SECOND AND THIRD ORDER RATIONAL DIFFERENCE EQUATIONS

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THESIS OF MASTER OF SCIENCE

İNCİ OKUMUŞ

SECOND AND THIRD ORDER RATIONAL DIFFERENCE EQUATIONS

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İnci OKUMUŞ

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İnci OKUMUŞ

ABSTRACT

Thesis of Master of Science

SECOND AND THIRD ORDER RATIONAL DIFFERENCE EQUATIONS

İnci OKUMUŞ

Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assoc. Prof. Yüksel SOYKAN
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In this thesis, we are primarily concerned with the boundedness nature of solutions and the stability of the equilibrium points of the second and third order rational difference equations.

The organization of this thesis is as follows:

In Chapter 1, we give the necessary preliminary results.

In Chapter 2, we present a collection of techniques for demonstrating boundedness of solutions of the difference equations.

In Chapter 3, we present some examples of the second and third order rational difference equations.

In Chapter 4, we focus on the study of stability and boundedness nature of the equilibrium points of the difference equation

ABSTRACT (continued)

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}}.$$

In Chapter 5, we investigate the study of stability of the equilibrium points of the difference equation

$$x_{n+1} = \frac{\alpha(x_n + x_{n-1}) + x_n x_{n-1}}{A + Bx_n x_{n-1}}.$$

Key Words: Difference equations, boundedness, equilibrium point, stability, local asymptotic stability, global asymptotic stability.

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ÖZET

Yüksek Lisans Tezi

İKİNCİ VE ÜÇÜNCÜ MERTEBEDEN RASYONEL FARK DENKLEMLERİ

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Bu tezde ikinci ve üçüncü mertebeden rasyonel fark denklemlerinin denge noktalarının kararlılığı ve çözümlerinin sınırlılığı ile ilgilenilmiştir.

Bu tezin organizasyonu aşağıdaki gibidir:

Birinci bölümde, konu ile ilgili gerekli ön bilgiler verilmiştir.

İkinci bölümde, fark denklemlerinin çözümlerinin sınırlılıklarını gösterme tekniklerinin bir derlemesi sunulmuştur.

Üçüncü bölümde, bazı ikinci ve üçüncü mertebeden rasyonel fark denklem örnekleri verilmiştir.

Dördüncü bölümde,

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{B x_n + C x_{n-1}}$$

ÖZET (devam ediyor)

fark denkleminin denge noktalarının kararlılığı ve sınırlılığı üzerine odaklanılmıştır.

Beşinci bölümde,

$$x_{n+1} = \frac{\alpha(x_n + x_{n-1}) + x_n x_{n-1}}{A + Bx_n x_{n-1}}$$

fark denkleminin denge noktalarının kararlılığı incelenmiştir.

Anahtar Sözcükler: Fark denklemleri, sınırlılık, denge noktası, kararlılık, yerel asimptotik kararlılık, global asimptotik kararlılık.

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FIELD OF STUDY

Major Field: Functional Analysis, Difference Equations

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

In this thesis we extensively use [1-2, 7, 19-22, 24, 26, 29, 34]. For some other basic results in the area of difference equations and systems, see [3-6, 8-18, 23, 25, 27-28, 30-33].
In this chapter we state some well known results.

1.1 DEFINITIONS OF STABILITY

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ... \tag{1.1}$$

A solution of Eq.(1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(1.1) for all $n \geq -k$.

Lemma 1.1 *For every set of initial conditions x_{-k}, $x_{-(k+1)}, ...,$ $x_0 \in I$, the difference equation (1.1) has a unique solution $\{x_n\}_{n=-k}^{\infty}$.*

As a special case of above lemma, for every set of initial conditions $x_0, x_{-1} \in I$, the second order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, ... \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$ and for every set of initial conditions $x_0, x_{-1}, x_{-2} \in I$, the third order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \qquad n = 0, 1, ... \tag{1.3}$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

Definition 1.1 *A solution of Eq.(1.1) that is constant for all $n \geq -k$ is called an equilibrium solution of Eq.(1.1). If*

$x_n = \overline{x}$, for all $n \geq -k$

is an equilibrium solution of Eq.(1.1), then $\overline{x}$ is called an equilibrium point, or simply an equilibrium of Eq.(1.1).

$\overline{x}$ is also called as a fixed point of f.

So a point $\overline{x} \in I$ is called an equilibrium point of Eq(1.1) if

$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x})$;

that is,

$x_n = \overline{x}$ for $n \geq -k$

is a solution of Eq.(1.1).

Definition 1.2 (Stability) *Let $\overline{x}$ an equilibrium point of Eq(1.1).*

(a) *An equilibrium point $\overline{x}$ of Eq.(1.1) is called locally stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1) with*

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

then

$$|x_n - \overline{x}| < \varepsilon, \quad \text{for all } n \geq -k.$$

(b) *An equilibrium point $\overline{x}$ of Eq.(1.1) is called locally asymptotically stable if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.1) with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

then we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) *An equilibrium point $\overline{x}$ of Eq.(1.1) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1), we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) *An equilibrium point $\overline{x}$ of Eq.(1.1) is called globally asymptotically stable if it is locally stable, and a global attractor.*

(e) *An equilibrium point $\overline{x}$ of Eq.(1.1) is called unstable if it is not locally stable.*

1.2 LINEARIZED STABILITY ANALYSIS

Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\overline{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x}), \quad \text{for } i = 0, 1, ..., k$$

denote the partial derivative of $f(u_0, u_1, ..., u_k)$ with respect to u_i evaluated at the equilibrium point $\overline{x}$ of Eq.(1.1).

Definition 1.3 *The equation*

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + ... + q_k y_{n-k}, \; n = 0, 1, ... \tag{1.4}$$

is called the linearized equation of Eq.(1.1) about the equilibrium point $\overline{x}$, *and the equation*

$$\lambda^{k+1} - q_0 \lambda^k - ... - q_{k-1}\lambda - q_k = 0 \tag{1.5}$$

is called the characteristic equation of Eq.(1.4) about $\overline{x}$.

Then the equation

$$y_{n+1} = q_0 y_n + q_1 y_{n-1}, \quad n = 0, 1, ... \tag{1.6}$$

is the linearized equation associated with Eq.(1.2) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^2 - q_0 \lambda - q_1 = 0$$

is the characteristic equation of Eq.(1.6) about $\overline{x}$.
Also, the equation

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + q_2 y_{n-2}, \quad n = 0, 1, ... \tag{1.7}$$

is the linearized equation associated with Eq.(1.3) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^3 - q_0 \lambda^2 - q_1 \lambda - q_2 = 0$$

is the characteristic equation of Eq.(1.7) about $\overline{x}$.
The following result, known as the *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Eq.(1.1).

Theorem 1.2 (The Linearized Stability Theorem) *([7], p.5)*
Assume that the function f is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:

(a) *When all the roots of Eq.(1.5) have absolute value less than one, then the equilibrium point $\overline{x}$ of Eq.(1.1) is locally asymptotically stable.*

(b) *If at least one root of Eq.(1.5) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Eq.(1.1) is unstable.*

The equilibrium point $\overline{x}$ of Eq.(1.1) is called *hyperbolic* if no root of Eq.(1.5) has absolute value equal to one. If there exists a root of Eq.(1.5) with absolute value equal to one, then the equilibrium $\overline{x}$ is called $non-hyperbolic$.
An equilibrium point $\overline{x}$ of Eq.(1.1) is called a *repeller* if all roots of Eq.(1.5) have absolute value greater than one.
As a special case of Theorem 1.2 we have the following corollary.

Corollary 1.1 (a) *If both roots of the characteristic equation (quadratic equation)*

$$\lambda^2 - q_0\lambda - q_1 = 0$$

of Eq.(1.6) lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of Eq(1.2) is locally asymptotically stable.

(b) *If all roots of the characteristic equation (qubic equation)*

$$\lambda^3 - q_0\lambda^2 - q_1\lambda - q_2 = 0$$

of Eq.(1.7) lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of Eq(1.3) is locally asymptotically stable.

The following two theorems state necessary and sufficient conditions for all the roots of a real polynomial of degree two or three, respectively, to have modulus less than one.

Theorem 1.3 *([7], p.6) Assume that a_1 and a_0 are real numbers. Then a necessary and sufficient condition for all roots of the equation*

$$\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$|a_1| < 1 + a_0 < 2.$

Theorem 1.4 ***([7], p.6)*** *Assume that a_2, a_1, and a_0 are real numbers. Then a necessary and sufficient condition for all roots of the equation*

$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$

to lie inside the unit disk is

$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1 \quad \text{and} \quad a_0^2 + a_1 - a_0a_2 < 1.$

Theorem 1.5 (Clark Theorem) ***([7], p.6)*** *Assume that $q_0, q_1, ..., q_k$ are real numbers such that*

$|q_0| + |q_1| + ... + |q_k| < 1$

Then all roots of Eq.(1.5) lie inside the unit disk.

Theorem 1.6 ***([26], p.9)*** *Consider the difference equation*

$$x_{n+1} = f_0\left(x_n, x_{n-1}\right) x_n + f_1\left(x_n, x_{n-1}\right) x_{n-1}, \quad n = 0, 1, ... \tag{1.8}$$

with nonnegative initial conditions and

$f_0, f_1 \in C\left[[0,\infty) \times [0,\infty), [0,1)\right].$

Assume that the following hypotheses hold:

(a) *f_0 and f_1 are non-increasing in each of their arguments;*

(b) *$f_0(x,x) > 0$ for all $x \geq 0$;*

(c) *$f_0(x,y) + f_1(x,y) < 1$ for all $x, y \in (0,\infty)$.*

Then the zero equilibrium of Eq(1.8) is globally asymptotically stable.

Theorem 1.7 ***([26], p.11)*** *Let $[a,b]$ be an interval of real numbers and assume that*

$f : [a,b] \times [a,b] \longrightarrow [a,b]$

is a continuous function satisfying the following properties:

(a) *$f(x,y)$ is non-decreasing in $x \in [a,b]$ for each $y \in [a,b]$, and $f(x,y)$ is non-increasing in $y \in [a,b]$ for each $x \in [a,b]$;*

(b) *If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system*

$$f(m,M) = m \quad \text{and} \quad f(M,m) = M,$$

then $m = M$.

Then Eq(1.2) has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of Eq(1.2) converges to $\overline{x}$.

Theorem 1.8 ***([26], p.12)*** *Let $[a,b]$ be an interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \longrightarrow [a,b]$$

is a continuous function satisfying the following properties:

(a) *$f(x,y)$ is non-increasing in $x \in [a,b]$ for each $y \in [a,b]$ and $f(x,y)$ is non-decreacing in $y \in [a,b]$ for each $x \in [a,b]$,*

(b) *The difference equation Eq(1.2) has no solutions of prime period two in $[a,b]$.*

Then Eq(1.2) has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of Eq(1.2) converges to $\overline{x}$.

Theorem 1.9 ***([26], p.13)*** *Let $[a,b]$ be an interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \longrightarrow [a,b]$$

is a continuous function satisfying the following properties:

(a) *$f(x,y)$ is non-increasing in each of its arguments;*

(b) *If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system*

$$f(m,m) = M \quad \text{and} \quad f(M,M) = m,$$

then $m = M$.

Then Eq(1.2) has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of Eq(1.2) converges to $\overline{x}$.

In the next theorem we make use of the following notation associated with a function $f(z_1, z_2)$ which is monotonic in both arguments.

For each pair of numbers (m, M) and for each $i \in \{1, 2\}$, define

$$M_i(m, M) = \begin{cases} M, & \text{if } f \text{ is increasing in } z_i \\ m, & \text{if } f \text{ is decreasing in } z_i \end{cases}$$

and

$$m_i(m, M) = M_i(M, m).$$

Theorem 1.10 *([1], **p.3**)Assume that $f \in C\left([0,\infty)^2, [0,\infty)\right)$ and $f(z_1, z_2)$ is either strictly increasing in z_1 and z_2, or strictly decreasing in z_1 and z_2, or strictly increasing in z_1 and strictly decreasing in z_2. Furthermore, assume that for every*

$$m \in (0,\infty) \text{ and } M > m,$$

either

$$[f(M_1(m,M), M_2(m,M)) - M][f(m_1(m,M), m_2(m,M)) - m] > 0$$

or

$$f(M_1(m,M), M_2(m,M)) = M \text{ and } f(m_1(m,M), m_2(m,M)) = m.$$

Then every solution of Eq.(1.2) which is bounded from above and from below by positive constants converges to a finite limit.

We now present two genereal global asymptotic stability results that apply to several special cases of the $(k+1)^{st}$-order rational difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} \beta_i x_{n-i}}{A + \sum_{i=0}^{k} B_i x_{n-i}}, \quad n = 0, 1, ... \tag{1.9}$$

with $A > 0$, the remaining parameters non-negative, with

$$\sum_{i=0}^{k} \beta_i \quad \text{and} \quad \sum_{i=0}^{k} B_i \in (0,\infty),$$

and with arbitrary non-negative initial conditions such that the denominator is always positive.

The characteristic equation of the linearized equation of Eq.(1.9) about an equilibrium point $\overline{x}$ is

$$\lambda^{k+1} + \frac{1}{A + \overline{x}.\sum_{i=0}^{k} B_i} \sum_{i=0}^{k} \left(B_i \overline{x} - \beta_i\right) \lambda^{k-i} = 0. \tag{1.10}$$

Zero is an equilibrium point Eq.(1.9) if and only if

$$\alpha = 0 \text{ and } A > 0. \tag{1.11}$$

As we will see later, when (1.11) holds, the zero equilibrium of Eq.(1.9) is globally asymptotically stable when

$$A > \sum_{i=0}^{k} \beta_i \tag{1.12}$$

and unstable when

$$A < \sum_{i=0}^{k} \beta_i.$$

Eq.(1.9) has a positive equilibrium point if and only if
either

$$\alpha > 0 \tag{1.13}$$

or

$$\alpha = 0 \text{ and } A < \sum_{i=0}^{k} \beta_i. \tag{1.14}$$

When (1.13) holds, the equation has the unique equilibrium point

$$\overline{x} = \frac{\beta - A + \sqrt{(\beta - A)^2 + 4\alpha B}}{2B}, \tag{1.15}$$

where for simplicity we use the notation,

$$\beta = \sum_{i=0}^{k} \beta_i \quad \text{and} \quad B = \sum_{i=0}^{k} B_i.$$

When (1.14) holds, Eq.(1.9) has the unique positive equilibrium point

$$\overline{x} = \frac{\beta - A}{B}.$$

Note that

$$\frac{1}{A + B\overline{x}} \sum_{i=0}^{k} |B_i\overline{x} - \beta_i| \leq \frac{1}{A + B\overline{x}}.\left(B\overline{x} - \beta\right). \tag{1.16}$$

Therefore, by Theorem 1.5 and 1.16, the equilibrium of Eq.(1.9) is locally asymptotically stable when (1.12) holds.

Theorem 1.11 ***([7], pp.150-151)*** *Assume that*

$$\beta = \sum_{i=0}^{k} \beta_i < A.$$

Then the following statements are true:

(a) *If*

$$\alpha = 0,$$

the zero equilibrium of Eq.(1.9) is globally asymptotically stable.

(b) *If*

$$\alpha > 0,$$

the positive equilibrium of Eq.(1.9) is globally asymptotically stable.

In the very special case when

$$A = \sum_{i=0}^{k} \beta_i > 0 \quad \text{and} \quad \alpha > 0,$$

the global character of solutions of Eq.(1.9) is completely described by the following result in [33]. In this case it is preferable to write the difference equation in the form

$$x_n = \frac{\alpha + \sum_{r=1}^{k} \beta_r x_{n-i_r}}{A + \sum_{t=1}^{m} B_j x_{n-j_t}}, \quad n = 1, 2, ... \tag{1.17}$$

Also, by making a change of variables, if necessary, we may and do assume that the greatest common divisor of all "delays" in the numerator and denominator is 1, that is,

$$\gcd\{i_1, ..., i_k, j_{1,...,}j_m\} = 1.$$

Theorem 1.12 ***([7], p.152)*** *Assume that*

$$\alpha, \beta_1, ..., \beta_k, B_1, ..., B_m \in (0, \infty) \quad \textit{and} \quad A = \sum_{i=0}^{k} \beta_i.$$

Then when the "delays" in the numerator

$i_1, ..., i_k$ *are all even*

and the "delays" in the denominator

$j_1, ..., j_m$ *are all odd,*

every solution of Eq.(1.17) converges to a period-two solution. In every other case of delays, every solution of Eq.(1.1) has a finite limit.

Theorem 1.13 ***([7], p.152)*** *Assume that*

$$\alpha = 0 \quad \textit{and} \quad \beta = \sum_{i=0}^{k} \beta_i = A$$

and that one of the following three conditons is satisfied:

(a) $\beta_i B_i > 0$ *for some* $i \in \{0, ..., k\}$.

(b) $\beta_0 > 0$.

(c) B_0 *and Eq.(1.9) has no period-two solutions.*

Then the zero equilibrium of Eq.(1.9) is globally asymptotically stable.

Theorem 1.14 ***([29], p.155)*** *Let* $l \in \{1, 2, ...\}$. *Suppose that on some interval* $I \subseteq \mathbb{R}$ *Eq.(1.1) has the linearization*

$$x_{n+l} = \sum_{i=1-l}^{m} g_i x_{n-i},$$

where the non-negative functions $g_i : I^{k+l} \to \mathbb{R}$ *are such that* $\sum\limits_{i=1-l}^{m} g_i = 1$ *is satisfied. Suppose that there exists* $A > 0$ *such that*

$$g_{1-l} \geq A, \quad n = 0, 1, ...$$

Then if $x_{l-1}, ...x_{-k} \in I$,

$$\lim_{n \to \infty} x_n = L \in I.$$

1.3 SOLVING QUADRATIC EQUATIONS AND INEQUALITIES

In this section we use the following web page: [19].

1.3.1 QUADRATIC EQUATIONS

A **quadratic equation** is one which can be written in the form

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

where a, b and c are given numbers and x is the unknown whose value(s) must be found. When it is difficult to factorise a quadratic equation, it may be possible to solve it using a formula which is used to calculate the roots. The formula is obtained by completing the square in the general quadratic $ax^2 + bx + c$. We proceed by removing the coefficient of a:

$$ax^2 + bx + c = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\} = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right\}.$$

Thus the solution of $ax^2 + bx + c = 0$ is the same as the solution to

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0.$$

So, solving:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

which leads to

$$x = -\frac{b}{2a} \pm \sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}}.$$

Simplifying this expression further we obtain the important result:

If $ax^2 + bx + c = 0$, $a \neq 0$, then the two solutions (roots) are

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

To apply the formula to a specific quadratic equation it is necessary to identify carefully the values of a, b and c, paying particular attention to the signs of these numbers. Substitution of these values into the formula then gives thee desired solutions.

Note that if the quantity $b^2 - 4ac$ (called the **discriminant**) is a positive number we can take its square root and the formula will produce two values known as **distinct real**

roots. If $b^2 - 4ac = 0$ there will be one value only known as a **repeated root** or **double root**. The value of this root is $x = -\frac{b}{2a}$. Finally, if $b^2 - 4ac$ is negative we say the equation possesses **complex roots**.
When finding roots of the quadratic equation $ax^2 + bx + c = 0$ first calculate the discriminant

$$b^2 - 4ac.$$

If $b^2 - 4ac > 0$, the quadratic has two real distinct roots.
If $b^2 - 4ac = 0$, the quadratic has two real and equal roots.
If $b^2 - 4ac < 0$, the quadratic has no real roots: there are two complex roots.

1.3.2 QUADRATIC INEQUALITIES

A quadratic inequality is just like a quadratic equation, except instead of an equal sign there's an inequality!
A quadratic inequality is one that can be written in one of the following standard forms:

$$ax^2 + bx + c < 0$$

or

$$ax^2 + bx + c \leq 0$$

or

$$ax^2 + bx + c > 0$$

or

$$ax^2 + bx + c \geq 0.$$

In other words, a quadratic inequality is in standard form when the inequality is set to 0. Just like in a quadratic equation, the degree of the polynomial expression is two.
To solve a quadratic inequality we must determine which part of the graph of a quadratic function lies above or below the -axis. An inequality can therefore be solved graphically using a graph or algebraically using a table of signs to determine where the function is positive and negative.

Many inequalities lead to finding the sign of a quadratic expression. Consider the quadratic function

$f(x) = ax^2 + bx + c.$

We know that

(1) If $b^2 - 4ac = 0$ (double root case), then we have

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2.$$

In this case, the function $f(x) = ax^2 + bx + c$ has the sign of the coefficient a.

(2) If $b^2 - 4ac > 0$ (two distinct real roots case). In this case, we have

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

where x_1 and x_2 are two roott with $x_1 < x_2$. Since $(x - x_1)(x - x_2)$ is always positive when $x < x_1$ and $x > x_2$, and always negative when $x_1 < x < x_2$, we get
$ax^2 + bx + c$ has same sign as the coefficient a when $x < x_1$ and $x > x_2$;
$ax^2 + bx + c$ has opposite sign as the coefficient a when $x_1 < x < x_2$.

(3) If $b^2 - 4ac < 0$ (complex roots case), then $ax^2 + bx + c$ has a constant sign same as the coefficient a.

Example 1.1 *Solve the inequality*

$x^2 - x - 2 \le 0.$

Solution. First let us find the root of the quadratic equation $x^2 - x - 2 = 0$. The quadratic formula gives

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-2)}}{2} = \frac{1 \pm 3}{2}$$

which yields $x = -1$ or $x = 2$. Therefore, the expression $x^2 - x - 2$ is negative or equal to 0 when $-1 \le x \le 2$. ■

1.4 DESCARTES' RULE OF SIGNS

In this section we use the following web pages: [20], [21] and [22].
In mathematics, Descartes' rule of signs, first described by René Descartes in his work La Géométrie, is a technique for determining the number of positive or negative real roots of a polynomial. The rule gives an upper bound number of positive or negative roots of a polynomial. In other words, it is a method of determining the maximum number of positive and negative real roots of a polynomial. However, it is not a complete criterion, and so, does not provide the exact number of positive or negative roots.
Descartes' Rule of Signs will not tell us where the polynomial's zeroes are (we'll need to use the Rational Roots Test and synthetic division, or draw a graph, to actually find the roots), but the Rule will tell us how many roots we can expect.

- For positive roots, start with the sign of the coefficient of the lowest (or highest) power. Count the number of sign changes n as you proceed from the lowest to the highest power (ignoring powers which do not appear). Then n is the maximum number of positive roots. Furthermore, the number of allowable roots is $n, n-2, n-4, \ldots$.

For example, consider the polynomial

$$f(x) = x^7 + 3x^6 - 6x^5 - 18x^4 + 9x^3 + 27x^2 - 4x - 12. \qquad (1.18)$$

Since there are three sign changes, there are a maximum of three possible positive roots.

- For negative roots, starting with a polynomial $f(x)$, write a new polynomial $f(-x)$ with the signs of all odd powers reversed, while leaving the signs of the even powers unchanged. Then proceed as before to count the number of sign changes n. Then n is the maximum number of negative roots.

For example, consider again the polynomial (1.18) and compute the new polynomial

$$f(-x) = -x^7 + 3x^6 + 6x^5 - 18x^4 - 9x^3 + 27x^2 + 4x - 12.$$

Since there are four sign changes, so there are a maximum of four negative roots.

In fact

$$f(x) = (x+1)^2(x-1)^2(x-2)(x+2)(x+3)$$

and

$$f(-x) = -(x-1)^2(x+1)^2(x-2)(x+2)(x-3).$$

So the zeros of f are $-3, -2, -1$ (twice), 1 (twice), 2. Thus f has exactly two positive roots and three negative roots.
For example, to find the number of negative roots of

$$f(x) = ax^3 + bx^2 + cx + d$$

we equivalently ask how many positive roots there are for $-x$ in

$$f(-x) = a(-x)^3 + b(-x)^2 + c(-x) + d = -ax^3 + bx^2 - cx + d \equiv g(x).$$

Using Descartes' rule of signs on $g(x)$ gives the number of positive roots x_i of g, and since

$$g(x) = f(-x)$$

it gives the number of positive roots $(-x_i)$ of f, which is the same as the number of negative roots x_i of f.

Example 1.2

The polynomial

$$f(x) = x^3 + x^2 - 21x - 45$$

has one sign change between the second and third terms (the sequence of pairs of successive signs is $++$, $+-$, $--$). Therefore it has exactly one positive root. Note that the leading sign needs to be considered although in this particular example it does not affect the answer. To find the number of negative roots, change the signs of the coefficients of the terms with odd exponents, i.e., apply Descartes' rule of signs to the polynomial $f(-x)$, to obtain a second polynomial

$$f(-x) = -x^3 + x^2 + 21x - 45.$$

This polynomial has two sign changes (the sequence of pairs of successive signs is $-+$, $++$, $+-$), meaning that this second polynomial has two or zero positive roots, thus the original polynomial has two or zero negative roots.
In fact, the factorization of the first polynomial is

$$f(x) = (x+3)^2(x-5)$$

so the roots are -3 (twice) and 5.
The factorization of the second polynomial is

$$f(-x) = -(x-3)^2(x+5)$$

So here, the roots are 3 (twice) and -5, the negation of the roots of the original polynomial.

Example 1.3

Using Descartes' Rule of Signs, determine the number of real solutions to

$$4x^7 + 3x^6 + x^5 + 2x^4 - x^3 + 9x^2 + x + 1 = 0.$$

We look first at the polynomial $f(x)$ (this is the "positive" case):

$$f(x) = +4x^7 + 3x^6 + x^5 + 2x^4 - x^3 + 9x^2 + x + 1.$$

There are two sign changes, so there are two or, counting down in pairs, zero positive solutions. Now We look at the polynomial $f(-x)$ (this is the "negative" case):

$$\begin{aligned} f(-x) &= 4(-x)^7 + 3(-x)^6 + (-x)^5 + 2(-x)^4 - (-x)^3 + 9(-x)^2 + (-x) + 1 \\ &= -4x^7 + 3x^6 - x^5 + 2x^4 + x^3 + 9x^2 - x + 1. \end{aligned}$$

There are five sign changes, so there are five or, counting down in pairs, there or one negative solutions.
There are two or zero positive solutions and five, there or one negative solutions.

Example 1.4

Use Descartes' Rule of Signs to find the number of real roots of

$$f(x) = x^5 + x^4 + 4x^3 + 3x^2 + x + 1.$$

We look first at $f(x)$:

$f(x) = +x^5 + x^4 + 4x^3 + 3x^2 + x + 1.$

There are no sign changes, so there are no positive roots. Now We look at $f(-x)$:

$$\begin{aligned} f(-x) &= (-x)^5 + (-x)^4 + 4(-x)^3 + 3(-x)^2 + (-x) + 1 \\ &= -x^5 + x^4 - 4x^3 + 3x^2 - x + 1. \end{aligned}$$

There are five sign changes, so there are as many as five negative roots.

Consequently, there are no positive roots and there are five, three or one negative roots.

Example 1.5

Use Descartes' Rule of Signs to determine the possible number of solutions to the equation

$2x^4 - x^3 + 4x^2 - 5x + 3 = 0.$

We look first at $f(x)$:

$f(x) = +2x^4 - x^3 + 4x^2 - 5x + 3.$

There are four sign changes, so there are $4, 2$ or 0 positive roots. Now I look at $f(-x)$:

$$\begin{aligned} f(-x) &= 2(-x)^4 - (-x)^3 + 4(-x)^2 - 5(-x) + 3 \\ &= +2x^4 + x^3 + 4x^2 + 5x + 3. \end{aligned}$$

There are no sign changes, so there are no negative roots.

As a result, there are four, two or zero positive roots and no negative roots.

Complex roots:

Any n^{th} degree polynomial has exactly n roots. So if $f(x)$ is a polynomial which does not have a root at 0 (which can be determined by inspection) then the minimum number of complex roots is equal to

$n - (p - q),$

where p denotes the maximum number of positive roots, q denotes the maximum number of negative roots (both of which can be found using Descartes' rule of signs), and n denotes the degree of the equation. A simple example is the polynomial

$f(x) = x^3 - 1,$

which has one sign change, so the maximum number of positive real roots is 1. From

$$f(-x) = -x^3 - 1,$$

we can tell that the polynomial has no negative real roots. So the minimum number of complex roots is

$$3 - (1 + 0) = 2.$$

Since complex roots of a polynomial with real coefficients must occur in conjugate pairs, note that $x^3 - 1$ has exactly 2 complex roots and 1 real (and positive) root.

CHAPTER 2

ON THE BOUNDEDNESS OF DIFFERENCE EQUATIONS

In this chapter we investigate the global character of the solutions of the rational difference equation of the second order.
We give a few methods to find the boundedness of the solutions of rational difference equations.

2.1 CONTRADICTION METHODS

2.1.1 The Case

$$x_{n+1} = \frac{\alpha + x_{n-1}}{(1 + Bx_n)x_{n-1}}, n = 0, 1, ... \tag{2.1}$$

We take this example from [2], see page [201-202].

Theorem 2.1 *Every solution of Eq.(2.1) is bounded.*

Proof. Suppose for the sake of contradiction that there exists a solution of Eq.(2.1) which is unbounded. There exists a sequence of indices $\{n_i\}$ such that

$$x_{n_i+1} \to \infty \text{ and } x_{n_i+1} > x_j \text{ for } j < n_i + 1. \tag{2.2}$$

Once

$$x_{n_i-1} \to 0,$$

because

$$x_{n_i+1} = \frac{\alpha + x_{n_i-1}}{(1 + Bx_{n_i})x_{n_i-1}}$$

From this and from

$$x_{n_i-1} = \frac{\alpha + x_{n_i-3}}{(1 + Bx_{n_i-2})x_{n_i-3}}$$

and

$$\begin{aligned} x_{n_i+1} &= \frac{\alpha + \frac{\alpha + x_{n_i-3}}{(1+Bx_{n_i-2})x_{n_i-3}}}{(1+Bx_{n_i})\frac{\alpha+x_{n_i-3}}{(1+Bx_{n_i-2})x_{n_i-3}}} \\ &= \frac{\alpha + \frac{\alpha + x_{n_i-3}}{(1+Bx_{n_i-2})x_{n_i-3}}}{(1+Bx_{n_i})(\alpha + x_{n_i-3})}(1+Bx_{n_i-2})x_{n_i-3} \\ &= \frac{\alpha x_{n_i-3} + \alpha B x_{n_i-2}x_{n_i-3} + \alpha + x_{n_i-3}}{(1+Bx_{n_i})(\alpha + x_{n_i-3})} \\ &= \frac{\alpha x_{n_i-3}}{(1+Bx_{n_i})(\alpha + x_{n_i-3})}(1+Bx_{n_i-2}) + \frac{1}{1+Bx_{n_i}} \end{aligned} \tag{2.3}$$

we have

$x_{n_i-2} \to \infty$ and x_{n_i}, $x_{n_i-3} \to 1$.

But then, (2.3) implies that, eventually,

$x_{n_i+1} < x_{n_i-2}$

which contradicts (2.2) and completes the proof. ■

2.1.2 The Case

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_{n-1}}, \ n = 0, 1, ... \tag{2.4}$$

The main result for Eq.(2.4) is the following. See **([2], pp.215-216).**

Theorem 2.2 (a) *Assume that*

$\beta \geq 1.$

Then every solution of Eq.(2.4) increases to ∞.

(b) *Assume that*

$\beta < 1.$

Then every solution of Eq.(2.4) converges to the positive equilibrium.

Proof.

(a) Obviously

$$x_{n+1} > \beta x_n \geq x_n$$

from which the result follows.

(b) The change of variables

$$x_n = \frac{1}{y_n}$$

transforms Eq.(2.4)

$$y_{n+1} = \frac{y_n}{\beta + y_n + \alpha y_n y_{n-1}}, \; n = 0, 1, ... \tag{2.5}$$

Clearly,

$y_n < 1$, for $n \geq 1$.

We also claim that every positive solution of Eq.(2.5) is bounded from below by a positive constant. To see this, suppose for the sake of contradiction that there exists a sequence of indices $\{n_i\}$ such that

$y_{n_i+1} \to 0$ and $y_{n_i+1} < y_j$, for $j < n_i + 1$.

From Eq.(2.5) we have

$y_{n_i}, \, y_{n_i-1} \to 0.$

Then eventually

$$y_{n_i+1} = \frac{y_{n_i}}{\beta + y_{n_i} + \alpha y_{n_i} y_{n_i-1}} > y_{n_i}$$

and this contradiction proves our assertion.
Define

$$I = \liminf_{n\to\infty} y_n \text{ and } S = \limsup_{n\to\infty} y_n.$$

Obviously,

$$S \leq \frac{S}{\beta + S + \alpha SI} \text{ and } I \geq \frac{I}{\beta + I + \alpha SI}$$

from which it folows that

$\beta + S + \alpha SI \le 1 \le \beta + I + \alpha SI$

and so

$S = I.$

This completes the proof. ■

2.1.3 The Case

$$x_{n+1} = \beta + \frac{x_{n-2}}{x_n}, \; n = 0, 1, ... \tag{2.6}$$

We consider the difference equation Eq.(2.6) with the parameter β positive and with arbitrary positive initial conditions x_{-2}, x_{-1}, x_0. See **([7], pp.46-48)**.

Theorem 2.3 *Every solution of Eq.(2.6) is bounded.*

Proof. First of all, we make the following useful general observations about the solutions of Eq.(2.6):

- $$x_{n+1} > \beta \text{ for } n \ge 0. \tag{2.7}$$

- $$x_{n+1} < \beta + \frac{1}{\beta} x_{n-2}, \text{ for } n \ge 1. \tag{2.8}$$

- $$x_{n+1} < \beta + \frac{1}{\beta}(\beta + \frac{x_{n-5}}{x_{n-3}}) < \beta + 1 + \frac{1}{\beta^2} x_{n-5}, \text{ for } n \ge 4. \tag{2.9}$$

- $$x_{n_i+1} \to \infty \Rightarrow x_{n_i-2} \to \infty. \tag{2.10}$$

- $$x_{n_i+1} \to \beta \Rightarrow x_{n_i} \to \infty. \tag{2.11}$$

Now suppose for the sake of contradiction that Eq.(2.6) has an unbounded solution $\{x_n\}$. Then there exists a sequence of indices $\{n_i\}$ such that

$$x_{n_i+1} \to \infty \tag{2.12}$$

and for every i,

$$x_{n_i+1} > x_j, \text{ for all } j < n_i + 1. \tag{2.13}$$

From (2.12) and (2.10) it follows that

$$x_{n_i-2} \to \infty,\ x_{n_i-5} \to \infty, \text{ and } x_{n_i-8} \to \infty. \tag{2.14}$$

Now we claim that the subsequence $\{x_{n_i-4}\}$ is bounded. Otherwise, there would exist a subsequence of $\{n_i\}$, which we still denote by $\{n_i\}$, such that

$$x_{n_i-4} \to \infty,\ x_{n_i-7} \to \infty, \text{ and } x_{n_i-10} \to \infty. \tag{2.15}$$

Note that, for every i,

$$x_{n_i-4} = \beta + \frac{x_{n_i-7}}{x_{n_i-5}}$$

and

$$x_{n_i-7} = \beta + \frac{x_{n_i-10}}{x_{n_i-8}}.$$

So, as a result of (2.14) and (2.15), we have eventually

$$x_{n_i-7} > x_{n_i-5} \text{ and } x_{n_i-10} > x_{n_i-8} \tag{2.16}$$

and

$$\frac{x_{n_i-7}}{x_{n_i-5}} \to \infty \text{ and } \frac{x_{n_i-10}}{x_{n_i-8}} \to \infty.$$

Hence, from (2.16) and (2.9), we note that eventually

$$\begin{aligned} x_{n_i+1} &< \beta + 1 + \frac{1}{\beta^2} x_{n_i-7} \\ &= \beta + 1 + \frac{1}{\beta^2}\left(\beta + \frac{x_{n_i-10}}{x_{n_i-8}}\right) \\ &= \beta + 1 + \frac{1}{\beta} + \frac{1}{\beta^2}.\left(\frac{x_{n_i-10}}{x_{n_i-8}}\right). \end{aligned}$$

Because of (2.15), it follows that the right-hand side of the above inequality is eventually less than x_{n_i-10}, which contradicts (2.13) and proves our claim that $\{x_{n_i-4}\}$ is bounded. From this and (2.14) we see

$$x_{n_i-1} = \beta + \frac{x_{n_i-4}}{x_{n_i-2}} \to \beta.$$

Furthermore,

$$\liminf_{i\to\infty} x_{n_i-3} > \beta.$$

Otherwise, a subsequence of $\{x_{n_i-3}\}$ would converge to β and therefore from (2.11), $\{x_{n_i-4}\}$ would be unbounded, which is not true.
Thus, eventually,

$$x_{n_i} = \beta + \frac{x_{n_i-3}}{x_{n_i-1}} > \beta + 1$$

and hence, for i sufficiently large,

$$x_{n_i+1} = \beta + \frac{x_{n_i-2}}{x_{n_i}} < \beta + \frac{x_{n_i-2}}{\beta+1} < x_{n_i-2},$$

which contradicts (2.13). This completes the proof. ■

2.2 INVARIANT INTERVAL METHODS

2.2.1 The Case

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}}, n = 0, 1, ... \tag{2.17}$$

We consider the difference equation (2.17).
We take this example from [1], see page [15-17]

Lemma 2.4 *Every positive solution of Eq.(2.17) is bounded.*

Proof. When

$$\gamma \le 1,$$

we have

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}} \le x_{n-1}$$

and thus the solutions of Eq.(2.17) are bounded. Now suppose that

$\gamma > 1$

and assume that $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2.17). Choose a positive number m such that

$$x_{-1},\ x_0 \in \left(m, \frac{\gamma - 1}{m}\right).$$

Define

$$f(x, y) = \frac{\gamma y}{1 + xy}.$$

$f(x,y)$ is decreasing in x. In fact,

$$f_x = \frac{0 - \gamma y.y}{(1 + xy)^2} = \frac{-\gamma y^2}{(1 + xy)^2}$$

and thus f is decreasing due to $\gamma > 1$.

$f(x,y)$ is increasing in y. In fact,

$$f_y = \frac{\gamma.(1 + xy) - \gamma y.x}{(1 + xy)^2} = \frac{\gamma}{(1 + xy)^2}$$

and so f is increasing because of $\gamma > 1$.

Therefore, by using the increasing character of f we find that

$$m = \frac{\gamma m}{1 + \frac{\gamma - 1}{m} m} < x_1 = \frac{\gamma x_{-1}}{1 + x_0 x_{-1}} < \frac{\gamma \frac{\gamma - 1}{m}}{1 + m\frac{\gamma - 1}{m}} = \frac{\gamma - 1}{m}$$

and hence by induction

$$x_n \in \left(m, \frac{\gamma - 1}{m}\right), \text{ for all } n \geq -1.$$

Consequently, $\{x_n\}_{n=-1}^{\infty}$ is bounded.

2.2.2 The Case

$$x_{n+1} = \beta x_n + \frac{1}{x_{n-1}},\ n = 0, 1, ... \tag{2.18}$$

We consider the difference equation (2.18). See **([1], p.22).**

Theorem 2.5 *Eq.(2.18) has bounded solutions, if and only if*

$$\beta < 1. \tag{2.19}$$

Proof. We see

$x_{n+1} > \beta x_n$

from which it follows that Eq.(2.18) has unbounded solutions for

$\beta \geq 1.$

On the other hand when (2.19) holds, we claim that every positive solution of Eq.(2.18) is bounded. In fact, if $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of Eq.(2.18) and if we choose positive numbers m and M such that

$$x_{-1},\, x_0 \in [m, M] \text{ and } mM = \frac{1}{1-\beta},$$

then

$$m = \frac{1}{(1-\beta)\,M} = \beta m + \frac{1}{M} \leq x_1 = \beta x_0 + \frac{1}{x_{-1}} \leq \beta M + \frac{1}{m} = \frac{1}{(1-\beta)\,m} = M$$

and inductively,

$x_n \in [m, M]$, for all $n \geq -1$

which proves our claim. ■

2.3 MIN-MAX METHODS

2.3.1 The case

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0, 1, ... \tag{2.20}$$

We consider the difference equation (2.20). See **([2], pp.217-218)**.

For Eq.(2.20) it can be seen that, for $n \geq 1$, every positive solution is bounded from below and from above by positive constants. In fact,

$$x_{n+1} \geq \frac{\alpha + \beta x_n x_{n-1}}{A + x_n x_{n-1}} \geq \frac{\min\{\alpha, \beta\}}{\max\{A, 1\}}$$

which shows that every solution of Eq.(2.20) is bounded from below, for $n \geq 1$, by the positive number

$$m = \frac{\min\{\alpha, \beta\}}{\max\{A, 1\}}.$$

So,

$$x_{n+1} < \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}} = \frac{\alpha}{x_n x_{n-1}} + \beta + \frac{1}{x_n} \leq \frac{\alpha}{m^2} + \beta + \frac{1}{m}$$

and thus every solution of Eq.(2.20) is also bounded from above, for $n \geq 2$, by the positive number

$$M = \frac{\alpha}{m^2} + \beta + \frac{1}{m}.$$

2.3.2 The Case

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, ... \tag{2.21}$$

We consider the difference equation (2.21). See **([7], pp.41-42)**.

Theorem 2.6 *Assume that α, $\beta \in [0, \infty)$ and γ, δ, C, $D \in (0, \infty)$. Then every positive solution of Eq.(2.21) is bounded from above and from below by positive numbers.*

Proof. We see that

$$x_{n+1} \geq \frac{\gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}} \geq \frac{\min\{\gamma, \delta\}}{\max\{C, D\}}$$

and then $\{x_n\}$ is bounded from below by the positive number

$$m = \frac{\min\{\gamma, \delta\}}{\max\{C, D\}}.$$

Moreover, for $n \geq 1$,

$$\begin{aligned}
x_{n+2} &= \frac{\alpha + \beta x_{n+1} + \gamma x_n + \delta x_{n-1}}{C x_n + D x_{n-1}} \\
&= \frac{\alpha}{C x_n + D x_{n-1}} + \frac{\gamma x_n + \delta x_{n-1}}{C x_n + D x_{n-1}} + \frac{\beta}{C x_n + D x_{n-1}} \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}} \\
&\leq \frac{\alpha}{(C+D)m} + \frac{\gamma x_n + \delta x_{n-1}}{C x_n + D x_{n-1}} + \frac{\beta}{C x_n + D x_{n-1}} \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}} \\
&= \frac{\alpha}{(C+D)m} + \frac{\gamma x_n + \delta x_{n-1}}{C x_n + D x_{n-1}} + \frac{\beta\alpha}{(C x_n + D x_{n-1})(C x_{n-1} + D x_{n-2})} + \\
&\quad \frac{\beta}{C x_n + D x_{n-1}} \frac{\gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1} + D x_{n-2}} + \frac{\beta^2 x_n}{(C x_n + D x_{n-1})(C x_{n-1} + D x_{n-2})} \\
&\leq \frac{\alpha}{(C+D)m} + \frac{\max\{\gamma, \delta\}}{\min\{C, D\}} + \frac{\beta\alpha}{(C+D)^2 m^2} + \frac{\beta}{(C+D)m} \frac{\max\{\gamma, \delta\}}{\min\{C, D\}} + \frac{\beta^2}{CDm}
\end{aligned}$$

and therefore the solution is also bounded from above. ■

2.3.3 The Case

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{B x_n x_{n-1} + x_{n-1}}, \ n = 0, 1, ... \tag{2.22}$$

We consider the difference equation Eq.(2.22). See **([2], p.219)**.

Eq.(2.22) is bounded from below and from above by positive constants. In fact for $n \geq 1$,

$$x_{n+1} \geq \frac{\beta x_n x_{n-1} + x_{n-1}}{B x_n x_{n-1} + x_{n-1}} = \frac{\beta x_n + 1}{B x_n + 1} \geq \frac{\min\{\beta, 1\}}{\max\{B, 1\}}.$$

Hence, for $n \geq 1$, every positive solution is bounded from below by

$$m = \frac{\min\{\beta, 1\}}{\max\{B, 1\}}.$$

So for $n \geq 2$,

$$\begin{aligned} x_{n+1} &= \frac{\alpha}{B x_n x_{n-1} + x_{n-1}} + \frac{\beta x_n + 1}{B x_n + 1} \\ &\leq \frac{\alpha}{B m^2 + m} + \frac{\max\{\beta, 1\}}{\min\{B, 1\}} \end{aligned}$$

which establishes our claim.

2.3.4 The Case

$$x_{n+1} = \frac{\alpha + x_n x_{n-1}}{A + x_n x_{n-1}}, \ n = 0, 1, ... \tag{2.23}$$

We consider the difference equation Eq.(2.23). See **([1], p.26)**.

Every solution of Eq.(2.23) is bounded from above and from below by positive constants. In fact for all $n \geq 0$,

$$\frac{\min\{\alpha, 1\}}{\max\{A, 1\}} < x_{n+1} = \frac{\alpha + x_n x_{n-1}}{A + x_n x_{n-1}} < \frac{\max\{\alpha, 1\}}{\min\{A, 1\}}.$$

The following case is an example both min-max method and invariant interval methods.

2.3.5 The Case

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} \beta_i x_{n-i}}{A + \sum_{i=0}^{k} B_i x_{n-i}}, \ n = 0, 1, ... \tag{2.24}$$

Consider the difference equation Eq.(2.24). See **([7], pp.34-37).**

Theorem 2.7 *Consider the $(k+1)^{st}$-order rational difference equation (2.24) with non-negative parameters*

$\alpha, A, \beta_0, .., \beta_k, B_0, ..., B_k$

and with arbitrary non-negative initial conditions $x_{-k}, ..., x_0$ such that the denominator is always positive. Suppose that for every $i \in \{0, 1, ..., k\}$ for which the parameter β_i in the numerator is positive, the corresponding parameter B_i in the denominator is also positive. Then every solution of Eq.(2.24) is bounded.

Proof. We denote by I and I_0 the following subsets of $\{0, 1, ..., k\}$:

$I = \{i \in \{0, 1, ..., k\} : \beta_i > 0 \text{ and } B_i > 0\}$

and

$I_0 = \{i \in \{0, 1, ..., k\} : \beta_i = 0 \text{ and } B_i > 0\}$.

Hence

$I \cup I_0 \subset \{0, 1, ..., k\}$

and Eq.(2.24) is equivalent to

$$x_{n+1} = \frac{\alpha + \sum_{i\in I} \beta_i x_{n-i}}{A + \sum_{i\in I} B_i x_{n-i} + \sum_{i\in I_0} B_i x_{n-i}}, \quad n = 0, 1, ... \tag{2.25}$$

with β_i, $B_i \in (0, \infty)$ for every $i \in I$ and with $B_i > 0$ for every $i \in I_0$. Of course, I or I_0, or both, may be empty sets.

First of all, we show that when

$A > 0$ or $\alpha = 0$,

every solution of Eq.(2.24) is bounded. In fact, when $A > 0$

$$x_{n+1} \leq \frac{\max_{i\in I}(\alpha, \beta_i)(1 + \sum_{i\in I} x_{n-i})}{\min_{i\in I}(A, B_i)(1 + \sum_{i\in I} x_{n-i})} = \frac{\max_{i\in I}(\alpha, \beta_i)}{\min_{i\in I}(A, B_i)}$$

and thus every solution of Eq.(2.24) is bounded.

In the above inequality by $\max_{i\in I}(\alpha, \beta_i)$, we mean α if $I = \emptyset$ and the maximum of α and $\max_{i\in I} \beta_i$ otherwise. Similarly for the minimum. Moreover, if $I = \emptyset$, we define

$\sum_{i\in I} x_{n-i} = 0.$

Next suppose that $\alpha = 0$. Hence the set I must be nonempty and

$$x_{n+1} \leq \frac{\sum_{i\in I} \beta_i x_{n-i}}{\sum_{i\in I} B_i x_{n-i}} \leq \frac{(\max_{i\in I} \beta_i) \sum_{i\in I} x_{n-i}}{(\min_{i\in I} B_i) \sum_{i\in I} x_{n-i}} = \frac{\max_{i\in I} \beta_i}{\min_{i\in I} B_i}$$

and every solution is bounded.

In the remaining part of the proof we suppose that

$A = 0$ and $\alpha > 0$.

Now the proof depends on whether I or I_0 is empty.

Case 1: $I_0 = \emptyset$. So, because $A = 0$, $I \neq \emptyset$ and

$$x_{n+1} = \frac{\alpha + \sum_{i\in I} \beta_i x_{n-i}}{\sum_{i\in I} B_i x_{n-i}} > \frac{\min_{i\in I} \beta_i}{\max_{i\in I} B_i}, \text{ for } n \geq 0.$$

Hence if we set

$$L = \frac{\min_{i\in I} \beta_i}{\max_{i\in I} B_i},$$

note that for $n \geq k$,

$$x_{n+1} \leq \frac{\alpha}{L \sum_{i\in I} B_i} + \frac{\max_{i\in I} \beta_i}{\min_{i\in I} B_i}$$

and every solution of Eq.(2.24) is bounded from below and from above. Indeed in this case the equation is permanent.

Case 2: $I = \emptyset$. Then $I_0 \neq \emptyset$. In this case the Eq.(2.24) reduces to

$$x_{n+1} = \frac{\alpha}{\sum_{i\in I_0} B_i x_{n-i}}, \ n = 0, 1, ... \tag{2.26}$$

with

$$\sum_{i\in I_0} B_i > 0.$$

We will show that every solution of Eq.(2.26) is bounded. To this end, let $\{x_n\}$ be a solution of Eq.(2.26) and suppose, without loss of generality, that the solution is positive for all $n \geq -k$. Let L, U be chosen in such a way that

$$x_{-k}, ..., x_0 \in (L, U)$$

and

$$LU = \frac{\alpha}{\sum_{i\in I_0} B_i}.$$

Hence

$$L = \frac{\alpha}{U\sum_{i\in I_0} B_i} < x_1 = \frac{\alpha}{\sum_{i\in I_0} B_i x_{-i}} < \frac{\alpha}{L\sum_{i\in I_0} B_i} = U.$$

Then,

$$x_1 \in (L, U)$$

and by induction

$$x_n \in (L, U) \text{ for } n \geq -k.$$

Case 3: Both I and I_0 are nonempty sets. In this case, as in case 2, we will suppose, without loss of generality, that a solution $\{x_n\}$ is positive and show that there exist an interval (L, U) that contains the entire solution.

To see how the interval is found note that

$$x_1 \in (L, U)$$

if and only if

$$L < \frac{\alpha + \sum_{i\in I} \beta_i x_{-i}}{\sum_{i\in I} B_i x_{-i} + \sum_{i\in I_0} B_i x_{-i}} < U$$

if and only if

$$\sum_{i\in I}(LB_i - \beta_i)x_{-i} + (L\sum_{i\in I_0} B_i x_{-i} - \alpha) < 0$$

and

$$\sum_{i\in I}(UB_i - \beta_i)x_{-i} + (U\sum_{i\in I_0} B_i x_{-i} - \alpha) > 0$$

if

$$L < \frac{\beta_i}{B_i} < U \text{ for all } i \in I$$

and

$$\frac{\alpha}{U} < \sum_{i\in I_0} B_i x_{-i} < \frac{\alpha}{L}.$$

But

$$L\sum_{i\in I_0} B_i < \sum_{i\in I_0} B_i x_{-i} < U\sum_{i\in I_0} B_i$$

and hence it suffices to choose L and U such that

$x_{-k}, ..., x_0 \in (L, U),$

$$L < \min_{i \in I}\left(\frac{\beta_i}{B_i}, \frac{B_i}{\beta_i} \frac{\alpha}{\sum_{j \in I_0} B_j}\right),$$

and

$$LU = \frac{\alpha}{\sum_{j \in I_0} B_j}.$$

With the above choice of (L, U), it is now easy to prove that

$x_1 \in (L, U)$

and then by induction

$x_n \in (L, U)$ for $n \geq -k$.

This completes the proof. ■

2.4 INVARIANT PRODUCT METHODS

2.4.1 The Case

$$x_{n+1} = \frac{\alpha + \beta x_n}{C x_{n-1}}, \ n = 0, 1, ... \tag{2.27}$$

We consider Eq.(2.27). See **([26], pp.70-71)**.

This equation is called **Lyness' Equation**. By the change of variables, Eq.(2.27) reduces to the equation

$$y_{n+1} = \frac{p + y_n}{y_{n-1}}, n = 0, 1, ... \tag{2.28}$$

where

$$p = \frac{\alpha C}{\beta^2}.$$

The special case of Eq.(2.28) where

$p = 1$

was discovered by Lyness in 1942 while he was working on a problem in Number Theory. In this special case, the equation becomes

$$y_{n+1} = \frac{1+y_n}{y_{n-1}}, \ n = 0, 1, ... \tag{2.29}$$

every solution of which is periodic with period five. Actually the solution of Eq.(2.29) with initial conditions y_{-1} and y_0 is the five-cycle:

$$y_{-1},\ y_0,\ \frac{1+y_0}{y_{-1}},\ \frac{1+y_{-1}+y_0}{y_{-1}y_0},\ \frac{1+y_{-1}}{y_0},....$$

Eq.(2.28) possesses the invariant

$$I_n = (p + y_{n-1} + y_n)(1 + \frac{1}{y_{n-1}})(1 + \frac{1}{y_n}) = constant \tag{2.30}$$

from which it follows that every solution of Eq.(2.28) is bounded from above and from below by positive constants.

In fact for $n \geq 0$

$$\begin{aligned} (p + y_n + y_{n+1})(1 + \frac{1}{y_n})(1 + \frac{1}{y_{n+1}}) &= (p + y_n + \frac{p+y_n}{y_{n-1}})(1 + \frac{1}{y_n})(1 + \frac{y_{n-1}}{p+y_n}) \\ &= (\frac{p+y_n}{p+y_n} + \frac{1}{y_{n-1}})(1 + \frac{1}{y_n})(p + y_n + y_{n-1}) \\ &= (p + y_{n-1} + y_n)(1 + \frac{1}{y_{n-1}})(1 + \frac{1}{y_n}). \end{aligned}$$

The proof follows by induction.

2.4.2 The Case

$$x_{n+1} = \frac{\alpha}{(1+x_n)x_{n-1}}, \ n = 0, 1, ... \tag{2.31}$$

We consider the difference equation Eq.(2.31). See **([1], p.8)**.

This equation has some similarities with **Lyness's** Equation,

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-1}}, \ n = 0, 1, ... \tag{2.32}$$

which is gifted with the **invariant** (see (2.30))**:**

$$(\alpha + x_{n-1} + x_n)(1 + \frac{1}{x_{n-1}})(1 + \frac{1}{x_n}) = constant, \forall n \geq 0.$$

In fact, as for Eq.(2.32), Eq.(2.31) possesses an invariant, namely,

$$x_{n-1} + x_n + x_{n-1}x_n + \alpha(\frac{1}{x_{n-1}} + \frac{1}{x_n}) = constant, \forall n \geq 0. \tag{2.33}$$

By using (2.33) we see that every positive solution of Eq.(2.31) is bounded from above and from below by positive constants.

In fact for $n \geq 0$

$$\begin{aligned} x_n + x_{n+1} + x_n x_{n+1} + \alpha(\frac{1}{x_n} + \frac{1}{x_{n+1}}) &= x_n + \frac{\alpha}{(1+x_n)x_{n-1}} + x_n \frac{\alpha}{(1+x_n)x_{n-1}} \\ &\quad + \alpha(\frac{1}{x_n} + \frac{(1+x_n)x_{n-1}}{\alpha}) \\ &= x_n + \frac{\alpha\,(1+x_n)}{(1+x_n)x_{n-1}} + \frac{\alpha}{x_n} + x_{n-1} + x_n x_{n-1} \\ &= x_{n-1} + x_n + x_{n-1}x_n + \alpha(\frac{1}{x_{n-1}} + \frac{1}{x_n}). \end{aligned}$$

The proof follows by induction.

2.5 INITIAL CONDITIONS METHODS

2.5.1 The Case

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, ... \tag{2.34}$$

We consider the difference equation (2.34). See **([1], pp.15-16)**.

When one of the initial conditions of a solution of Eq.(2.34) is zero, Eq.(2.34) reduces to the linear equation

$$x_{n+1} = \gamma x_{n-1}$$

with one initial condition equal to zero. If the other initial condition of a solution is ϕ, then the solution of the equation is

..., 0, ϕ, 0, $\gamma\phi$, 0, $\gamma^2\phi$, ...

Therefore the solution converges to zero when

$\gamma < 1$.

When

$\gamma = 1$,

the solution is the period-two sequence

..., 0, ϕ, 0, ϕ, 0, ϕ, ...

and when

$\gamma > 1$ and $\phi > 0$,

the solution is **unbounded**.

2.5.2 The Case

$$x_{n+1} = \frac{(1+\beta x_n)\, x_{n-1}}{A + x_n x_{n-1}}, \ \ n = 0, 1, ... \tag{2.35}$$

We consider the difference equation (2.35). See **([2], pp.202-204)**.
When one of the initial conditions of a solution of Eq.(2.35) is zero, Eq.(2.35) reduces to the linear equation

$$x_{n+1} = \frac{1}{A} x_{n-1}$$

with one initial condition equal to zero. If the other initial condition of a solution is ϕ, then the solution of the equation is

$$..., 0,\ \phi,\ 0,\ \frac{1}{A}\phi,\ 0,\ \frac{1}{A^2}\phi,\ ....$$

So the solution converges to zero when

$A > 1$.

When

$A = 1$,

the solution is the (not necessarily prime) period-two sequence:

$$..., 0,\ \phi,\ 0,\ \phi,\ ...$$

and when

$A < 1$ and $\phi > 0$,

the solution is **unbounded**.

CHAPTER 3

EXAMPLES

In this chapter we investigate the local asymptotic stability and global asymptotic stability of some difference equations.

3.1 EXAMPLES OF THE SECOND ORDER DIFFERENCE EQUATIONS

Example 3.1 ***([7], pp.189-190)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n}{C x_{n-1}}, \quad n = 0, 1, ... \tag{3.1}$$

(a) *The normalized form of Eq.(3.1) is*

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-1}}, \quad n = 0, 1, ... \tag{3.2}$$

(b) *Equilibrium point of Eq.(3.2) is*

$$\overline{x} = \frac{1 + \sqrt{1 + 4\alpha}}{2}.$$

(c) *The linearized equation of Eq.(3.2) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} - \frac{2}{1 + \sqrt{1 + 4\alpha}} z_n + z_{n-1} = 0 \tag{3.3}$$

and the corresponding charasteristic equation of Eq.(3.3) is

$$\lambda^2 - \frac{2}{1 + \sqrt{1 + 4\alpha}} \lambda + 1 = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(3.2) is non-hyperbolic for every $\alpha > 0$.*

Solution.

(a) Using the change of variables

$$x_n = \frac{\beta}{C} y_n,$$

Eq.(3.1) can be written in the normalized form

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-1}}, \quad n = 0, 1, ...$$

with positive parameter α and with arbitrary non-negative initial conditions x_{-1}, x_0 such that the denominator is always positive.

(b) The equilibrium point of Eq.(3.2) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \overline{x}}{\overline{x}}$$

or equivalently

$$\overline{x}^2 - \overline{x} - \alpha = 0. \tag{3.4}$$

Then the only equilibrium point Eq.(3.2) is

$$\overline{x} = \frac{1 + \sqrt{1 + 4\alpha}}{2}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha + x_n}{x_{n-1}}.$$

Hence, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \left[\frac{1}{x_{n-1}}\right](\overline{x}, \overline{x}) = \frac{1}{\overline{x}} = \frac{2}{1 + \sqrt{1 + 4\alpha}}$$

and from (3.4)

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{0 - (\alpha + x_n).1}{(x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{-\alpha + \overline{x}}{\overline{x}^2} = -1.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.2), then the linearized equation associated with Eq.(3.2) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} - \frac{2}{1 + \sqrt{1 + 4\alpha}} z_n + z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.2) about the equilibrium $\overline{x}$ is

$$\lambda^2 - \frac{2}{1 + \sqrt{1 + 4\alpha}} \lambda + 1 = 0.$$

(d) From (c), the characteristic roots are

$$\lambda_1 = \frac{1 + i\sqrt{1 + 4\alpha + 2\sqrt{1 + 4\alpha}}}{1 + \sqrt{1 + 4\alpha}} \text{ and } \lambda_2 = \frac{1 - i\sqrt{1 + 4\alpha + 2\sqrt{1 + 4\alpha}}}{1 + \sqrt{1 + 4\alpha}}.$$

For every $\alpha > 0$, it holds

$$|\lambda_1| = |\lambda_2| = 1$$

and so $\overline{x}$ is a non-hyperbolic equilibrium point.

Example 3.2 ***([7], p.168)*** *Consider the difference equation*

$$x_{n+1} = \frac{\beta x_n}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, ... \tag{3.5}$$

(a) *The normalized form of Eq.(3.5) is*

$$x_{n+1} = \frac{x_n}{Bx_n + x_{n-1}}, \quad n = 0, 1, ... \tag{3.6}$$

(b) *Equilibrium point of Eq.(3.6) is*

$$\overline{x} = \frac{1}{B + 1}.$$

(c) *The linearized equation of Eq.(3.6) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} - \frac{1}{B + 1} z_n + \frac{1}{B + 1} z_{n-1} = 0 \tag{3.7}$$

and the corresponding charasteristic equation of Eq.(3.7) is

$$\lambda^2 - \frac{1}{B + 1}\lambda + \frac{1}{B + 1} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(3.6) is locally asymptotically stable when*

$$B > 0.$$

Solution.

(a) Using the change of variables

$$x_n = \frac{\beta}{C} y_n,$$

Eq.(3.5) can be written in the normalized form

$$x_{n+1} = \frac{x_n}{Bx_n + x_{n-1}}, \quad n = 0, 1, ...$$

with positive parameter B and with arbitrary positive initial conditions x_{-1}, x_0.

(b) The equilibrium point of Eq.(3.6) is the non-negative solution of the equation

$$\overline{x} = \frac{\overline{x}}{B\overline{x} + \overline{x}}.$$

So, (3.6) has the unique equilibrium point

$$\overline{x} = \frac{1}{B+1}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{x_n}{Bx_n + x_{n-1}}.$$

Thus, we observe that

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \left[\frac{1.(Bx_n + x_{n-1}) - x_n.B}{(Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{\overline{x}}{(B\overline{x} + \overline{x})^2} = \frac{1}{B+1}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{0 - x_n.1}{(Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{-\overline{x}}{(B\overline{x} + \overline{x})^2} = \frac{-1}{B+1}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.6), then the linearized equation associated with Eq.(3.6) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} - \frac{1}{B+1} z_n + \frac{1}{B+1} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.6) about the equilibrium $\overline{x}$ is

$$\lambda^2 - \frac{1}{B+1}\lambda + \frac{1}{B+1} = 0.$$

(d) From (c) and Theorem 1.3, $\overline{x}$ is locally asymptotically stable, as long as $B > 0$. ∎

Example 3.3 ***([7], pp.246-247)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + C x_{n-1}}, \quad n = 0, 1, ... \tag{3.8}$$

(a) *The normalized form of Eq.(3.8) is*

$$x_{n+1} = \frac{\beta x_n + x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, ... \tag{3.9}$$

(b) $\overline{x} = 0$ *is always an equilibrium points of Eq.(3.9) and when*

$$A < \beta + 1$$

holds, Eq.(3.9) has also the unique positive equilibrium point

$$\overline{x} = \beta + 1 - A.$$

(c) *The linearized equation of Eq.(3.9) about the equilibrium point* $\overline{x} = 0$ *is*

$$z_{n+1} - \frac{\beta}{A} z_n - \frac{1}{A} z_{n-1} = 0 \tag{3.10}$$

and the corresponding charasteristic equation of Eq.(3.10) is

$$\lambda^2 - \frac{\beta}{A}\lambda - \frac{1}{A} = 0.$$

(d) *The linearized equation of Eq.(3.9) about the equilibrium point* $\overline{x} = \beta + 1 - A$ *is*

$$z_{n+1} - \frac{\beta}{\beta + 1} z_n + \frac{\beta - A}{\beta + 1} z_{n-1} = 0 \tag{3.11}$$

and the corresponding charasteristic equation of Eq.(3.11) is

$$\lambda^2 - \frac{\beta}{\beta + 1}\lambda + \frac{\beta - A}{\beta + 1} = 0.$$

(e) *The zero equilibrium of Eq.(3.9) is globally asymptotically stable when*

$$A \geq \beta + 1$$

and unstable when

$$A < \beta + 1.$$

(f) *The equilibrium point* $\overline{x} = \beta + 1 - A$ *of Eq.(3.9) is locally asymptotically stable for all positive values of the parameters, as long as* $A < \beta + 1$.

Solution.

(a) By the change of variables

$$x_n = \frac{\gamma}{C} y_n,$$

Eq.(3.8) can be written in the normalized form

$$x_{n+1} = \frac{\beta x_n + x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, ...$$

with positive parameters β, A and with arbitrary positive initial conditions x_{-1}, x_0.

(b) The equilibrium points of Eq.(3.9) are the non-negative solution of the equation

$$\overline{x} = \frac{\beta \overline{x} + \overline{x}}{A + \overline{x}}$$

or equivalently

$$\overline{x}\,(\overline{x} - \beta - 1 + A) = 0.$$

Thus,

$$\overline{x} = 0$$

is always an equilibrium point of Eq.(3.9) and when $A < \beta + 1$, Eq.(3.9) has also the unique positive equilibrium point

$$\overline{x} = \beta + 1 - A.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\beta x_n + x_{n-1}}{A + x_{n-1}}.$$

Hence, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \frac{\beta}{A + \overline{x}}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{1.(A + x_{n-1}) - (\beta x_n + x_{n-1}).1}{(A + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{A - \beta\overline{x}}{(A + \overline{x})^2}.$$

So, for $\overline{x} = 0$, the linearized equation associated with Eq.(3.9) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} - \frac{\beta}{A} z_n - \frac{1}{A} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.9) about the equilibrium $\overline{x} = 0$ is

$$\lambda^2 - \frac{\beta}{A}\lambda - \frac{1}{A} = 0. \qquad (3.12)$$

(d) From (c), for $\overline{x} = \beta + 1 - A$, the linearized equation associated with Eq.(3.9) about the equilibrium point $\overline{x}$ is

$$z_{n+1} - \frac{\beta}{\beta + 1} z_n + \frac{\beta - A}{\beta + 1} z_{n-1} = 0$$

and the characteristic equation of the linearized equation of Eq.(3.9) about the equilibrium $\overline{x} = \beta + 1 - A$ is

$$\lambda^2 - \frac{\beta}{\beta + 1}\lambda + \frac{\beta - A}{\beta + 1} = 0. \qquad (3.13)$$

(e) From (3.12) and by Theorem 1.3, Theorem 1.11 and Theorem 1.13, it follows that the zero equilibrium of Eq.(3.9) is globally asymptotically stable when

$$A \geq \beta + 1$$

and unstable when

$$A < \beta + 1.$$

(f) From (3.13) and Theorem 1.3, it follows that the positive equilibrium $\overline{x}$ of Eq.(3.9) is locally asymptotically stable for all positive values of the parameters, as long as $A < \beta + 1$. ■

Example 3.4 *([1], pp.22-23) Consider the second order difference equation*

$$x_{n+1} = \beta x_n + \frac{1}{x_{n-1}}, \quad n = 0, 1, ... \tag{3.14}$$

(a) *Equilibrium point of Eq.(3.14) is*

$$\overline{x} = \frac{1}{\sqrt{1-\beta}}.$$

(b) *The linearized equation of Eq.(3.14) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} - \beta z_n + (1-\beta) z_{n-1} = 0 \tag{3.15}$$

and the corresponding charasteristic equation of Eq.(3.15) is

$$\lambda^2 - \beta\lambda + 1 - \beta = 0.$$

(c) *Eq.(3.14) has bounded solutions if and only if $\beta < 1$.*

(d) *The equilibrium point $\overline{x}$ of Eq.(3.14) is locally asymptotically stable for all values of the parameter β.*

(e) *The equilibrium point $\overline{x}$ of Eq.(3.14) is globally asymptotically stable when*

$$\beta < 1.$$

(f) *The equilibrium point $\overline{x}$ of Eq.(3.14) is not globally asymptotically stable when*

$$\beta \geq 1.$$

Solution.

(a) The equilibrium point of Eq.(3.14) is the non-negative solution of the equation

$$\overline{x} = \beta\overline{x} + \frac{1}{\overline{x}}$$

or

$$(1-\beta)\,\overline{x}^2 - 1 = 0. \tag{3.16}$$

So, when

$$\beta < 1, \tag{3.17}$$

Eq.(3.14) has the unique positive equilibrium point

$$\overline{x} = \frac{1}{\sqrt{1-\beta}}.$$

(b) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \beta x_n + \frac{1}{x_{n-1}}.$$

So, from (3.16), we obtain

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \beta$$

and

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = -\frac{1}{\overline{x}^2} = -(1-\beta).$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.14), then the linearized equation associated with Eq.(3.14) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} - \beta z_n + (1-\beta) z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.14) about the equilibrium $\overline{x}$ is

$$\lambda^2 - \beta\lambda + 1 - \beta = 0. \tag{3.18}$$

(c) It was shown subsection (2.5).

(d) From (3.18) and by Theorem 1.3 the required results follows.

(e) To complete the proof it remains to show that when (3.17) holds, every solution of Eq.(3.14) converges to the equilibrium $\overline{x}$. This follows now by applying Theorem 1.10. Indeed for every $m \in (0, \infty)$ and $M > m$,

$$\left(\beta M + \frac{1}{m} - M\right)\left(\beta m + \frac{1}{M} - m\right) = \left(\frac{(\beta - 1) mM + 1}{m}\right)\left(\frac{(\beta - 1) mM + 1}{M}\right)$$

and the hypotheses of Theorem 1.10 are satisfied. The proof is complete.

(f) From (e), the required results follows. ■

Example 3.5 ***([2], pp.202-203)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{(1 + \beta x_n) x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0, 1, ... \tag{3.19}$$

(a) $\overline{x} = 0$ *is always an equilibrium points of Eq.(3.19) and when*

$$\beta^2 < 4 (A - 1), \tag{3.20}$$

zero is the only equilibrium of the Eq.(3.19). When

$$A \leq 1 \tag{3.21}$$

and also when

$$A > 1 \quad and \quad \beta^2 = 4 (A - 1), \tag{3.22}$$

Eq.(3.19) has the unique positive equilibrium point

$$\overline{x} = \frac{\beta + \sqrt{\beta^2 - 4 (A - 1)}}{2}.$$

Moreover, when

$$A > 1 \quad and \quad \beta^2 > 4 (A - 1), \tag{3.23}$$

Eq.(3.19) has two positive equilibrium points, namely,

$$\overline{x}_1 = \frac{\beta - \sqrt{\beta^2 - 4 (A - 1)}}{2} \quad and \quad \overline{x}_2 = \frac{\beta + \sqrt{\beta^2 - 4 (A - 1)}}{2}.$$

(b) *The linearized equation of Eq.(3.19) about the equilibrium point* $\overline{x} = 0$ *is*

$$z_{n+1} - \frac{1}{A} z_{n-1} = 0 \tag{3.24}$$

and the corresponding charasteristic equation of Eq.(3.24) is

$$\lambda^2 - \frac{1}{A} = 0.$$

(c) *The linearized equation of Eq.(3.19) about a positive equilibrium point* $\overline{y}$*, which takes place* $\overline{x}$ *or* $\overline{x}_1$ *or* $\overline{x}_2$*, is*

$$z_{n+1} + \frac{\overline{y}\left(\overline{y} - A\beta\right)}{\left(1 + \beta\overline{y}\right)^2} z_n - \frac{A}{1 + \beta\overline{y}} z_{n-1} = 0 \tag{3.25}$$

and the corresponding charasteristic equation of Eq.(3.25) is

$$\lambda^2 + \frac{\overline{y}\left(\overline{y} - A\beta\right)}{\left(1 + \beta\overline{y}\right)^2} \lambda - \frac{A}{1 + \beta\overline{y}} = 0.$$

(d) *The equilibrium point* $\overline{x} = 0$ *of Eq.(3.19) is locally asymptotically stable when*

$$A > 1.$$

When

$$A = 1,$$

the zero equilibrium is non-hyberbolic and when

$$A < 1$$

the zero equilibrium is a repeller.

(e) *The positive equilibrium point* $\overline{x} = \frac{\beta + \sqrt{\beta^2 - 4(A-1)}}{2}$ *of Eq.(3.19) is locally asymptotically stable when (3.21) holds, and non-hyperbolic when (3.22) holds. Also, when (3.23) holds, the positive equilibrium point* $\overline{x}_1$ *of Eq.(3.19) is unstable and the positive equilibrium point* $\overline{x}_2$ *of Eq.(3.19) is locally asymptotically stable.*

Solution.

(a) The equilibrium points of Eq.(3.19) are the non-negative solution of the equation

$$\overline{x} = \frac{(1+\beta\overline{x})\,\overline{x}}{A+\overline{x}^2}$$

or equivalently

$$\overline{x}\left(\overline{x}^2 - \beta\overline{x} + A - 1\right) = 0.$$

Thus, we have

$$\overline{x} = 0$$

or

$$\overline{x}^2 - \beta\overline{x} + A - 1 = 0. \tag{3.26}$$

Hence, zero is always an equilibrium point of Eq.(3.19) and when (3.20) holds, zero is the only equilibrium of the Eq.(3.19).

In addition to the zero equilibrium, from (3.26), when (3.21) and (3.22) holds, Eq.(3.19) has the unique positive equilibrium point

$$\overline{x} = \frac{\beta + \sqrt{\beta^2 - 4(A-1)}}{2}.$$

Also, when (3.23) holds, Eq.(3.19) has two positive equilibrium points:

$$\overline{x}_1 = \frac{\beta - \sqrt{\beta^2 - 4(A-1)}}{2} \quad \text{and} \quad \overline{x}_2 = \frac{\beta + \sqrt{\beta^2 - 4(A-1)}}{2}.$$

(b) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{(1+\beta x_n)\,x_{n-1}}{A + x_n x_{n-1}}.$$

Then, we have

$$\begin{aligned} q_0 &= \frac{\partial f}{\partial x_n}(\overline{x},\overline{x}) = \left[\frac{\beta x_{n-1}.\,(A + x_n x_{n-1}) - (1+\beta x_n)\,x_{n-1}.x_{n-1}}{(A + x_n x_{n-1})^2}\right](\overline{x},\overline{x}) \\ &= \frac{\overline{x}\,(A\beta - \overline{x})}{(A+\overline{x}^2)^2} \end{aligned}$$

and

$$
\begin{aligned}
q_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x},\overline{x}) = \left[\frac{(1+\beta x_n).(A+x_n x_{n-1}) - (1+\beta x_n)\,x_{n-1}.x_n}{(A+x_n x_{n-1})^2}\right](\overline{x},\overline{x}) \\
&= \frac{A(1+\beta\overline{x})}{(A+\overline{x}^2)^2}.
\end{aligned}
$$

From this, for $\overline{x}=0$, the linearized equation associated with Eq.(3.19) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_1 z_{n-1}$$

or

$$z_{n+1} - \frac{1}{A} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.19) about the equilibrium $\overline{x}=0$ is

$$\lambda^2 - \frac{1}{A} = 0. \tag{3.27}$$

(c) From (b) and (3.26), the linearized equation of Eq.(3.19) about a positive equilibrium point $\overline{x}$ is

$$z_{n+1} + \frac{\overline{y}(\overline{y}-A\beta)}{(1+\beta\overline{y})^2} z_n - \frac{A}{1+\beta\overline{y}} z_{n-1} = 0$$

and the corresponding charasteristic equation of Eq.(3.25) is

$$\lambda^2 + \frac{\overline{y}(\overline{y}-A\beta)}{(1+\beta\overline{y})^2}\lambda - \frac{A}{1+\beta\overline{y}} = 0. \tag{3.28}$$

(d) From (3.27) and by Theorem 1.3, the required results follows.

(e) From (3.28) and by Theorem 1.3 the required results follows. ■

Example 3.6 ***([2], p.219)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{\alpha+\beta x_n x_{n-1} + x_{n-1}}{B x_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, ... \tag{3.29}$$

(a) *Eq.(3.29) has a unique equilibrium $\overline{x}$, and $\overline{x}$ is the unique positive solution of the cubic equation*

$$B\overline{x}^3 + (1-\beta)\,\overline{x}^2 - \overline{x} - \alpha = 0.$$

(b) *The linearized equation of Eq.(3.29) about the equilibrium point* $\overline{x}$ *is*

$$z_{n+1} + \frac{\alpha B - (\beta - B)\,\overline{x}}{\overline{x}\,(B\overline{x}+1)^2} z_n + \frac{\alpha}{\overline{x}^2\,(B\overline{x}+1)} z_{n-1} = 0 \tag{3.30}$$

and the corresponding charasteristic equation of Eq.(3.30) is

$$\lambda^2 + \frac{\alpha B - (\beta - B)\,\overline{x}}{\overline{x}\,(B\overline{x}+1)^2}\lambda + \frac{\alpha}{\overline{x}^2\,(B\overline{x}+1)} = 0.$$

(c) *Every positive solution of Eq.(3.29) is bounded.*

(d) *The equilibrium point* $\overline{x}$ *of Eq.(3.29) is locally asymptotically stable for all values of the parameters* α*,* β *and* B*.*

Solution.

(a) The equilibrium point of Eq.(3.29) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \beta\overline{x}^2 + \overline{x}}{B\overline{x}^2 + \overline{x}}$$

or equivalently

$$B\overline{x}^3 + (1-\beta)\,\overline{x}^2 - \overline{x} - \alpha = 0. \tag{3.31}$$

It follows that, Eq.(3.29) has a unique equilibrium $\overline{x}$, and $\overline{x}$ is the unique positive root of the cubic equation (3.31).

(b) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{B x_n x_{n-1} + x_{n-1}}.$$

From (3.31), we have

$$\begin{aligned}
q_0 &= \frac{\partial f}{\partial x_n}(\overline{x},\overline{x}) \\
&= \left[\frac{\beta x_{n-1}.\,(B x_n x_{n-1} + x_{n-1}) - (\alpha + \beta x_n x_{n-1} + x_{n-1})\,.B x_{n-1}}{(B x_n x_{n-1} + x_{n-1})^2}\right](\overline{x},\overline{x}) \\
&= \frac{-\alpha B + (\beta - B)\,\overline{x}}{\overline{x}\,(B\overline{x}+1)^2}
\end{aligned}$$

and

$$\begin{aligned} q_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x},\overline{x}) \\ &= \left[\frac{(\beta x_n+1).(Bx_nx_{n-1}+x_{n-1})-(\alpha+\beta x_nx_{n-1}+x_{n-1}).(\beta x_n+1)}{(Bx_nx_{n-1}+x_{n-1})^2}\right](\overline{x},\overline{x}) \\ &= \frac{-\alpha}{\overline{x}^2(B\overline{x}+1)}. \end{aligned}$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.29), then the linearized equation associated with Eq.(3.29) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} + \frac{\alpha B-(\beta-B)\overline{x}}{\overline{x}(B\overline{x}+1)^2} z_n + \frac{\alpha}{\overline{x}^2(B\overline{x}+1)} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.29) about the equilibrium $\overline{x}$ is

$$\lambda^2 + \frac{\alpha B-(\beta-B)\overline{x}}{\overline{x}(B\overline{x}+1)^2}\lambda + \frac{\alpha}{\overline{x}^2(B\overline{x}+1)} = 0. \tag{3.32}$$

(c) It was shown subsection (2.22).

(d) From (3.32) and by Theorem 1.3, the required results. ∎

Example 3.7 ***([1], pp.22-23 and [29], p.156)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{\alpha}{1+x_nx_{n-1}}, \quad n=0,1,.... \tag{3.33}$$

(a) *Eq.(3.33) has a unique equilibrium $\overline{x}$, and $\overline{x}$ is the unique positive root of the cubic equation:*

$$\overline{x}^3 + \overline{x} - \alpha = 0.$$

(b) *The linearized equation of Eq.(3.33) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + \frac{\alpha-\overline{x}}{\alpha} z_n + \frac{\alpha-\overline{x}}{\alpha} z_{n-1} = 0 \tag{3.34}$$

and the corresponding charasteristic equation of Eq.(3.34) is

$$\lambda^2 + \frac{\alpha-\overline{x}}{\alpha}\lambda + \frac{\alpha-\overline{x}}{\alpha} = 0.$$

(c) *The equilibrium point $\overline{x}$ of Eq.(3.33) is locally asymptotically stable for all values of the parameter α.*

(d) *Every solution of Eq.(3.33) is bounded.*

(e) *The unique positive equilibrium of Eq.(3.33) is globally asymptotically stable for all values of the parameter α.*

Solution.

(a) The equilibrium point of Eq.(3.33) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha}{1+\overline{x}^2}$$

or

$$\overline{x}^3 + \overline{x} - \alpha = 0. \qquad (3.35)$$

From this, Eq.(3.33) has a unique equilibrium $\overline{x}$, and $\overline{x}$ is the unique positive root of the cubic equation (3.35).

(b) Let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha}{1+x_n x_{n-1}}.$$

Then, from (3.35), we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x},\overline{x}) = \left[\frac{0-\alpha.x_{n-1}}{(1+x_n x_{n-1})^2}\right](\overline{x},\overline{x}) = \frac{-\alpha\overline{x}}{(1+\overline{x}^2)^2} = \frac{\overline{x}-\alpha}{\alpha}$$

and

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x},\overline{x}) = \left[\frac{0-\alpha.x_{n}}{(1+x_n x_{n-1})^2}\right](\overline{x},\overline{x}) = \frac{-\alpha\overline{x}}{(1+\overline{x}^2)^2} = \frac{\overline{x}-\alpha}{\alpha}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.33), then the linearized equation associated with Eq.(3.33) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} + \frac{\alpha-\overline{x}}{\alpha} z_n + \frac{\alpha-\overline{x}}{\alpha} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.33) about the equilibrium $\overline{x}$ is

$$\lambda^2 + \frac{\alpha - \overline{x}}{\alpha}\lambda + \frac{\alpha - \overline{x}}{\alpha} = 0. \tag{3.36}$$

(c) From (3.36) and by Theorem 1.3, the required results.

(d) Note that

$$\frac{\alpha}{1+\alpha^2} \leq x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}} \leq \alpha, \quad \text{for all } n \geq 1.$$

So, every solution of Eq.(3.33) is bounded.

(e) Every solution of Eq.(3.33) satisfies

$$\frac{\alpha}{1+\alpha^2} \leq x_n \leq \alpha, \quad n = 1, 2, ... \tag{3.37}$$

Every solution of Eq.(3.33) satisfies $\alpha = x_{n+1}(1 + x_n x_{n-1})$ and so by using it in the first iterate of Eq.(3.33) we obtain

$$x_{n+2} = \frac{x_{n+1}(1 + x_n x_{n-1})}{1 + x_n x_{n+1}}$$

which means that a solution of Eq.(3.33) satisfies the following equation on the interval $I = [0, \infty)$

$$x_{n+2} = \frac{1}{1 + x_n x_{n+1}} x_{n+1} + \frac{x_n x_{n+1}}{1 + x_n x_{n+1}} x_{n-1} = g_{-1} x_{n+1} + g_1 x_{n-1}, \quad n = 0, 1, ...$$

where

$$g_1 = \frac{x_n x_{n+1}}{1 + x_n x_{n+1}}, \quad g_{-1} = \frac{1}{1 + x_n x_{n+1}}.$$

Clearly $g_{-1} + g_1 = 1$. By using the estimate (3.37) we obtain

$$g_{-1} = \frac{1}{1 + x_n x_{n+1}} \geq \frac{1}{1+\alpha^2} > 0,$$

and so all conditions of Theorem 1.14 are satisfied on the interval $I = [0, \infty)$, which implies that the unique positive equilibrium $\overline{x}$ is global attractor and because it is locally asymptotically stable, it is also globally asymptotically stable. ■

Example 3.8 ***([2], p.214 and [29], p.157-158)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}}, \quad n = 0, 1, ... \tag{3.38}$$

(a) *Eq.(3.38) has a unique equilibrium* $\overline{x}$, *and* $\overline{x}$ *is the unique positive solution of the cubic equation*

$$\overline{x}^3 - \beta\overline{x}^2 - \overline{x} - \alpha = 0.$$

(b) *The linearized equation of Eq.(3.38) about the equilibrium point* $\overline{x}$ *is*

$$z_{n+1} + \frac{\overline{x} - \beta}{\overline{x}} z_n + \frac{\alpha}{\overline{x}^3} z_{n-1} = 0 \tag{3.39}$$

and the corresponding charasteristic equation of Eq.(3.39) is

$$\lambda^2 + \frac{\overline{x} - \beta}{\overline{x}} \lambda + \frac{\alpha}{\overline{x}^3} = 0.$$

(c) *The equilibrium point* $\overline{x}$ *of Eq.(3.38) is locally asymptotically stable for all values of the parameters* α *and* β.

(d) *Every solution of Eq.(3.38) is bounded.*

(e) *The unique positive equilibrium of Eq.(3.38) is globally asymptotically stable for all values of the parameter* α.

Solution.

(a) The equilibrium point of Eq.(3.38) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \beta\overline{x}^2 + \overline{x}}{\overline{x}^2}$$

or equivalently

$$\overline{x}^3 - \beta\overline{x}^2 - \overline{x} - \alpha = 0. \tag{3.40}$$

Hence, the equilibrium point $\overline{x}$ of Eq.(3.38) is the unique positive solution of the cubic equation (3.40).

(b) Now, let I be some interval of real numbers and let $f : I \times I \rightarrow I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}}.$$

From (3.40), we have

$$\begin{aligned} q_0 &= \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) \\ &= \left[\frac{\beta x_{n-1}.x_n x_{n-1} - (\alpha + \beta x_n x_{n-1} + x_{n-1}).x_{n-1}}{(x_n x_{n-1})^2} \right] (\overline{x}, \overline{x}) \\ &= \frac{-(\alpha + \overline{x})}{\overline{x}^3} = \frac{(\beta - \overline{x})}{\overline{x}} \end{aligned}$$

and

$$\begin{aligned} q_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) \\ &= \left[\frac{(\beta x_n + 1).x_n x_{n-1} - (\alpha + \beta x_n x_{n-1} + x_{n-1}).x_n}{(x_n x_{n-1})^2} \right] (\overline{x}, \overline{x}) \\ &= \frac{-\alpha}{\overline{x}^3}. \end{aligned}$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.38), then the linearized equation associated with Eq.(3.38) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} + \frac{\overline{x} - \beta}{\overline{x}} z_n + \frac{\alpha}{\overline{x}^3} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.38) about the equilibrium $\overline{x}$ is

$$\lambda^2 + \frac{\overline{x} - \beta}{\overline{x}} \lambda + \frac{\alpha}{\overline{x}^3} = 0. \tag{3.41}$$

(c) From (3.41) and by Theorem 1.3, the required results follows.

(d) Every solution of Eq.(3.38) is bounded from above and below by positive constants. Indeed for $n \geq 0$,

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}} > \frac{\beta x_n x_{n-1}}{x_n x_{n-1}} = \beta$$

and

$$x_{n+1} \le \frac{\alpha}{\beta^2} + \beta + \frac{1}{\beta}.$$

(e) Every solution of Eq.(3.38) satisfies

$$\beta \le x_n \le \beta + \frac{\alpha}{\beta^2} + \frac{1}{\beta}, \quad n = 1, 2, ... \tag{3.42}$$

Every solution of Eq.(3.38) $\alpha = x_{n-1}x_nx_{n+1} - \beta x_nx_{n-1} - x_{n-1}$ and so by replacing it in Eq.(3.38), after one iteration, we obtain

$$x_{n+2} = \frac{x_{n-1}x_nx_{n+1} - \beta x_nx_{n-1} - x_{n-1} + \beta x_{n+1}x_n}{x_nx_{n+1}}$$

which means that a solution of Eq.(3.38) satisfies the following embedded third order difference equation on the interval $I = (0, \infty)$ for $n = 0, 1, ...$

$$x_{n+2} = \frac{\beta x_n}{x_nx_{n+1}}x_{n+1} + \frac{1}{x_nx_{n+1}}x_n + \frac{x_nx_{n+1} - \beta x_n - 1}{x_nx_{n+1}}x_{n-1} = g_{-1}x_{n+1} + g_0x_n + g_1x_{n-1}, \tag{3.43}$$

where

$$g_{-1} = \frac{\beta x_n}{x_nx_{n+1}}, \quad g_0 = \frac{1}{x_nx_{n+1}}, \quad g_1 = \frac{x_nx_{n+1} - \beta x_n - 1}{x_nx_{n+1}}.$$

Note that $g_{-1} + g_0 + g_1 = 1$. By using the estimate (3.42) we obtain

$$g_{-1} = \frac{\beta x_n}{x_nx_{n+1}} \ge \frac{\beta}{\beta + \frac{\alpha}{\beta^2} + \frac{1}{\beta}} = \frac{\beta^3}{\alpha + \beta^3 + \beta} > 0.$$

Furthermore, $g_0 > 0$ and $g_1 \ge 0$ if and only if $x_nx_{n+1} - \beta x_n - 1 \ge 0$, which immediately follows from Eq.(3.38). Thus, all conditions of Theorem 1.14 are satisfied on the interval $I = (0, \infty)$ for Eq.(3.43), which implies that every solution of that equation converges to a finite limit. This implies that the unique positive equilibrium $\overline{x}$ of Eq.(3.38) is global attractor and because it is locally asymptotically stable, it is also globally asymptotically stable. ■

3.2 EXAMPLES OF THE THIRD ORDER DIFFERENCE EQUATIONS

Example 3.9 ***([7], p.184)*** *Consider the third order difference equation*

$$x_{n+1} = \frac{\delta x_{n-2}}{Bx_n + Dx_{n-2}}, \quad n = 0, 1, ... \tag{3.44}$$

(a) *The normalized form of Eq.(3.44) is*

$$x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-2}}, \qquad n = 0, 1, ... \tag{3.45}$$

(b) *Equilibrium point of Eq.(3.45) is*

$$\overline{x} = \frac{1}{B+1}.$$

(c) *The linearized equation of Eq.(3.45) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + \frac{B}{B+1} z_n - \frac{B}{B+1} z_{n-2} = 0 \tag{3.46}$$

and the corresponding charasteristic equation of Eq.(3.46) is

$$\lambda^3 + \frac{B}{B+1}\lambda^2 - \frac{B}{B+1} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(3.45) is locally asymptotically stable when*

$$B < 1 + \sqrt{2}$$

and unstable when

$$B > 1 + \sqrt{2}$$

and non-hyperbolic when

$$B = 1 + \sqrt{2}.$$

Solution.

(a) Using the change of variables

$$x_n = \frac{\delta}{D} y_n,$$

Eq.(3.44) can be written in the normalized form

$$x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-2}}, \qquad n = 0, 1, ...$$

with positive parameter B and with arbitrary non-negative initial conditions x_{-2}, x_{-1}, x_0 such that the denominator is always positive.

(b) The equilibrium point of Eq.(3.45) is the non-negative solution of the equation

$$\overline{x} = \frac{\overline{x}}{B\overline{x} + \overline{x}}$$

or equivalently the only equilibrium point Eq.(3.45) is

$$\overline{x} = \frac{1}{B+1}.$$

(c) Now, let I be some interval of real numbers and let $f : I^3 \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \frac{x_{n-2}}{Bx_n + x_{n-2}}.$$

Therefore, we observe that

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{0 - x_{n-2}B}{(Bx_n + x_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{-B}{B+1}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = 0$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{1.(Bx_n + x_{n-2}) - x_{n-2}.1}{(Bx_n + x_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{B}{B+1}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.45), then the linearized equation associated with Eq.(3.45) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} + \frac{B}{B+1} z_n - \frac{B}{B+1} z_{n-2} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.45) about the equilibrium $\overline{x}$ is

$$\lambda^3 + \frac{B}{B+1}\lambda^2 - \frac{B}{B+1} = 0.$$

(d) From (c) and Theorem 1.4 it follows that the positive equilibrium $\overline{x}$ of Eq.(3.45) is locally asymptotically stable when

$$B < 1 + \sqrt{2}$$

and unstable when

$B > 1 + \sqrt{2}$.

When

$B = 1 + \sqrt{2}$,

$\overline{x}$ is a non-hyperbolic equilibrium. In fact, the eigenvalues of the corresponding characteristic equation are

$$\lambda_1 = \frac{\sqrt{2}}{2}, \quad \lambda_2 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \quad \lambda_3 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

Note that λ_2 and λ_3 are eighth roots of unity. ■

Example 3.10 ***([7], p.227)*** *Consider the difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n}{Bx_n + Dx_{n-2}}, \quad n = 0, 1, ... \tag{3.47}$$

(a) *The normalized form of Eq.(3.47) is*

$$x_{n+1} = \frac{\alpha + x_n}{x_n + Dx_{n-2}}, \quad n = 0, 1, ... \tag{3.48}$$

(b) *Equilibrium point of Eq.(3.48) is*

$$\overline{x} = \frac{1 + \sqrt{1 + 4\alpha(D+1)}}{2(D+1)}.$$

(c) *The linearized equation of Eq.(3.48) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + \frac{\overline{x} - 1}{\overline{x}(1+D)} z_n + \frac{D}{1+D} z_{n-2} = 0 \tag{3.49}$$

and the corresponding charasteristic equation of Eq.(3.49) is

$$\lambda^3 + \frac{\overline{x} - 1}{\overline{x}(1+D)} \lambda^2 + \frac{D}{1+D} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(3.48) is locally asymptotically stable when*

$0 < D \leq 1 + \sqrt{2}$

or

$D > 1 + \sqrt{2}$ *and* $\alpha < \dfrac{D(D^2 - 2D - 1)}{(3D+1)^2}$

and unstable when

$D > 1 + \sqrt{2}$ *and* $\alpha < \dfrac{D(D^2 - 2D - 1)}{(3D+1)^2}$.

Solution.

(a) Using the change of variables

$$x_n = \frac{\beta}{B} y_n,$$

Eq.(3.47) can be written in the normalized form

$$x_{n+1} = \frac{\alpha + x_n}{x_n + Dx_{n-2}}, \quad n = 0, 1, ...$$

with positive parameters α, D and with arbitrary positive initial conditions x_{-2}, x_{-1}, x_0.

(b) The equilibrium point of Eq.(3.48) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \overline{x}}{\overline{x} + D\overline{x}}$$

or equivalently

$$(D+1)\,\overline{x}^2 - \overline{x} - \alpha = 0. \tag{3.50}$$

Therefore, the only equilibrium point of Eq.(3.48) is

$$\overline{x} = \frac{1 + \sqrt{1 + 4\alpha(D+1)}}{2(D+1)}.$$

(c) Now, let I be some interval of real numbers and let $f : I^3 \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \frac{\alpha + x_n}{x_n + Dx_{n-2}}.$$

From this and (3.50) we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{1.(x_n + Dx_{n-2}) - (\alpha + x_n).1}{(x_n + Dx_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{D\overline{x} - \alpha}{\overline{x}^2(1+D)^2} = \frac{1 - \overline{x}}{\overline{x}(1+D)}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = 0$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{0 - (\alpha + x_n)D}{(x_n + Dx_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{-D\alpha - D\overline{x}}{\overline{x}^2(1+D)^2} = \frac{-D}{1+D}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.48), then the linearized equation associated with Eq.(3.48) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} + \frac{\overline{x}-1}{\overline{x}(1+D)} z_n + \frac{D}{1+D} z_{n-2} = 0$$

The characteristic equation of the linearized equation of Eq.(3.48) about the equilibrium $\overline{x}$ is

$$\lambda^3 + \frac{\overline{x}-1}{\overline{x}(1+D)}\lambda^2 + \frac{D}{1+D} = 0.$$

(d) From (c) and Theorem 1.4 it follows that the positive equilibrium $\overline{x}$ of Eq.(3.48) is locally asymptotically stable when

$$0 < D \leq 1 + \sqrt{2}$$

or

$$D > 1 + \sqrt{2} \text{ and } \alpha > \frac{D(D^2 - 2D - 1)}{(3D+1)^2}$$

and unstable when

$$D > 1 + \sqrt{2} \text{ and } \alpha < \frac{D(D^2 - 2D - 1)}{(3D+1)^2}. \quad \blacksquare$$

Example 3.11 ***([7], pp.234-235)*** *Consider the difference equation*

$$x_{n+1} = \frac{\alpha + \delta x_{n-2}}{A + C x_{n-1}}, \quad n = 0, 1, ... \tag{3.51}$$

(a) *The normalized form of Eq.(3.51) is*

$$x_{n+1} = \frac{\alpha + x_{n-2}}{A + x_{n-1}}, \quad n = 0, 1, ... \tag{3.52}$$

(b) *Equilibrium point of Eq.(3.52) is*

$$\overline{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha}}{2}.$$

(c) *The linearized equation of Eq.(3.52) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + \frac{\overline{x}}{A+\overline{x}} z_{n-1} - \frac{1}{A+\overline{x}} z_{n-2} = 0 \tag{3.53}$$

and the corresponding charasteristic equation of Eq.(3.53) is

$$\lambda^3 + \frac{\overline{x}}{A+\overline{x}}\lambda - \frac{1}{A+\overline{x}} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(3.52) is locally asymptotically stable when*

$$A \geq 1$$

or

$$A < 1 \text{ and } \alpha > \frac{(A-1)^2 (A+1)}{A^2}$$

and unstable when

$$A < 1 \text{ and } \alpha < \frac{(A-1)^2 (A+1)}{A^2}.$$

(e) *The equilibrium point $\overline{x}$ of Eq.(3.52) is globally asymptotically stable when*

$$A \geq 1.$$

Solution.

(a) By the change of variables

$$x_n = \frac{\delta}{C} y_n,$$

Eq.(3.51) can be written in the normalized form

$$x_{n+1} = \frac{\alpha + x_{n-2}}{A + x_{n-1}}, \quad n = 0, 1, ...$$

with positive parameters α, A and with arbitrary positive initial conditions x_{-2}, x_{-1}, x_0.

(b) The equilibrium point of Eq.(3.52) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \overline{x}}{A + \overline{x}}$$

or equivalently the only equilibrium point Eq.(3.52) is

$$\overline{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha}}{2}.$$

(c) Now, let I be some interval of real numbers and let $f : I^3 \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \frac{\alpha + x_{n-2}}{A + x_{n-1}}.$$

From this, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = 0$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{0 - (\alpha + x_{n-2}).1}{(A + x_{n-1})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{-\alpha + \overline{x}}{(A + \overline{x})^2} = \frac{-\overline{x}}{A + \overline{x}}$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{1}{A + x_{n-1}}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{1}{A + \overline{x}}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.52), then the linearized equation associated with Eq.(3.52) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} + \frac{\overline{x}}{A + \overline{x}} z_{n-1} - \frac{1}{A + \overline{x}} z_{n-2} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.52) about the equilibrium $\overline{x}$ is

$$\lambda^3 + \frac{\overline{x}}{A + \overline{x}}\lambda - \frac{1}{A + \overline{x}} = 0.$$

(d) From (c) and Theorem 1.4 it follows that the positive equilibrium $\overline{x}$ of Eq.(3.52) is locally asymptotically stable when

$$A \geq 1$$

or

$$A < 1 \text{ and } \alpha > \frac{(A-1)^2 (A+1)}{A^2}$$

and unstable when

$$A < 1 \text{ and } \alpha < \frac{(A-1)^2 (A+1)}{A^2}.$$

(e) By Theorems 1.11 and 1.12 it follows that when

$$A \geq 1,$$

the equilibrium point $\overline{x}$ of Eq.(3.52) is globally asymptotically stable. ■

CHAPTER 4

DIFFERENCE EQUATION:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, ...$$

In this chapter, we investigate the global character of the solutions of the rational difference equation of the second order

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}}, \; n = 0, 1, .. \tag{4.1}$$

where the parameters α, β, γ, B and C are non-negative real numbers, and the initial conditions $x_{-1,}$ x_0 are arbitrary non-negative real numbers such that the denominator of Eq.(4.1) is never zero.

4.1 LINEARIZED STABILITY ANALYSIS

Lemma 4.1 **(a)** *Eq.(4.1) can be written in the normalized form*

$$x_{n+1} = \frac{\alpha + x_n + \gamma x_{n-1}}{x_n + Cx_{n-1}}, \; n = 0, 1, ... \tag{4.2}$$

with positive parameters α, γ, C and with arbitrary positive initial conditions $x_{-1,}$ x_0.

(b) *Equilibrium point of Eq.(4.2) is*

$$\overline{x} = \frac{1 + \gamma + \sqrt{(1+\gamma)^2 + 4\alpha(1+C)}}{2(1+C)}.$$

(c) *The linearized equation of Eq.(4.2) about its positive equilibrium $\overline{x}$ is*

$$z_{n+1} + \frac{\overline{x} - 1}{\overline{x}(1+C)} z_n + \frac{C\overline{x} - \gamma}{\overline{x}(1+C)} z_{n-1} = 0.$$

Proof.

(a) The Eq.(4.1) which by the change of variables

$$x_n = \frac{\beta}{B} y_n$$

reduces to the difference equation

$$x_{n+1} = \frac{\alpha + x_n + \gamma x_{n-1}}{x_n + C x_{n-1}}, n = 0, 1, ...$$

where

$$\alpha := \frac{\alpha B}{\beta^2}, \quad \gamma := \frac{\gamma}{\beta}, \quad C := \frac{C}{B}.$$

(b) The equilibrium points of Eq.(4.2) are the non-negative solutions of the equation

$$\overline{x} = \frac{\alpha + \overline{x} + \gamma \overline{x}}{\overline{x} + C\overline{x}}$$

or equivalently

$$(1 + C)\overline{x}^2 - (1 + \gamma)\overline{x} - \alpha = 0. \tag{4.3}$$

Hence, the solutions of Eq.(4.3) are

$$\overline{x} = \frac{1 + \gamma + \sqrt{(1 + \gamma)^2 + 4\alpha(1 + C)}}{2(1 + C)} \tag{4.4}$$

and

$$\overline{x} = \frac{1 + \gamma - \sqrt{(1 + \gamma)^2 + 4\alpha(1 + C)}}{2(1 + C)}$$

So, the positive equilibrium point of Eq.(4.2) is unique and is given by (4.4).

(c) Now, let I be some interval of real numbers and let

$$f : I \times I \to I$$

be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha + x_n + \gamma x_{n-1}}{x_n + C x_{n-1}}.$$

From Eq.(4.3), we obtain that

$$\begin{aligned} q_0 &= \frac{\partial f}{\partial x_n}(\overline{x},\overline{x}) = \left[\frac{1.(x_n + Cx_{n-1}) - (\alpha + x_n + \gamma x_{n-1}).1}{(x_n + Cx_{n-1})^2}\right](\overline{x},\overline{x}) \\ &= \frac{\overline{x} + C\overline{x} - \alpha - \overline{x} - \gamma\overline{x}}{\overline{x}^2(1+C)^2} = \frac{-\alpha + (C-\gamma)\overline{x}}{\overline{x}^2(1+C)^2} = \frac{1-\overline{x}}{\overline{x}(1+C)} \end{aligned}$$

and

$$\begin{aligned} q_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x},\overline{x}) = \left[\frac{\gamma.(x_n + Cx_{n-1}) - (\alpha + x_n + \gamma x_{n-1}).C}{(x_n + Cx_{n-1})^2}\right](\overline{x},\overline{x}) \\ &= \frac{\gamma\overline{x} + \gamma C\overline{x} - \alpha C - C\overline{x} - \gamma C\overline{x}}{\overline{x}^2(1+C)^2} = \frac{-\alpha C + (\gamma - C)\overline{x}}{\overline{x}^2(1+C)^2} = \frac{\gamma - C\overline{x}}{\overline{x}(1+C)}. \end{aligned}$$

If $\overline{x}$ denotes an equilibrium point of Eq.(4.2), then the linearized equation associated with Eq.(4.2) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} + \frac{\overline{x}-1}{\overline{x}(1+C)}z_n + \frac{C\overline{x}-\gamma}{\overline{x}(1+C)}z_{n-1} = 0. \ \blacksquare \tag{4.5}$$

Lemma 4.2 *Every solution $\{x_n\}$ of Eq.(4.1) is bounded.*

Proof. Here for $n \geq 0$,

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}} > \frac{\beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}} > \frac{\min\{\beta,\gamma\}(x_n + x_{n-1})}{\max\{B,C\}(x_n + x_{n-1})} = \frac{\min\{\beta,\gamma\}}{\max\{B,C\}}.$$

Set

$$L = \frac{\min\{\beta,\gamma\}}{\max\{B,C\}}.$$

Then for $n \geq 2$,

$$x_{n+1} = \frac{\alpha}{Bx_n + Cx_{n-1}} + \frac{\beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}} \leq \frac{\alpha}{(B+1)L} + \frac{\max\{\beta,\gamma\}}{\min\{B,C\}}$$

and consequently, every solution is bounded. $\blacksquare$

Lemma 4.3 *The positive equilibrium $\overline{x}$ of Eq.(4.2) is locally asymptotically stable when*

$$\frac{(\gamma-1)^2 - C(\gamma+3)(\gamma-1)}{4C^2} < \alpha. \tag{4.6}$$

Proof. The characteristic equation of the linearized equation of Eq.(4.2) about the equilibrium $\overline{x}$ is

$$\lambda^2 + \frac{\overline{x}-1}{\overline{x}(1+C)}\lambda + \frac{C\overline{x}-\gamma}{\overline{x}(1+C)} = 0.$$

From Theorem 1.3 we observe that

$$\left|\frac{\overline{x}-1}{\overline{x}(1+C)}\right| < 1 + \frac{C\overline{x}-\gamma}{\overline{x}(1+C)} < 2.$$

Then, we have

$$\left|\frac{\overline{x}-1}{\overline{x}(1+C)}\right| < \frac{(2C+1)\overline{x}-\gamma}{\overline{x}(1+C)} \tag{4.7}$$

and

$$\frac{C\overline{x}-\gamma}{\overline{x}(1+C)} < 1. \tag{4.8}$$

So, from (4.7) we obtain

$$\frac{\gamma-(2C+1)\overline{x}}{\overline{x}(1+C)} < \frac{\overline{x}-1}{\overline{x}(1+C)} < \frac{(2C+1)\overline{x}-\gamma}{\overline{x}(1+C)}$$

so that

$$\frac{1+\gamma}{2(1+C)} < \overline{x} \text{ and } \frac{\gamma-1}{2C} < \overline{x}.$$

In case $\frac{1+\gamma}{2(1+C)} < \overline{x}$, from Eq.(4.4) we have

$$\frac{-(\gamma+1)^2}{4(C+1)} < \alpha. \tag{4.9}$$

Because of the fact that α, γ, and C are positive, (4.9) is always true for all positive values of the parameters.

In case $\frac{\gamma-1}{2C} < \overline{x}$, from Eq.(4.4) we get

$$\begin{aligned}
\frac{\gamma-1}{2C} &< \frac{1+\gamma+\sqrt{(1+\gamma)^2+4\alpha(1+C)}}{2(1+C)} \\
&\Rightarrow (\gamma-1)(1+C) < C(1+\gamma+\sqrt{(1+\gamma)^2+4\alpha(1+C)}) \\
&\Rightarrow \gamma-2C-1 < C\sqrt{(1+\gamma)^2+4\alpha(1+C)} \\
&\Rightarrow (\gamma-2C-1)^2-(C+C\gamma)^2 < 4\alpha C^2(1+C) \\
&\Rightarrow (\gamma-C-1+C\gamma)(\gamma-3C-1-C\gamma) < 4\alpha C^2(1+C) \\
&\Rightarrow (\gamma-1)((\gamma-1)-C(\gamma+3)) < 4\alpha C^2 \\
&\Rightarrow (\gamma-1)^2-C(\gamma+3)(\gamma-1) < 4\alpha C^2 \\
&\Rightarrow \frac{(\gamma-1)^2-C(\gamma+3)(\gamma-1)}{4C^2} < \alpha.
\end{aligned}$$

Finally, from (4.8) we obtain

$C\overline{x} - \gamma < \overline{x}(1 + C)$

therefore, we have

$$-\gamma < \overline{x} \tag{4.10}$$

In case $-\gamma < \overline{x}$, (4.10) is always true for all values of γ due to the fact that γ is positive.

■

We can have some result of the above lemma using a different theorem, namely, Clark Theorem 1.5.

Lemma 4.4 *The positive equilibirium $\overline{x}$ of Eq.(4.2) is locally asymtotically stable when either*

$$\frac{(\gamma - 1)^2 - C(\gamma + 3)(\gamma - 1)}{4C^2} < \alpha$$

or

$$\frac{(C + 3)\gamma^2 - 2(C + 1)\gamma + C - 1}{4} < \alpha.$$

Proof. From Theorem 1.5 it follows all roots of Eq.(4.2) lie in an open disc $|\lambda| < 1$, if

$|q_0| + |q_1| < 1.$

This implies that

$$\left|\frac{\overline{x} - 1}{\overline{x}(1 + C)}\right| + \left|\frac{C\overline{x} - \gamma}{\overline{x}(1 + C)}\right| < 1.$$

Hence

$$|\overline{x} - 1| + |C\overline{x} - \gamma| < \overline{x}(1 + C) \tag{4.11}$$

and so we have four cases for (4.11).

Case 1: $\overline{x} > 1$ and $\overline{x} > \frac{\gamma}{C}$.

Then

$\overline{x} - 1 + C\overline{x} - \gamma < \overline{x} + C\overline{x}$

and so

$$-1 < \gamma. \tag{4.12}$$

From hypothesis on γ, (4.12) is always true.

Case 2: $\overline{x} > 1$ and $\overline{x} < \frac{\gamma}{C}$.

Hence

$$\overline{x} - 1 + \gamma - C\overline{x} < \overline{x} + C\overline{x}$$

and thus $\frac{\gamma-1}{2C} < \overline{x}$. From Eq.(4.4) we obtain that

$$\frac{(\gamma-1)^2 - C(\gamma+3)(\gamma-1)}{4C^2} < \alpha.$$

Case 3: $\overline{x} < 1$ and $\overline{x} < \frac{\gamma}{C}$.

Thus

$$1 - \overline{x} + \gamma - C\overline{x} < \overline{x} + C\overline{x}$$

and so we get

$$\frac{1+\gamma}{2(1+C)} < \overline{x}.$$

From Eq.(4.4) we obtain that

$$\frac{-(\gamma+1)^2}{4(C+1)} < \alpha.$$

This inequality is always true since α, γ, and C are positive.

Case 4: $\overline{x} < 1$ and $\overline{x} > \frac{\gamma}{C}$.

Therefore

$$1 - \overline{x} + C\overline{x} - \gamma < \overline{x} + C\overline{x}$$

and hence, we have

$$\frac{1-\gamma}{2} < \overline{x}.$$

From Eq.(4.4), we observe that

$$\begin{aligned}
\frac{1-\gamma}{2} &< \frac{1+\gamma+\sqrt{(1+\gamma)^2+4\alpha(1+C)}}{2(1+C)} \\
&\Rightarrow (1-\gamma)(1+C) < 1+\gamma+\sqrt{(1+\gamma)^2+4\alpha(1+C)} \\
&\Rightarrow C - C\gamma - 2\gamma < \sqrt{(1+\gamma)^2+4\alpha(1+C)} \\
&\Rightarrow (C - C\gamma - 2\gamma)^2 - (1+\gamma)^2 < 4\alpha(1+C) \\
&\Rightarrow (C+3)\gamma^2 - 2(C+1)\gamma + C - 1 < 4\alpha.
\end{aligned}$$

Consequently, when

$$\frac{(C+3)\gamma^2 - 2(C+1)\gamma + C - 1}{4} < \alpha$$

the positive equilibirium $\overline{x}$ of Eq.(4.2) is locally asymtotically stable. ■

4.2 INVARIANT INTERVALS

Here we present some results about invariant intervals for Eq.(4.2). We consider the cases $\gamma = C$, $\gamma > C$, and $\gamma < C$.

Lemma 4.5 *Eq.(4.2) possesses the following invariant intervals:*

(a) *$[a, b]$ when $\gamma = C$ and a and b are positive numbers such that*

$$\alpha + (1+\gamma)a \leq (1+C)ab \leq \alpha + (1+\gamma)b \tag{4.13}$$

(b)

$$\left[\frac{\gamma}{C}, \frac{C\alpha}{\gamma - C}\right] \text{ when } \gamma > C \text{ and } \frac{\gamma^2 - \gamma C}{C^2} < \alpha;$$
$$\left[\frac{C\alpha}{\gamma - C}, \frac{\gamma}{C}\right] \text{ when } \gamma > C \text{ and } \frac{\gamma^2 - \gamma C}{C^2} > \alpha.$$

(c)

$$\left[1, \frac{\alpha}{C-\gamma}\right] \text{ when } \gamma < C < \gamma + \alpha;$$
$$\left[\frac{\alpha}{C-\gamma}, 1\right] \text{ when } C > \gamma + \alpha.$$

Proof.

(a) It is easy to see that when $\gamma = C$ the function

$$f(x, y) = \frac{\alpha + x + \gamma y}{x + Cy}$$

is decreasing in both arguments. Hence

$$a \leq \frac{\alpha + (1+\gamma)b}{(1+C)b} = f(b, b) \leq f(x, y) \leq f(a, a) = \frac{\alpha + (1+\gamma)a}{(1+C)a} \leq b.$$

(b) Clearly, the function $f(x,y)$ is decreasing in both arguments when $x < \frac{C\alpha}{\gamma - C}$, and it is decreasing in x and increasing in y for $x \geq \frac{C\alpha}{\gamma - C}$.

First assume that $\gamma > C$ and $\frac{\gamma^2 - \gamma C}{C^2} < \alpha$. Then for $x, y \in \left[\frac{\gamma}{C}, \frac{C\alpha}{\gamma - C}\right]$ we obtain

$$\frac{\gamma}{C} = f(\frac{C\alpha}{\gamma - C}, \frac{C\alpha}{\gamma - C}) \leq f(x,y) \leq f(\frac{\gamma}{C}, \frac{\gamma}{C}) = \frac{C\alpha + \gamma(1+\gamma)}{\gamma(1+C)} \leq \frac{C\alpha}{\gamma - C}.$$

The inequalities

$$\frac{C\alpha + \gamma(1+\gamma)}{\gamma(1+C)} \leq \frac{C\alpha}{\gamma - C} \text{ and } \frac{\gamma}{C} < \frac{C\alpha}{\gamma - C}$$

are equivalent to the inequality $\frac{\gamma^2 - \gamma C}{C^2} < \alpha$.

Next assume that $\gamma > C$ and $\frac{\gamma^2 - \gamma C}{C^2} > \alpha$. For $x, y \in \left[\frac{C\alpha}{\gamma - C}, \frac{\gamma}{C}\right]$, we obtain

$$\frac{\gamma^2(\gamma - C) + \alpha\gamma C}{C^2 + C\gamma(\gamma - C)} = f(\frac{C\alpha}{\gamma - C}, \frac{\gamma}{C}) \leq f(x,y) \leq f(\frac{C\alpha}{\gamma - C}, \frac{C\alpha}{\gamma - C}) = \frac{\gamma}{C}.$$

The inequalities

$$\frac{\gamma^2(\gamma - C) + \alpha\gamma C}{C^2 + C\gamma(\gamma - C)} \geq \frac{C\alpha}{\gamma - C} \text{ and } \frac{\gamma}{C} > \frac{C\alpha}{\gamma - C}$$

follow from the inequality $\frac{\gamma^2 - \gamma C}{C^2} > \alpha$.

(c) It is clear that the function $f(x,y)$ is decreasing in both arguments for $y < \frac{\alpha}{C - \gamma}$, and it is increasing in x and decreasing in y for $y \geq \frac{\alpha}{C - \gamma}$.

First, assume that $\gamma < C < \gamma + \alpha$. Using the decreasing character of f, we obtain

$$1 = f(\frac{\alpha}{C - \gamma}, \frac{\alpha}{C - \gamma}) \leq f(x,y) \leq f(1,1) = \frac{\alpha + 1 + \gamma}{1 + C} \leq \frac{\alpha}{C - \gamma}.$$

The inequalities

$$1 < \frac{\alpha}{C - \gamma} \text{ and } \frac{\alpha + 1 + \gamma}{1 + C} < \frac{\alpha}{C - \gamma}$$

are equivalent to the inequality $C < \gamma + \alpha$.

Next assume that $C > \gamma + \alpha$. Using the increasing character of f in x and the decreasing character of f in y, we obtain

$$\frac{(\gamma - C)(\alpha + \gamma) + \alpha}{\alpha + C(C - \gamma)} = f(\frac{\alpha}{C - \gamma}, 1) \leq f(x,y) \leq f(1, \frac{\alpha}{C - \gamma}) = 1.$$

The inequalities

$$\frac{(\gamma - C)(\alpha + \gamma) + \alpha}{\alpha + C(C - \gamma)} \geq \frac{\alpha}{C - \gamma} \text{ and } \frac{\alpha}{C - \gamma} < 1$$

follow from the inequality $C > \gamma + \alpha$. ■

4.3 CONVERGENCE OF SOLUTIONS

Here we obtain some convergence results for Eq.(4.2).

Theorem 4.6 **(a)** *Assume that* $\gamma = C$. *Then every solution of Eq.(4.2) with initial conditions in the invariant interval* $[a, b]$, *where* $0 < a < b$ *satisfy (4.13) converges to the equilibrium* $\overline{x}$.

(b) *Assume that* $\gamma > C$ *and* $\alpha > \frac{\gamma^2 - \gamma C}{C^2}$. *Then every solution of Eq.(4.2) with initial conditions in the invariant interval* $\left[\frac{\gamma}{C}, \frac{C\alpha}{\gamma - C}\right]$ *converges to the equilibrium* $\overline{x}$.

(c) *Assume that* $\gamma < C < \gamma + \alpha$. *Then every solution of Eq.(4.2) with initial conditions in the invariant interval* $\left[1, \frac{\alpha}{C - \gamma}\right]$ *converges to the equilibrium* $\overline{x}$.

Proof. The proof is an immediate consequence of Lemma 4.5 and Theorem 1.9. ■

CHAPTER 5

DIFFERENCE EQUATION:

$$x_{n+1} = \frac{\alpha(x_n + x_{n-1}) + x_n x_{n-1}}{A + Bx_n x_{n-1}}, n = 0, 1, ...$$

In this section we investigate the global character of the solutions of the difference equation of the second order

$$x_{n+1} = \frac{\alpha(x_n + x_{n-1}) + x_n x_{n-1}}{A + Bx_n x_{n-1}}, \ n = 0, 1, ... \tag{5.1}$$

where the parameters α, A and B are non-negative real numbers, and the initial conditions $x_{-1},\ x_0$ are arbitrary non-negative real numbers such that the denominator of Eq.(5.1) is never zero.

Lemma 5.1 **(a)** *Eq.(5.1) can be written in the normalized form*

$$x_{n+1} = \frac{\alpha(x_n + x_{n-1}) + x_n x_{n-1}}{A + x_n x_{n-1}}, \ n = 0, 1, ... \tag{5.2}$$

with non-negative parameters α A and with arbitrary non-negative initial conditions $x_{-1},\ x_0$.

(b) $\overline{x} = 0$

is always an equilibrium point of Eq.(5.2) for all parameters and, when $\frac{4A-1}{8} < \alpha < \frac{A}{2}$ the others positive equilibrium points are

$$\overline{x}_1 = \frac{1 - \sqrt{1 + 8\alpha - 4A}}{2}$$

and

$$\overline{x}_2 = \frac{1 + \sqrt{1 + 8\alpha - 4A}}{2}$$

and, if $\alpha = \frac{4A-1}{8}$, then the other equilibrium point is

$$\overline{x} = \frac{1}{2}.$$

(c) *The linearized equation of Eq.(5.2) about zero equilibrium is*

$$z_{n+1} - \frac{\alpha}{A} z_n - \frac{\alpha}{A} z_{n-1} = 0 \tag{5.3}$$

and the corresponding characteristic equation of Eq.(5.3) is

$$\lambda^2 - \frac{\alpha}{A}\lambda - \frac{\alpha}{A} = 0.$$

(d) *The linearized equation of Eq.(5.2) about its positive equilibrium* $\{\overline{x}_i\}_{i=1,2}$ *is*

$$z_{n+1} + \frac{\alpha - A}{A + \overline{x}_i^2} z_n + \frac{\alpha - A}{A + \overline{x}_i^2} z_{n-1} = 0 \tag{5.4}$$

and the corresponding characteristic equation of Eq.(5.4) is

$$\lambda^2 + \frac{\alpha - A}{A + \overline{x}_i^2}\lambda + \frac{\alpha - A}{A + \overline{x}_i^2} = 0.$$

(e) *The linearized equation of Eq.(5.2) about* $\overline{x} = \frac{1}{2}$ *equilibrium is*

$$z_{n+1} + \frac{4(\alpha - A)}{4A + 1} z_n + \frac{4(\alpha - A)}{4A + 1} z_{n-1} = 0 \tag{5.5}$$

and the corresponding characteristic equation of Eq.(5.5) is

$$\lambda^2 + \frac{4(\alpha - A)}{4A + 1}\lambda + \frac{4(\alpha - A)}{4A + 1} = 0.$$

Proof.

(a) The Eq.(5.1) which by the change of variables

$$x_n = \frac{1}{B} y_n$$

reduces to the difference equation

$$x_{n+1} = \frac{\alpha(x_n + x_{n-1}) + x_n x_{n-1}}{A + x_n x_{n-1}}, \; n = 0, 1, \ldots$$

where

$$\alpha := \alpha B, \quad A := AB.$$

(b) The equilibrium points of Eq.(5.2) are the non-negative solutions of the equation

$$\overline{x} = \frac{\alpha(\overline{x} + \overline{x}) + \overline{x}^2}{A + \overline{x}^2}$$

or equivalently

$$\overline{x}^3 - \overline{x}^2 + (A - 2\alpha)\overline{x} = 0.$$

Thus,

$$\overline{x}(\overline{x}^2 - \overline{x} + A - 2\alpha) = 0$$

and so we have

$$\overline{x} = 0$$

or

$$\overline{x}^2 - \overline{x} + A - 2\alpha = 0. \tag{5.6}$$

Hence, zero is always an equilibrium point of Eq.(5.2). In addition to the zero equilibrium, when

$$\frac{4A - 1}{8} < \alpha < \frac{A}{2}$$

from the solution of Eq.(5.6), Eq.(5.2) has two positive equilibrium points:

$$\overline{x}_1 = \frac{1 - \sqrt{1 + 8\alpha - 4A}}{2} \text{ and } \overline{x}_2 = \frac{1 + \sqrt{1 + 8\alpha - 4A}}{2}.$$

Moreover, when

$$\alpha = \frac{4A - 1}{8}$$

from the solution of Eq.(5.6), Eq.(5.2) has the unique positive equilibrium point

$$\overline{x} = \frac{1}{2}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha(x_n + x_{n-1}) + x_n x_{n-1}}{A + x_n x_{n-1}}.$$

Hence, we establish

$$\begin{aligned}
\frac{\partial f}{\partial x_n} &= \frac{(\alpha + x_{n-1}).(A + x_n x_{n-1}) - (\alpha(x_n + x_{n-1}) + x_n x_{n-1}).x_{n-1}}{(A + x_n x_{n-1})^2} \\
&= \frac{\alpha A + A x_{n-1} + \alpha x_n x_{n-1} + x_n x_{n-1}^2 - \alpha x_n x_{n-1} - \alpha x_{n-1}^2 - x_n x_{n-1}^2}{(A + x_n x_{n-1})^2} \\
&= \frac{\alpha A + A x_{n-1} - \alpha x_{n-1}^2}{(A + x_n x_{n-1})^2} \qquad (5.7)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial x_{n-1}} &= \frac{(\alpha + x_n).(A + x_n x_{n-1}) - (\alpha(x_n + x_{n-1}) + x_n x_{n-1}).x_n}{(A + x_n x_{n-1})^2} \\
&= \frac{\alpha A + A x_n + \alpha x_n x_{n-1} + x_n^2 x_{n-1} - \alpha x_n^2 - \alpha x_n x_{n-1} - x_n^2 x_{n-1}}{(A + x_n x_{n-1})^2} \\
&= \frac{\alpha A + A x_n - \alpha x_n^2}{(A + x_n x_{n-1})^2}. \qquad (5.8)
\end{aligned}$$

Therefore, for the equilibrium point $\overline{x} = 0$ we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \frac{\alpha A}{A^2} = \frac{\alpha}{A}$$

and

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \frac{\alpha A}{A^2} = \frac{\alpha}{A}.$$

So, the linearized equation associated with Eq.(5.2) about the zero equilibrium is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} - \frac{\alpha}{A} z_n - \frac{\alpha}{A} z_{n-1} = 0$$

and the characteristic equation of the linearized equation about zero equilibrium of Eq.(5.2) is

$$\lambda^2 - \frac{\alpha}{A}\lambda - \frac{\alpha}{A} = 0. \qquad (5.9)$$

(d) From Eq.(5.6), (5.7) and (5.8), for the equilibrium point $\{\overline{x}_i\}_{i=1,2}$, it follows that

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \frac{\alpha A + A\overline{x} - \alpha\overline{x}^2}{(A+\overline{x}^2)^2} = \frac{A-\alpha}{A+\overline{x}^2}$$

because we have

$$\begin{aligned} \alpha A + A\overline{x} - \alpha\overline{x}^2 &= (A-\alpha)(A+\overline{x}^2) \\ &\Leftrightarrow \alpha A + A\overline{x} - \alpha\overline{x}^2 = A^2 + A\overline{x}^2 - \alpha A - \alpha\overline{x}^2 \\ &\Leftrightarrow \alpha A + A\overline{x} = A^2 + A\overline{x}^2 - \alpha A \\ &\Leftrightarrow \alpha + \overline{x} = A + \overline{x}^2 - \alpha \\ &\Leftrightarrow \overline{x}^2 - \overline{x} + A - 2\alpha = 0 \end{aligned}$$

and

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \frac{\alpha A + A\overline{x} - \alpha\overline{x}^2}{(A+\overline{x}^2)^2} = \frac{A-\alpha}{A+\overline{x}^2}.$$

If $\{\overline{x}_i\}_{i=1,2}$ denotes an equilibrium point of Eq.(5.2), then the linearized equation associated with Eq.(5.2) about the equilibrium point $\{\overline{x}_i\}_{i=1,2}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} = 0$$

or

$$z_{n+1} + \frac{\alpha - A}{A+\overline{x}_i^2} z_n + \frac{\alpha - A}{A+\overline{x}_i^2} z_{n-1} = 0.$$

So, the characteristic equation of the linearized equation about a positive equilibrium $\{\overline{x}_i\}_{i=1,2}$ of Eq.(5.2) is

$$\lambda^2 + \frac{\alpha - A}{A+\overline{x}_i^2}\lambda + \frac{\alpha - A}{A+\overline{x}_i^2} = 0. \tag{5.10}$$

(e) For $\overline{x} = \frac{1}{2}$ (5.10) also holds so the linearized equation of Eq.(5.2) about $\overline{x} = \frac{1}{2}$ equilibrium is

$$z_{n+1} + \frac{4(\alpha - A)}{4A+1} z_n + \frac{4(\alpha - A)}{4A+1} z_{n-1} = 0$$

and the characteristic equation of the linearized equation about a positive equilibrium $\overline{x} = \frac{1}{2}$ of Eq.(5.2) is

$$\lambda^2 + \frac{4(\alpha - A)}{4A+1}\lambda + \frac{4(\alpha - A)}{4A+1} = 0. \quad \blacksquare$$

Lemma 5.2 *The zero equilibrium of Eq.(5.2) is locally asymptotically stable when*

$$\alpha < \frac{A}{2}$$

and unstable when

$$\alpha > \frac{A}{2}$$

and non-hyperbolic when

$$\alpha = \frac{A}{2}.$$

Proof. From Theorem 1.3, a necessary and sufficient condition for all roots of the Eq.(5.9) to lie inside the unit disk is

$$\left|-\frac{\alpha}{A}\right| < 1 - \frac{\alpha}{A} < 2.$$

Then we have

$$\frac{\alpha}{A} < \frac{A-\alpha}{A} \text{ and } -\frac{\alpha}{A} < 1$$

or

$$\alpha < \frac{A}{2} \text{ and } -\alpha < A.$$

$-\alpha < A$ is always true for all positive values of the parameters due to the fact that α and A are non-negative.

Consequently, when

$$\alpha < \frac{A}{2},$$

the zero equilibrium is locally asymptotically stable and when

$$\alpha > \frac{A}{2},$$

the zero equilibrium is unstable.

While

$$\alpha = \frac{A}{2},$$

$\overline{x}$ is a non-hyperbolic equilibrium. In fact, in this case the two roots of the corresponding characteristic equation are:

$\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. ■

Lemma 5.3 *The following statements are true:*

(a) *The equilibrium point* $\overline{x}_1$ *of Eq.(5.2) is unstable for all positive values of the parameters satisfying*

$$\frac{4A-1}{8} < \alpha < \frac{A}{2}. \tag{5.11}$$

(b) *The equilibrium point* $\overline{x}_2$ *of Eq.(5.2) is locally asymptotically stable for all positive values of the parameters satisfying (5.11).*

(c) *The equilibrium point* $\overline{x} = \frac{1}{2}$ *of Eq.(5.2) is locally asymptotically stable for all positive values of the parameters satisfying* $\alpha = \frac{4A-1}{8}$.

Proof. From Theorem 1.3, a necessary and sufficient condition for all roots of the Eq.(5.10) to lie inside the unit disk is

$$\left|\frac{\alpha - A}{A + \overline{x}_i^2}\right| < 1 + \frac{\alpha - A}{A + \overline{x}_i^2} < 2.$$

Then we have

$$\left|\frac{\alpha - A}{A + \overline{x}_i^2}\right| < \frac{\overline{x}_i^2 + \alpha}{A + \overline{x}_i^2} \tag{5.12}$$

and

$$\frac{\alpha - A}{A + \overline{x}_i^2} < 1$$

or equivalently

$$\alpha - 2A < \overline{x}_i^2.$$

This inequality is always true for all positive values of the parameters due to the $\alpha < \frac{A}{2}$.So, from (5.12) we have

$$-\left(\frac{\overline{x}_i^2 + \alpha}{A + \overline{x}_i^2}\right) < \frac{\alpha - A}{A + \overline{x}_i^2} < \frac{\overline{x}_i^2 + \alpha}{A + \overline{x}_i^2}$$

so that

$$A - 2\alpha < \overline{x}_i^2 \tag{5.13}$$

and

$$-A < \overline{x}_i^2.$$

$-A < \overline{x}_i^2$ is always true for all positive values of the parameters because of the fact that α and A are non-negative.

Hence, all roots of the Eq.(5.10) are inside the unit disk if and only if for $i = 1, 2$, $\overline{x}_i$ holds (5.13).

(a) Consider $\overline{x}_1$. Then we get

$$\begin{aligned} A - 2\alpha &< \overline{x}_1^2 \Rightarrow A - 2\alpha < \left(\frac{1 - \sqrt{1 + 8\alpha - 4A}}{2}\right)^2 \\ &\Rightarrow A - 2\alpha < \frac{1 - 2\sqrt{1 + 8\alpha - 4A} + 1 + 8\alpha - 4A}{4} \\ &\Rightarrow 4\left(A - 2\alpha\right) < -2\sqrt{1 + 8\alpha - 4A} + 2 + 8\alpha - 4A \\ &\Rightarrow \sqrt{1 + 8\alpha - 4A} < 1 + 8\alpha - 4A. \end{aligned} \tag{5.14}$$

(5.14) is always true for $\frac{A}{2} < \alpha$. Indeed, since $\sqrt{m} < m \Leftrightarrow m > 1$ so that m is a real number, a necessary and sufficient condition for the inequality (5.14) to be always true is

$$1 < 1 + 8\alpha - 4A \Leftrightarrow 0 < 8\alpha - 4A \Leftrightarrow \frac{A}{2} < \alpha.$$

But this condition contradicts with $\overline{x}_1$ being the positive. On account of this, $\overline{x}_1$ is unstable for all values of the parameters.

(b) Consider $\overline{x}_2$. Then we have

$$\begin{aligned} A - 2\alpha &< \overline{x}_2^2 \Rightarrow A - 2\alpha < \left(\frac{1 + \sqrt{1 + 8\alpha - 4A}}{2}\right)^2 \\ &\Rightarrow A - 2\alpha < \frac{1 + 2\sqrt{1 + 8\alpha - 4A} + 1 + 8\alpha - 4A}{4} \\ &\Rightarrow 4\left(A - 2\alpha\right) < 2\sqrt{1 + 8\alpha - 4A} + 2 + 8\alpha - 4A \\ &\Rightarrow -\left(1 + 8\alpha - 4A\right) < \sqrt{1 + 8\alpha - 4A}. \end{aligned} \tag{5.15}$$

By the assumption on parameters, (5.15) is always true, hence, the required result follows.

(c) It can be easily proved as in (a) taking $\overline{x} = \frac{1}{2}$ i.e., $\alpha = \frac{4A-1}{8}$. ■

Note that when

$$\alpha < \frac{4A-1}{8} \tag{5.16}$$

$\overline{x} = 0$ is the only equilibrium point of Eq.(5.2). In this case we can mention the globally asymptotically stable of the zero equilibrium point. Therefore, we state that

Open Problem: Assume that $\alpha < \frac{4A-1}{8}$ and $A > 0$. Then the zero equilibrium point of Eq.(5.2) is globally asymptotically stable.

If all solutions of Eq.(5.2) is bounded, the proof of the open problem is as follows.

Note that $\alpha < \frac{4A-1}{8} < \frac{A}{2}$. The linearized equation associated with Eq.(5.2) about the zero equilibrium point is

$$z_{n+1} = \frac{\alpha}{A} z_n + \frac{\alpha}{A} z_{n-1}.$$

In the notations of Theorem 1.6, we have $f_0(x_n, x_{n-1}) = \frac{\alpha}{A}$ and $f_1(x_n, x_{n-1}) = \frac{\alpha}{A}$ such that $f_0, f_1 \in C\left[[0,\infty) \times [0,\infty), [0,1)\right]$. Then the following hold:

(a) f_0 and f_1 are non-increasing in each of their arguments due to the fact that parameters are non-negative;

(b) $f_0(x,x) = \frac{\alpha}{A} > 0$ for all $x \geq 0$;

(c) $f_0(x,y) + f_1(x,y) = \frac{\alpha}{A} + \frac{\alpha}{A} = \frac{2\alpha}{A} < 1$, that is, $\alpha < \frac{A}{2}$ for all $x, y \in (0,\infty)$.

Hence the hypotheses of Theorem 1.6 hold, so it follows that that Theorem that the zero equilibrium of Eq.(5.2) is globally asymptotically stable under the assumptions of the parameters.

Lemma 5.4 *Assume that $A = 0$. Then every solution $\{x_n\}$ of Eq.(5.2) is bounded from above and from below by positive numbers.*

Proof. We see that

$$x_{n+1} = \frac{\alpha(x_n + x_{n-1}) + x_n x_{n-1}}{x_n x_{n-1}} \geq \frac{x_n x_{n-1}}{x_n x_{n-1}} = 1$$

and then every solution $\{x_n\}$ of Eq.(5.2) is bounded from below, for $n \geq 1$, by the positive number 1.

So,

$$x_{n+1} = \frac{\alpha}{x_{n-1}} + \frac{\alpha}{x_n} + 1 \leq 2\alpha + 1$$

and thus every solution $\{x_n\}$ of Eq.(5.2) is also bounded from above by the positive number $m = 2\alpha + 1$. ■

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