0$ containing no equilibrium points of (2.1) other than $x = 0$ itself, so that $f(0) = 0$.

To characterize the stability of an equilibrium point of nonlinear differential equations, two well-known methods are due to Lyapunov: Lyapunov indirect method and Lyapunov direct method are mainly used, (Zubov (1964)).

2.1 Lyapunov indirect method (First method)

By the Lyapunov indirect method, we can infer the stability behavior of an equilibrium point of system (2.1) from the study of the stability behavior of a linear system, under certain conditions. In that method, we examine each equilibrium point, seperately, if there is more than one. An equilibrium point, if it is not at the origin, can be translated to the origin by an appropriate coordinate transformation.

To draw a conclusion about the stability behaviour of the equilibrium point $x = 0$, in addition to the above assumptions, we assume that f is continuously differentiable with respect to its arguments. So, we can write a Taylor series of $f(x)$ about the point $x = 0$ and separate the linear and nonlinear terms

$$\dot{x} = Jx + g_1(x). \tag{2.2}$$

Here $J = \left(\frac{\partial f(x)}{\partial x}\right)_{x=0}$ is the Jacobian evaluated at $x = 0$, and that $g_1(x)$ consists of higher order terms in the component of x which satisfies two conditions:

$$g_1(0) = 0$$
$$\forall \epsilon \geq 0, \exists \eta \geq 0 : |x| \leq \eta \implies |g_1(x)| \leq \epsilon |x|.$$

We call the equation

$$\dot{x} = Jx \tag{2.3}$$

the linearized system of nonlinear system (2.1). The indirect method gives stability conditions for $x = 0$ in (2.1) using stability results in (2.3).

Theorem 2.1.1. *(Poincaré-Lyapunov's Theorem, Poincaré (1881)). The equilibrium $x = 0$ of (2.3) is asymptotically stable, if all the eigenvalues of the Jacobian matrix J have negative real parts, and it is unstable, if at least one eigenvalue of the Jacobian matrix J has a positive real part.*

Using the Lyapunov indirect method, we can conclude that whether the equilibrium $x = 0$ of (2.1) is asymptotically stable, but we can not give any conclusion on the extent of asymptotic stability using this method. In order to assess the extend of asymptotic stability we need Lyapunov's direct method.

2.2 Lyapunov Direct Method (Second Method)

In order to find out the stability properties of an equilibrium point of system (2.1) without solving it, an effectice method is the Lyapunov direct method. It also gives information on the extent of stability. Lyapunov direct method is based on the energy idea in a physical system. The stability of an equilibrium point is settled from the rate of change of the stored energy of the physical system, along the motions of the system. On the other hand, to find an expression for energy for a complex system may be difficult. As it was proved by Lyapunov, if we can find a function $V(x)$ having the similar properties of energy, then the stability of an equilibrium point of system (2.1) can be determined using the sign definiteness of the time derivative of $V(x)$.

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} \cdot f_i(x) = \nabla V^T \cdot f(x) \tag{2.4}$$

The function $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function (LF) if it is assumed to be continuously differentiable with respect to all of its arguments.

Definition 2.2.1. When $x = 0$ is asymptotically stable the region of attraction of the origin is defined as ROA$(0) = \{x_0 | \lim_{t\to\infty} \Phi(t; x_0) = 0\}$, where $\Phi(t; x_0)$ is a solution of Eq. (2.1) that starts at initial state x_0.

Assume that $D \subset \mathbb{R}^n$ is an arbitrary domain and the equilibrium x_0 of system (2.1) lies in the interior of D. Then we state three important theorems of the second method of Lyapunov.

Theorem 2.2.2. *For all $x \in D$, if there exists a continuously differentiable function $V(x)$ such that $V(0) = 0$, $V(x) > 0$ and its time derivative, $\dot{V}(x) \leq 0$ on D, then the equilibrium $x = 0$ of system (2.1) is called stable.*

Theorem 2.2.3. *For all $x \in D$, if we can find a continuously differentiable function $V(x)$ such that $V(0) = 0$, $V(x) > 0$ and its time derivative, $\dot{V}(x) < 0$ on $D \setminus \{0\}$, then the equilibrium $x = 0$ of the system (2.1) is said to be asymptotically stable.*

Theorem 2.2.4. *For all $x \in \mathbb{R}^n$, if there exists a positive definite and radially unbounded function $V(x)$ such that $\dot{V}(x)$ is negative definite, then the equilibrium $x = 0$ of (2.1) is globally asymptotically stable.*

Let D be the domain given by Theorem 2.3 and also let Ω be a domain defined by

$$\Omega = \{x \in \mathbb{R}^n | V(x) \leq c, c \geq 0\}. \tag{2.5}$$

Now we can show that if it is wholly contained in D, then the domain Ω will be included in the actual ROA of $x = 0$, ROA(0), and it can be used as an estimate of ROA(0). If we can find the maximum value of $c > 0$ such that $\Omega \subset D$, then Ω will be the largest estimate of ROA(0), obtained by that particular $V(x)$.

The basic and most important characteristic of the Lyapunov's direct method is that for a particular solution of a given nonlinear system defined by (2.1) definite stability properties can be proved by only finding a suitable LF. On the other hand, once a LF is known then it settles the local problem of stability, and also allows us to determine certain regions of the system under consideration (Krasovskii (1963)). An LF may determine a region which is the exact ROA. But, using Lyapunov's direct method only a part of the exact ROA can be estimated. In general, this method does not afford any information about the real size of the ROA.

2.3 LaSalle's Invariance Principle

LaSalle's invariance principle was developed by LaSalle (1960). In fact, it was first, introduced by Barbashin and Krasovskii in a special case in (Barbashin and Krasovskii (1952)), and later by Krasovskii in the general case in (Krasovskii (1959)). LaSalle's invariance principle attempts to relax the constraints imposed on the candidate LF for the conditions of asymptotic stability, (Khalil (2002)).

The principle basically says that if we have a LF defined in a neighbourhood of the origin, with a negative semi-definite time derivative along the trajectories of system (2.1), and additionally, if we can establish that any trajectory can not stay identically at points where $V(x) = 0$, except at the origin, then the origin is asymptotically stable.

Proposition 2.3.1. *(LaSalle's Principle, (LaSalle (1960))) Let $V : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable function and consider that*

$$\Sigma_d = \{x \in \mathbb{R}^n | V(x) \leq d\}$$

is bounded and that $\dot{V}(x) \leq 0$ for all $x \in \Sigma_d$. Define $T \subset \Sigma_d$ by $T = \{x \in \Sigma_d | \dot{V}(x) = 0\}$ and let K be the largest invariant set in T. Then whenever $x_0 \in \Sigma_d, \Phi(t, x_0)$ approaches K as $t \to \infty$.

Theorem 2.3.2. *(LaSalle's invariance principle for asymptotic stability, LaSalle (1960)) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally positive-definite function on the compact set*

$$\Sigma_d = \{x \in \mathbb{R}^n | V(x) \leq d\}.$$

Then we have $\dot{V}(x) = 0$. Define $T = \{x \in \Sigma_d | \dot{V}(x) = 0\}$. The trajectory tends to the largest invariant set inside T, as $t \to \infty$. In particular, if T contains no invariant sets other than $x = 0$, then 0 is asymptotically stable.

Theorem 2.3.3. *(LaSalle's invariance principle for global asymptotic stability, LaSalle (1960)) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a globally positive-definite function and $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. In addition, let the set $T = \{x \in \Sigma_d | \dot{V}(x) = 0\}$ contain no nontrivial trajectories. Then the equilibrium point $x = 0$ is globally asymptotically stable.*

3. ON THE WORKED SYSTEMS AND THEIR ROAs

In this section, we introduce some dynamical properties of the systems under consideration. We present some useful facts on the characterization of the boundary of ROA and prove that the ROAs of the worked systems are unbounded.

3.1 Worked Systems

3.1.1 The Genesio System

The Genesio system that is proposed by Genesio and Tesi (1992) possesses many features of chaotic systems. It includes a quadratic nonlinear term and consists of three positive parameters. The dynamical equations of the system is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -ax - by - cz + x^2. \end{aligned} \tag{3.1}$$

Genesio system has two equilibria, $(0,0,0)$ and $(a,0,0)$ for every values of the parameters. By linearization, we can easily show that the origin, $(0,0,0)$ is a stable focus when $bc > a$ and is a saddle focus when $bc < a$ while the point $(a,0,0)$ is a saddle focus for all values of the parameters.

E.g., system (3.1) becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -x - 3y - z + x^2 \end{aligned} \tag{3.2}$$

when $a = 1, b = 3, c = 1$. We can easily see that origin is a stable focus and the point $(1,0,0)$ is a type-1 saddle, by using linearization analysis. Since the Jacobian matrix at the origin

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -1 \end{pmatrix}$$

has three eigenvalues with negative real parts, namely

$$\lambda_1 = -0.3611, \lambda_2 = -0.3194 + 1.6332i, \lambda_3 = -0.3194 - 1.6332i$$

We can conclude that the origin $(0, 0, 0)$ is a stable focus which is asymptotically stable by Theorem 2.1.

Similarly, the Jacobian matrix

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 0.2956, \lambda_2 = -0.6478 + 1.7214i, \lambda_3 = -0.6478 - 1.7214i$. Since one eigenvalue has a positive real part and the remaining two have negative real parts, we can conclude that the point $(1, 0, 0)$ is an unstable focus (or saddle focus) which is of type-1 saddle.

3.1.2 The Lorenz System

The Lorenz system, first studied by Edward N. Lorenz around 1963, was derived from a simplified model of convection in the earth's atmosphere. It also arises naturally in models of lasers and dynamos. The system is defined by 3 coupled nonlinear differential equations

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz, \end{aligned} \tag{3.3}$$

where the parameters σ, r and b are positive numbers. $(0, 0, 0)$ is an equilibrium point for all values of the parameters. It is the only equilibrium point when $0 < r < 1$.

E.g., system (3.3) becomes

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= \frac{1}{2}x - y - xz \\ \dot{z} &= xy - \frac{8}{3}z, \end{aligned} \tag{3.4}$$

when $\sigma = 10, r = 1/2, b = 8/3$. Linearization analysis says that the origin $(0, 0, 0)$ is a

stable node which is globally stable. Since we find the Jacobian matrix at that point as

$$J = \begin{pmatrix} -10 & 10 & 0 \\ 1/2 & -1 & 1 \\ 0 & 0 & -8/3 \end{pmatrix}$$

with three eigenvalues having negative real parts, namely $\lambda_1 = -10.5249$, $\lambda_2 = -0.4751$, $\lambda_3 = -2.6667$, by linearizing the system about the origin, we can conclude that the origin is a stable node. There is also a pair of equilibrium points $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$, for $r > 1$. These coalesce with the origin as $r \to 1^+$ in a pitchfork bifurcation. As r increases Lorenz system displays a wide range of chaotic behaviors.

3.1.3 The Rössler System

The Rossler system which exhibits chaotic dynamics with the fractal properties of the attractor, is originally studied by Otto Rossler. The system is described by the system of equations

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{3.5}$$

It has two equilibrium points $E_1 = (\frac{c-\sqrt{c^2-4ab}}{2}, \frac{-c+\sqrt{c^2-4ab}}{2a}, \frac{c-\sqrt{c^2-4ab}}{2a})$ and $E_2 = (\frac{c+\sqrt{c^2-4ab}}{2}, \frac{-c-\sqrt{c^2-4ab}}{2a}, \frac{c+\sqrt{c^2-4ab}}{2a})$ if $c^2 > 4ab$, only one equilibrium point $(\frac{c}{2}, \frac{-c}{2a}, \frac{c}{2a})$ if $c^2 = 4ab$ and no equilibrium point if $c^2 < 4ab$.

The system

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + y \\ \dot{z} &= 2.22 + z(x - 3) \end{aligned} \tag{3.6}$$

has two equilibrium points for $a = 1$, $b = 2.22$ and $c = 3$: $E_1 = (1.32679, -1.32679, 1.32679)$, and $E_2 = (1.67321, -1.67321, 1.67321)$. Using linearization analysis, we compute the Jacobian matrix at the point E_1 as

$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 1.32679 & 0 & -1.67321 \end{pmatrix}$$

with eigenvalues $\lambda_1 = -0.047351 + 0.772374i, \lambda_2 = -0.047351 - 0.77237i$ and $\lambda_3 = -0.578503$. Since all eigenvalues have negative real parts, we can conclude that the equilibrium point $E_1 = (1.32679, -1.32679, 1.32679)$ is a stable focus which is asymptotically stable. Similarly, the Jacobian matrix at the point E_2 is

$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 1.67321 & 0 & -1.32679 \end{pmatrix}$$

with eigenvalues $\lambda_1 = -0.280591 + 1.18288i, \lambda_2 = -0.280591 - 1.18288i, \lambda_3 = 0.234387$. Since it has two eigenvalues with negative real parts and one eigenvalue with positive real part the equilibrium point $E_2 = (1.67321, -1.67321, 1.67321)$ is of type-1 saddle.

3.1.4 The Belousov-Zhabotinsky Reaction

A Belousov-Zhabotinsky reaction (or BZ reaction) is one of a class of reactions that serve as a classical example of non-equilibrium thermodynamics, resulting in the establishment of a nonlinear chemical oscillator. The phenomenon is first, observed by Boris Belousov in 1951 and then, Anatol Zhabotinsky confirmed his results in 1961.

The simplest realistic model of the chemical dynamics of the oscillatory Belousov-Zhabotinsky reaction which was created by Richard Field and Richard M. Noyes, is called the oreganator. In the following, we consider an oregonator model of BZ reaction

$$\begin{aligned} \epsilon\dot{x} &= qy - xy + x(1-x) \\ \delta\dot{y} &= -qy - xy + fz \\ \dot{z} &= x - z \end{aligned} \tag{3.7}$$

Its equilibrium points are $(0,0,0)$, and $(\frac{1}{2}(1-f-q \pm \sqrt{1-2f+f^2+2q+6fq+q^2}),$ $\frac{1}{4q}(q+3fq \pm \sqrt{1-2f+f^2+2q+6fq+q^2}), (1-f-q \pm \sqrt{1-2f+f^2+2q+6fq+q^2}))$.

E.g., system (3.7) becomes

$$\begin{aligned} 10^{-2}\dot{x} &= 10^{-4}y - xy + x(1-x) \\ 10^{-5}\dot{y} &= -10^{-4}y - xy + \frac{1}{2}z \\ \dot{z} &= x - z. \end{aligned} \tag{3.8}$$

when $q = 10^{-4}, f = 1/2, \epsilon = 10^{-2}, \delta = 10^{-5}$. The equilibrium points are $E_1 = (0,0,0)$, $E_2 = (-0.00029988, 0.75015, -0.00029988)$ and $E_3 = (0.5002, 0.4999, 0.5002)$. Lineariz-

ing the system about the point E_1, we compute the Jacobian matrix at that point as

$$J = \begin{pmatrix} 100 & 0.001 & 0 \\ 0 & -10 & 50000 \\ 1 & 0 & -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 100.045, \lambda_2 = -9.46007, \lambda_3 = -1.5849$. Since it has three eigenvalues with one positive and two negative real parts, the origin is of type-1 saddle which is unstable. We now linearize the system about the point E_2 and compute the Jacobian matrix at that point as

$$J = \begin{pmatrix} 25.045 & 0.039988 & 0 \\ -75015 & 19.988 & 50000 \\ 1 & 0 & -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 22.2325 + 54.5911i, \lambda_2 = 22.2325 - 54.5911i, \lambda_3 = -0.431979$. Since it has eigenvalues having two positive and one negative real parts the point E_2 is a type-2 saddle which is unstable. We finally linearize the system about the point E_3 and compute the Jacobian matrix at that point as

$$J = \begin{pmatrix} -50.03 & -50.01 & 0 \\ -49990 & -50030 & 50000 \\ 1 & 0 & -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = -50080, \lambda_2 = -0.529452 + 7.05051i, \lambda_3 = -0.529452 - 7.05051i$. Since all the eigenvalues have negative real part, E_3 is a stable spiral.

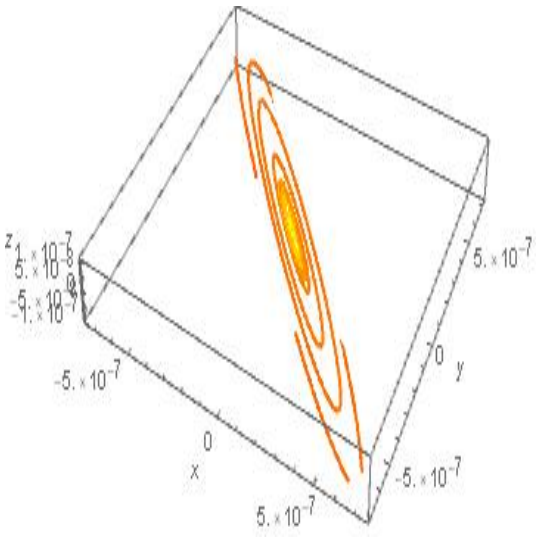


Figure 3.1: BZ reaction phase plot

3.1.5 A nonlinear system from (Hachicho and Tibken (2007))

The system of equations are

$$\begin{aligned} \dot{x} &= -x + yz^2 \\ \dot{y} &= -y + xy \\ \dot{z} &= -z. \end{aligned} \tag{3.9}$$

$(0, 0, 0)$ is the only equilibrium point of this system. Using linearization analysis, we compute the Jacobian matrix at the origin as

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = -1$ and the corresponding eigenvectors are $v_1 = (1, 0, 0)^T$, $v_2 = (0, 1, 0)^T$ and $v_3 = (0, 0, 1)^T$. Since all eigenvalues are negative and the corresponding eigenvectors span the stable eigenspace of the linearized system, which is $\mathbb{R}^3$, we can conclude that the origin is asymptotically stable node.

In the rest of this chapter, we consider the ROA and its boundary.

3.2 Characterization of the ROA

First we state a well-known theorem:

Theorem 3.2.1. *(Yerugin (1951a,b); Hahn (1967)) The boundary of the ROA is formed by whole trajectories.*

Next we consider the characterization of the boundary of the ROA and see that hyperbolic equilibrium points play an important role in this characterization.

An equilibrium point $\hat{x}$ of system (2.1) is said to be hyperbolic if, in local coordinates, any of eigenvalues of the Jacobian matrix J at $\hat{x}$ has no zero real part. For a hyperbolic equilibrium point $\hat{x}$, the tangent space of the state space of system can be decomposed as a direct sum of the stable and unstable eigenspaces. Consequently, one of the important features of a hyperbolic equilibrium point $\hat{x}$ is that its stable and unstable manifolds intersect transversely at $\hat{x}$. Transversal intersection is important because it remains same under perturbation of the vector field.

Under the following assumptions concerning the vector field:

(i) all the equilibrium points on the boundary of ROA are hyperbolic,

(ii) the stable and unstable manifolds of equilibrium points on the boundary satisfy the transversality condition, and

(iii) every trajectory on the boundary approaches one of the equilibrium points as $t \to \infty$.

Theorem 4.3 in Chiang et al. (1988) states a necessary condition for the existence of certain types of equilibrium points on the boundary of ROA which is bounded. The contrapositive of this theorem is the following.

Corollary 3.2.2. *(Chiang et al. (1988)) (Sufficient conditions for ROA to be unbounded). For the nonlinear autonomous system (2.1), if the assumptions (i) to (iii) are satisfied and if the boundary contains no source, then the ROA is unbounded.*

Using Corollary 3.1, we can state the following theorem.

Theorem 3.2.3. *Systems (3.2), (3.4), (3.6), (3.8) and (3.9) have unbounded ROAs.*

Proof. We first note that any of these systems has a source.

We start with the proof of our assertion for systems (3.2) and (3.6). Since each of systems, (3.2) and (3.6) has an asymptotically stable focus and an unstable focus, a type-1 saddle point, and these points are of hyperbolic type, and moreover there exists a Lyapunov function for the asymptotically stable equilibrium point of the systems we can conclude that systems (3.2) and (3.6) have an unbounded ROA, by Corollary 3.2.2.

We now prove the assertion for system (3.8). This system has an asymptotically stable node and an unstable node, a type-1 saddle, and an unstable focus, a type-2 saddle. These equilibrium points are all hyperbolic type. The transversal intersection of the stable and unstable manifolds of the saddle points is a generic property of the autonomous systems with hyperbolic equilibria we can assume such an intersection. Furthermore, there exists a Lyapunov function for the asymptotically stable focus. Thus, we can conclude that system (3.8) has an unbounded ROA by Corollary 3.2.2.

Finally, we prove the statement for systems (3.4) and (3.9). Each system has only one equilibrium point which is an asymptotically stable node and there exists a polynomial

Lyapunov function for each stable node which is radially unbounded. So, the ROA of the each system is $\mathbb{R}^3$. Hence, they have unbounded ROA. □

We note that to prove the assertion we use the fact that a Lyapunov function is an energy function with decreasing properties, and if such a function exists then each trajectory of the system converges to one of the equilibrium point.

4. LYAPUNOV METHODS

Lyapunov methods are based on Lyapunov's stability theory and its various extensions. They can give sufficient conditions to confirm stability of dynamical systems using the LFs, and approach the problem of estimation of ROA by characterizing the trajectories that lie on the boundary of ROA.

Almost all Lyapunov methods used for estimating ROA are based on the search for a LF, $V(x)$ and a positive scalar c such that time derivative of $V(x)$, $\dot{V}(x)$ is negative over sub-level set $\{x|V(x) \leq c\}$, Hahn (1967). Given such a function V and a positive scalar c, it can be proven that the connected component of that set containing the equilibrium point is an inner approximation to the ROA.

Such techniques have been successfully implemented for estimating the ROAs of various nonlinear systems, but they have several disadvantages by limiting their applicability in practice, (Chesi (2011)): First of all, the approach depend upon the existence of suitable LFs. Since Lyapunov theory has a non-constructive nature, computation of such LFs for nonlinear systems are extremely difficult. In other words, the related theorem only ensures the existence of a ROA, but it does not provide a systematic way for finding an initial feasible LF. It is possible to construct a quadratic LF easily by solving the Lyapunov equation of the linearized system, but this function only captures the local behaviour of the nonlinear systems around the equilibrium point. Then, the most of the existing methods are limited to polynomial systems (Hachicho and Tibken (2007), Chesi et al. (2005)). In the case of the non-polynomial systems, the equations of motion are first, approximated by using the Taylor's expansion and then, ROA is computed based on the approximated polynomial equations. Finally, the Lyapunov methods generally yield conservative estimations of ROA, since the techniques often relax the optimization problem to maximize the sub-level set of the LF, and the available methods are usually computationally time consuming.

In this chapter, we shall use the methods of Zubov, Groebner bases and SOS programming for estimating the ROAs of the specific examples of nonlinear autonomous systems.

4.1 The Zubov's Method

There are a very few methods for computing the entire ROA, in the literature. Their applications are impractical since these methods have serious computational problems. Zubov's method is the well-known of them. It provides necessary and sufficient conditions to characterize areas which are thought as a ROA around an asymptotically stable equilibrium point.

The following theorem due to Zubov which states that for a given asymptotically stable equilibrium point, it is always possible to find a LF, defined on the ROA that satisfies the partial differential equation, (PDE) (4.1), (Zubov (1961, 1962), Hahn (1967)).

Theorem 4.1.1. *The origin of system (2.1) is asymptotically stable on a set $D \subseteq \mathbb{R}^n$ if, and only if there exist functions $V, \varphi \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying*

(i) *V is continuous, and positive definite in $D, 0 \leq V < 1$, and $\lim_{|x| \to \partial D} V(x) = 1$,*

(ii) *φ is continuous and positive definite; and*

(iii) *the partial differential equation*

$$\dot{V} = [\nabla V(x)]^T f(x) = -\varphi(x)(1 - V(x))\sqrt{1 + f^T(x) f(x)} \tag{4.1}$$

is satisfied.

Assuming that system (2.1) does not have a finite escape time, we can write equation (4.1) as

$$[\nabla V(x)]^T f(x) = -\phi(x)(1 - V(x)). \tag{4.2}$$

First we take a positive definite function $\varphi(x)$ which is continuous. Then, solving (4.2) for $V(x)$ with the boundary condition $V(0) = 0$, we find the true ROA, defined by $0 \leq V(x) < 1$. We note that it is enough to take $\varphi(x)$ as quadratic form, not identically zero, along a trajectory in that region.

Theorem 4.1.2. *(Zubov (1962)) As the solution of the PDE (4.2), the function $V(x)$ is a LF that establishes the asymptotic stability of the equilibrium $x = 0$ of system (2.1).*

By making the substitution

$$W(x) = -\ln(1 - V(x)), \tag{4.3}$$

where $0 \leq V(x) < 1$, we obtain another PDE

$$[\nabla W(x)]^T f(x) = -\varphi(x, y, z). \tag{4.4}$$

We note that similar statements and equations as for V can also be given for W.

Theorem 4.1.3. *(Zubov (1962)) If $x \in D$, where D is the ROA of system (2.1), then $0 \leq V(x) < 1$.*

Theorem 4.1.4. *(Zubov (1962)) If ξ is a point lying on the boundary of domain D then $\lim_{x\to\xi} V(x) = 1$ from the interior of D.*

Theorem 4.1.5. *(Zubov (1962)) If exists, the curve $V(x) = 1$, is an integral curve of system (2.1).*

Theorem 4.1.6. *(Zubov (1962)) For a fixed x, the solution to system (2.1) is uniquely determined in interior of D.*

Theorem 4.1.7. *(Zubov (1962)) The boundary of domain D is a family of curves $V(x) = 1, \forall x$.*

Theorem 4.1.8. *(Zubov (1962)) In order for the equilibruim of system (2.1) to be asymptotically stable globally, it is necessary and sufficient that $V(x) < 1$, $\forall x$.*

4.1.9 The Zubov's Recursive Procedure

In general, we can not find an analytic solution of the Zubov's PDE. However, we can use a recursive procedure to construct the solution, (Margolis and Vogt (1963)) for some classes of systems. We start the procedure by assuming the right hand side of system (2.1) possesses continuous partial derivatives of all orders. Then we can write the Taylor series of $f(x)$ about $x = 0$ which yields

$$\dot{x} = Jx + h(x), \tag{4.5}$$

where

$$J = \left(\frac{\partial f(x)}{\partial x}\right)\Big|_{x=0} \tag{4.6}$$

is the Jacobian matrix of the linearized equations and the nonlinear part $h(x)$ has a power series representation

$$h(x) = h_1(x) + h_2(x) + \cdots, \tag{4.7}$$

where all terms $h_m(x)$ are homogeneous functions of degree $m \geq 2$, that is,

$$h_m(x) = \Sigma_{\sum n_j} c_{n_1,\ldots,n_\alpha} x_1^{n_1} x_2^{n_2} \cdots x_{n_\alpha}^{n_{n_\alpha}}, \tag{4.8}$$

where $n_1 + n_2 + \ldots + n_{n_\alpha} = m$. Choosing $\varphi(x)$ as a quadratic function, we can write the solution of Zubov's PDE

$$\frac{\partial V}{\partial x}[Jx + h(x)] = -\varphi(x)(1 - V(x)) \tag{4.9}$$

using the boundary condition $V(0) = 0$, as an infinite power series

$$V(x) = V_2(x) + V_3(x) + \cdots . \tag{4.10}$$

Here, we assume that the functions $V_m(x)$ are homogeneous functions of degree m

$$V_m(x) = \Sigma_{\sum n_j} b_{n_1,\ldots,n_\alpha} x_1^{n_1} x_2^{n_2} \cdots x_{n_\alpha}^{n_{n_\alpha}} \tag{4.11}$$

We can calculate the coefficients of each $V_m(x)$ by replacing (4.7) and (4.10) in equation (4.9) to obtain

$$\left(\sum_{n=2} \frac{\partial V_n}{\partial x}\right) = (Jx + \sum_{n=2} h_n(x) + \cdots) = -\varphi(x)\left(1 - \sum_{n=2} V\right). \tag{4.12}$$

In order to determine $V(x)$ as maximal LF, we first choose $\varphi(x)$ as a quadratic function, then we expand both sides of (4.12) and equate the same degree terms to get the recursive relations:

$$\begin{aligned}
\frac{\partial V_2}{\partial x} Jx &= -\varphi(x) && (4.13)\\
\frac{\partial V_3}{\partial x} Jx &= \frac{\partial V_2}{\partial x} h_2(x) && (4.14)\\
\frac{\partial V_4}{\partial x} Jx &= \varphi(x) V_2(x) h_3(x) - \frac{\partial V_3}{\partial x} h_2(x) && (4.15)\\
&\vdots \\
\frac{\partial V_m}{\partial x} Jx &= \varphi(x) V_{m-2}(x) h_{m-1}(x) - \sum_{j=2}^{m-1} \frac{\partial V_2 j}{\partial x} h_{m+1-j}(x) && (4.16)
\end{aligned}$$

Substituting $V_2(x)$ (from equation (4.11) with $m = 2$) in equation (4.13), and equating coefficients of the corresponding terms with the left and right sides we get a system of equations for $V_2(x)$. After determining $V_2(x)$ and using the equation (4.14) we can determine the coefficients of $V_3(x)$. We repeat this process until every element of the power series of $V(x)$ is calculated. By this way, we reduce the problem of determination of the coefficients of V_m to the problem of finding the solution of $m + 1$ linear equations which can be solved, easily. If we can write the power series that represents a function $V(x)$ in closed form, then the exact ROA is defined by $0 \leq V(x) \leq 1$. This is another way for representing the ROA, but it is obtained taking several terms from the series of LF, $V(x)$.

4.1.10 The Optimization Procedure

The optimization process (Walter and Thomas (1997)) can be explained as follows: For an equilibrium at the origin, we choose a LF $V(x)$ which is positive definite and its time derivative, $\dot{V}(x)$ is negative definite in some deleted neighborhood of the origin. Next, we solve the problem:

$$\min_x V(x) \text{ subject to } \dot{V}(x) \leq 0. \tag{4.17}$$

The region $V(x) < \tilde{V}$ will be the largest estimate for the ROA, based on the specified LF $V(x)$ provided that the problem (4.17) has a nonzero solution $\tilde{x}$, and the region defined by $V(x) < \tilde{V} = V(\tilde{x})$ is bounded, with the assumption that $\dot{V}(x) \leq 0$ there. We also impose that $\dot{V}(x)$ is not zero at each point x except the point zero.

By using the first-order necessary optimality conditions, we can compute the solutions of the problem (4.17). With the assumption $\frac{\partial V(\tilde{x})}{\partial x} \neq 0$, we can reduce the conditions to

$$\dot{V}(\tilde{x}) = 0 \tag{4.18}$$

$$\frac{\partial V(\tilde{x})}{\partial x} = \mu \frac{\partial \dot{V}(\tilde{x})}{\partial x}, \tag{4.19}$$

where $\mu > 0$. The solutions of (4.18) and (4.19) imply the tangency of the surfaces $V(x) = \tilde{V}$ and $\dot{V}(\tilde{x}) = 0$ at $\tilde{x}$.

4.1.11 The problems arise in the applications of the Zubov's recursive procedure

(1) The implementation of the power series approximation of the Zubov's PDE is usually restricted to the lower order systems due to computational difficulties and due to the fact that the computed power series is not uniformly convergent. For second-order systems, Margolis and Vogt (1963) developed a solution procedure. Unfortunately, this procedure can not immediately be generalized to higher-order systems. To obtain the solution of the recursive equations for V_N, we need all the previously calculated values, namely $V_2, V_3, \ldots, V_{N-1}$. The total number of coefficients to be stored will increase rapidly as N increases. For instance, in a third-order system for $N = 20$, the total number of terms in the final evaluation of the function V is 1770.

(2) Zubov's method has a nonmonotonic nature.

(3) There is no systematic way for selection of positive definite function $\varphi(x)$ and number of terms N of the series to be taken to optimize the stability boundary estimate. Different choices of $\varphi(x)$ (also N) yield different order estimations.

(4) Another problem arises in the determination of a constant c in the optimization problem so that $V - c$ gives the boundary of ROA. To determine c, we solve the minimization problem

$$\min_{c,x} c \text{ subject to } V - c = 0, \quad \dot{V} = 0, \quad c > 0$$

4.1.12 Applications

In this section, we apply the Zubov's recursion scheme (4.13)–(4.16) for estimating ROA of systems of Genesio (3.2), Lorenz (3.4), Rössler (3.6), and BZ reaction (3.8) and system (3.9). In order to implement the recursive scheme, we choose the positive definite function $\varphi(x, y, z) = \lambda(x^2 + y^2 + z^2)$, where λ is a positive real scalar in each application. From our trials we have observed that for some systems, the estimations can be computed for a wide range of the values of λ. However, for some systems this can be achieved for a small range of the values of λ.

In each application, we compute finite number of terms of the following power series

$$V(x) = \sum_{i=2}^{\infty} V_i(x) = V_2(x) + V_3(x) + \cdots + V_N(x) + \cdots, \tag{4.20}$$

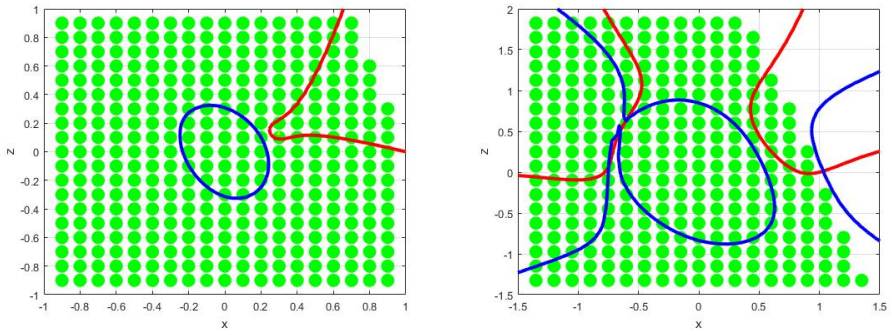
where, for each N, $V_N(x)$ is defined as

$$V_N(x, y, z) = \sum_{i=2}^{N} \sum_{j=0}^{i} \sum_{k=0}^{i-j} a_{i-j-k,j,k} x^{i-j-k} y^j z^k. \tag{4.21}$$

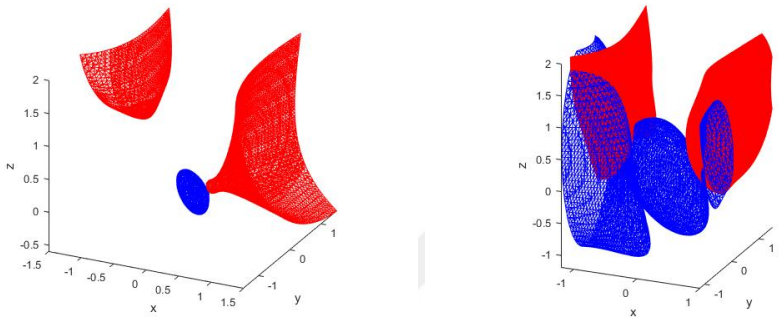
Example 1. The Zubov's PDE for the Genesio system (3.1) is

$$\frac{\partial V}{\partial x}(y) + \frac{\partial V}{\partial y}(z) + \frac{\partial V}{\partial z}(-ax - by - cz + x^2) = \varphi(x, y, z)(1 - V(x, y, z)). \tag{4.22}$$

Applying the recursion scheme (4.13)–(4.16) to (4.22) for the values of parameters $a = 1, b = 3, c = 1$, and taking $N = 6$ in (4.20) we compute estimations of the ROA for system (3.2). For many different λ values, cumbersome calculations were carried out until the degree of V has reached at most 20 by trial and error. We illustrate two of them in Figure 4.1. For $\lambda = 10$ we arrive at the sublevel set $V_6 = 0.7737$ in Figure 4.1 (a), (c), and for $\lambda = 1$ we get $V_6 = 0.6085$ in Figure 4.1 (b), (d). The blue colored surfaces represent $V_6 = c$ while the red colored regions for $\dot{V}_6 = 0$.



(a) $\lambda = 10$ (b) $\lambda = 1$



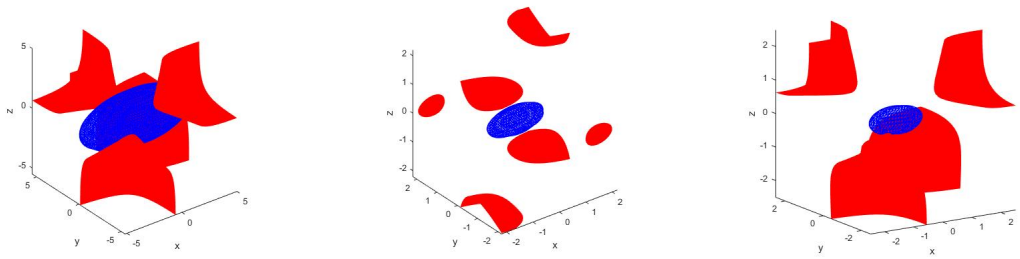
(c) $\lambda = 10$ (d) $\lambda = 1$

Figure 4.1: Genesio system ROA estimation

Example 2. The Zubov's PDE for the Lorenz system (3.3) is

$$\frac{\partial V}{\partial x}(\sigma(y-z)) + \frac{\partial V}{\partial y}(rx - y - xz) + \frac{\partial V}{\partial z}(xy - bz) = \varphi(x,y,z)(1 - V(x,y,z)). \quad (4.23)$$

For the values of parameters $\sigma = 10, r = \frac{1}{2}, b = \frac{8}{3}$ and for $N = 6$, and $\lambda = 10$, applying the recursion scheme (4.13)–(4.16) to (4.23) we obtain estimations to ROA of the origin of Lorenz system (3.4). In Figure 4.2 sublevels of (a) a second degree, (b) a third degree and (c) a sixth degree V functions, $V = 11.1576$, $V = 0.5178$ and $V = 0.3107$, respectively, are plotted.



(a) $\lambda = 10, V_{\text{deg}} = 2$ (b) $\lambda = 10, V_{\text{deg}} = 3$ (c) $\lambda = 10, V_{\text{deg}} = 6$

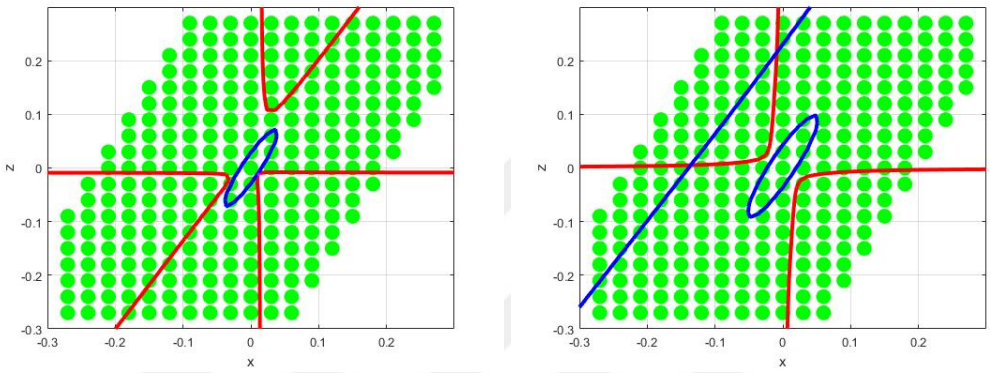
Figure 4.2: Lorenz system ROA estimation

From graphs we can conclude that the best estimation are obtained by using the second degree LF.

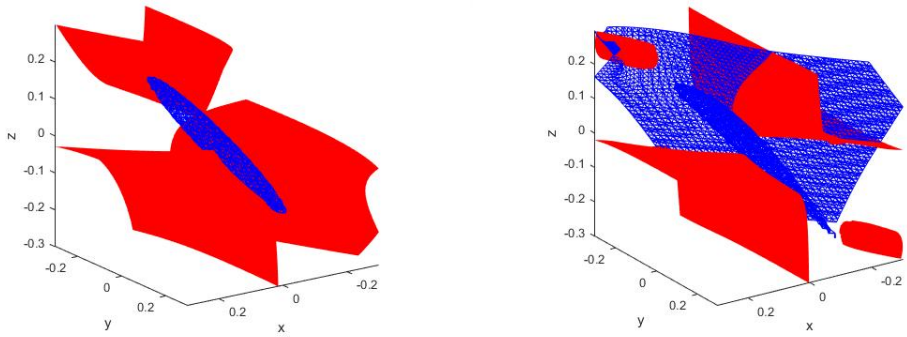
Example 3. The Zubov's PDE for the Rössler system (3.5) is

$$\frac{\partial V}{\partial x}(-y-z)+\frac{\partial V}{\partial y}(x+ay)+\frac{\partial V}{\partial z}(b+xz-cz)=\varphi(x,y,z)(1-V(x,y,z)). \quad (4.24)$$

For the values $a = 1, b = 2.22, c = 3$ and for $N = 2, 3$, and $\lambda = 5$, applications of scheme (4.13)–(4.16) yield estimations for ROA of the asymptotically stable equilibrium point E_1 of Rossler system (3.6). Two and three dimensional estimations $V_2(x) = 0.1545$ and $V_3(x) = 0.2697$, are plotted in Figure 4.3 (a), (b) in $2D$, (c), (d) in $3D$, respectively. In this example, the LF with degree three yields a better estimation than the LF with degree two.



(a) $V = 0.1545$ (b) $V = 0.2697$



(c) $V = 0.1545$ (d) $V = 0.2697$

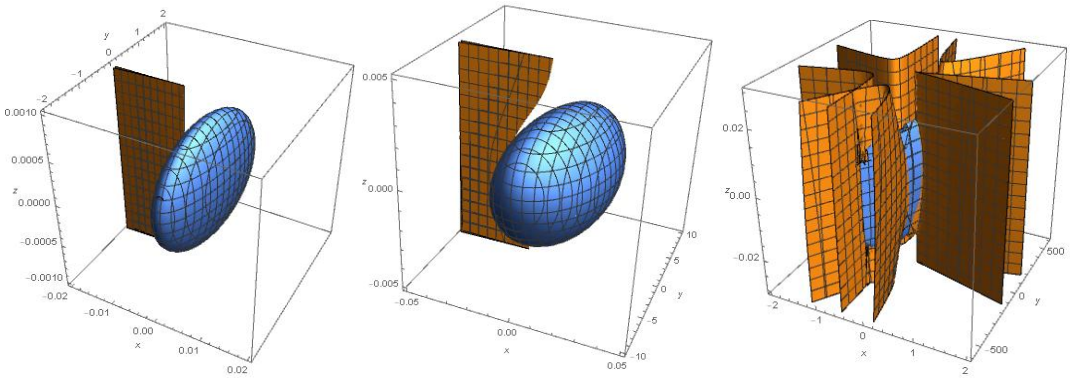
Figure 4.3: Rössler system ROA estimation

Example 4. The Zubov's PDE for the BZ reaction (3.7) is

$$\frac{\partial V}{\partial x}(\frac{q}{\epsilon}y-\frac{1}{\epsilon}xy+\frac{1}{\epsilon}x-\frac{1}{\epsilon}x^2)+\frac{\partial V}{\partial y}(-\frac{q}{\delta}y-\frac{1}{\delta}xy+\frac{1}{\delta}z)+\frac{\partial V}{\partial z}(x-z)=\varphi(x,y,z)(1-V(x,y,z)). \quad (4.25)$$

For the values of the parameters $q = 10^{-4}, f = 1/2, \epsilon = 10^{-2}, \delta = 10^{-}5$ and for $N = 2, 4$ and 6 and, $\lambda = 10$ with scheme (4.13)–(4.16) we obtain estimations to ROA of the asymptotically stable equilibrium point E_3 of BZ reaction (3.8). Figure 4.4 (a)-(c) display 3-dimensional views of the estimations for $\lambda = 10$, $V_2(x) = 2.6036 \times 10^{-4}, V_4(x) = 0.0056$

and $V_6(x) = 5.4314$, respectively (blue), and the surfaces $\dot{V} = 0$ (light brown). From figures, we can observe that, as degree of V increases from 2 to 6 we get better estimations.



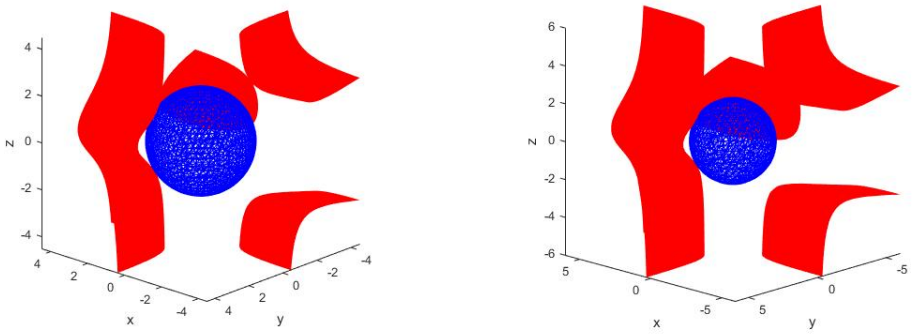
(a) $V = 2.6036 \times 10^{-4}$ (b) $V = 0.0056$ (c) $V = 5.4314$

Figure 4.4: BZ reaction ROA estimation

Example 5. The Zubov's PDE for system (3.9) is

$$\frac{\partial V}{\partial x}(-x + yz^2) + \frac{\partial V}{\partial y}(-y + xy) + \frac{\partial V}{\partial z}(-z) = \varphi(x, y, z)(1 - V(x, y, z)). \tag{4.26}$$

For $N = 6$ and $N = 10$ and $\lambda = 600$, scheme (4.13)–(4.16) yields the estimation for ROA of the origin of system (3.9), $V_6 = 5.3342 \times 10^8$ and $V_{10}(x) = 5.7782 \times 10^{13}$as depicted in Figure 4.5. From the figures we can conclude that the better estimation is obtained when degree of V is 10.



(a) (b)

Figure 4.5: HT system ROA estimation

4.2 The Groebner Bases

Groebner bases, (GB) have been used seriously for solving difficult problems in commutative algebra by mathematicians since the mid of 70's . Buchberger (1976), Robbiano (1989), Beveniste et. al.,(1989) made significant contributions in the improvement

of the GB algorithm. The algorithms used to construct GB are implemented on computers (Char et.al.1988).

The solution of a system of nonlinear algebraic equations in several variables arising in different areas is important. To solve such systems with polynomial type nonlinearities we can use a general method based on GB. We can find the solutions of these systems by using the method of GB by triangulating them. In this procedure, the minimal dimension of the solution becomes apparent which yields an equation in one variable only, if the solution is zero dimensional.

In the following, we shall give the criteria for the existence of the solution of an elimination problem:

Let $\mathfrak{k}[x_1, x_2, \ldots, x_n]$ be the polynomial ring, where $\mathfrak{k}$ is an arbitrary but, fixed field. And, let $\mathcal{I}$ be the ideal generated by a set of polynomials $\mathcal{P} = \{p_1, \ldots, p_m\}$, that is

$$\mathcal{I} = \left\langle \sum_{i=1}^{m} \gamma_i p_i \right\rangle$$

where $\gamma_i \in \mathfrak{k}[x_1, x_2, \ldots, x_n]$.

Definition 4.2.1. (Forsman and Glad (1990)) The manifold the ideal $\mathcal{I}$ is the set of solutions

$$\mathcal{Z}(\mathcal{I}) = \{x \in \mathbb{C}^n | \forall p \in \mathcal{I} : p(x) = 0\}$$

Thus a system of equations $\mathcal{P} = 0$ has a finite number of solutions iff, $\mathcal{Z}(\mathcal{P})$ is zero dimensional.

We eliminate variables one by one in the triangulation process of a system. To determine in what order, we should eliminate the variables and we use a ranking of the variables. For example the ranking

$$x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n \tag{4.27}$$

means that x_k is eliminated before x_l if $k > l$.

We have able to compare polynomials for the elimination of the variables. It can be done, if we have an ordering of the monomials of $\mathfrak{k}[x_1, x_2, \ldots, x_n]$. This ordering is achieved, if we say that a high ranking monomial should have a high degree in a high ranking variable.

A Groebner base is a generating set for an ideal. It has certain properties which are attracting algorithmically.

What makes GB so useful?

For a given set $\mathcal{P}$ of polynomials, there exists an algorithm which calculates the Groebner base for the ideal generated by $\mathcal{P}$ in a finite number of steps (Buchberger (1976)). It is implemented on computers.

Theorem 4.2.2. *(Forsman and Glad (1990)) Let $\mathfrak{G}$ be a Groebner basis for the ideal $\mathfrak{I}$ with respect to the ranking (4.27). Then the following applies:*

1. $\mathcal{Z}(\mathfrak{I}) = \emptyset \iff \mathfrak{G} = \{1\}$
2. *dim*$(\mathcal{Z}(\mathfrak{I})) = 0 \iff$ *Each variable appears alone in the highest ranking monomial of a polynomial in* $\mathfrak{G}$.
3. $\mathfrak{I} \cap \mathfrak{k}[x_1] = \{0\} \iff \mathfrak{G} \cap \mathfrak{k}[x_1] = \emptyset$

Proof. The proof can be found in (Buchberger (1970) and Pauer and Pfeifhofer (1988)) □

4.2.3 Some Structure Theorems

We say that $p \in \mathfrak{k}[x_1, x_2, \ldots, x_n]$ is "regular" with respect to x_j if lm$p \in \mathfrak{k}[x_j]$ using a plex term ordering which ranks x_j the highest. (Here lm denotes "leading monomial").

The following states how GB can be used to solve systems of polynomial equations.

Theorem 4.2.4. *(Forsman (1991)) Let $\mathfrak{I}$ be an ideal in $\mathfrak{k}[x_1, x_2, \ldots, x_n]$. Then $\dim \mathfrak{I} = 0$ if and only if there is a $p \in GB(\mathfrak{I})$ such that p is regular with respect to x_i, for all i.*

Proof. See (Gianni and Mora (1989)). □

This theorem does not say that there are n elements in the Groebner base, e.g., the set x_1^2, x_1x_2, x_2^2 in a Groebner base of a zero dimensional ideal in $\mathfrak{k}[x_1, x_2]$.

In Gianni and Mora (1989), it is also proved that generally the $(n-1)$st elements of the GB are linear in the leading variable. Thus, the generic look of a plex-Groebner base for a zero dimensional ideal with respect to $x_n \prec x_{n-1} \prec \cdots \prec x_2 \prec x_1$ is

$$\{x_1 - p_1, x_2 - p_2, \ldots, x_n - p_n\}, \tag{4.28}$$

where $p_j \in \mathfrak{k}[x_n]$ for all j, and $\deg p_n > \deg p_j$ for $j = 1, \ldots, n-1$.

Definition 4.2.5. An arbitrary ideal that has a Groebner base of type (4.28) with respect to the plex term-ordering is said to be in "generic position".

Thus, an ideal which is zero dimensional is in generic position provided that for two different zeroes $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_n)$, we have $c_n \neq d_n$.

"To be in generic position" is an important property. Since for a plex-Groebner base with respect to a term ordering, if the zero dimensional ideal is in generic property then the GB technique can be applied effectively.

Whenever the ideal is not zero dimensional other techniques like Boege, Gebouer and Kredel algorithm, (Böge et al. (1986)) or Robinovich trick, (Rabinowitsch (1929)) can be used to find the solution of the problem.

Even if the great advantage of computation of GB for an ideal generated by a finite number of polynomials is by using an algorithm in a finite number of steps with help of the computers, the main drawback of the method comes from the fact that their computational complexity is in general, very high. In this case, other techniques such as Boege, Gebouer and Kredel algotirhm, (Böge et al. (1986)), characteristic sets Chen (1988) or resultants, Hodge and Pedoe (1953) can be used.

4.2.6 Groebner Bases in Lyapunov Theory

GB have important applications in the Lyapunov stability theory. Here, we use GB technique for estimating ROA of nonlinear autonomous systems.

Definition 4.2.7. (Forsman (1991)) A polynomial $p \in \mathfrak{k}[x_1, x_2, \ldots, x_n]$ is positive if $p(0) = 0$ and $\forall x \in \mathbb{R}^n \setminus 0 : p(x) > 0$. p is non-negative if $p(0) = 0$ and $\forall x \in \mathbb{R}^n : p(x) \geq 0$. We say that p is locally positive if $p(0) = 0$ and there is a neighborhood $\mathcal{H}$ of the origin such that $\forall x \in \mathcal{H}0 : p(x) > 0$, p is locally non-negative if $p(0) = 0$ and there is a neighborhood $\mathcal{H}$ of the origin such that $\forall x \in \mathcal{H}0 : p(x) \geq 0$.

We define negative etc. analogously.

We consider a nonlinear autonomous dynamical system defined by (2.1) and assume that $x = 0$ is an asymptotically stable equilibrium point of this system. We also suppose that the components of the vector field f are polynomials in x: $f_j \in \mathfrak{k}[x_1, x_2, \ldots, x_n], \forall j$. Since f is polynomial, the solution of (2.1) is uniquely determined by an initial value $x_0 \in \mathbb{R}^n$

(Coddington and Levinson, 1985). We write $\varphi(t, x_0)$ for the solution starting at x_0; so φ is a function $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 4.2.8. (Forsman (1991)) Given a $V \in \mathbb{k}[x_1, x_2, \ldots, x_n]$ the regions $\Pi, \bar{\Pi}$ and G_c are defined by $\Pi = \{x \in \mathbb{R}^n | \dot{V} = \nabla V \cdot f < 0\} \cup \{0\} \quad \bar{\Pi} = \{x \in \mathbb{R}^n | \dot{V} = \nabla V \cdot f \leq 0\}$ $G_c = \{x \in \mathbb{R}^n | V(x) \leq c\}$ where $\dot{V} = \nabla V \cdot f = \sum f \cdot \frac{\partial V}{\partial x_i}$ is the time derivative of V along a trajectory of system (2.1). This means that the boundary ∂G_c is a level surface of V for each c.

From definition (4.2.8), we can easily see that

$$a \leq b \implies G_a \subseteq G_b \tag{4.29}$$

for $a, b \in \mathbb{R}$.

Definition 4.2.9. (Forsman (1991)) A function $V \in \mathbb{k}[x_1, x_2, \ldots, x_n]$ is a polynomial LF for system (2.1) if $\dot{V}$ is non-positive.

Definition 4.2.10. (Forsman (1991)) A function $V \in \mathbb{k}[x_1, x_2, \ldots, x_n]$ is a local LF for system (2.1) if $\dot{V}$ is a locally non-positive and $\exists c \in \mathbb{R} : G_c \subseteq \bar{\Pi}$.

If V is a local LF for (2.1) and c is such that $G_c \subseteq \bar{\Pi}$, then $x_0 \in G_c \implies \varphi(t, x_0) \in G_c, \forall t \geq 0$. In other words the set G_c is invariant if V is a local LF. In particular, if V is a LF for (2.1) then G_c is invariant for all c. G_c is not necessarily connected; make a partition

$$G_c = \bigcup_i G_c^i \tag{4.30}$$

Since φ is continuous in t we have the following result.

Theorem 4.2.11. *(Forsman (1991)) If V is a local LF for (2.1) and $G_c^i \subseteq \bar{\Pi}$. Then $x_0 \in G_c^i \implies \forall t \geq 0 : \Phi(t, x_0) \in G_c^i$.*

For a polynomial function V the partition (4.30) consists of a finite number of components. The number of its connected components of G_c depends on the value of c.

The most important application of Lyapunov theory is stability theory. In order to establish stability using LFs we need one more concept.

Definition 4.2.12. (Forsman (1991)) A function $p : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded if $p(x) \to \infty$ as $\|x\| \to \infty$.

Thus, G_c is bounded for all c if and only if V is radially unbounded.

Theorem 4.2.13. *(Forsman (1991)) If V is positive, radially unbounded, local LF for (2.1) and $G_c^0 \subset \Pi$, then $x_0 \in G_c^0 \implies \lim_{t\to\infty} \Phi(t, x_0) = 0$, where $0 \in G_c^0$. Thus the origin of system (2.1) a is locally asymptotically stable equilibrium.*

4.2.14 Determining Critical Levels

Having defined local LF and explained their connection to stability theory above, the following important problem immediately arises: How can we find a constant c in order to get G_c as large as possible, while G_c is still inside of Π ?

If $G_c \subset \Pi$, then we can guarantee stability in G_c according to Theorem 4.12, e.g., in (Chiang and Thorp (1989)) to solve such problem, an effective method is given.

Here, we shall introduce an algorithm for solving this problem using GB, in the case of polynomial nonlinearities.

To study the problem, we introduce the set $D \subseteq \mathbb{R}$:

$$D = \{c \in \mathbb{R} | G_c \subseteq \Pi\}. \tag{4.31}$$

This set is obviously the projection of $\mathcal{Z}(\langle V - c, \nabla V \cdot f\rangle)$ on the c-axis $(\mathcal{Z}(\mathcal{I}))$ is the set of real zeros of the ideal $\mathcal{I}$ in $\mathbb{R}^n$, thus it is semi-algebraic. It is a simple consequence of property (4.29) that D is connected, i.e.,consists of a single interval. In view of the property (4.29) we may reformulate the problem as: which is the smallest $c > 0$ such that

$$\partial G_c \cap \partial\Pi \neq \emptyset$$

If we view the problem as one of optimization we can formulate it as follows: we wish to minimize V subject to $\nabla V \cdot f = 0$. This observation is also made in e.g., Shields and Storey (1975).

Summing up we get n new equations while introducing one new variable, the Lagrange multiplier λ. So, we get a system of $n + 2$ equations in $n + 2$ variables:

$$V - c = 0, \quad \nabla V \cdot f = 0, \quad \nabla(\nabla V \cdot f) - \lambda \nabla V = 0,$$

i.e., we should study the ideal

$$\langle V - c, \nabla V \cdot f, \frac{\partial V}{\partial x_1} - \lambda \frac{\partial(\nabla V \cdot f)}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} - \lambda \frac{\partial(\nabla V \cdot f)}{\partial x_n} \rangle$$

in $\mathfrak{k}[x_1, x_2, \ldots, x_n, \lambda, c]$. As mentioned, we presuppose that we do not have $\nabla V \neq 0 = \nabla(V \cdot f)$ at the relevant point. It is thus sufficient that the hypersurface $V - c$ is not singular for the critical c.

A plex GB for the ideal $\mathfrak{I}$ with respect to a term-ordering of the type

$$\{x_1, x_2, \ldots, x_n, \lambda\} > c$$

gives us the contraction $\mathfrak{I} \cap \mathfrak{k}[c]$. If $\mathfrak{I} \cap \mathfrak{k}[c] \neq \{0\}$, we thus obtain a polynomial p in c only, derived from the original equations.

4.2.15 Applications

Example 6. By applying the Groebner bases method to Genesio system (3.2) by using the local LF

$$V(x) = 2.25x^2 + 3.5xy + xz + 4y^2 + 1.5yz + 1.25z^2$$

and the corresponding ideal

$$\begin{aligned} I \;=\; & \langle 2.25x^2 + 3.5xy + xz + 4y^2 + 1.5yz + 1.25z^2 - c, \\ & -x^2 + x^3 + 1.5x^2y - y^2 + 2.5x^2z - z^2, \\ & 4.5x + 3.5y + z - \lambda(-2x + 3x^2 + 3xy + 5xz), \\ & 3.5x + 8y + 1.5z - \lambda(1.5x^2 - 2y), \\ & x + 1.5y + 2.5z - \lambda(2.5x^2 - 2z)\rangle \end{aligned}$$

we get the polynomial

$$\begin{aligned} & 793317013596830016c - 5682312568021093098192c^2 \\ & +473296516411713954392041 20c^3 - 1109836128619944537408750 15c^4 \\ & -399171735948270788641820484c^5 + 170212660568541882096685 6097c^6 \\ & +310987132309460568655455 6733c^7 - 665214209921613497086952 9000c^8 \\ & +211690874181470527587925 0000c^9 \end{aligned}$$

which has three real solutions $c = 0, c = 0.943864,$ and $, c = 2.40866.$ Choice of the positive minimum value of c yields a level surface of V, that is $V = 0.943864$ as depicted in Figure 4.6.

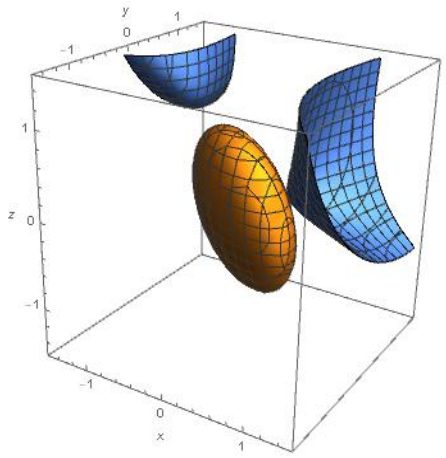


Figure 4.6: Genesio system ROA estimation

Example 7. Using the function

$$V(x) = 0.05682x^2 + 0.2727xy + 1.8636y^2 + 0.1875z^2$$

as a LF for the origin of Lorenz system (3.4) and the corresponding ideal

$$\begin{aligned} I \;=\; & \langle 0.05682x^2 + 0.2727xy + 1.8636y^2 + 0.1875z^2 - c, \\ & -1.00005x^2 + 0.0003xy - 1.0002y^2 - 0.2727x^2z - 3.3522xyz - z^2, \\ & 0.11364x + 0.2727y - \lambda(-2.0001x + 0.0003y - 0.5454xz - 3.3522yz), \\ & 0.2727x + 3.7272y - \lambda(0.0003x - 2.0004y - 3.3522xz), \\ & 0.375z - \lambda(-0.2727x^2 - 3.3522xy - 2z)\rangle \end{aligned}$$

we get the level surface, $V = 1.11574$, which is illustrated in Figure 4.7.

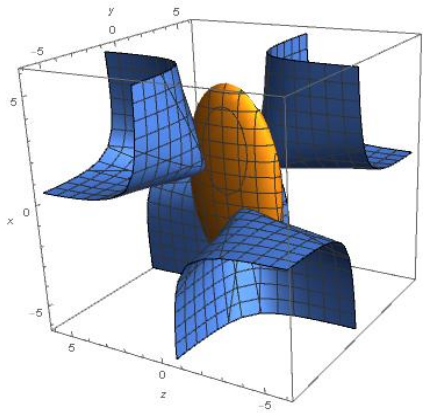


Figure 4.7: Lorenz system ROA estimation

Example 8. The local LF for Rössler system (3.6)

$$V = 114.934x^2 + 151.451xy - 114.901xz + 75.226y^2 - 54.293yz + 34.635z^2$$

and the corresponding ideal is

$$\begin{aligned} I \;=\; & \langle 114.934x^2 + 151.451xy - 114.901xz + 75.226y^2 - 54.293yz + 34.635z^2 - c, \\ & -0.99965x^2 - 0.00095xy - 0.99899y^2 - 0.00121xz - 114.901x^2z + 0.000047yz \\ & -54.293xyz - 1.001564z^2 + 69.27xz^2, 229.868x + 151.451y - 114.901z \\ & -\lambda(-1.999294x - 0.000952y - 0.0012108z - 229.802xz - 54.293yz + 69.27z^2), \\ & 151.451x + 150.452y - 54.293z - \lambda(-0.000952x - 1.99799y + 0.0000476z \\ & -54.293xz), -114.901x - 54.293y + 69.27z - \lambda(-0.0012108x - 114.901x^2 \\ & +0.0000476y - 54.293xy - 2.003128z + 138.54xz)\rangle \end{aligned}$$

Application of the Groebner bases method results the level surface $V = 0.0309072$. As it can be seen in Figure 4.8, it is very tiny region.

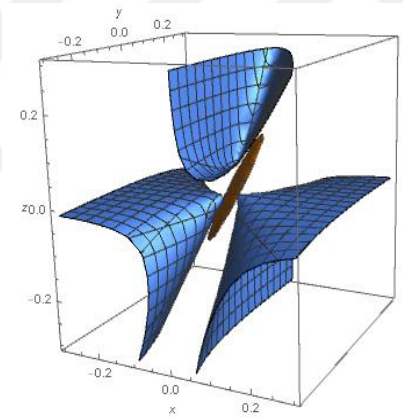


Figure 4.8: Rossler system ROA estimation

Example 9. For BZ reaction (3.8) the local LF is

$$V = 9.6058x^2 - 0.0194032xy + 0.000109638y^2 - 18.8119xz + 0.0190232yz + 480.579z^2$$

and resulting Lyapunov surface, $V = 0.0000984541$, is depicted in Figure 4.9.

Example 10. For system (3.9) the LF

$$V = 0.5x^2 + 0.5y^2 + 0.5z^2.$$

Its computed level set, $V = 2.61595$ is illustrated in Figure 4.10.

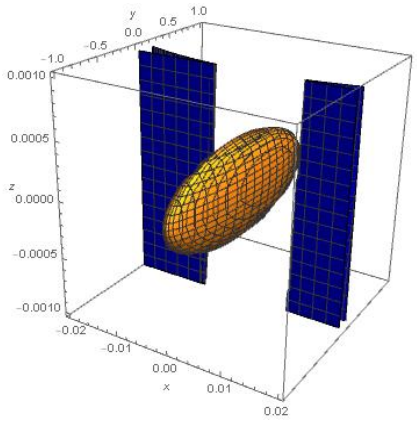


Figure 4.9: BZ reaction ROA estimation

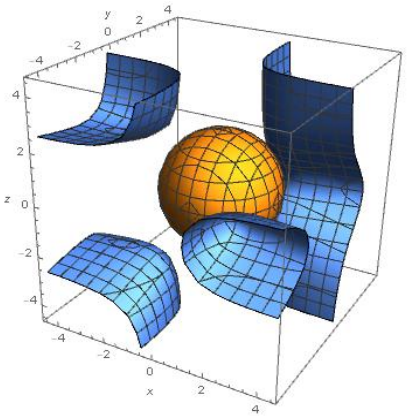


Figure 4.10: HT system ROA estimation

4.3 Sum of Squares Programming Method

The sum of squares, (SOS) programming method is applicable to nonlinear autonomous dynamical systems with polynomial vector fields. It depends upon the connections between sum of squares polynomials and positive semidefinite matrices. That connection was first, made by Parrilo, (Parrilo (2000)). Then, the researches on computational tools for estimating ROAs have been led.

The estimations of the ROA of the asymptotically stable equilibrium point can be computed using SOS programming technique. The basic problem in such a problem is to find a suitable LF which requires to test nonlinear functions for positive definiteness on a certain region of state space. The advantage of the SOS programming method is to solve the problem for polynomial functions with a restriction. In point of fact, the polynomial is checked whether a sum of square representation exists which is naturally positive semidefinite. Moreover, the LFs of degree larger than 2 can be constructed systematically using SOS programming method. Such LFs may give better estimates for the ROA when compared to

quadratic LFs.

4.3.1 Sum of squares polynomials

Let the set of all polynomials in n variables with real coefficients be denoted by $\mathfrak{R}_n$ and as a subset of $\mathfrak{R}_n$ we let $\mathfrak{R}_{n,\alpha}$ be the set of all polynomials in n variables that have maximum degree α.

Definition 4.3.2. For $x \in \mathbb{R}^n$, a polynomial $p(x)$ is referred to as a SOS polynomial if there are polynomials $s_i \in \mathfrak{R}_n, i = 1, \ldots, m$, such that $p(x)$ can be written as

$$p(x) = \sum_{i=1}^{m} s_i^2(x)$$

We denote the set of SOS polynomials by

$$\Sigma_n := \left\{ p \in \mathfrak{R}_n \, | p(x) = \sum_{i=1}^{m} s_i^2(x), \text{ for } s_i \in \mathfrak{R}_n, i = 1, \ldots, m \right\}$$

Obviously, if $p \in \Sigma_n$ then $p(x) \geq 0, \text{ for all } x \in \mathbb{R}^n$. But, the converse of this is not true, that is, there are globally nonnegative polynomials which are not SOS polynomials (see Parrilo (2003)).

We also let $\Sigma_{n,\alpha}$ be the set of all SOS polynomials in n variables that have maximum degree α.

An analogous identification of the set of SOS polynomials is as follows:

Proposition 4.3.3. *(Parrilo (2000)) A polynomial p of degree 2α is SOS if and only if there is a positive semidefinite matrix G and a vector of monomials v_n^α containing monomials in x of degree $\leq \alpha$ such that*

$$p(x) = (v_n^\alpha)^T G v_n^\alpha$$

where $v_n^\alpha = [1, x_1 \ldots, x_n, \ldots, x_n^\alpha]^T$ Here, the matrix G is called the Gram matrix.

We note that if the matrix G is positive semidefinite then p has a SOS decomposition, and hence, is nonnegative. We can find a set of affine relations in the elements of G by expanding $(v_n^\alpha)^T G v_n^\alpha$ and equating its coefficients to the corresponding monomials of $p(x)$. That matrix G is not unique. It forms an algebraic set of a linear subspace of the space of symmetric matrices. If the intersection of this affine subspace with the convex cone of positive semidefinite matrix is not equal to empty set then p is SOS. The problem of finding such intersection is a semidefinite program (SDP).

Definition 4.3.4 (Semidefinite Programming)**.** A semidefinite program has the form: Minimize a linear function of k variables $x = (x_1, \ldots, x_k)$ subject to a matrix inequality constraint.

$$\begin{cases} \text{minimize } \gamma^T x \\ \text{subject to } H(x) \succeq 0 \\ \text{where } H(x) := H_0 + x_1 H_1 + \ldots + x_k H_k. \end{cases}$$

The matrix inequality $H(x) \succeq 0$ means that the symmetric matrix $H(x)$ is positive semidefinite that is, $x^T H x$, $\forall x \mathbb{R}^m$, $H \in \mathbb{R}^{m \times m}$. The symmetric matrices $H_0, H_1, \ldots, H_k \in \mathbb{R}^{m \times m}$ and the vector $\gamma \in \mathbb{R}^k$ are given data. The vector $x \in \mathbb{R}^k$ is the decision variable and the constraint, $H_0 + x_1 H_1 + \ldots + x_k H_k \succeq 0$ is called a linear matrix inequality (LMI).

Because of the existing of the available effective algorithms and softwares it will be very beneficial to formulate problems using LMI. The problem of calculation of a matrix G certificates $p(x)$ is a SOS polynomial, and it can be recast as a SDP. This problem was observed first time by Parrilo, (Parrilo (2000)).

Theorem 4.3.5 (Parrilo (2003), Theorem 3.3)**.** *Given $p \in \mathfrak{R}_{n,2\alpha}$, find the relevant affine subspace $G_p = \{G_0 + \sum_i \theta_i G_i | \theta_i \in \mathbb{R}\}$. $p \in \sum_{n,2\alpha}$ if and only if the following LMI is feasible*

$$\begin{aligned} &\exists \theta_i \\ &\textit{subject to} \quad G_0 + \sum \theta_i G_i \succeq 0 \end{aligned}$$

Parrilo also realized and introduced the following

Theorem 4.3.6 (Parrilo (2003), §3.2)**.** *For a given finite set, of polynomials $\{p_i\}_{i=0}^s \in \mathfrak{R}_n$, if there exist $\{\sigma_i\}_{i=0}^s \in \mathbb{R}$ such that*

$$p_0 + \sum_{i=1}^{s} \sigma_i p_i \in \Sigma_n$$

Then, it is called as an LMI feasibility problem.

Semidefinite programming is a generalization of linear programming, and a useful tool in polynomial optimization. It allows one to compute approximate solutions in polynomial time. Even if, semidefinite programs are much more general than linear programs, they are not much harder to solve. They has been implemented by using MATLAB. There

is a collection of software tools for solving SDP problems: DSDP, SDPLR, CSDP, SDPT3, SDPA-M, SeDuMi, LMILAB and PENSDP.

Employing Positivstellensatz and the sum of squares decomposition, the S-procedure can be improved to get less conservative conditions.

4.3.7 The Positivstellensatz

Given sets of polynomials $\{f_1, \ldots, f_m\}$, $\{g_1, \ldots, g_k\}$ and $\{h_1, \ldots, h_t\}$ we define:

The multiplicative monomid $(\mathcal{M})$ generated by f_i's is the set of the collection of polynomials of the form $\prod_{i=1}^{m} f_i^{\tau_i}$, where each τ_i is a nonnegative integer.

The cone, $(\mathcal{P})$ generated by g_i's is

$$\mathcal{P} = \left\{ s_0 + \sum_{i=1}^{k} s_i t_i | s_i \in \Sigma_n, t_i \in \mathcal{M} \right\}$$

where $\mathcal{M}$ is the multiplicative monoid generated by g_i's. The cone is the smallest set of polynomials that includes all SOS polynomials which is closed under addition and multiplication.

The ideal, $(\mathcal{I})$ generated by h_i's is

$$\mathcal{I} = \left\{ \sum_{k=1}^{t} h_k p_k | p_k \in \mathfrak{R}_n \right\}$$

Theorem 4.3.8 (Positivstellensatz)**.** *(Bochnak et al. (1998)) Let* R *be a real closed field. Let* $(f_i)_{i=1,\ldots,m}$, $(g_j)_{j=1,\ldots,k}$ *and* $(h_l)_{l=1,\ldots,t}$ *be given finite sets of polynomials in* $\mathfrak{R}_n$*. Then the following properties are equivalent:*

(i) The set

$$\{x \in \mathrm{R}^n | f_i(x) \geq 0, i = 1, \ldots, m, g_j(x) \neq 0, j = 1, \ldots, k, h_l(x) = 0, l = 1, \ldots, t\}$$

is empty.

(ii) There exist $f \in \mathcal{M}, g \in \mathcal{P}$ *and* $h \in \mathcal{I}$ *such that* $f + g^2 + h = 0$.

Positivstellansatz, (P-satz) yields a characterization of the stability of polynomial equations and inequalities over the reals. The P-satz provides an alternative system of polynomial inequalities such that the first system has no solution if and only if the second system

has a solution. This means that if a system of polynomial inequalities cannot be satisfied, it is possible to demonstrate, or certify, its infeasibility.

Replacing the nonnegativity constraints of semidefinite program by SOS conditions we have the following SOS optimization problem:

Definition 4.3.9. (Sum of Squares Programming, (Prajna et al. (2002))) A SOS program is an optimization problem with a linear cost. On the decision variables SOS constraints minimize

$$\begin{aligned}&\text{minimize } e_1 w_1 + \ldots + e_n w_n \\ &\text{subject to } b_{m,0}(x) + b_{m,1}(x)w_1 + \ldots + b_{m,n}(x)w_n \in \Sigma_n, \quad m = 1, \ldots, K_s\end{aligned}$$

In this definition, the polynomials $b_{m,i}$ are given data, and $w_i \in \mathbb{R}^n, i = 1, \ldots, n$ are decision variables of the optimization problem.

Proposition 4.3.3 is a link for converting an SOS program into a semidefinite-programming problem. E.g., we can write the constraint $b_{m,0}(x) + b_{m,1}(x)w_1 + \ldots + b_{m,n}(x)w_n \in \Sigma_n$ in the above definition equivalently as:

$$b_{m,0}(x) + b_{m,1}(x)w_1 + \ldots + b_{m,n}(x)w_n = v_d^T G v_d \tag{4.32}$$

$$G \succeq 0 \tag{4.33}$$

Here G is a new matrix of decision variables. It is needed to introduce when an SOS constraint is converted to an LMI constraint. The linear equality constraints on the decision variables $w_i, i = 1, \ldots, n$ and G is obtained by equating the coefficients of $v_d^T G v_d$ and $b_{m,0}(x) + b_{m,1}(x)w_1 + \ldots + b_{m,n}(x)w_n$. Hence, it can be possible to write equation (4.32) as a set of linear equality constraints on the decision variables. So, we can replace all SOS constraints in SOS program in Definition 4.3.9 with linear equality constraints and LMI constraints, in this manner. As a result, the SOS program can be written in the SDP dual form. Equivalently, given an SOS polynomial $f \in \mathfrak{R}_{n,2\alpha}$; the identity $f(x) = v_d(x)^T G v_d(x)$ for all x that provides linear equations that coefficients of the matrix G must satisfy. Hence, by writing

$$v_d(x)v_d(x)^T = \sum_{\alpha \in \mathbb{N}^n} \mathbf{T}_\alpha x^\alpha$$

for appropriate $r_{v_d} \times r_{v_d}$ real symmetric matrices $(\mathbf{T}_\alpha)$, we can check whether the polynomial $x \to f(x) = \Sigma_\alpha f_\alpha x^\alpha$ is SOS. By this way, our problem is reduced to semidefinite optimization problem:

Find $G \in \mathbb{R}^{r_{v_d} \times r_{v_d}}$ such that:
$G = G^T, \quad G \succeq 0, \quad \langle G, \mathbf{T}_\alpha \rangle = f_\alpha, \quad \forall \alpha \in \mathbb{N}^n$

This conversion can be performed using available packages, such as SOSTOOLS Prajna et al. (2002), Yalmip Lofberg (2004), and SOSOPT Seiler (2013). These are freely available MATLAB toolboxes. They provide the specification of the polynomial constraints using a symbolic toolbox and then convertion of the SOS optimization into an SDP. The final problem can be solved with SeDuMi (Sturm (1999, 2001)) or another freely available SDP solver.

4.3.10 Estimating ROA

Consider a nonlinear autonomous dynamical system described by (2.1) and assume that the point $x = 0$ is an asymptotically stable equilibrium of this system. The method for estimating of ROA of the equilibrium $x = 0$ which is restricted to ellipsoidal approximations is explained as follows:

We define the shape function $p(x) = x^T E x$ and level set $\Psi = \{x \in \mathbb{R}^n | p(x) \leq \delta\}$ for a given $n \times n$ matrix $E = E^T > 0$. Here, the shape of the ellipsoid, Ψ_δ is defined by $p(x)$ and its size is determined by δ. Problem dependent choice of p reflects dimensional scaling information, and the importance of certain directions in the state space. For given shape function p, we intend to compute the largest ellipsoid Ψ_δ contained in the ROA. The size of that ellipsoid is determined by solving the optimization problem:

$$\begin{aligned} \delta^* &= \max \delta \\ &\text{subject to: } \Psi_\delta \subset \text{ ROA } (0) \end{aligned} \tag{4.34}$$

But, the problem of determination of the best ellipsoidal approximation for ROA has not been solved completely, yet. Hence, for our problem we compute lower and upper bounds for δ^* satisfying $\delta_{lb} \leq \delta^* \leq \delta^{ub}$. Whenever, the largest ellipsoid level set, defined by Equation (4.34), is approximately determined when we compute the lower and upper bounds close enough.

The upper bounds are computed by a search for initial conditions leading to divergent trajectories. But, since we want to determine estimations for the ROA we mainly concern with the convergent trajectories. Hence, depending on our purpose it is enough to compute

lower bounds.

To compute the lower bounds of δ^* we need the following lemma.

Lemma 4.3.11. *If we can find a number μ and a polynomial $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \neq 0 \tag{4.35}$$

$$\mathfrak{D}_\mu = \{x \in \mathbb{R}^n | V(x) \leq \mu\} \text{ is bounded} \tag{4.36}$$

$$\mathfrak{D}_\mu \subset \{x \in \mathbb{R}^n | \nabla V(x) \cdot f(x) < 0\} \cup \{0\} \tag{4.37}$$

then solution of (2.1) exists, and satisfies, $\Phi(t, x) \in \mathfrak{D}_\mu$, for all $x \in \mathfrak{D}_\mu$, for all $t \geq 0$, and $\mathfrak{D}_\mu \subset$ ROA(0).

A function V, satisfying the hypotheses of Lemma 4.3.11 is called a LF. It provides an estimation to the ROA. By linearizing system (2.1) about the asymptotically stable equilibrium $x = 0$ we can compute a LF: Let $J := \frac{\partial f}{\partial x}\|_{x=0}$ be the Jacobian matrix of the linearized system of (2.1) at $x = 0$. Then, by solving the Lyapunov equation $J^T P + PJ = -I$ for P, we can find $P > 0$ and hence we can construct $V_{lin}(x) := xTPx$ as a quadratic LF. This function satisfies the conditions of Lemma 4.3.11 for sufficiently small $\mu > 0$ and can be used to compute a lower bound on δ^* by solving two maximization problems:

$$\begin{aligned} \mu^* &:= \max \mu \\ &\quad \text{subject to: } \mathfrak{D}_\mu \subset \{x \in \mathbb{R}^n | \nabla V_{lin}(x) f(x) < 0\} \end{aligned} \tag{4.38}$$

$$\begin{aligned} \delta_{lb} &:= \max \delta \\ &\quad \text{subject to: } \Psi_\delta \subset \mathfrak{D}_{\mu^*} \end{aligned} \tag{4.39}$$

The largest level set of $V_{lin}, \mathfrak{D}_{\mu^*}$, satisfying the Lemma 4.3.11 can be determined from the first maximization (4.38), and the largest ellipsoid $\Psi_{\delta_{lb}}$ contained within $\mathfrak{D}_{\mu^*}$ can be computed from the second maximization.

The set containment constraints can be replaced with a sufficient condition involving nonnegative functions as given in (Tan (2006)). That sufficient condition is provided by the following Lemma, which is a generalization of the S-procedure.

Lemma 4.3.12. *(Tan (2006)) Define two sets $\mathcal{C} := \{x \in \mathbb{R}^n | f_\mathcal{C}(x) \geq 0\}$ and $\mathcal{E} := \{x \in \mathbb{R}^n | f_\mathcal{E}(x) \geq 0\}$. If there exists a function $s(x) \geq 0 \forall x$ such that $f_\mathcal{E}(x) - f_\mathcal{C}(x)s(x) \geq 0 \forall x$ then $\mathcal{C} \subseteq \mathcal{E}$.*

Here, the function s is called a multiplier. A simplification of the most general Positivstellensatz conditions is stated as the condition in Lemma 4.3.12. It needs much less computation

than the most general Positivstellensatz conditions.

Applying this Lemma to $\Psi_\delta \subset \mathfrak{D}_{\mu^*}$ in optimization (4.39) gives the optimization:

$$\begin{aligned} \delta_{lb} \quad := \quad & \max_{\delta, s(x)} \delta \\ & \text{subject to: } s(x) \geq 0 \forall x \\ & \qquad -(\delta - p(x))s(x) + (\mu^* - V_{lin}(x)) \geq 0 \forall x \end{aligned} \tag{4.40}$$

The function $s(x)$ is a decision variable of the optimization in equation (4.40), that is, its coefficients are decision variables and computed as a part of the optimization. A sufficient condition for the set containment condition in Optimization (4.39) arises from the two non-negativity conditions in Optimization (4.40). These constraints involving the non-negativity of polynomial functions can be obtained by restricting $s(x)$ to be a polynomial. Restriction of a polynomial to be SOS is a sufficient condition for the polynomial to be nonnegative. With the replacement of the non-negativity conditions in Optimization (4.40) with SOS constraints yields an SOS optimization problem:

$$\begin{aligned} \delta_{lb} \quad := \quad & \max \delta \\ & \text{subject to: } s(x) \in \Sigma_n \\ & \qquad -(\delta - p(x))s(x) + (\mu^* - V_{lin}(x)) \in \Sigma_n \end{aligned} \tag{4.41}$$

In order to solve such SOS optimization problems there is an available software.

We use $\delta_{lb,lin}$ to denote the lower bound obtained from optimization (4.41) using the quadratic LF obtained from the linearized system.

It is possible to compute better lower bounds δ_{lb} by optimizing the choice of the LF. This results a bilinear optimization problem in the LF and a multiplier function. The set containment condition in equation (4.38) gives us the following SOS constraint:

$$-(\mu - V)s_2 - (\nabla V \cdot f + l_2) \in \Sigma_n \tag{4.42}$$

where s_2 is also a SOS multiplier function and $l_2(x) = -\epsilon_2 x^T x$ where ϵ_2 is a small positive constant on the order of 10^{-6}. When we vary V and s_2 together in the term Vs_2 then this constraint yields a bilinear problem involving the unknown coefficients of V and s_2. As a result, the set containment multipliers is not a convex problem. Therefore, heuristic solution methods are needed. There are various methods for computation of better LFs, including V-s iterations (Wloszek (2003); Wloszek et al. (2003); Tan and Packard (2004); Wloszek et al. (2005)), bilinear optimization (Tan (2006)), and the use of simulation data (Topcu et al. (2007, 2008)).

In our applications we use the V-s iteration. The details of that iteration are:

We start the iteration by initializing the LF with the linearized LF, V_{lin}. We use the functions $l_1(x) = -\epsilon_2 x^T x$ and$l_2(x) = -\epsilon_2 x^T x$ where ϵ_1 and ϵ_2 are small positive constants on the order of 10^{-6} when implementing the iteration. The steps of the V-s iteration algorithm are explained as:

1. μ Step: Hold V fixed and solve for s_2 and μ^*

$$\mu^* := \max_{\mu, s_2 \in \Sigma_n} \mu \quad \text{s.t.} \quad -(\mu - V)s_2 - (\nabla V \cdot f + l_2) \in \Sigma_n$$

2. δ Step: Hold V, μ^* fixed and solve for s_1 and δ_{lb}

$$\delta_{lb} := \max_{\delta, s_1(x)} \delta \quad \text{s.t.} \quad -(\delta - p)s_1 + (\mu^* - V) \in \Sigma_n$$

3. V Step: Hold $s_1, s_2, \underline{\delta}, \mu^*$ fixed and solve for V satisfying:

$$\begin{aligned} &-(\mu^* - V)s_2 - (\nabla V \cdot f + l_2) \in \Sigma_n \\ &-(\delta - p)s_1 + (\mu^* - V) \in \Sigma_n \\ &V - l_1 \in \Sigma_n, V(0) = 0 \end{aligned}$$

4. Repeat as long as the lower bound δ_{lb} continues to increase.

The basic idea of the iteration is to avoid the bilinearity in Vs_2 by holding either s_2 or V fixed. When we achieve this in each step, the problem becomes a linear SOS optimization and can be solved using an available software. The LFs can be computed as a polynomial with degree greater than two in this iteration. We can improve the lower bound by increasing the degree of the LF even if this causes the complexity in the computation. Obviously, this results increase in the computational time grows rapidly. For this iteration we can use the simulation data to find a good initial candidate LF, V (Tan et al. (2008)).

In the V-step, a feasible solution to the LMI feasibility problem can be computed by using an interior-point solver is used. The LF, V used in the μ and δ steps will be feasible for the constraints in the V step. Therefore, the same LF that was used in the μ and δ steps can be calculated in this step. Also computation of a different V is also possible which causes the increase in the values of both μ and δ at the next iteration.

4.3.13 Applications

In this section, we present some illustrative examples to find the largest level surface of a LF for enlarging the guaranteed ROA.

Example 11. We consider the Genesio system (3.2). In order to initialize the algorithm we, first solve the Lyapunov equation and form the initial quadratic LF, $V(x) = x^T Px$ and the shape function, $p(x) = x^T Px$. This initial LF certifies the locally asymptotic stability of system (3.2). Then we apply the V-s algorithm together with trace minimization. Trace minimization serves the optimal parameters for LF. V-s algorithm is bilinear in decision variables, so it implies bisection. On the other hand, trace minimization does not depend on bisection. Hence, we apply both V-s algorithm and trace minimization taking advantage of practicality, and get

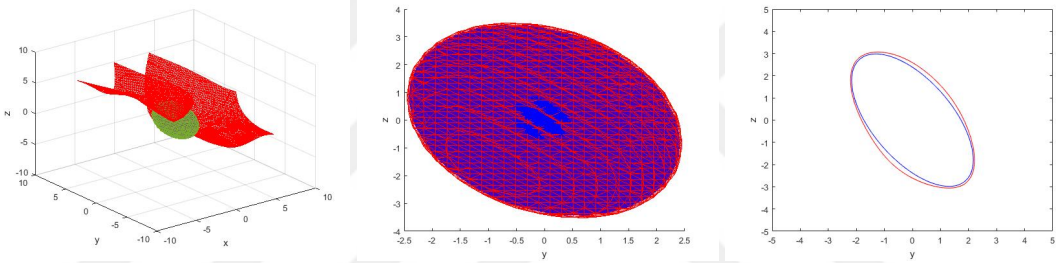
$$\begin{aligned}
V(x) &= 2.25x^2 + 3.5xy + xz + 4y^2 + 1.5yz + 1.25z^2 \\
p(x) &= 2.25x^2 + 3.5xy + xz + 4y^2 + 1.5yz + 1.25z^2 \\
\beta &= 1.8931 \\
V(x) &= 1.375398x^2 + 1.070970xy + 0.807866xz + 0.458202y^2 + 0.400794yz \\
&\quad +0.242336z^2 \\
\gamma &= 1.3716 \\
s_1(x) &= 0.7245400932904933
\end{aligned}$$

$$\begin{aligned}
s_2(x) &= 0.085491x^4 - 0.009749x^3y + 0.003377x^3z + 0.075020x^2y^2 \\
&\quad -0.007758x^2yz + 0.059941x^2z^2 + 0.006970xy^3 + 0.007981xy^2z \\
&\quad +0.00926xyz^2 + 0.023307xz^3 + 0.027119y^4 + 0.027989y^3z \\
&\quad +0.028141y^2z^2 + 0.015755yz^3 + 0.009105z^4 - 0.071682x^3 \\
&\quad -0.051889x^2y - 0.068776x^2z + 0.011240xy^2 + 0.042580xyz \\
&\quad +0.032297xz^2 - 0.001596y^3 - 0.006505y^2z - 0.005399yz^2 \\
&\quad -0.001281z^3 + 0.285952x^2 + 0.010873xy + 0.064859xz \\
&\quad +0.032129y^2 + 0.031028yz + 0.02040z^2
\end{aligned}$$

Now we refer the V function as a new shape function

$$\begin{aligned}
p(x) &= 1.375398x^2 + 1.070970xy + 0.807866xz + 0.458202y^2 \\
&\quad +0.400794yz + 0.242336z^2
\end{aligned}$$

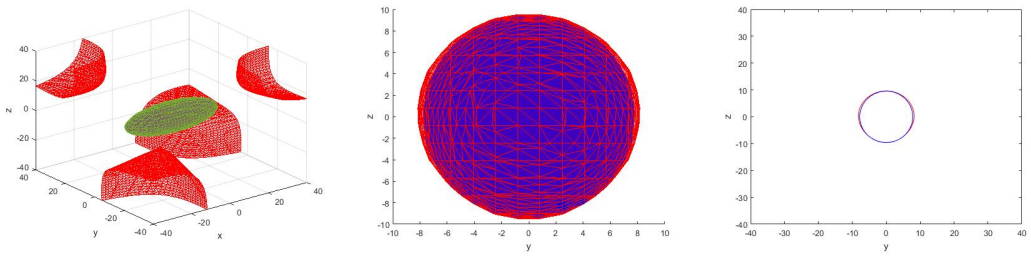
Then we search for higher degree LF, V to enrich and enlarge the estimation, as long as our optimization problem is feasible, the provable ROA. By choosing the new shape function p, we initialize the V-s iteration algorithm and at each iteration, we replace the quadratic part of the computed LF, V as a new shape function p for the next step. Setting the number of V-s iterations and trace minimization iterations as 10 and 40, respectively, we get the results of V-s iteration algorithm supressed at iteration 1 yielding $\beta = 1.000214$, $\gamma = 1.000214$ and trace minimization iterations ended at iteration 37 (at 38th iteration $V - s_1$ step is infeasible) to produce $\beta = 1.0023$, $\gamma = 1.0016$. In this example we set the bisection tolerance to $\tau = 10^{-3}$, the degree of V to 8 and positive definite l_1 and l_2 functions in V and γ step in algorithm, respectively, as $l_1 = 10^{-6}(x^2 + y^2 + z^2)$ and $l_2 = 10^{-6}(x^2 + y^2 + z^2)$. In Figure 4.11, resulting ROA is plotted (a) together with $p(x) = 1.0023$ (purple, enclosed by $V(x) = 0.0016$ (green)) and $\dot{V}(x) = 0$ (red) in 3D (b) as a 2D projection of (a) with $x = 0$ (c) as 2D contours of polynomials p and V.



(a) (b) (c)

Figure 4.11: Genesio system ROA estimation

Example 12. For system (3.4), the first initializing step of the V-s iteration ends with $\beta = 1.115417, \gamma = 1.115417$, then trace minimization ends with $\beta = 2.0412$ and $\gamma = 1.1154$. Using sixth degree LF and setting the stopping criteria of bisection to $\tau = 10^{-2}$ we arrive at the estimation parameters $\beta = 5.0933$ and $\gamma = 5.2666$ since because the maximum γ step returns infeasible at sixth iteration. However, the resulting estimations are quite close and satisfactory, can be seen in Figure 4.12.



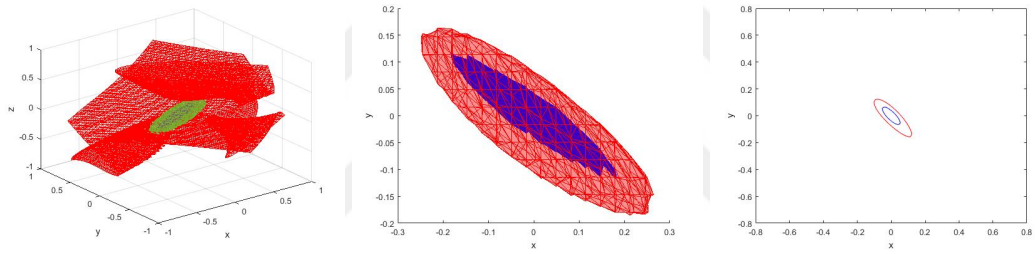
(a) (b) (c)

Figure 4.12: Lorenz system ROA estimation

Example 13. Consider system (3.6). We employed a quartic LF to estimate the ROA of this system. Resulting LF is as follows

$$\begin{aligned} V(x) \;=\; & 10.288266x^4 + 36.541077x^3y + 9.120070x^3z + 78.229269x^2y^2 \\ & +76.664737x^2yz + 18.031402x^2z^2 + 48.068389xy^3 + 98.796792xy^2z \\ & +53.763837xyz^2 + 9.601077xz^3 + 44.724080y^4 + 95.235858y^3z \\ & +82.880441y^2z^2 + 34.418163yz^3 + 6.093231z^4 + 85.739017x^3 \\ & +304.071846x^2y + 28.902033x^2z + 255.406788xy^2 + 140.792645xyz \\ & +38.878563xz^2 + 146.858683y^3 + 210.374463y^2z - -30.293369yz^2 \\ & -46.221122z^3 + 288.032890x^2 + 387.821183xy - 325.334015xz \\ & +183.600348y^2 - 178.601235yz + 111.196313z^2. \end{aligned}$$

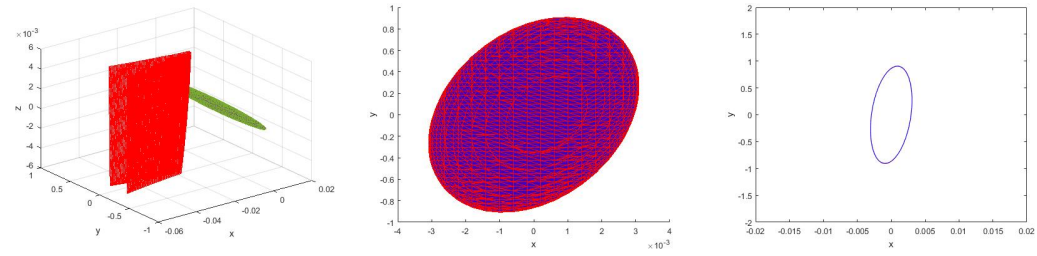
This problem was concluded in 3 iterations. Figure 4.13 shows in (a) ROA estimation of system (3.6) in 3D (b) $x = 0$ projection (c) contour of polynomials $p(x) = 1.0464$ (blue) and $V(x) = 0.9995$ (red).



(a) (b) (c)

Figure 4.13: Rossler system ROA estimation

Example 14. For the system of the BZ reaction defined in (3.8), a quadratic estimation, $V(x) = 36910.517x^2 - 74.557xy - 72285.239xz + 0.421y^2 + 73.097yz + 1846636.293z^2 = 0.3171$, is shown in Figure 4.14.



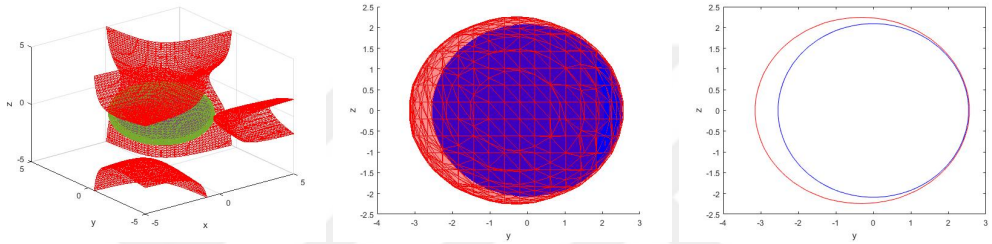
(a) (b) (c)

Figure 4.14: BZ reaction ROA estimation

Example 15. Using a quartic LF for system (3.9), the resulting ROA is illustrated in Figure 4.15, which is consistent with estimation obtained in the Zubov method, as in the all

considered systems. Resulting LF is as follows.

$$
\begin{aligned}
V(x) \;=\; & 0.016218x^4 + 0.000370x^3y + 9.9880 \times 10^{-12}x^3z + 0.033717x^2y^2 \\
& +4.3571 \times 10^{-11}x^2yz + 0.046113x^2z^2 + 0.001371xy^3 - 4.1070 \times 10^{-11}xy^2z \\
& +0.004175xyz^2 - 2.444150 \times 10^{-11}xz^3 + 0.017299y^4 + 3.475946 \times 10^{-11}y^3z \\
& +0.049876y^2z^2 + 1.3313 \times 10^{-11}yz^3 + 0.035367z^4 + 0.019058x^3 \\
& +0.019286x^2y + 4.9330 \times 10^{-11}x^2z + 0.023902xy^2 - 6.4902 \times 10^{-11}xyz \\
& +0.031783xz^2 + 0.020622y^3 - 1.0806 \times 10^{-12}y^2z + 0.033182yz^2 \\
& +1.3530 \times 10^{-11}x3^3 + 0.007656x^2 + 0.014995xy - 5.5610 \times 10^{-12}xz \\
& +0.008426y^2 - 3.1292 \times 10^{-11}yz + 0.000666z^2
\end{aligned}
$$



(a) (b) (c)

Figure 4.15: HT system ROA estimation

5. NON-LYAPUNOV METHODS

The non-Lyapunov methods are constructed on the elements of the Lyapunov stability theory, but they do not explicitly employ LFs. Such methods focus on the topological properties of ROA.

Early contributions to the non-Lyapunov methods were oriented to second order systems. They are based on a geometrical interpretation of system equations in the light of the theory of flows, (Infante and Clark (1964), Jocic (1982)), and on the tracking function approach, (Hewit and Storey (1967)), and on the prediction of the existence of limit cycles, (Wang et al. (2013)). A numerical method for second order systems was developed for determining a segment of the ROA boundary from boundary point, (Davison and Cowan (1969)). An approach given in Loparo and Blankenship (1978) permits the system solution's expression in terms of Volterra series and provides an iterative procedure for enlargement of the initial ROA estimate according to a fixed approximation error.

Then, in order to reduce conservativeness in the estimations of the ROA, backward integration method was proposed in (Genesio and Vicino (1984)). This method synthesizes the estimated boundary of ROA using a number of system trajectories calculated by backward integration, and is known as the trajectory reversing method (Genesio and Vicino (1984), Genesio et al. (1985), Stacey and Stonier (1998)). In Genesio et al. (1985), the trajectory reversing method is shown to be efficient in the enlargement procedure of the ROA of an asymptotically stable equilibrium point from an initial estimate of ROA.

Some modifications of the trajectory reversing method have been presented for a fairly large class of nonlinear time invariant systems, (Noldus and Loccufier (1995),Loccufier and Noldus (1995), Chiang et al. (1988)). E.g., in Chiang et al. (1988), it is shown that the entire ROA can be found by using the proposed method when it is feasible. This method is based on a complete characterization of the stability boundary using the stable manifolds. For low-dimensional nonlinear systems the stable manifolds may be determined by numerical methods. However, for high dimensional systems, the current computational methods are not adequate for the derivation of the stable manifolds.

The non-Lyapunov methods mainly rely on solution of differential equation from

finitely many points chosen from the boundary of the ROA. So, these methods work well for lower dimensional systems. For higher dimensional systems, the ROA can exhibit complex boundary shape and it is hard to capture the shape by just solving differential equations from finitely many points on the boundary.

In this chapter, we apply a procedure due to Stacey and Stonier (1998), trajectory reversing method in Genesio and Vicino (1984) and Genesio et al. (1985) and the method of eigenvectors in Lee and Han (1998, 2000) to Genesio system, Rossler system, Lorenz system, BZ reaction and a specific example of nonlinear autonomous systems.

5.1 Stacey and Stonier Method

Several methods which estimate the ROA are based on the backward evolution of the dynamical system. They are primarily iterative computational algorithms starting with a finite number of points close to the equilibrium and then, propagating them backwards in time. Such an approach is useful for two dimensional systems, however it is less satisfactory for higher dimensional systems. Because tracking only a finite number of points may not be sufficient to describe the behaviour of the system, accurately.

Stacey and Stonier, (Stacey and Stonier (1998)) proposed a procedure which back propagates an initial surface surrounding the asymptotically stable equilibrium point to produce an analytic estimate for the boundary of the ROA. The algorithm is started by defining the initial surface $x_0(r, s)$ parametrically and then, back propagated an arbitrary point on that surface to produce a new surface $x_n(r, s)$ defined parametrically by using a numerical method and a symbolic mathematics package, such as Mathematica or Matlab.

We note that this method can be applied to two-and three-dimensional systems.

5.1.1 Analytic Backpropagation Algorithm

The algorithm is started with a closed surface enclosing the asymptotically stable equilibrium point and contained within the ROA. The initial surface is defined parametrically. Typically, it is started with a sphere, or with an ellipsoid about the origin. Then, the entire surface is propagated backwards in time by applying a numerical scheme to an arbitrary parameter dependent point. This is accomplished by first writing Eq.(2.1) backward in time and then integrating it numerically forward in time. In the first iteration, all

of the points on the initial surface is simultaneously propagated to a new surface. We note that the functional form of each computed surface changes as the surface is pushed out and deformed by the flow of the vector field.

The algorithm is:

1. Define an initial surface

   For three-dimensional problems with the equilibrium point at the origin we use the parameterised sphere $x(r,s) = \rho cosrsins, y(r,s) = \rho sinrsins, z(r,s) = \rho coss$ where ρ is the radius of the initial surface and r and s are the parameters taking all values between 0 and 2π, and 0 and π, respectively.

2. Choose a numerical integration method

   Any numerical method can be used for backward propagation of the initial surface. The time step Δt is a parameter of the algorithm. In order to get better results it can be varied. As an output of this procedure a new parametrically defined surface is obtained.

3. Discard the small terms

   During the numerical calculations, the number of terms may exceed the capacity of the computer. To overcome this problem, the terms are removed after each iteration if they are too small.

4. Check the resulting surface whether lies within the ROA.

   The resulting surface from either Steps 2 or 3 above could possibly be outside the ROA of the system. In step 2 a time step, Δt may be chosen too large which causes the bounding surface out past the boundary of the ROA. In practice, this will only be a problem when we are iterating close to the boundary. By using an adaptive time step we can solve this problem. In step 3 discarding terms may also cause distortion of the bounding surface so that part of it lies outside the boundary of the ROA. However, if the resulting computed surface is within the ROA distortion of the bounding surface during the execution of the process is not of itself a problem.

To check that the iterated surface whether pass outside the boundary of the ROA, we check that the flow of the system given by Eq.(2.1) which is directed into the surface at all points on the new surface. This is done by observing tangent vectors to the flow on the

final surface of dotted with the outward normal vectors to the surface. If this dot product is negative at all points on the surface we conclude that the vector field is flowing in.

5.1.2 Applications

In each application of the procedure we start with a sphere with parametric equations $x(r,s) = \rho cosrsins$, $y(r,s) = \rho sinrsins$, $z(r,s) = \rho coss$ around the origin. We choose the values of the radius of the initial surface small enough so that such spheres lie entirely within the ROA.

For backward propagation of the initial surface, due to its simplicity we prefer Euler method

$$x_{n+1}(r,s) = x_n(r,s) + f(x_n(r,s))\Delta t$$

where $x_n(r,s)$ is the parametric representation of the surface after n iterations and Δt is the time step chosen for the Euler method.

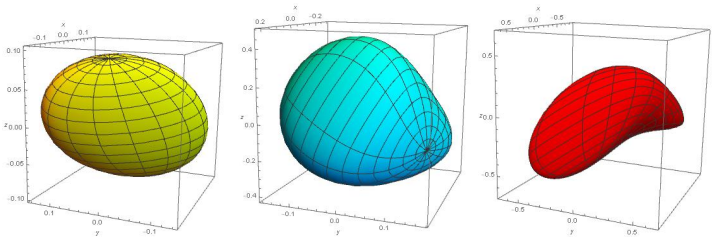
All numerical solutions are performed using mathematics package Mathematica.

Example 16. Consider the nonlinear continuous autonomous system (3.2). The origin of the system is asymptotically stable. To implement the method, we consider an initial sphere around the origin and within the ROA. Choosing a radius of sphere 0.05 the parametric equations of the sphere are $x(r,s) = 0.05cosrsins, y(r,s) = 0.05sinrsins$, $z(r,s) = 0.05coss$. Taking the time step as 0.05 evaluation of the surface is computed backward in time up to $t = 6$. After the time is evolved backwards to $t = 2$ the initial sphere was deformed to the surface:

$$\begin{aligned}
x(r,s) &= 0.0717882coss + 0.113881cosrsins + 0.0550463sinrsins \\
y(r,s) &= -0.0178131coss - 0.0762707cosrsins - 0.108484sinrsins \\
z(r,s) &= -0.0902093coss + 0.0197746cosrsins - 0.00637934cosrcosssins \\
&\quad -0.016974sinrsins - 0.00248268cos^2rsin^2s - 0.00113375cosrsinrsin^2s
\end{aligned}$$

which is plotted in Figure 5.1 (a).

At time $t = 4.5$ and $t = 6$ numerically computed surfaces are as follows, depicted in Figure 5.1 (b) and (c), respectively.



(a) $t = 2$ (b) $t = 4.5$ (c) $t = 6$

Figure 5.1: Genesio system ROA estimation

At $t = 4.5$

$$\begin{aligned}
x(r,s) &= 0.0510104coss + 0.216905cosrsins - 0.0390205cosrcosssins \\
&\quad -0.0777762sinrsins - 0.0164809cosssinrsins \\
&\quad -0.0572024cos^2rsin^2s - 0.0571138cosrsinrsin^2s \\
&\quad -0.00815452sin^2rsin^2s \\
y(r,s) &= -0.133044coss - 0.0534377cosrsins + 0.0096834cosrcosssins \\
&\quad +0.0503179sinrsins + 0.0103648cosssinrsins \\
&\quad +0.052676cos^2rsin^2s - 0.00183373cos^2rcosssin^2s \\
&\quad +0.0341007cosrsinrsin^2s + 0.0129145sin^2rsin^2s \\
&\quad -0.00178263cos^3rsin^3s - 0.0025615cos^2rsinrsin^3s \\
z(r,s) &= 0.183491coss + 0.139859cosrsins + 0.0406474cosrcosssins \\
&\quad +0.346394sinrsins + 0.0237372cosssinrsins \\
&\quad +0.00223591cos^2rsin^2s + 0.0145663cos^2rcosssin^2s \\
&\quad +0.0616832cosrsinrsin^2s + 0.00524654sin^2rsin^2s \\
&\quad +0.0143642cos^3rsin^3s + 0.0152301cos^2rsinrsin^3s
\end{aligned}$$

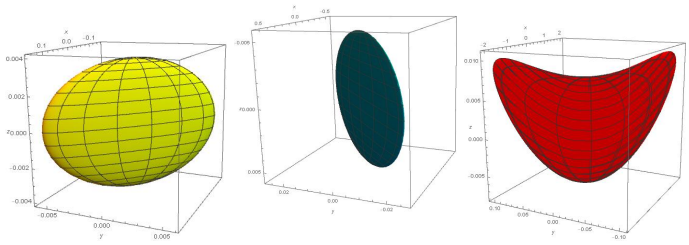
At $t = 6$

$$\begin{aligned}
x(r,s) &= 0.274046coss - 0.0052311cos^2s + 0.49758cosrsins \\
&\quad -0.055534cosrcosssins + 0.260873sinrsins \\
&\quad +0.00519544cosssinrsins - 0.126323cos^2rsin^2s \\
&\quad +0.0364016cos^2rcosssin^2s - 0.00780427cosrsinrsin^2s \\
&\quad +0.00650491cosrcosssinrsin^2s - 0.00914971sin^2rsin^2s \\
&\quad +0.0369067cos^3rsin^3s + 0.0276393cos^2rsinrsin^3s
\end{aligned}$$

$$
\begin{aligned}
y(r,s) \;=\; & -0.0181884 coss + 0.0253515 cos^2 s - 0.29171 cosrsins \\
& +0.08227 cosrcosssins - 0.00831128 cosrcos^2 ssins \\
& -0.356121 sinrsins - 0.0211264 cosssinrsins \\
& +0.0634158 cos^2 rsin^2 s - 0.0629016 cos^2 rcosssin^2 s \\
& -0.0618782 cosrsinrsin^2 s - 0.0231205 cosrcosssinrsin^2 s \\
& -0.0120256 sin^2 rsin^2 s - 0.0639169 cos^3 rsin^3 s \\
& +0.00781934 cos^3 rcosssin^3 s - 0.0418466 cos^2 rsinrsin^3 s \\
& +0.00132169 cos^2 rcosssinrsin^3 s - 0.002658 cosrsin^2 rsin^3 s \\
& +0.00639074 cos^4 rsin^4 s + 0.00998688 cos^3 rsinrsin^4 s
\end{aligned}
$$

$$
\begin{aligned}
z(r,s) \;=\; & -0.335206 coss - 0.059356 cos^2 s + 0.0225752 cosrsins \\
& -0.23166 cosrcosssins + 0.0243293 cosrcos^2 ssins \\
& -0.21411 sinrsins - 0.0644939 cosssinrsins \\
& -0.107703 cos^2 rsin^2 s + 0.0668914 cos^2 rcosssin^2 s \\
& -0.00683016 cos^2 rcos^2 ssin^2 s - 0.121078 cosrsinrsin^2 s \\
& +0.0352447 cosrcosssinrsin^2 s - 0.00476749 sin^2 rsin^2 s \\
& +0.0683014 cos^3 rsin^3 s - 0.0323654 cos^3 rcosssin^3 s \\
& +0.0335966 cos^2 rsinrsin^3 s - 0.0156848 cos^2 rcosssinrsin^3 s \\
& +0.0157036 cosrsin^2 rsin^3 s - 0.0274548 cos^4 rsin^4 s \\
& +0.00122992 cos^4 rcosssin^4 s - 0.0288154 cos^3 rsinrsin^4 s \\
& -0.00356144 cos^2 rsin^2 rsin^4 s
\end{aligned}
$$

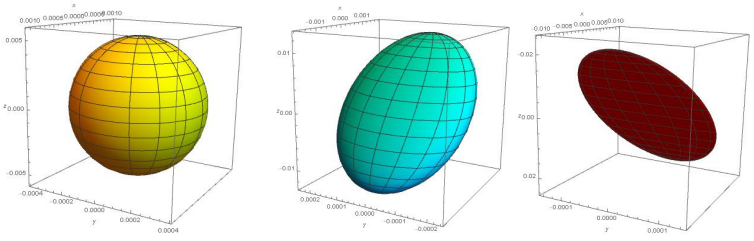
Example 17. Consider the dynamical system defined in (3.4). For this example taking $\Delta t = 0.025$, radius of sphere as $\rho = 0.001$ and computing backwards up to $t = 0.5, 0.625$, and $t = 0.75$ the results obtained are as in Figure 5.2 (a), (b) and (c), respectively.



(a) $t = 0.5$ (b) $t = 0.625$ (c) $t = 0.75$

Figure 5.2: Lorenz system ROA estimation

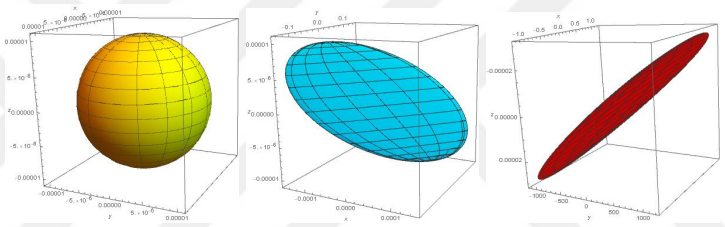
Example 18. The Rossler system (3.6) is examined using the values $\Delta t = 0.01$, $\rho = 0.001$. The progression of the initial estimate over $t = 1$, $t = 1.5$ and $t = 2$ are depicted in Figure 5.3.



(a) $t = 1$ (b) $t = 1.5$ (c) $t = 2$

Figure 5.3: Rossler system ROA estimation

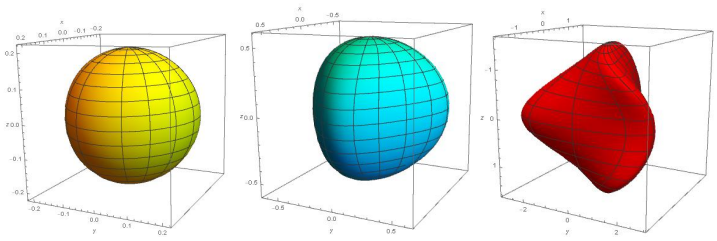
Example 19. For the system defined in (3.8), $\Delta t = 0.0025$ and $\rho = 10^{-5}$ were chosen and resulting estimates including initial estimates are illustrated in Figure 5.4.



(a) $t = 0$ (b) $t = 0.0025$ (c) $t = 0.005$

Figure 5.4: BZ reaction ROA estimation

Example 20. Finally, we consider system (3.9). Allowing the time step $\Delta t = 0.01$ and initial sphere $x(r, s) = 0.01cosrsins$, $y(r, s) = 0.01sinrsins$, $z(r, s) = 0.01coss$, the evaluation of the surface is computed backward in time for a total of 5 units. In Figure 5.5 (a), (b), and (c) the resultant surfaces are displayed for $t = 3$, $t = 4$ and $t = 5$, respectively.



(a) $t = 3$ (b) $t = 4$ (c) $t = 5$

Figure 5.5: HT system ROA estimation

5.2 Trajectory Reversing Method

The trajectory reversing method which is based on the LaSalle extension principle of the Lyapunov stability theory. It provides an iterative procedure for obtaining the global attracting region for multidimensional systems without conditions on the topological nature of the asymptotically stable point under study. The method performs a numerical backward integration of Eq. (2.1) to obtain an estimation to ROA. This is accomplished by forward integration of the related system

$$\dot{x} = -f(x) \tag{5.1}$$

System (5.1) is characterized by the same trajectories in state space of Eq. (2.1), only the arrows on the trajectories are reversed. The origin now becomes an unstable equilibrium point of Eq. (5.1).

The general formulation of the trajectory reversing method as described in Genesio and Vicino (1984) and Genesio et al. (1985) and involves the enlargment of an initial arbitrary small estimate of ROA. Sufficient conditions for such an enlargement are given by the following theorem:

Theorem 5.2.1. *If the origin is asymptotically stable equilibrium of system (2.1) with continuous f, in other words, if there exists a positive definite Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) $\Omega_0 = \{x | V(x) < M$ *is simply connected with boundary* $\gamma_0\}$

(ii) $\dot{V}(x) < 0, \forall x \in \{x | V(x) < M\}, x \neq 0$

then the ROA may be approximated by means of a convergent sequence of simply connected domains generated by the backward integration technique, starting from the initial estimate Ω_0.

Proof. (Genesio et al. (1985)) □

Since $f(x)$ is continuous for any $x \in \gamma_0$, the backward mapping

$$\int_{t_0}^{t_1} (-f(x))\, dt : x(t_0) \to x(t_1), t_1 > t_0$$

is a homeomorphism. It ensures that transformation of γ_0 onto γ_1 is one-to-one and also the domain Ω_1, which is bounded by γ_1 is simply connected (Guillemin and Pollack (1974)).

Furthermore, it is guaranteed that no point of γ_0 be taken by the backward mapping into the domain $V(x) \leq M$, from hypothesis (i). As a consequence, we obtain that

$$\Omega_1 \supset \Omega_0$$

The definition of ROA and the way in which γ_1 has been obtained to ensure that γ_1 bounds a domain entirely contained in the true ROA. Now, letting t_1 increase to $t_2, t_3, \ldots, t_i, \ldots$ we obtain simply connected nested surfaces. From the uniqueness of the solutions of (2.1), these surfaces are such that

$$\Omega_{i+1} \supset \Omega_i$$

They define domains entirely lying in the exact ROA. Ω_i approximates the ROA boundary, in the sense that along every direction from the origin, every point of the ROA boundary is the limit of the successions made of points on the boundaries of Ω_i. If the ROA is unbounded, there is a succession of points on the boundaries of Ω_i tending to infinity. So, we have the result.

The Theorem 5.2.1 implies that the ROA of an asymptotically stable equilibrium point must necessarily be simply connected. Infact, this result confirms a theorem,Wilson (1967), Bhatia and Szego (1970), which states that the ROA of system (2.1) is homeomorphic to $\mathbb{R}^n$.

Using the trajectory reversing method, the ROA (or its boundary) is approximated by a sequence of estimates consisting of certain surfaces around the asymptotically stable equilibrium point. Starting from an initial estimate Ω_0 inside the true stability region, we perform a backward integration by choosing points randomly on the surface γ_0 and obtain a new estimate Ω_1. If γ_0 denote boundary surface of Ω_0, the backward integration maps the points of γ_0 along the trajectories of the system into a new surface γ_1 which bounds the new estimate Ω_1. By repeating this process we obtain a set of new estimates $\{\Omega_k\}$ with boundaries $\{\gamma_k\}$ which forms a strictly monotonically increasing sequence. As it is proved in Guttalu and Flashner (1988) the sequence $\{\gamma_k\}$ converges to the boundary surface of the true ROA.

The algorithm involves the following steps:

(1) We compute Jacobian matrix at the asymptotically stable equilibrium point assuming the origin is such a point.

(2) We compute the positive definite matrix P by solving the Lyapunov equation

$$J^T P + PJ = -I \tag{5.2}$$

(3) We form the LF $V(x) = x^T Px$ for the linearized system about the equilibrium point.

(4) We solve the optimization problem

$$c = \min V(x) \text{ subject to } \dot{V}(x) = 0 \tag{5.3}$$

(5) We form an ellipsoid $V(x) = c$ enclosing the equilibruim, which is an initial estimation for ROA.

(6) We pick up points randomly on that surface close to the equilibruim point.

(7) We perform the forward numerical integration of the backward system (5.1) using those random points as initial points and obtain a new estimation for the ROA.

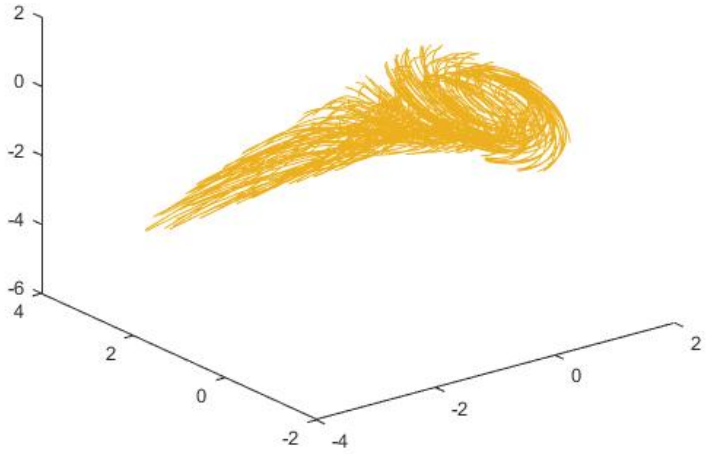
(8) We repeat the last step to find the better estimation of the ROA.

5.2.2 Applications

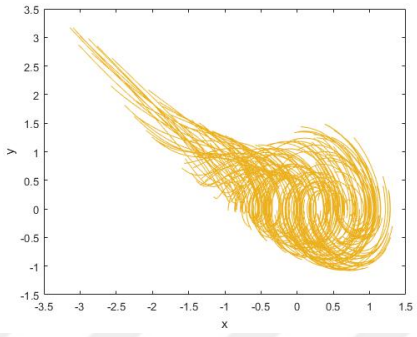
Example 21. The origin is an asymptotically stable point of the Genesio system (3.2). We apply the above algorithm and find the LF for the linearized system

$$V = x^T Px = 2.25x^2 + 3.5xy + xz + 4y^2 + 1.5yz + 1.25z^2$$

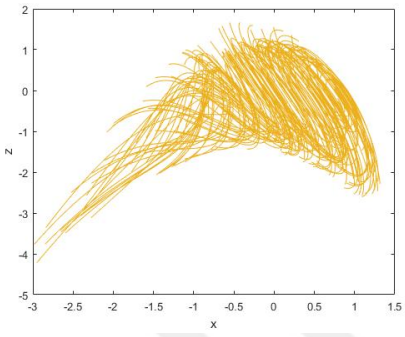
where P is a positive definite matrix computed from the Lyapunov equation (5.2). Solution of the optimization problem (5.3) yields $c = 0.9439$. Hence, the initial estimation is obtained as $V(x) = 0.9439$. Then forward and backward integration of the Genesio system with randomly chosen 200 initial points yields the estimation at $t = 2$ as shown in Figure 5.6.



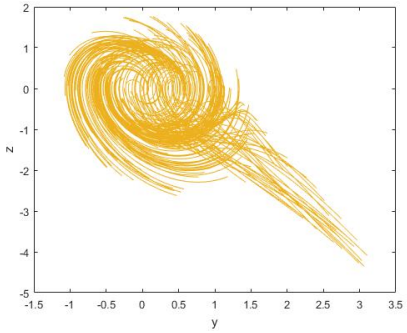
(a) 3D view



(b) 2D projection, $z = 0$



(c) 2D projection, $y = 0$



(d) 2D projection, $x = 0$

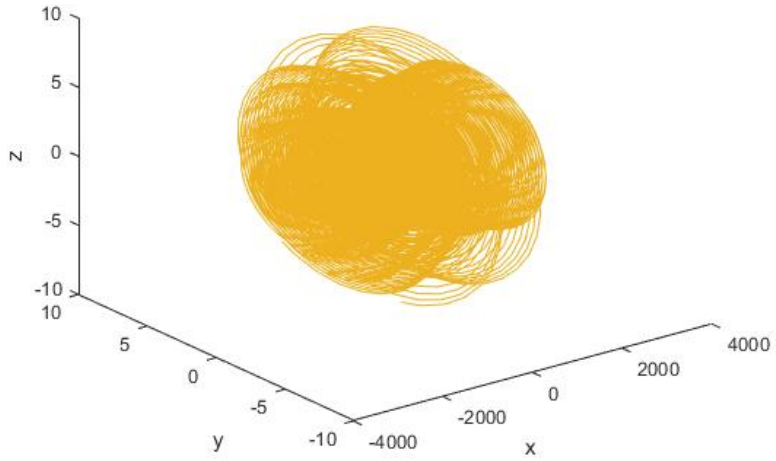
Figure 5.6: Genesio system ROA estimation

Example 22. The origin is a globally asymptotically stable point of the Lorenz system (3.4). Applying the above algorithm we find that

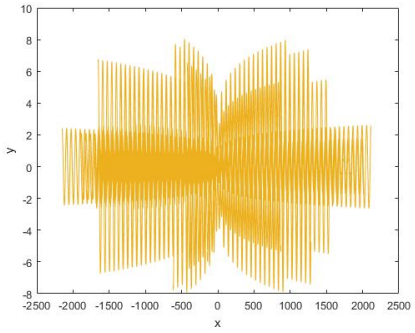
$$V = 0.056818x^2 + 0.272727xy + 1.863636y^2 + 0.1875z^2$$

The solution of the optimization problem (5.3) yields $c = 1.1158$. Then choosing 30 random

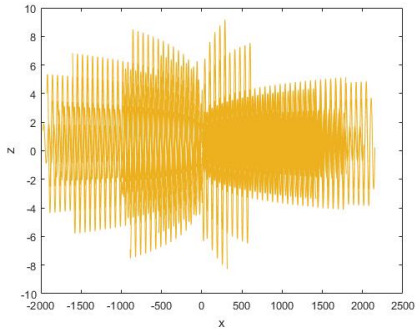
initial points on the ellipsoid, $V(x) = 1.1158$, backward integration of the Lorenz system yields the estimation at $t = 0.6$ as shown in Figure 5.7.



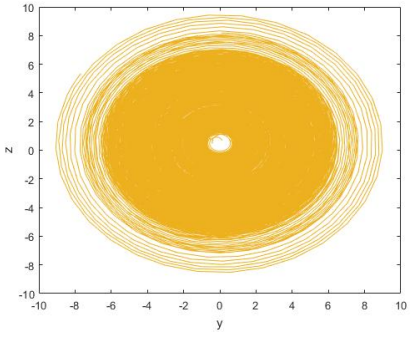
(a) 3D view



(b) 2D projection, $z = 0$



(c) 2D projection, $y = 0$



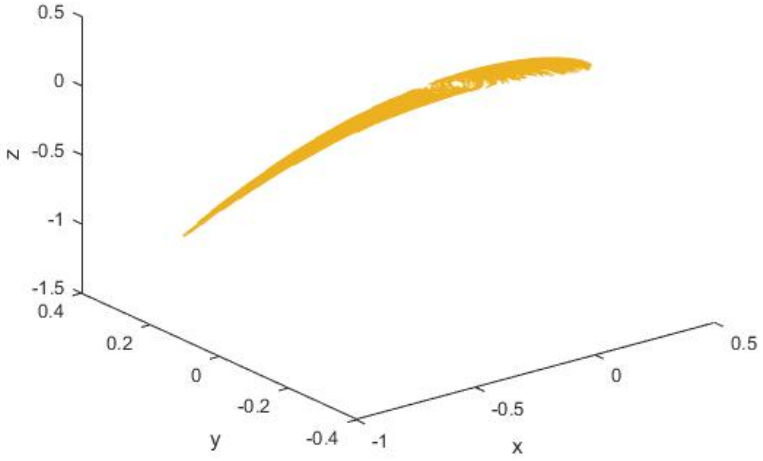
(d) 2D projection, $x = 0$

Figure 5.7: Lorenz system ROA estimation

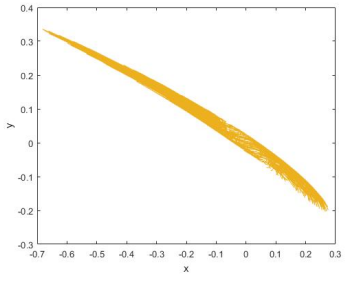
Example 23. E_1 is an asymptotically stable point of the Rossler system (3.6). We first, shift that point to the origin by an appropriate coordinate transformation. Then applying the algorithm, we compute a LF,

$$V(x) = 114.9335x^2 + 151.4512xy - 114.9014xz + 75.2256y^2 - 54.2926yz + 34.6346z^2$$

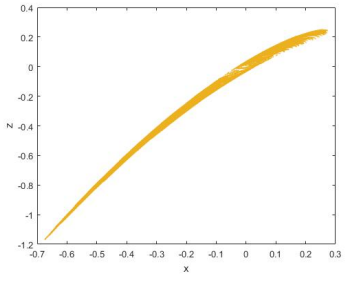
and an ellipsoid that is used to pick initial points $V(x) = 0.0309$ yields the estimation at $t = 2.4$ as displayed in Figure 5.8 with 3D and 2D views.



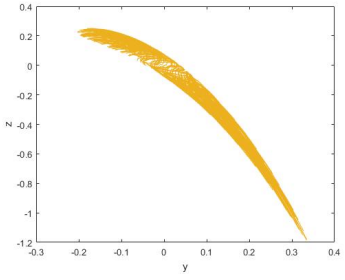
(a) 3D view



(b) 2D projection, $z = 0$



(c) 2D projection, $y = 0$

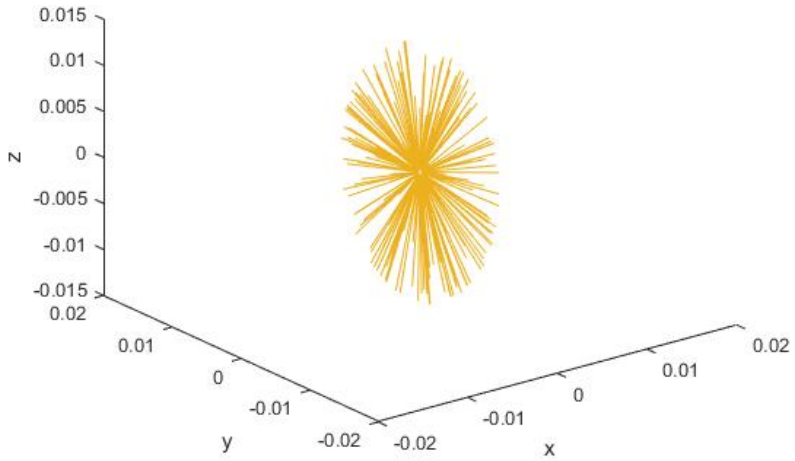


(d) 2D projection, $x = 0$

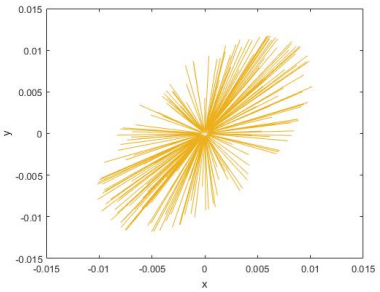
Figure 5.8: Rossler system ROA estimation

Example 24. The origin is a globally asymptotically stable equilibrium point of system

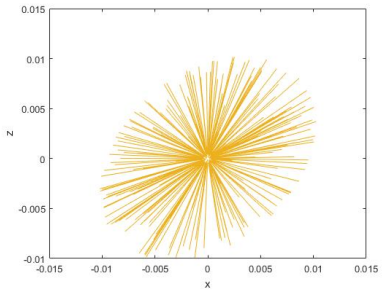
(3.9). Solving the Lyapunov equation (5.2) for this system by taking $Q = I$ we get $V(x) = 0.5(x^2+y^2+z^2)$ which has local minimum at $c \approx 2.45938$. Integrating backwards, at $t = 3$, we obtain an estimation of ROA, as shown in Figure 5.9.



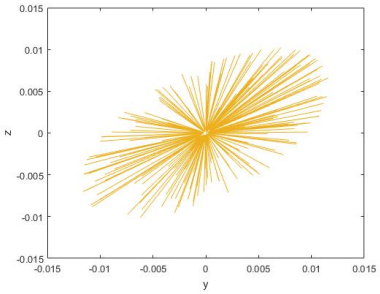
(a) 3D view



(b) 2D projection, $z = 0$



(c) 2D projection, $y = 0$



(d) 2D projection, $x = 0$

Figure 5.9: HT system ROA estimation

5.3 Estimation of ROA using Eigenvectors

The estimations of ROAs can be determined by using the eigenvectors, (Lee and Han (2000)). As an extension of the trajectory reversing method, this approach aims to find an accurate boundary of ROA. To achieve this, given system is first linearized at an unstable equilibrium point. Then the trajectories are depicted starting from a chosen set of initial points very near the unstable equilibrium point such as a saddle point and on the surface spanned by the stable right eigenvector of the linearization around that point. Finally, forward and backward integration methods are used to calculate the trajectories which form the boundary of ROA approximately with a computer.

5.3.1 Saddle Points and the Boundary of ROA

We consider the nonlinear autonomous system described by (2.1) in 3-dimensional space, where $x \in \mathbb{R}^3$ is the state vector, and $f : \mathbb{R}^3 \to \mathbb{R}^3$. We assume that f is sufficiently smooth so that system (2.1) has a unique solution. Linearizing the system at an equilibrium point, x^*, that is, $f(x^*) = 0$, we get

$$\dot{z} = Jz \tag{5.4}$$

where $z = x - x^*$ and $J = \left(\frac{\partial f_i}{\partial x}\right)_{x=x^*}, \quad i = 1, 2, 3$, is a Jacobian matrix, evaluated at $x = x^*$.

For a hyperbolic equilibrium point, the Hartman-Grobman theorem in Nayfeh and Balachandran (1995) says that the equilibrium point $x = x^*$ of a nonlinear system is asymptotically stable (unstable) if the equilibrium point $z = 0$ of the linearized system is asymptotically stable (unstable). In other words, the local nonlinear dynamic characteristics near a hyperbolic equilibrium $x = x^*$ are qualitatively similar to the linear dynamic characteristics near $z = 0$ since a hyperbolic equilibrium point is isolated and remains same under the perturbation of the vector field.

We let λ_{n_i} and $\lambda_{p_i}, (i = 1, 2, 3)$ be the eigenvalues of the Jacobian matrix J for system (5.4) with negative and positive real parts, respectively. We also let v_{n_i} and $v_{p_i}, (i = 1, 2, 3)$ be the corresponding eigenvectors for λ_{n_i} and λ_{p_i} respectively. The equilibrium point is called "type 1-saddle" if the eigenvalues of matrix J have one positive and two negative real parts, and,"type-2" if they have one-negative and two-positive real parts. If the equilibrium is of type-1 then the eigenvectors v_{n_1} and v_{n_2} corresponding to the eigenvalues with

negative real parts span the stable eigenspace. Each trajectory starting from the points on this plane will stay there and converge to the saddle point, as time goes to infinity. Therefore, these trajectories form a boundary of ROA. If the equilibrium is of type 2, then the number of stable eigenvalue is one. So, only one trajectory will converge to the saddle pointi and it cannot form a boundary of ROA in a 3-dimensional space, however there is also an unstable subspace due to the saddle point of this type (Genesio et al. (1985)).

5.3.2 Eigenvalues and Eigenvectors of Type-1 Saddle Points and the Boundary of ROA

We consider system (5.4) and assume that at $x = x^*$. Then v_i^r and v_i^l are the column and row vectors, respectively, and satisfy

$$Jv_i^r = \lambda_i v_i^r \tag{5.5}$$

$$v_i^l J = \lambda_i v_i^l \tag{5.6}$$

where $i = 1, 2, 3$. If $\lambda_i, i = 1, 2, 3$ is the eigenvalue of J, and v_i^r and v_i^l are the right and left eigenvectors corresponding to the eigenvalue λ_i, respectively. Let $v^r = [v_1^r, \ldots, v_n^r]$ be the right eigenmatrix of J, and let $v^l = \left[(v_1^l)^T, \ldots, (v_n^l)^T\right]^T$ be the left eigenmatrix of J. Using these we can rewrite Eqs. (5.5) and (5.6) as

$$Jv^r = v^r D \tag{5.7}$$

$$v^l J = D v^l \tag{5.8}$$

where D is a diagonal matrix if J has different eigenvalues, if not, it is a Jordan matrix. From (5.7) and (5.8) we get

$$Jv^r = v^r D \implies (v^r)^{-1} J = D(v^r)^{-1} \tag{5.9}$$

Comparison of (5.7) and (589) with (5.9), gives

$$v^l = (v^r)^{-1} \implies v^l v^r = I \implies \begin{cases} v_i^l v^r = 1, & i = j \\ v_i^l v^r = 0, & i \neq j \end{cases} \tag{5.10}$$

Suppose that the eigenspace including the point $z = 0$ has the normal vector v^T. Then we can show that

$$v^T z = 0 \tag{5.11}$$

Since all the states on the eigenspace must satisfy the linearized equation (5.4), by differentiating (5.11), we obtain

$$\frac{d}{dt}\left(v^T z\right) = 0 \implies v^T\left(\frac{dz}{dt}\right) = 0 \implies v^T J z = 0 \tag{5.12}$$

Considering (5.10) and letting Eq.(5.12) be equal to Eq.(5.11), here μ is a constant, it can be shown that if we want to satisfy both (5.11) and (5.12) at the same time, then

$$v^T J y = \mu v^T y \implies v^T J = \mu v^T \tag{5.13}$$

(5.13) shows that v^T which is related to the positive eigenvalue of J must be the left eigenvector of J and the plane with v^T is in the normal vector of the plane which is spanned by the two right eigenvectors that are related to the negative values.

5.3.3 The Procedure used for Estimation of the Boundaries of ROA

The algorithm is:

Step 1. We compute all equilibrium points by solving $f(x) = 0$.

Step 2. We compute the eigenvalues of the linearized nonlinear system, and the corresponding eigenvectors and also, the associated eigenplanes at each saddle point.

Step 3. We select a set of initial points on a small circle around the saddle in each of the converging planes, and in the neighborhood of the saddle points.

Step 4. We perform the forward integrations of the backwarded system equations by using all the initial points defined in the previous step. Then we plot all the reversing trajectories to estimate the boundary of ROA.

5.3.4 Applications

We apply the above algorithm to Genesio and Rossler systems and BZ reaction since they have type-1 saddle points.

Example 25. We apply the above algorithm to the Genesio system (3.2). The origin is an asymptotically stable focus and $(1, 0, 0)$ is a type-1 saddle point for these values. The eigenvalues of the Jacobian matrix at $(1, 0, 0)$ and corresponding right and left eigenvectors

are

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -1 \end{pmatrix}$$

$\lambda_1 = 0.2956$, $\lambda_2 = -0.6478 + 1.7214i$, $\lambda_3 = -0.6478 - 1.7214i$ and $v_1^r = (-0.9556, -0.2825, -0.0835)$, $v_2^r = (-0.1890 + 0.1657i, -0.1628 - 0.4327i, 0.8503)$, $v_3^r = (-0.1890 - 0.1657i, -0.1628 + 0.4327i, 0.8503)$ and $v_1^l = (-0.9002, -0.3448, -0.2661)$, $v_2^l = (-0.2565 - 0.0409i, 0.8393, 0.0957 + 0.4680i)$, $v_3^l = (-0.2565 + 0.0409i, 0.8393, 0.0957 - 0.4680i)$, respectively. The vectors v_1^2 and v_2^2 span the stable eigenspace of the linearized system at $(1, 0, 0)$ which is a plane and v_1^l spans the unstable eigenspace which is a line. That line is the normal line and its direction vector is in the direction of the outer normal of the stable eigenspace. Choosing randomly points in a circle lying in this plane we perform forward and backward integration of system (3.2). The computed estimation are depicted in Figure 5.10 for $t = 9$.

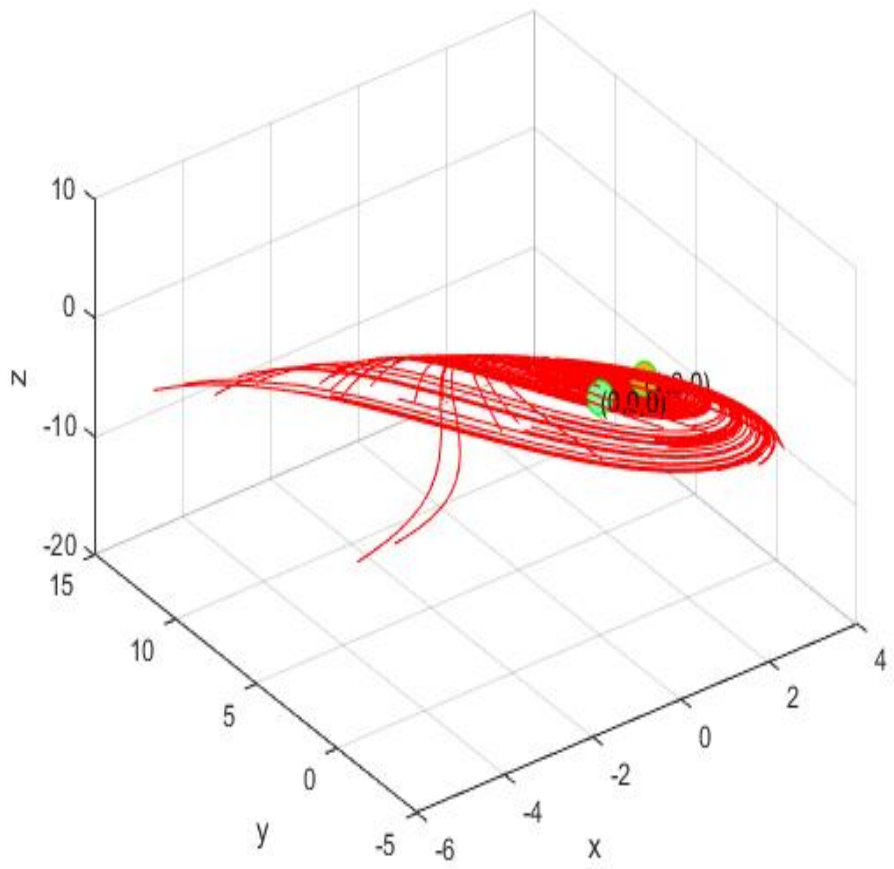


Figure 5.10: Genesio system ROA estimation

Example 26. For system (3.6) we consider the case that asymptotic stable equilibrium point

shifted to the origin. Then E_2 is a type-1 saddle point. The eigenvalues of the Jacobian matrix at $E_2 = (0.3464, -0.3464, 0.3464)$ and corresponding right and left eigenvectors are

$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 1.6732 & 0 & -1.3268 \end{pmatrix}$$

$-0.2806 + 1.1829i$, $-0.2806 - 1.1829i$, 0.2344 and $v_1^r = (0.4231 + 0.4784i, 0.0079 - 0.3663i, 0.6767)$, $v_2^r = (0.4231 - 0.4784i, 0.0079 + 0.3663i, 0.6767)$, $v_3^r = (0.5093, -0.6653, 0.5459)$ and $v_1^l = (0.7603, 0.3204 - 0.2959i, -0.3190 - 0.3606i)$, $v_2^l = (0.7603, 0.3204 + 0.2959i, -0.3190 + 0.3606i)$, $v_3^l = (-0.5665, -0.7399, 0.3629)$, respectively. Solving the system with initial conditions chosen within a small neighbourhood of the saddle point on stable plane, desired estimation result is achieved as shown in Figure 5.11.

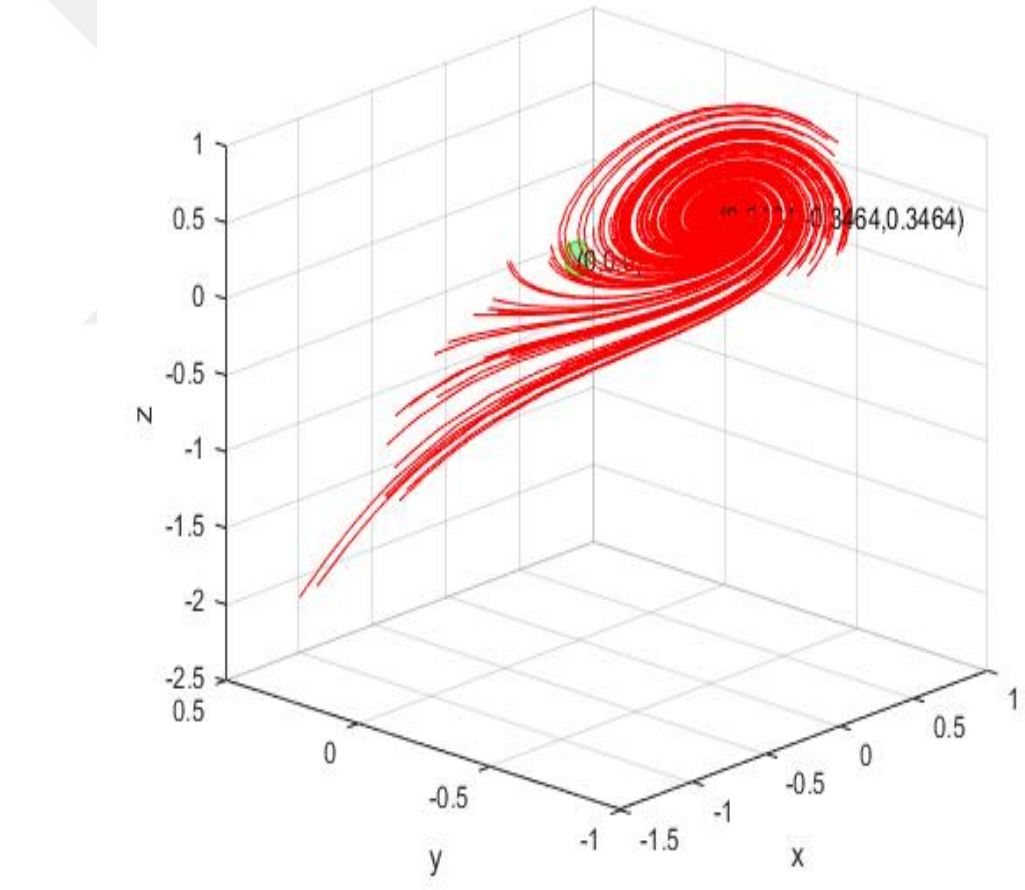


Figure 5.11: Rossler system ROA estimation

Example 27. Finally, we consider system (3.8). We first shift the asymptotically stable point $E_3 = (0.5002, 0.4999, 0.5002)$ to the origin and then apply the above algorithm. The Jacobian matrix at a new saddle point shifted, $(-0.5002, -0.4999, -0.5002)$, and the right and left eigenvectors of the Jacobian matrix are computed as

$$J = \begin{pmatrix} 100 & 0.00998801 & 0 \\ 0.00599396 & -10.012 & 50000 \\ 1 & 0 & -1 \end{pmatrix}.$$

$v_1^r = (-0.2171, -0.9761, -0.0021)$, $v_2^r = (-0.0001, 1, 0.0002)$, $v_3^r = (0.0001, -1, 0)$ and $v_1^l = (-0.9990, -0.0001, -0.0449)$, $v_2^l = (-0.0098, 0, 1)$, $v_3^l = (-0.0091, -0.0002, 1)$, respectively. Solution of the system with the initial points chosen in the vicinity of that point yields the result in Figure 5.12.

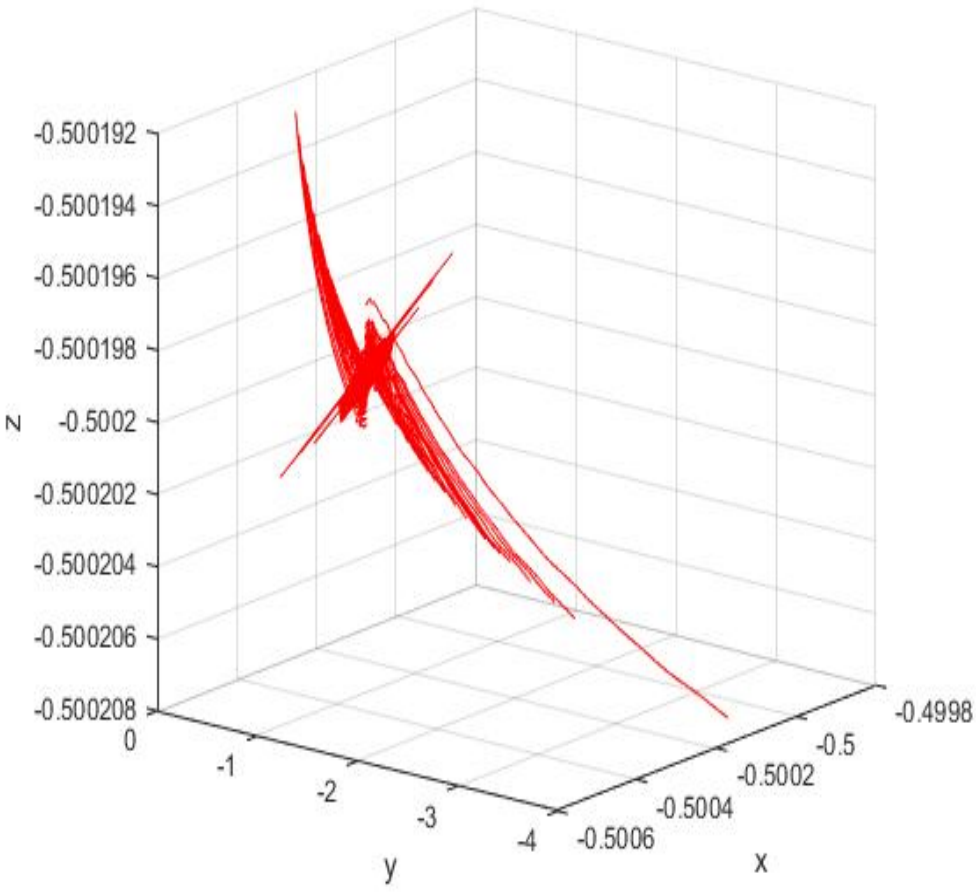


Figure 5.12: BZ reaction ROA estimation

6. CONCLUSION AND RECOMMENDATION

Throughout the thesis we apply Lyapunov and non-Lyapunov techniques for determining the estimations for the ROAs of the specific examples of polynomial nonlinear autonomous dynamical systems. When we use Lyapunov methods we obtain subsets of the ROA. But, when we apply non-Lyapunov method, by performing backward evaluation of the system we get the pictures of the trajectories lying both in the ROA and its boundary, so are estimations for ROA and its boundary.

As the future study, we intend to implement a method proposed in Chiang et.al.(1988). That method, when feasible, finds the exact ROA, rather than a subset of it. Even if, the algorithms in this method are rather complicated we plan to implement them. By this way we expect to determine the real pictures of the ROAs of some nonlinear systems and also to make some perturbations to the related theory.

Moreover we will study to make some perturbations to the related theory.

7. RESULTS AND DISCUSSIONS

As the first example of the Lyapunov methods we apply the Zubov method to the worked systems. In each application, we choose the positive definite function $\varphi(x) = \lambda(x^2 + y^2 + z^2)$ where λ is a positive scalar. In theory the choice of that function does not affect the exact solution of the Zubov PDE. But, in applications different values of λ result different estimations. In our applications, we have observed that for some values of λ numerical calculations yield good estimations for the ROA but for some values, the numerical solutions may not be possible. The most important result obtained from our experiences is that, some of the worked systems are extremely sensitive to the value of λ. For these systems estimations of ROA can be computed for a small range of this scalar. But, for others, the Zubov procedure yields estimations of ROA for wide range of λ. Of course, the degree of the LF, that is the number of terms taken from the power series of LF affects the estimations. When we increase the degree of LFs up to some value, we get better estimates. But, when we continue to increase the degree of LF from that value we can not improve the estimations, on the contrary, the resulting estimations are smaller than the previous estimations. This is, due to the nonmonotonic convergence nature of the Zubov procedure.

Groebner bases method is used as the second example of Lyapunov methods. In these applications we can find solutions to our problems using only quadratic LFs. For higher degree LFs Groebner bases method did not yield any solutions. Using this method we compute most conservative estimations for ROA in three of the Lyapunov methods.

In the applications of the SOS programming technique, the estimations and their enlargements are obtained by using V-s algorithm together with trace minimization algorithm we obtain enlarged estimations for ROA that are provable regions. This method yields the best estimations in all three Lyapunov methods. However, these three techniques give similar estimated regions for the ROA.

In the applications of the non-Lyapunov methods, we study Stacey- Stonier procedure as the first example. Since the idea in this procedure lies in backward propagation of a parametrically defined initial surface (a closed surface) to a new parametrically defined surface it results conservative estimations. In each example, we choose a sphere with different radius enclosing the asymptotically stable equilibrium as an initial closed surface. Also, in

numerical integration of the backwarded systems we use Euler method, the time increments, Δt are taken with different values. The final estimations are computed as the parametrically defined surface for each system. In each example, we perform different number of iterations depending on the properties of ROA. For example, the final estimations for Genesio and Lorenz systems are obtained after 125 and 200 iterations, respectively. On the other hand the final estimations for Rossler and system (3.9) are obtained after 30 and 50 iterations, respectively. The worst result is obtained for BZ reaction. We can perform only two steps for the final estimation using Euler method. To use another numerical integration method as Runge- Kutta method may result better estimations.

Trajectory reversing method is the second example of the non- Laypunov methods. The idea in this method depends on the backward propagation of the randomly chosen points on an initial surface by integrating numerically in forward time. It results better estimations than the previous methods. All applications yield estimations of the boundary of ROA also.

As the last example of the non-Lyapunov methods, we use the procedure involving the calculations of the eigenvectors of the linearization of the system (2.1) around the type-1 saddle. Since for the determination of the stability boundary of the worked system existence of a type-1 saddle point is necessary we can apply this method to our systems having such a point. That is, we apply the method to Genesio system, Rossler system and BZ reaction. In each application, the presence of the type-1 saddle and the part of the stability boundary can be seen clearly from figures.

10. REFERENCES

Barbashin E (1951) “The method of sections in the theory of dynamical systems”, Mat. Sb. (N.S.), 29(71):233–280.

Barbashin E and Krasovskii N (1952) “On stability of motion in the large”, (Russion) Doklady Akad. Nauk SSSR (N.S.), 86:453–456.

Bhatia N and Szego G (1970) Stability Theory of Dynamical Systems, Springer Verlag, New York.

Bochnak J, Coste M, and Roy M (1998) Real Algebraic Geometry, Springer Verlag, Berlin.

Buchberger B (1970) “An algorithmical criterion for the solvability of algebraic systems of equations”, Aequ. Math., 4:374–382.

Buchberger B (1976) “Some properties of gröbner-base for polynomial ideal”, ACM-SIGSAM Bull., 10(4):19–24.

Böge W, Gebauer R, and Kredel H (1986) “Some examples for solving systems of algebraic equations by calculating gröbner bases”, J. Symb. Comp., 1:83–98.

Chesi G (2011) Domain of Attraction: Analysis and Control via SOS Programming, Springer, Berlin.

Chesi G, Garulli A, Tesi A, and Vicino A (2005) “Lmi-based computation of optimal quadratic lyapunov functions for odd polynomial systems”, International Journal of Robust Nonlinear Control, 15:35–49.

Chiang H, Hirsch M, and Wu F (1988) “Stability regions of nonlinear autonomous dynamical systems”, IEEE Transactions on Automatic Control, 33:16–27.

Chiang H and Thorp J (1989) “Stability regions of nonlinear dynamical systems: A constructive methodology”, IEEE Transactions on Automatic Control, 34:1229–1241.

Davidson E and Kurak E (1971) “A computational method for determining quadratic lyapunov functions for nonlinear systems”, Automatica, 7:627–636.

Davison E and Cowan K (1969) “A computational method for determining the stability region of a second-order non-linear autonomous system”, Znt. J. Contr, 9:349–357.

Forsman K (1991) “Applications of groebner bases to nonlinear systems”, In Proc. First European Control Conf, pages 164–169.

Forsman K and Glad S (1990) “Constructive algebraic geometry in nonlinear control”, In Proc. 29th. CDC, IEEE, 5:2825–2827.

Genesio R, Tartaglia M, and Vicino A (1985) “On the estimation of asymptotic stability regions: State of the art and new proposals”, IEEE Transactions on Automatic Control, 30:747–755.

Genesio R and Tesi A (1992) “Harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems”, Automatica, 28:531–531.

Genesio R and Vicino A (1984) “New techniques for constructing asymptotic stability regions for nonlinear systems”, IEEE Transactions on Circuits and Systems, 31:574–581.

Gianni P and Mora T (1989) “Algebraic solution of systems of polynomial equations using gröbner bases”, Applied Algebra, Algebraic Algorithms and Error Correcting Codes, Proceedings of AAECC-5, 356:247–257.

Giesl P (2008) “Construction of a local and global lyapunov function using radial basis functions”, IMA Journal of Applied Mathematics, 73:782–802.

Guillemin V and Pollack A (1974) Differential Topology, Prentice-Hall.

Guttalu R and Flashner H (1988) “A numerical method for computing domains of attraction for dynamical systems”, Int.J.Num.Meth.in Engineering, 26(4):875–890.

Hachicho O and Tibken B (2007) “A novel lmi-based optimization algorithm for the guaranteed estimation of the domain of attraction using rational lyapunov functions”, J. Frankl. Inst., 344(5):535–552.

Hahn W (1967) Stability of Motion, Springer Berlin Heidelberg.

Hewit J and Storey C (1967) “Computer program for obtaining regions of practical stability for second-order autonomous systems”, Proc. Inst. Elect. Eng., 114:1347–1350.

Hodge W and Pedoe D (1953) Methods of Algebraic Geometry, Cambridge Univ. Press, Cambridge.

Infante E and Clark L (1964) “A method for the determination of the domain of stability of second-order nonlinear autonomous systems”, J. Appl. Mech., 86:315–320.

Jocic L (1982) “Planar regions of attraction”, IEEE Trans. Automat. Contr, 27:708–710.

Johansen T (2000) “Computation of lyapunov functions for smooth nonlinear systems using convex optimization”, Automatica, 36:1617–1626.

Kaslik E, Balint A, and Balint S (2005) “Methods for determination and approximation of the domain of attraction”, Nonlinear Analysis, 60:703–717.

Khalil H (2002) Nonlinear Systems, Prentice Hall, Third Edition.

Kirin N, Nelepin R, and Baidaev V (1981) “Construction of the attraction region by zubov’s method”, Differential Equations, 17:871–880.

Kormanik J and Li C (1972) “Decision surface estimate of nonlinear system stability domain by lie series method”, IEEE Transactions on Automatic Control, 17:666–669.

Krasovskii N (1959) “Certain problems in the theory of stability of motion”, (Russian) Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow,, pages 211–211.

Krasovskii N (1963) Stability of Motion, Stanford University Press.

Krasovskii N and Barbashin E (1954) “On the existence of lyapunov functions in the case of asymptotic stability in the whole”, Prikle, Kat. Mekh., 17:345–350.

LaSalle J (1960) “Some extensions of liapunov’s second method”, IRE Trans. Circuit Theory, 7:520–527.

LaSalle J (1967) “An invariance principle in the theory of stability”, Differential Equations and Dynamical Systems, pages 277–286.

Lee H and Han K (1998) “Estimation of asymptotic stability regions of nonlinear systems by use of eigen-vectors”, Nonlinear Problem in Aviation and Aerospace, pages 113–122.

Lee H and Han K (2000) “Analysis of nonlinear reactor systems by forward and backward integration methods”, IEEE Transaction on Nuclear Science, 47(6):2693–2698.

Loccufier M and Noldus E (1995) “On the estimation of asymptotic stability regions for autonomous nonlinear systems”, IMA J. Math. Control Inform, 12:91–109.

Loccufier M and Noldus E (2000) “New trajectory reversing method for estimating stability regions of autonomous nonlinear systems”, Nonlinear Dynamics, 21:265–288.

Lofberg J (2004) “Yalmip: A toolbox for modeling and optimization in matlab”, In Proceedings of the CACSD Conference, Taipei, Taiwan, pages 602–608.

Loparo K and Blankenship G (1978) “Estimating the domain of attraction of nonlinear feedback systems”, IEEE Trans. Automat. Corm., 23:602–608.

Margolis S and Vogt W (1963) “Control engineering applications of v.i. zubov’s construction procedure for lyapunov functions”, IEEE Trans. Automatic Control, 8:104–113.

Nayfeh A and Balachandran B (1995) Applied Nonlinear Dynamics, New York, Wiley.

Noldus E and Loccufier M (1995) “A new trajectory reversing method for the estimation of asymptotic stability regions”, Int. J. Control, 61(4):917–932.

Parrilo P (2000) “Structered semidefinite programs and semialgebraic geometry methods in robustness and optimization”, Ph.D thesis, California Institute of Technology, Pasadena.

Parrilo P (2003) “Semidefinite programming relaxations for semialgebraic problems”, Mathematical Programming Series B, 96(2):293–320.

Pauer F and Pfeifhofer M (1988) “The theory of gröbner bases”, L’Enseignement Math. t, 34:215–232.

Poincaré H (1881) “Mémoire sur les courbes définies par une équation différentielle”, J. Math. Pures et Appl., 7(3):375–422.

Prajna S, Papachristodoulou A, and Parrilo P (2002) “Introducing sostools: A general purpose sum of squares programming solver”, Proc. IEEE Conf. on Decision and Control, pages 293–320.

Rabinowitsch J (1929) “Zum Hilbertschen Nullstellensatz”, Math. Ann., 102(1):520.

Rezaiee-Pajand M and Moghaddasie B (2012) “Estimating the region of attraction via collocation for autonomous nonlinear systems”, Structural Engineering and Mechanics, 41:263–284.

Robbiano L (1989) Computational aspects of commutative algebra (special issue of Journal of Symbolic Computation), Academic Press, London.

Rodden J (1964) “Numerical applications of lyapunov stability theory”, Proceedings of the Joint Automatic Control Conference, pages 261–268.

Sanchez W, Castellanos J, and Alexandrov V (2009) “A computational method for the determination of attraction regions”, 6th International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE), pages 102–108.

Seiler P (2013) “Sosopt: A toolbox for polynomial optimization version 2.00”, pages 102–108.

Shields D and Storey C (1975) “The behavior of optimal lyapunov functions”, Int. J. Control, 21(4):561–573.

Stacey A and Stonier R (1998) “Analytic estimates for the boundary of the region of asymptotic attraction”, Dynamics and Control, 8:177–189.

Sturm J (1999) “Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones”, Optimization Methods and Software, 11(12):625–653.

Sturm J (2001) “Sedumi version 1.05”, .

Tan W (2006) “Nonlinear control analysis and synthesis using sum-of-squares programming”, Ph. D. thesis, University of California, Berkeley.

Tan W and Packard A (2004) “Searching for control lyapunov functions using sums of squares programming.” In: 42nd Annual Allerton Conference on Communications, Control and Computing, pages 210–219.

Tan W, Topcu U, Seiler P, Balas G, and Packard A (2008) “Simulation-aided reachability and local gain analysis for nonlinear dynamical systems”, In: Proceedings of the IEEE Conference on Decision and Control,, pages 4097–4102.

Tibken B and Hachicho O (2000) “Estimation of the domain of attraction for polynomial systems using multidimensional grids”, IEEE Conf. on Decision and Control, Sydney, Australia, pages 3870–3874.

Topcu U, Packard A, and Seiler P (2008) “Local stability analysis using simulations and sum-of-squares programming”, Automatica, 44(10):2669–2675.

Topcu U, Packard A, Seiler P, and Wheeler T (2007) “Stability region analysis using simulations and sum-of-squares programming.” In: Proceedings of the American Control Conference, pages 6009–6014.

Vannelli A and Vidyasagar M (1985) “Maximal lyapunov functions and domains of attraction for autonomous nonlinear systems”, Automatica, 21:69–80.

Walter J and Thomas L (1997) Nonlinear and Optimal Control Systems, John Wiley and Sons, New-York.

Wang T, Lall S, and West M (2013) “Polynomial level-set method for polynomial system reachable set estimation”, IEEE Transactions on Automatic Control, 58(10):2508–2521.

Wilson F (1967) “The structure of the level surfaces of a lyapunov function”, J. Differential Equations, 3:323–239.

Wloszek Z (2003) “Lyapunov based analysis and controller synthesis for polynomial systems using sum-of-squares optimization”, Ph. D. thesis, University of California, Berkeley.

Wloszek Z, Feeley R, Tan W, Sun K, and Packard A (2003) “Some controls applications of sum of squares programming.” In: Proceedings of the 42nd IEEE Conference on Decision and Control, 5:4676–4681.

Wloszek Z, Feeley R, Tan W, Sun K, and Packard A (2005) “Controls applications of sum of squares programming.” In: Positive Polynomials in Control. LNCIS, 312:3–22.

Yerugin N (1951a) “On the continuation of solutions of differential equations”, Prikl. Math. Mekh., 15:55–58.

Yerugin N (1951b) “Some general questions in the theory of the stability of motion”, Prikl. Math. Mekh., 15(2):227–236.

Zubov V (1961) Methods of A.M. Lyapunov and Their Applications. Translated from the Russian Edition of 1957., P. Noordhoff Ltd., Groningen, The Netherlands.

Zubov V (1962) Mathematical methods for the study of automatic control systems, Israel: Jarusalem Academic Press.

Zubov V (1964) Methods of A.M. Lyapunov and Their Application. Translation prepared by under the auspices of the United States Atomic Energy Commission: edited by Leo F. Boron, P.Noordhoff Ltd. Groningen.

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