ON THE SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS

A Thesis Submitted to

The Graduate School of Natural and Applied Sciences

of

Bülent Ecevit University

By

Miraç GÜNEYSU

In Partial Fulfillment of The Requirements

for

The Degree of Master of Science

In

Mathematics

ZONGULDAK

June 2015

APPROVAL OF THE THESIS:

The thesis entitled "ON THE SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS" submitted by Miraç GÜNEYSU has been examined and accepted by the jury as a Master of Science of Philosophy thesis in Department of Mathematics, Graduate School of Natural and Applied Sciences, Bülent Ecevit University. 22/06/2015

Chair : Assist. Prof. Dr. Melih GÖCEN
Bülent Ecevit University
....

Member : Assoc. Prof. Dr. Yüksel SOYKAN
Bülent Ecevit University
....

Member : Assist. Prof. Dr. Ömer KİŞİ
Bartın University
....

Approved by the Graduate School of Natural and Applied Sciences. ..../..../20....

Prof. Dr. Kemal BÜYÜKGÜZEL
Director

"I here by declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work."

Miraç GÜNEYSU

ABSTRACT

Thesis of Master of Science

ON THE SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS

Miraç GÜNEYSU

Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Asst. Prof. Melih GÖCEN
June 2015, 73 Pages

In this thesis, we investigate the asymptotic behaviors of the solutions of some rational difference equations and the global attractivity of the equilibrium points of certain rational difference equations.

This thesis consists of five chapters.

In Chapter 1, we give some basic definitions and theorems needed in this thesis.

In Chapter 2, we exhibit some examples of the local asymptotic stability of the some second order rational difference equations.

In Chapter 3, we present some examples regarding the asymptotic behavior of some third order rational difference equations.

ABSTRACT (continued)

In Chapter 4, we study the periodicity, stability and global attractivity of certain rational difference equations.

In Chapter 5, we investigate the local and global stability of the rational difference equation $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{C x_{n-1}+D x_{n-2}}$.

Key Words: Difference equations, equilibrium point, local asymptotic stability, global attractor.

Science Code: 403.03.01

ÖZET

Yüksek Lisans Tezi

BAZI RASYONEL FARK DENKLEMLERİNİN ÇÖZÜMLERİ ÜZERİNE

Miraç GÜNEYSU

Bülent Ecevit Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı

Tez Danışmanı: Yrd. Doç. Dr. Melih GÖCEN

Haziran 2015, 73 sayfa

Bu tezde bazı rasyonel fark denklemlerinin çözümlerinin asimptotik davranışları ve belirli rasyonel fark denklemlerinin denge noktalarının global çekiciliği araştırılmıştır.

Bu tez beş bölümden oluşmaktadır.

Birinci bölümde, tezde gerekli olan bazı temel tanımlar ve teoremler verilmiştir.

İkinci bölümde, bazı ikinci mertebeden rasyonel fark denklemlerinin yerel asimptotik kararlılığı ile ilgili bazı örnekler gösterilmiştir.

Üçüncü bölümde, bazı üçüncü mertebeden rasyonel fark denklemlerinin asimptotik davranışları ile ilgili bazı örnekler sunulmuştur.

Dördüncü bölümde, belirli rasyonel fark denklemlerinin periyodikliliği, kararlılığı ve global çekiciliği çalışılmıştır.

ÖZET (devam ediyor)

Beşinci bölümde,

$$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{Cx_{n-1}+Dx_{n-2}}$$

rasyonel fark denkleminin yerel ve global kararlılığı araştırılmıştır.

Anahtar Sözcükler: Fark denklemleri, denge noktası, yerel asimptotik kararlılık, global çekicilik.

Bilim Kodu: 403.03.01

ACKNOWLEDGMENTS

I would like to express the deepest appreciation to my Msc advisor Assist. Professor Melih Göcen, who has the attitude and substance of a genius: he continually and convincingly conveyed a spirit of adventure in regard to research and scholarship, and an excitement in regard to teaching. I would like to thank to him for supporting me during these past three years. Without his guidance and persistent help this dissertation would not have been possible.

Besides my advisor, I would like to thank to Assoc. Professor Yüksel Soykan for his helping, suggestions and helpful career advice in general. The good advice, support and friendship of him has been invaluable on both an academic and a personal level for which I am extremely grateful.

My friends were sources of laughter, joy, and support. Special thanks go to Yalçın Girgin, Inci Okumus, Özge Bayrak, Ayşe Olcay for providing support and friendship that I needed.

Last, but by no means least, I would like to thank my parents Mustafa Güneysu and Hacer Güneysu, for giving birth to me at the first place and supporting me spiritually throughout my life. Furthermore, I would like to thank to my sisters Sevim Akyel, Serap Güneysu, and Seçil Güneysu, my sister's husband Emrah Akyel and my nephew Yusuf Berat Akyel, for their support and encouragement throughout. I owe them everything and wish I could show them just how much I love and appreciate them.

CONTENTS

Page

APPROVAL ii
ABSTRACT iii
ÖZET........ v
ACKNOWLEDGMENTS... vii
CONTENTS ix

CHAPTER 1 INTRODUCTION AND PRELIMINARIES... 1

1.1 DEFINITIONS OF STABILITY... 1
1.2 LINEARIZED STABILITY ANALYSIS 3

CHAPTER 2 EXAMPLES OF THE SECOND ORDER DIFFERENCE EQUATIONS.... …15

CHAPTER 3 EXAMPLES OF THE THIRD ORDER DIFFERENCE EQUATIONS 29

CHAPTER 4 BEHAVIOR OF THE SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS..41

4.1 THE EQUATION$x_{n+1} = ax_n + \frac{bx_{n-1}+cx_{n-2}}{dx_{n-1}+ex_{n-2}}$...41

4.1.1 Local Stability of the Equilibrium Point of Equation (4.1) 41
4.1.2 Boundedness of Solutions of Equation (4.1)...43
4.1.3 Existence of Periodic Solutions...44
4.1.4 Global Attractivity of the Equilibrium Point of Equation (4.1).............................48

CONTENTS(continued)

Page

4.2 THE EQUATION $x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s}-b} + a$ 50

4.2.1 Periodic Solutions..........50

4.2.2 Local Stability of Equation (4.13)..........53

4.2.3 Global Attractivity of the Equilibrium Point of Equation (4.13)..........55

4.3 THE EQUATION $x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p}-cx_{n-q}}$ 56

4.3.1 Local Stability of Equation (4.21)..........56

4.3.2 Global Attractivity of the Equilibrium Point of Equation (4.21)..........57

CHAPTER 5 THE DIFFERENCE EQUATION $x_{n+1} = \frac{a+\beta x_n+\gamma x_{n-1}}{Cx_{n-1}+Dx_{n-2}}$..........61

5.1 LINEARIZED STABILITY ANALYSIS..........61

5.2 GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQUATION (5.2)..........65

BIBLIOGRAPHY..........69

CV..........73

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

In this thesis we considerably use [1,2,7,13,14,18, 22, 24, 27, 32]. For some other basic results in the area of difference equations and systems, see [3-6, 8-20, 21, 23, 25-26, 28-31]. We present some well known definitions, results and theorems in this chapter.

1.1 DEFINITIONS OF STABILITY

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ... \tag{1.1}$$

A solution of Eq.(1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(1.1) for all $n \geq -k$.

Lemma 1.1 *For every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation (1.1) has a unique solution $\{x_n\}_{n=-k}^{\infty}$.*

As a special case of above lemma, for every set of initial conditions $x_0, x_{-1} \in I$, the second order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, ... \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$ and for every set of initial conditions $x_0, x_{-1}, x_{-2} \in I$, the third order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \qquad n = 0, 1, ... \tag{1.3}$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

Definition 1.1 *A point $\overline{x} \in I$ is called an equilibrium point of Eq.(1.1) If*

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is

$x_n = \overline{x}$ *for* $n \geq 0$

is a solution of Eq.(1.1) or equivalently, $\overline{x}$ *is a fixed point of* f.

Definition 1.2 (Stability) (a) *The equilibrium point* $\overline{x}$ *of Eq.(1.1) is locally stable if for every* $\varepsilon > 0$*; there exists* $\delta > 0$ *such that for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \varepsilon, \quad \text{for all } n \geq -k.$$

(b) *The equilibrium point* $\overline{x}$ *of Eq.(1.1) is locally asymptotically stable if* $\overline{x}$ *is locally stable solution of Eq.(1.1)and there exists* $\gamma > 0$*, such that for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ *with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) *The equilibrium point* $\overline{x}$ *of Eq.(1.1) is global attractor if for all* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) *The equilibrium point* $\overline{x}$ *of Eq.(1.1) is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq.(1.1).*

(e) *The equilibrium point* $\overline{x}$ *of Eq.(1.1) is unstable if* $\overline{x}$ *is not locally stable.*

1.2 LINEARIZED STABILITY ANALYSIS

Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\overline{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x}), \quad \text{for } i = 0, 1, ..., k$$

denote the partial derivative of $f(u_0, u_1, ..., u_k)$ with respect to u_i evaluated at the equilibrium point $\overline{x}$ of Eq.(1.1).

Definition 1.3 *The equation*

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + ... + q_k y_{n-k} = \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x}) y_{n-i}, \quad n = 0, 1, ... \tag{1.4}$$

is called the linearized equation of Eq.(1.1) about the equilibrium point $\overline{x}$ *and the equation*

$$\lambda^{k+1} - q_0 \lambda^k - ... - q_{k-1}\lambda - q_k = 0 \tag{1.5}$$

is called the characteristic equation of Eq.(1.4) about $\overline{x}$.

Then the equation

$$y_{n+1} = q_0 y_n + q_1 y_{n-1}, \quad n = 0, 1, ... \tag{1.6}$$

is the linearized equation associated with Eq.(1.2) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^2 - q_0 \lambda - q_1 = 0$$

is the characteristic equation of Eq.(1.6) about $\overline{x}$.

Also, the equation

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + q_2 y_{n-2}, \quad n = 0, 1, ... \tag{1.7}$$

is the linearized equation associated with Eq.(1.3) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^3 - q_0 \lambda^2 - q_1 \lambda - q_2 = 0$$

is the characteristic equation of Eq.(1.7) about $\overline{x}$.

The following result, known as the *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Eq.(1.1).

Theorem 1.2 (The Linearized Stability Theorem) ***([7], p.5)***
Assume that the function f is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:

(a) *When all the roots of Eq.(1.5) have absolute value less than one, then the equilibrium point $\overline{x}$ of Eq.(1.1) is locally asymptotically stable.*

(b) *If at least one root of Eq.(1.5) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Eq.(1.1) is unstable.*

The equilibrium point $\overline{x}$ of Eq.(1.1) is called *hyperbolic* if no root of Eq.(1.5) has absolute value equal to one. If there exists a root of Eq.(1.5) with absolute value equal to one, then the equilibrium $\overline{x}$ is called $non-hyperbolic$.
An equilibrium point $\overline{x}$ of Eq.(1.1) is called a *repeller* if all roots of Eq.(1.5) have absolute value greater than one.
As a special case of Theorem 1.2 we have the following corollary.

Corollary 1.1 **(a)** *If both roots of the characteristic equation (quadratic equation)*

$$\lambda^2 - q_0\lambda - q_1 = 0$$

of Eq.(1.6) lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of Eq.(1.2) is locally asymptotically stable.

(b) *If all roots of the characteristic equation (qubic equation)*

$$\lambda^3 - q_0\lambda^2 - q_1\lambda - q_2 = 0$$

of Eq.(1.7) lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of Eq.(1.3) is locally asymptotically stable.

The following two theorems state necessary and sufficient conditions for all the roots of a real polynomial of degree two or three, respectively, to have modulus less than one.

Theorem 1.3 ***([7], p.6)*** *Assume that a_1 and a_0 are real numbers. Then a necessary and sufficient condition for all roots of the equation*

$$\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$|a_1| < 1 + a_0 < 2.$

Theorem 1.4 ***([7], p.6)*** *Assume that a_2, a_1 and a_0 are real numbers. Then a necessary and sufficient condition for all roots of the equation*

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1 \quad \text{and} \quad a_0^2 + a_1 - a_0 a_2 < 1.$$

Theorem 1.5 (Clark Theorem) ***([7], p.6)*** *Assume that $q_0, q_1, ..., q_k$ are real numbers such that*

$$\sum_{i=1}^{k} |q_i| = |q_0| + |q_1| + ... + |q_k| < 1.$$

Then all roots of Eq.(1.5) lie inside the unit disk.

Using The Linearized Stability Theorem and Clark Theorem we have the following result.

Theorem 1.6 *([13],**p.863**)* *Assume that $q_i \in \mathbb{R}$, $i = 1, 2, ...$ and $k \in \{0, 1, 2, ...\}$. Then*

$$\sum_{i=1}^{k} |q_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation,

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, n = 0, 1, \cdots.$$

Theorem 1.7 ***([24], p.9)*** *Consider the difference equation*

$$x_{n+1} = f_0(x_n, x_{n-1}) x_n + f_1(x_n, x_{n-1}) x_{n-1}, \quad n = 0, 1, ... \tag{1.8}$$

with nonnegative initial conditions and

$$f_0, f_1 \in C[[0, \infty) \times [0, \infty), [0, 1)].$$

Assume that the following hypothesis hold:

(a) *f_0 and f_1 are non-increasing in each of their arguments;*

(b) $f_0(x,x) > 0$ *for all* $x \geq 0$*;*

(c) $f_0(x,y) + f_1(x,y) < 1$ *for all* $x, y \in (0,\infty)$*.*

Then the zero equilibrium of Eq.(1.8) is globally asymptotically stable.

Theorem 1.8 ***([24], p.11)*** *Let* $[a,b]$ *be an interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \longrightarrow [a,b]$$

is a continuous function satisfying the following properties:

(a) $f(x,y)$ *is non-decreasing in* $x \in [a,b]$ *for each* $y \in [a,b]$*, and* $f(x,y)$ *is non-increasing in* $y \in [a,b]$ *for each* $x \in [a,b]$*;*

(b) *If* $(m,M) \in [a,b] \times [a,b]$ *is a solution of the system*

$$f(m,M) = m \quad \textit{and} \quad f(M,m) = M,$$

then $m = M$*.*

Then Eq.(1.2) has a unique equilibrium $\overline{x} \in [a,b]$ *and every solution of Eq.(1.2) converges to* $\overline{x}$*.*

Theorem 1.9 ***([24], p.12)*** *Let* $[a,b]$ *be an interval of real numbers and assume that*

$$f : [a,b] \times [a,b] \longrightarrow [a,b]$$

is a continuous function satisfying the following properties:

(a) $f(x,y)$ *is non-increasing in* $x \in [a,b]$ *for each* $y \in [a,b]$ *and* $f(x,y)$ *is non-decreacing in* $y \in [a,b]$ *for each* $x \in [a,b]$*,*

(b) *The difference equation Eq.(1.2) has no solutions of prime period two in* $[a,b]$*.*

Then Eq.(1.2) has a unique equilibrium $\overline{x} \in [a,b]$ *and every solution of Eq.(1.2) converges to* $\overline{x}$*.*

Theorem 1.10 *([24], p.13) Let $[a,b]$ be an interval of real numbers and assume that*

$f:[a,b]\times[a,b]\longrightarrow[a,b]$

is a continuous function satisfying the following properties:

(a) *$f(x,y)$ is non-increasing in each of its arguments;*

(b) *If $(m,M)\in[a,b]\times[a,b]$ is a solution of the system*

$$f(m,m)=M \quad \text{and} \quad f(M,M)=m,$$

then $m=M$.

Then Eq.(1.2) has a unique equilibrium $\overline{x}\in[a,b]$ and every solution of Eq.(1.2) converges to $\overline{x}$.

In the next theorem we make use of the following notation associated with a function $f(z_1,z_2)$ which is monotonic in both arguments.
For each pair of numbers (m,M) and for each $i\in\{1,2\}$, define

$$M_i(m,M)=\begin{cases} M, & \text{if } f \text{ is increasing in } z_i \\ m, & \text{if } f \text{ is decreasing in } z_i \end{cases}$$

and

$$m_i(m,M)=M_i(M,m).$$

Theorem 1.11 *([1], p.3)Assume that $f\in C\left([0,\infty)^2,[0,\infty)\right)$ and $f(z_1,z_2)$ is either strictly increasing in z_1 and z_2, or strictly decreasing in z_1 and z_2, or strictly increasing in z_1 and strictly decreasing in z_2. Furthermore, assume that for every*

$m\in(0,\infty)$ *and* $M>m$,

either

$$[f(M_1(m,M),M_2(m,M))-M][f(m_1(m,M),m_2(m,M))-m]>0$$

or

$$f(M_1(m,M),M_2(m,M))=M \text{ and } f(m_1(m,M),m_2(m,M))=m.$$

Then every solution of Eq.(1.2) which is bounded from above and from below by positive constants converges to a finite limit.

We now present two genereal global asymptotic stability results that apply to several special cases of the $(k+1)^{st}$-order rational difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} \beta_i x_{n-i}}{A + \sum_{i=0}^{k} B_i x_{n-i}}, \quad n = 0, 1, ... \tag{1.9}$$

with $A > 0$, the remaining parameters non-negative, with

$$\sum_{i=0}^{k} \beta_i \quad \text{and} \quad \sum_{i=0}^{k} B_i \in (0, \infty),$$

and with arbitrary non-negative initial conditions such that the denominator is always positive.

The characteristic equation of the linearized equation of Eq.(1.9) about an equilibrium point $\overline{x}$ is

$$\lambda^{k+1} + \frac{1}{A + \overline{x}.\sum_{i=0}^{k} B_i} \sum_{i=0}^{k} \left(B_i \overline{x} - \beta_i\right) \lambda^{k-i} = 0. \tag{1.10}$$

Zero is an equilibrium point Eq.(1.9) if and only if

$$\alpha = 0 \text{ and } A > 0. \tag{1.11}$$

As we will see later, when (1.11) holds, the zero equilibrium of Eq.(1.9) is globally asymptotically stable when

$$A > \sum_{i=0}^{k} \beta_i \tag{1.12}$$

and unstable when

$$A < \sum_{i=0}^{k} \beta_i.$$

Eq.(1.9) has a positive equilibrium point if and only if either

$$\alpha > 0 \tag{1.13}$$

or

$$\alpha = 0 \text{ and } A < \sum_{i=0}^{k} \beta_i. \tag{1.14}$$

When (1.13) holds, the equation has the unique equilibrium point

$$\overline{x} = \frac{\beta - A + \sqrt{(\beta - A)^2 + 4\alpha B}}{2B}, \tag{1.15}$$

where for simplicity we use the notation,

$$\beta = \sum_{i=0}^{k} \beta_i \quad \text{and} \quad B = \sum_{i=0}^{k} B_i.$$

When (1.14) holds, Eq.(1.9) has the unique positive equilibrium point

$$\overline{x} = \frac{\beta - A}{B}.$$

Note that

$$\frac{1}{A + B\overline{x}} \sum_{i=0}^{k} |B_i\overline{x} - \beta_i| \leq \frac{1}{A + B\overline{x}} \left(B\overline{x} - \beta\right). \tag{1.16}$$

Therefore, by Theorem 1.5 and 1.16, the equilibrium of Eq.(1.9) is locally asymptotically stable when (1.12) holds.

Theorem 1.12 ***([7], pp.150-151)*** *Assume that*

$$\beta = \sum_{i=0}^{k} \beta_i < A.$$

Then the following statements are true:

(a) *If*

$$\alpha = 0,$$

the zero equilibrium of Eq.(1.9) is globally asymptotically stable.

(b) *If*

$$\alpha > 0,$$

the positive equilibrium of Eq.(1.9) is globally asymptotically stable.

In a special case when

$$A = \sum_{i=0}^{k} \beta_i > 0 \quad \text{and} \quad \alpha > 0,$$

the global character of solutions of Eq.(1.9) is completely described by the following result in [31]. In this case it is preferable to write the difference equation in the form

$$x_n = \frac{\alpha + \sum\limits_{r=1}^{k} \beta_r x_{n-i_r}}{A + \sum\limits_{t=1}^{m} B_j x_{n-j_t}}, \quad n = 1, 2, ... \tag{1.17}$$

Also, by making a change of variables, if necessary, we may and do assume that the greatest common divisor of all "delays" in the numerator and denominator is 1, that is,

$$\gcd\{i_1, ..., i_k, j_{1,...,}j_m\} = 1.$$

Theorem 1.13 ***([7], p.152)*** *Assume that*

$$\alpha = 0 \quad \textit{and} \quad \beta = \sum_{i=0}^{k} \beta_i = A$$

and that one of the following three conditons is satisfied:

(a) $\beta_i B_i > 0$ *for some* $i \in \{0, ..., k\}$.

(b) $\beta_0 > 0$.

(c) B_0 *and Eq.(1.9) has no period-two solutions.*

Then the zero equilibrium of Eq.(1.9) is globally asymptotically stable.

Theorem 1.14 ***([27], p.155)*** *Let* $l \in \{1, 2, ...\}$. *Suppose that on some interval* $I \subseteq \mathbb{R}$ *Eq.(1.1) has the linearization*

$$x_{n+l} = \sum_{i=1-l}^{m} g_i x_{n-i},$$

where the non-negative functions $g_i : I^{k+l} \to \mathbb{R}$ *are such that* $\sum\limits_{i=1-l}^{m} g_i = 1$ *is satisfied. Suppose that there exists* $A > 0$ *such that*

$$g_{1-l} \geq A, \quad n = 0, 1, ...$$

Then if $x_{l-1}, ...x_{-k} \in I$,

$$\lim_{n\to\infty} x_n = L \in I.$$

Theorem 1.15 *([24], p.205) Let $[a,b]$ be an interval of real numbers and assume that*

$f : [a,b] \times [a,b] \times [a,b] \longrightarrow [a,b]$

is a continuous function satisfying the following properties:

(a) *$f(x,y,z)$ is non-decreasing in x for each y and $z \in [a,b]$ and is non-increasing in y and z for each $x \in [a,b]$ of its arguments;*

(b) *If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system*

$$M = f(M,m,m) \quad \text{and} \quad m = f(m,M,M),$$

then $m = M$.

Then Eq.(1.2) has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of Eq.(1.2) converges to $\overline{x}$.

Theorem 1.16 *([24], p.202) Let $[a,b]$ be an interval of real numbers and assume that*

$f : [a,b] \times [a,b] \times [a,b] \longrightarrow [a,b]$

is a continuous function satisfying the following properties:

(a) *$f(x,y,z)$ is non-decreasing in x and $y \in [a,b]$ for each $z \in [a,b]$, and is non-increasing in $z \in [a,b]$ for each x and $y \in [a,b]$*

(b) *If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system*

$$m = f(m,m,M) \quad \text{and} \quad M = f(M,M,m),$$

then $m = M$.

Then Eq.(1.2) has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of Eq.(1.2) converges to $\overline{x}$.

Theorem 1.17 *([7], p.331) Let $\{x_n\}$ be any solution of equation*

$$x_{n+1} = \frac{\alpha + x_{n-m}}{A + Mx_{n-m} + Lx_{n-1}}, n = 0, 1, ...$$

Then the following statements are true:

(i) When

$$0 \leq A < 1 \quad \text{and} \quad \frac{(L-M)(1-A)^2}{4M^2} \leq \alpha < \frac{A}{M} + \frac{L}{M^2},$$

the solution $\{x_n\}$ eventually enters the interval $\left[\frac{\alpha M - A}{L}, \frac{1}{M}\right]$ and the function $f(x_{n-m}, x_{n-1})$ is eventually increasing in x_{n-m} and strictly decreasing in x_{n-1}.Furthermore, the solution $\{x_n\}$ converges to the equilibrium.

(ii) When

$$0 \leq A < 1 \quad \text{and} \quad \alpha > \frac{A}{M} + \frac{L}{M^2},$$

the solution $\{x_n\}$ eventually enters the interval $\left[\frac{1}{M}, \frac{\alpha M - A}{L}\right]$ and the function $f(x_{n-m}, x_{n-1})$ is eventually strictly decreasing in x_{n-m} and x_{n-1}. Furthermore, the solution $\{x_n\}$ converges to the equilibrium.

(iii) When

$$0 \leq A < 1 \quad \text{and} \quad \alpha = \frac{A}{M} + \frac{L}{M^2},$$

the solution $\{x_n\}$ converges to the equilibrium.

Theorem 1.18 ***([7], p.331)*** *Assume that the following conditions hold:*

(i) $f \in C\left[(0,\infty) \times (0,\infty), (0,\infty)\right]$.

(ii) $f(x,y)$ is decreasing in x and strictly decreasing in y.

(iii) $xf(x,x)$ is strictly increasing in x.

(iv) The equation

$$x_{n+1} = x_n f(x_n, x_{n-1}), n = 0, 1, ... \tag{1.18}$$

has a unique positive equilibrium $\overline{x}$.

Then $\overline{x}$ is a global attractor of all positive solutions of Eq.(1.18).

Theorem 1.19 ***([24], p.202)*** *Let $[\alpha, \beta]$ be an interval of real numbers and assume that*

$$g : [\alpha, \beta]^3 \to [\alpha, \beta],$$

is a continuous function satisfying the following properties:

(a) $g(x,y,z)$ is non-decreasing in x and z in $[\alpha,\beta]$ for each $y \in [\alpha,\beta]$, and is non-increasing in $y \in [\alpha,\beta]$ for each x and z in $[\alpha,\beta]$;
(b) If $(m,M) \in [\alpha,\beta] \times [\alpha,\beta]$ is a solution of the system

$$M = g(M,m,M) \quad \text{and} \quad m = g(m,M,m),$$

then

$$m = M.$$

Then

$$x_{n+1} = g(x_n, x_{n-1}, x_{n-2}). \tag{1.19}$$

has a unique equilibrium $\overline{x} \in [\alpha,\beta]$ and every solution of Eq.(1.19) converges to $\overline{x}$.

Theorem 1.20 ***([24], p.202)*** *Let $[a,b]$ be an inerval of real numbers and assume that*

$$g : [a,b]^3 \to [a,b]$$

is a continuous function satisfying the following properties :

(a) $g(x,y,z)$ is non-increasing in all three variables $x,y,z \in [a,b]$.
(b) If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system

$$M = g(m,m,m) \quad \text{and} \quad m = g(M,M,M)$$

then

$$m = M.$$

Then Eq.(1.19) has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of Eq.(1.19) converges to $\overline{x}$.

CHAPTER 2

EXAMPLES OF THE SECOND ORDER DIFFERENCE EQUATIONS

In this chapter we investigate the local asymptotic stability of some second order difference equations.

Example 2.1 ***([7], pp.335-336)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.1}$$

(a) *The normalized form of Eq.(2.1) is*

$$x_{n+1} = \frac{\alpha + x_{n-1}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.2}$$

(b) *Equilibrium point of Eq.(2.2) is*

$$\overline{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha(1+B)}}{2(1+B)}.$$

(c) *The linearized equation of Eq.(2.2) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + \frac{B\overline{x}}{A + (1+B)\overline{x}} z_n + \frac{\overline{x} - 1}{A + (1+B)\overline{x}} z_{n-1} = 0 \tag{2.3}$$

and the corresponding charasteristic equation of Eq.(2.3) is

$$\lambda^2 + \frac{B\overline{x}}{A + (1+B)\overline{x}} \lambda + \frac{\overline{x} - 1}{A + (1+B)\overline{x}} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(2.2) is locally asymptotically stable when*

$$\overline{x} > \frac{1-A}{2},$$

which is equivalent to

$$A \geq 1,$$

or

$$0 \le A < 1 \quad \textit{and} \quad B \le 1,$$

or

$$0 \le A < 1, \quad B > 1, \quad \textit{and} \quad \alpha > \frac{(B-1)(1-A)^2}{4},$$

and unstable when

$$0 \le A < 1, \quad B > 1, \quad \textit{and} \quad \alpha < \frac{(B-1)(1-A)^2}{4}.$$

Solution.

(a) Using the change of variables

$$x_n = \frac{\gamma}{C} y_n,$$

Eq.(2.1) can be written in the normalized form

$$x_{n+1} = \frac{\alpha + x_{n-1}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, \cdots.$$

with positive parameter α and B and with arbitrary non-negative initial conditions x_{-1}, x_0 such that the denominator is always positive. Throughout this example we allow the parameter A to be nonnegative.

(b) The equilibrium point of Eq.(2.2) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \overline{x}}{A + B\overline{x} + \overline{x}}$$

or equivalently

$$\overline{x}^2(B+1) + \overline{x}(A-1) - \alpha = 0. \tag{2.4}$$

Then the only equilibrium point Eq.(2.2) is

$$\overline{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha(1+B)}}{2(1+B)}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha + x_{n-1}}{A + Bx_n + x_{n-1}}.$$

Hence, we have from (2.4)

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \left[\frac{-(\alpha + x_{n-1})B}{(A + Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{-B\overline{x}}{A + \overline{x}(1 + B)}$$

and from (2.4)

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{1.\,(A + Bx_n + x_{n-1}) - (\alpha + x_{n-1}).1}{(A + Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{1 - \overline{x}}{A + (1 + B)\overline{x}}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(2.2), then the linearized equation associated with Eq.(2.2) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} + \frac{B\overline{x}}{A + (1 + B)\overline{x}} z_n + \frac{\overline{x} - 1}{A + (1 + B)\overline{x}} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(2.2) about the equilibrium $\overline{x}$ is

$$\lambda^2 + \frac{B\overline{x}}{A + (1 + B)\overline{x}} \lambda + \frac{\overline{x} - 1}{A + (1 + B)\overline{x}} = 0.$$

(d) From (c) and Theorem 1.3 it follows that the equilibrium $\overline{x}$ of Eq.(2.2) is locally asymptotically stable when

$$\overline{x} > \frac{1 - A}{2},$$

which is equivalent to

$$A \geq 1,$$

or

$$0 \leq A < 1 \quad \text{and} \quad B \leq 1,$$

or

$$0 \leq A < 1, B > 1, \quad \text{and} \quad \alpha > \frac{(B-1)(1-A)^2}{4}$$

and unstable when

$$0 \leq A < 1, B > 1, \quad and \quad \alpha < \frac{(B-1)(1-A)^2}{4}.$$

Example 2.2 ***([7], pp.223-224)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + C x_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.5}$$

(a) *The normalized form of Eq.(2.5) is*

$$x_{n+1} = \frac{\alpha + x_n}{A + x_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.6}$$

(b) *Equilibrium point of Eq.(2.6) is*

$$\overline{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha}}{2}.$$

(c) *The linearized equation of Eq.(2.6) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} - \frac{1}{A + \overline{x}} z_n + \frac{\overline{x}}{A + \overline{x}} z_{n-1} = 0 \tag{2.7}$$

and the corresponding charasteristic equation of Eq.(2.7) is

$$\lambda^2 - \frac{1}{A + \overline{x}} \lambda + \frac{\overline{x}}{A + \overline{x}} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(2.6) is locally asymptotically stable for all positive values of the parameters.*

Solution.

(a) Using the change of variables

$$x_n = \frac{\beta}{C} y_n,$$

Eq.(2.5) can be written in the normalized form

$$x_{n+1} = \frac{\alpha + x_n}{A + x_{n-1}}, \quad n = 0, 1, \cdots.$$

with positive parameter α, A and with arbitrary non-negative initial conditions x_{-1}, x_0 such that the denominator is always positive.

(b) The equilibrium point of Eq.(2.6) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \overline{x}}{A + \overline{x}}$$

or equivalently

$$\overline{x}^2 - (1 - A)\overline{x} - \alpha = 0. \tag{2.8}$$

Then the only equilibrium point Eq.(2.6) is

$$\overline{x} = \frac{1 - A + \sqrt{(1 - A)^2 + 4\alpha}}{2}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha + x_n}{A + x_{n-1}}.$$

Hence, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \left[\frac{1}{A + x_{n-1}}\right](\overline{x}, \overline{x}) = \frac{1}{A + \overline{x}} = \frac{2}{1 + A + \sqrt{(1 - A)^2 + 4\alpha}}$$

and from (2.8)

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{0 - (\alpha + x_n).1}{(A + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{-\overline{x}}{A + \overline{x}}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(2.6), then the linearized equation associated with Eq.(2.6) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} - \frac{1}{A + \overline{x}} z_n + \frac{\overline{x}}{A + \overline{x}} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(2.6) about the equilibrium $\overline{x}$ is

$$\lambda^2 - \frac{1}{A + \overline{x}}\lambda + \frac{\overline{x}}{A + \overline{x}} = 0.$$

(d) From (c) and Theorem 1.3 it follows that the equilibrium $\overline{x}$ of Eq.(2.5) is locally asymptotically stable for all values of the parameters.

Example 2.3 ***([7], p.231)*** *Consider the difference equation*

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.9}$$

(a) *The normalized form of Eq.(2.9) is*

$$x_{n+1} = \frac{\alpha + x_{n-1}}{Bx_n + x_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.10}$$

(b) *Equilibrium point of Eq.(2.10) is*

$$\overline{x} = \frac{1 + \sqrt{1 + 4\alpha(B+1)}}{2(B+1)}.$$

(c) *The linearized equation of Eq.(2.10) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + \frac{B}{1+B} z_n + \frac{\overline{x} - 1}{(B+1)\overline{x}} z_{n-1} = 0 \tag{2.11}$$

and the corresponding charasteristic equation of Eq.(2.11) is

$$\lambda^2 + \frac{B}{1+B}\lambda + \frac{\overline{x} - 1}{(B+1)\overline{x}} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(2.10) is locally asymptotically stable when*

$$\alpha > \frac{B-1}{4}$$

and unstable when

$$\alpha < \frac{B-1}{4}.$$

Solution.

(a) Using the change of variables

$$x_n = \frac{\gamma}{C} y_n,$$

Eq.(2.9) can be written in the normalized form

$$x_{n+1} = \frac{\alpha + x_{n-1}}{Bx_n + x_{n-1}}, \quad n = 0, 1, \cdots.$$

with positive parameter α, B and with arbitrary positive initial conditions x_{-1}, x_0.

(b) The equilibrium point of Eq.(2.10) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \overline{x}}{B\overline{x} + \overline{x}}.$$

So, (2.10) has the only equilibrium point

$$\overline{x} = \frac{1 + \sqrt{1 + 4\alpha(B+1)}}{2(B+1)}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha + x_{n-1}}{Bx_n + x_{n-1}}.$$

Thus, we observe that

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \left[\frac{0.(Bx_n + x_{n-1}) - (\alpha + x_{n-1}).B}{(Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{-B\overline{x}}{B\overline{x} + \overline{x}} = \frac{-B}{1+B}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{1.(Bx_n + x_{n-1}) - (\alpha + x_{n-1}).1}{(Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{1 - \overline{x}}{B\overline{x} + \overline{x}} = \frac{1 - \overline{x}}{(B+1)\overline{x}}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(2.10), then the linearized equation associated with Eq.(2.10) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} + \frac{B}{1+B} z_n + \frac{\overline{x} - 1}{(B+1)\overline{x}} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(2.10) about the equilibrium $\overline{x}$ is

$$\lambda^2 + \frac{B}{1+B}\lambda + \frac{\overline{x} - 1}{(B+1)\overline{x}} = 0.$$

(d) From (c) and Theorem 1.3 $\overline{x}$ is locally asymptotically stable,when

$$\alpha > \frac{B-1}{4}$$

and unstable when

$$\alpha < \frac{B-1}{4}.$$

Example 2.4 ***([7], pp.251-252)*** *Consider the second order difference equation*

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.12}$$

(a) *The normalized form of Eq.(2.12) is*

$$x_{n+1} = \frac{\beta x_n + x_{n-1}}{Bx_n + x_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.13}$$

(b) *Equilibrium point of Eq.(2.13) is*

$$\overline{x} = \frac{\beta + 1}{B + 1}.$$

(c) *The linearized equation of Eq.(2.13) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} - \frac{\beta - B}{(\beta + 1)(B + 1)} z_n + \frac{\beta - B}{(\beta + 1)(B + 1)} z_{n-1} = 0 \tag{2.14}$$

and the corresponding charasteristic equation of Eq.(2.14) is

$$\lambda^2 - \frac{\beta - B}{(\beta + 1)(B + 1)} \lambda + \frac{\beta - B}{(\beta + 1)(B + 1)} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(2.13) is locally asymptotically stable when*

$$\beta > B$$

or

$$\beta < B \text{ and } B < 3\beta + \beta B + 1$$

and unstable when

$$\beta < B \text{ and } B > 3\beta + \beta B + 1.$$

Solution.

(a) By the change of variables

$$x_n = \frac{\gamma}{C} y_n,$$

Eq.(2.12) can be written in the normalized form

$$x_{n+1} = \frac{\beta x_n + x_{n-1}}{Bx_n + x_{n-1}}, \quad n = 0, 1, \cdots .$$

with positive parameters β, B and with arbitrary positive initial conditions x_{-1}, x_0. We also assume that $\beta \neq B$ because otherwise the equation eventually becomes trivial.

(b) The equilibrium point of Eq.(2.13) is the non-negative solution of the equation.

$$\overline{x} = \frac{\beta\overline{x} + \overline{x}}{B\overline{x} + \overline{x}}$$

So, by solving the equation

$$\overline{x}^2(B+1) - \overline{x}(\beta+1) = 0,$$

(2.13) has the only equilibrium point

$$\overline{x} = \frac{\beta+1}{B+1}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\beta x_n + x_{n-1}}{Bx_n + x_{n-1}}.$$

Hence, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \left[\frac{\beta.\,(Bx_n + x_{n-1}) - (\beta x_n + x_{n-1}).B}{(Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{\beta - B}{(\beta+1)(B+1)}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{1.\,(Bx_n + x_{n-1}) - (\beta x_n + x_{n-1}).1}{(Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{-(\beta - B)}{(\beta+1)(B+1)}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(2.13), then the linearized equation associated with Eq.(2.13) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} - \frac{\beta - B}{(\beta+1)(B+1)} z_n + \frac{(\beta - B)}{(\beta+1)(B+1)} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(2.13) about the equilibrium $\overline{x}$ is

$$\lambda^2 - \frac{\beta - B}{(\beta+1)(B+1)}\lambda + \frac{(\beta - B)}{(\beta+1)(B+1)} = 0. \tag{2.15}$$

(d) From (c) and Theorem 1.3 it follows that the equilibrium $\overline{x}$ of Eq.(2.13) is locally asymptotically stable when

$\beta > B$

or

$\beta < B$ and $B < 3\beta + \beta B + 1$

and unstable when

$\beta < B$ and $B > 3\beta + \beta B + 1$.

We give another example.

Example 2.5 ***([7], pp.422-423)*** *Consider the difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.16}$$

(a) The normalized form of Eq.(2.16) is

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.17}$$

(b) Equilibrium point of Eq.(2.17) is

$$\overline{x} = \frac{(\beta + 1 - A) + \sqrt{(A - \beta - 1)^2 + 4\alpha(B + 1)}}{2(B + 1)}.$$

(c) The linearized equation of Eq.(2.17) about the equilibrium point $\overline{x}$ is

$$z_{n+1} + \frac{B\overline{x} - \beta}{A + (B+1)\overline{x}} z_n + \frac{\overline{x} - 1}{A + (B+1)\overline{x}} z_{n-1} = 0 \tag{2.18}$$

and the corresponding charasteristic equation of Eq.(2.18) is

$$\lambda^2 + \frac{B\overline{x} - \beta}{A + (B+1)\overline{x}} \lambda + \frac{\overline{x} - 1}{A + (B+1)\overline{x}} = 0 \ .$$

(d) The equilibrium point $\overline{x}$ of Eq.(2.17) is locally asymptotically stable when

$$\frac{(1 - A - \beta)\left[(1 - A)(B - 1) - \beta(3 + B)\right]}{4} < \alpha$$

and

$$\frac{-(A-\beta-1)^2}{4(B+1)} < \alpha$$

and

$$\frac{(A+1)\left[B(2+\beta)+(A+1)\right]}{B^2} < \alpha.$$

Solution.

(a) By the change of variables

$$x_n = \frac{\gamma}{C} y_n,$$

Eq.(2.16) can be written in the normalized form

$$x_{n+1} = \frac{\alpha+\beta x_n + x_{n-1}}{A+Bx_n+x_{n-1}}, \quad n=0,1,\cdots.$$

with positive parameters α, A, β, B and with arbitrary positive initial conditions x_{-1}, x_0.

(b) The equilibrium point of Eq.(2.17) is the non-negative solution of the equation.

$$\overline{x} = \frac{\alpha+\beta\overline{x}+\overline{x}}{A+B\overline{x}+\overline{x}}$$

So, by solving the equation

$$\overline{x}^2(B+1) - \overline{x}(\beta+1-A) - \alpha = 0.$$

So (2.17) has the only equilibrium point

$$\overline{x} = \frac{(\beta+1-A)+\sqrt{(A-\beta-1)^2+4\alpha(B+1)}}{2(B+1)}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}) = \frac{\alpha+\beta x_n + x_{n-1}}{A+Bx_n+x_{n-1}}.$$

Hence, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x},\overline{x}) = \left[\frac{\beta.\left(A+Bx_n+x_{n-1}\right)-(\alpha+\beta x_n+x_{n-1}).B}{(A+Bx_n+x_{n-1})^2}\right](\overline{x},\overline{x}) = \frac{\beta-B\overline{x}}{A+(B+1)\overline{x}}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x},\overline{x}) = \left[\frac{1.(A+Bx_n+x_{n-1})-(\alpha+\beta x_n+x_{n-1}).1}{(A+Bx_n+x_{n-1})^2}\right](\overline{x},\overline{x}) = \frac{1-\overline{x}}{A+(B+1)+\overline{x}}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(2.17) then the linearized equation associated with Eq.(2.17) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} + \frac{B\overline{x}-\beta}{A+(B+1)} z_n + \frac{\overline{x}-1}{A+(B+1)\overline{x}} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(2.17) about the equilibrium $\overline{x}$ is

$$\lambda^2 + \frac{B\overline{x}-\beta}{A+(B+1)\overline{x}}\lambda + \frac{\overline{x}-1}{A+(B+1)\overline{x}} = 0. \tag{2.19}$$

(d) From (c) and Theorem 1.3 it follows that the equilibrium $\overline{x}$ of Eq.(2.17) is locally asymptotically stable when

$$\frac{(1-A-\beta)\left[(1-A)(B-1)-\beta(3+B)\right]}{4} < \alpha$$

and

$$\frac{-(A-\beta-1)^2}{4(B+1)} < \alpha$$

and

$$\frac{(A+1)\left[B(2+\beta)+(A+1)\right]}{B^2} < \alpha.$$

Example 2.6 ***([7], pp.284-285)*** *Consider the difference equation*

$$x_{n+1} = \frac{\beta x_n}{A+Bx_n+Cx_{n-1}}, n = 0, 1, \cdots. \tag{2.20}$$

(a) The normalized form of Eq.2.20) is

$$x_{n+1} = \frac{\beta x_n}{1+Bx_n+1x_{n-1}}, \quad n = 0, 1, \cdots. \tag{2.21}$$

(b) Equilibrium point of Eq.(2.21) is

$$\overline{x} = \frac{\beta - 1}{B + 1}.$$

(c) The linearized equation of Eq.(2.21) about the equilibrium point $\overline{x}$ is

$$z_{n+1} - \frac{\beta + B}{\beta (B + 1)} z_n + \frac{\beta - 1}{\beta (B + 1)} z_{n-1} = 0 \tag{2.22}$$

and the corresponding charasteristic equation of Eq.(2.22) is

$$\lambda^2 - \frac{\beta + B}{\beta (B + 1)} \lambda + \frac{\beta - 1}{\beta (B + 1)} = 0.$$

(d) The equilibrium point $\overline{x}$ of Eq.(2.21) is locally asymptotically stable when

$$\beta > 1$$

and unstable when

$$\beta < 1.$$

Solution.

(a) By the change of variables

$$x_n = \frac{A}{C} y_n,$$

Eq.(2.20) can be written in the normalized form

$$x_{n+1} = \frac{\beta x_n}{1 + B x_n + x_{n-1}}, \quad n = 0, 1, \cdots.$$

with positive parameters β, B and with arbitrary nonnegative initial conditions x_{-1}, x_0.

(b) The equilibrium point of Eq.(2.21) is the non-negative solution of the equation

$$\overline{x} = \frac{\beta \overline{x}}{1 + B\overline{x} + \overline{x}}$$

or equivalently the only equilibrium point Eq.(2.21) is

$$\overline{x} = \frac{\beta - 1}{B + 1}.$$

(c) From this, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = \left[\frac{\beta.(1 + Bx_n + x_{n-1}) - \beta x_n.B}{(1 + Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{\beta + B}{\beta(B+1)}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{0.(1 + Bx_n + x_{n-1}) - \beta x_n.1}{(1 + Bx_n + x_{n-1})^2}\right](\overline{x}, \overline{x}) = \frac{1 - \beta}{\beta(B+1)}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(2.21) then the linearized equation associated with Eq.(2.21) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1}$$

or

$$z_{n+1} - \frac{\beta + B}{\beta(B+1)} z_n + \frac{\beta - 1}{\beta(B+1)} z_{n-1} = 0.$$

The characteristic equation of the linearized equation of Eq.(2.21) about the equilibrium $\overline{x}$ is

$$\lambda^2 - \frac{\beta + B}{\beta(B+1)}\lambda + \frac{\beta - 1}{\beta(B+1)} = 0.$$

(d) From (c) and Theorem 1.3 it follows that the positive equilibrium $\overline{x}$ of Eq.(2.21) is locally asymptotically stable when

$$\beta > 1$$

and unstable when

$$\beta < 1.$$

CHAPTER 3

EXAMPLES OF THE THIRD ORDER DIFFERENCE EQUATIONS

In this chapter we investigate the local asymptotic stability of some third order difference equations.

Example 3.1 ***([7], pp.186-187)*** *Consider the third order difference equation*

$$x_{n+1} = \frac{\delta x_{n-2}}{Cx_{n-1} + Dx_{n-2}} \cdots. \tag{3.1}$$

(a) *The normalized form of Eq.(3.1) is*

$$x_{n+1} = \frac{x_{n-2}}{Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \cdots. \tag{3.2}$$

(b) *Equilibrium point of Eq.(3.2) is*

$$\overline{x} = \frac{1}{C+1}.$$

(c) *The linearized equation of Eq.(3.2) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + \frac{C}{C+1} z_{n-1} - \frac{C}{C+1} z_{n-2} = 0 \tag{3.3}$$

and the corresponding charasteristic equation of Eq.(3.3) is

$$\lambda^3 + \frac{C}{C+1}\lambda - \frac{C}{C+1} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(3.2) is locally asymptotically stable when*

$$C < \frac{1+\sqrt{5}}{2}$$

and unstable when

$$C > \frac{1+\sqrt{5}}{2}.$$

(e) *The equilibrium point* $\overline{x}$ *of Eq.(3.2) is nonhyperbolic when*

$$C = \frac{1+\sqrt{5}}{2}.$$

Solution.

(a) By the change of variables

$$x_n = \frac{\delta}{D} y_n,$$

Eq.(3.1) can be written in the normalized form

$$x_{n+1} = \frac{x_{n-2}}{Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \cdots.$$

with positive parameter C and with arbitrary nonnegative initial conditions x_{-2},x_{-1},x_0 such that denominator is always positive.

(b) The equilibrium point of Eq.(3.2) is the non-negative solution of the equation

$$\overline{x} = \frac{\overline{x}}{C\overline{x} + \overline{x}}$$

or

$$\overline{x}^2(C+1) - \overline{x} = 0. \tag{3.4}$$

The only equilibrium point of Eq.(3.2) is

$$\overline{x} = \frac{1}{C+1}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \frac{x_{n-2}}{Cx_{n-1} + x_{n-2}}.$$

So, from (3.4), we obtain

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}) = 0$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}) = \left[\frac{0.\,(Cx_{n-1} + x_{n-2}) - (x_{n-2}).C}{(Cx_{n-1} + x_{n-2})^2}\right](\overline{x}, \overline{x}) = \frac{-C}{(C+1)}$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}) = \left[\frac{1.(Cx_{n-1} + x_{n-2}) - (x_{n-2}).1}{(Cx_{n-1} + x_{n-2})^2}\right](\overline{x}, \overline{x}) = \frac{C}{(C+1)}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.2), then the linearized equation associated with Eq.(3.2) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} + \frac{C}{C+1} z_{n-1} - \frac{C}{C+1} z_{n-2} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.2) about the equilibrium $\overline{x}$ is

$$\lambda^3 + \frac{C}{C+1}\lambda - \frac{C}{C+1} = 0.$$

(d) From (c) and Theorem 1.4 it follows that the equilibrium $\overline{x}$ of Eq.(3.2) is locally asymptotically stable when

$$C < \frac{1+\sqrt{5}}{2}$$

and unstable when

$$C > \frac{1+\sqrt{5}}{2}$$

(e) When

$$C = \frac{1+\sqrt{5}}{2},$$

$\overline{x}$ is a non-hyperbolic equilibrium. In fact, the eigenvalues of the corresponding characteristic equation are

$$\lambda_1 = \frac{-1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}-i\sqrt{10+2\sqrt{5}}}{4}, \quad \lambda_3 = \frac{1-\sqrt{5}+i\sqrt{10+2\sqrt{5}}}{4}.$$

Example 3.2 ***([7], p.199-200)*** *Consider the third order difference equation*

$$x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{B x_n}, \quad n = 0, 1, \cdots. \tag{3.5}$$

(a) The normalized form of Eq.(3.5) is

$$x_{n+1} = \beta + \frac{x_{n-2}}{x_n}, \quad n = 0, 1, \cdots. \tag{3.6}$$

(b) Equilibrium point of Eq.(3.6) is

$$\overline{x} = \beta + 1.$$

(c) The linearized equation of Eq.(3.6) about the equilibrium point $\overline{x}$ is

$$z_{n+1} + \frac{1}{\beta+1} z_n - \frac{1}{\beta+1} z_{n-2} = 0 \tag{3.7}$$

and the corresponding charasteristic equation of Eq.(3.7) is

$$\lambda^3 + \frac{1}{\beta+1}\lambda^2 - \frac{1}{\beta+1} = 0.$$

(d) The equilibrium point $\overline{x}$ of Eq.(3.6) is locally asymptotically stable when

$$\beta > -1 + \sqrt{2}$$

and unstable when

$$B < -1 + \sqrt{2}.$$

Solution.

(a) Using the change of variables

$$x_n = \frac{\delta}{B} y_n,$$

Eq.(3.5) can be written in the normalized form

$$x_{n+1} = \beta + \frac{x_{n-2}}{x_n}, \quad n = 0, 1, \cdots.$$

with positive parameter β and with arbitrary non-negative initial conditions x_{-2}, x_{-1}, x_0 such that the denominator is always positive.

(b) The equilibrium point of Eq.(3.6) is the non-negative solution of the equation

$$\overline{x} = \beta + \frac{\overline{x}}{\overline{x}}$$

or equivalently the only equilibrium point Eq.(3.6) is

$$\overline{x} = \beta + 1.$$

(c) Now, let I be some interval of real numbers and let $f : I^3 \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \beta + \frac{x_{n-2}}{x_n}.$$

Therefore, we observe that

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{0 - x_{n-2}.1}{{x_n}^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{-1}{\beta + 1}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = 0$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{1.x_n - (\beta x_n + x_{n-2}).0}{x_n^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{1}{\beta + 1}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.6), then the linearized equation associated with Eq.(3.6) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} + \frac{1}{\beta + 1} z_n - \frac{1}{\beta + 1} z_{n-2} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.6) about the equilibrium $\overline{x}$ is

$$\lambda^3 + \frac{1}{\beta + 1}\lambda^2 - \frac{1}{\beta + 1} = 0.$$

(d) From (c) and Theorem 1.4 it follows that the positive equilibrium $\overline{x}$ of Eq.(3.6) is locally asymptotically stable when

$$\beta > -1 + \sqrt{2}$$

and unstable when

$$B < -1 + \sqrt{2}.$$

Example 3.3 ***([7], p.209-210)*** *Consider the difference equation*

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{B x_n}, \quad n = 0, 1, \cdots. \tag{3.8}$$

(a) *The normalized form of Eq.(3.8) is*

$$x_{n+1} = \frac{\gamma x_{n-1} + x_{n-2}}{x_n}, \quad n = 0, 1, \cdots . \tag{3.9}$$

(b) *Equilibrium point of Eq.(3.9) is*

$$\overline{x} = \gamma + 1.$$

(c) *The linearized equation of Eq.(3.9) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + z_n - \frac{\gamma}{\gamma+1} z_{n-1} - \frac{1}{\gamma+1} = 0 \tag{3.10}$$

and the corresponding charasteristic equation of Eq.(3.10) is

$$\lambda^3 + \lambda^2 - \frac{\gamma}{\gamma+1}\lambda - \frac{1}{\gamma+1} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(3.9) is locally asymptotically stable when*

$$\frac{\sqrt{3}-1}{2} < \gamma < 1$$

and unstable when

$$\gamma < \frac{\sqrt{3}-1}{2}.$$

Solution.

(a) Using the change of variables

$$x_n = \frac{\delta}{B} y_n,$$

Eq.(3.8) can be written in the normalized form

$$x_{n+1} = \frac{\gamma x_{n-1} + x_{n-2}}{x_n}, \quad n = 0, 1, \cdots .$$

with positive parameters γ and with arbitrary positive initial conditions x_{-2}, x_{-1}, x_0.

(b) The equilibrium point of Eq.(3.9) is the non-negative solution of the equation

$$\overline{x} = \frac{\gamma\overline{x} + \overline{x}}{\overline{x}}$$

or equivalently

$$\overline{x}^2 - (\gamma+1)\overline{x} = 0. \tag{3.11}$$

Therefore, the only equilibrium point of Eq.(3.9) is

$$\overline{x} = \gamma + 1.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = x_{n+1} = \frac{\gamma x_{n-1} + x_{n-2}}{x_n}.$$

From this and (3.11) we have

$$\begin{aligned} q_0 &= \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{0.x_n - (\gamma x_{n-1} + x_{n-2}).1}{{x_n}^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{-(\gamma\overline{x} + \overline{x})}{\overline{x}^2} = \frac{-(\gamma+1)}{\overline{x}} \\ &= \frac{-(\gamma+1)}{\gamma+1} = -1 \end{aligned}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{\gamma.x_n - (\gamma x_{n-1} + x_{n-2}).0}{{x_n}^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{\gamma\overline{x}}{\overline{x}^2} = \frac{\gamma}{\gamma+1}$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{1.x_n - (\gamma x_{n-1} + x_{n-2}).0}{{x_n}^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{\overline{x}}{\overline{x}^2} = \frac{1}{\gamma+1}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.9), then the linearized equation associated with Eq.(3.9) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} + z_n - \frac{\gamma}{\gamma+1} z_{n-1} - \frac{1}{\gamma+1} z_{n-2} = 0\ .$$

The characteristic equation of the linearized equation of Eq.(3.9) about the equilibrium $\overline{x}$ is

$$\lambda^3 + \lambda^2 - \frac{\gamma}{\gamma+1}\lambda - \frac{1}{\gamma+1} = 0.$$

(d) From (c) and Theorem 1.4 it follows that the positive equilibrium $\overline{x}$ of Eq.(3.9) is locally asymptotically stable when

$$\frac{\sqrt{3}-1}{2} < \gamma < 1$$

and unstable when

$$\gamma < \frac{\sqrt{3}-1}{2}.$$

Example 3.4 ***([7], pp.210-211)*** *Consider the difference equation*

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_{n-2}}{C x_{n-1}}, \quad n = 0, 1, \cdots. \tag{3.12}$$

(a) *The normalized form of Eq.(3.12) is*

$$x_{n+1} = \gamma + \frac{x_{n-2}}{x_{n-1}}, \quad n = 0, 1, \cdots. \tag{3.13}$$

(b) *Equilibrium point of Eq.(3.13) is*

$$\overline{x} = \gamma + 1.$$

(c) *The linearized equation of Eq.(3.13) about the equilibrium point $\overline{x}$ is*

$$z_{n+1} + \frac{1}{\gamma+1} z_{n-1} - \frac{1}{\gamma+1} z_{n-2} = 0 \tag{3.14}$$

and the corresponding charasteristic equation of Eq.(3.14) is

$$\lambda^3 + \frac{1}{\gamma+1}\lambda - \frac{1}{\gamma+1} = 0.$$

(d) *The equilibrium point $\overline{x}$ of Eq.(3.13) is locally asymptotically stable when*

$$\gamma > \frac{-1+\sqrt{5}}{2}$$

and unstable when

$$\gamma < \frac{-1+\sqrt{5}}{2}.$$

Solution.

(a) By the change of variables

$$x_n = \frac{\delta}{C} y_n,$$

Eq.(3.12) can be written in the normalized form

$$x_{n+1} = \gamma + \frac{x_{n-2}}{x_{n-1}}, \quad n = 0, 1, \cdots.$$

with positive parameters γ and with arbitrary positive initial conditions x_{-2}, x_{-1}, x_0.

(b) The equilibrium point of Eq.(3.13) is the non-negative solution of the equation

$$\overline{x} = \gamma + \frac{\overline{x}}{\overline{x}}$$

or equivalently the only equilibrium point Eq.(3.13) is

$$\overline{x} = \gamma + 1.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \gamma + \frac{x_{n-2}}{x_{n-1}}.$$

From this, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = 0$$

$$\begin{aligned} q_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{\gamma.x_{n-1} - (\gamma.x_{n-1} + x_{n-2}).1}{(x_{n-1})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{\gamma\overline{x} - (\gamma\overline{x} + \overline{x})}{(\overline{x})^2} \\ &= \frac{-1}{\overline{x}} = \frac{-1}{\gamma + 1} \end{aligned}$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{1.x_{n-1} - (\gamma.x_{n-1} + x_{n-2}).0}{(x_{n-1})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{1}{\overline{x}} = \frac{1}{\gamma + 1}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.13), then the linearized equation associated with Eq.(3.13) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} + \frac{1}{\gamma + 1} z_{n-1} - \frac{1}{\gamma + 1} z_{n-2} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.13) about the equilibrium $\overline{x}$ is

$$\lambda^3 + \frac{1}{\gamma + 1}\lambda - \frac{1}{\gamma + 1} = 0.$$

(d) From (c) and Theorem 1.4 it follows that the positive equilibrium $\overline{x}$ of Eq.(3.13) is locally asymptotically stable when

$$\gamma > \frac{-1+\sqrt{5}}{2}$$

and unstable when

$$\gamma < \frac{-1+\sqrt{5}}{2}.$$

Example 3.5 ***([7], pp.225-226)*** *Consider the difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + D x_{n-2}}, \quad n = 0, 1, \cdots. \tag{3.15}$$

(a) The normalized form of Eq.(3.15) is

$$x_{n+1} = \frac{\alpha + x_n}{A + x_{n-2}}, n = 0, 1, \cdots. \tag{3.16}$$

(b) Equilibrium point of Eq.(3.16) is

$$\overline{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha}}{2}.$$

(c) The linearized equation of Eq.(3.16) about the equilibrium point $\overline{x}$ is

$$z_{n+1} - \frac{1}{A+\overline{x}} z_n + \frac{\overline{x}}{A+\overline{x}} z_{n-2} = 0 \tag{3.17}$$

and the corresponding charasteristic equation of Eq.(3.17) is

$$\lambda^3 - \frac{1}{A+\overline{x}}\lambda^2 + \frac{\overline{x}}{A+\overline{x}} = 0.$$

(d) The equilibrium point $\overline{x}$ of Eq.(ceydaa) is locally asymptotically stable when either

$$A \geq \frac{1}{2}$$

or

$$\frac{1}{3} < A < \frac{1}{2} \text{ and } \alpha < \frac{A^2(-A^2+3A-1)}{(2A-1)^2}$$

and unstable when

$$\frac{1}{3} < A < \frac{1}{2} \text{ and } \alpha > \frac{A^2(-A^2+3A-1)}{(2A-1)^2}$$

or

$$0 < A < \frac{1}{3}.$$

Solution.

(a) By the change of variables

$$x_n = \frac{\beta}{D} y_n,$$

Eq.(3.15) can be written in the normalized form

$$x_{n+1} = \frac{\alpha + x_n}{A + x_{n-2}}, \quad n = 0, 1, \cdots.$$

with positive parameters α, A and with arbitrary positive initial conditions x_{-2}, x_{-1}, x_0.

(b) The equilibrium point of Eq.(3.16) is the non-negative solution of the equation

$$\overline{x} = \frac{\alpha + \overline{x}}{A + \overline{x}}$$

or equivalently the only equilibrium point Eq.(3.16) is

$$\overline{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha}}{2}.$$

(c) Now, let I be some interval of real numbers and let $f : I \times I \times I \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \frac{\alpha + x_n}{A + x_{n-2}}.$$

From this, we have

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{1.(A + x_{n-2}) - (\alpha + x_n).0}{(A + x_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{1}{A + \overline{x}}$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = 0$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{0.(A + x_{n-2}) - (\alpha + x_n).1}{(A + x_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) = \frac{-(\alpha + \overline{x})}{(A + \overline{x})^2} = \frac{-\overline{x}}{A + \overline{x}}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(3.16), then the linearized equation associated with Eq.(3.16) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} - \frac{1}{A + \overline{x}} z_n + \frac{\overline{x}}{A + \overline{x}} z_{n-2} = 0.$$

The characteristic equation of the linearized equation of Eq.(3.16) about the equilibrium $\overline{x}$ is

$$\lambda^3 - \frac{1}{A+\overline{x}}\lambda^2 + \frac{\overline{x}}{A+\overline{x}} = 0.$$

(d) From (c) and Theorem 1.4 it follows that the positive equilibrium $\overline{x}$ of Eq.(3.16) is locally asymptotically stable when either

$$A \geq \frac{1}{2}$$

or

$$\frac{1}{3} < A < \frac{1}{2} \text{ and } \alpha < \frac{A^2(-A^2+3A-1)}{(2A-1)^2}$$

and unstable when

$$\frac{1}{3} < A < \frac{1}{2} \text{ and } \alpha > \frac{A^2(-A^2+3A-1)}{(2A-1)^2}$$

or

$$0 < A < \frac{1}{3}.$$

CHAPTER 4

BEHAVIOR OF THE SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS

In this chapter[3] we investigate the periodicity, local stability and global attractivity of the following three rational difference equation which was studied in [13], [14] and [18].

4.1 Equation $x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2}}{dx_{n-1} + ex_{n-2}}$, $\quad n = 0, 1, \cdots$.

In this section we concerned with the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2}}{dx_{n-1} + ex_{n-2}}, \qquad n = 0, 1, \cdots \tag{4.1}$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-2}, x_{-1} and x_0 are positive real numbers.

4.1.1 Local Stability of the Equilibrium Point of Equation (4.1)

Here, we deal with the local stability character of the equilibrium point of Eq.(4.1). Eq.(4.1) has equilibrium point and is given by

$$\overline{x} = a\overline{x} + \frac{b + c}{d + e}.$$

If $a < 1$, then the only positive equilibrium point of Eq.(4.1) is given by

$$\overline{x} = \frac{b + c}{(1 - a)(d + e)}.$$

Let $f : (0, \infty)^3 \rightarrow (0, \infty)$ be a continuous function defined by

$$f(u, v, w) = au + \frac{bv + cw}{dv + ew}.$$

Therefore it follows that

$$\frac{\partial f(u, v, w)}{\partial u} = a,$$

$$\frac{\partial f(u,v,w)}{\partial v} = \frac{(be-dc)w}{(dv+ew)^2},$$
$$\frac{\partial f(u,v,w)}{\partial w} = \frac{(dc-be)u}{(dv+ew)^2}.$$

Then we see that

$$\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial u} = a = -a_2,$$

$$\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial v} = \frac{(be-dc)}{(d+e)^2\overline{x}} = \frac{(be-dc)(1-a)}{(d+e)(b+c)} = -a_1,$$
$$\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial w} = \frac{(dc-be)}{(d+e)^2\overline{x}} = \frac{(dc-be)(1-a)}{(d+e)(b+c)} = -a_0.$$

Then the linearized equation of Eq.(4.1) about $\overline{x}$ is

$$y_{n+1} + a_2 y_n + a_1 y_{n-1} + a_0 y_{n-2} = 0, \tag{4.2}$$

whose characteristic equation is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{4.3}$$

Theorem 4.1 *Assume that*

$$2\left|(be-dc)\right| < (d+e)(b+c).$$

Then the equilibrium point of Eq.(4.1) is locally asymptotically stable.

Proof: It follows by Theorem 1.6 that, Eq.(4.2) is asymptotically stable if all roots of Eq.(4.3) lie in the open disc $|\lambda| < 1$ that is if

$$|a_2| + |a_1| + |a_0| < 1,$$

$$|a| + \left|\frac{(be-dc)(1-a)}{(d+e)(b+c)}\right| + \left|\frac{(dc-be)(1-a)}{(d+e)(b+c)}\right| < 1,$$

and so

$$2\left|\frac{(be-dc)(1-a)}{(d+e)(b+c)}\right| < (1-a), \quad a < 1,$$

or

$$2\left|be-dc\right| < (d+e)(b+c).$$

The proof is complete.

4.1.2 Boundedness of Solutions of Equation (4.1)

Here we study the boundedness of the solutions of Eq.(4.1).

Theorem 4.2 *Every solution of Eq.(4.1) is bounded if* $a < 1$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(4.1). It follows from Eq.(4.1) that

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2}}{dx_{n-1} + ex_{n-2}} = ax_n + \frac{bx_{n-1}}{dx_{n-1} + ex_{n-2}} + \frac{cx_{n-2}}{dx_{n-1} + ex_{n-2}}.$$

Then

$$x_{n+1} \leq ax_n + \frac{bx_{n-1}}{dx_{n-1}} + \frac{cx_{n-2}}{ex_{n-2}} = ax_n + \frac{b}{d} + \frac{c}{e} \qquad \text{for all } n \geq 1.$$

By using a comparsion, we can write the right hand side as follows

$$y_{n+1} = ay_n + \frac{b}{d} + \frac{c}{e},$$

then

$$y_n = a^n y_0 + cons\tan t,$$

and this equation is locally asmptotically stable because $a < 1$ and converges to the equilibrium point $\overline{y} = \dfrac{be + cd}{de(1-a)}$.

Therefore

$$\limsup_{n\to\infty} x_n \leq \frac{be + cd}{de(1-a)}.$$

Hence the solutions is bounded.

Theorem 4.3 *Every solution of Eq.(4.1) is bounded if* $a > 1$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(4.1). Then from Eq.(4.1) we see that

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2}}{dx_{n-1} + ex_{n-2}} > ax_n \qquad \text{for all} \qquad n \geq 1.$$

We see that the right hand side can be written as follows

$$y_{n+1} = ay_n \Rightarrow y_n = a^n y_0,$$

and this equation is unstable because $a > 1$ and $\lim_{n\to\infty} y_n = \infty$. Then by using the ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above.

4.1.3 Existence of Periodic Solutions

In this subsection we study the existence of periodic solutions of Eq.(4.1).The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 4.4 *Eq.(4.1) has positive prime period two solutions if and only if*

$$(b-c)(d-e)(1+a)+4(bae+cd)>0, \quad d>e, \qquad b>c. \tag{4.4}$$

Proof: First suppose that there exists a prime period two solution

$$..., p, q, p, q, ...,$$

of Eq.(4.1). We will prove that (4.4) holds.
We see from Eq.(4.1) that

$$p = aq + \frac{bp+cq}{dp+eq}$$

and

$$q = ap + \frac{bq+cp}{dq+ep}.$$

Then

$$dp^2 + epq = adpq + aeq^2 + bp + cq, \tag{4.5}$$

and

$$dq^2 + epq = adpq + aep^2 + bq + cp. \tag{4.6}$$

Substracting (4.5) from (4.6) gives

$$d(p^2-q^2) = -ae(p^2-q^2) + (b-c)(p-q).$$

Since $p \neq q$, it follows that

$$p+q = \frac{(b-c)}{(d+ae)}. \tag{4.7}$$

Again, adding (4.5) and (4.6) yields

$$d(p^2+q^2) + 2epq = 2adpq + ae(p^2+q^2) + (b+c)(p+q),$$

$$(d-ae)(p^2+q^2)+2(e-ad)pq=(b+c)(p+q). \tag{4.8}$$

It follows by (4.7), (4.8) and the relation

$$p^2+q^2=(p+q)^2-2pq \qquad \text{for all} \quad p,q\in R,$$

that

$$2(e-d)(1+a)pq=\frac{2(bae+cd)(b-c)}{(d+ae)^2}.$$

Thus

$$pq=\frac{(bae+cd)(b-c)}{(d+ae)^2(e-d)(1+a)}. \tag{4.9}$$

Now it is clear from Eq.(4.7) and Eq.(4.9) that p and q are the two distinct roots of the quadratic equation

$$t^2-\left(\frac{(b-c)}{(d+ae)}\right)t+\left(\frac{(bae+cd)(b-c)}{(d+ae)^2(e-d)(1+a)}\right)=0$$

$$(d+ae)t^2-(b-c)t+\left(\frac{(bae+cd)(b-c)}{(d+ae)(e-d)(1+a)}\right)=0 \tag{4.10}$$

and so

$$[b-c]^2-\frac{4(bae+cd)(b-c)}{(e-d)(1+a)}>0,$$

or

$$[b-c]^2+\frac{4(bae+cd)(b-c)}{(d-e)(1+a)}>0.$$

$$(b-c)(d-e)(1+a)+4(bae+cd)>0.$$

Therefore inequalities (4.4) holds.

Secondly suppose that inequalities (4.4) are true. We will show that Eq.(4.1) has a prime period two solution.

Assume that

$$p=\frac{b-c+\zeta}{2(d+ae)},$$

and so

$$q=\frac{b-c-\zeta}{2(d+ae)},$$

where

$$\zeta = \sqrt{[b-c]^2 - \frac{4(bae+cd)(b-c)}{(e-d)(1+a)}}.$$

We see from inequalities (4.4) that

$$(b-c)(d-e)(1+a) + 4(bae+cd) > 0, b > c, d > e,$$

which is equivalent to

$$(b-c)^2 > \frac{4(bae+cd)(b-c)}{(e-d)(1+a)}.$$

Therefore p and q are distinct real numbers.

Set

$$x_2 = q, x_{-1} = p \text{ and } x_0 = q.$$

We wish to show that

$$x_1 = x_{-1} = p \qquad \text{and} \qquad x_2 = x_0 = q.$$

It follows from Eq.(4.1) that

$$x_1 = aq + \frac{bp+cq}{dp+eq} = a\left(\frac{b-c-\zeta}{2(d+ae)}\right) + \frac{b\left(\frac{b-c+\zeta}{2(d+ae)}\right) + c\left(\frac{b-c-\zeta}{2(d+ae)}\right)}{d\left(\frac{b-c+\zeta}{2(d+ae)}\right) + e\left(\frac{b-c-\zeta}{2(d+ae)}\right)}.$$

Dividing the denominator and numerator by $2(d+ae)$ gives

$$\begin{aligned} x_1 &= \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{b(b-c+\zeta)+c(b-c-\zeta)}{d(b-c+\zeta)+e(b-c-\zeta)} \\ &= \frac{ab-ac-a\zeta}{2(d+ae)} + \frac{(b-c)\left[(b+c)+\zeta\right]}{(d+e)(b-c)+(d-e)\zeta}. \end{aligned}$$

Multiplying the denominator and numerator of the right side by $(d+e)(b-c)-(d-e)\zeta$ gives

$$\begin{aligned}
x_1 &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)\left[(b+c)+\zeta\right]\left[(d+e)(b-c)-(d-e)\zeta\right]}{\left[(d+e)(b-c)+(d-e)\zeta\right]\left[(d+e)(b-c)-(d-e)\zeta\right]}\\
&= \frac{ab-ac-a\zeta}{2(d+ae)}\\
&\quad+\frac{(b-c)\left\{(d+e)(b^2-c^2)+\zeta\left[(d+e)(b-c)-(d-e)(b+c)-(d-e)\zeta^2\right]\right\}}{(d+e)^2(b-c)^2-(d-e)^2\zeta^2}\\
&= \frac{ab-ac-a\zeta}{2(d+ae)}\\
&\quad+\frac{(b-c)\left\{(d+e)(b^2-c^2)+2\zeta(eb-cd)-(d-e)\left([b-c]^2-\frac{4(bae+cd)(b-c)}{(e-d)(1+a)}\right)\right\}}{(d+e)^2(b-c)^2-(d-e)^2\left([b-c]^2-\frac{4(bae+cd)(b-c)}{(e-d)(1+a)}\right)}\\
&= \frac{ab-ac-a\zeta}{2(d+ae)}\\
&\quad+\frac{(b-c)\left\{(d+e)(b^2-c^2)+2\zeta(eb-cd)-(d-e)\,(b-c)^2-\frac{4(bae+cd)(b-c)}{(1+a)}\right\}}{(d+e)^2(b-c)^2-(d-e)^2\left([b-c]^2-\frac{4(bae+cd)(b-c)}{(e-d)(1+a)}\right)}\\
&= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)\left\{2(b-c)\left[dc+eb-\frac{2(bae+cd)}{(1+a)}\right]+2\zeta(eb-cd)\right\}}{4(b-c)\left[ed(b-c)+\frac{(e-d)(bae+cd)}{(1+a)}\right]}.
\end{aligned}$$

Multiplying the denominator and numerator of the right side by $(1+a)$ we obtain

$$\begin{aligned}
x_1 &= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)\left[(dc+eb)(1+a)-2(bae+cd)\right]+\zeta(1+a)(eb-cd)}{2\left[ed(b-c)(1+a)+(e-d)(bae+cd)\right]}\\
&= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)(eb-dc)(1-a)+\zeta(1+a)(eb-cd)}{2\left[ed(b-c)(1+a)+(e-d)(bae+cd)\right]}\\
&= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(eb-dc)\left\{(b-c)(1-a)+\zeta(1+a)\right\}}{2(eb-cd)(d+ae)}\\
&= \frac{ab-ac-a\zeta}{2(d+ae)}+\frac{(b-c)(1-a)+\zeta(1+a)}{2(d+ae)}\\
&= \frac{ab-ac-a\zeta+(b-c)(1-a)+\zeta(1+a)}{2(d+ae)}=\frac{b-c+\zeta}{2(d+ae)}=p.
\end{aligned}$$

Similarly as before one can show that

$$x_2=q.$$

Then it follows by induction that

$$x_{2n}=q \qquad \text{and} \qquad x_{2n+1}=p \qquad \text{for all} \quad n\geq -1.$$

Then Eq.(4.1) has prime period two solution

$$..., p, q, p, q, ...,$$

where p and q are the distinct roots of the quadratic equation (4.10) and the proof is complete.

4.1.4 Global Attractivity of the Equilibrium Point of Equation (4.1)

In this subsection we investigate the global asymptotic stability of Eq.(4.1).

Theorem 4.5 *The equilibrium point $\overline{x}$ is a global attractor of Eq.(4.1) if one of the following statements holds:*

$$(1) \quad be \geq dc \quad and \quad c \geq b, \tag{4.11}$$

$$(2) \quad be \leq dc \quad and \quad c \leq b. \tag{4.12}$$

Proof. Let α and β be real numbers and assume that $g : [\alpha, \beta]^3 \rightarrow [\alpha, \beta]$ is a function defined by

$$g(u, v, w) = au + \frac{bv + cw}{dv + ew}.$$

Then

$$\frac{\partial g(u, v, w)}{\partial u} = a,$$

$$\frac{\partial g(u, v, w)}{\partial v} = \frac{(be - dc)w}{(dv + ew)^2},$$

$$\frac{\partial g(u, v, w)}{\partial w} = \frac{(dc - be)u}{(dv + ew)^2}.$$

We consider two cases:

Case 1. Assume that (4.11) is true, then we can easily see that the function $g(u, v, w)$ is increasing in u, v and decreasing in w.

Suppose that (m, M) is a solution of the system $M = g(M, M, m)$ and $m = g(m, m, M)$. Then from Eq.(4.1), we see that

$$M = aM + \frac{bM + cm}{dM + em}, \quad m = am + \frac{bm + cM}{dm + eM},$$

or

$$M(1-a)=\frac{bM+cm}{dM+em},\quad m(1-a)=\frac{bm+cM}{dm+eM}.$$

then

$$d(1-a)M^2+e(1-a)Mm=bM+cm,\ d(1-a)m^2+e(1-a)Mm=bm+cM.$$

Substraction this two equations we obtain

$$(M-m)\left\{d(1-a)(M+m)+(c-b)\right\}=0,$$

under the conditions $c\geq b$, $a<1$, we see that

$$M=m.$$

It follows by Theorem 1.16 that $\overline{x}$ is a global attractor of Eq.(4.1) and then the proof is complete.

Case 2. Assume that (4.12) is true, let α and β be real numbers and assume that $g:[\alpha,\beta]^3\to[\alpha,\beta]$ is a function defined by $g(u,v,w)=au+\dfrac{bv+cw}{dv+ew}$, then we can easily see that function $g(u,v,w)$ is increasing in u,w and decreasing in v.

Suppose that (m,M) is a solution of the system $M=g(M,m,M)$ and $m=g(m,M,m)$. Then from Eq.(4.1), we see that

$$M=aM+\frac{bm+cM}{dm+eM},\qquad m=am+\frac{bM+cm}{dM+em},$$

or

$$M(1-a)=\frac{bm+cM}{dm+eM},\quad m(1-a)=\frac{bM+cm}{dM+em}.$$

then

$$d(1-a)Mm+e(1-a)M^2=bm+cM,\ d(1-a)mM+e(1-a)m^2=bM+cm.$$

Substracting we obtain

$$(M-m)\left\{e(1-a)(M+m)+(b-c)\right\}=0,$$

under the conditions $b\geq c, a<1$ we see that

$$M=m.$$

It follows by Theorem 1.19 that $\overline{x}$ is a global attractor of Eq.(4.1) and then the proof is complete.

4.2 Equation $x_{n+1} = \dfrac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a, \quad n = 0, 1, \cdots$.

In this section we deal with some properties of the solutions of the recursive sequence

$$x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a, \quad n = 0, 1, \cdots \tag{4.13}$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, ..., x_0$ are arbitrary positive real numbers with $x_i \neq \frac{b}{c}$ for $i = -r, -r+1, ...0$, $a > b/c$, $r = max\{l, k, s\}$ is nonnegative integer and a, b, c, d are positive constants.

4.2.1 Periodic Solutions

In this subsection we study the existence of periodic solutions of Eq.(4.13). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions.

Theorem 4.6 *Eq.(4.13) has positive prime period two solutions if and only if one of the following conditions is satisfied:*

(i) If l, k, s are even and $(b - ac)^2 > \frac{4d(b^2 - abc - abd)}{(c+d)}$, $b > a(c + d)$.

(ii) If l, k, s are odd and $(ac + b)^2 > 4ab(c - d)$, $c > d$.

(iii) If $l, k-$ even, $s-$odd and $(ac + b)^2 > \frac{4(abc^2 + abd^2 - a^2c^2d - b^2d)}{(c-d)}$, $ab(c^2 + d^2) > d(a^2c^2 - b^2)$, $c \neq d$.

(iv) If $l-$even, $k, s-$odd and $(ac + b)^2 > \frac{4abc^2}{(c+d),}$, $c > d$.

(v) If $k-$even, $l, s-$odd and $(ac + b)^2 > \frac{4abc^2}{(c+d)}$.

(vi) If $s-$even, $l, k-$odd and $(ac - b)^2 > \frac{4d(a^2c^2 - abc - abd)}{(c+d)}$, $ac^2 > b(c + d)$.

Proof. We will prove the theorem when Case (i) is true. The proof of other cases is similiar.

First suppose that there exists a prime period two solution

$..., p, q, p, q, ...$

of Eq.(4.13). We will prove that condition (i) holds.

When $l, k, s-$even, we see from Eq.(4.13) that

$$p = \frac{dq^2}{cq - b} + a$$

and

$$q = \frac{dp^2}{cp - b} + a.$$

Then

$$cpq - bp = dq^2 + acq - ab, \tag{4.14}$$

and

$$cpq - bq = dp^2 + acp - ab. \tag{4.15}$$

Substracting (4.14) from (4.15) gives

$$b(q - b) = d(q^2 - p^2) + ac(q - p).$$

Since $p \neq q$, it follows that

$$p + q = \frac{(b - ac)}{d}. \tag{4.16}$$

Also, since p and q are positive, $(b - ac)$ should be positive.

Again, adding (4.14) and (4.15) yields

$$2cpq - b(p + q) = d(p^2 + q^2) + ac(p + q) - 2ab. \tag{4.17}$$

It follows by (4.16), (4.17) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \quad \text{for all} \quad p, q \in R,$$

that

$$pq = \frac{b^2 - abc - abd}{d(c + d)}. \tag{4.18}$$

It is clear now from Eq.(4.16) and Eq.(4.18) that p and q are the two positive distinct roots of the quadratic equation

$$dt^2 - (b - ac)t + \frac{b^2 - abc - abd}{(c + d)} = 0 \tag{4.19}$$

and so

$$(b-ac)^2 > \frac{4d(b^2-abc-abd)}{(c+d)}.$$

So inequality (i) holds.

Secondly suppose that (i) is true. We will show that Eq.(4.13) has a prime period two solution.

Assume that

$$p = \frac{(b-ac) - \sqrt{(b-ac)^2 - \dfrac{4d(b^2-abc-abd)}{(c+d)}}}{2d}$$

and

$$q = \frac{(b-ac) - \sqrt{(b-ac)^2 - \dfrac{4d(b^2-abc-abd)}{(c+d)}}}{2d}.$$

We see from (i) that

$$(b-ac)^2 > \frac{4d(b^2-abc-abd)}{(c+d)}.$$

Therefore p and q are distinct real numbers.

Set

$$x_{-r} = p, x_{-r+1} = q, ..., \quad and \quad x_{-1} = q, x_0 = p.$$

We wish to show that

$$x_1 = x_{-1} = q \quad and \quad x_2 = x_0 = p.$$

It follows from Eq.(4.13) that

$$\begin{aligned} x_1 &= \frac{dx_{-l}x_{-k}}{cx_{-s}-b} + a = \frac{dp^2}{cp-b} + a \\ &= \frac{d\left[\dfrac{(b-ac) - \sqrt{(b-ac)^2 - \dfrac{4d(b^2-abc-abd)}{(c+d)}}}{2d}\right]^2}{c\left[\dfrac{(b-ac) - \sqrt{(b-ac)^2 - \dfrac{4d(b^2-abc-abd)}{(c+d)}}}{2d}\right] - b} + a, \end{aligned}$$

or,

$$x_1 = \frac{\left[2b^2 - 2abc - \frac{4b^2d}{(c+d)} - 2b\sqrt{(b-ac)^2 - \frac{4d(b^2-abc-abd)}{(c+d)}}\right]}{2\left[bc - ac^2 - 2bd - c\sqrt{(b-ac) - \frac{4d(b^2-abc-abd)}{(c+d)}}\right]}.$$

Multiplying the denominator and numerator by

$$(c+d)\left[bc - ac^2 - 2bd + c\sqrt{(b-ac)^2 - \frac{4d(b^2-abc-abd)}{(c+d)}}\right]$$

we get

$$x_1 = \frac{\left[4b^3d^2 - 4ab^2cd^2 + 4b^2d^2\sqrt{(b-ac)^2 - \frac{4d(b^2-abc-abd)}{(c+d)}}\right]}{8b^2d^3}$$

or,

$$x_1 = \frac{\left[b - ac + \sqrt{(b-ac)^2 - \frac{4d(b^2-abc-abd)}{(c+d)}}\right]}{2d} = q.$$

Similarly as before one can easily show that

$$x_2 = p.$$

Then it follows by induction that

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all } n \geq -r.$$

Thus Eq.(4.13) has the positive prime period two solution

$$..., p, q, p, q, ...$$

where p and q are the distinct roots of the quadratic Eq.(4.19) and then the proof is complete.

4.2.2 Local Stability of Equation (4.13)

In this subsection we study the local stability character of the equilibrium point of Eq.(4.13) in the case $c = d$.

The equilibrium points of Eq.(4.13) are given by the relation

$$\overline{x} = \frac{d\overline{x}^2}{c\overline{x} - b} + a.$$

If $c = d$, then the only equilibrium point of Eq.(4.13) is given by

$$x = \frac{ab}{ac + b}.$$

Let $g : (0, \infty)^3 \rightarrow (0, \infty)$ be a function defined by

$$g(u, v, w) = \frac{cuv}{cw - b} + a.$$

Therefore

$$\begin{aligned}
\frac{\partial g(u, v, w)}{\partial u} &= \frac{cv}{cw - b}, \\
\frac{\partial g(u, v, w)}{\partial v} &= \frac{cu}{cw - b}, \\
\frac{\partial g(u, v, w)}{\partial w} &= -\frac{c^2 uv}{(cw - b)^2}.
\end{aligned}$$

Then we see that

$$\begin{aligned}
\frac{\partial g(\overline{x}, \overline{x}, \overline{x})}{\partial u} &= \frac{-ac}{b} = -c_0, \\
\frac{\partial g(\overline{x}, \overline{x}, \overline{x})}{\partial v} &= \frac{-ac}{b} = -c_1, \\
\frac{\partial g(\overline{x}, \overline{x}, \overline{x})}{\partial w} &= \frac{-a^2 c^2}{b^2} = -c_2.
\end{aligned}$$

Then the linearized equation of Eq.(4.13) about $\overline{x}$ is

$$y_{n+1} + c_0 y_{n-l} + c_1 y_{n-k} + c_2 y_{n-s} = 0. \qquad (4.20)$$

Theorem 4.7 *Assume that*

$$\sqrt{2}b > (ac + b).$$

Then the positive equilibrium point of Eq.(4.13) is locally asymptotically stable.

Proof. It is follows by Theorem (1.6) that, Eq.(4.20) is asymptotically stable if

$$|c_2| + |c_1| + |c_0| < 1,$$

$$\left|\frac{-ac}{b}\right| + \left|\frac{-ac}{b}\right| + \left|\frac{-a^2c^2}{b^2}\right| < 1, \quad \text{or} \quad \frac{2ac}{b} + \frac{a^2c^2}{b^2} < 1,$$

and so

$$a^2c^2 + 2abc + b^2 < 2b^2 \Rightarrow \sqrt{2}b > (ac + b).$$

The proof is complete.

4.2.3 Global Attractivity of the Equilibrium Point of Equation (4.13)

In this subsection we investigate the global attractivity of the equilibrium point of Eq.(4.13).

Theorem 4.8 . *If $b = ac$, $c = d$, then the equilibrium point $\overline{x}$ of Eq.(4.13) is global attractor.*

Proof. Let p, q be real numbers and assume that $g : [p,q]^3 \to [p,q]$ is a function defined by

$$g(u,v,w) = \frac{cuv}{cw-b} + a.$$

Therefore

$$\frac{\partial g(u,v,w)}{\partial u} = \frac{cv}{cw-b},$$

$$\frac{\partial g(u,v,w)}{\partial v} = \frac{cu}{cw-b},$$

$$\frac{\partial g(u,v,w)}{\partial w} = -\frac{c^2uv}{(cw-b)^2}.$$

Case i. If $cw - b > 0$, then we can easily see that the function $g(u,v,w)$ increasing in u, v and decreasing in w.

Suppose that (m, M) is a solution of the system

$$m = g(m,m,M) \quad \text{and} \quad M = g(M,M,m).$$

Then from Eq.(4.13), we see that

$$m = \frac{cm^2}{cM-b} + a, \quad M = \frac{cM^2}{cm-b} + a,$$

$$cMm - bm = cm^2 + acM - ab, \quad cMm - bM = cM^2 + acm - ab,$$

then

$$b(M-m) = c(m^2 - M^2) + ac(M-m), \quad b = ac.$$

Thus $M = m$.

It follows by the Theorem (1.16) that $\overline{x}$ is a global attractor of Eq.(4.13).

Case ii. If $cw - b < 0$, then we can easily see that the function $g(u,v,w)$ decreasing in u, v, w.

Suppose that (m, M) is a solution of the system

$$M = g(m, m, m) \quad and \quad m = g(M, M, M).$$

Then from Eq.(4.13), we see that

$$M = \frac{cm^2}{cm - b} + a, \quad m = \frac{cM^2}{cM - b} + a$$

$$cMm - bM = cm^2 + acm - ab, \quad cMm - bm = cM^2 + acM - ab,$$

then

$$b(m - M) = c(m^2 - M^2) + ac(m - M), \quad b = ac.$$

Thus $M = m$.

It follows by the Theorem 1.20 that $\overline{x}$ is a global attractor of Eq.(4.13) and then the proof is complete.

4.3 Equation $x_{n+1} = \dfrac{ax_{n-l}x_{n-k}}{bx_{n-p} - cx_{n-q}}, \quad n = 0, 1, \cdots.$

In this section we investigate the global attractivity of the equilibrium point, and the asymptotic behavior of the solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} - cx_{n-q}}, \quad n = 0, 1, \cdots \tag{4.21}$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, ..., x_0$ are arbitrary positive real numbers, $r = \max\{l, k, p, q\}$ is nonnegative integer and a, b, c are positive constants.

4.3.1 Local Stability of Equation (4.21)

In this subsection we investigate the local stability character of the solutions of Eq.(4.21). Eq.(4.21) has a uniqe positive equilibrium point and is given by

$$\overline{x} = \frac{a\overline{x}^2}{b\overline{x} - c\overline{x}}.$$

If $a \neq b - c, b \neq c$, then the unique equilibrium point is $\overline{x} = 0$.

Let $f : (0, \infty)^4 \to (0, \infty)$ be a function defined by

$$f(u, v, w, s) = \frac{auv}{bw - cs}, \tag{4.22}$$

Therefore, it follows that

$$\begin{aligned} f_u(u,v,w,s) &= \frac{av}{(bw-cs)}, \qquad f_v(u,v,w,s)=\frac{au}{(bw-cs)}, \\ f_w(u,v,w,s) &= \frac{-bauv}{(bw-cs)^2}, \qquad f_s(u,v,w,s)=\frac{cauv}{(bw-cs)^2}, \end{aligned}$$

we see that

$$\begin{aligned} f_u(\overline{x},\overline{x},\overline{x},\overline{x}) &= \frac{a}{(b-c)}, \qquad f_v(\overline{x},\overline{x},\overline{x},\overline{x})=\frac{a}{(b-c)}, \\ f_w(\overline{x},\overline{x},\overline{x},\overline{x}) &= \frac{-ab}{(b-c)^2}, \qquad f_s(\overline{x},\overline{x},\overline{x},\overline{x})=\frac{ac}{(b-c)^2}, \end{aligned}$$

The linearized eqation of Eq.(4.21) about $\overline{x}$ is

$$\gamma_{n+1}+\frac{a}{(b-c)}\gamma_{n-1}+\frac{a}{(b-c)}\gamma_{n-k}-\frac{ab}{(b-c)^2}\gamma_{n-p}+\frac{ac}{(b-c)^2}\gamma_{n-q}=0. \tag{4.23}$$

Theorem 4.9 *Assume that*

$$a(3\zeta-\eta)<(b-c)^2,$$

where $\zeta=\max\{b,c\}, \eta=\min\{b,c\}$. Then the equilibrium point of Eq.(4.21) is locally asymptotically stable.

Proof: It follows by Theorem 1.6 that Eq.(4.23) is asymptotically stable if

$$\left|\frac{a}{(b-c)}\right|+\left|\frac{a}{(b-c)}\right|+\left|\frac{ab}{(b-c)^2}\right|+\left|\frac{ac}{(b-c)^2}\right|<1,$$

or

$$\left|\frac{2a}{(b-c)}\right|+\left|\frac{a(b+c)}{(b-c)^2}\right|<1,$$

and so

$$2a\left|b-c\right|+a(b+c)<(b-c)^2.$$

The proof is complete.

4.3.2 Global Attractivity of the Equlibrium Point of Equation (4.21)

In this subsection we investigate the global attractivty of the equilibrium point of Eq.(4.21). We give the following two theorems which is a minor modification of Theorem 1.16.

Theorem 4.10 *Let $[a,b]$ be an interval of real numbers and that*

$f:[a,b]^{k+1}\rightarrow[a,b]$,

is a continuous function satisfying the following properties:

(i) $f(x_1,x_2,...,x_{k+1})$ is non-increasing in one component (for example x_t) for each $x_r(r\neq t)$ in $[a,b]$ and non-decreasing in the remaining components for each x_t in $[a,b]$.

(ii) if $(m,M)\in[a,b]\times[a,b]$ is a solution of the system

$$M=f(M,M,...,M,m,M,...,M,M) \text{ and } m=f(m,m,...,m,M,m,...,m,m)$$

implies $m=M$.

Then Equation

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-k}),n=0,1,... \qquad (4.24)$$

has a unique equilibrium $\overline{x}\in[a,b]$ and every solution of Eq.(4.24) converges to $\overline{x}$.

Proof: Set

$m_0=a$ and $M_0=b$,

and for each $i=1,2,...$ set

$m_i=f(m_{i-1},m_{i-1},...,m_{i-1},M_{i-1},m_{i-1},...,m_{i-1},m_{i-1})$,

and

$M_i=f(M_{i-1},M_{i-1},...,M_{i-1},m_{i-1},M_{i-1},...,M_{i-1},M_{i-1})$.

Now observe that for each $i\geq 0$,

$a=m_0\leq m_1\leq...\leq m_i\leq M_i\leq...M_1\leq M_0=b$,

and

$m_i\leq x_p\leq M_i$ for $p\geq(k+1)i+1$.

Set

$m=\lim\limits_{i\rightarrow\infty}m_i$ and $M=\lim\limits_{i\rightarrow\infty}M_i$.

Then

$$M \geq \lim_{i\to\infty} \sup x_i \geq \lim_{i\to\infty} \inf x_i \geq m.$$

and by the continuity of f,

$$M = f(M, M, ..., M, m, M, ..., M, M) \text{ and } m = f(m, m, ..., m, M, m, ..., m, m).$$

In view of **(ii)**,

$$m = M = \overline{x},$$

from which the result follows.

Theorem 4.11

Let $[a, b]$ be an interval of real numbers and assume that

$$f : [a, b]^{k+1} \to [a, b],$$

is a continuous function satisfying the following properties:
(i) $f(x_1, x_2, ..., x_{k+1})$ is non-increasing in one component (for example x_t) for each $x_r (r \neq t)$ in $[a, b]$ and non-increasing in the remaining components for each x_t in $[a, b]$.
(ii) if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = f(m, m, ..., m, M, m, ..., m, m) \text{ and } m = f(M, M, ..., M, m, M, ..., M, M)$$

implies $m = M$.
Then Eq.(4.24) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq.(4.24) converges to $\overline{x}$
Proof: As the proof of Theorem 4.10 and will be omitted.

Theorem 4.12

The equilibrium point $\overline{x}$ of Equation 4.21 is global attractor if $c \neq a$.
Proof: Let p, q are a real numbers and assume that $f : [p, q]^4 \to [p, q]$ be a function defined by Eq.(4.22), then we can easily see that the function $f(u, v, w, s)$ increasing in s and decreasing in w.

Case 1. If $bw - cs > 0$, then we can easily that the function $f(u, v, w, s)$ increasing in u, v, s and decreasing in w.

Suppose that (m, M) is a solution of the system

$M = f(m, m, M, m)$ and $M = f(M, M, m, M)$.

Then from Eq.(4.21), we see that

$$m = \frac{am^2}{bM - cm}, \quad M = \frac{aM^2}{bm - cM},$$

$$bM = cm + am, \quad bm = cM + aM,$$

then

$$(M - m)(b + c + d) = 0$$

Thus

$$M = m.$$

It follows by Theorem 4.10 that $\overline{x}$ is a global attractor of Eq.(4.21) and then the proof is complete.

Case 2. If $bw - cs < 0$, then we can easily that the function $f(u, v, w, s)$ decreasing in u, v, w and increasing in s.

Suppose that (m, M) is a solution of the system

$M = f(m, m, m, M)$ and $M = f(M, M, M, m)$.

Then from Eq.(4.21), we see that

$$M = \frac{am^2}{bm - cM}, \quad m = \frac{aM^2}{bM - cm},$$

$$bmM - cM^2 = am^2, \quad bmM - cm^2 = aM^2,$$

then

$$(M^2 - m^2)(c - a) = 0, \quad a \neq c.$$

Thus,

$$M = m.$$

It follows by Theorem 4.11 that $\overline{x}$ is a global attractor of Eq.(4.21) and then the proof is complete.

CHAPTER 5

DIFFERENCE EQUATION:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{C x_{n-1} + D x_{n-2}}, \quad n = 0, 1, \cdots .$$

In this chapter, we investigate the global character of the solutions of the rational difference equation of the third order

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{C x_{n-1} + D x_{n-2}}, \ n = 0, 1, \cdots \tag{5.1}$$

where the parameters $\alpha, \beta, \gamma,\ D$ and C are non-negative real numbers and the initial conditions $x_{-2}, x_{-1,}\ x_0$ are arbitrary non-negative real numbers such that the denominator of Eq.(5.1) is never zero.

5.1 LINEARIZED STABILITY ANALYSIS

Lemma 5.1 **(a)** *Eq.(5.1) can be written in the normalized form*

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + D x_{n-2}}, \ n = 0, 1, \cdots \tag{5.2}$$

with positive parameters α, β, D and with arbitrary positive initial conditions $x_{-2}, x_{-1,}$ x_0.

(b) *Equilibrium point of Eq.(5.2) is*

$$\overline{x} = \frac{(\beta + 1) + \sqrt{(\beta + 1)^2 + 4\alpha(1 + D)}}{2(1 + D)}.$$

(c) *The linearized equation of Eq.(5.2) about its positive equilibrium $\overline{x}$ is*

$$z_{n+1} - \frac{\beta}{(1 + D)\overline{x}} z_n + \frac{\overline{x} - 1}{(1 + D)\overline{x}} z_{n-1} + \frac{D}{(1 + D)} z_{n-2} = 0. \tag{5.3}$$

and the corresponding characteristic equation of Eq.(5.3) is

$$\lambda^3 - \frac{\beta}{(1 + D)\overline{x}} \lambda^2 + \frac{\overline{x} - 1}{(1 + D)\overline{x}} \lambda + \frac{D}{(1 + D)} = 0.$$

Proof.

(a) The Eq.(5.1) which by the change of variables

$$x_n = \frac{\gamma}{C} y_n,$$

reduces to the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + D x_{n-2}}, n = 0, 1, \cdots$$

where

$$\alpha := \frac{\alpha C}{\gamma^2}, \quad \beta := \frac{\beta}{\gamma}, \quad D := \frac{D}{C}.$$

(b) The equilibrium points of Eq.(5.2) are the non-negative solutions of the equation

$$\overline{x} = \frac{\alpha + \beta\overline{x} + \overline{x}}{\overline{x} + D\overline{x}}$$

or equivalently

$$(1 + D)\overline{x}^2 - (1 + \beta)\overline{x} - \alpha = 0. \tag{5.4}$$

Hence, the solutions of Eq.(5.4) are

$$\overline{x} = \frac{1 + \beta + \sqrt{(1 + \beta)^2 + 4\alpha(1 + D)}}{2(1 + D)} \tag{5.5}$$

and

$$\overline{x} = \frac{1 + \beta - \sqrt{(1 + \beta)^2 + 4\alpha(1 + D)}}{2(1 + D)}.$$

So, the positive equilibrium point of Eq.(5.2) is unique and is given by (5.5).

(c) Now, let I be some interval of real numbers and let

$$f : I \times I \times I \to I$$

be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + D x_{n-2}}.$$

From Eq.(5.4), we obtain that

$$\begin{aligned} q_0 &= \frac{\partial f}{\partial x_n}(\overline{x},\overline{x},\overline{x}) = \left[\frac{\beta.(x_{n-1}+Dx_{n-2})-(\alpha+\beta x_n+x_{n-1}).0}{(x_{n-1}+Dx_{n-2})^2}\right](\overline{x},\overline{x},\overline{x}) \\ &= \frac{\beta.(\overline{x}+D\overline{x})}{\overline{x}^2(1+D)^2} = \frac{\beta\overline{x}(1+D)}{\overline{x}^2(1+D)^2} = \frac{\beta}{\overline{x}(1+D)}, \end{aligned}$$

$$\begin{aligned} q_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x},\overline{x},\overline{x}) = \left[\frac{1.(x_{n-1}+Dx_{n-2})-(\alpha+\beta x_n+x_{n-1}).1}{(x_{n-1}+Dx_{n-2})^2}\right](\overline{x},\overline{x},\overline{x}) \\ &= \frac{\overline{x}+D\overline{x}-\overline{x}.(\overline{x}+D\overline{x})}{\overline{x}^2(1+D)^2} = \frac{(\overline{x}+D\overline{x})(1-\overline{x})}{\overline{x}^2(1+D)^2} = \frac{1-\overline{x}}{\overline{x}(1+D)}, \end{aligned}$$

and

$$\begin{aligned} q_2 &= \frac{\partial f}{\partial x_{n-2}}(\overline{x},\overline{x},\overline{x}) = \left[\frac{0.(x_{n-1}+Dx_{n-2})-(\alpha+\beta x_n+x_{n-1}).D}{(x_{n-1}+Dx_{n-2})^2}\right](\overline{x},\overline{x},\overline{x}) \\ &= \frac{-(\alpha+\beta\overline{x}+\overline{x}).D}{(\overline{x}+D\overline{x})^2} = \frac{-\overline{x}.(\overline{x}+D\overline{x}).D}{(\overline{x}+D\overline{x})^2} = \frac{-D}{(1+D)}. \end{aligned}$$

If $\overline{x}$ denotes an equilibrium point of Eq.(5.2), then the linearized equation associated with Eq.(5.2) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1+} q_2 z_{n-2}$$

or

$$z_{n+1} - \frac{\beta}{(1+D)\overline{x}} z_n + \frac{\overline{x}-1}{(1+D)\overline{x}} z_{n-1} + \frac{D}{(1+D)} z_{n-2} = 0. \tag{5.6}$$

Lemma 5.2 *The positive equilibirium $\overline{x}$ of Eq.(5.2) is locally asymptotically stable when $\frac{(1+\beta)^2(D-1)}{4} < \alpha$ and $\beta < 1$.*

Proof. From Theorem 1.5 it follows all roots of Eq.(5.2) lie in an open disc $|\lambda| < 1$, if

$$|q_0| + |q_1| + |q_2| < 1.$$

This implies that

$$\left|\frac{\beta}{(1+D)\overline{x}}\right| + \left|\frac{1-\overline{x}}{(1+D)\overline{x}}\right| + \left|\frac{-D}{1+D}\right| < 1.$$

Hence

$$\begin{aligned} \frac{\beta}{(1+D)\overline{x}} + \frac{|1-\overline{x}|}{(1+D)\overline{x}} + \frac{D\overline{x}}{(1+D)\,\overline{x}} &< 1 \qquad (5.7) \\ \beta + D\overline{x} + |1-\overline{x}| &< (1+D)\overline{x} \\ |1-\overline{x}| &< \overline{x} + D\overline{x} - \beta - D\overline{x} \\ |1-\overline{x}| &< \overline{x} - \beta \qquad (5.8) \end{aligned}$$

and so we have two cases for (5.7).

Case 1:

Then

$$
\begin{aligned}
1-\overline{x}<\overline{x}-\beta \quad &\Rightarrow \quad \frac{1+\beta}{2}<\overline{x} \\
&\Rightarrow \quad \frac{1+\beta}{2}<\frac{(\beta+1)+\sqrt{(\beta+1)^2+4\alpha(1+D)}}{2(1+D)} \\
&\Rightarrow \quad \frac{(1+\beta)^2(D-1)}{4}<\alpha.
\end{aligned}
$$

Case 2:

Hence

$$-\overline{x}+\beta<1-\overline{x} \Rightarrow \beta<1.$$

So the positive equilibrium $\overline{x}$ of Eq.(5.2) is locally asymptotically stable.

Lemma 5.3 *The positive equilibrium $\overline{x}$ of Eq.(5.2) is locally asymptotically stable when* $\frac{[(D+3)\beta-(D-1)](\beta-1)}{4}<\alpha$ *and* $\frac{-(1+\beta)^2}{4(1+D)}<\alpha$

and

$\frac{(\beta+1)^2(7D+3)}{4(3D+1)^2}<\alpha$ *and* $\frac{[(D-1)\beta-(D+3)](\beta-1)}{4}<\alpha.$

Proof. The characteristic equation of the linearized equation of Eq.(5.2) about the equilibrium $\overline{x}$ is

$$\lambda^3-\frac{\beta}{(1+D)\overline{x}}\lambda^2+\frac{\overline{x}-1}{(1+D)\overline{x}}\lambda+\frac{D}{(1+D)}=0.$$

From Theorem 1.4 using

$$a_2=\frac{-\beta}{(1+D)\overline{x}}, a_1=\frac{\overline{x}-1}{(1+D)\overline{x}}, a_0=\frac{D}{(1+D)}$$

we observe that

$$
\begin{aligned}
|a_2+a_0|<1+a_1 \quad &\Rightarrow \quad \left|\frac{-\beta}{(1+D)\overline{x}}+\frac{D}{(1+D)}\right|<1+\frac{\overline{x}-1}{(1+D)\overline{x}} \\
&\Rightarrow \quad \left|\frac{-\beta+D\overline{x}}{(1+D)\overline{x}}\right|<\frac{\overline{x}+D\overline{x}+\overline{x}-1}{(1+D)\overline{x}} \\
&\Rightarrow \quad \frac{1-\beta}{2}<\overline{x} \text{ and } \frac{1+\beta}{2(D+1)}<\overline{x} \\
&\Rightarrow \quad \frac{1-\beta}{2}<\frac{(\beta+1)+\sqrt{(\beta+1)^2+4\alpha(1+D)}}{2(1+D)} \\
&\qquad \text{and} \\
&\qquad \frac{1+\beta}{2(D+1)}<\frac{(\beta+1)+\sqrt{(\beta+1)^2+4\alpha(1+D)}}{2(1+D)} \\
&\Rightarrow \quad \frac{[(D+3)\beta-(D-1)](\beta-1)}{4}<\alpha \text{ and } \frac{-(1+\beta)^2}{4(1+D)}<\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
|a_2 - 3a_0| < 3 - a_1 \quad &\Rightarrow \quad \left|\frac{-\beta}{(1+D)\overline{x}} - \frac{3D}{(1+D)}\right| < 3 - \frac{\overline{x}-1}{(1+D)\overline{x}} \\
&\Rightarrow \quad \left|\frac{-\beta - 3D\overline{x}}{(1+D)\overline{x}}\right| < \frac{3\overline{x} + 3D\overline{x} - \overline{x} + 1}{(1+D)\overline{x}} \\
&\Rightarrow \quad \frac{\beta - 1}{2} < \overline{x} \text{ and } \frac{-\beta - 1}{6D+2} < \overline{x} \\
&\Rightarrow \quad \frac{-\beta - 1}{6D+2} < \frac{(\beta+1) + \sqrt{(\beta+1)^2 + 4\alpha(1+D)}}{2(1+D)} \\
&\Rightarrow \quad \text{and } \frac{\beta - 1}{2} < \frac{(\beta+1) + \sqrt{(\beta+1)^2 + 4\alpha(1+D)}}{2(1+D)} \\
&\Rightarrow \quad \frac{(\beta+1)^2(7D+3)}{4(3D+1)^2} < \alpha \text{ and } \frac{[(D-1)\beta - (D+3)](\beta-1)}{4} < \alpha
\end{aligned}
$$

and also

$$
\begin{aligned}
a_0^2 + a_1 - a_0 a_2 < 1 \quad &\Rightarrow \quad \left(\frac{D}{1+D}\right)^2 + \frac{\overline{x}-1}{(1+D)\overline{x}} - \left(\frac{D}{1+D}\right)\left(\frac{-\beta}{(1+D)\overline{x}}\right) < 1 \\
&\Rightarrow \quad \frac{D^2}{(1+D)^2} + \frac{\overline{x}-1}{(1+D)\overline{x}} + \frac{D\beta}{(1+D)^2\overline{x}} < 1 \\
&\Rightarrow \quad \frac{-1}{D} - 1 + \beta < \overline{x} \\
&\Rightarrow \quad \frac{-1}{D} - 1 + \beta < \frac{(\beta+1) + \sqrt{(\beta+1)^2 + 4\alpha(1+D)}}{2(1+D)} \\
&\Rightarrow \quad \frac{[D(\beta-1) - 1]\,[D(\beta-1) + \beta]}{1+D} < \alpha.
\end{aligned}
$$

So the positive equilibirium $\overline{x}$ of Eq.(5.2) is locally asymptotically stable.

5.2 GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQUATION (5.2)

Lemma 5.4 *The equilibrium point $\overline{x}$ is a global attractor of Eq.(5.2) if one of the following statements holds:*

(1) $Dw < \alpha + \beta u$ *and* $\beta \neq 1.$ (5.9)

(2) $Dw > \alpha + \beta u$ *and* $D > 1,\ \dfrac{(\beta+1)^2(D-1)}{4} < \alpha.$ (5.10)

Proof. Let α and β be real numbers and assume that $g : [\alpha, \beta]^3 \to [\alpha, \beta]$ is a function defined by

$$g(u, v, w) = \frac{\alpha + \beta u + v}{v + Dw}.$$

Then

$$\frac{\partial g\left(u,v,w\right)}{\partial u}=\frac{\beta}{v+Dw},$$

$$\frac{\partial g\left(u,v,w\right)}{\partial v}=\frac{Dw-\alpha-\beta u}{\left(v+Dw\right)^{2}},$$

$$\frac{\partial g\left(u,v,w\right)}{\partial w}=\frac{-(\alpha+\beta u+v)D}{\left(v+Dw\right)^{2}}.$$

We consider two cases:

Case 1 If $Dw < \alpha+\beta u$ then we can easily see that the function $g\left(u,v,w\right)$ is increasing in u and decreasing in v, w.

Suppose that (m, M) is a solution of the system $M = g(M, m, m)$ and $m = g(m, M, M)$ then from Eq.(5.2) we see that

$$M=\frac{\alpha+\beta M+m}{m+Dm}, \qquad m=\frac{\alpha+\beta m+M}{M+DM}.$$

Since

$$Mm+DMm-\beta M-m-\alpha=0,$$

$$Mm+DMm-\beta m-M-\alpha=0$$

we have

$$(m-M)(\beta-1)=0.$$

When $\beta\neq 1$, we have

$$M=m$$

which the result follows.

It follows by Theorem 1.15 that $\overline{x}$ is global attractor of Eq.(5.1) and then the proof is complete.

Case 2 If $Dw > \alpha+\beta u$, then we can easily see that the function $g\left(u,v,w\right)$ is increasing in u, v and decreasing in w.

Suppose that (m, M) is a solution of the system $M = g(M, M, m)$ and $m = g(m, m, M)$. Then from (5.2), we see that

$$M=\frac{\alpha+\beta M+M}{M+Dm}, \qquad m=\frac{\alpha+\beta m+m}{m+DM}.$$

Since

$$M^2 + DMm - M(\beta + 1) - \alpha = 0,$$

$$m^2 + DMm - m(\beta + 1) - \alpha = 0$$

we have

$$(m - M)((m + M) - (\beta + 1)) = 0$$

with simple calculations. Now if $m + M \neq \beta + 1$, then $M = m$. On the other hand if $m + M = \beta + 1$, then m and M satisfy the equation

$$m^2 + Dm\,(\beta + 1 - m) = \alpha + \beta m + m$$

and so

$$m^2\,(1 - D) + (\beta + 1)\,(D - 1)\,m - \alpha = 0. \tag{5.11}$$

The discriminant of the Eq.(5.11)

$$\begin{aligned} \Delta &= [(\beta + 1)\,(D - 1)]^2 + 4\,(1 - D)\,\alpha \\ &= (D - 1)\left[(\beta + 1)^2\,(D - 1) - 4\alpha\right] \end{aligned}$$

is negative when

$$D > 1 \text{ and } (\beta + 1)^2\,(D - 1) < 4\alpha$$

then we have

$$M = m$$

which the result follows.

It follows by Theorem 1.16 that $\overline{x}$ is global attractor of Eq.(5.2) and then the proof is complete.

BIBLIOGRAPHY

[1] **Amleh A M, Camouzis E and Ladas G** (2008) On the Dynamics of a Rational Difference Equation, Part 1. *International Journal of Difference Equations (IJDE)*, 3 (1): 1-35.

[2] **Amleh A M, Camouzis E and Ladas G** (2008) On the Dynamics of a Rational Difference Equation, Part 2. *International Journal of Difference Equations (IJDE)*, 3 (2): 195-225.

[3] **Amleh A M, Camouzis E and Ladas G** (2007) On Second Order Rational Difference Equations, I. *J. Difference Equ. Appl.,* 13 (11): 969-1004.

[4] **Amleh A M, Camouzis E and Ladas G** (2008) On Second Order Rational Difference Equations, II. *J. Difference Equ. Appl.,* 14: 215-228.

[5] **Camouzis E and Ladas G** (2006) When Does Local Asymptotic Stability Imply Global Attractivity in Rational Equations. *J. Difference Equ. Appl.,* 12 (8): 863-885.

[6] **Camouzis E and Ladas G** (2007) Global Convergence in Difference Equations. *Comm. Appl. Nonlinear Anal.,* 14 (4): 1-16.

[7] **Camouzis E and Ladas G** (2008) *Dynamics of Third-Order Rational Difference Equations with Open Problems and Conjecture,* Chapman & Hall/CRC, London.

[8] **Camouzis E, Kulenovic M R S, Ladas G, and Merino O** (2009) Rational Systems In The Plane. *J. Difference Equ. Appl.,* 15: 303-323.

[9] **Cima A, Gasull A, Armengol and Manosa V** (2008) Studying Discrete Dynamical Systems through Differential Equations. *J. Differential Equations,* 244 (3): 630-648.

[10] **Chatterjee E** (2013) On The Global Character Of The Solutions Of $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-k}}{A+x_{n-k}}$. *International Journal of Applied Mathematics,* 26 (1): 9-17.

[11] **Cushing J M, Levarge S, Chitnis N and Henson S M** (2004) Some Discrete Competition Models and the Competitive Exclusion Principle. *J. Difference Equ. Appl.*, 10 (13-15): 1139-1151.

[12] **Drymonis E, Kostrov Y and Kudlak Z** (2012) On Rational Difference Equations with Nonnegative Periodic Coefficients. *International Journal of Difference Equations*, 7 (1): 19-34.

[13] **Elabbasy E M, El-Metwally H and Elsayed E M** (2007) Qualitative Behavior of Higher Order Difference Equation. *Soochow Journal of Mathematics,* 4 (33): 861-873.

[14] **Elabbasy E M, El-Metwally H and Elsayed E M (**2011) Global Behavior of the Solution of Some Difeerence Equations. *Advances in Difference Equations,* 1 (28): 1-16.

[15] **Elaydi S** (2005) *An Introduction to Difference Equations,* Springer, New York.

[16] **El-Metwally H, Grove E A and Ladas G** (2000) A Global Convergence Result with Applications to Periodic Solutions. *J. Math. Anal. Appl.*, 245 (1): 161-170.

[17] **El-Metwally H, Grove E A, Ladas G and Voulov H D** (2001) On the Global Attractivity and the Periodic Character of Some Difference Equations. *J. Differ. Equations. Appl.,* 7 (6): 837-850.

[18] **Elsayed E M** (2010) On the Global Attractivity and the Periodic Character of a Recursive Sequence. *Opuscula Mathematica,* 4 (30): 431-446.

[19] **Grove E A, Janowski E J, Kent C M and Ladas G** (1994) On the Rational Recursive Sequence $x_{n+1} = \frac{\alpha x_n + \beta}{(\gamma x_n + \delta) x_{n-1}}$. *Comm. Appl. Nonlinear Anal.,* 1 (3): 61-72.

[20] **Grove E A and Ladas G** (2005) *Periodicities in Nonlinear Difference Equations,* Advances in Discrete Mathematics and Applications, Volume 4, Chapman & Hall/CRC, London.

[21] **Janowski E A and Kulenovic M R S** (2009) Attractivity and Global Stability for Linearizable Difference Equations. *Comput. Math. Appl.*, 57: 1592-1607.

[22] **Kocic V L and Ladas G** (1993) *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications,* Chapman & Hall/CRC, London.

[23] **Kulenovic M R S, Ladas G and Sizer W S** (1998) On the Recursive Sequence $x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\gamma x_n + \delta x_{n-1}}$. *Math. Sci. Res. Hot-Line,* 2 (5): 1-16.

[24] **Kulenovic M R S and Ladas G** (2002) *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjecture,* Chapman & Hall/CRC, London.

[25] **Kulenovic E A and Merino O** (2002) *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman & Hall/CRC, London.

[26] **Kulenovic E A and Merino O** (2006) A Global Attractivity Result for Maps with Invarıant Boxes. *Discrete Contin. Dyn. Syst. Ser. B*, 6: 97-110.

[27] **Kulenovic E A and Mehuljic M** (2012) Global Behavior of Some Rational Second Order Difference Equation. *International Journal of Difference Equations (IJDE)*, 7 (2): 153-162.

[28] **Ladas G, Lugo and Pallodino F J** (2012) Open Problems and Conjectures on Rational Systems in Three Dimensions. *Sarajevo J. Math.*, 8: 311-321.

[29] **Liu P and Elaydi S N** (2001) Discrete Competitive and Cooperative Models of Lotka-Volterra Type. *J. Comput. Anal. Appl.*, 3 (1): 53-73.

[30] **Nussbaum R L** (2007) Global Stability, Two Conjectures ad Maple. *Nonlinear Analysis TMA,* 66: 1064-1090.

[31] **Stevic S** (2007) On the Recursive Sequence $x_{n+1} = \frac{\alpha + \sum_{i=1}^{k} \alpha_i x_{n-p_i}}{1 + \sum_{j=1}^{m} \beta_j x_{n-q_j}}$. *J Difference Equ. Appl.*, 13: 41-46.

[32] **Yazlik Y** (2013) On the Solutions and Behavior of Rational Difference Equations. *J. Computational Analysis and Applications,* 17 (3): 584-594.

CURRICULUM VITAE

PERSONAL INFORMATION

Name, surname: Miraç GÜNEYSU
Nationality: Turkish (T.C.)
Date of Birth: 02.01.1991
Date of Place: Zonguldak
E-mail: gnsu_mrc@hotmail.com
Address: Meşrutiyet Mahallesi Rüzgarlımeşe Sokak No: 51/5.
Merkez / ZONGULDAK
Tel: (0543) 761 54 88

EDUCATION

Degree	Institution	Year of Graduation
MSc	Bülent Ecevit University, Mathematics	2012-2015
BS	Bülent Ecevit University, Mathematics	2008-2012
High School	Atatürk Anadolu Lisesi, Zonguldak	2004-2008

FOREIGN LANGUAGE

English

FIELD OF STUDY

Major Field: Functional Analysis, Difference Equations