PERIODICITY OF NON-LINEAR DIFFERENCE EQUATIONS

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Özge BAYRAK

ABSTRACT

M. Sc. Thesis

PERIODICITY OF NON-LINEAR DIFFERENCE EQUATIONS

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Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assoc. Prof. Yüksel SOYKAN

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In this thesis, we are primarily concerned with the periodicity of non-linear difference equations which are rational, max-type, special-type and the system of rational difference equations.

The organization of this thesis is as follows:

In Chapter 1, we give the necessary preliminary definitions and some examples of linear and non-linear difference equations.

In Chapter 2, we present some examples of the periodicity of some rational difference equations.

In Chapter 3, we exhibit some examples of the periodicity of some max-type non-linear difference equations.

ABSTRACT (continued)

In Chapter 4, we present some examples of the periodicity of some special-type difference equations.

In Chapter 5, we investigate the study of some special max-type difference equations with eventually periodic solutions.

In Chapter 6, we present some examples of the periodicity of the system of rational difference equations.

Key Words: Difference equations, periodicity, periodic solutions, eventually periodic solutions.

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ÖZET

Yüksek Lisans Tezi

LİNEER OLMAYAN FARK DENKLEMLERİNİN PERİYODİKLİĞİ

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Bu tezde rasyonel, maksimum tipli, özel tipli lineer olmayan fark denklemleri ve rasyonel fark denklem sistemlerinin periyodikliği ile ilgilenilmiştir.

Bu tezin organizasyonu aşağıdaki gibidir:

Birinci bölümde, gerekli tanımlar, lineer ve lineer olmayan fark denklemleri ile ilgili bazı örnekler verilmiştir.

İkinci bölümde, bazı rasyonel fark denklemlerinin periyodikliği ile ilgili örnekler sunulmuştur.

Üçüncü bölümde, bazı maksimum tipli lineer olmayan fark denklemlerinin periyodikliği ile ilgili örnekler verilmiştir.

ÖZET (devam ediyor)

Dördüncü bölümde, bazı özel tipli fark denklemlerinin periyodikliği ile ilgili örnekler sunulmuştur.

Beşinci bölümde, eninde sonunda periyodik çözümleri olan bazı maksimum tipli fark denklemleri incelenmiştir.

Altıncı bölümde, bazı rasyonel fark denklem sistemlerinin periyodikliği ile ilgili örnekler sunulmuştur.

Anahtar Sözcükler: Fark denklemleri, periyodiklik, periyodik çözümler, eninde sonunda periyodik çözümler

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English

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Major Field: Functional Analysis, Difference Equations

CHAPTER 1

PRELIMINARIES

1.1 INTRODUCTION

In this thesis we extensively use [1-28]. In this chapter we present some definitions and supple some examples of linear and non-linear difference equations.

1.2 DEFINITIONS OF PERIODICITY

A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ... \tag{1.1}$$

where f is a continious function which maps some set J^{k+1} into J. The set J is usually an interval of real numbers, or a union of intervals, but it may even be a discrete set such as the set of integers $\mathbb{Z} = \{..., -1, 0, 1, ...\}$.

A solution of Eq. (1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies Eq. (1.1) for all $n \geq 0$. If we prescribe a set of $(k+1)$ initial conditions

$$x_{-k}, x_{-k+1}, ..., x_0 \in J$$

then

$$\begin{aligned} x_1 &= f(x_0, x_{-1}, ..., x_{-k}) \\ x_2 &= f(x_1, x_0, ..., x_{-k+1}) \\ &\vdots \end{aligned}$$

and so the solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (1.1) exists for all $n \geq -k$ and is uniquely determined by the initial conditions.

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (1.1) is called *periodic with period* p (or a period p solution) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \quad \text{for all } n \geq -k \tag{1.2}$$

We say that the solution is periodic with *prime period* p if p is the smallest positive integer for which Eq. (1.2) holds. In this case, a p-tuple

$$(x_{n+1}, x_{n+2}, ..., x_{n+p})$$

of any p consecutive values of the solution is called a *p-cycle* of Eq. (1.1).
A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (1.1) is called *eventually periodic with period* p if there exists an integer $N \geq -k$ such that $\{x_n\}_{n=N}^{\infty}$ is periodic with period p; that is,

$$x_{n+p} = x_n \quad \text{for all } n \geq N.$$

The following lemma describes when a solution of Eq. (1.1) converges to a periodic solution of Eq. (1.1).

Lemma 1.1 *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1.1), and let $p \geq 1$ be a positive integer. Suppose there exist real numbers $l_0, l_1, ..., l_{p-1} \in J$ such that*

$$\lim_{n\to\infty} x_{pn+j} = l_j \quad \textit{for all} \;\; j = 0, 1, ..., p-1.$$

Finally, let $\{y_n\}_{n=-k}^{\infty}$ be the period-p sequence of real numbers in J such that for every integer j with $0 \leq j \leq p-1$, we have

$$y_{pn+j} = l_j \quad \textit{for all} \;\; n = 0, 1, ...$$

Then the following statements are true:

(a) *$\{y_n\}_{n=-k}^{\infty}$ is a period-p solution of Eq. (1.1).*

(b) $\lim_{n\to\infty} x_{pn+j} = y_j$ *for every $j \geq -k$.*

Proof. It suffices to show that $\{y_n\}_{n=-k}^{\infty}$ is a solution of Eq. (1.1). Note that for $j \geq 0$, we have

$$\begin{aligned} y_{j+1} &= \lim_{n\to\infty} x_{pn+j+1} = \lim_{n\to\infty} f(x_{pn+j}, x_{pn+j-1}, ..., x_{pn+j-k}) \\ &= f(y_j, y_{j-1}, ..., y_{j-k}). \end{aligned}$$

1.3 LINEAR AND NON-LINEAR DIFFERENCE EQUATIONS

The normal form of a k.order nonhomogeneous linear difference equation is given by

$$x_{n+k} + p_{1_n} x_{n+k-1} + ... + p_{k_n} x_n = g_n, \tag{1.3}$$

where p_{i_n} and g_n are real-valued functions defined for $n \geq n_0$ and $p_{k_n} \neq 0$ for all $n \geq n_0$. If g_n is identically zero, then Eq. (1.3) is said to be a homogeneous equation. Eq. (1.3) may be written in the form

$$x_{n+k} = -p_{1_n} x_{n+k-1} - ... - p_{k_n} x_n + g_n. \tag{1.4}$$

By letting $n = 0$ in Eq. (1.4), we obtain x_k in terms of $x_{k-1}, x_{k-2}, ..., x_0$. Explicitly, we have

$$x_k = -p_{1_0} x_{k-1} - p_{2_0} x_{k-2} - ... - p_{k_0} x_0 + g_0.$$

Once x_k is computed, we can go to the next step and evaluate x_{k+1} by letting $n = 1$ in Eq. (1.4). This yields

$$x_{k+1} = -p_{1_1} x_k - p_{2_1} x_{k-1} - ... - p_{k1} x_1 + g_1.$$

By repeating the above process, it is possible to evaluate all x_n for $n \geq k$.

If a difference equation is not linear then we say that it is non-linear.

1.4 EXAMPLES

Example 1.1 *We give some examples of linear difference equations:*

$$\begin{aligned}
x_{n+1} - (\sin n) x_n &= e^n \\
\frac{2n}{n+1} x_{n+2} + 3x_{n+1} - 4x_n &= n; \\
x_{n+3} + \frac{n}{n+2} x_{n+2} + x_{n+1} + \frac{n+1}{2n+3} x_n &= 3n+5; \\
x_{n+4} + e^{n+6} x_{n+2} + 6x_n &= 0.
\end{aligned}$$

Example 1.2 *We give some examples of non-linear rational type difference equations:*

$$\begin{aligned}
x_{n+2} &= \frac{x_{n+1} - x_n}{3x_{n+1}}; \\
x_{n+3} &= \frac{(n+1)x_{n+2} - (3n-1)x_{n+1}}{3x_{n+2} + x_{n+1}}; \\
x_{n+4} &= \frac{(n-2)\, x_{n+3} + 2x_{n+2}}{x_n}.
\end{aligned}$$

Example 1.3 *We give some examples of non-linear min-max type difference equations:*

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_{n-1}};$$
$$x_{n+2} = \frac{\max\{x_n, 1\}}{x_n^2 x_{n-1}};$$
$$x_{n+3} = \frac{\min\{x_n^2, 1\}}{x_n x_{n-1}};$$
$$x_{n+4} = \frac{\min\{x_n^2, 1\}}{x_n^3 x_{n-1}}.$$

Example 1.4 *We give some examples of non-linear systems of rational type difference equations:*

$$x_{n+1} = \frac{x_{n-1} + y_n}{y_n x_{n-1} - 1}, \ y_{n+1} = \frac{y_{n-1} + x_n}{x_n y_{n-1} - 1};$$
$$x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2} x_{n-1} y_n}, \ y_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2} y_{n-1} x_n}, z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 + x_{n-2} y_{n-1} x_n};$$
$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, z_{n+1} = \frac{1}{y_n z_{n-1} - 1};$$
$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, z_{n+1} = \frac{1}{y_n z_{n-1} - 1}.$$

CHAPTER 2

ON THE PERIODICITY OF SOLUTIONS OF SOME RATIONAL DIFFERENC EQUATIONS

In this chapter, we investigate peridiocities of the following non-linear difference equations (see **([7], pp. 23-25)**). Those present a wealth of examples of some rational-types difference equations with the property that every solution of each equation is periodic with the same period.

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, ... \tag{2.1}$$

$$x_{n+1} = \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, ... \tag{2.2}$$

$$x_{n+1} = \frac{1}{x_{n-1}}, \quad n = 0, 1, ... \tag{2.3}$$

$$x_{n+1} = \frac{1+x_n}{x_{n-1}}, \quad n = 0, 1, ... \tag{2.4}$$

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ... \tag{2.5}$$

$$x_{n+1} = \frac{1+x_n+x_{n-1}}{x_{n-2}}, \quad n = 0, 1, ... \tag{2.6}$$

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} x_{n-3}}, \quad n = 0, 1, ... \tag{2.7}$$

When one tries to investigate the periodicity of solutions of a difference equation, a key question of fundamental importance is the following. What is it that makes every solution of a difference equation periodic with the same period? Is there an easily verifiable necessary and sufficient condition that can be used to test for this property?

Note: The answer is that every solution of each of the above equations is periodic with the different period. However, there is some difference equation which is not periodic or eventually periodic (see Chapter 5). Also generally, there is no easy way to test whether a difference equation is periodic or not (see Remark 2.1).

2.1 FIRST ORDER NON-LINEAR RATIONAL DIFFERENCE EQUATIONS

2.1.1 THE CASE

$$x_{n+1} = \frac{1}{x_n} \tag{2.8}$$

Example 2.1 *Every solution of the equation*

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, 2, ... \tag{2.9}$$

is periodic with period 2*.*

Solution. If the initial condition of Eq. (2.9) is a non-zero real number denoted by

$x_0 = \alpha$

then we have

$$\begin{aligned}
x_1 &= \frac{1}{x_0} = \frac{1}{\alpha}, \\
x_2 &= \frac{1}{x_1} = \frac{1}{\frac{1}{\alpha}} = \alpha, \\
x_3 &= \frac{1}{x_2} = \frac{1}{\alpha}, \\
x_4 &= \alpha, \\
x_5 &= \frac{1}{\alpha}, \\
&\vdots
\end{aligned}$$

So the solution of Eq. (2.9) is the period-2 sequence

$\alpha, \frac{1}{\alpha}, \alpha, \frac{1}{\alpha}, \alpha, \frac{1}{\alpha}, \alpha, \frac{1}{\alpha}, \alpha, \frac{1}{\alpha}, \alpha, \frac{1}{\alpha}, ...$ ■

2.2 SECOND ORDER NON-LINEAR RATIONAL DIFFERENCE EQUATIONS

2.2.1 THE CASE

$$x_{n+1} = \frac{1}{x_n x_{n-1}} \tag{2.10}$$

Example 2.2 *Every solution of the equation*

$$x_{n+1} = \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, 2, ... \tag{2.11}$$

is periodic with period 3*.*

Solution. Suppose that the initial conditions are non-zero real numbers denoted by

$x_{-1} = \alpha$ and $x_0 = \beta$.

Then we obtain

$$
\begin{aligned}
x_1 &= \frac{1}{x_0 x_{-1}} = \frac{1}{\alpha\beta}, \\
x_2 &= \frac{1}{x_1 x_0} = \frac{1}{\frac{1}{\alpha\beta}\beta} = \alpha, \\
x_3 &= \frac{1}{x_2 x_1} = \frac{1}{\alpha\frac{1}{\alpha\beta}} = \beta, \\
x_4 &= \frac{1}{x_3 x_2} = \frac{1}{\alpha\beta}, \\
x_5 &= \frac{1}{x_4 x_3} = \frac{1}{\frac{1}{\alpha\beta}\beta} = \alpha, \\
x_6 &= \frac{1}{x_5 x_4} = \frac{1}{\alpha\frac{1}{\alpha\beta}} = \beta, \\
&\vdots
\end{aligned}
$$

Therefore the solution of Eq. (2.11) is the period-3 sequence

$$\alpha, \beta, \frac{1}{\alpha\beta}, \alpha, \beta, \frac{1}{\alpha\beta}, \alpha, \beta, \frac{1}{\alpha\beta}, \alpha, \beta, \frac{1}{\alpha\beta}, ... \blacksquare$$

2.2.2 THE CASE

$$x_{n+1} = \frac{1}{x_{n-1}} \tag{2.12}$$

Example 2.3 *Every solution of the equation*

$$x_{n+1} = \frac{1}{x_{n-1}}, \quad n = 0, 1, 2, ... \tag{2.13}$$

is periodic with period 4.

Solution. Let the initial conditions are non-zero real numbers denoted by

$x_{-1} = \alpha$ and $x_0 = \beta$.

Hence we obtain

$$\begin{aligned}
x_1 &= \frac{1}{x_{-1}} = \frac{1}{\alpha}, \\
x_2 &= \frac{1}{x_0} = \frac{1}{\beta}, \\
x_3 &= \frac{1}{x_1} = \frac{1}{\frac{1}{\alpha}} = \alpha, \\
x_4 &= \frac{1}{x_2} = \frac{1}{\frac{1}{\beta}} = \beta, \\
x_5 &= \frac{1}{x_3} = \frac{1}{\alpha}, \\
x_6 &= \frac{1}{x_4} = \frac{1}{\beta}, \\
x_7 &= \frac{1}{x_5} = \frac{1}{\frac{1}{\alpha}} = \alpha, \\
x_8 &= \frac{1}{x_6} = \frac{1}{\frac{1}{\beta}} = \beta, \\
&\vdots
\end{aligned}$$

Then the solution of Eq. (2.13) is the period-4 sequence

$$\alpha, \beta, \frac{1}{\alpha}, \frac{1}{\beta}, \alpha, \beta, \frac{1}{\alpha}, \frac{1}{\beta}, ... \blacksquare$$

2.2.3 THE CASE

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}} \tag{2.14}$$

Example 2.4 *Every solution of the equation*

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, 2, ... \tag{2.15}$$

is periodic with period 5*.*

Solution. Assume that the initial conditions are non-zero real numbers denoted by

$x_{-1} = \alpha$ and $x_0 = \beta$.

Then we obtain

$$x_1 = \frac{1 + x_0}{x_{-1}} = \frac{1 + \beta}{\alpha},$$

$$x_2 = \frac{1+x_1}{x_0} = \frac{1+\frac{1+\beta}{\alpha}}{\beta} = \frac{1+\alpha+\beta}{\alpha\beta},$$

$$\begin{aligned} x_3 &= \frac{1+x_2}{x_1} = \frac{1+\frac{1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}} = \frac{\alpha\beta+1+\alpha+\beta}{\alpha\beta}\frac{\alpha}{1+\beta} \\ &= \frac{\alpha(1+\beta)+(1+\beta)}{\beta}.\frac{1}{1+\beta} = \frac{(1+\beta)(1+\alpha)}{\beta}.\frac{1}{1+\beta}, \end{aligned}$$

$$x_4 = \frac{1+x_3}{x_2} = \frac{1+\frac{1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha\beta}} = \frac{1+\alpha+\beta}{\frac{1+\alpha+\beta}{\alpha}} = \alpha,$$

$$x_5 = \frac{1+x_4}{x_3} = \frac{1+\alpha}{\frac{1+\alpha}{\beta}} = \beta,$$

$$x_6 = \frac{1+x_5}{x_4} = \frac{1+\beta}{\alpha}$$

$$x_7 = \frac{1+x_6}{x_5} = \frac{1+\frac{1+\beta}{\alpha}}{\beta} = \frac{1+\alpha+\beta}{\alpha\beta},$$

$$\begin{aligned} x_8 &= \frac{1+x_7}{x_6} = \frac{1+\frac{1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}} = \frac{\alpha\beta+1+\alpha+\beta}{\alpha\beta}\frac{\alpha}{1+\beta} \\ &= \frac{\alpha(1+\beta)+(1+\beta)}{\beta}.\frac{1}{1+\beta} = \frac{(1+\beta)(1+\alpha)}{\beta}.\frac{1}{1+\beta}, \end{aligned}$$

$$x_9 = \frac{1+x_8}{x_7} = \frac{1+\frac{1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha\beta}} = \frac{1+\alpha+\beta}{\frac{1+\alpha+\beta}{\alpha}} = \alpha,$$

$$\begin{aligned} x_{10} &= \frac{1+x_9}{x_8} = \frac{1+\alpha}{\frac{1+\alpha}{\beta}} = \beta, \\ &\vdots \end{aligned}$$

Consequently the solution of Eq. (2.15) is the period-5 sequence

$$\alpha, \beta, \frac{1+\beta}{\alpha}, \frac{1+\alpha+\beta}{\alpha\beta}, \frac{1+\alpha}{\beta}, \alpha, \beta, \frac{1+\beta}{\alpha}, \frac{1+\alpha+\beta}{\alpha\beta}, \frac{1+\alpha}{\beta} .... \blacksquare$$

2.2.4 THE CASE

$$x_{n+1} = \frac{x_n}{x_{n-1}} \tag{2.16}$$

Example 2.5 *Every solution of the equation*

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \quad n = 0, 1, 2, ... \tag{2.17}$$

is periodic with period 6.

Solution. If the initial conditions are non-zero real numbers denoted by

$x_{-1} = \alpha$ and $x_0 = \beta$,

then we get

$$x_1 = \frac{x_0}{x_{-1}} = \frac{\beta}{\alpha},$$

$$x_2 = \frac{x_1}{x_0} = \frac{\frac{\beta}{\alpha}}{\beta} = \frac{1}{\alpha},$$

$$x_3 = \frac{x_2}{x_1} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}} = \frac{1}{\beta},$$

$$x_4 = \frac{x_3}{x_2} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha}} = \frac{\alpha}{\beta},$$

$$x_5 = \frac{x_4}{x_3} = \frac{\frac{\alpha}{\beta}}{\frac{1}{\beta}} = \alpha,$$

$$x_6 = \frac{x_5}{x_4} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$x_7 = \frac{x_6}{x_5} = \frac{\beta}{\alpha},$$

$$x_8 = \frac{x_7}{x_6} = \frac{\frac{\beta}{\alpha}}{\beta} = \frac{1}{\alpha},$$

$$x_9 = \frac{x_8}{x_7} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}} = \frac{1}{\beta},$$

$$x_{10} = \frac{x_9}{x_8} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha}} = \frac{\alpha}{\beta},$$

$$x_{11} = \frac{x_{10}}{x_9} = \frac{\frac{\alpha}{\beta}}{\frac{1}{\beta}} = \alpha,$$

$$x_{12} = \frac{x_{11}}{x_{10}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$\vdots$$

So the solution of Eq. (2.17) is the period-6 sequence

$$\alpha, \beta, \frac{\beta}{\alpha}, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{\alpha}{\beta}, \alpha, \beta, \frac{\beta}{\alpha}, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{\alpha}{\beta} \ldots . \blacksquare$$

2.3 THIRD ORDER NON-LINEAR RATIONAL DIFFERENCE EQUATIONS

2.3.1 THE CASE

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}} \tag{2.18}$$

Example 2.6 *Every solution of the equation*

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}}, \quad n = 0, 1, 2, ... \tag{2.19}$$

is periodic with period 8.

Solution. Suppose that the initial conditions are non-zero real numbers denoted by

$x_{-2} = \alpha$, $x_{-1} = \beta$ and $x_0 = \gamma$.

Then as a first term we have

$$x_1 = \frac{1 + x_0 + x_{-1}}{x_{-2}} = \frac{1 + \beta + \gamma}{\alpha},$$

the second term is

$$x_2 = \frac{1 + x_1 + x_0}{x_{-1}} = \frac{1 + \frac{1+\beta+\gamma}{\alpha} + \gamma}{\beta} = \frac{1 + \alpha + \beta + \gamma + \alpha\gamma}{\alpha\beta},$$

the third term is

$$\begin{aligned} x_3 &= \frac{1 + x_2 + x_1}{x_0} = \frac{1 + \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta} + \frac{1+\beta+\gamma}{\alpha}}{\gamma} = \frac{\alpha\beta + 1 + \alpha + \beta + \gamma + \alpha\gamma + \beta + \beta^2 + \beta\gamma}{\alpha\beta\gamma} \\ &= \frac{(1+\beta)^2 + \alpha(1+\beta) + \gamma(1+\alpha+\beta)}{\alpha\beta\gamma} = \frac{(1+\beta)(1+\alpha+\beta) + \gamma(1+\alpha+\beta)}{\alpha\beta\gamma} \\ &= \frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}, \end{aligned}$$

the fourth term is

$$\begin{aligned} x_4 &= \frac{1 + x_3 + x_2}{x_1} = \frac{1 + \frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma} + \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta}}{\frac{1+\beta+\gamma}{\alpha}} \\ &= \frac{\alpha\beta\gamma + (1+\alpha+\beta)(1+\beta+\gamma) + \gamma + \alpha\gamma + \beta\gamma + \gamma^2 + \alpha\gamma^2}{\beta\gamma(1+\beta+\gamma)} \\ &= \frac{(1+\alpha+\beta)(1+\beta+\gamma) + \alpha\gamma(1+\beta+\gamma) + \gamma(1+\beta+\gamma)}{\beta\gamma(1+\beta+\gamma)} \\ &= \frac{(1+\beta+\gamma)(1+\alpha+\beta+\gamma+\alpha\gamma)}{\beta\gamma(1+\beta+\gamma)} = \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}, \end{aligned}$$

the fifth term is

$$\begin{aligned} x_5 &= \frac{1+x_4+x_3}{x_2} = \frac{1+\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}+\frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}}{\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta}} \\ &= \frac{\alpha\beta\gamma+\alpha+\alpha^2+\beta\alpha+\gamma\alpha+\alpha^2\gamma+(1+\alpha+\beta)(1+\beta+\gamma)}{\gamma(1+\alpha+\beta+\gamma+\alpha\gamma)} \\ &= \frac{\alpha\beta\gamma+\alpha+\alpha^2+\beta\alpha+\gamma\alpha+\alpha^2\gamma+(1+\alpha+\beta)(1+\beta+\gamma)}{\gamma(1+\alpha+\beta+\gamma+\alpha\gamma)} \\ &= \frac{(1+\alpha+\beta)(\alpha+\alpha\gamma+1+\beta+\gamma)}{\gamma(1+\alpha+\beta+\gamma+\alpha\gamma)} = \frac{1+\alpha+\beta}{\gamma}, \end{aligned}$$

the sixth term is

$$\begin{aligned} x_6 &= \frac{1+x_5+x_4}{x_3} = \frac{1+\frac{1+\alpha+\beta}{\gamma}+\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}}{\frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}} \\ &= \frac{\alpha\left(\beta\gamma+\beta+\alpha\beta+\beta^2+1+\alpha+\beta+\gamma+\alpha\gamma\right)}{(1+\alpha+\beta)(1+\beta+\gamma)} \\ &= \frac{\alpha((1+\alpha+\beta)(1+\beta+\gamma))}{(1+\alpha+\beta)(1+\beta+\gamma)} = \alpha, \end{aligned}$$

and the seventh term is

$$x_7 = \frac{1+x_6+x_5}{x_4} = \frac{1+\alpha+\frac{1+\alpha+\beta}{\gamma}}{\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}} = \frac{\gamma+\alpha\gamma+1+\alpha+\beta}{\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta}} = \beta,$$

and the eighth term is

$$x_8 = \frac{1+x_7+x_6}{x_5} = \frac{1+\beta+\alpha}{\frac{1+\alpha+\beta}{\gamma}} = \gamma,$$

and the nineth term is

$$x_9 = \frac{1+x_8+x_7}{x_6} = \frac{1+\beta+\gamma}{\alpha},$$

and the tenth term is

$$x_{10} = \frac{1+x_9+x_8}{x_7} = \frac{1+\frac{1+\beta+\gamma}{\alpha}+\gamma}{\beta} = \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta},$$

and the eleventh term is

$$\begin{aligned} x_{11} &= \frac{1+x_{10}+x_9}{x_8} = \frac{1+\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta}+\frac{1+\beta+\gamma}{\alpha}}{\gamma} = \frac{\alpha\beta+1+\alpha+\beta+\gamma+\alpha\gamma+\beta+\beta^2+\beta\gamma}{\alpha\beta\gamma} \\ &= \frac{(1+\beta)^2+\alpha(1+\beta)+\gamma(1+\alpha+\beta)}{\alpha\beta\gamma} = \frac{(1+\beta)(1+\alpha+\beta)+\gamma(1+\alpha+\beta)}{\alpha\beta\gamma} \\ &= \frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}, \end{aligned}$$

and the twelfth

$$
\begin{aligned}
x_{12} &= \frac{1+x_{11}+x_{10}}{x_9} = \frac{1+\frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}+\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta}}{\frac{1+\beta+\gamma}{\alpha}} \\
&= \frac{\alpha\beta\gamma+(1+\alpha+\beta)(1+\beta+\gamma)+\gamma+\alpha\gamma+\beta\gamma+\gamma^2+\alpha\gamma^2}{\beta\gamma(1+\beta+\gamma)} \\
&= \frac{(1+\alpha+\beta)(1+\beta+\gamma)+\alpha\gamma(1+\beta+\gamma)+\gamma(1+\beta+\gamma)}{\beta\gamma(1+\beta+\gamma)} \\
&= \frac{(1+\beta+\gamma)(1+\alpha+\beta+\gamma+\alpha\gamma)}{\beta\gamma(1+\beta+\gamma)} = \frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma},
\end{aligned}
$$

and the thirteenth term is

$$
\begin{aligned}
x_{13} &= \frac{1+x_{12}+x_{11}}{x_{10}} = \frac{1+\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}+\frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}}{\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta}} \\
&= \frac{\alpha\beta\gamma+\alpha+\alpha^2+\beta\alpha+\gamma\alpha+\alpha^2\gamma+(1+\alpha+\beta)(1+\beta+\gamma)}{\gamma(1+\alpha+\beta+\gamma+\alpha\gamma)} \\
&= \frac{\alpha\beta\gamma+\alpha+\alpha^2+\beta\alpha+\gamma\alpha+\alpha^2\gamma+(1+\alpha+\beta)(1+\beta+\gamma)}{\gamma(1+\alpha+\beta+\gamma+\alpha\gamma)} \\
&= \frac{(1+\alpha+\beta)(\alpha+\alpha\gamma+1+\beta+\gamma)}{\gamma(1+\alpha+\beta+\gamma+\alpha\gamma)} = \frac{1+\alpha+\beta}{\gamma},
\end{aligned}
$$

and the fourteenth term is

$$
\begin{aligned}
x_{14} &= \frac{1+x_{13}+x_{12}}{x_{11}} = \frac{1+\frac{1+\alpha+\beta}{\gamma}+\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}}{\frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma}} \\
&= \frac{\alpha\left(\beta\gamma+\beta+\alpha\beta+\beta^2+1+\alpha+\beta+\gamma+\alpha\gamma\right)}{(1+\alpha+\beta)(1+\beta+\gamma)} \\
&= \frac{\alpha((1+\alpha+\beta)(1+\beta+\gamma))}{(1+\alpha+\beta)(1+\beta+\gamma)} = \alpha,
\end{aligned}
$$

and the fifteenth term is

$$
x_{15} = \frac{1+x_{14}+x_{13}}{x_{12}} = \frac{1+\alpha+\frac{1+\alpha+\beta}{\gamma}}{\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma}} = \frac{\gamma+\alpha\gamma+1+\alpha+\beta}{\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta}} = \beta,
$$

and the sixteenth term is

$$
\begin{aligned}
x_{16} &= \frac{1+x_{15}+x_{14}}{x_{13}} = \frac{1+\beta+\alpha}{\frac{1+\alpha+\beta}{\gamma}} = \gamma, \\
&\vdots
\end{aligned}
$$

Then the solution of Eq. (2.19) is the period-8 sequence

$$
\alpha,\beta,\gamma,\frac{1+\beta+\gamma}{\alpha},\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\alpha\beta},\frac{(1+\alpha+\beta)(1+\beta+\gamma)}{\alpha\beta\gamma},\frac{1+\alpha+\beta+\gamma+\alpha\gamma}{\beta\gamma},\frac{1+\alpha+\beta}{\gamma},\ldots. \blacksquare
$$

2.4 FOURTH ORDER NON-LINEAR RATIONAL DIFFERENCE EQUATIONS

2.4.1 THE CASE

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} x_{n-3}} \tag{2.20}$$

Example 2.7 *Every solution of the equation*

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} x_{n-3}}, \quad n = 0, 1, 2, ... \tag{2.21}$$

is periodic with period 10*.*

Solution. If the initial conditions are non-zero real numbers denoted by

$x_{-3} = \alpha$, $x_{-2} = \beta$, $x_{-1} = \gamma$ and $x_0 = \delta$

then we have

$$x_1 = \frac{x_0 x_{-2}}{x_{-1} x_{-3}} = \frac{\delta\beta}{\gamma\alpha},$$

$$x_2 = \frac{x_1 x_{-1}}{x_0 x_{-2}} = \frac{\frac{\delta\beta}{\gamma\alpha}\gamma}{\delta\beta} = \frac{1}{\alpha},$$

$$x_3 = \frac{x_2 x_0}{x_1 x_{-1}} = \frac{\frac{1}{\alpha}\delta}{\frac{\delta\beta}{\gamma\alpha}\gamma} = \frac{1}{\beta},$$

$$x_4 = \frac{x_3 x_1}{x_2 x_0} = \frac{\frac{1}{\beta}\frac{\delta\beta}{\gamma\alpha}}{\frac{1}{\alpha}\delta} = \frac{1}{\gamma},$$

$$x_5 = \frac{x_4 x_2}{x_3 x_1} = \frac{\frac{1}{\gamma}\frac{1}{\alpha}}{\frac{1}{\beta}\frac{\delta\beta}{\gamma\alpha}} = \frac{1}{\delta},$$

$$x_6 = \frac{x_5 x_3}{x_4 x_2} = \frac{\frac{1}{\delta}\frac{1}{\beta}}{\frac{1}{\gamma}\frac{1}{\alpha}} = \frac{\gamma\alpha}{\delta\beta},$$

$$x_7 = \frac{x_6 x_4}{x_5 x_3} = \frac{\frac{\gamma\alpha}{\delta\beta}\frac{1}{\gamma}}{\frac{1}{\delta}\frac{1}{\beta}} = \alpha,$$

$$x_8 = \frac{x_7 x_5}{x_6 x_4} = \frac{\alpha\frac{1}{\delta}}{\frac{\gamma\alpha}{\delta\beta}\frac{1}{\gamma}} = \beta,$$

$$x_9 = \frac{x_8 x_6}{x_7 x_5} = \frac{\beta\frac{\gamma\alpha}{\delta\beta}}{\alpha\frac{1}{\delta}} = \gamma,$$

$$x_{10}=\frac{x_9x_7}{x_8x_6}=\frac{\gamma\alpha}{\beta\frac{\gamma\alpha}{\delta\beta}}=\delta,$$

$$x_{11}=\frac{x_{10}x_8}{x_9x_7}=\frac{\delta\beta}{\gamma\alpha},$$

$$x_{12}=\frac{x_{11}x_9}{x_{10}x_8}=\frac{\frac{\delta\beta}{\gamma\alpha}\gamma}{\delta\beta}=\frac{1}{\alpha},$$

$$x_{13}=\frac{x_{12}x_{10}}{x_{11}x_9}=\frac{\frac{1}{\alpha}\delta}{\frac{\delta\beta}{\gamma\alpha}\gamma}=\frac{1}{\beta},$$

$$x_{14}=\frac{x_{13}x_{11}}{x_{12}x_{10}}=\frac{\frac{1}{\beta}\frac{\delta\beta}{\gamma\alpha}}{\frac{1}{\alpha}\delta}=\frac{1}{\gamma},$$

$$x_{15}=\frac{x_{14}x_{12}}{x_{13}x_{11}}=\frac{\frac{1}{\gamma}\frac{1}{\alpha}}{\frac{1}{\beta}\frac{\delta\beta}{\gamma\alpha}}=\frac{1}{\delta},$$

$$x_{16}=\frac{x_{15}x_{13}}{x_{14}x_{12}}=\frac{\frac{1}{\delta}\frac{1}{\beta}}{\frac{1}{\gamma}\frac{1}{\alpha}}=\frac{\gamma\alpha}{\delta\beta},$$

$$x_{17}=\frac{x_{16}x_{14}}{x_{15}x_{13}}=\frac{\frac{\gamma\alpha}{\delta\beta}\frac{1}{\gamma}}{\frac{1}{\delta}\frac{1}{\beta}}=\alpha,$$

$$x_{18}=\frac{x_{17}x_{15}}{x_{16}x_{14}}=\frac{\alpha\frac{1}{\delta}}{\frac{\gamma\alpha}{\delta\beta}\frac{1}{\gamma}}=\beta,$$

$$x_{19}=\frac{x_{18}x_{16}}{x_{17}x_{15}}=\frac{\beta\frac{\gamma\alpha}{\delta\beta}}{\alpha\frac{1}{\delta}}=\gamma,$$

$$\begin{aligned}x_{20} &= \frac{x_{19}x_{17}}{x_{18}x_{16}}=\frac{\gamma\alpha}{\beta\frac{\gamma\alpha}{\delta\beta}}=\delta,\\ &\vdots\end{aligned}$$

Therefore the solution of Eq. (2.21) is the period-10 sequence

$$\alpha,\beta,\gamma,\delta,\frac{\beta\delta}{\alpha\gamma},\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{\delta},\frac{\alpha\gamma}{\beta\delta},\alpha,\beta,\gamma,\delta,\frac{\beta\delta}{\alpha\gamma},\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{\delta},\frac{\alpha\gamma}{\beta\delta},\ldots. \blacksquare$$

What is that makes every solution of a difference equation periodic with the same period? Is there an easily verifiable test that we can apply to determine whether or not is ture? The following Lemma gives a partial answer to this questions. In fact, it is interesting to note that Eq. (2.1) and Eq. (2.2) follow a pattern (see (a) of the following Lemma) and also. Note that Eq. (2.17) and Eq. (2.21) also follows a pattern (see (b) of the following Lemma).

Lemma 2.1 **(a)** *For every $k \in \{0, 1, ...\}$, every solution of the difference equation*

$$x_{n+1} = \frac{1}{x_n x_{n-1} ... x_{n-k}}, \quad n = 0, 1, ... \tag{2.22}$$

is periodic with period $(k+2)$.

(b) *Given a non-negative integer $k \geq 0$, every solution of the equation*

$$x_{n+1} = \frac{x_n x_{n-2} ... x_{n-2k}}{x_{n-1} x_{n-3} ... x_{n-(2k+1)}}, \quad n = 0, 1, .. \tag{2.23}$$

is periodic with period $(4k+6)$.

Proof.

(a) If the initial conditions of Eq. (2.22) are the non-zero real numbers $x_{-k}, x_{-k+1}, ..., x_0$, then the solution of Eq. (2.22) is the period-$(k+2)$ sequence

$$x_{-k}, x_{-k+1}, ..., x_0, \frac{1}{x_0 x_{-1} ... x_{-k}}, ...$$

(b) If the initial conditions of Eq. (2.23) are the non-zero real numbers $x_{-2k-1}, x_{-2k}, ..., x_0$, then the solution of Eq. (2.23) is the period-$(4k+6)$ sequence

$$x_{-2k-1}, x_{-2k}, ..., x_0, \frac{x_0 x_{-2} ... x_{-2k}}{x_{-1} x_{-3} ... x_{-2k-1}}, \frac{1}{x_{-2k-1}}, \frac{1}{x_{-2k}}, ..., \frac{1}{x_0}, \frac{x_{-1} x_{-3} ... x_{-2k-1}}{x_0 x_{-2} ... x_{-2k}}, ...$$

Remark 2.1 *In view of the periodicities Eq. (2.15), and Eq. (2.19), we may be misled into believing that, Eq. (2.15), and Eq. (2.19) also follows a pattern and, in particular, that every solution of the non-linear difference equation*

$$x_{n+1} = \frac{1 + x_n + x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n = 0, 1, ... \tag{2.24}$$

is periodic with period 11. Unfortunately,3 Eq. (2.15), and Eq. (2.19) do not follow any obvious pattern. If $\{x_n\}_{n=-3}^{\infty}$ is the solution of Eq. (2.24) with initial conditions $x_{-3} = x_{-2} = x_{-1} = x_0 = 1$, we see that the first 12 terms of $\{x_n\}_{n=-3}^{\infty}$ are

$$1, 1, 1, 1, 4, 7, 13, 25, \frac{23}{2}, \frac{101}{14}, \frac{313}{91}, \frac{29,498}{31,850}$$

and so $\{x_n\}_{n=-3}^{\infty}$ is not periodic with period 11.

CHAPTER 3

MAX-TYPE NON-LINEAR DIFFERENCE EQUATIONS WITH PERIODIC SOLUTIONS

In this chapter, we investigate peridiocities of some non-linear difference equations (see (**[7], pp. 27-28**)). Those present a wealth of examples of some max-types difference equations with the property that every solution of each equation is periodic with the same period.

3.1 SECOND ORDER MAX-TYPE DIFFERENCE EQUATIONS

In this section, we consider peridiocities of the following second order max-type difference equations (see (**[7], pp. 27-28**)).

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_{n-1}} \quad , n = 0, 1, ... \tag{3.1}$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n x_{n-1}} \quad , n = 0, 1, ... \tag{3.2}$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n^2 x_{n-1}} \quad , n = 0, 1, ... \tag{3.3}$$

$$x_{n+1} = \frac{\max\left\{x_n^2, 1\right\}}{x_n x_{n-1}} \quad , n = 0, 1, ... \tag{3.4}$$

$$x_{n+1} = \frac{\max\left\{x_n^2, 1\right\}}{x_n^3 x_{n-1}} \quad , n = 0, 1, ... \tag{3.5}$$

The answer is that every positive solution of each of the above difference equation is periodic with the same period.

- Every solution of Eq. (3.1) is periodic with period 5.
- Every solution of Eq. (3.2) is periodic with period 7.
- Every solution of Eq. (3.3) is periodic with period 8.
- Every solution of Eq. (3.4) is periodic with period 9.

- Every solution of Eq. (3.5) is periodic with period 12.

We now investigate in detail the periodicity of all above max-type difference equations.

3.1.1 THE CASE

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_{n-1}} \tag{3.6}$$

Example 3.1 *Every positive solution of Eq.(3.6) is periodic with period* 5. *We illustrate the result in the following table.*

Case 1	**Case 2**	**Case 3**	**Case 4**
$x_{-1} = \alpha \leq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \leq 1$ $x_0 = \beta \geq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \geq 1$
$x_1 = \frac{1}{\alpha}$	$x_1 = \frac{1}{\alpha}$	$x_1 = \frac{\beta}{\alpha}$	$x_1 = \frac{\beta}{\alpha}$
$x_2 = \frac{1}{\alpha\beta}$	$x_2 = \frac{1}{\beta}$	$x_2 = \frac{1}{\alpha}$	$x_2 = \max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha}\right\}$
$x_3 = \frac{1}{\beta}$	$x_3 = \frac{\alpha}{\beta}$	$x_3 = \frac{1}{\beta}$	$x_3 = \frac{\alpha}{\beta}$
$x_4 = \alpha$	$x_4 = \alpha$	$x_4 = \alpha$	$x_4 = \alpha$
$x_5 = \beta$	$x_5 = \beta$	$x_5 = \beta$	$x_5 = \beta$
$x_6 = \frac{1}{\alpha}$	$x_6 = \frac{1}{\alpha}$	$x_6 = \frac{\beta}{\alpha}$	$x_6 = \frac{\beta}{\alpha}$
$x_7 = \frac{1}{\alpha\beta}$	$x_7 = \frac{1}{\beta}$	$x_7 = \frac{1}{\alpha}$	$x_7 = \max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha}\right\}$
$x_8 = \frac{1}{\beta}$	$x_8 = \frac{\alpha}{\beta}$	$x_8 = \frac{1}{\beta}$	$x_8 = \frac{\alpha}{\beta}$
$x_9 = \alpha$	$x_9 = \alpha$	$x_9 = \alpha$	$x_9 = \alpha$
$x_{10} = \beta$	$x_{10} = \beta$	$x_{10} = \beta$	$x_{10} = \beta$

Solution.

Case 1:

For

$$x_{-1} = \alpha \leq 1$$
$$x_0 = \beta \leq 1$$

we get the first term is

$$x_1 = \frac{\max\{x_0, 1\}}{x_{-1}} = \frac{\max\{\beta, 1\}}{\alpha} = \frac{1}{\alpha},$$

the second term is

$$x_2 = \frac{\max\{x_1, 1\}}{x_0} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\beta} = \frac{\frac{1}{\alpha}}{\beta} = \frac{1}{\alpha\beta},$$

the third term is

$$x_3 = \frac{\max\{x_2, 1\}}{x_1} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha}} = \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha}} = \frac{1}{\beta},$$

the fourth term is

$$x_4 = \frac{\max\{x_3, 1\}}{x_2} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\alpha\beta}} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha\beta}} = \alpha,$$

and the fifth term is

$$x_5 = \frac{\max\{x_4, 1\}}{x_3} = \frac{\max\{\alpha, 1\}}{\frac{1}{\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

and the sixth term is

$$x_6 = \frac{\max\{x_5, 1\}}{x_4} = \frac{\max\{\beta, 1\}}{\alpha} = \frac{1}{\alpha},$$

and the seventh term is

$$x_7 = \frac{\max\{x_6, 1\}}{x_5} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\beta} = \frac{\frac{1}{\alpha}}{\beta} = \frac{1}{\alpha\beta},$$

and the eighth term is

$$x_8 = \frac{\max\{x_7, 1\}}{x_6} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha}} = \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha}} = \frac{1}{\beta},$$

and the nineth term is

$$x_9 = \frac{\max\{x_8, 1\}}{x_7} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\alpha\beta}} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha\beta}} = \alpha,$$

and the tenth term is

$$x_{10} = \frac{\max\{x_9, 1\}}{x_8} = \frac{\max\{\alpha, 1\}}{\frac{1}{\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 2:

For

$$x_{-1} = \alpha \geq 1$$
$$x_0 = \beta \leq 1$$

we get the first term is

$$x_1 = \frac{\max\{x_0, 1\}}{x_{-1}} = \frac{\max\{\beta, 1\}}{\alpha} = \frac{1}{\alpha},$$

the second term is

$$x_2 = \frac{\max\{x_1, 1\}}{x_0} = \frac{\max\{\frac{1}{\alpha}, 1\}}{\beta} = \frac{1}{\beta},$$

the third term is

$$x_3 = \frac{\max\{x_2, 1\}}{x_1} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\alpha}} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha}} = \frac{\alpha}{\beta},$$

the fourth term is

$$x_4 = \frac{\max\{x_3, 1\}}{x_2} = \frac{\max\left\{\frac{\alpha}{\beta}, 1\right\}}{\frac{1}{\beta}} = \frac{\frac{\alpha}{\beta}}{\frac{1}{\beta}} = \alpha,$$

and the fifth term is

$$x_5 = \frac{\max\{x_4, 1\}}{x_3} = \frac{\max\{\alpha, 1\}}{\frac{\alpha}{\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

and the sixth term is

$$x_6 = \frac{\max\{x_5, 1\}}{x_4} = \frac{\max\{\beta, 1\}}{\alpha} = \frac{1}{\alpha},$$

and the seventh term is

$$x_7 = \frac{\max\{x_6, 1\}}{x_5} = \frac{\max\{\frac{1}{\alpha}, 1\}}{\beta} = \frac{1}{\beta},$$

and the eighth term is

$$x_8 = \frac{\max\{x_7, 1\}}{x_6} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\alpha}} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha}} = \frac{\alpha}{\beta},$$

and the nineth term is

$$x_9 = \frac{\max\{x_8, 1\}}{x_7} = \frac{\max\left\{\frac{\alpha}{\beta}, 1\right\}}{\frac{1}{\beta}} = \frac{\frac{\alpha}{\beta}}{\frac{1}{\beta}} = \alpha,$$

and the tenth term is

$$x_{10} = \frac{\max\{x_9,1\}}{x_8} = \frac{\max\{\alpha,1\}}{\frac{\alpha}{\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$\vdots$$

Case 3:

For

$$x_{-1} = \alpha \le 1$$
$$x_0 = \beta \ge 1$$

we obtain the first term is

$$x_1 = \frac{\max\{x_0,1\}}{x_{-1}} = \frac{\max\{\beta,1\}}{\alpha} = \frac{\beta}{\alpha},$$

the second term is

$$x_2 = \frac{\max\{x_1,1\}}{x_0} = \frac{\max\left\{\frac{\beta}{\alpha},1\right\}}{\beta} = \frac{\frac{\beta}{\alpha}}{\beta} = \frac{1}{\alpha},$$

the third term is

$$x_3 = \frac{\max\{x_2,1\}}{x_1} = \frac{\max\left\{\frac{1}{\alpha},1\right\}}{\frac{\beta}{\alpha}} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}} = \frac{1}{\beta},$$

the fourth term is

$$x_4 = \frac{\max\{x_3,1\}}{x_2} = \frac{\max\left\{\frac{1}{\beta},1\right\}}{\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

and the fifth term is

$$x_5 = \frac{\max\{x_4,1\}}{x_3} = \frac{\max\{\alpha,1\}}{\frac{1}{\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

and the sixth term is

$$x_6 = \frac{\max\{x_5,1\}}{x_4} = \frac{\max\{\beta,1\}}{\alpha} = \frac{\beta}{\alpha},$$

and the seventh term is

$$x_7 = \frac{\max\{x_6,1\}}{x_5} = \frac{\max\left\{\frac{\beta}{\alpha},1\right\}}{\beta} = \frac{\frac{\beta}{\alpha}}{\beta} = \frac{1}{\alpha},$$

and the eighth term is

$$x_8 = \frac{\max\{x_7,1\}}{x_6} = \frac{\max\left\{\frac{1}{\alpha},1\right\}}{\frac{\beta}{\alpha}} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}} = \frac{1}{\beta},$$

and the nineth term is

$$x_9 = \frac{\max\{x_8, 1\}}{x_7} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

and the tenth term is

$$x_{10} = \frac{\max\{x_9, 1\}}{x_8} = \frac{\max\{\alpha, 1\}}{\frac{1}{\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 4:

For

$$x_{-1} = \alpha \geq 1$$
$$x_0 = \beta \geq 1$$

the first term is

$$x_1 = \frac{\max\{x_0, 1\}}{x_{-1}} = \frac{\max\{\beta, 1\}}{\alpha} = \frac{\beta}{\alpha},$$

the second term is

$$x_2 = \frac{\max\{x_1, 1\}}{x_0} = \frac{\max\left\{\frac{\beta}{\alpha}, 1\right\}}{\beta} = \max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha}\right\},$$

the third term is

$$x_3 = \frac{\max\{x_2, 1\}}{x_1} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{\beta}{\alpha}} = \frac{1}{\frac{\beta}{\alpha}} = \frac{\alpha}{\beta},$$

the fourth term is

$$x_4 = \frac{\max\{x_3, 1\}}{x_2} = \frac{\max\left\{\frac{\alpha}{\beta}, 1\right\}}{\frac{1}{\beta}} = \frac{\frac{\alpha}{\beta}}{\frac{1}{\beta}} = \alpha,$$

and the fifth term is

$$x_5 = \frac{\max\{x_4, 1\}}{x_3} = \frac{\max\{\alpha, 1\}}{\frac{\alpha}{\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

and the sixth term is

$$x_6 = \frac{\max\{x_5, 1\}}{x_4} = \frac{\max\{\beta, 1\}}{\alpha} = \frac{\beta}{\alpha},$$

and the seventh term is

$$x_7 = \frac{\max\{x_6, 1\}}{x_5} = \frac{\max\left\{\frac{\beta}{\alpha}, 1\right\}}{\beta} = \max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha}\right\},$$

and the eighth term is

$$x_8 = \frac{\max\{x_7, 1\}}{x_6} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{\beta}{\alpha}} = \frac{1}{\frac{\beta}{\alpha}} = \frac{\alpha}{\beta},$$

and the nineth term is

$$x_9 = \frac{\max\{x_8, 1\}}{x_7} = \frac{\max\left\{\frac{\alpha}{\beta}, 1\right\}}{\frac{1}{\beta}} = \frac{\frac{\alpha}{\beta}}{\frac{1}{\beta}} = \alpha,$$

and the tenth term is

$$x_{10} = \frac{\max\{x_9, 1\}}{x_8} = \frac{\max\{\alpha, 1\}}{\frac{\alpha}{\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$\vdots$$

3.1.2 THE CASE

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n x_{n-1}} \tag{3.7}$$

Example 3.2 *Every positive solution of Eq.(3.7) is periodic with period* 7*. We illustrate the result in the following table.*

Case 1	***Case 2***	***Case 3***	***Case 4***
$x_{-1} = \alpha \leq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \leq 1$ $x_0 = \beta \geq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \geq 1$
$x_1 = \frac{1}{\alpha\beta}$	$x_1 = \frac{1}{\alpha\beta}$	$x_1 = \frac{1}{\alpha}$	$x_1 = \frac{1}{\alpha}$
$x_2 = \frac{1}{\beta}$	$x_2 = \max\left\{\alpha, \frac{1}{\beta}\right\}$	$x_2 = \frac{1}{\beta}$	$x_2 = \frac{\alpha}{\beta}$
$x_3 = \alpha\beta$	$x_3 = \alpha\beta$	$x_3 = \alpha\beta$	$x_3 = \max\{\alpha, \beta\}$
$x_4 = \frac{1}{\alpha}$	$x_4 = \frac{1}{\alpha}$	$x_4 = \max\left\{\beta, \frac{1}{\alpha}\right\}$	$x_4 = \frac{\beta}{\alpha}$
$x_5 = \frac{1}{\alpha\beta}$	$x_5 = \frac{1}{\beta}$	$x_5 = \frac{1}{\alpha\beta}$	$x_5 = \frac{1}{\beta}$
$x_6 = \alpha$	$x_6 = \alpha$	$x_6 = \alpha$	$x_6 = \alpha$
$x_7 = \beta$	$x_7 = \beta$	$x_7 = \beta$	$x_7 = \beta$
$x_8 = \frac{1}{\alpha\beta}$	$x_8 = \frac{1}{\alpha\beta}$	$x_8 = \frac{1}{\alpha}$	$x_8 = \frac{1}{\alpha}$
$x_9 = \frac{1}{\beta}$	$x_9 = \max\left\{\alpha, \frac{1}{\beta}\right\}$	$x_9 = \frac{1}{\beta}$	$x_9 = \frac{\alpha}{\beta}$
$x_{10} = \alpha\beta$	$x_{10} = \alpha\beta$	$x_{10} = \alpha\beta$	$x_{10} = \max\{\alpha, \beta\}$
$x_{11} = \frac{1}{\alpha}$	$x_{11} = \frac{1}{\alpha}$	$x_{11} = \max\left\{\beta, \frac{1}{\alpha}\right\}$	$x_{11} = \frac{\beta}{\alpha}$
$x_{12} = \frac{1}{\alpha\beta}$	$x_{12} = \frac{1}{\beta}$	$x_{12} = \frac{1}{\alpha\beta}$	$x_{12} = \frac{1}{\beta}$
$x_{13} = \alpha$	$x_{13} = \alpha$	$x_{13} = \alpha$	$x_{13} = \alpha$
$x_{14} = \beta$	$x_{14} = \beta$	$x_{14} = \beta$	$x_{14} = \beta$

In fact, the solution of Eq.(3.7) is:

Case 1:

For;

$$x_{-1} = \alpha \leq 1$$
$$x_0 = \beta \leq 1$$

we obtain that

$$x_1 = \frac{\max\{x_0, 1\}}{x_0 x_{-1}} = \frac{\max\{\beta, 1\}}{\beta\alpha} = \frac{1}{\alpha\beta},$$

$$x_2 = \frac{\max\{x_1, 1\}}{x_1 x_0} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha\beta}\beta} = \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha}} = \frac{1}{\beta},$$

$$x_3 = \frac{\max\{x_2, 1\}}{x_2 x_1} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\beta}\frac{1}{\alpha\beta}} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha\beta^2}} = \alpha\beta,$$

$$x_4 = \frac{\max\{x_3, 1\}}{x_3 x_2} = \frac{\max\{\alpha\beta, 1\}}{\alpha\beta\frac{1}{\beta}} = \frac{1}{\alpha},$$

$$x_5 = \frac{\max\{x_4, 1\}}{x_4 x_3} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\frac{1}{\alpha}\alpha\beta} = \frac{\frac{1}{\alpha}}{\beta} = \frac{1}{\alpha\beta},$$

$$x_6 = \frac{\max\{x_5, 1\}}{x_5 x_4} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha\beta}\frac{1}{\alpha}} = \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha^2\beta}} = \alpha,$$

$$x_7 = \frac{\max\{x_6, 1\}}{x_6 x_5} = \frac{\max\{\alpha, 1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$x_8 = \frac{\max\{x_7, 1\}}{x_7 x_6} = \frac{\max\{\beta, 1\}}{\beta\alpha} = \frac{1}{\alpha\beta},$$

$$x_9 = \frac{\max\{x_8, 1\}}{x_8 x_7} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha\beta}\beta} = \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha}} = \frac{1}{\beta},$$

$$x_{10} = \frac{\max\{x_9, 1\}}{x_9 x_8} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\beta}\frac{1}{\alpha\beta}} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha\beta^2}} = \alpha\beta,$$

$$x_{11} = \frac{\max\{x_{10}, 1\}}{x_{10} x_9} = \frac{\max\{\alpha\beta, 1\}}{\alpha\beta\frac{1}{\beta}} = \frac{1}{\alpha},$$

$$x_{12} = \frac{\max\{x_{11}, 1\}}{x_{11} x_{10}} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\frac{1}{\alpha}\alpha\beta} = \frac{\frac{1}{\alpha}}{\beta} = \frac{1}{\alpha\beta},$$

$$x_{13} = \frac{\max\{x_{12}, 1\}}{x_{12} x_{11}} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha\beta}\frac{1}{\alpha}} = \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha^2\beta}} = \alpha,$$

$$x_{14} = \frac{\max\{x_{13},1\}}{x_{13}x_{12}} = \frac{\max\{\alpha,1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 2:

For;

$$x_{-1} = \alpha \geq 1$$

$$x_0 = \beta \leq 1$$

we obtain that

$$x_1 = \frac{\max\{x_0,1\}}{x_0x_{-1}} = \frac{\max\{\beta,1\}}{\beta\alpha} = \frac{1}{\alpha\beta},$$

$$x_2 = \frac{\max\{x_1,1\}}{x_1x_0} = \frac{\max\left\{\frac{1}{\alpha\beta},1\right\}}{\frac{1}{\alpha\beta}\beta} = \max\left\{\alpha,\frac{1}{\beta}\right\},$$

$$x_3 = \frac{\max\{x_2,1\}}{x_2x_1} = \frac{\max\left\{\max\left\{\alpha,\frac{1}{\beta}\right\},1\right\}}{\max\left\{\alpha,\frac{1}{\beta}\right\}\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_4 = \frac{\max\{x_3,1\}}{x_3x_2} = \frac{\max\{\alpha\beta,1\}}{\alpha\beta\max\left\{\alpha,\frac{1}{\beta}\right\}} = \frac{1}{\alpha},$$

$$x_5 = \frac{\max\{x_4,1\}}{x_4x_3} = \frac{\max\left\{\frac{1}{\alpha},1\right\}}{\frac{1}{\alpha}\alpha\beta} = \frac{1}{\beta},$$

$$x_6 = \frac{\max\{x_5,1\}}{x_5x_4} = \frac{\max\left\{\frac{1}{\beta},1\right\}}{\frac{1}{\beta}\frac{1}{\alpha}} = \frac{\frac{1}{\beta}}{\frac{1}{\beta}\frac{1}{\alpha}} = \alpha,$$

$$x_7 = \frac{\max\{x_6,1\}}{x_6x_5} = \frac{\max\{\alpha,1\}}{\alpha\frac{1}{\beta}} = \frac{\alpha}{\alpha\frac{1}{\beta}} = \beta,$$

$$x_8 = \frac{\max\{x_7,1\}}{x_7x_6} = \frac{\max\{\beta,1\}}{\beta\alpha} = \frac{1}{\alpha\beta},$$

$$x_9 = \frac{\max\{x_8,1\}}{x_8x_7} = \frac{\max\left\{\frac{1}{\alpha\beta},1\right\}}{\frac{1}{\alpha\beta}\beta} = \max\left\{\alpha,\frac{1}{\beta}\right\},$$

$$x_{10} = \frac{\max\{x_9,1\}}{x_9x_8} = \frac{\max\left\{\max\left\{\alpha,\frac{1}{\beta}\right\},1\right\}}{\max\left\{\alpha,\frac{1}{\beta}\right\}\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_{11} = \frac{\max\{x_{10},1\}}{x_{10}x_9} = \frac{\max\{\alpha\beta,1\}}{\alpha\beta\max\left\{\alpha,\frac{1}{\beta}\right\}} = \frac{1}{\alpha},$$

$$x_{12} = \frac{\max\{x_{11},1\}}{x_{11}x_{10}} = \frac{\max\left\{\frac{1}{\alpha},1\right\}}{\frac{1}{\alpha}\alpha\beta} = \frac{1}{\beta},$$

$$x_{13} = \frac{\max\{x_{12},1\}}{x_{12}x_{11}} = \frac{\max\left\{\frac{1}{\beta},1\right\}}{\frac{1}{\beta}\frac{1}{\alpha}} = \frac{\frac{1}{\beta}}{\frac{1}{\beta}\frac{1}{\alpha}} = \alpha,$$

$$x_{14} \quad = \quad \frac{\max\{x_{13},1\}}{x_{13}x_{12}} = \frac{\max\{\alpha,1\}}{\alpha\frac{1}{\beta}} = \frac{\alpha}{\alpha\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 3:

For

$$x_{-1} = \alpha \leq 1$$
$$x_0 = \beta \geq 1$$

we get that

$$x_1 = \frac{\max\{x_0,1\}}{x_0x_{-1}} = \frac{\max\{\beta,1\}}{\beta\alpha} = \frac{\beta}{\alpha\beta} = \frac{1}{\alpha},$$

$$x_2 = \frac{\max\{x_1,1\}}{x_1x_0} = \frac{\max\left\{\frac{1}{\alpha},1\right\}}{\frac{1}{\alpha}\beta} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}} = \frac{1}{\beta},$$

$$x_3 = \frac{\max\{x_2,1\}}{x_2x_1} = \frac{\max\left\{\frac{1}{\beta},1\right\}}{\frac{1}{\beta}\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_4 = \frac{\max\{x_3,1\}}{x_3x_2} = \frac{\max\{\alpha\beta,1\}}{\alpha\beta\frac{1}{\beta}} = \max\left\{\beta,\frac{1}{\alpha}\right\},$$

$$x_5 = \frac{\max\{x_4,1\}}{x_4x_3} = \frac{\max\left\{\max\left\{\beta,\frac{1}{\alpha}\right\},1\right\}}{\max\left\{\beta,\frac{1}{\alpha}\right\}\alpha\beta} = \frac{1}{\alpha\beta},$$

$$x_6 = \frac{\max\{x_5,1\}}{x_5x_4} = \frac{\max\left\{\frac{1}{\alpha\beta},1\right\}}{\frac{1}{\alpha\beta}\max\left\{\beta,\frac{1}{\alpha}\right\}} = \alpha,$$

$$x_7 = \frac{\max\{x_6,1\}}{x_6x_5} = \frac{\max\{\alpha,1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$x_8 = \frac{\max\{x_7,1\}}{x_7x_6} = \frac{\max\{\beta,1\}}{\beta\alpha} = \frac{\beta}{\alpha\beta} = \frac{1}{\alpha},$$

$$x_9 = \frac{\max\{x_8,1\}}{x_8x_7} = \frac{\max\left\{\frac{1}{\alpha},1\right\}}{\frac{1}{\alpha}\beta} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}} = \frac{1}{\beta},$$

$$x_{10} = \frac{\max\{x_9,1\}}{x_9x_8} = \frac{\max\left\{\frac{1}{\beta},1\right\}}{\frac{1}{\beta}\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_{11} = \frac{\max\{x_{10}, 1\}}{x_{10}x_9} = \frac{\max\{\alpha\beta, 1\}}{\alpha\beta\frac{1}{\beta}} = \max\left\{\beta, \frac{1}{\alpha}\right\},$$

$$x_{12} = \frac{\max\{x_{11}, 1\}}{x_{11}x_{10}} = \frac{\max\left\{\max\left\{\beta, \frac{1}{\alpha}\right\}, 1\right\}}{\max\left\{\beta, \frac{1}{\alpha}\right\}\alpha\beta} = \frac{1}{\alpha\beta},$$

$$x_{13} = \frac{\max\{x_{12}, 1\}}{x_{12}x_{11}} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha\beta}\max\left\{\beta, \frac{1}{\alpha}\right\}} = \alpha,$$

$$x_{14} = \frac{\max\{x_{13}, 1\}}{x_{13}x_{12}} = \frac{\max\{\alpha, 1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 4:

For

$$x_{-1} = \alpha \geq 1$$
$$x_0 = \beta \geq 1$$

we have that

$$x_1 = \frac{\max\{x_0, 1\}}{x_0x_{-1}} = \frac{\max\{\beta, 1\}}{\beta\alpha} = \frac{\beta}{\alpha\beta} = \frac{1}{\alpha},$$

$$x_2 = \frac{\max\{x_1, 1\}}{x_1x_0} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\frac{1}{\alpha}\beta} = \frac{1}{\frac{\beta}{\alpha}} = \frac{\alpha}{\beta},$$

$$x_3 = \frac{\max\{x_2, 1\}}{x_2x_1} = \frac{\max\left\{\frac{\alpha}{\beta}, 1\right\}}{\frac{\alpha}{\beta}\frac{1}{\alpha}} = \max\{\alpha, \beta\},$$

$$x_4 = \frac{\max\{x_3, 1\}}{x_3x_2} = \frac{\max\{\max\{\alpha, \beta\}, 1\}}{\max\{\alpha, \beta\}\frac{\alpha}{\beta}} = \frac{\beta}{\alpha},$$

$$x_5 = \frac{\max\{x_4, 1\}}{x_4x_3} = \frac{\max\left\{\frac{\beta}{\alpha}, 1\right\}}{\frac{\beta}{\alpha}\max\{\alpha, \beta\}} = \frac{1}{\beta},$$

$$x_6 = \frac{\max\{x_5, 1\}}{x_5x_4} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\beta}\frac{\beta}{\alpha}} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_7 = \frac{\max\{x_6, 1\}}{x_6x_5} = \frac{\max\{\alpha, 1\}}{\alpha\frac{1}{\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$x_8 = \frac{\max\{x_7, 1\}}{x_7x_6} = \frac{\max\{\beta, 1\}}{\beta\alpha} = \frac{\beta}{\alpha\beta} = \frac{1}{\alpha},$$

$$x_9 = \frac{\max\{x_8, 1\}}{x_8x_7} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\frac{1}{\alpha}\beta} = \frac{1}{\frac{\beta}{\alpha}} = \frac{\alpha}{\beta},$$

$$x_{10} = \frac{\max\{x_9, 1\}}{x_9 x_8} = \frac{\max\left\{\frac{\alpha}{\beta}, 1\right\}}{\frac{\alpha}{\beta}\frac{1}{\alpha}} = \max\{\alpha, \beta\},$$

$$x_{11} = \frac{\max\{x_{10}, 1\}}{x_{10} x_9} = \frac{\max\{\max\{\alpha, \beta\}, 1\}}{\max\{\alpha, \beta\}\frac{\alpha}{\beta}} = \frac{\beta}{\alpha},$$

$$x_{12} = \frac{\max\{x_{11}, 1\}}{x_{11} x_{10}} = \frac{\max\left\{\frac{\beta}{\alpha}, 1\right\}}{\frac{\beta}{\alpha}\max\{\alpha, \beta\}} = \frac{1}{\beta},$$

$$x_{13} = \frac{\max\{x_{12}, 1\}}{x_{12} x_{11}} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\beta}\frac{\beta}{\alpha}} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_{14} = \frac{\max\{x_{13}, 1\}}{x_{13} x_{12}} = \frac{\max\{\alpha, 1\}}{\alpha\frac{1}{\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$\vdots$$

3.1.3 THE CASE

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n^2 x_{n-1}} \tag{3.8}$$

Example 3.3 *Every positive solution of Eq.(3.8) is periodic with period* 8. *We illustrate the result in the following table.*

Case 1	**Case 2**	**Case 3**	**Case 4**
$x_{-1} = \alpha \leq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \leq 1$ $x_0 = \beta \geq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \geq 1$
$x_1 = \frac{1}{\alpha\beta^2}$	$x_1 = \frac{1}{\alpha\beta^2}$	$x_1 = \frac{1}{\alpha\beta}$	$x_1 = \frac{1}{\alpha\beta}$
$x_2 = \alpha\beta$	$x_2 = \alpha\beta$	$x_2 = \alpha$	$x_2 = \alpha^2\beta$
$x_3 = \frac{1}{\alpha}$	$x_3 = \max\left\{\beta, \frac{1}{\alpha}\right\}$	$x_3 = \frac{\beta}{\alpha}$	$x_3 = \frac{1}{\alpha}$
$x_4 = \frac{1}{\beta}$	$x_4 = \max\left\{\frac{1}{\alpha\beta^3}, \frac{\alpha}{\beta}\right\}$	$x_4 = \frac{1}{\beta}$	$x_4 = \frac{1}{\beta}$
$x_5 = \alpha\beta$	$x_5 = \beta$	$x_5 = \alpha\beta$	$x_5 = \alpha\beta^2$
$x_6 = \frac{1}{\alpha^2\beta}$	$x_6 = \frac{1}{\alpha\beta}$	$x_6 = \frac{1}{\alpha^2\beta}$	$x_6 = \frac{1}{\alpha\beta}$
$x_7 = \alpha$	$x_7 = \alpha$	$x_7 = \alpha$	$x_7 = \alpha$
$x_8 = \beta$	$x_8 = \beta$	$x_8 = \beta$	$x_8 = \beta$
$x_9 = \frac{1}{\alpha\beta^2}$	$x_9 = \frac{1}{\alpha\beta^2}$	$x_9 = \frac{1}{\alpha\beta}$	$x_9 = \frac{1}{\alpha\beta}$
$x_{10} = \alpha\beta$	$x_{10} = \alpha\beta$	$x_{10} = \alpha$	$x_{10} = \alpha^2\beta$
$x_{11} = \frac{1}{\alpha}$	$x_{11} = \max\left\{\beta, \frac{1}{\alpha}\right\}$	$x_{11} = \frac{\beta}{\alpha}$	$x_{11} = \frac{1}{\alpha}$
$x_{12} = \frac{1}{\beta}$	$x_{12} = \max\left\{\frac{1}{\alpha\beta^3}, \frac{\alpha}{\beta}\right\}$	$x_{12} = \frac{1}{\beta}$	$x_{12} = \frac{1}{\beta}$
$x_{13} = \alpha\beta$	$x_{13} = \beta$	$x_{13} = \alpha\beta$	$x_{13} = \alpha\beta^2$
$x_{14} = \frac{1}{\alpha^2\beta}$	$x_{14} = \frac{1}{\alpha\beta}$	$x_{14} = \frac{1}{\alpha^2\beta}$	$x_{14} = \frac{1}{\alpha\beta}$
$x_{15} = \alpha$	$x_{15} = \alpha$	$x_{15} = \alpha$	$x_{15} = \alpha$
$x_{16} = \beta$	$x_{16} = \beta$	$x_{16} = \beta$	$x_{16} = \beta$

Solution.

Case 1:

For;

$$x_{-1} = \alpha \leq 1$$
$$x_0 = \beta \leq 1$$

we have that

$$x_1 = \frac{\max\{x_0, 1\}}{x_0^2 x_{-1}} = \frac{\max\{\beta, 1\}}{\beta^2\alpha} = \frac{1}{\alpha\beta^2},$$

$$x_2 = \frac{\max\{x_1, 1\}}{x_1^2 x_0} = \frac{\max\left\{\frac{1}{\alpha\beta^2}, 1\right\}}{\frac{1}{\alpha^2\beta^4}\beta} = \frac{\frac{1}{\alpha\beta^2}}{\frac{1}{\alpha^2\beta^3}} = \alpha\beta,$$

$$x_3 = \frac{\max\{x_2, 1\}}{x_2^2 x_1} = \frac{\max\{\alpha\beta, 1\}}{\alpha^2\beta^2\frac{1}{\alpha\beta^2}} = \frac{1}{\alpha},$$

$$x_4 = \frac{\max\{x_3, 1\}}{x_3^2 x_2} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\frac{1}{\alpha^2}\alpha\beta} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}} = \frac{1}{\beta},$$

$$x_5 = \frac{\max\{x_4, 1\}}{x_4^2 x_3} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\beta^2}\frac{1}{\alpha}} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha\beta^2}} = \alpha\beta,$$

$$x_6 = \frac{\max\{x_5, 1\}}{x_5^2 x_4} = \frac{\max\{\alpha\beta, 1\}}{\alpha^2\beta^2\frac{1}{\beta}} = \frac{1}{\alpha^2\beta},$$

$$x_7 = \frac{\max\{x_6, 1\}}{x_6^2 x_5} = \frac{\max\left\{\frac{1}{\alpha^2\beta}, 1\right\}}{\frac{1}{\alpha 4\beta^2}\alpha\beta} = \frac{\frac{1}{\alpha^2\beta}}{\frac{1}{\alpha^3\beta}} = \alpha,$$

$$x_8 = \frac{\max\{x_7, 1\}}{x_7^2 x_6} = \frac{\max\{\alpha, 1\}}{\alpha^2\frac{1}{\alpha^2\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$x_9 = \frac{\max\{x_8, 1\}}{x_8^2 x_7} = \frac{\max\{\beta, 1\}}{\beta^2\alpha} = \frac{1}{\alpha\beta^2},$$

$$x_{10} = \frac{\max\{x_9, 1\}}{x_9^2 x_8} = \frac{\max\left\{\frac{1}{\alpha\beta^2}, 1\right\}}{\frac{1}{\alpha^2\beta^4}\beta} = \frac{\frac{1}{\alpha\beta^2}}{\frac{1}{\alpha^2\beta^3}} = \alpha\beta,$$

$$x_{11} = \frac{\max\{x_{10}, 1\}}{x_{10}^2 x_9} = \frac{\max\{\alpha\beta, 1\}}{\alpha^2\beta^2\frac{1}{\alpha\beta^2}} = \frac{1}{\alpha},$$

$$x_{12} = \frac{\max\{x_{11}, 1\}}{x_{11}^2 x_{10}} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\frac{1}{\alpha^2}\alpha\beta} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}} = \frac{1}{\beta},$$

$$x_{13} = \frac{\max\{x_{12}, 1\}}{x_{12} x_{11}} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\beta^2}\frac{1}{\alpha}} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha\beta^2}} = \alpha\beta,$$

$$x_{14} = \frac{\max\{x_{13}, 1\}}{x_{13}^2 x_{12}} = \frac{\max\{\alpha\beta, 1\}}{\alpha^2\beta^2\frac{1}{\beta}} = \frac{1}{\alpha^2\beta},$$

$$x_{15} = \frac{\max\{x_{14}, 1\}}{x_{14}^2 x_{13}} = \frac{\max\left\{\frac{1}{\alpha^2\beta}, 1\right\}}{\frac{1}{\alpha 4\beta^2}\alpha\beta} = \frac{\frac{1}{\alpha^2\beta}}{\frac{1}{\alpha^3\beta}} = \alpha,$$

$$x_{16} = \frac{\max\{x_{15},1\}}{x_{15}^2 x_{14}} = \frac{\max\{\alpha,1\}}{\alpha^2 \frac{1}{\alpha^2\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 2:

For;

$$x_{-1} = \alpha \geq 1$$
$$x_0 = \beta \leq 1$$

we get that

$$x_1 = \frac{\max\{x_0,1\}}{x_0^2 x_{-1}} = \frac{\max\{\beta,1\}}{\beta^2\alpha} = \frac{1}{\alpha\beta^2},$$

$$x_2 = \frac{\max\{x_1,1\}}{x_1^2 x_0} = \frac{\max\left\{\frac{1}{\alpha\beta^2},1\right\}}{\frac{1}{\alpha^2\beta^4}\beta} = \frac{\frac{1}{\alpha\beta^2}}{\frac{1}{\alpha^2\beta^3}} = \alpha\beta,$$

$$x_3 = \frac{\max\{x_2,1\}}{x_2^2 x_1} = \frac{\max\{\alpha\beta,1\}}{\alpha^2\beta^2\frac{1}{\alpha\beta^2}} = \frac{\max\{\alpha\beta,1\}}{\alpha} = \max\left\{\beta,\frac{1}{\alpha}\right\},$$

$$x_4 = \frac{\max\{x_3,1\}}{x_3^2 x_2} = \frac{\max\left\{\max\left\{\beta,\frac{1}{\alpha}\right\},1\right\}}{\max\left\{\beta,\frac{1}{\alpha}\right\}^2\alpha\beta} = \max\left\{\frac{1}{\alpha\beta^3},\frac{\alpha}{\beta}\right\},$$

$$x_5 = \frac{\max\{x_4,1\}}{x_4^2 x_3} = \frac{\max\left\{\max\left\{\frac{1}{\alpha\beta^3},\frac{\alpha}{\beta}\right\},1\right\}}{\max\left\{\frac{1}{\alpha\beta^3},\frac{\alpha}{\beta}\right\}^2\max\left\{\beta,\frac{1}{\alpha}\right\}} = \beta,$$

$$x_6 = \frac{\max\{x_5,1\}}{x_5^2 x_4} = \frac{\max\{\beta,1\}}{\max\left\{\frac{1}{\alpha\beta^3},\frac{\alpha}{\beta}\right\}^2\max\left\{\frac{1}{\alpha\beta^3},\frac{\alpha}{\beta}\right\}} = \frac{1}{\alpha\beta},$$

$$x_7 = \frac{\max\{x_6,1\}}{x_6^2 x_5} = \frac{\max\left\{\frac{1}{\alpha\beta},1\right\}}{\frac{1}{\alpha^2\beta^2}\alpha\beta^2} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_8 = \frac{\max\{x_7,1\}}{x_7^2 x_6} = \frac{\max\{\alpha,1\}}{\alpha^2\frac{1}{\alpha\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$x_9 = \frac{\max\{x_8,1\}}{x_8^2 x_7} = \frac{\max\{\beta,1\}}{\beta^2\alpha} = \frac{1}{\alpha\beta^2},$$

$$x_{10} = \frac{\max\{x_9,1\}}{x_9^2 x_8} = \frac{\max\left\{\frac{1}{\alpha\beta^2},1\right\}}{\frac{1}{\alpha^2\beta^4}\beta} = \frac{\frac{1}{\alpha\beta^2}}{\frac{1}{\alpha^2\beta^3}} = \alpha\beta,$$

$$x_{11} = \frac{\max\{x_{10},1\}}{x_{10}^2 x_9} = \frac{\max\{\alpha\beta,1\}}{\alpha^2\beta^2\frac{1}{\alpha\beta^2}} = \frac{\max\{\alpha\beta,1\}}{\alpha} = \max\left\{\beta,\frac{1}{\alpha}\right\},$$

$$x_{12} = \frac{\max\{x_{11},1\}}{x_{11}^2 x_{10}} = \frac{\max\left\{\max\left\{\beta,\frac{1}{\alpha}\right\},1\right\}}{\max\left\{\beta,\frac{1}{\alpha}\right\}^2\alpha\beta} = \max\left\{\frac{1}{\alpha\beta^3},\frac{\alpha}{\beta}\right\},$$

$$x_{13} = \frac{\max\{x_{12}, 1\}}{x_{12}^2 x_{11}} = \frac{\max\left\{\max\left\{\frac{1}{\alpha\beta^3}, \frac{\alpha}{\beta}\right\}, 1\right\}}{\max\left\{\frac{1}{\alpha\beta^3}, \frac{\alpha}{\beta}\right\}^2 \max\left\{\beta, \frac{1}{\alpha}\right\}} = \beta,$$

$$x_{14} = \frac{\max\{x_{13}, 1\}}{x_{13}^2 x_{12}} = \frac{\max\{\beta, 1\}}{\max\left\{\frac{1}{\alpha\beta^3}, \frac{\alpha}{\beta}\right\}^2 \max\left\{\frac{1}{\alpha\beta^3}, \frac{\alpha}{\beta}\right\}} = \frac{1}{\alpha\beta},$$

$$x_{15} = \frac{\max\{x_{14}, 1\}}{x_{14}^2 x_{13}} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha^2\beta^2}\alpha\beta^2} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_{16} = \frac{\max\{x_{15}, 1\}}{x_{15}^2 x_{14}} = \frac{\max\{\alpha, 1\}}{\alpha^2 \frac{1}{\alpha\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$\vdots$$

Case 3:

For

$$x_{-1} = \alpha \leq 1$$
$$x_0 = \beta \geq 1$$

we get that

$$x_1 = \frac{\max\{x_0, 1\}}{x_0^2 x_{-1}} = \frac{\max\{\beta, 1\}}{\beta^2 \alpha} = \frac{\beta}{\alpha\beta^2} = \frac{1}{\alpha\beta},$$

$$x_2 = \frac{\max\{x_1, 1\}}{x_1^2 x_0} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha^2\beta^2}\beta} = \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha^2\beta}} = \alpha,$$

$$x_3 = \frac{\max\{x_2, 1\}}{x_2^2 x_1} = \frac{\max\{\alpha, 1\}}{\alpha^2 \frac{1}{\alpha\beta}} = \frac{1}{\frac{\alpha}{\beta}} = \frac{\beta}{\alpha},$$

$$x_4 = \frac{\max\{x_3, 1\}}{x_3^2 x_2} = \frac{\max\left\{\frac{\beta}{\alpha}, 1\right\}}{\frac{\beta^2}{\alpha^2}\alpha} = \frac{\frac{\beta}{\alpha}}{\frac{\beta^2}{\alpha}} = \frac{1}{\beta},$$

$$x_5 = \frac{\max\{x_4, 1\}}{x_4^2 x_3} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\beta^2}\frac{\beta}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_6 = \frac{\max\{x_5, 1\}}{x_5^2 x_4} = \frac{\max\{\alpha\beta, 1\}}{\alpha^2\beta^2\frac{1}{\beta}} = \frac{1}{\alpha^2\beta},$$

$$x_7 = \frac{\max\{x_6, 1\}}{x_6^2 x_5} = \frac{\max\left\{\frac{1}{\alpha^2\beta}, 1\right\}}{\frac{1}{\alpha^4\beta^2}\alpha\beta} = \frac{\frac{1}{\alpha^2\beta}}{\frac{1}{\alpha^3\beta}} = \alpha,$$

$$x_8 = \frac{\max\{x_7, 1\}}{x_7^2 x_6} = \frac{\max\{\alpha, 1\}}{\alpha^2\frac{1}{\alpha^2\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$x_9 = \frac{\max\{x_8,1\}}{x_8^2 x_7} = \frac{\max\{\beta,1\}}{\beta^2\alpha} = \frac{\beta}{\alpha\beta^2} = \frac{1}{\alpha\beta},$$

$$x_{10} = \frac{\max\{x_9,1\}}{x_9^2 x_8} = \frac{\max\left\{\frac{1}{\alpha\beta},1\right\}}{\frac{1}{\alpha^2\beta^2}\beta} = \frac{\frac{1}{\alpha\beta}}{\frac{1}{\alpha^2\beta}} = \alpha,$$

$$x_{11} = \frac{\max\{x_{10},1\}}{x_{10}^2 x_9} = \frac{\max\{\alpha,1\}}{\alpha^2\frac{1}{\alpha\beta}} = \frac{1}{\frac{\alpha}{\beta}} = \frac{\beta}{\alpha},$$

$$x_{12} = \frac{\max\{x_{11},1\}}{x_{11}^2 x_{10}} = \frac{\max\left\{\frac{\beta}{\alpha},1\right\}}{\frac{\beta^2}{\alpha^2}\alpha} = \frac{\frac{\beta}{\alpha}}{\frac{\beta^2}{\alpha}} = \frac{1}{\beta},$$

$$x_{13} = \frac{\max\{x_{12},1\}}{x_{12}^2 x_{11}} = \frac{\max\left\{\frac{1}{\beta},1\right\}}{\frac{1}{\beta^2}\frac{\beta}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_{14} = \frac{\max\{x_{13},1\}}{x_{13}^2 x_{12}} = \frac{\max\{\alpha\beta,1\}}{\alpha^2\beta^2\frac{1}{\beta}} = \frac{1}{\alpha^2\beta},$$

$$x_{15} = \frac{\max\{x_{14},1\}}{x_{14}^2 x_{13}} = \frac{\max\left\{\frac{1}{\alpha^2\beta},1\right\}}{\frac{1}{\alpha^4\beta^2}\alpha\beta} = \frac{\frac{1}{\alpha^2\beta}}{\frac{1}{\alpha^3\beta}} = \alpha,$$

$$x_{16} = \frac{\max\{x_{15},1\}}{x_{15}^2 x_{14}} = \frac{\max\{\alpha,1\}}{\alpha^2\frac{1}{\alpha^2\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 4:

For

$$x_{-1} = \alpha \geq 1$$
$$x_0 = \beta \geq 1$$

we have that

$$x_1 = \frac{\max\{x_0,1\}}{x_0^2 x_{-1}} = \frac{\max\{\beta,1\}}{\beta^2\alpha} = \frac{\beta}{\alpha\beta^2} = \frac{1}{\alpha\beta},$$

$$x_2 = \frac{\max\{x_1,1\}}{x_1^2 x_0} = \frac{\max\left\{\frac{1}{\alpha\beta},1\right\}}{\frac{1}{\alpha^2\beta^2}\beta} = \frac{1}{\frac{1}{\alpha^2\beta}} = \alpha^2\beta,$$

$$x_3 = \frac{\max\{x_2,1\}}{x_2^2 x_1} = \frac{\max\{\alpha^2\beta,1\}}{\alpha^4\beta^2\frac{1}{\alpha\beta}} = \frac{\alpha^2\beta}{\alpha^3\beta} = \frac{1}{\alpha},$$

$$x_4 = \frac{\max\{x_3,1\}}{x_3^2 x_2} = \frac{\max\left\{\frac{1}{\alpha},1\right\}}{\frac{1}{\alpha^2}\alpha^2\beta} = \frac{1}{\beta},$$

$$x_5 = \frac{\max\{x_4,1\}}{x_4^2 x_3} = \frac{\max\left\{\frac{1}{\beta},1\right\}}{\frac{1}{\beta^2}\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta^2}} = \alpha\beta^2,$$

$$x_6 = \frac{\max\{x_5, 1\}}{x_5^2 x_4} = \frac{\max\left\{\alpha\beta^2, 1\right\}}{\alpha^2\beta^4\frac{1}{\beta}} = \frac{\alpha\beta^2}{\alpha^2\beta^3} = \frac{1}{\alpha\beta},$$

$$x_7 = \frac{\max\{x_6, 1\}}{x_6^2 x_5} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha^2\beta^2}\alpha\beta^2} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_8 = \frac{\max\{x_7, 1\}}{x_7^2 x_6} = \frac{\max\{\alpha, 1\}}{\alpha^2\frac{1}{\alpha\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta,$$

$$x_9 = \frac{\max\{x_8, 1\}}{x_8^2 x_7} = \frac{\max\{\beta, 1\}}{\beta^2\alpha} = \frac{\beta}{\alpha\beta^2} = \frac{1}{\alpha\beta},$$

$$x_{10} = \frac{\max\{x_9, 1\}}{x_9^2 x_8} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha^2\beta^2}\beta} = \frac{1}{\frac{1}{\alpha^2\beta}} = \alpha^2\beta,$$

$$x_{11} = \frac{\max\{x_{10}, 1\}}{x_{10}^2 x_9} = \frac{\max\{\alpha^2\beta, 1\}}{\alpha^4\beta^2\frac{1}{\alpha\beta}} = \frac{\alpha^2\beta}{\alpha^3\beta} = \frac{1}{\alpha},$$

$$x_{12} = \frac{\max\{x_{11}, 1\}}{x_{11}^2 x_{10}} = \frac{\max\left\{\frac{1}{\alpha}, 1\right\}}{\frac{1}{\alpha^2}\alpha^2\beta} = \frac{1}{\beta},$$

$$x_{13} = \frac{\max\{x_{12}, 1\}}{x_{12}^2 x_{11}} = \frac{\max\left\{\frac{1}{\beta}, 1\right\}}{\frac{1}{\beta^2}\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta^2}} = \alpha\beta^2,$$

$$x_{14} = \frac{\max\{x_{13}, 1\}}{x_{13}^2 x_{12}} = \frac{\max\left\{\alpha\beta^2, 1\right\}}{\alpha^2\beta^4\frac{1}{\beta}} = \frac{\alpha\beta^2}{\alpha^2\beta^3} = \frac{1}{\alpha\beta},$$

$$x_{15} = \frac{\max\{x_{14}, 1\}}{x_{14}^2 x_{13}} = \frac{\max\left\{\frac{1}{\alpha\beta}, 1\right\}}{\frac{1}{\alpha^2\beta^2}\alpha\beta^2} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$\begin{aligned} x_{16} &= \frac{\max\{x_{15}, 1\}}{x_{15}^2 x_{14}} = \frac{\max\{\alpha, 1\}}{\alpha^2\frac{1}{\alpha\beta}} = \frac{\alpha}{\frac{\alpha}{\beta}} = \beta, \\ &\vdots \end{aligned}$$

3.1.4 THE CASE

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n x_{n-1}} \tag{3.9}$$

Example 3.4 *Every positive solution of Eq.(3.8) is periodic with period* 9*. We illustrate the result in the following table.*

Case 1	**Case 2**	**Case 3**	**Case 4**
$x_{-1} = \alpha \leq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \leq 1$	$x_{-1} = \alpha \leq 1$ $x_0 = \beta \geq 1$	$x_{-1} = \alpha \geq 1$ $x_0 = \beta \geq 1$
$x_1 = \frac{1}{\alpha\beta}$	$x_1 = \frac{1}{\alpha\beta}$	$x_1 = \frac{\beta}{\alpha}$	$x_1 = \frac{\beta}{\alpha}$
$x_2 = \frac{1}{\alpha\beta^2}$	$x_2 = \alpha$	$x_2 = \frac{1}{\alpha}$	$x_2 = \max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\}$
$x_3 = \frac{1}{\beta}$	$x_3 = \alpha^2\beta$	$x_3 = \frac{1}{\beta}$	$x_3 = \max\left\{\beta, \frac{\alpha^2}{\beta}\right\}$
$x_4 = \alpha\beta$	$x_4 = \alpha\beta$	$x_4 = \alpha\beta$	$x_4 = \max\left\{\frac{\beta^3}{\alpha}, \frac{\alpha^3}{\beta}\right\}$
$x_5 = \frac{1}{\alpha}$	$x_5 = \frac{1}{\alpha}$	$x_5 = \alpha\beta^2$	$x_5 = \max\left\{\frac{\beta^2}{\alpha}, \alpha\right\}$
$x_6 = \frac{1}{\alpha^2\beta}$	$x_6 = \frac{1}{\beta}$	$x_6 = \beta$	$x_6 = \max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\}$
$x_7 = \frac{1}{\alpha\beta}$	$x_7 = \frac{\alpha}{\beta}$	$x_7 = \frac{1}{\alpha\beta}$	$x_7 = \frac{\alpha}{\beta}$
$x_8 = \alpha$	$x_8 = \alpha$	$x_8 = \alpha$	$x_8 = \alpha$
$x_9 = \beta$	$x_9 = \beta$	$x_9 = \beta$	$x_9 = \beta$
$x_{10} = \frac{1}{\alpha\beta}$	$x_{10} = \frac{1}{\alpha\beta}$	$x_{10} = \frac{\beta}{\alpha}$	$x_{10} = \frac{\beta}{\alpha}$
$x_{11} = \frac{1}{\alpha\beta^2}$	$x_{11} = \alpha$	$x_{11} = \frac{1}{\alpha}$	$x_{11} = \max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\}$
$x_{12} = \frac{1}{\beta}$	$x_{12} = \alpha^2\beta$	$x_{12} = \frac{1}{\beta}$	$x_{12} = \max\left\{\beta, \frac{\alpha^2}{\beta}\right\}$
$x_{13} = \alpha\beta$	$x_{13} = \alpha\beta$	$x_{13} = \alpha\beta$	$x_{13} = \max\left\{\frac{\beta^3}{\alpha}, \frac{\alpha^3}{\beta}\right\}$
$x_{14} = \frac{1}{\alpha}$	$x_{14} = \frac{1}{\alpha}$	$x_{14} = \alpha\beta^2$	$x_{14} = \max\left\{\frac{\beta^2}{\alpha}, \alpha\right\}$
$x_{15} = \frac{1}{\alpha^2\beta}$	$x_{15} = \frac{1}{\beta}$	$x_{15} = \beta$	$x_{15} = \max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\}$
$x_{16} = \frac{1}{\alpha\beta}$	$x_{16} = \frac{\alpha}{\beta}$	$x_{16} = \frac{1}{\alpha\beta}$	$x_{16} = \frac{\alpha}{\beta}$
$x_{17} = \alpha$	$x_{17} = \alpha$	$x_{17} = \alpha$	$x_{17} = \alpha$
$x_{18} = \beta$	$x_{18} = \beta$	$x_{18} = \beta$	$x_{18} = \beta$

Solution.

Case 1:

For;

$$x_{-1} = \alpha \leq 1$$
$$x_0 = \beta \leq 1$$

we have that

$$x_1 = \frac{\max\{x_0^2, 1\}}{x_0 x_{-1}} = \frac{\max\{\beta^2, 1\}}{\beta\alpha} = \frac{1}{\alpha\beta},$$

$$x_2 = \frac{\max\{x_1^2, 1\}}{x_1 x_0} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha\beta}\beta} = \frac{\frac{1}{\alpha^2\beta^2}}{\frac{1}{\alpha}} = \frac{1}{\alpha\beta^2},$$

$$x_3 = \frac{\max\{x_2^2, 1\}}{x_2 x_1} = \frac{\max\left\{\frac{1}{\alpha^2\beta^4}, 1\right\}}{\frac{1}{\alpha\beta^2}\frac{1}{\alpha\beta}} = \frac{\frac{1}{\alpha^2\beta^4}}{\frac{1}{\alpha^2\beta^3}} = \frac{1}{\beta},$$

$$x_4 = \frac{\max\{x_3^2, 1\}}{x_3 x_2} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta}\frac{1}{\alpha\beta^2}} = \frac{\frac{1}{\beta^2}}{\frac{1}{\alpha\beta^3}} = \alpha\beta,$$

$$x_5 = \frac{\max\{x_4^2, 1\}}{x_4 x_3} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha\beta\frac{1}{\beta}} = \frac{1}{\alpha},$$

$$x_6 = \frac{\max\{x_5^2, 1\}}{x_5 x_4} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha}\alpha\beta} = \frac{\frac{1}{\alpha^2}}{\beta} = \frac{1}{\alpha^2\beta},$$

$$x_7 = \frac{\max\{x_6^2, 1\}}{x_6 x_5} = \frac{\max\left\{\frac{1}{\alpha^4\beta^2}, 1\right\}}{\frac{1}{\alpha^2\beta}\frac{1}{\alpha}} = \frac{\frac{1}{\alpha^4\beta^2}}{\frac{1}{\alpha^3\beta}} = \frac{1}{\alpha\beta},$$

$$x_8 = \frac{\max\{x_7^2, 1\}}{x_7 x_6} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha\beta}\frac{1}{\alpha^2\beta}} = \frac{\frac{1}{\alpha^2\beta^2}}{\frac{1}{\alpha^3\beta^2}} = \alpha,$$

$$x_9 = \frac{\max\{x_8^2, 1\}}{x_8 x_7} = \frac{\max\left\{\alpha^2, 1\right\}}{\alpha\frac{1}{\alpha\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$x_{10} = \frac{\max\{x_9^2, 1\}}{x_9 x_8} = \frac{\max\left\{\beta^2, 1\right\}}{\beta\alpha} = \frac{1}{\alpha\beta},$$

$$x_{11} = \frac{\max\{x_{10}^2, 1\}}{x_{10} x_9} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha\beta}\beta} = \frac{\frac{1}{\alpha^2\beta^2}}{\frac{1}{\alpha}} = \frac{1}{\alpha\beta^2},$$

$$x_{12} = \frac{\max\{x_{11}^2, 1\}}{x_{11} x_{10}} = \frac{\max\left\{\frac{1}{\alpha^2\beta^4}, 1\right\}}{\frac{1}{\alpha\beta^2}\frac{1}{\alpha\beta}} = \frac{\frac{1}{\alpha^2\beta^4}}{\frac{1}{\alpha^2\beta^3}} = \frac{1}{\beta},$$

$$x_{13} = \frac{\max\{x_{12}^2, 1\}}{x_{12} x_{11}} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta}\frac{1}{\alpha\beta^2}} = \frac{\frac{1}{\beta^2}}{\frac{1}{\alpha\beta^3}} = \alpha\beta,$$

$$x_{14} = \frac{\max\{x_{13}^2, 1\}}{x_{13} x_{12}} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha\beta\frac{1}{\beta}} = \frac{1}{\alpha},$$

$$x_{15} = \frac{\max\{x_{14}^2, 1\}}{x_{14} x_{13}} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha}\alpha\beta} = \frac{\frac{1}{\alpha^2}}{\beta} = \frac{1}{\alpha^2\beta},$$

$$x_{16} = \frac{\max\{x_{15}^2, 1\}}{x_{15} x_{14}} = \frac{\max\left\{\frac{1}{\alpha^4\beta^2}, 1\right\}}{\frac{1}{\alpha^2\beta}\frac{1}{\alpha}} = \frac{\frac{1}{\alpha^4\beta^2}}{\frac{1}{\alpha^3\beta}} = \frac{1}{\alpha\beta},$$

$$x_{17} = \frac{\max\{x_{16}^2, 1\}}{x_{16} x_{15}} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha\beta}\frac{1}{\alpha^2\beta}} = \frac{\frac{1}{\alpha^2\beta^2}}{\frac{1}{\alpha^3\beta^2}} = \alpha,$$

$$x_{18} = \frac{\max\{x_{17}^2,1\}}{x_{17}x_{16}} = \frac{\max\{\alpha^2,1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$
$$\vdots$$

Case 2:

For

$$x_{-1} = \alpha \geq 1$$
$$x_0 = \beta \leq 1$$

we get that

$$x_1 = \frac{\max\{x_0^2,1\}}{x_0x_{-1}} = \frac{\max\{\beta^2,1\}}{\beta\alpha} = \frac{1}{\alpha\beta},$$

$$x_2 = \frac{\max\{x_1^2,1\}}{x_1x_0} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2},1\right\}}{\frac{1}{\alpha\beta}\beta} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_3 = \frac{\max\{x_2^2,1\}}{x_2x_1} = \frac{\max\{\alpha^2,1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{\alpha^2}{\frac{1}{\beta}} = \alpha^2\beta,$$

$$x_4 = \frac{\max\{x_3^2,1\}}{x_3x_2} = \frac{\max\{\alpha^4\beta^2,1\}}{\alpha^2\beta\alpha} = \frac{\alpha^4\beta^2}{\alpha^3\beta} = \alpha\beta,$$

$$x_5 = \frac{\max\{x_4^2,1\}}{x_4x_3} = \frac{\max\{\alpha^2\beta^2,1\}}{\alpha\beta\alpha^2\beta} = \frac{\alpha^2\beta^2}{\alpha^3\beta^2} = \frac{1}{\alpha},$$

$$x_6 = \frac{\max\{x_5^2,1\}}{x_5x_4} = \frac{\max\left\{\frac{1}{\alpha^2},1\right\}}{\frac{1}{\alpha}\alpha\beta} = \frac{1}{\beta},$$

$$x_7 = \frac{\max\{x_6^2,1\}}{x_6x_5} = \frac{\max\left\{\frac{1}{\beta^2},1\right\}}{\frac{1}{\beta}\frac{1}{\alpha}} = \frac{\frac{1}{\beta^2}}{\frac{1}{\alpha\beta}} = \frac{\alpha}{\beta},$$

$$x_8 = \frac{\max\{x_7^2,1\}}{x_7x_6} = \frac{\max\left\{\frac{\alpha^2}{\beta^2},1\right\}}{\frac{\alpha}{\beta}\frac{1}{\beta}} = \frac{\frac{\alpha^2}{\beta^2}}{\frac{\alpha}{\beta^2}} = \alpha,$$

$$x_9 = \frac{\max\{x_8^2,1\}}{x_8x_7} = \frac{\max\{\alpha^2,1\}}{\alpha\frac{\alpha}{\beta}} = \frac{\alpha^2}{\frac{\alpha^2}{\beta}} = \beta,$$

$$x_{10} = \frac{\max\{x_9^2,1\}}{x_9x_8} = \frac{\max\{\beta^2,1\}}{\beta\alpha} = \frac{1}{\alpha\beta},$$

$$x_{11} = \frac{\max\{x_{10}^2,1\}}{x_{10}x_9} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2},1\right\}}{\frac{1}{\alpha\beta}\beta} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_{12} = \frac{\max\{x_{11}^2,1\}}{x_{11}x_{10}} = \frac{\max\{\alpha^2,1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{\alpha^2}{\frac{1}{\beta}} = \alpha^2\beta,$$

$$x_{13} = \frac{\max\{x_{12}^2, 1\}}{x_{12}x_{11}} = \frac{\max\left\{\alpha^4\beta^2, 1\right\}}{\alpha^2\beta\alpha} = \frac{\alpha^4\beta^2}{\alpha^3\beta} = \alpha\beta,$$

$$x_{14} = \frac{\max\{x_{13}^2, 1\}}{x_{13}x_{12}} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha\beta\alpha^2\beta} = \frac{\alpha^2\beta^2}{\alpha^3\beta^2} = \frac{1}{\alpha},$$

$$x_{15} = \frac{\max\{x_{14}^2, 1\}}{x_{14}x_{13}} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha}\alpha\beta} = \frac{1}{\beta},$$

$$x_{16} = \frac{\max\{x_{15}^2, 1\}}{x_{15}x_{14}} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta}\frac{1}{\alpha}} = \frac{\frac{1}{\beta^2}}{\frac{1}{\alpha\beta}} = \frac{\alpha}{\beta},$$

$$x_{17} = \frac{\max\{x_{16}^2, 1\}}{x_{16}x_{15}} = \frac{\max\left\{\frac{\alpha^2}{\beta^2}, 1\right\}}{\frac{\alpha}{\beta}\frac{1}{\beta}} = \frac{\frac{\alpha^2}{\beta^2}}{\frac{\alpha}{\beta^2}} = \alpha,$$

$$x_{18} = \frac{\max\{x_{17}^2, 1\}}{x_{17}x_{16}} = \frac{\max\{\alpha^2, 1\}}{\alpha\frac{\alpha}{\beta}} = \frac{\alpha^2}{\frac{\alpha^2}{\beta}} = \beta,$$

$$\vdots$$

Case 3:

For

$$x_{-1} = \alpha \leq 1$$
$$x_0 = \beta \geq 1$$

we get that

$$x_1 = \frac{\max\{x_0^2, 1\}}{x_0x_{-1}} = \frac{\max\left\{\beta^2, 1\right\}}{\beta\alpha} = \frac{\beta^2}{\alpha\beta} = \frac{\beta}{\alpha},$$

$$x_2 = \frac{\max\{x_1^2, 1\}}{x_1x_0} = \frac{\max\left\{\frac{\beta^2}{\alpha^2}, 1\right\}}{\frac{\beta}{\alpha}\beta} = \frac{\frac{\beta^2}{\alpha^2}}{\frac{\beta^2}{\alpha}} = \frac{1}{\alpha},$$

$$x_3 = \frac{\max\{x_2^2, 1\}}{x_2x_1} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha}\frac{\beta}{\alpha}} = \frac{\frac{1}{\alpha^2}}{\frac{\beta}{\alpha^2}} = \frac{1}{\beta},$$

$$x_4 = \frac{\max\{x_3^2, 1\}}{x_3x_2} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta}\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_5 = \frac{\max\{x_4^2, 1\}}{x_4x_3} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha\beta\frac{1}{\beta}} = \frac{\alpha^2\beta^2}{\alpha} = \alpha\beta^2,$$

$$x_6 = \frac{\max\{x_5^2, 1\}}{x_5x_4} = \frac{\max\left\{\alpha^2\beta^4, 1\right\}}{\alpha\beta^2\alpha\beta} = \frac{\alpha^2\beta^4}{\alpha^2\beta^3} = \beta,$$

$$x_7 = \frac{\max\{x_6^2, 1\}}{x_6x_5} = \frac{\max\left\{\beta^2, 1\right\}}{\beta\alpha\beta^2} = \frac{\beta^2}{\alpha\beta^3} = \frac{1}{\alpha\beta},$$

$$x_8 = \frac{\max\{x_7^2, 1\}}{x_7 x_6} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha\beta}\beta} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_9 = \frac{\max\{x_8^2, 1\}}{x_8 x_7} = \frac{\max\{\alpha^2, 1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$x_{10} = \frac{\max\{x_{10}^2, 1\}}{x_{10} x_9} = \frac{\max\left\{\beta^2, 1\right\}}{\beta\alpha} = \frac{\beta^2}{\alpha\beta} = \frac{\beta}{\alpha},$$

$$x_{11} = \frac{\max\{x_{10}^2, 1\}}{x_{10} x_9} = \frac{\max\left\{\frac{\beta^2}{\alpha^2}, 1\right\}}{\frac{\beta}{\alpha}\beta} = \frac{\frac{\beta^2}{\alpha^2}}{\frac{\beta^2}{\alpha}} = \frac{1}{\alpha},$$

$$x_{12} = \frac{\max\{x_{11}^2, 1\}}{x_{11} x_{10}} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha}\frac{\beta}{\alpha}} = \frac{\frac{1}{\alpha^2}}{\frac{\beta}{\alpha^2}} = \frac{1}{\beta},$$

$$x_{13} = \frac{\max\{x_{12}^2, 1\}}{x_{12} x_{11}} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta}\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_{14} = \frac{\max\{x_{13}^2, 1\}}{x_{13} x_{12}} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha\beta\frac{1}{\beta}} = \frac{\alpha^2\beta^2}{\alpha} = \alpha\beta^2,$$

$$x_{15} = \frac{\max\{x_{14}^2, 1\}}{x_{14} x_{13}} = \frac{\max\left\{\alpha^2\beta^4, 1\right\}}{\alpha\beta^2\alpha\beta} = \frac{\alpha^2\beta^4}{\alpha^2\beta^3} = \beta,$$

$$x_{16} = \frac{\max\{x_{15}^2, 1\}}{x_{15} x_{14}} = \frac{\max\left\{\beta^2, 1\right\}}{\beta\alpha\beta^2} = \frac{\beta^2}{\alpha\beta^3} = \frac{1}{\alpha\beta},$$

$$x_{17} = \frac{\max\{x_{16}^2, 1\}}{x_{16} x_{15}} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha\beta}\beta} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_{18} = \frac{\max\{x_{17}^2, 1\}}{x_{17} x_{16}} = \frac{\max\{\alpha^2, 1\}}{\alpha\frac{1}{\alpha\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 4:

When

$$x_{-1} = \alpha \geq 1$$
$$x_0 = \beta \geq 1$$

we have that

$$x_1 = \frac{\max\{x_0^2, 1\}}{x_0 x_{-1}} = \frac{\max\left\{\beta^2, 1\right\}}{\beta\alpha} = \frac{\beta^2}{\alpha\beta} = \frac{\beta}{\alpha},$$

$$x_2 = \frac{\max\{x_1^2, 1\}}{x_1 x_0} = \frac{\max\left\{\frac{\beta^2}{\alpha^2}, 1\right\}}{\frac{\beta}{\alpha}\beta} = \frac{\max\left\{\frac{\beta^2}{\alpha^2}, 1\right\}}{\frac{\beta^2}{\alpha}} = \max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\},$$

$$x_3 = \frac{\max\{x_2^2, 1\}}{x_2 x_1} = \frac{\max\left\{\max\left\{\frac{\alpha^2}{\beta^4}, \frac{1}{\alpha^2}\right\}, 1\right\}}{\max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\}\frac{\beta}{\alpha}} = \frac{1}{\max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\}} = \max\left\{\beta, \frac{\alpha^2}{\beta}\right\},$$

$$x_4 = \frac{\max\{x_3^2, 1\}}{x_3 x_2} = \frac{\max\left\{\max\left\{\beta^2, \frac{\alpha^4}{\beta^2}\right\}, 1\right\}}{\max\left\{\beta, \frac{\alpha^2}{\beta}\right\}\max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\}} = \frac{\max\left\{\beta^2, \frac{\alpha^4}{\beta^2}\right\}}{\frac{\alpha}{\beta}} = \max\left\{\frac{\beta^3}{\alpha}, \frac{\alpha^3}{\beta}\right\},$$

$$x_5 = \frac{\max\{x_4^2, 1\}}{x_4 x_3} = \frac{\max\left\{\max\left\{\frac{\beta^6}{\alpha^2}, \frac{\alpha^6}{\beta^2}\right\}, 1\right\}}{\max\left\{\frac{\beta^3}{\alpha}, \frac{\alpha^3}{\beta}\right\}\max\left\{\beta, \frac{\alpha^2}{\beta}\right\}} = \frac{\max\left\{\frac{\beta^6}{\alpha^2}, \frac{\alpha^6}{\beta^2}\right\}}{\max\left\{\frac{\beta^4}{\alpha}, \frac{\alpha^5}{\beta^2}\right\}} = \max\left\{\frac{\beta^2}{\alpha}, \alpha\right\},$$

$$x_6 = \frac{\max\{x_5^2, 1\}}{x_5 x_4} = \frac{\max\left\{\max\left\{\frac{\beta^4}{\alpha^2}, \alpha^2\right\}, 1\right\}}{\max\left\{\frac{\beta^2}{\alpha}, \alpha\right\}\max\left\{\frac{\beta^3}{\alpha}, \frac{\alpha^3}{\beta}\right\}} = \frac{\max\left\{\frac{\beta^4}{\alpha^2}, \alpha^2\right\}}{\max\left\{\frac{\beta^5}{\alpha^2}, \frac{\alpha^4}{\beta}\right\}} = \max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\},$$

$$x_7 = \frac{\max\{x_6^2, 1\}}{x_6 x_5} = \frac{\max\left\{\max\left\{\frac{1}{\beta^2}, \frac{\beta^2}{\alpha^4}\right\}, 1\right\}}{\max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\}\max\left\{\frac{\beta^2}{\alpha}, \alpha\right\}} = \frac{1}{\frac{\beta}{\alpha}} = \frac{\alpha}{\beta},$$

$$x_8 = \frac{\max\{x_7^2, 1\}}{x_7 x_6} = \frac{\max\left\{\frac{\alpha^2}{\beta^2}, 1\right\}}{\frac{\alpha}{\beta}\max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\}} = \frac{\max\left\{\frac{\alpha^2}{\beta^2}, 1\right\}}{\max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\}} = \alpha,$$

$$x_9 = \frac{\max\{x_8^2, 1\}}{x_8 x_7} = \frac{\max\{\alpha^2, 1\}}{\alpha\frac{\alpha}{\beta}} = \frac{\alpha^2}{\frac{\alpha^2}{\beta}} = \beta,$$

$$x_{10} = \frac{\max\{x_9^2, 1\}}{x_9 x_8} = \frac{\max\{\beta^2, 1\}}{\beta\alpha} = \frac{\beta^2}{\alpha\beta} = \frac{\beta}{\alpha},$$

$$x_{11} = \frac{\max\{x_{10}^2, 1\}}{x_{10} x_9} = \frac{\max\left\{\frac{\beta^2}{\alpha^2}, 1\right\}}{\frac{\beta}{\alpha}\beta} = \frac{\max\left\{\frac{\beta^2}{\alpha^2}, 1\right\}}{\frac{\beta^2}{\alpha}} = \max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\},$$

$$x_{12} = \frac{\max\{x_{11}^2, 1\}}{x_{11} x_{10}} = \frac{\max\left\{\max\left\{\frac{\alpha^2}{\beta^4}, \frac{1}{\alpha^2}\right\}, 1\right\}}{\max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\}\frac{\beta}{\alpha}} = \frac{1}{\max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\}} = \max\left\{\beta, \frac{\alpha^2}{\beta}\right\},$$

$$x_{13} = \frac{\max\{x_{12}^2, 1\}}{x_{12} x_{11}} = \frac{\max\left\{\max\left\{\beta^2, \frac{\alpha^4}{\beta^2}\right\}, 1\right\}}{\max\left\{\beta, \frac{\alpha^2}{\beta}\right\}\max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\}} = \frac{\max\left\{\beta^2, \frac{\alpha^4}{\beta^2}\right\}}{\frac{\alpha}{\beta}} = \max\left\{\frac{\beta^3}{\alpha}, \frac{\alpha^3}{\beta}\right\},$$

$$x_{14} = \frac{\max\{x_{13}^2, 1\}}{x_{13} x_{12}} = \frac{\max\left\{\max\left\{\frac{\beta^6}{\alpha^2}, \frac{\alpha^6}{\beta^2}\right\}, 1\right\}}{\max\left\{\frac{\beta^3}{\alpha}, \frac{\alpha^3}{\beta}\right\}\max\left\{\beta, \frac{\alpha^2}{\beta}\right\}} = \frac{\max\left\{\frac{\beta^6}{\alpha^2}, \frac{\alpha^6}{\beta^2}\right\}}{\max\left\{\frac{\beta^4}{\alpha}, \frac{\alpha^5}{\beta^2}\right\}} = \max\left\{\frac{\beta^2}{\alpha}, \alpha\right\},$$

$$x_{15} = \frac{\max\{x_{14}^2, 1\}}{x_{14} x_{13}} = \frac{\max\left\{\max\left\{\frac{\beta^4}{\alpha^2}, \alpha^2\right\}, 1\right\}}{\max\left\{\frac{\beta^2}{\alpha}, \alpha\right\}\max\left\{\frac{\beta^3}{\alpha}, \frac{\alpha^3}{\beta}\right\}} = \frac{\max\left\{\frac{\beta^4}{\alpha^2}, \alpha^2\right\}}{\max\left\{\frac{\beta^5}{\alpha^2}, \frac{\alpha^4}{\beta}\right\}} = \max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\},$$

$$x_{16} = \frac{\max\{x_{15}^2, 1\}}{x_{15} x_{14}} = \frac{\max\left\{\max\left\{\frac{1}{\beta^2}, \frac{\beta^2}{\alpha^4}\right\}, 1\right\}}{\max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\}\max\left\{\frac{\beta^2}{\alpha}, \alpha\right\}} = \frac{1}{\frac{\beta}{\alpha}} = \frac{\alpha}{\beta},$$

$$x_{17} = \frac{\max\{x_{16}^2, 1\}}{x_{16}x_{15}} = \frac{\max\left\{\frac{\alpha^2}{\beta^2}, 1\right\}}{\frac{\alpha}{\beta}\max\left\{\frac{1}{\beta}, \frac{\beta}{\alpha^2}\right\}} = \frac{\max\left\{\frac{\alpha^2}{\beta^2}, 1\right\}}{\max\left\{\frac{\alpha}{\beta^2}, \frac{1}{\alpha}\right\}} = \alpha,$$

$$x_{18} = \frac{\max\{x_{17}^2, 1\}}{x_{17}x_{16}} = \frac{\max\{\alpha^2, 1\}}{\alpha\frac{\alpha}{\beta}} = \frac{\alpha^2}{\frac{\alpha^2}{\beta}} = \beta,$$

$$\vdots$$

3.1.5 THE CASE

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n^3 x_{n-1}} \tag{3.10}$$

Example 3.5 *Every positive solution of Eq.(3.10) is periodic with period 12. We illustrate the result in the following table.*

Case 1	***Case 2***	***Case 3***	***Case 4***
$x_{-1} = \alpha \le 1$ $x_0 = \beta \le 1$	$x_{-1} = \alpha \ge 1$ $x_0 = \beta \le 1$	$x_{-1} = \alpha \le 1$ $x_0 = \beta \ge 1$	$x_{-1} = \alpha \ge 1$ $x_0 = \beta \ge 1$
$x_1 = \frac{1}{\alpha\beta^3}$	$x_1 = \frac{1}{\alpha\beta^3}$	$x_1 = \frac{1}{\alpha\beta}$	$x_1 = \frac{1}{\alpha\beta}$
$x_2 = \alpha\beta^2$	$x_2 = \alpha\beta^2$	$x_2 = \alpha$	$x_2 = \alpha^3\beta^2$
$x_3 = \frac{1}{\alpha^2\beta^3}$	$x_3 = \frac{1}{\alpha^2\beta^3}$	$x_3 = \frac{\beta}{\alpha^2}$	$x_3 = \frac{1}{\alpha^2\beta}$
$x_4 = \alpha\beta$	$x_4 = \alpha\beta$	$x_4 = \frac{\alpha}{\beta}$	$x_4 = \alpha^3\beta$
$x_5 = \frac{1}{\alpha}$	$x_5 = \frac{1}{\alpha}$	$x_5 = \frac{\beta^2}{\alpha}$	$x_5 = \frac{1}{\alpha}$
$x_6 = \frac{1}{\beta}$	$x_6 = \frac{\alpha^2}{\beta}$	$x_6 = \frac{1}{\beta}$	$x_6 = \frac{1}{\beta}$
$x_7 = \alpha\beta$	$x_7 = \frac{\beta}{\alpha}$	$x_7 = \alpha\beta$	$x_7 = \alpha\beta^3$
$x_8 = \frac{1}{\alpha^3\beta^2}$	$x_8 = \frac{\alpha}{\beta^2}$	$x_8 = \frac{1}{\alpha^3\beta^2}$	$x_8 = \frac{1}{\alpha\beta^2}$
$x_9 = \alpha^2\beta$	$x_9 = \beta$	$x_9 = \alpha^2\beta$	$x_9 = \alpha^2\beta^3$
$x_{10} = \frac{1}{\alpha^3\beta}$	$x_{10} = \frac{1}{\alpha\beta}$	$x_{10} = \frac{1}{\alpha^3\beta}$	$x_{10} = \frac{1}{\alpha\beta}$
$x_{11} = \alpha$	$x_{11} = \alpha$	$x_{11} = \alpha$	$x_{11} = \alpha$
$x_{12} = \beta$	$x_{12} = \beta$	$x_{12} = \beta$	$x_{12} = \beta$
$x_{13} = \frac{1}{\alpha\beta^3}$	$x_{13} = \frac{1}{\alpha\beta^3}$	$x_{13} = \frac{1}{\alpha\beta}$	$x_{13} = \frac{1}{\alpha\beta}$
$x_{14} = \alpha\beta^2$	$x_{14} = \alpha\beta^2$	$x_{14} = \alpha$	$x_{14} = \alpha^3\beta^2$
$x_{15} = \frac{1}{\alpha^2\beta^3}$	$x_{15} = \frac{1}{\alpha^2\beta^3}$	$x_{15} = \frac{\beta}{\alpha^2}$	$x_{15} = \frac{1}{\alpha^2\beta}$
$x_{16} = \alpha\beta$	$x_{16} = \alpha\beta$	$x_{16} = \frac{\alpha}{\beta}$	$x_{16} = \alpha^3\beta$
$x_{17} = \frac{1}{\alpha}$	$x_{17} = \frac{1}{\alpha}$	$x_{17} = \frac{\beta^2}{\alpha}$	$x_{17} = \frac{1}{\alpha}$
$x_{18} = \frac{1}{\beta}$	$x_{18} = \frac{\alpha^2}{\beta}$	$x_{18} = \frac{1}{\beta}$	$x_{18} = \frac{1}{\beta}$
$x_{19} = \alpha\beta$	$x_{19} = \frac{\beta}{\alpha}$	$x_{19} = \alpha\beta$	$x_{19} = \alpha\beta^3$
$x_{20} = \frac{1}{\alpha^3\beta^2}$	$x_{20} = \frac{\alpha}{\beta^2}$	$x_{20} = \frac{1}{\alpha^3\beta^2}$	$x_{20} = \frac{1}{\alpha\beta^2}$
$x_{21} = \alpha^2\beta$	$x_{21} = \beta$	$x_{21} = \alpha^2\beta$	$x_{21} = \alpha^2\beta^3$
$x_{22} = \frac{1}{\alpha^3\beta}$	$x_{22} = \frac{1}{\alpha\beta}$	$x_{22} = \frac{1}{\alpha^3\beta}$	$x_{22} = \frac{1}{\alpha\beta}$
$x_{23} = \alpha$	$x_{23} = \alpha$	$x_{23} = \alpha$	$x_{23} = \alpha$
$x_{24} = \beta$	$x_{24} = \beta$	$x_{24} = \beta$	$x_{24} = \beta$

Solution.

Case 1:

For;

$$x_{-1} = \alpha \leq 1$$
$$x_0 = \beta \leq 1$$

we have that

$$x_1 = \frac{\max\{x_0^2, 1\}}{x_0^3 x_{-1}} = \frac{\max\{\beta^2, 1\}}{\beta^3 \alpha} = \frac{1}{\alpha\beta^3},$$

$$x_2 = \frac{\max\{x_1^2, 1\}}{x_1^3 x_0} = \frac{\max\left\{\frac{1}{\alpha^2\beta^6}, 1\right\}}{\frac{1}{\alpha^3\beta^9}\beta} = \frac{\frac{1}{\alpha^2\beta^6}}{\frac{1}{\alpha^3\beta^8}} = \alpha\beta^2,$$

$$x_3 = \frac{\max\{x_2^2, 1\}}{x_2^3 x_1} = \frac{\max\{\alpha^2\beta^4, 1\}}{\alpha^3\beta^6\frac{1}{\alpha\beta^3}} = \frac{1}{\alpha^2\beta^3},$$

$$x_4 = \frac{\max\{x_3^2, 1\}}{x_3^3 x_2} = \frac{\max\left\{\frac{1}{\alpha^4\beta^6}, 1\right\}}{\frac{1}{\alpha^6\beta^9}\alpha\beta^2} = \frac{\frac{1}{\alpha^4\beta^6}}{\frac{1}{\alpha^5\beta^7}} = \alpha\beta,$$

$$x_5 = \frac{\max\{x_4^2, 1\}}{x_4^3 x_3} = \frac{\max\{\alpha^2\beta^2, 1\}}{\alpha^3\beta^3\frac{1}{\alpha^2\beta^3}} = \frac{1}{\alpha},$$

$$x_6 = \frac{\max\{x_5^2, 1\}}{x_5^3 x_4} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha^3}\alpha\beta} = \frac{\frac{1}{\alpha^2}}{\frac{\beta}{\alpha^2}} = \frac{1}{\beta},$$

$$x_7 = \frac{\max\{x_6^2, 1\}}{x_6^3 x_5} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta^3}\frac{1}{\alpha}} = \frac{\frac{1}{\beta^2}}{\frac{1}{\alpha\beta^3}} = \alpha\beta,$$

$$x_8 = \frac{\max\{x_7^2, 1\}}{x_7^3 x_6} = \frac{\max\{\alpha^2\beta^2, 1\}}{\alpha^3\beta^3\frac{1}{\beta}} = \frac{1}{\alpha^3\beta^2},$$

$$x_9 = \frac{\max\{x_8^2, 1\}}{x_8^3 x_7} = \frac{\max\left\{\frac{1}{\alpha^6\beta^4}, 1\right\}}{\frac{1}{\alpha^9\beta^6}\alpha\beta} = \frac{\frac{1}{\alpha^6\beta^4}}{\frac{1}{\alpha^8\beta^5}} = \alpha^2\beta,$$

$$x_{10} = \frac{\max\{x_9^2, 1\}}{x_9^3 x_8} = \frac{\max\{\alpha^4\beta^2, 1\}}{\alpha^6\beta^3\frac{1}{\alpha^3\beta^2}} = \frac{1}{\alpha^3\beta},$$

$$x_{11} = \frac{\max\{x_{10}^2, 1\}}{x_{10}^3 x_9} = \frac{\max\left\{\frac{1}{\alpha^6\beta^2}, 1\right\}}{\frac{1}{\alpha^9\beta^3}\alpha^2\beta} = \frac{\frac{1}{\alpha^6\beta^2}}{\frac{1}{\alpha^7\beta^2}} = \alpha,$$

$$x_{12} = \frac{\max\{x_{11}^2, 1\}}{x_{11}^3 x_{10}} = \frac{\max\{\alpha^2, 1\}}{\alpha^3\frac{1}{\alpha^3\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$x_{13}=\frac{\max\left\{x_{12}^2,1\right\}}{x_{12}^3x_{11}}=\frac{\max\left\{\beta^2,1\right\}}{\beta^3\alpha}=\frac{1}{\alpha\beta^3},$$

$$x_{14}=\frac{\max\left\{x_{13}^2,1\right\}}{x_{13}^3x_{12}}=\frac{\max\left\{\frac{1}{\alpha^2\beta^6},1\right\}}{\frac{1}{\alpha^3\beta^9}\beta}=\frac{\frac{1}{\alpha^2\beta^6}}{\frac{1}{\alpha^3\beta^8}}=\alpha\beta^2,$$

$$x_{15}=\frac{\max\left\{x_{14}^2,1\right\}}{x_{14}^3x_{13}}=\frac{\max\left\{\alpha^2\beta^4,1\right\}}{\alpha^3\beta^6\frac{1}{\alpha\beta^3}}=\frac{1}{\alpha^2\beta^3},$$

$$x_{16}=\frac{\max\left\{x_{16}^2,1\right\}}{x_{16}^3x_{15}}=\frac{\max\left\{\frac{1}{\alpha^4\beta^6},1\right\}}{\frac{1}{\alpha^6\beta^9}\alpha\beta^2}=\frac{\frac{1}{\alpha^4\beta^6}}{\frac{1}{\alpha^5\beta^7}}=\alpha\beta,$$

$$x_{17}=\frac{\max\left\{x_{17}^2,1\right\}}{x_{17}^3x_{16}}=\frac{\max\left\{\alpha^2\beta^2,1\right\}}{\alpha^3\beta^3\frac{1}{\alpha^2\beta^3}}=\frac{1}{\alpha},$$

$$x_{18}=\frac{\max\left\{x_{17}^2,1\right\}}{x_{17}^3x_{16}}=\frac{\max\left\{\frac{1}{\alpha^2},1\right\}}{\frac{1}{\alpha^3}\alpha\beta}=\frac{\frac{1}{\alpha^2}}{\frac{\beta}{\alpha^2}}=\frac{1}{\beta},$$

$$x_{19}=\frac{\max\left\{x_{18}^2,1\right\}}{x_{18}^3x_{17}}=\frac{\max\left\{\frac{1}{\beta^2},1\right\}}{\frac{1}{\beta^3}\frac{1}{\alpha}}=\frac{\frac{1}{\beta^2}}{\frac{1}{\alpha\beta^3}}=\alpha\beta,$$

$$x_{20}=\frac{\max\left\{x_{19}^2,1\right\}}{x_{19}^3x_{18}}=\frac{\max\left\{\alpha^2\beta^2,1\right\}}{\alpha^3\beta^3\frac{1}{\beta}}=\frac{1}{\alpha^3\beta^2},$$

$$x_{21}=\frac{\max\left\{x_{20}^2,1\right\}}{x_{20}^3x_{19}}=\frac{\max\left\{\frac{1}{\alpha^6\beta^4},1\right\}}{\frac{1}{\alpha^9\beta^6}\alpha\beta}=\frac{\frac{1}{\alpha^6\beta^4}}{\frac{1}{\alpha^8\beta^5}}=\alpha^2\beta,$$

$$x_{22}=\frac{\max\left\{x_{21}^2,1\right\}}{x_{21}^3x_{20}}=\frac{\max\left\{\alpha^4\beta^2,1\right\}}{\alpha^6\beta^3\frac{1}{\alpha^3\beta^2}}=\frac{1}{\alpha^3\beta},$$

$$x_{23}=\frac{\max\left\{x_{22}^2,1\right\}}{x_{22}^3x_{21}}=\frac{\max\left\{\frac{1}{\alpha^6\beta^2},1\right\}}{\frac{1}{\alpha^9\beta^3}\alpha^2\beta}=\frac{\frac{1}{\alpha^6\beta^2}}{\frac{1}{\alpha^7\beta^2}}=\alpha,$$

$$x_{24}=\frac{\max\left\{x_{23}^2,1\right\}}{x_{23}^3x_{22}}=\frac{\max\left\{\alpha^2,1\right\}}{\alpha^3\frac{1}{\alpha^3\beta}}=\frac{1}{\frac{1}{\beta}}=\beta,$$

$$\vdots$$

Case 2:

For;

$$x_{-1}=\alpha\geq 1$$
$$x_0=\beta\leq 1$$

we get that

$$x_1=\frac{\max\left\{x_0^2,1\right\}}{x_0^3x_{-1}}=\frac{\max\left\{\beta^2,1\right\}}{\beta^3\alpha}=\frac{1}{\alpha\beta^3},$$

$$x_2 = \frac{\max\{x_1^2, 1\}}{x_1^3 x_0} = \frac{\max\left\{\frac{1}{\alpha^2\beta^6}, 1\right\}}{\frac{1}{\alpha^3\beta^9}\beta} = \frac{\frac{1}{\alpha^2\beta^6}}{\frac{1}{\alpha^3\beta^8}} = \alpha\beta^2,$$

$$x_3 = \frac{\max\{x_2^2, 1\}}{x_2^3 x_1} = \frac{\max\left\{\alpha^2\beta^4, 1\right\}}{\alpha^3\beta^6\frac{1}{\alpha\beta^3}} = \frac{1}{\alpha^2\beta^3},$$

$$x_4 = \frac{\max\{x_3^2, 1\}}{x_3^3 x_2} = \frac{\max\left\{\frac{1}{\alpha^4\beta^6}, 1\right\}}{\frac{1}{\alpha^6\beta^9}\alpha\beta^2} = \frac{\frac{1}{\alpha^4\beta^6}}{\frac{1}{\alpha^5\beta^7}} = \alpha\beta,$$

$$x_5 = \frac{\max\{x_4^2, 1\}}{x_4^3 x_3} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha^3\beta^3\frac{1}{\alpha^2\beta^3}} = \frac{1}{\alpha},$$

$$x_6 = \frac{\max\{x_5^2, 1\}}{x_5^3 x_4} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha^3}\alpha\beta} = \frac{1}{\frac{\beta}{\alpha^2}} = \frac{\alpha^2}{\beta},$$

$$x_7 = \frac{\max\{x_6^2, 1\}}{x_6^3 x_5} = \frac{\max\left\{\frac{\alpha^4}{\beta^2}, 1\right\}}{\frac{\alpha^6}{\beta^3}\frac{1}{\alpha}} = \frac{\frac{\alpha^4}{\beta^2}}{\frac{\alpha^5}{\beta^3}} = \frac{\beta}{\alpha},$$

$$x_8 = \frac{\max\{x_7^2, 1\}}{x_7^3 x_6} = \frac{\max\left\{\frac{\beta^2}{\alpha^2}, 1\right\}}{\frac{\beta^3}{\alpha^3}\frac{\alpha^2}{\beta}} = \frac{1}{\frac{\beta^2}{\alpha}} = \frac{\alpha}{\beta^2},$$

$$x_9 = \frac{\max\{x_8^2, 1\}}{x_8^3 x_7} = \frac{\max\left\{\frac{\alpha^2}{\beta^4}, 1\right\}}{\frac{\alpha^3}{\beta^6}\frac{\beta}{\alpha}} = \frac{\frac{\alpha^2}{\beta^4}}{\frac{\alpha^2}{\beta^5}} = \beta,$$

$$x_{10} = \frac{\max\{x_9^2, 1\}}{x_9^3 x_8} = \frac{\max\left\{\beta^2, 1\right\}}{\beta^3\frac{\alpha}{\beta^2}} = \frac{1}{\alpha\beta},$$

$$x_{11} = \frac{\max\{x_{10}^2, 1\}}{x_{10}^3 x_9} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha^3\beta^3}\beta} = \frac{\frac{1}{\alpha^2\beta^2}}{\frac{1}{\alpha^3\beta^2}} = \alpha,$$

$$x_{12} = \frac{\max\{x_{11}^2, 1\}}{x_{11}^3 x_{10}} = \frac{\max\{\alpha^2, 1\}}{\alpha^3\frac{1}{\alpha\beta}} = \frac{\alpha^2}{\frac{\alpha^2}{\beta}} = \beta,$$

$$x_{13} = \frac{\max\{x_{12}^2, 1\}}{x_{12}^3 x_{11}} = \frac{\max\left\{\beta^2, 1\right\}}{\beta^3\alpha} = \frac{1}{\alpha\beta^3},$$

$$x_{14} = \frac{\max\{x_{13}^2, 1\}}{x_{13}^3 x_{12}} = \frac{\max\left\{\frac{1}{\alpha^2\beta^6}, 1\right\}}{\frac{1}{\alpha^3\beta^9}\beta} = \frac{\frac{1}{\alpha^2\beta^6}}{\frac{1}{\alpha^3\beta^8}} = \alpha\beta^2,$$

$$x_{15} = \frac{\max\{x_{14}^2, 1\}}{x_{14}^3 x_{13}} = \frac{\max\left\{\alpha^2\beta^4, 1\right\}}{\alpha^3\beta^6\frac{1}{\alpha\beta^3}} = \frac{1}{\alpha^2\beta^3},$$

$$x_{16} = \frac{\max\{x_{16}^2, 1\}}{x_{16}^3 x_{15}} = \frac{\max\left\{\frac{1}{\alpha^4\beta^6}, 1\right\}}{\frac{1}{\alpha^6\beta^9}\alpha\beta^2} = \frac{\frac{1}{\alpha^4\beta^6}}{\frac{1}{\alpha^5\beta^7}} = \alpha\beta,$$

$$x_{17} = \frac{\max\{x_{17}^2, 1\}}{x_{17}^3 x_{16}} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha^3\beta^3\frac{1}{\alpha^2\beta^3}} = \frac{1}{\alpha},$$

$$x_{18}=\frac{\max\{x_{17}^2,1\}}{x_{17}^3x_{16}}=\frac{\max\left\{\frac{1}{\alpha^2},1\right\}}{\frac{1}{\alpha^3}\alpha\beta}=\frac{1}{\frac{\beta}{\alpha^2}}=\frac{\alpha^2}{\beta},$$

$$x_{19}=\frac{\max\{x_{18}^2,1\}}{x_{18}^3x_{17}}=\frac{\max\left\{\frac{\alpha^4}{\beta^2},1\right\}}{\frac{\alpha^6}{\beta^3}\frac{1}{\alpha}}=\frac{\frac{\alpha^4}{\beta^2}}{\frac{\alpha^5}{\beta^3}}=\frac{\beta}{\alpha},$$

$$x_{20}=\frac{\max\{x_{19}^2,1\}}{x_{19}^3x_{18}}=\frac{\max\left\{\frac{\beta^2}{\alpha^2},1\right\}}{\frac{\beta^3}{\alpha^3}\frac{\alpha^2}{\beta}}=\frac{1}{\frac{\beta^2}{\alpha}}=\frac{\alpha}{\beta^2},$$

$$x_{21}=\frac{\max\{x_{20}^2,1\}}{x_{20}^3x_{19}}=\frac{\max\left\{\frac{\alpha^2}{\beta^4},1\right\}}{\frac{\alpha^3}{\beta^6}\frac{\beta}{\alpha}}=\frac{\frac{\alpha^2}{\beta^4}}{\frac{\alpha^2}{\beta^5}}=\beta,$$

$$x_{22}=\frac{\max\{x_{21}^2,1\}}{x_{21}^3x_{20}}=\frac{\max\left\{\beta^2,1\right\}}{\beta^3\frac{\alpha}{\beta^2}}=\frac{1}{\alpha\beta},$$

$$x_{23}=\frac{\max\{x_{22}^2,1\}}{x_{22}^3x_{21}}=\frac{\max\left\{\frac{1}{\alpha^2\beta^2},1\right\}}{\frac{1}{\alpha^3\beta^3}\beta}=\frac{\frac{1}{\alpha^2\beta^2}}{\frac{1}{\alpha^3\beta^2}}=\alpha,$$

$$x_{24}=\frac{\max\{x_{23}^2,1\}}{x_{23}^3x_{22}}=\frac{\max\{\alpha^2,1\}}{\alpha^3\frac{1}{\alpha\beta}}=\frac{\alpha^2}{\frac{\alpha^2}{\beta}}=\beta,$$

$$\vdots$$

Case 3:

For

$$x_{-1}=\alpha\le 1$$
$$x_0=\beta\ge 1$$

we get that

$$x_1=\frac{\max\{x_0^2,1\}}{x_0^3x_{-1}}=\frac{\max\left\{\beta^2,1\right\}}{\beta^3\alpha}=\frac{\beta^2}{\alpha\beta^3}=\frac{1}{\alpha\beta},$$

$$x_2=\frac{\max\{x_1^2,1\}}{x_1^3x_0}=\frac{\max\left\{\frac{1}{\alpha^2\beta^2},1\right\}}{\frac{1}{\alpha^3\beta^3}\beta}=\frac{\frac{1}{\alpha^2\beta^2}}{\frac{1}{\alpha^3\beta^2}}=\alpha,$$

$$x_3=\frac{\max\{x_2^2,1\}}{x_2^3x_1}=\frac{\max\{\alpha^2,1\}}{\alpha^3\frac{1}{\alpha\beta}}=\frac{1}{\frac{\alpha^2}{\beta}}=\frac{\beta}{\alpha^2},$$

$$x_4=\frac{\max\{x_3^2,1\}}{x_3^3x_2}=\frac{\max\left\{\frac{\alpha^2}{\beta^2},1\right\}}{\frac{\alpha^3}{\beta^3}\frac{\beta}{\alpha^2}}=\frac{\frac{\beta^2}{\alpha^4}}{\frac{\beta^3}{\alpha^5}}=\frac{\alpha}{\beta},$$

$$x_5=\frac{\max\{x_4^2,1\}}{x_4^3x_3}=\frac{\max\left\{\alpha^2\beta^2,1\right\}}{\alpha^3\beta^3\frac{1}{\alpha^2\beta^3}}=\frac{\beta^2}{\alpha},$$

$$x_6 = \frac{\max\{x_5^2, 1\}}{x_5^3 x_4} = \frac{\max\left\{\frac{\beta^4}{\alpha^2}, 1\right\}}{\frac{\beta^6}{\alpha^3}\frac{\alpha}{\beta}} = \frac{\frac{\beta^4}{\alpha^2}}{\frac{\beta^5}{\alpha^2}} = \frac{1}{\beta},$$

$$x_7 = \frac{\max\{x_6^2, 1\}}{x_6^3 x_5} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta^3}\frac{\beta^2}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_8 = \frac{\max\{x_7^2, 1\}}{x_7^3 x_6} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha^3\beta^3\frac{1}{\beta}} = \frac{1}{\alpha^3\beta^2},$$

$$x_9 = \frac{\max\{x_8^2, 1\}}{x_8^3 x_7} = \frac{\max\left\{\frac{1}{\alpha^6\beta^4}, 1\right\}}{\frac{1}{\alpha^9\beta^6}\alpha\beta} = \frac{\frac{1}{\alpha^6\beta^4}}{\frac{1}{\alpha^8\beta^5}} = \alpha^2\beta,$$

$$x_{10} = \frac{\max\{x_9^2, 1\}}{x_9^3 x_8} = \frac{\max\left\{\alpha^4\beta^2, 1\right\}}{\alpha^6\beta^3\frac{1}{\alpha^3\beta^2}} = \frac{1}{\alpha^3\beta},$$

$$x_{11} = \frac{\max\{x_{10}^2, 1\}}{x_{10}^3 x_9} = \frac{\max\left\{\frac{1}{\alpha^6\beta^2}, 1\right\}}{\frac{1}{\alpha^9\beta^3}\alpha^2\beta} = \frac{\frac{1}{\alpha^6\beta^2}}{\frac{1}{\alpha^7\beta^2}} = \alpha,$$

$$x_{12} = \frac{\max\{x_{11}^2, 1\}}{x_{11}^3 x_{10}} = \frac{\max\{\alpha^2, 1\}}{\alpha^3\frac{1}{\alpha^3\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$x_{13} = \frac{\max\{x_{12}^2, 1\}}{x_{12}^3 x_{11}} = \frac{\max\left\{\beta^2, 1\right\}}{\beta^3\alpha} = \frac{\beta^2}{\alpha\beta^3} = \frac{1}{\alpha\beta},$$

$$x_{14} = \frac{\max\{x_{13}^2, 1\}}{x_{13}^3 x_{12}} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha^3\beta^3}\beta} = \frac{\frac{1}{\alpha^2\beta^2}}{\frac{1}{\alpha^3\beta^2}} = \alpha,$$

$$x_{15} = \frac{\max\{x_{14}^2, 1\}}{x_{14}^3 x_{13}} = \frac{\max\{\alpha^2, 1\}}{\alpha^3\frac{1}{\alpha\beta}} = \frac{1}{\frac{\alpha^2}{\beta}} = \frac{\beta}{\alpha^2},$$

$$x_{16} = \frac{\max\{x_{16}^2, 1\}}{x_{16}^3 x_{15}} = \frac{\max\left\{\frac{\alpha^2}{\beta^2}, 1\right\}}{\frac{\alpha^3}{\beta^3}\frac{\beta}{\alpha^2}} = \frac{\frac{\beta^2}{\alpha^4}}{\frac{\beta^3}{\alpha^5}} = \frac{\alpha}{\beta},$$

$$x_{17} = \frac{\max\{x_{17}^2, 1\}}{x_{17}^3 x_{16}} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha^3\beta^3\frac{1}{\alpha^2\beta^3}} = \frac{\beta^2}{\alpha},$$

$$x_{18} = \frac{\max\{x_{17}^2, 1\}}{x_{17}^3 x_{16}} = \frac{\max\left\{\frac{\beta^4}{\alpha^2}, 1\right\}}{\frac{\beta^6}{\alpha^3}\frac{\alpha}{\beta}} = \frac{\frac{\beta^4}{\alpha^2}}{\frac{\beta^5}{\alpha^2}} = \frac{1}{\beta},$$

$$x_{19} = \frac{\max\{x_{18}^2, 1\}}{x_{18}^3 x_{17}} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta^3}\frac{\beta^2}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta}} = \alpha\beta,$$

$$x_{20} = \frac{\max\{x_{19}^2, 1\}}{x_{19}^3 x_{18}} = \frac{\max\left\{\alpha^2\beta^2, 1\right\}}{\alpha^3\beta^3\frac{1}{\beta}} = \frac{1}{\alpha^3\beta^2},$$

$$x_{21} = \frac{\max\{x_{20}^2, 1\}}{x_{20}^3 x_{19}} = \frac{\max\left\{\frac{1}{\alpha^6\beta^4}, 1\right\}}{\frac{1}{\alpha^9\beta^6}\alpha\beta} = \frac{\frac{1}{\alpha^6\beta^4}}{\frac{1}{\alpha^8\beta^5}} = \alpha^2\beta,$$

$$x_{22} = \frac{\max\{x_{21}^2, 1\}}{x_{21}^3 x_{20}} = \frac{\max\{\alpha^4\beta^2, 1\}}{\alpha^6\beta^3\frac{1}{\alpha^3\beta^2}} = \frac{1}{\alpha^3\beta},$$

$$x_{23} = \frac{\max\{x_{22}^2, 1\}}{x_{22}^3 x_{21}} = \frac{\max\left\{\frac{1}{\alpha^6\beta^2}, 1\right\}}{\frac{1}{\alpha^9\beta^3}\alpha^2\beta} = \frac{\frac{1}{\alpha^6\beta^2}}{\frac{1}{\alpha^7\beta^2}} = \alpha,$$

$$x_{24} = \frac{\max\{x_{23}^2, 1\}}{x_{23}^3 x_{22}} = \frac{\max\{\alpha^2, 1\}}{\alpha^3\frac{1}{\alpha^3\beta}} = \frac{1}{\frac{1}{\beta}} = \beta,$$

$$\vdots$$

Case 4:

When

$$x_{-1} = \alpha \geq 1$$
$$x_0 = \beta \geq 1$$

we have that

$$x_1 = \frac{\max\{x_0^2, 1\}}{x_0^3 x_{-1}} = \frac{\max\{\beta^2, 1\}}{\beta^3\alpha} = \frac{\beta^2}{\alpha\beta^3} = \frac{1}{\alpha\beta},$$

$$x_2 = \frac{\max\{x_1^2, 1\}}{x_1^3 x_0} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha^3\beta^3}\beta} = \frac{1}{\frac{1}{\alpha^3\beta^2}} = \alpha^3\beta^2,$$

$$x_3 = \frac{\max\{x_2^2, 1\}}{x_2^3 x_1} = \frac{\max\{\alpha^6\beta^4, 1\}}{\alpha^9\beta^6\frac{1}{\alpha\beta}} = \frac{1}{\alpha^2\beta},$$

$$x_4 = \frac{\max\{x_3^2, 1\}}{x_3^3 x_2} = \frac{\max\left\{\frac{1}{\alpha^4\beta^2}, 1\right\}}{\frac{1}{\alpha^6\beta^3}\alpha^3\beta^2} = \frac{1}{\frac{1}{\alpha^5\beta}} = \alpha^3\beta,$$

$$x_5 = \frac{\max\{x_4^2, 1\}}{x_4^3 x_3} = \frac{\max\{\alpha^6\beta^2, 1\}}{\alpha^9\beta^3\frac{1}{\alpha^2\beta}} = \frac{\alpha^6\beta^2}{\alpha^7\beta^2} = \frac{1}{\alpha},$$

$$x_6 = \frac{\max\{x_5^2, 1\}}{x_5^3 x_4} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha^3}\alpha^3\beta} = \frac{1}{\beta},$$

$$x_7 = \frac{\max\{x_6^2, 1\}}{x_6^3 x_5} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta^3}\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta^3}} = \alpha\beta^3,$$

$$x_8 = \frac{\max\{x_7^2, 1\}}{x_7^3 x_6} = \frac{\max\{\alpha^2\beta^6, 1\}}{\alpha^3\beta^9\frac{1}{\beta}} = \frac{\alpha^2\beta^6}{\alpha^3\beta^8} = \frac{1}{\alpha\beta^2},$$

$$x_9 = \frac{\max\{x_8^2, 1\}}{x_8^3 x_7} = \frac{\max\left\{\frac{1}{\alpha^2\beta^4}, 1\right\}}{\frac{1}{\alpha^3\beta^6}\alpha\beta^3} = \frac{1}{\frac{1}{\alpha^2\beta^5}} = \alpha^2\beta^3,$$

$$x_{10} = \frac{\max\{x_9^2, 1\}}{x_9^3 x_8} = \frac{\max\{\alpha^4\beta^6, 1\}}{\alpha^6\beta^9\frac{1}{\alpha\beta^2}} = \frac{\alpha^4\beta^6}{\alpha^5\beta^7} = \frac{1}{\alpha\beta},$$

$$x_{11} = \frac{\max\{x_{10}^2, 1\}}{x_{10}^3 x_9} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha^3\beta^3}\alpha^2\beta^3} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_{12} = \frac{\max\{x_{11}^2, 1\}}{x_{11}^3 x_{10}} = \frac{\max\{\alpha^2, 1\}}{\alpha^3\frac{1}{\alpha\beta}} = \frac{\alpha^2}{\frac{\alpha^2}{\beta}} = \beta,$$

$$x_{13} = \frac{\max\{x_{12}^2, 1\}}{x_{12}^3 x_{11}} = \frac{\max\{\beta^2, 1\}}{\beta^3\alpha} = \frac{\beta^2}{\alpha\beta^3} = \frac{1}{\alpha\beta},$$

$$x_{14} = \frac{\max\{x_{13}^2, 1\}}{x_{13}^3 x_{12}} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha^3\beta^3}\beta} = \frac{1}{\frac{1}{\alpha^3\beta^2}} = \alpha^3\beta^2,$$

$$x_{15} = \frac{\max\{x_{14}^2, 1\}}{x_{14}^3 x_{13}} = \frac{\max\{\alpha^6\beta^4, 1\}}{\alpha^9\beta^6\frac{1}{\alpha\beta}} = \frac{1}{\alpha^2\beta},$$

$$x_{16} = \frac{\max\{x_{16}^2, 1\}}{x_{16}^3 x_{15}} = \frac{\max\left\{\frac{1}{\alpha^4\beta^2}, 1\right\}}{\frac{1}{\alpha^6\beta^3}\alpha^3\beta^2} = \frac{1}{\frac{1}{\alpha^5\beta}} = \alpha^3\beta,$$

$$x_{17} = \frac{\max\{x_{17}^2, 1\}}{x_{17}^3 x_{16}} = \frac{\max\{\alpha^6\beta^2, 1\}}{\alpha^9\beta^3\frac{1}{\alpha^2\beta}} = \frac{\alpha^6\beta^2}{\alpha^7\beta^2} = \frac{1}{\alpha},$$

$$x_{18} = \frac{\max\{x_{17}^2, 1\}}{x_{17}^3 x_{16}} = \frac{\max\left\{\frac{1}{\alpha^2}, 1\right\}}{\frac{1}{\alpha^3}\alpha^3\beta} = \frac{1}{\beta},$$

$$x_{19} = \frac{\max\{x_{18}^2, 1\}}{x_{18}^3 x_{17}} = \frac{\max\left\{\frac{1}{\beta^2}, 1\right\}}{\frac{1}{\beta^3}\frac{1}{\alpha}} = \frac{1}{\frac{1}{\alpha\beta^3}} = \alpha\beta^3,$$

$$x_{20} = \frac{\max\{x_{19}^2, 1\}}{x_{19}^3 x_{18}} = \frac{\max\{\alpha^2\beta^6, 1\}}{\alpha^3\beta^9\frac{1}{\beta}} = \frac{\alpha^2\beta^6}{\alpha^3\beta^8} = \frac{1}{\alpha\beta^2},$$

$$x_{21} = \frac{\max\{x_{20}^2, 1\}}{x_{20}^3 x_{19}} = \frac{\max\left\{\frac{1}{\alpha^2\beta^4}, 1\right\}}{\frac{1}{\alpha^3\beta^6}\alpha\beta^3} = \frac{1}{\frac{1}{\alpha^2\beta^5}} = \alpha^2\beta^3,$$

$$x_{22} = \frac{\max\{x_{21}^2, 1\}}{x_{21}^3 x_{20}} = \frac{\max\{\alpha^4\beta^6, 1\}}{\alpha^6\beta^9\frac{1}{\alpha\beta^2}} = \frac{\alpha^4\beta^6}{\alpha^5\beta^7} = \frac{1}{\alpha\beta},$$

$$x_{23} = \frac{\max\{x_{22}^2, 1\}}{x_{22}^3 x_{21}} = \frac{\max\left\{\frac{1}{\alpha^2\beta^2}, 1\right\}}{\frac{1}{\alpha^3\beta^3}\alpha^2\beta^3} = \frac{1}{\frac{1}{\alpha}} = \alpha,$$

$$x_{24} = \frac{\max\{x_{23}^2, 1\}}{x_{23}^3 x_{22}} = \frac{\max\{\alpha^2, 1\}}{\alpha^3\frac{1}{\alpha\beta}} = \frac{\alpha^2}{\frac{\alpha^2}{\beta}} = \beta,$$

$$\vdots$$

CHAPTER 4

SOME SPECIAL-TYPE DIFFERENCE EQUATIONS WITH PERIODIC SOLUTIONS

In this chapter we use [**7**] to supply some special-type difference equations with periodic solutions. Those examples show us enrichment and appreciation of the fascinating world of difference equations and their richness in periodicities.

4.1 LYNESS' EQUATION

Lyness' equation is defined by

$$x_{n+1} = \frac{a + x_n}{x_{n-1}}, \quad n = 0, 1, , , \tag{4.1}$$

and was introduced by Lyness in 1942, while he was working on a problem in number theory.

As we mentioned in Chapter 2, every positive solution of equation

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, , , \tag{4.2}$$

which is a special case of Lyness' equation is periodic with period 5. Actually if

$$x_{-1} = \alpha \ \text{ and } \ x_0 = \beta$$

are positive initial conditions, then the solution $\{x_n\}_{n=-1}^{\infty}$ is the period-5 sequence

$$\alpha, \beta, \frac{1+\beta}{\alpha}, \frac{1+\alpha+\beta}{\alpha\beta}, \frac{1+\alpha}{\beta}, ... \tag{4.3}$$

Remark 4.1 *It is a fascinating fact that Eqs. (3.1) and (2.5) have great similarities. The solution of Eq. (3.1) with positive initial conditions*

$$x_{-1} = \alpha \ \text{ and } x_0 = \beta$$

*is the period-*5 *sequence*

$$\alpha, \beta, \frac{\max\{1,\beta\}}{\alpha}, \frac{\max\{1,\alpha,\beta\}}{\alpha\beta}, \frac{\max\{1,\alpha\}}{\beta}, ...$$

*Compare this with the period-*5 *solution in Eq. (4.3).*

4.1.1 FRIZE PATTERNS

An infinite strip with a repeating pattern is called a frieze pattern, or sometimes a border pattern or an infinite strip pattern. The term "frieze" is from architecture, where a frieze refers to a decorative carving or pattern that runs horizontally just below a roofline or ceiling.

Since the patterns repeat, we show only a finite portion, but you should keep in mind that these pictures should extend infinitely far in both directions. Although most of the frieze patterns in most book will be horizontal, there's no reason a frieze pattern cannot be vertical, or even set at an angle.

Eq. (4.2) arises in frize patterns, see [4] and [3].

The idea of a frieze pattern is most quickly carried by means of an example, such as the following pattern of order 7 :

0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 ...

... 1 2 2 3 1 2 4 1 2 2 3

1 3 5 2 1 7 3 1 3 5 ...

... 2 1 7 3 1 3 5 2 1 7 3

1 2 4 1 2 2 3 1 2 4 ...

... 1 1 1 1 1 1 1 1 1 1 1

0 0 0 0 0 0 0 0 0 0 ...

Another example of a frieze pattern is the following basic one.

... 1 1 1 1 1 ...

... 1 3 1 3 ...

... 2 2 2 2 2 ...

... 3 1 3 1 ...

The property which defines a frieze pattern is that except for possible borders of zeros and ones, every four adjacent numbers forming a rhombus. In other words, apart from the borders of zeros and ones, the essential property is that every four adjacent numbers forming a square

$$\begin{array}{ccc} & q & \\ p & & r \\ & s & \end{array}$$

are positive and satisfy the unimodular equation

$pr - qs = 1.$

Moreover, the insist that all the numbers (except the borders of zeros) shall be positive. The astounding consequence is that every such pattern is periodic. More precisely, it is symmetrical by a glide: the product of a horizontal translation and a horizontal reflection. Coxeter, see [4], has shown that every frieze pattern is periodic. For instance, the following frieze pattern is periodic with period 5:

$$\begin{array}{ccccccccccccc}
... & 1 & & 1 & & 1 & & 1 & & 1 & & & ... \\
... & & x_1 & & x_3 & & x_5 & & x_2 & & x_4 & & ... \\
... & x_5 & & x_2 & & x_4 & & x_1 & & x_3 & & & ... \\
... & & 1 & & 1 & & 1 & & 1 & & 1 & & ...
\end{array}$$

If $x_1 = \alpha$ and $x_2 = \beta$ are arbitrary positive numbers, then from the definiton of frieze patterns it follows that

$$x_3 = \frac{1+\beta}{\alpha}, \ x_4 = \frac{1+\alpha+\beta}{\alpha\beta}, \quad \text{and } x_5 = \frac{1+\alpha}{\beta}$$

because

$$\begin{aligned}
x_1x_3 - x_2 &= 1 \Rightarrow \alpha x_3 - \beta = 1 \Rightarrow x_3 = \frac{1+\beta}{\alpha}, \\
x_2x_4 - x_3 &= 1 \Rightarrow \beta x_4 - \frac{1+\beta}{\alpha} = 1 \Rightarrow x_4 = \frac{1+\alpha+\beta}{\alpha\beta}, \\
x_4x_1 - x_5 &= 1 \Rightarrow \frac{1+\alpha+\beta}{\alpha\beta}\alpha - x_5 = 1 \Rightarrow x_5 = \frac{1+\alpha}{\beta}.
\end{aligned}$$

Therefore, the above pattern is generated by Eq. (4.2).

4.2 TODD'S EQUATION

Eq. (2.6), that is, the equation

$$x_{n+1} = \tfrac{1+x_n+x_{n-1}}{x_{n-2}}, \quad n = 0, 1, ... \tag{4.4}$$

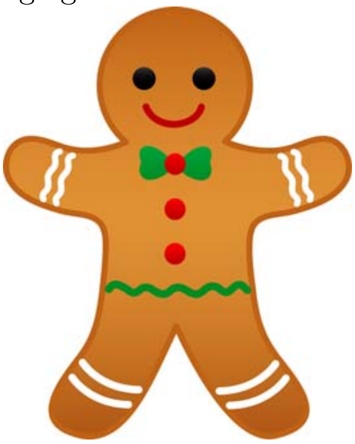
is called Todd's Equation. As we saw in Example 2.6, every positive solution of Eq. (4.4) is periodic with period 8.

4.3 THE GINGERBREADMAN EQUATION

The gingerbreadman difference equation is linear difference equation

$$x_{n+1} = |x_n| - x_{n-1} + 1, \quad n = 0, 1, ... \tag{4.5}$$

which is piecewise. It was investigated by Devaney, see [5]. The name of this equation is due to the fact that the orbits of certain points in the plane fill a region that looks like a "gingerbreadman." As an example, we illustrate the following picture:



If someone uses a computer to plot the orbit of the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (4.5) with initial conditions

$$(x_{-1}, x_0) = \left(-\frac{1}{10}, 0\right)$$

the computer may predict that after 100.000 iterations, the solution is still not periodic. See [24]. Although a computer may be fooled due to round-off and truncation errors, one can show that the orbit of the solution of Eq. (4.5) with initial condition

$$(x_{-1}, x_0) = \left(-\frac{1}{10}, 0\right)$$

is periodic with period 126. An easy way to see this is to make the substitution

$$x_n = \frac{1}{10} y_n.$$

Then Eq. (4.5) is transformed into the difference equation

$$y_{n+1} = |y_n| - y_{n-1} + 10, \quad n = 0, 1, ... \tag{4.6}$$

and the initial conditions $(x_{-1}, x_0) = \left(-\frac{1}{10}, 0\right)$ of the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (4.5) are transformed into

$$(y_{-1}, y_0) = (-1, 0).$$

Let $\{y_n\}_{n=-1}^{\infty}$ be the solution of Eq. (4.6) with initial conditions $(y_{-1}, y_0) = (-1, 0)$. Then the values of y_n for $-1 \leq n \leq 126$ are given as follows:

$y_{-1} = -1$	$y_0 = 0$					
$y_1 = 11$	$y_2 = 21$	$y_3 = 20$	$y_4 = 9$	$y_5 = -1$	$y_6 = 2$	$y_7 = 13$
$y_8 = 21$	$y_9 = 18$	$y_{10} = 7$	$y_{11} = -1$	$y_{12} = 4$	$y_{13} = 15$	$y_{14} = 21$
$y_{15} = 16$	$y_{16} = 5$	$y_{17} = -1$	$y_{18} = 6$	$y_{19} = 17$	$y_{20} = 21$	$y_{21} = 14$
$y_{22} = 3$	$y_{23} = -1$	$y_{24} = 8$	$y_{25} = 19$	$y_{26} = 21$	$y_{27} = 12$	$y_{28} = 1$
$y_{29} = -1$	$y_{30} = 10$	$y_{31} = 21$	$y_{32} = 21$	$y_{33} = 10$	$y_{34} = -1$	$y_{35} = 1$
$y_{36} = 12$	$y_{37} = 21$	$y_{38} = 19$	$y_{39} = 8$	$y_{40} = -1$	$y_{41} = 3$	$y_{42} = 14$
$y_{43} = 21$	$y_{44} = 17$	$y_{45} = 6$	$y_{46} = -1$	$y_{47} = 5$	$y_{48} = 16$	$y_{49} = 21$
$y_{50} = 15$	$y_{51} = 4$	$y_{52} = -1$	$y_{53} = 7$	$y_{54} = 18$	$y_{55} = 21$	$y_{56} = 13$
$y_{57} = 2$	$y_{58} = -1$	$y_{59} = 9$	$y_{60} = 20$	$y_{61} = 21$	$y_{62} = 11$	$y_{63} = 0$
$y_{64} = -1$	$y_{65} = 11$	$y_{66} = 22$	$y_{67} = 21$	$y_{68} = 9$	$y_{69} = -2$	$y_{70} = 3$
$y_{71} = 15$	$y_{72} = 22$	$y_{73} = 17$	$y_{74} = 5$	$y_{75} = -2$	$y_{76} = 7$	$y_{77} = 19$
$y_{78} = 22$	$y_{79} = 13$	$y_{80} = 1$	$y_{81} = -2$	$y_{82} = 11$	$y_{83} = 23$	$y_{84} = 22$
$y_{85} = 9$	$y_{86} = -3$	$y_{87} = 4$	$y_{88} = 17$	$y_{89} = 23$	$y_{90} = 16$	$y_{91} = 3$
$y_{92} = -3$	$y_{93} = 10$	$y_{94} = 23$	$y_{95} = 23$	$y_{96} = 10$	$y_{97} = -3$	$y_{98} = 3$
$y_{99} = 16$	$y_{100} = 23$	$y_{101} = 17$	$y_{102} = 4$	$y_{103} = -3$	$y_{104} = 9$	$y_{105} = 22$
$y_{106} = 23$	$y_{107} = 11$	$y_{108} = -2$	$y_{109} = 1$	$y_{110} = 13$	$y_{111} = 22$	$y_{112} = 19$
$y_{113} = 7$	$y_{114} = -2$	$y_{115} = 5$	$y_{116} = 17$	$y_{117} = 22$	$y_{118} = 15$	$y_{119} = 3$
$y_{120} = -2$	$y_{121} = 9$	$y_{122} = 21$	$y_{123} = 22$	$y_{124} = 11$		
$y_{125} = -1$	$y_{126} = 0.$					

So, the sequence $\{y_n\}_{n=-1}^{\infty}$ (and therefore also $\{x_n\}_{n=-1}^{\infty}$) is periodic with prime period 126.

It is interesting to note that the gingerbreadman difference equation is a special case of the max difference equation

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}}, \quad n = 0, 1, ... \tag{4.7}$$

Indeed the change of variables, see [16],

$$x_n = \begin{cases} A^{\frac{1+y_n}{2}} & \text{if} \quad A > 1 \\ e^{\frac{y_n}{2}} & \text{if} \quad A = 1 \\ A^{\frac{1-y_n}{2}} & \text{if} \quad 0 < A < 1 \end{cases}$$

reduces Eq. (4.7) to the piecewise linear difference equation

$$y_{n+1} = |y_n| - y_{n-1} + \delta, \quad n = 0, 1, ...$$

where

$$\delta = \begin{cases} -1 & \text{if} \quad A > 1 \\ 0 & \text{if} \quad A = 1 \\ 1 & \text{if} \quad A < 1. \end{cases}$$

To see this, note that if $\alpha, \beta \in \mathbb{R}$, then

$$\min\{\alpha, \beta\} = \frac{1}{2}(\alpha + \beta - |\alpha - \beta|) \text{ and } \max\{\alpha, \beta\} = \frac{1}{2}(\alpha + \beta + |\alpha - \beta|).$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (4.7) and assume that $0 < A < 1$. Then

$$\begin{aligned} A^{\frac{1-y_{n+1}}{2}} &= \frac{\max\{A^{1-y_n}, A\}}{A^{\frac{2-y_n-y_{n-1}}{2}}} = \frac{A^{\min\{1-y_n,1\}}}{A^{\frac{2-y_n-y_{n-1}}{2}}} = \frac{A^{\frac{1}{2}(2-y_n-|y_n|)}}{A^{\frac{2-y_n-y_{n-1}}{2}}} \\ &= A^{\frac{1}{2}(-|y_n|+y_{n-1})} \end{aligned}$$

and thus

$$y_{n+1} = |y_n| - y_{n-1} + 1.$$

The proof in the other cases can be proved similarly.

Observe that Eq. (4.7) with

$$A \in (0, 1)$$

reduces to the gingerbreadman difference equation Eq. (4.5).

When

$$A = 1$$

Eq. (4.7) reduces to Eq. (3.4) which by the above change of variables is transformed into the piecewise linear difference equation

$$y_{n+1} = |y_n| - y_{n-1}, \quad n = 0, 1, ... \tag{4.8}$$

Therefore every solution of Eq. (4.8) is periodic with period 9.

4.4 THE GENERALİZED LOZI EQUATION

Lozi's map is the system of difference equations

$$\begin{cases} x_{n+1} = 1 - a\left|x_n\right| + y_n \\ \qquad\qquad\qquad\qquad , \quad n = 0, 1, ... \\ y_{n+1} = bx_n \end{cases}$$

introduced by Lozi, see [21], in 1978 as a piecewise linear analogue of the Henon map

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + y_n \\ \qquad\qquad\qquad\qquad , \quad n = 0, 1, ... \\ y_{n+1} = bx_n \end{cases}$$

The Henon map was introduced by the theoretical astronomer Henon, see [9], in 1976 to illuminate the strange attractor which was observed by the meteorologist Lorenz, see [20], in 1963 in the simple-looking non-linear system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 10(y - x) \\ \dfrac{dy}{dt} = x(28 - z) - y \\ \dfrac{dz}{dt} = xy - \dfrac{8}{3}z \end{cases}$$

which Lorenz used in his research to model weather patterns.

When Lorenz used Euler's method to integrate numerically in his royal-McBee LGP-30 computer, the solutions of this system exhibited extremely complicated behavior. The solutions exhibited sensitive dependence upon initial conditions about which Lorenz coined the phrase butterfly effect. If a butterfly flaps its wings in Tokyo, Japan, this may cause it to rain in Kingston, Rhode Island four days later. This is bad news for numerical methods, and means that we should be suspicious of what we "see in the computer" until we set up it by a rigorous proof.

The solutions oscillate irregularly, never exactly repeating but always remaining in a bounded region in the (x, y, z) space, and they settle onto a complicated set resembling an owl's mask or a pair of butterfly wings, which we now call a *strange attractor,* strange because its boundary is a fractal (with dimension between 2 and 3). All solutions approach the attractor quite rapidly, and there are no periodic or asymptotically periodic solutions. The term *strange attractor* was coined by Ruelle and Takens, see [25], in 1971.

By eliminating the variable y_n, Lozi's map reduces to the second-order piecewise linear difference equation

$$x_{n+1} = 1 - a\,|x_n| + bx_{n-1}, \quad n = 0, 1, ... \tag{4.9}$$

where a and b are real numbers.

Several of the equations which has been recently investigated, and which exhibit an interesting periodic character, are of the form

$$x_{n+1} = \frac{\max\left\{x_n^k, A\right\}}{x_n^l x_{n-1}^m}, \quad n = 0, 1, ... \tag{4.10}$$

where

$$k, l, m \in \mathbb{Z} \text{ and } A, x_{-1}, x_0 \in (0, \infty).$$

Some special cases of this equation were investigated in [2], [10],[17], and [16] and were found to have every interesting dynamics.

As we have seen in previous chapters, when $A = 1$ and $m = 1$, every solution (4.10) is periodic with period

3	if	$k = 0$	and	$l = 1$,	(see (2.11))
4	if	$k = 0$	and	$l = 0$,	(see (2.13))
5	if	$k = 1$	and	$l = 0$,	(see (3.6))
6	if	$k = 0$	and	$l = -1$,	(see (2.17))
7	if	$k = 1$	and	$l = 1$,	(see (3.7))
8	if	$k = 1$	and	$l = 2$,	(see (3.8))
9	if	$k = 2$	and	$l = 1$,	(see (3.9))
12	if	$k = 2$	and	$l = 3$,	(see (3.10)).

4.5 INVESTIGATION OF THE SECOND-ORDER DIFFERENCE EQUATION $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$

Consider the non-linear second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, ... \tag{4.11}$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are non-negative real numbers with $B + C > 0$, and where the initial conditions x_{-1} and x_0 are non-negative real numbers such that the right-hand side of Eq. (4.11) is well defined for all $n \geq 0$. When is every solution of Eq. (4.11) periodic with the same period?

The following four special examples of Eq. (4.11)

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, ... \tag{4.12}$$

$$x_{n+1} = \frac{1}{x_{n-1}}, \quad n = 0, 1, ... \tag{4.13}$$

$$x_{n+1} = \frac{1+x_n}{x_{n-1}}, \quad n = 0, 1, ... \tag{4.14}$$

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ... \tag{4.15}$$

are remarkable in the following sense.

Every positive solution of Eq. (4.12) is periodic with period 2 (see (2.9)).

Every positive solution of Eq. (4.13) is periodic with period 4 (see (2.13)).

Every positive solution of Eq. (4.14) is periodic with period 5 (see (2.15)).

Every positive solution of Eq. (4.15) is periodic with period 6 (see (2.17)).

The following result characterizes all possible special cases of equations of the form of Eq. (4.11) with the property that every solution of the equation is periodic with the same period. See [13].

Theorem 4.1 *Assume that $p \geq 2$ be a positive integer, and suppose that every positive solution of Eq. (4.11) is periodic with period p. Then the following statements are true:*

(a) *Let $C > 0$. Then $A = B = \gamma = 0$.*

(b) *Let $C = 0$. Then $\gamma(\alpha + \beta) = 0$.*

Proof. Consider the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (4.11) with

$$x_{-1} = 1 \text{ and } x_0 \in (0, \infty).$$

So clearly

$$x_{p-1} = x_{-1} = 1 \text{ and } x_p = x_0$$

and therefore by Eq. (4.11)

$$x_0 = x_p = \frac{\alpha + \beta + \gamma x_{p-2}}{A + B + C x_{p-2}}.$$

Hence it follows that

$$(A + B)\, x_0 + (C x_0 - \gamma)\, x_{p-2} = \alpha + \beta. \tag{4.16}$$

(a) Let $C > 0$. We claim that

$$A = B = 0. \tag{4.17}$$

If not, $A + B > 0$. So by choosing

$$x_0 > \max\left\{\frac{\alpha+\beta}{A+B}, \frac{\gamma}{C}\right\}$$

we see that Eq. (4.16) is impossible. Therefore Eq. (4.17) is true. In addition to Eq. (4.17), we now also claim that

$$\gamma = 0. \tag{4.18}$$

Otherwise, then $\gamma > 0$. So by choosing

$$x_0 < \min\left\{\frac{\alpha+\beta}{A+B}, \frac{\gamma}{C}\right\}$$

it follows again that Eq. (4.16) is impossible. Thus Eq. (4.18) also holds.

(b) Suppose that $C = 0$ and for the sake of contradiction, let $\gamma(\alpha+\beta) > 0$. Then again by choosing x_0 sufficiently small, we see that Eq. (4.16) is impossible. ■

The following corollary of 4.1 states that Eq. (4.12), Eq. (4.13), (4.14), and Eq. (4.15) are the only special cases of Eq. (4.11) with the property that every positive solution is periodic with the same period. See [13].

Corollary 4.1 *Let $p \in \{2, 3, 4, 5, 6\}$. Suppose that $B + C > 0$, and that every positive solution of Eq. (4.11) is periodic with period p. Then making a change of variables of the form*

$$x_n = \lambda y_n$$

Eq. (4.11) reduces to one of the Eq. (4.12), Eq. (4.13), (4.14), and Eq. (4.15).

4.5.1 PERIOD-2 SOLUTIONS OF $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$

Consider the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, ... \tag{4.19}$$

with non-negative parameters and non-negative initial conditions. To avoid degenerate cases, we shall assume that

$$\alpha+\beta+\gamma,\ B+C,\ \beta+B,\ \gamma+C\in(0,\infty)$$

We also suppose that the parameters and initial conditions are chosen in such a way that the denominator of Eq. (4.19) is always positive. See [7] and [8].
There are now 30 with positive parameters which are included in Eq. (4.19) as special cases.
For some choices of the non-zero parameters of Eq. (4.19), six of these equations have a multitude of prime period-two solutions, six have a unique two-cycle, and two have one or the other of the above properties, depending upon the particular values of the non-zero parameters. See [8] and [13].

4.5.2 THE CASE $C = 0$.

In the case $C = 0$, Eq. (4.19) supposes the form

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n},\quad n=0,1,... \tag{4.20}$$

and shows the following trichotomy character when $B > 0$.

$$\begin{cases}\gamma<\beta+A\Rightarrow & \text{every solution converges;}\\ \gamma=\beta+A\Rightarrow & \text{every solution converges to a period-2 solution;}\\ \gamma>\beta+A\Rightarrow & \text{there exists unbounded solutions.}\end{cases}$$

When

$$\gamma=\beta+A$$

every prime period-2 solution of Eq. (4.20) is given by

$$B\phi\psi=\alpha+\beta\left(\phi+\psi\right)\ \text{with}\ \phi,\psi\in[0,\infty)\ \text{and}\ \phi\neq\psi.$$

Remark 4.2 *In the sequel, when we say that "every solution of a difference equation converges to a periodic solution with period* p*," we mean that every solution of the equation converges to a periodic solution of the equation with (not necessarily prime) period* p*, and that there exists solutions of the equation with prime period* p*.*

4.5.3 THE CASE $C > 0$.

In the case $C > 0$, a necessary condition for Eq. (4.19) to have a prime period-2 solution is

$\gamma > 0$

and so we can rewrite Eq. (4.19) in the normalized form

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, ... \tag{4.21}$$

with non-negative parameters and non-negative initial conditions.

SUBCASE (a)

Let $\alpha = \beta = 0$.
In this case Eq. (4.21) is the equation

$$x_{n+1} = \frac{x_{n-1}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, ... \tag{4.22}$$

with

$$A \in [0, \infty) \text{ and } B > 0. \tag{4.23}$$

Eq. (4.22) has prime period-2 solutions if and only if

$A < 1$.

Moreover when Eq. (4.23) holds, the following statements are true:

(a) When

$B = 1$

every prime period-2 solution

$..., \phi, \psi, ...$

of Eq. (4.22) is given by

$\phi + \psi = 1 - A$ with $\phi, \psi \in [0, \infty)$ and $\phi \neq \psi$.

(b) When

$$B \neq 1$$

Eq. (4.22) has the unique period-2 solution

$$..., 0, 1 - A, ... \tag{4.24}$$

and the solution Eq. (4.24) of Eq. (4.22) is locally asymptotically stable when

$$B > 1$$

and is unstable when

$$0 < B < 1.$$

SUBCASE (b)

Assume that

$$\alpha + \beta > 0. \tag{4.25}$$

In this case Eq. (4.21) has prime period-2 solutions if and only if

$$\beta + A < 1, \quad B > 1, \quad \text{and} \quad 4\alpha < (1 - \beta - A) [B (1 - \beta - A) - (1 + 3\beta - A)]. \tag{4.26}$$

Moreover, when Eq. (4.25) and Eq. (4.26) hold, Eq. (4.21) has the unique prime period-2 solution

$$..., \phi, \psi, ... \tag{4.27}$$

where the values of ϕ and ψ are the two positive and distinct roots of the quadratic equation

$$t^2 - (1 - \beta - A) t + \frac{\alpha + \beta (1 - \beta - A)}{B - 1} = 0. \tag{4.28}$$

4.5.4 EQUATIONS WITH A UNIQUE PRIME PERIOD-TWO SOLUTION

It follows from 4.5.2 and 4.5.3 that after some obvious renormalizations, the only equations of the type of Eq. (4.19) with a unique prime period-2 solutions are the following nine equations with positive parameters. See also [8] and [13]

Case 1 (See [13], p.18)

$$y_{n+1} = \frac{y_{n-1}}{p + y_{n-1}}, \quad n = 0, 1, ... \tag{4.29}$$

Case 2 (See [13], p.60)

$$y_{n+1} = \frac{y_{n-1}}{py_n + y_{n-1}}, \quad n = 0, 1, ... \tag{4.30}$$

Case 3 (See [11] and [13], p.92)

$$y_{n+1} = \frac{p + y_{n-1}}{qy_n + y_{n-1}}, \quad n = 0, 1, ... \tag{4.31}$$

Case 4 (See [13], p.113 and [12])

$$y_{n+1} = \frac{py_n + y_{n-1}}{qy_n + y_{n-1}}, \quad n = 0, 1, ... \tag{4.32}$$

Case 5 (See [13], p.133)

$$y_{n+1} = \frac{y_{n-1}}{p + qy_n + y_{n-1}}, \quad n = 0, 1, ... \tag{4.33}$$

Case 6 (See [13], p.149)

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + y_n + ry_{n-1}}, \quad n = 0, 1, ... \tag{4.34}$$

Case 7 (See [13], p.158)

$$y_{n+1} = \frac{py_n + y_{n-1}}{r + qy_n + y_{n-1}}, \quad n = 0, 1, ... \tag{4.35}$$

Case 8 (See [13], p.175)

$$y_{n+1} = \frac{r + py_n + y_{n-1}}{qy_n + y_{n-1}}, \quad n = 0, 1, ... \tag{4.36}$$

Case 9 (See [13], p.184)

$$y_{n+1} = \frac{p + qy_n + y_{n-1}}{r + sy_n + y_{n-1}}, \quad n = 0, 1, ... \tag{4.37}$$

4.6 THE RICCATI DIFFERENCE EQUATION

The Riccati difference equation is the difference equation

$$z_{n+1} = \frac{a + bz_n}{c + dz_n}, \quad n = 0, 1, ... \tag{4.38}$$

where the parameters a, b, c, d are given real numbers, and initial condition z_0 is an arbitrary real number.

To avoid degenerate cases, we suppose throughout this section without further mention that

$$|a| + |b| \neq 0 \text{ and } |c| + |d| \neq 0.$$

We shall also assume throughout this section, unless otherwise stated, that

$$d \neq 0 \text{ and } bc - ad \neq 0.$$

In fact, when $d = 0$, Eq. (4.38) is a linear equation, while if

$$d \neq 0 \text{ and } bc - ad = 0$$

Eq. (4.38) reduces to the trivial non-linear difference equation

$$z_{n+1} = \frac{\frac{bc}{d} + bz_n}{c + dz_n} = \frac{b\,(c + dz_n)}{d\,(c + dz_n)} = \frac{b}{d}, \quad n = 0, 1, ...$$

Assume that $\overline{z}$ is an equilibrim point of Eq. (4.38). Then

$$d\overline{z}^2 + (c - b)\,\overline{z} - a = 0.$$

Therefore we have that Eq. (4.38) has exactly two equilibrim points if $(b - c)^2 + 4ad > 0$, exactly one equilibrim point if $(b - c)^2 + 4ad = 0$, and no equilibrim points if $(b - c)^2 + 4ad < 0$.

For the results given in this section, see [6].

Theorem 4.2 *The following statements are true:*

(a) *Eq. (4.38) has a prime period-2 solution if and only if $b + c = 0$.*

(b) *Assume that $b + c = 0$. Then every solution $\{z_n\}_{n=0}^{\infty}$ of Eq. (4.38) with $z_0 \neq \frac{c}{d}$ is periodic with period-2.*

Let $\mathcal{G}$ be the set of all initial conditiions $z_0 \in \mathbb{R}$ such that the solution $\{z_n\}_{n=0}$ of Eq. (4.38) exists for all $n \geq 0$, and set $\mathcal{F} = \mathbb{R} - \mathcal{G}$.

Therefore $\mathcal{F}$ is the set of all $z_0 \in \mathbb{R}$ such that the solution of Eq. (4.38) with initial conditiion z_0 fails to exist after a finite number of terms. $\mathcal{G}$ is called the good set of Eq. (4.38), and $\mathcal{F}$ is called forbidden set of Eq. (4.38).

When $b + c = 0$, the forbidden set of Eq. (4.38) is the singleton

$$\mathcal{F} = \left\{-\frac{c}{d}\right\}$$

while in the degenerate case $d\,(bc - ad) = 0$, the forbidden set of Eq. (4.38) is the empty set.

Throughout the remainder of this section we shall suppose that

$$d \neq 0,\ \ bc - ad \neq 0,\ \ \text{and } b + c \neq 0. \tag{4.39}$$

The change of variables

$$z_n = \frac{b+c}{d} w_n - \frac{c}{d}$$

transforms Eq. (4.38) into the difference equation with one parameter

$$w_{n+1} = 1 - \frac{\mathcal{R}}{w_n}, \quad n = 0, 1, ... \tag{4.40}$$

where the parameter $\mathcal{R}$, which we call the Riccati number of Eq. (4.38), is the non-zero real number

$$\mathcal{R} = \frac{bc - ad}{(b+c)^2}$$

and where the initial condition w_0 of Eq. (4.40) is

$$w_0 = \frac{dz_0 + c}{b + c}.$$

We make the further change of variables

$$\begin{cases} w_n = \dfrac{u_{n+1}}{u_n} & \text{for } n = 0, 1, ... \\ u_0 = 1 \end{cases}$$

which reduces Eq. (4.40) to the second order linear difference equation

$$u_{n+2} - u_{n+1} + \mathcal{R} u_n = 0, \quad n = 0, 1, ... \tag{4.41}$$

with initial conditions

$u_0 = 1$ and $u_1 = w_0$.

The characteristic equation of Eq.(4.41) is

$$\lambda^2 - \lambda + \mathcal{R} = 0$$

with characteristic roots

$$\lambda_1 = \frac{1 - \sqrt{1 - 4\mathcal{R}}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 + \sqrt{1 - 4\mathcal{R}}}{2}.$$

CHAPTER 5

SOME SPECIAL MAX-TYPE DIFFERENCE EQUATIONS WITH EVENTUALLY PERIODIC SOLUTIONS

5.1 The Equation $x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_{n-1}}\right\}$

In this section, we consider the periodic character of solutions of the max-type difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_{n-1}}\right\}, \quad n = 0, 1, ... \tag{5.1}$$

where the parameter A and the initial conditions x_{-1} and x_0 are nonzero real numbers. The following theorem shows that every solution of Eq. (5.1) is eventually periodic with period 2, 3 or 4, and it determines the period in terms of A and the initial conditions. We take this theorem from [1], see page [402-403]. See also **([7], pp.62-64).**

Theorem 5.1 *Suppose that the parameter A and the initial conditions x_{-1} and x_0 are nonzero real numbers. Then every solution of Eq. (5.1) is eventually periodic with* 2, 3, *or* 4. *More precisely, the following are true.*

(a) *Suppose that $A < 0$. Then every solution of Eq. (5.1) is eventually periodic with period two and is of the form $\{p, 1/p\}$, for some positive number p which depends on A and the initial conditions x_{-1} and x_0.*

(b) *Suppose that $A > 0$ and the initial conditions x_{-1} and x_0 are not both negative numbers. Then every solution of Eq. (5.1) is eventually periodic with period*

$$\begin{cases} 2 & \text{if} \quad A < 1 \\ 3 & \text{if} \quad A = 1 \\ 4 & \text{if} \quad A > 1. \end{cases}$$

(c) *Suppose that $A > 0$ and the initial conditions x_{-1} and x_0 are both negative numbers. Then every solution of Eq. (5.1) is negative and eventually periodic with period*

$$\begin{cases} 2 & \text{if} \quad A > 1 \\ 3 & \text{if} \quad A = 1 \\ 4 & \text{if} \quad A < 1. \end{cases}$$

Before we give the proof of Theorem 5.1, we make the following obversation.

Remark 5.1 *The periodic character of solutions of Eq. (5.1) depends upon whether the parameter A is dominated 1 (i.e., when $A < 1$), equals 1 (i.e., when $A = 1$), or dominates 1 (i.e., when $A > 1$). In all cases it seems that the dominant term in Eq. (5.1) determines the period of the solutions.*

For example, when $A < 1$ (when A is dominated by 1), every solution of Eq. (5.1) is eventually periodic with period 2. Note that the "dominating" difference equation

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, ... \tag{5.2}$$

has the property that every solution is periodic with period 2. When $A > 1$ (when A dominates 1), every solution of Eq. (5.1) is eventually periodic with period 4, and every solution of the "dominating" difference equation

$$x_{n+1} = \frac{A}{x_{n-1}}, \quad n = 0, 1, ... \tag{5.3}$$

has period 4. Lastly, when $A = 1$, every solution of Eq. (5.1) is eventually periodic with period 3, the avarage of the periods of the solutions of Eq. (5.2) and Eq. (5.3).

Proof of Theorem 5.1

(a) It is easy to see that in this case, every solution of Eq. (5.1 is eventually positive. So, Eq. (5.1 eventually becomes

$$x_{n+1} = \frac{1}{x_n},$$

from which the result follows.

(b) We give the proof as the following 3 cases:

(i) Let $0 < A < 1$. For each $n \geq -1$, put $x_n = A^{y_n}$. Then Eq. (5.1) is transformed into the min-type difference equation

$$y_{n+1} = \min\{-y_n, 1 - y_{n-1}\}, \quad n = 0, 1, ... \tag{5.4}$$

where $y_{-1} = \frac{1}{A}\ln x_{-1}$ and $y_0 = \frac{1}{A}\ln x_0$ are real numbers. It suffices to show that $\{y_n\}_{n=-1}^{\infty}$ is eventually periodic with period 2. Put

$$\mathbf{S} = \left\{(\alpha, -\alpha) : -\frac{1}{2} \leq \alpha \leq \frac{1}{2}\right\}.$$

Case 1. Suppose $\{y_{-1}, y_0\} \in \mathbf{S}$. Then $\{y_n\}_{n=-1}^{\infty}$ is clearly periodic with period 2.Case 2. Suppose $\{y_{-1}, y_0\} \notin \mathbf{S}$. Note that $\{y_n\}_{n=-1}^{\infty}$ oscillates about zero, and so, without loss of generality, we may assume that $y_{-1} \geq 0$ and $y_0 \leq 0$. Set

$$\mathbf{L} = \{(\alpha, -\alpha) : \alpha \in \mathbb{R}\}.$$

We claim there exists $N \in \{0, 1, ...\}$ such that $(y_N, y_{N+1}) \in \mathbf{L}$. For the sake of contradiction, suppose this claim is false. Then $\{y_n\}_{n=-1}^{\infty}$ satisfies the difference equation

$$y_{n+1} = 1 - y_{n-1}, \quad n = 0, 1, ... \tag{5.5}$$

as well as the difference inequality

$$-y_n > 1 - y_{n-1}, \quad n = 0, 1, ... \tag{5.6}$$

It follows from Eq. (5.5) that $\{y_n\}_{n=-1}^{\infty}$ is periodic with period 4, while it follows from inequality Eq. (5.6) that

$$\lim_{n\to\infty} y_n = -\infty.$$

This is a contradiction, and so we obtain that there does exist $N \geq 0$ so that $(y_N, y_{N+1}) \in \mathbf{L}$.The proof follows from Case 1 if $(y_N, y_{N+1}) \in \mathbf{S}$. Therefore suppose that $(y_N, y_{N+1}) \in \mathbf{L} - \mathbf{S}$. Note that if $y_N < -\frac{1}{2}$, then $y_{N+1} > \frac{1}{2}$ and thus

$$y_{N+2} = \min\{-y_{N+1}, 1 - y_N\} = -y_{N+1}.$$

Then it follows that without loss of generality we may assume that $y_{-1} > \frac{1}{2}$, and that $(y_{-1}, y_0) \in \mathbf{L}$. The first five terms of $\{y_n\}_{n=-1}^{\infty}$ are

$$y_{-1}, -y_{-1}, 1 - y_{-1}, y_{-1} - 1, 1 - y_{-1}.$$

Hence (y_{-1}, y_0) and (y_2, y_3) are elements of $\mathbf{L}$, and the distance between (y_{-1}, y_0) and (y_2, y_3) is $\sqrt{2}$. Now $y_{-1} > \frac{1}{2}$ and the length of $\mathbf{S}$ is $\sqrt{2}$, and so we can easily see that (y_2, y_3) is closer to $\mathbf{S}$ than (y_{-1}, y_0) is. Therefore as the length of $\mathbf{S}$ is $\sqrt{2}$, it follows that there exists $M \geq 0$ such that $(y_M, y_{M+1}) \in \mathbf{S}$, and now the proof follows from Case 1.

(ii) Let $A = 1$. It can be easily seen that $\{x_n\}_{n=-1}^{2}$ contains two consecutive terms each greater than or equal to 1, and hence without loss of generality we may assume that $x_{-1} \geq 1$ and $x_0 \geq 1$. Then $\{x_n\}_{n=-1}^{\infty}$ is either

$$x_{-1}, x_0, \frac{1}{x_{-1}}, x_{-1}, x_{-1}, \frac{1}{x_{-1}}, x_{-1}, \ldots$$

or

$$x_{-1}, x_0, \frac{1}{x_0}, x_0, x_0, \frac{1}{x_0}, x_0, \ldots$$

Hence in either case $\{x_n\}_{n=-1}^{\infty}$ is periodic with period 3.

(iii) Let $A > 1$. For each $n \geq -1$, put $x_n = A^{y_n + \frac{1}{2}}$. Then Eq. (5.1) is transformed into the max-type difference equation

$$y_{n+1} = \max\{-1 - y_n, -y_{n-1}\}, \quad n = 0, 1, \ldots \tag{5.7}$$

where $y_{-1} = \frac{\ln x_{-1}}{\ln A} - \frac{1}{2}$ and $y_0 = \frac{\ln x_0}{\ln A} - \frac{1}{2}$ are real numbers.

Put

$$\mathbf{B} = \{(\alpha, \beta) : |\alpha| + |\beta| \leq 1\}$$

and

$$\mathbf{T} = \{(\alpha, -\alpha - 1) : \alpha \leq 0\}.$$

It can be easily seen that $\{y_n\}_{n=-1}^{\infty}$ is of the form

$$y_{-1}, y_0, -y_{-1}, -y_0, y_{-1}, \ y_0 \cdots$$

if and only if $(y_{-1}, y_0) \in \mathbf{B}$, so it suffices to consider the case where $(y_{-1}, y_0) \notin \mathbf{B}$.It can be easily seen that $\{y_n\}_{n=-1}^{3}$ contains two consecutive non-negative terms, and so without loss of generality we may assume that $y_{-1} \geq 0$ and $y_0 \geq 0$. So since $y_{-1}+y_0 > 1$, we see that either $y_1 = -1-y_0$ and $y_2 = -1-y_1$ or $y_1 = -y_{-1}$ and $y_2 = -1-y_1$. In either case $(y_1, y_2) \in \mathbf{T}-\mathbf{B}$. Finally, suppose that $(y_N, y_{N+1}) \in \mathbf{T} - \mathbf{B}$ for some $N \geq 1$. Clearly $y_N < -1$, and then

$$y_{N+2} = -y_N, \; y_{N+3} = 1 + y_N, \; y_{N+4} = -2 - y_N.$$

Note that $(y_{N+3}, y_{N+4}) \in \mathbf{T}$, and that the distance between (y_N, y_{N+1}) and (y_{N+3}, y_{N+4}) is $\sqrt{2}$. Furthermore, the point $(y_{N+3}, y_{N+4}) = (1 + y_N, -2 - y_N)$ is closer to $\mathbf{B}$ than the point $(y_N, y_{N+1}) = (y_N, -1 - y_N)$ is. Consequently, as the length of $\mathbf{B} \cap \mathbf{T}$ is $\sqrt{2}$, it follows that there exists $n_0 \geq 4$ such that $(y_{n_0}, y_{n_0+1}) \in \mathbf{B} \cap \mathbf{T}$.

(c) In this case, the change of variables $x_n = -y_0$ reduces Eq. (5.1) to

$$y_{n+1} = \min\left\{\frac{1}{y_n}, \frac{A}{y_{n-1}}\right\}, \quad n = 0, 1, ...,$$

with positive initial conditions. The remaining part of the proof is similar to that given in part (b) and will be omitted.

Remark 5.2 *It is interesting to note that the more general equation*

$$x_{n+1} = \max\left\{\frac{a}{x_n}, \frac{A}{x_{n-1}}\right\}, \quad n = 0, 1, ..., \tag{5.8}$$

where a, A, and the initial conditions x_{-1} and x_0 are nonzero real numbers, does not have the property that every solution is eventually periodic. In fact, if $a = -1$ and $A = -2$, then the solution with the initial conditions $x_{-1} = 2$ and $x_0 = -1/2$ is unbounded and is given by

$$2, -\frac{1}{2}, 2, ..., 2^n, -\frac{1}{2^n}, 2^n, ...$$

In general, one can show that if a and A are both negative and not equal, then every solution of Eq. (5.8) with initial conditions in $\mathbb{R} - \{0\}$ is unbounded.

5.2 The Equation $x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, ..., \frac{A_k}{x_{n-k}}\right\}$

In this section, we consider the periodic character of solutions of the max-type difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, ..., \frac{A_k}{x_{n-k}}\right\} \quad (n \geq k) \tag{5.9}$$

with initial conditions

$$x_0 = a_0, \; x_1 = a_1, ..., \; x_k = a_k. \tag{5.10}$$

where k is any fixed natural number; $A_0, A_1, ..., A_k$ and $a_0, a_1, ..., a_k$ are fixed real numbers such that $A_k \neq 0$.

It is useful to rewrite the recursion Eq. (5.9) as

$$x_{n+1} = \min \frac{1}{\{x_n/A_0, ((x_{n-1})/A_1), ..., ((x_{n-k})/A_k)\}} \quad (n \geq k). \tag{5.11}$$

Some features of this sequence are listed and conjectured in [18]. The case when all the coefficients A_i are negative is completely open. The main conjecture is the following. For any coefficients $A_i \in \mathbb{R}$ and for initial values $a_i \in \mathbb{R}$ $(0 \leq i \leq k)$ and $k \in \mathbb{N}$ this sequence is eventually periodic if and only if it is bounded, and furthermore it is always bounded for positive numbers $A_i \in \mathbb{R}$ and $a_i \in \mathbb{R}$ $(0 \leq i \leq k)$. The case $k = 1$ is considered in [1]. In this section we give a complete characterization of the behaviour of the sequences satisfying Eq. (5.9) and Eq. (5.10) in the case when all the coefficients A_i $(0 \leq i \leq k)$ are negative (see theorem 5.2). In Theorem 5.2 we also give a simple argument for an already stated result for the case when all the coefficients have the same fixed positive value (see Theorem 5.3).

5.2.1 THE NEGATIVE COEFFICIENTS CASE

The following Theorem completely describe the behaviour of the sequnce when all the coefficients are negative: $A_i < 0$, $A_k \neq 0$, but $a_i \in \mathbb{R}$ are arbitrary real numbers for $i \leq k$. We take this theorem from [26], see page [25-29].

Theorem 5.2 *For any $k \geq 0$ and $A_i < 0$, $A_k \neq 0$, $a_i \in \mathbb{R}$ $(i \leq k)$ the following statements are equivalent:*

(a) *The sequence (x_n) is periodic;*

(b) *The sequence* (x_n) *is periodic with period* $k+2$;

(c) $A_i = A_{k-i}$ *for* $0 \le i \le k$;

(d) *The sequence* (x_n) *is bounded.*

Proof. We observe first that A_i/x_{n-i} and hence x_{n+1} are positive if and only if x_{n-i} is negative for some $i < k$. This implies that enlarging n step by step we leave behind all the negative elements of the sequence. In other words, we reach to an n_0 such that

$$x_{n_0-i} > 0 \quad \text{for } 0 \le i \le k. \tag{5.12}$$

Let

$$z_0 = x_{n_0+1}.$$

Clearly z_0 is negative, and by Eq. (5.12), the previous $k+1$ elements of the sequence are positive. This by Eq. (5.12) implies that the next $k+1$ elements of the sequence are

$$x_{n_0+2+i} = \frac{A_i}{z_0} \quad \text{for } 0 \le i \le k \tag{5.13}$$

and all of them are positive.

Now it can be easily seen that

$$z_1 = x_{n_0+k+3} = z_0.\max\left\{\frac{A_i}{A_{k-i}} : 0 \le i \le k\right\}.$$

A repeated argument shows that for every natural number $t \in \mathbb{N}$

$$z_t = x_{n_0+1+t.(k+2)} = z_0.K^t,$$

where

$$K = \max\left\{\frac{A_i}{A_{k-i}} : i \le k\right\}.$$

This clearly shows (c)⇔(d).

Checking now the terms between z_t and z_{t+1} we obtain (c)⇒(a)+(b).

Since (a)⇒(d) is obvious, theorem is proved.

Observe also that $A_i \ne 0$ must hold for $i \le k$ if there is no positive term among these coeeficients.

The argument given above shows that for $n \ge k+2$ the solution consists of positive semicycles of length $k+1$, followed by negative semicycles of length 1, etc., or the other way around (i.e. replace positive by negative), hence there exists an $N \in \{1, 2, ..., k+2\}$ such that $x_N < 0$.

5.2.2 THE SAME POSITIVE COEFFICIENTS CASE

Assume that now that all the coefficents $A_i \in \mathbb{R}$ have the same fixed positive value $A_i = A$. It ca now be proved that the sequence is periodic also in this case, using an argument similar to the proof of Theorem 5.2.

Theorem 5.3 (([26], pp.25-29])) *In the case* $A_i = A > 0$ $(i \leq k)$ *where* A *is any fixed real number, the sequence* (x_n) *is periodic with period* $k + 2$.

Proof Let $\alpha = \sqrt{A}$. We observe first that $A_i/x_{n-i} > \alpha$ and $x_{n+1} > \alpha$ hold exactly in the case if $x_{n-i} < \alpha$ for some $i < k$. This implies that step by step enlarging n we reach an n_0 such that

$$x_{n_0-i} > \alpha \quad \text{for } i \leq k. \tag{5.14}$$

The above inequality clearly implies

$$x_{n_0+1} < \alpha$$

and so the next $k + 1$ elements of the sequence are

$$x_{n_0+2+i} = \frac{1}{x_{n_0+1}} \quad \text{for } i \leq k. \tag{5.15}$$

In other words, all they have the same value which is greater than α. Then it can be easily seen that

$$x_{n_0+k+3} = \frac{1}{x_{n_0+2}} = x_{n_0+1}$$

and also that the sequence is periodic with period $k + 2$.
Consequently, Theorem is proved.

CHAPTER 6

EXAMPLES ON THE PERIODICITY OF SOLUTIONS OF THE SYSTEM OF RATIONAL DIFFERENCE EQUATIONS

In this chapter, we give some examples of the periodicity of solutions of the system of rational difference equations.

Example 6.1 ***([14], p.411)*** *Suppose that $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$ be arbitrary real numbers and assume that $\{x_n, y_n\}$ be a solution of the system*

$$x_{n+1} = \frac{x_{n-1} + y_n}{y_n x_{n-1} - 1}, \; y_{n+1} = \frac{y_{n-1} + x_n}{x_n y_{n-1} - 1} \tag{6.1}$$

Also, let $ad \neq 1$ and $cb \neq 1$. All solutions of (6.1) are given as follows:

$$x_n = \begin{cases} \frac{d+a}{ad-1} & , \quad n = 6k+1 \\ b & , \quad n = 6k+2 \\ a & , \quad n = 6k+3 \\ \frac{b+c}{cb-1} & , \quad n = 6k+4 \\ d & , \quad n = 6k+5 \\ c & , \quad n = 6k+6 \end{cases} , \; k = 0, 1, 2, ... \tag{6.2}$$

$$y_n = \begin{cases} \frac{b+c}{cb-1} & , \quad n = 6k+1 \\ d & , \quad n = 6k+2 \\ c & , \quad n = 6k+3 \\ \frac{d+a}{ad-1} & , \quad n = 6k+4 \\ b & , \quad n = 6k+5 \\ a & , \quad n = 6k+6 \end{cases} , \; k = 0, 1, 2, ... \tag{6.3}$$

Solution. *For $n = 0, 1, 2, 3, 4, 5,$ we have*

$$x_1 = \frac{x_{-1} + y_0}{y_0 x_{-1} - 1} = \frac{d+a}{ad-1},$$
$$y_1 = \frac{y_{-1} + x_0}{x_0 y_{-1} - 1} = \frac{b+c}{cb-1},$$

$$x_2 = \frac{x_0+y_1}{y_1x_0-1} = \frac{c+\frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c-1} = \frac{\frac{c(cb-1)+b+c}{cb-1}}{\frac{c^2+1}{cb-1}} = \frac{c^2b+b}{c^2+1} = b,$$

$$y_2 = \frac{y_0+x_1}{x_1y_0-1} = \frac{a+\frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a-1} = \frac{\frac{a(ad-1)+d+a}{ad-1}}{\frac{a^2+1}{ad-1}} = \frac{a^2d+d}{a^2+1} = d,$$

$$x_3 = \frac{x_1+y_2}{y_2x_1-1} = \frac{\frac{d+a}{ad-1}+d}{d\frac{d+a}{ad-1}-1} = \frac{\frac{a\left(1+d^2\right)}{ad-1}}{\frac{d^2+1}{ad-1}} = a,$$

$$y_3 = \frac{y_1+x_2}{x_2y_1-1} = \frac{\frac{b+c}{cb-1}+b}{b\frac{b+c}{cb-1}-1} = \frac{c\left(1+b^2\right)}{b^2+1} = c,$$

$$x_4 = \frac{x_2+y_3}{y_3x_2-1} = \frac{b+c}{cb-1},$$

$$y_4 = \frac{y_2+x_3}{x_3y_2-1} = \frac{d+a}{ad-1},$$

$$x_5 = \frac{x_3+y_4}{y_4x_3-1} = \frac{a+\frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a-1} = \frac{a^2d-a+d+a}{ad+a^2-ad+1} = d,$$

$$y_5 = \frac{y_3+x_4}{x_4y_3-1} = \frac{c+\frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c-1} = \frac{b+bc^2}{c^2+1} = b,$$

and

$$x_6 = \frac{x_4+y_5}{y_5x_4-1} = \frac{\frac{b+c}{cb-1}+b}{b\frac{b+c}{cb-1}-1} = \frac{c\left(b^2+1\right)}{b^2+1} = c,$$

$$y_6 = \frac{y_4+x_5}{x_5y_4-1} = \frac{\frac{d+a}{ad-1}+d}{d\frac{d+a}{ad-1}-1} = \frac{ad^2+a}{d^2+1} = a.$$

For $n = 6,7,8,9,10,11,$ *we have*

$$x_7 = \frac{x_5+y_6}{y_6x_5-1} = \frac{d+a}{ad-1} = x_1,$$

$$y_7 = \frac{y_5+x_6}{x_6y_5-1} = \frac{b+c}{cb-1} = y_1,$$

$$x_8 = \frac{x_6+y_7}{y_7x_6-1} = \frac{c+\frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c-1} = \frac{b\left(c^2+1\right)}{c^2+1} = b = x_2,$$

$$y_8 = \frac{y_6+x_7}{x_7y_6-1} = \frac{a+\frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a-1} = \frac{a^2d+d}{a^2+1} = d = y_2,$$

$$x_9 = \frac{x_7+y_8}{y_8x_7-1} = \frac{\frac{d+a}{ad-1}}{d\frac{d+a}{ad-1}-1} = \frac{ad^2+1}{d^2+1} = a = x_3,$$

$$y_9 = \frac{y_7+x_8}{x_8y_7-1} = \frac{\frac{b+c}{cb-1}}{b\frac{b+c}{cb-1}-1} = \frac{c+cb^2}{b^2+1} = c = y_3,$$

$$\begin{aligned} x_{10} &= \frac{x_8+y_9}{y_9x_8-1} = \frac{b+c}{cb-1} = x_4, \\ y_{10} &= \frac{y_8+x_9}{x_9y_8-1} = \frac{d+a}{ad-1} = y_4, \end{aligned}$$

$$\begin{aligned} x_{11} &= \frac{x_9+y_{10}}{y_{10}x_{10}-1} = \frac{a+\frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a-1} = \frac{a^2d+d}{a^2+1} = d = x_5, \\ y_{11} &= \frac{y_9+x_{10}}{x_{10}y_9-1} = \frac{c+\frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c-1} = \frac{c^2b+b}{c^2+1} = b = y_5 \end{aligned}$$

and

$$\begin{aligned} x_{12} &= \frac{x_{10}+y_{11}}{y_{11}x_{10}-1} = \frac{\frac{b+c}{cb-1}+b}{b\frac{b+c}{cb-1}-1} = \frac{c+cb^2}{1+b^2} = c = x_6, \\ y_{12} &= \frac{y_{10}+x_{11}}{x_{11}y_{10}-1} = \frac{\frac{d+a}{ad-1}+d}{d\frac{d+a}{ad-1}-1} = \frac{ad^2+a}{d^2+1} = a = y_6. \end{aligned}$$

Also, we get

$$\begin{aligned} x_1 &= \frac{d+a}{ad-1} = x_7 = x_{13} = ... = x_{6n+1},\ n=0,1,2,3,... \\ x_2 &= b = x_8 = x_{14} = ... = x_{6n+2},\ n=0,1,2,3,... \\ x_3 &= a = x_9 = x_{15} = ... = x_{6n+3},\ n=0,1,2,3,... \\ x_4 &= \frac{b+c}{cb-1} = x_{10} = x_{16} = ... = x_{6n+4},\ n=0,1,2,3,... \\ x_5 &= d = x_{11} = x_{17} = ... = x_{6n+5},\ n=0,1,2,3,... \\ x_6 &= c = x_{12} = x_{18} = ... = x_{6n+6},\ n=0,1,2,3,... \end{aligned}$$

and

$$\begin{aligned} y_1 &= \frac{b+c}{cb-1} = y_7 = y_{13} = ... = y_{6n+1},\ n=0,1,2,3,... \\ y_2 &= d = y_8 = y_{14} = ... = y_{6n+2},\ n=0,1,2,3,... \\ y_3 &= c = y_9 = y_{15} = ... = y_{6n+3},\ n=0,1,2,3,... \\ y_4 &= \frac{d+a}{ad-1} = y_{10} = y_{16} = ... = y_{6n+4},\ n=0,1,2,3,... \\ y_5 &= b = y_{11} = y_{17} = ... = y_{6n+5},\ n=0,1,2,3,... \\ y_6 &= a = y_{12} = y_{18} = ... = y_{6n+6},\ n=0,1,2,3,... \end{aligned}$$

Remark 6.1 *From Example 6.1 we have the following two observations:*

(a) *Let $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$ be arbitrary real numbers and let $\{x_n, y_n\}$ be a solution of the system Eq. (6.1). Also, assume that $ad \neq 1$ and $cb \neq 1$. The solutions of x_n and y_n are six periodic.*

(b) *If* $n \in \mathbb{N}$ *then* $x_{6n+k} = y_{6n+k+3}$.

Example 6.2 ***([23], p.1)*** *Suppose that* $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, $x_{-2} = f$, $z_0 = k$, $z_{-1} = p$, *and* $z_{-2} = q$ *be arbitrary real numbers, and let* $\{x_n, y_n, z_n\}$ *be a solution of the system*

$$x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}y_n},\ y_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2}y_{n-1}x_n}, z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 + x_{n-2}y_{n-1}x_n}, \quad n \in \mathbb{N}_0. \tag{6.4}$$

Also, assume that $b \neq 0$, $e \neq 0$, $fbd \neq 1$, *and* $cea \neq 1$. *Then, all six-period solutions of (6.4) are as follows:*

$$\begin{array}{lll} x_{6n+1} = \dfrac{c}{1 - cea}, & y_{6n+1} = \dfrac{f}{fbd - 1}, & z_{6n+1} = -\dfrac{f + c}{fbd - 1}, \\ x_{6n+2} = b\left(fbd - 1\right), & y_{6n+2} = e\left(cea - 1\right), & z_{6n+2} = -\left(e + b\right)\left(cea + 1\right), \\ x_{6n+3} = \dfrac{a}{1 - cea}, & y_{6n+3} = \dfrac{d}{fbd - 1}, & z_{6n+3} = -\dfrac{d + a}{fbd + 1}, \\ x_{6n+4} = f, & y_{6n+4} = c, & z_{6n+4} = \dfrac{c\left(fbd + 1\right) + f\left(cea + 1\right)}{fbd + 1}, \quad n \in \mathbb{N}_0. \\ x_{6n+5} = e, & y_{6n+5} = b, & z_{6n+5} = \dfrac{b\left(fbd + 1\right) + e\left(cea + 1\right)}{fbd + 1}, \\ x_{6n+6} = d, & y_{6n+6} = a, & z_{6n+6} = \dfrac{a\left(fbd + 1\right) + d\left(cea + 1\right)}{fbd + 1}, \end{array} \tag{6.5}$$

Solution. For $n = 0, 1, 2, 3, 4, 5,$ we have

$$\begin{aligned}
x_1 &= \frac{y_{-2}}{-1+y_{-2}x_{-1}y_0} = \frac{c}{-1+cea},\\
y_1 &= \frac{x_{-2}}{-1+x_{-2}y_{-1}x_0} = \frac{f}{-1+fbd},\\
z_1 &= \frac{x_{-2}+y_{-2}}{-1+x_{-2}y_{-1}x_0} = \frac{f+c}{-1+fbd},\\
x_2 &= \frac{y_{-1}}{-1+y_{-1}x_0y_1} = \frac{b}{-1+bd\left(f/\left(fbd-1\right)\right)} = \frac{b}{1/\left(fbd-1\right)} = b\left(fbd-1\right),\\
y_2 &= \frac{x_{-1}}{-1+x_{-1}y_0x_1} = \frac{e}{-1+ea\left(c/\left(cea-1\right)\right)} = \frac{e}{1/\left(cea-1\right)} = e\left(cea-1\right),\\
z_2 &= \frac{x_{-1}+y_{-1}}{-1+x_{-1}y_0x_1} = \frac{e+b}{-1+ea\left(c/\left(cea-1\right)\right)} = \frac{e+b}{1/\left(cea-1\right)} = \left(e+b\right)\left(cea-1\right),\\
x_3 &= \frac{y_0}{-1+y_0x_1y_2} = \frac{a}{-1+a\left(c/\left(cea-1\right)e\left(cea-1\right)\right)} = \frac{a}{cea-1},\\
y_3 &= \frac{x_0}{-1+x_0y_1x_2} = \frac{d}{-1+d\left(f/\left(fbd-1\right)b\left(fbd-1\right)\right)} = \frac{d}{fbd-1},\\
z_3 &= \frac{x_0+y_0}{-1+x_0y_1x_2} = \frac{d+a}{-1+d\left(f/\left(fbd-1\right)b\left(fbd-1\right)\right)} = \frac{d+a}{fbd-1},\\
x_4 &= \frac{y_1}{-1+y_1x_2y_3} = \frac{f/\left(fbd-1\right)}{-1+\left(f/\left(fbd-1\right)\right)b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)}\\
&= \frac{f/\left(fbd-1\right)}{-1+\left(fbd/\left(fbd-1\right)\right)} = \frac{f/\left(fbd-1\right)}{1/\left(fbd-1\right)} = f,\\
y_4 &= \frac{x_1}{-1+x_1y_2x_3} = \frac{c/\left(cea-1\right)}{-1+\left(c/\left(cea-1\right)\right)e\left(cea-1\right)\left(a/\left(cea-1\right)\right)}\\
&= \frac{c/\left(cea-1\right)}{-1+\left(cea/\left(cea-1\right)\right)} = \frac{c/\left(cea-1\right)}{1/\left(cea-1\right)} = c,\\
z_4 &= \frac{x_1+y_1}{-1+x_1y_2x_3} = \frac{\left(c/\left(cea-1\right)\right)+\left(f/\left(fbd-1\right)\right)}{-1+\left(c/\left(cea-1\right)\right)e\left(cea-1\right)\left(a/\left(cea-1\right)\right)}\\
&= \frac{c\left(fbd-1\right)+f\left(cea-1\right)}{fbd-1},\\
x_5 &= \frac{y_2}{-1+y_2x_3y_4} = \frac{e\left(cea-1\right)}{-1+e\left(cea-1\right)\left(a/\left(cea-1\right)\right)c} = e,\\
y_5 &= \frac{x_2}{-1+x_2y_3x_4} = \frac{b\left(fbd-1\right)}{-1+b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)f} = b,\\
z_5 &= \frac{x_2+y_2}{-1+x_2y_3x_4} = \frac{b\left(fbd-1\right)+e\left(cea-1\right)}{-1+b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)f} = \frac{b\left(fbd-1\right)+e\left(cea-1\right)}{fbd-1},\\
x_6 &= \frac{y_3}{-1+y_3x_4y_5} = \frac{d/\left(fbd-1\right)}{-1+\left(d/\left(fbd-1\right)\right)fb} = d,\\
y_6 &= \frac{x_3}{-1+x_3y_4x_5} = \frac{a/\left(cea-1\right)}{-1+\left(a/\left(cea-1\right)\right)ce} = a,\\
z_6 &= \frac{x_3+y_3}{-1+x_3y_4x_5} = \frac{\left(a/\left(cea-1\right)\right)+\left(d/\left(fbd-1\right)\right)}{1+\left(a/\left(cea-1\right)ce\right)} = \frac{a\left(fbd-1\right)+d\left(cea-1\right)}{fbd-1}
\end{aligned}$$

For $n=6,7,8,9,10,11,$ we get

$$\begin{aligned}
x_7 &= \frac{y_4}{-1+y_4x_5y_6} = \frac{c}{-1+cea} = x_1, \qquad (6.6)\\
y_7 &= \frac{x_4}{-1+x_4y_5x_6} = \frac{f}{-1+fbd} = y_1,\\
z_7 &= \frac{x_4+y_4}{-1+x_4y_5x_6} = \frac{f+c}{-1+fbd} = z_1,\\
x_8 &= \frac{y_5}{-1+y_5x_6y_7} = \frac{b}{-1+bd\left(f/\left(-1+fbd\right)\right)} = b\left(fbd-1\right) = x_2,\\
y_8 &= \frac{x_5}{-1+x_5y_6x_7} = \frac{e}{-1+ea\left(c/\left(-1+cea\right)\right)} = e\left(cea-1\right) = y_2,\\
z_8 &= \frac{x_5+y_5}{-1+x_5y_6x_7} = \frac{e+b}{-1+ea\left(c/\left(-1+cea\right)\right)} = \left(e+b\right)\left(cea-1\right) = z_2,\\
x_9 &= \frac{y_6}{-1+y_6x_7y_8} = \frac{a}{-1+a\left(c/\left(-1+cea\right)\right)e\left(cea-1\right)} = \frac{a}{cea-1} = x_3,\\
y_9 &= \frac{x_6}{-1+x_6y_7x_8} = \frac{d}{-1+d\left(f/\left(-1+fbd\right)\right)b\left(fbd-1\right)} = \frac{d}{fbd-1} = y_3,\\
z_9 &= \frac{x_6+y_6}{-1+x_6y_7x_8} = \frac{d+a}{-1+d\left(f/\left(-1+fbd\right)\right)b\left(fbd-1\right)} = \frac{d+a}{fbd-1} = z_3,\\
x_{10} &= \frac{y_7}{-1+y_7x_8y_9} = \frac{f/\left(-1+fbd\right)}{-1+\left(f/\left(-1+fbd\right)\right)b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)}\\
&= \frac{f/\left(-1+fbd\right)}{-1+\left(fbd/\left(-1+fbd\right)\right)} = f = x_4,\\
y_{10} &= \frac{x_7}{-1+x_7y_8x_9} = \frac{c/\left(-1+cea\right)}{-1+\left(c/\left(-1+cea\right)\right)e\left(cea-1\right)\left(a/\left(cea-1\right)\right)}\\
&= \frac{c/\left(-1+cea\right)}{-1+\left(cea/\left(-1+cea\right)\right)} = c = y_4,\\
z_{10} &= \frac{x_7+y_7}{-1+x_7y_8x_9} = \frac{\left(c/\left(-1+cea\right)\right)+\left(f/\left(-1+fbd\right)\right)}{-1+\left(c/\left(-1+cea\right)e\left(cea-1\right)a/\left(cea-1\right)\right)}\\
&= \frac{c\left(fbd-1\right)+f\left(cea-1\right)}{fbd-1} = z_4,\\
x_{11} &= \frac{y_8}{-1+y_8x_9y_{10}} = \frac{e\left(cea-1\right)}{-1+e\left(cea-1\right)\left(a/\left(cea-1\right)\right)c} = e = x_5,\\
y_{11} &= \frac{x_8}{-1+x_8y_9x_{10}} = \frac{b\left(fbd-1\right)}{-1+b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)f} = b = y_5,\\
z_{11} &= \frac{x_8+y_8}{-1+x_8y_9x_{10}} = \frac{b\left(fbd-1\right)+e\left(cea-1\right)}{-1+b\left(fbd-1\right)\left(d/\left(fbd-1\right)\right)f}\\
&= \frac{b\left(fbd-1\right)+e\left(cea-1\right)}{fbd-1} = z_5,\\
x_{12} &= \frac{y_9}{-1+y_9x_{10}y_{11}} = \frac{d/\left(fbd-1\right)}{-1+\left(d/\left(fbd-1\right)\right)fb} = d = x_6,\\
y_{12} &= \frac{x_9}{-1+x_9y_{10}x_{11}} = \frac{a/\left(cea-1\right)}{-1+\left(a/\left(cea-1\right)\right)ce} = a = y_6,\\
z_{12} &= \frac{x_9+y_9}{-1+x_9y_{10}x_{11}} = \frac{\left(a/\left(cea-1\right)\right)+\left(d/\left(fbd-1\right)\right)}{-1+\left(a/\left(cea-1\right)\right)ce}\\
&= \frac{a\left(fbd-1\right)+d\left(cea-1\right)}{fbd-1} = z_6,
\end{aligned}$$

are true. Also, we have for $n \in \mathbb{N}_0$

$$\begin{aligned}
x_1 &= \frac{c}{cea-1} = x_7 = x_{13} = ... = x_{6n+1}, \\
x_2 &= b\,(fbd-1) = x_8 = x_{14} = ... = x_{6n+2}, \\
x_3 &= \frac{a}{cea-1} = x_9 = x_{15} = ... = x_{6n+3}, \\
x_4 &= f = x_{10} = x_{16} = ... = x_{6n+4}, \\
x_5 &= e = x_{11} = x_{17} = ... = x_{6n+5}, \\
x_6 &= d = x_{12} = x_{18} = ... = x_{6n+6},
\end{aligned}$$

$$\begin{aligned}
y_1 &= \frac{f}{fbd-1} = y_7 = y_{13} = ... = y_{6n+1}, \\
y_2 &= e\,(cea-1) = y_8 = y_{14} = ... = y_{6n+2}, \\
y_3 &= \frac{d}{fbd-1} = y_9 = y_{15} = ... = y_{6n+3}, \\
y_4 &= c = y_{10} = y_{16} = ... = y_{6n+4}, \\
y_5 &= b = y_{11} = y_{17} = ... = y_{6n+5}, \\
y_6 &= a = y_{12} = y_{18} = ... = y_{6n+6},
\end{aligned}$$

$$\begin{aligned}
z_1 &= \frac{f+c}{fbd-1} = z_7 = z_{13} = ... = z_{6n+1}, \\
z_2 &= (e+b)\,(cea-1) = z_8 = z_{14} = ... = z_{6n+2}, \\
z_3 &= \frac{d+a}{fbd-1} = z_9 = z_{15} = ... = z_{6n+3}, \\
z_4 &= \frac{c\,(fbd-1) + f\,(cea-1)}{fbd-1} = z_{10} = z_{16} = ... = z_{6n+4}, \\
z_5 &= \frac{b\,(fbd-1) + e\,(cea-1)}{fbd-1} = z_{11} = z_{17} = ... = z_{6n+5}, \\
z_6 &= \frac{a\,(fbd-1) + d\,(cea-1)}{fbd-1} = z_{12} = z_{18} = ... = z_{6n+6}.
\end{aligned}$$

Remark 6.2 *The following conclusions are valid for $n \in \mathbb{N}$:*

(a) $x_{6n+2}y_{6n+3} = x_{6n+6}y_{6n+5},$

(b) $x_{6n+1}y_{6n+2} = x_{6n+5}y_{6n+6},$

(c) $x_{6n+1}y_{6n+6} = x_{6n+3}y_{6n+4},$

(d) $x_{6n+4}y_{6n+3} = x_{6n+6}y_{6n+1}.$

Example 6.3 *([23], p.4) Let* $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, $x_{-2} = f$, $z_0 = k$, $z_{-1} = p$, *and* $z_{-2} = q$ *be arbitrary real numbers, and suppose that* $\{x_n, y_n, z_n\}$ *be a solution of the system*

$$x_{n+1} = \frac{y_{n-2}}{-1 - y_{n-2}x_{n-1}y_n}, \; y_{n+1} = \frac{x_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \; z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \quad n \in \mathbb{N}_0. \tag{6.7}$$

Also, assume that $b \neq 0$, $e \neq 0$, $fbd \neq 1$, *and* $cea \neq 1$. *Then, all six-period solutions of (6.7) are as follows:*

$$\begin{array}{lll} x_{6n+1} = -\frac{c}{1+cea}, & y_{6n+1} = -\frac{f}{fbd+1}, & z_{6n+1} = -\frac{-f(1+fbd)+c}{fbd+1}, \\ x_{6n+2} = -b\left(fbd+1\right), & y_{6n+2} = -e\left(cea+1\right), & z_{6n+2} = e - b\left(1+cea\right), \\ x_{6n+3} = -\frac{a}{1+cea}, & y_{6n+3} = -\frac{d}{fbd+1}, & z_{6n+3} = -\frac{-d(fbd+1)+a}{fbd+1}, \\ x_{6n+4} = f, & y_{6n+4} = c, & z_{6n+4} = \frac{-c(fbd+1)+f\left(1+cea+c^2e^2a^2\right)}{(fbd+1)(cea+1)}, \\ x_{6n+5} = e, & y_{6n+5} = b, & z_{6n+5} = \frac{-b\left(1+2fbd+f^2b^2d^2\right)+e(1+cea)}{fbd+1}, \\ x_{6n+6} = d, & y_{6n+6} = a, & z_{6n+6} = \frac{-a(fbd+1)+d\left(1+cea+c^2e^2a^2\right)}{(1+cea)(fbd+1)}, \end{array} \quad n \in \mathbb{N}_0. \tag{6.8}$$

Solution. For $n = 0, 1, 2, 3, 4, 5,$ we have

$$\begin{aligned}
x_1 &= \frac{y_{-2}}{-1 - y_{-2}x_{-1}y_0} = \frac{c}{-1 - cea} = -\frac{c}{1 + cea}, \qquad (6.9)\\
y_1 &= \frac{x_{-2}}{-1 - x_{-2}y_{-1}x_0} = \frac{f}{-1 - fbd} = -\frac{f}{fbd + 1},\\
z_1 &= \frac{x_{-2} + y_{-2}}{-1 - x_{-2}y_{-1}x_0} = \frac{f + c}{-1 - fbd} = -\frac{f + c}{1 + fbd},\\
x_2 &= \frac{y_{-1}}{-1 - y_{-1}x_0y_1} = \frac{b}{-1 - bd\left(-\left(f/\left(fbd + 1\right)\right)\right)} = \frac{b}{-\left(1/\left(fbd + 1\right)\right)} = -b\left(fbd + 1\right),\\
y_2 &= \frac{x_{-1}}{-1 - x_{-1}y_0x_1} = \frac{e}{-1 - ea\left(-\left(c/\left(1 + cea\right)\right)\right)} = \frac{e}{-\left(1/\left(1 + cea\right)\right)} = -e\left(cea + 1\right),\\
z_2 &= \frac{x_{-1} + y_{-1}}{-1 - x_{-1}y_0x_1} = \frac{e + b}{-1 - ea\left(-\left(c/\left(1 + cea\right)\right)\right)} = \frac{e + b}{-\left(1/\left(1 + cea\right)\right)} = -\left(e + b\right)\left(1 + cea\right),\\
x_3 &= \frac{y_0}{-1 - y_0x_1y_2} = \frac{a}{-1 - a\left(-\left(c/\left(1 + cea\right)\right)\right) - \left(e\left(1 + cea\right)\right)} = -\frac{a}{1 + cea},\\
y_3 &= \frac{x_0}{-1 - x_0y_1x_2} = \frac{d}{-1 - d\left(-\left(f/\left(1 + fbd\right)\right)\right)\left(-b\left(1 + fbd\right)\right)} = -\frac{d}{1 + fbd},\\
z_3 &= \frac{x_0 + y_0}{-1 - x_0y_1x_2} = \frac{d + a}{-1 - d\left(-\left(f/\left(1 + fbd\right)\right)\right)\left(-b\left(1 + fbd\right)\right)} = -\frac{d + a}{1 + fbd},\\
x_4 &= \frac{y_1}{-1 - y_1x_2y_3} = \frac{-\left(f/\left(1 + fbd\right)\right)}{-1 + \left(f/\left(1 + fbd\right)\right)b\left(1 + fbd\right)\left(d/\left(1 + fbd\right)\right)}\\
&= \frac{-\left(f/\left(1 + fbd\right)\right)}{-1 + \left(fbd/\left(1 + fbd\right)\right)} = \frac{-\left(f/\left(1 + fbd\right)\right)}{-\left(1/\left(1 + fbd\right)\right)} = f,\\
y_4 &= \frac{x_1}{-1 - x_1y_2x_3} = \frac{-\left(c/\left(1 + cea\right)\right)}{-1 + \left(c/\left(1 + cea\right)\right)e\left(1 + cea\right)\left(a/\left(1 + cea\right)\right)}\\
&= \frac{-\left(c/\left(1 + cea\right)\right)}{-1 + \left(cea/\left(1 + cea\right)\right)} = \frac{-\left(c/\left(1 + cea\right)\right)}{-\left(1/\left(1 + cea\right)\right)} = c,\\
z_4 &= \frac{x_1 + y_1}{-1 - x_1y_2x_3} = \frac{-\left(c/\left(1 + cea\right)\right) - \left(f/\left(1 + fbd\right)\right)}{-1 + \left(c/\left(1 + cea\right)\right)e\left(1 + cea\right)\left(a/\left(1 + cea\right)\right)}\\
&= \frac{c\left(1 + fbd\right) + f\left(1 + cea\right)}{1 + fbd},\\
x_5 &= \frac{y_2}{-1 - y_2x_3y_4} = \frac{-e\left(1 + cea\right)}{-1 - e\left(1 + cea\right)\left(a/\left(1 + cea\right)\right)c} = e,\\
y_5 &= \frac{x_2}{-1 - x_2y_3x_4} = \frac{-b\left(1 + fbd\right)}{-1 - b\left(1 + fbd\right)\left(d/\left(1 + fbd\right)\right)f} = b,\\
z_5 &= \frac{x_2 + y_2}{-1 - x_2y_3x_4} = \frac{-b\left(1 + fbd\right) - e\left(1 + cea\right)}{-1 - b\left(1 + fbd\right)\left(d/\left(1 + fbd\right)\right)f} = \frac{b\left(1 + fbd\right) + e\left(1 + cea\right)}{1 + fbd},\\
x_6 &= \frac{y_3}{-1 - y_3x_4y_5} = \frac{-\left(d/\left(1 + fbd\right)\right)}{-1 + \left(d/\left(1 + fbd\right)\right)fb} = d,\\
y_6 &= \frac{x_3}{-1 - x_3y_4x_5} = \frac{-\left(a/\left(1 + cea\right)\right)}{-1 + \left(a/\left(1 + cea\right)\right)ce} = a,\\
z_6 &= \frac{x_3 + y_3}{-1 - x_3y_4x_5} = \frac{-\left(a/\left(1 + cea\right)\right) - \left(d/\left(1 + fbd\right)\right)}{-1 + \left(a/\left(1 + cea\right)\right)ce} = \frac{a\left(1 + fbd\right) + d\left(1 + cea\right)}{1 + fbd}.
\end{aligned}$$

For $n = 6, 7, 8, 9, 10, 11,$ we get

$$\begin{aligned}
x_7 &= \frac{y_4}{-1-y_4x_5y_6} = \frac{c}{-1-cea} = -\frac{c}{1+cea} = x_1, \\
y_7 &= \frac{x_4}{-1-x_4y_5x_6} = \frac{f}{-1-fbd} = -\frac{f}{1+fbd} = y_1, \\
z_7 &= \frac{x_4+y_4}{-1-x_4y_5x_6} = \frac{f+c}{-1-fbd} = -\frac{f+c}{1+fbd} = z_1, \\
x_8 &= \frac{y_5}{-1-y_5x_6y_7} = \frac{b}{-1+bd\left(f/\left(1+fbd\right)\right)} = -b\left(1+fbd\right) = x_2, \\
y_8 &= \frac{x_5}{-1-x_5y_6x_7} = \frac{e}{-1+ea\left(c/\left(1+cea\right)\right)} = -e\left(1+cea\right) = y_2, \\
z_8 &= \frac{x_5+y_5}{-1-x_5y_6x_7} = \frac{e+b}{-1+ea\left(c/\left(1+cea\right)\right)} = -\left(e+b\right)\left(1+cea\right) = z_2, \\
x_9 &= \frac{y_6}{-1-y_6x_7y_8} = \frac{a}{-1-a\left(c/\left(1+cea\right)\right)e\left(1+cea\right)} = -\frac{a}{1+cea} = x_3, \\
y_9 &= \frac{x_6}{-1-x_6y_7x_8} = \frac{d}{-1-d\left(f/\left(1+fbd\right)\right)b\left(1+fbd\right)} = -\frac{d}{1+fbd} = y_3, \\
z_9 &= \frac{x_6+y_6}{-1-x_6y_7x_8} = \frac{d+a}{-1-d\left(f/\left(1+fbd\right)\right)b\left(1+fbd\right)} = -\frac{d+a}{1+fbd} = z_3, \\
x_{10} &= \frac{y_7}{-1-y_7x_8y_9} = \frac{-\left(f/\left(1+fbd\right)\right)}{-1+\left(f/\left(1+fbd\right)\right)b\left(1+fbd\right)\left(d/\left(1+fbd\right)\right)} \\
&= \frac{-\left(f/\left(1+fbd\right)\right)}{-1/\left(1+fbd\right)} = f = x_4, \\
y_{10} &= \frac{x_7}{-1-x_7y_8x_9} = \frac{-\left(c/\left(1+cea\right)\right)}{-1+\left(c/\left(1+cea\right)\right)e\left(1+cea\right)\left(a/\left(1+cea\right)\right)} \\
&= \frac{-\left(c/\left(1+cea\right)\right)}{-1/\left(1+cea\right)} = c = y_4, \\
z_{10} &= \frac{x_7+y_7}{-1-x_7y_8x_9} = \frac{-\left(c/\left(1+cea\right)\right)-\left(f/\left(1+fbd\right)\right)}{-1+\left(c/\left(1+cea\right)\right)e\left(1+cea\right)\left(a/\left(1+cea\right)\right)} \\
&= \frac{c\left(1+fbd\right)+f\left(1+cea\right)}{1+fbd} = z_4, \\
x_{11} &= \frac{y_8}{-1-y_8x_9y_{10}} = \frac{-e\left(1+cea\right)}{-1-e\left(1+cea\right)\left(a/\left(1+cea\right)\right)c} = e = x_5, \\
y_{11} &= \frac{x_8}{-1-x_8y_9x_{10}} = \frac{-b\left(1+fbd\right)}{-1-b\left(1+fbd\right)\left(d/\left(1+fbd\right)\right)f} = b = y_5, \\
z_{11} &= \frac{x_8+y_8}{-1-x_8y_9x_{10}} = \frac{-b\left(1+fbd\right)-e\left(1+cea\right)}{-1-b\left(1+fbd\right)\left(d/\left(1+fbd\right)\right)f} \\
&= \frac{b\left(1+fbd\right)+e\left(1+cea\right)}{1+fbd} = z_5, \\
x_{12} &= \frac{y_9}{-1-y_9x_{10}y_{11}} = \frac{-\left(d/\left(1+fbd\right)\right)}{-1+\left(d/\left(1+fbd\right)\right)fb} = d = x_6, \\
y_{12} &= \frac{x_9}{-1-x_9y_{10}x_{11}} = \frac{-\left(a/\left(1+cea\right)\right)}{-1+\left(a/\left(1+cea\right)\right)ce} = a = y_6, \\
z_{12} &= \frac{x_9+y_9}{-1-x_9y_{10}x_{11}} = \frac{-\left(a/\left(1+cea\right)\right)-\left(d/\left(1+fbd\right)\right)}{-1+\left(a/\left(1+cea\right)\right)ce} \\
&= \frac{a\left(1+fbd\right)+d/\left(1+cea\right)}{1+fbd} = z_6.
\end{aligned} \tag{6.10}$$

are true. Also, we have for $n \in \mathbb{N}_0$

$$\begin{aligned}
x_1 &= -\frac{c}{1+cea} = x_7 = x_{13} = ... = x_{6n+1}, \\
x_2 &= -b\left(1+fbd\right) = x_8 = x_{14} = ... = x_{6n+2}, \\
x_3 &= -\frac{a}{1+cea} = x_9 = x_{15} = ... = x_{6n+3}, \\
x_4 &= f = x_{10} = x_{16} = ... = x_{6n+4}, \\
x_5 &= e = x_{11} = x_{17} = ... = x_{6n+5}, \\
x_6 &= d = x_{12} = x_{18} = ... = x_{6n+6},
\end{aligned}$$

$$\begin{aligned}
y_1 &= -\frac{f}{1+fbd} = y_7 = y_{13} = ... = y_{6n+1}, \\
y_2 &= -e\left(1+cea\right) = y_8 = y_{14} = ... = y_{6n+2}, \\
y_3 &= -\frac{d}{1+fbd} = y_9 = y_{15} = ... = y_{6n+3}, \\
y_4 &= c = y_{10} = y_{16} = ... = y_{6n+4}, \\
y_5 &= b = y_{11} = y_{17} = ... = y_{6n+5}, \\
y_6 &= a = y_{12} = y_{18} = ... = y_{6n+6},
\end{aligned}$$

$$\begin{aligned}
z_1 &= -\frac{f+c}{1+fbd} = z_7 = z_{13} = ... = z_{6n+1}, \\
z_2 &= -\left(e+b\right)\left(1+cea\right) = z_8 = z_{14} = ... = z_{6n+2}, \\
z_3 &= -\frac{d+a}{1+fbd} = z_9 = z_{15} = ... = z_{6n+3}, \\
z_4 &= \frac{c\left(1+fbd\right)+f\left(1+cea\right)}{1+fbd} = z_{10} = z_{16} = ... = z_{6n+4}, \\
z_5 &= \frac{b\left(1+fbd\right)+e\left(1+cea\right)}{1+fbd} = z_{11} = z_{17} = ... = z_{6n+5}, \\
z_6 &= \frac{a\left(1+fbd\right)+d\left(1+cea\right)}{1+fbd} = z_{12} = z_{18} = ... = z_{6n+6}.
\end{aligned}$$

Remark 6.3 *The following conclusions are valid for $n \in \mathbb{N}$:*

(a) $x_{6n+1}y_{6n+6} = x_{6n+3}y_{6n+4}$,

(b) $x_{6n+6}y_{6n+1} = x_{6n+4}y_{6n+3}$,

(c) $x_{6n+3}y_{6n+2} = x_{6n+5}y_{6n+6}$,

(d) $x_{6n+1}y_{6n+2} = x_{6n+5}y_{6n+4}$.

Example 6.4 ([19], p.107) *Suppose that a, b, c, d, e and f are real numbers such that $(ad-1)(cb-1) \neq 0$, $abef \neq 0$. We suppose that the initial values to be $y_0 = a$, $x_0 = c$, $y_{-1} = b$, $x_{-1} = d$, $z_0 = e$, $z_{-1} = f$. Assume that $\{x_n, y_n, z_n\}$ be a solution of the system*

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, z_{n+1} = \frac{1}{y_n z_{n-1}}, \quad n \in \mathbb{N}_0. \tag{6.11}$$

Then, all solutions of (6.11) are as follows:

$$x_n = \begin{cases} \dfrac{d}{(ad-1)^k} & , \ n = 2k-1 \\ c(cb-1)^k & , \quad n = 2k \end{cases}, \quad k = 1, 2, ... \tag{6.12}$$

$$y_n = \begin{cases} \dfrac{b}{(cb-1)^k} & , \ n = 2k-1 \\ a(ad-1)^k & , \quad n = 2k \end{cases}, \quad k = 1, 2, ... \tag{6.13}$$

$$z_n = \begin{cases} \dfrac{1}{af(ad-1)^{k-1}} & , \ n = 4(k-1)+1 \\ \dfrac{(cb-1)^k}{be} & , \ n = 4(k-1)+2 \\ \dfrac{f}{(ad-1)^k} & , \ n = 4(k-1)+3 \\ e(bc-1)^k & , \ n = 4(k-1)+4 \end{cases}, \quad k = 1, 2, ... \tag{6.14}$$

Solution *It is obvious to show (6.12) and (6.13) and referred to [15]. Here, we only focus on (6.14). First, for $k = 1$, from 6.11 and 6.13, we easily check that*

$$\begin{aligned} z_1 &= \frac{1}{y_0 z_{-1}} = \frac{1}{af}, \\ z_2 &= \frac{1}{y_1 z_0} = \frac{1}{\frac{b}{(cb-1)} e} = \frac{cb-1}{be}, \\ z_3 &= \frac{1}{y_2 z_1} = \frac{f}{ad-1}, \\ z_4 &= \frac{1}{y_3 z_2} = e(cb-1). \end{aligned}$$

Now, we assume the conclusion is true for k, that is, (6.14) holds. Then, for $k+1$, we confirm it. Indeed, from (6.11) and (6.13) and (6.14), we obtain

$$\begin{aligned} z_{4k+1} &= \frac{1}{y_{4k} z_{4(k-1)+3}} = \frac{1}{a(ad-1)^{2k} \times \frac{f}{(ad-1)^k}} = \frac{1}{af(ad-1)^k}, \\ z_{4k+2} &= \frac{1}{y_{4k+1} z_{4k}} = \frac{1}{\frac{b}{(cb-1)^{2k+1}} \times e(cb-1)^k} = \frac{(cb-1)^{k+1}}{be}, \\ z_{4k+3} &= \frac{1}{y_{4k+2} z_{4k+1}} = \frac{1}{a(ad-1)^{2k+1} \times \frac{1}{af(ad-1)^k}} = \frac{f}{(ad-1)^{k+1}}, \\ z_{4k+4} &= \frac{1}{y_{4k+3} z_{4k+2}} = \frac{1}{\frac{b}{(cb-1)^{2k+2}} \times \frac{(cb-1)^{k+1}}{be}} = e(cb-1)^{k+1} \end{aligned}$$

Example 6.5 ***([19], p.109)*** *Suppose that the assumption of Example 6.4 holds. Also, if* $ad = cb = 2$, *then all solutions of (6.11) are periodic.*

Solution. In this case, from (6.12), (6.13) and (6.14), we get

$$x_n = \begin{cases} d \quad , \quad n = 2k-1 \\ c \quad , \quad n = 2k \end{cases} \quad , \quad k = 1, 2, ... \tag{6.15}$$

$$y_n = \begin{cases} b \quad , \quad n = 2k-1 \\ a \quad , \quad n = 2k \end{cases} \quad , \quad k = 1, 2, ... \tag{6.16}$$

$$z_n = \begin{cases} \frac{1}{af} \quad , \quad n = 4(k-1)+1 \\ \frac{1}{be} \quad , \quad n = 4(k-1)+2 \\ f \quad , \quad n = 4(k-1)+3 \\ e \quad , \quad n = 4(k-1)+4 \end{cases} \quad , \quad k = 1, 2, ... \tag{6.17}$$

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