CS-MODULES AND GENERALIZATIONS OF CS-MODULES

by

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THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OF

THE ABANT İZZET BAYSAL UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF MATHEMATICS

JANUARY 2009

Approval of the Graduate School of Natural and Applied Sciences.

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ABSTRACT

CS-MODULES AND GENERALIZATIONS OF CS-MODULES

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January 2009, 55 pages

This study contains CS-modules (extending modules),and P-extending and EF-extending modules which are generalizations of CS-modules.

This study consists of three sections: In section 1, we present some definitions and theorems which will be used in the following sections. Section 2 contains a general characterization of CS-modules. It is known that every direct summand of a CS-module is a CS-module too. However, the direct sum of CS-modules may not be a CS-module. In this section, it is given under which conditions the direct sum of CS-modules are CS-modules.

In section, after giving some characterizations and features of principally injective modules, the following results of the P-extending and EF-extending modules which are the generalizations of principally injective modules are studied.

Let M be a quasi-principally injective module and $S = \mathrm{End}(M)$ and $K, H \leq M$. If $K \cong H$,then $SH = SK$.

If M has the condition (PC_2),then M has the property (PC_3).

Under which conditions, direct sums of P-extending modules is P-extending is given.

Some examples regarding converse of the implication which is not true are given.

Under which conditions, an ef-extending module is extending is given.

Definitions of EC-submodules and EC-injective modules are given and by means of these definitions, under which conditions the module $M = M_1 \oplus M_2$ is P-extending is given.
Keywords: essential submodules, complement submodules, injective modules,
CS-modules, ef-extending and P-extending modules.

ÖZET

CS-MODÜLLER VE CS-MODÜLLERİN GENELLEMELERİ

Öztürk, Hakan

Master Tezi, Matematik Bölümü

Tez Yöneticisi: Yard. Doç. Dr. Cesim Çelik

Ocak 2009, 55 sayfa

Üç bölümden oluşan bu çalışma, CS-modülleri (extending modules) ve bu modullerin genellemeleri olan P-extending, EF-extending modüllerin karakterizasyonunu içermektedir.

Birinci bölüm, diğer bölümlerde kullanılan temel tanım ve teoremlerden oluşmaktadır.

İkinci bölüm, CS-modüllerin genel bir karakterizasyonunu içermektedir. Bir CS-modülün her dik toplananınında bir CS-modül olduğu bilinmektedir. Ancak, CS-modüllerin dik toplamları her zaman CS-modül değildir. Bu bölümde, CS-modüllerin hangi koşullar altında yine CS-modül olduğu verilmiştir.

Üçüncü bölümde, temel injektif (principally injective) modüllerin bazı karakterizasyonları ve özellikleri verildikten sonra, temel injektif modüllerin birer genellemeleri olan P-extending ve EF-extending modüllerin karakterizasyonuyla ilgili aşağıgaki sonuçlar incelenmiştir.

M yarı temel injektif (quasi-principally injective) modül, K, $H \leq M$ ve $S = \mathrm{End}(M)$ olmak üzere, $K \cong H$ ise $SH = SK$.

M, (PC_2)'yi sağlıyor ise M, (PC_3) özelliğini sağlar.

P-extending modüllerin dik toplamı ne zaman P-extending modüldür.

"extending $\Rightarrow$ ef-extending $\Rightarrow$ uniform-extending" önermesinin tersinin doğru olmadığına dair örnekler verildi.

Bir ef-extending modülün ne zaman extending modül olduğu verildi.

EC-altmodül ve EC-injektif modül tanımları verilip, bu tanımlar yardımıyla, $M = M_1 \oplus M_2$ modülünün hangi koşullar altında P-extending modül olduğu verilmiştir.

Anahtar Kelimeler: esas altmodül, injektif modül, CS-modül, ef-extending modül, P-extending modül.

To My Family,

ACKNOWLEGMENTS

I would like to express my sincere appreciation to my supervisor Assist. Prof. Dr. Cesim Çelik who has guided, encouraged and helped me in many ways throughout the thesis.

I would also like to thank to the other examining committee members for their sugesstions.

Finally I thank to my family and friends for their support and encouragement.

TABLE OF CONTENTS

CHAPTER 1

INRODUCTION AND PRELIMINARIES

1.1 Essential and Complement Submodules

Definition 1.1 *Let M be a right R-module and N be a submodule of M. N is called essential submodule of M ($N \leq_e M$) if $N \cap K \neq 0$ for any submodule K of M with $K \neq 0$.*

Definition 1.2 *Let M be a right R-module and $A, B \leq M$. A is called complement of B in M if A is maximal with respect to the property $A \cap B = 0$. If a submodule N of M is complement submodule in M, then it is denoted by $N \leq_c M$.*

Proposition 1.3 *Let M be a right R-module.*

(i)$N \leq_e M$ if and only if $N \cap mR \neq 0$ for every $0 \neq m \in M$.

(ii) Let $K \leq N \leq M$. $K \leq_e M$ if and only if $K \leq_e N$ and $N \leq_e M$.

(iii) Let $N \leq_e M$ and $K \leq M$. Then $N \cap K \leq_e K$.

(iv) Let $N_i \leq_e K_i$ for $1 \leq i \leq t$. Then $N_1 \cap N_2 \cap ...... \cap N_t \leq_e K_1 \cap K_2 \cap ...... \cap K_t$.

(v) Let $K \leq N \leq M$. If $(N/K) \leq_e (M/K)$, then $N \leq_e M$.

(vi) If $K \leq_c N \leq_e M$ then $(N/K) \leq_e (M/K)$.

(vii) Let $N \leq_e M$ and $m \in M$. $(N : m) = \{r \in R : mr \in N\} \leq_e R_R$.

(viii) Let $N_i \leq_e M_i (i \in I)$ for a nonempty index set I. Then $\oplus_I N_i \leq_e \oplus_I M_i$.

Lemma 1.4 *Let M be a right R-module and $A, B \leq M$. If $A \cap B = 0$, there exists a complement C of B such that $A \leq_e C$ and $C \oplus B \leq_e M$.*

There are two kinds of complement definitions in literature. The first one is above. At the same time this definition is known as complement in Faith meaning. The second one is complement in Harada meaning : Let R be a ring and let M be an R-module. For $N \leq M$, the submodule $Cl_M(N) = \{m \in M : (N : m) \leq_e R\}$ is called the closure of N in M. If $Cl_M(N) = N$, N is called the complement in Harada meaning.

Every complement submodule in Harada meaning is complement submodule in Faith meaning, but in general, the converse of the above implication is not true.

Example 1.5 *Let Z be a Z-module and* $E = E(Z_Z)$*(where* $E = E(Z_Z)$ *is the minimal injective Z-module contains* Z_Z *as essential). Let p be a prime integer and let* $M = E \oplus Z_p$. $Cl_M(E) = E$ *and* $Cl_M(Z_p) = Z_p$. *Let* $K \leq_c E \oplus Z_p$. *For each* $x \in K$*, there exists* $x' \in E$ *and* $n' \in Z_p$ *such that* $x = (x', n')$. *If* $K < E$ *or* $K < Z_p$, $Cl_M(K) = E \neq K$ *or* $Cl_M(K) = Z_p \neq K$. *Let* $K \nleq E$ *and* $K \nleq Z_p$. *For* $0 \neq x \in K$, $x = (x', n') : 0 \neq x' \in E$, $0 \neq n' \in Z_p$. $Zx' \leq K$ *and* $Zn' \leq K$*, also* $x' \in E$ *and* $n' \in Z_p$ *then* $Zx' \leq_e E$ *and* $Zn' \leq_e Z_p$. *For each* $x \in E$, $(Zx' : x) \leq_e Z$ *and for each* $n \in Z_p$, $(Zn' : n) \leq_e Z$. $(x, n) \in E \oplus Z_p$ *and*

$$I = (Zx' : x) \cap (Zn' : n) \leq_e Z$$

since $I(x, n) \leq K, (x, n) \in Cl_M(K)$. *Hence* $Cl_M(K) = E \oplus Z_p \neq K$.

Definition 1.6 *Let M be a right R-module. Then the submodule of M*

$$Z(M) = \{m \in M : r_R(m) \leq_e R\}.$$

is called singular submodule of M. If $Z(M) = M$*, (*$Z(M) = 0$*), then M is called singular (nonsingular) R-module.*

$$Z_2(M) = \{m \in M : m + Z(m) \in Z(M/Z(M))\}.$$

$Z_2(M)$ is a submodule of M and it is the largest singular submodule of M. Also $Z(M) \leq_e Z_2(M)$. In fact, let $m \in Z_2(M)$. Then $m + Z(m) \in Z(M/Z(M))$. This implies that there exists an essential ideal I in R such that $mI \leq Z(M)$. Hence $Z(M) \leq_e Z_2(M)$.

Lemma 1.7 *Let M be a nonsingular right R-module and let N be a submodule of M. Then ;*

(i) $N \leq_e M$ if and only if $Z(M/N) = M/N$.

(ii) $Z_2(M) \leq_c M$.

Proposition 1.8 *Let M be a nonsingular right R-module. The submodule K of M is the complement in Harada meaning if and only if K is the complement in Faith meaning.*

Definition 1.9 *Let M be a right R-module and $N \leq M$. K is called essential closure of N in M such that $N \leq_e K \leq_c M$.*

Proposition 1.10 *Let M be a right R-module and $N \leq K \leq M$. Then*

(i) $N \leq_c M$ if and only if the essential closure of N in M is itself.

(ii) $N \leq_c K \leq_c M$ then $N \leq_c M$ and if $N \leq_c M$ then $N \leq_c K$.

(iii) If L is the complement of N in M and U is the complement of L in M with $N \leq U$, then $N \leq_e U$.

(iv) L is essential closure of N in M if and only if L is the maximal submodule with respect to the property $N \leq_e L$ if and only if L is the minimal submodule of the complement submodules which contain N in M.

1.2 Semi-simple Modules

Definition 1.11 *Let M be a right R-module. The submodule*

$$Soc(M) = \bigcap\{N \leq M : N \text{ is essential submodule }\}$$

$$= \sum\{N \leq M : N \text{ is simple submodule }\}$$

is called socle of M.

Lemma 1.12 *Let M be a right R-module. $Soc(M)$ is direct summand of simple submodules of M. i.e. $Soc(M) = \bigoplus_{i\in I} M_i$ where M_i is simple submodule of M for all $i \in I$.*

Theorem 1.13 *Let M be a right R-module. The followings are equivalent.*

(i) Every submodule of M is a sum of the simple submodules of M.

(ii) M is a sum of simple submodules of M.

(iii) M is a direct sum of simple submodules of M.

(iv) Every submodule of M is a direct summand of M.

Definition 1.14 *Let M be a right R-module. M is called a semi-simple module if M satisfies one of the conditions of Theorem 1.13.*

Corollary 1.15 *(i) Every submodule of a semi-simple module is semi-simple.*

(ii) Homomorphic image of every semi-simple module is semi-simple.

(iii) Every sum of semi-simple modules is semi-simple.

Lemma 1.16 *Let $\{M_i : i \in I\}$ be a family of modules. Then*

$$\bigoplus_{i\in I} Soc(M_i) = Soc(\bigoplus_{i\in I} M_i).$$

1.3 Finite Uniform Dimension Modules

Definition 1.17 *Let M be a right R-module. M is called uniform module if every submodule of M is essential in M.*

Definition 1.18 *Let M be a right R-module. Then we call M has a finite uniform dimension (finite Goldie dimension) if there exists an independent sequence $H_1, H_2, ..., H_n$ $(n < \infty)$ of uniform submodules of M with $H_1 \oplus H_2 \oplus .... \oplus H_n \leq_e M$. Also it is denoted by $ud(M) = n < \infty$*

Proposition 1.19 *Let M be a right R-module and $A \leq M$.*

(i) M has a finite uniform dimension if and only if every submodule of M has a finite uniform dimension.

(ii) If $A \leq_c M$ has a finite uniform dimension then (M/A) has a finite uniform dimension.

(iii) If $A_1, A_2, ..., A_n \leq M$ and for each i, A_i has a finite uniform dimension then $A_1 \oplus A_2 \oplus .... \oplus A_n$ has a finite uniform dimension.

(iv) If $A \leq_e M$ and A has a finite uniform dimension then M has a finite uniform dimension.

Lemma 1.20 *Let M be a right R-module.*

(i) If $A_1, A_2, ..., A_n \leq M$ then

$$ud(A_1 \oplus A_2 \oplus .... \oplus A_n) = ud(A_1) + ud(A_2) + ... + ud(A_n).$$

(ii) Let $A \leq M$ and A has a finite uniform dimension. Then $A \leq_e M$ if and only if $ud(M) = ud(A)$.

Proposition 1.21 *Let M be a right R-module and $A \leq M$.*

(i) If $A \leq_c M$ then $ud(M) = ud(A) + ud(M/A)$.

(ii) Let M has a finite uniform dimension. If $ud(M) = ud(A) + ud(M/A)$ then $A \leq_c M$.

1.4 Injective Modules

Definition 1.22 *Let R be a ring.Let M and A be R-modules with identity. If every homomorphism from a submodule X of A to M extend from A to M then M is said to be A-injective. For every R-module A if M is A-injective then M is called injective module. If M is M-injective then M is called quasi-injective module. M and A are called relatively injective if M is A-injective and A is M-injective.*

Note : If M is R_R injective then M is injective.

Proposition 1.23 *Let $\{M_i : i \in I\}$ be a family of R-modules. $\prod_{i \in I} M_i$ is injective if and only if for each $i \in I$, M_i is injective.*

Proposition 1.24 *Let M be a right R-module.*

(i) M is injective if and only if M is a direct summand of every R-module which contains M.

(ii) Let A be an R-module and B be a submodule of A. If M is A-injective then M is A/B and B-injective.

Proof. It is clear that M is B-injective. Let $X \leq A$ and X/B be a submodule of A/B and $\varphi : X/B \to M$ be a homomorphism. Let $\pi : A \to A/B$ be projection map and $\pi' = \pi|_X$. Since M is A-injective, there exists a homomorphism $\theta : A \to M$ that extends $\varphi\pi'$. Now $\theta(B) = (\varphi\pi')(B) = \varphi(0) = 0$. Hence $Ker\pi \leq Ker\theta$. Hence there exists a homomorphism $\psi : A/B \to M$ such that $\psi\pi = \theta$. For every $x \in X$

$$\psi(x + B) = \psi(\pi(x)) = \theta(x) = \varphi\pi'(x) = \varphi(x + B).$$

Thus ψ extends φ, and therefore N is A/B-injective. □

Proposition 1.25 *A module M is $(\bigoplus_{i\in I} A_i)$-injective if and only if M is A_i-injective for every $i \in I$.*

Proof. Assume that M is A_i-injective for all $i \in I$. Let $A = \bigoplus_{i\in I} A_i$, $X \leq A$ and consider a homomorphism $\varphi : X \to M$. We may assume, by Zorn's Lemma, that φ cannot be extended to a homomorphism $X' \to M$ for any submodule X' of A which contains X properly. Then $X \leq_e A$. We claim that $X = A$. Suppose not. Then there exists $j \in I$ and $a \in A_j$ such that a is not an element of X. Since M is A_j-injective, M is aR-injective. Let $K = \{r \in R : ar \in X\}$. K is an ideal of R and aK is a submodule of aR and also $aK \leq X$. $M = \varphi|_{aK} : aK \to M$ is a homomorphism and extends to a homomorphism $\beta : aR \to M$. Let $\psi : X + aR \to M$ be defined by $\psi(x + ar) = \varphi(x + \beta(ar))$. $\psi|_X = \varphi$. This is a contradiction by maximality of φ. Then $X = A$. □

Definition 1.26 *Let M be a right R-module. The injective module which contains M as essential is called the injective hull of M and it is denoted by $E(M)$.*

Proposition 1.27 *Let M be a right R-module. The following are equivalent.*

(i) The injective hull of M is $E(M)$.

(ii) $E(M)$ is the maximal module of the modules which contains M as essential.

(iii) $E(M)$ is the minimal module of the injective modules which contain M.

1.5 Continuous Modules

Definition 1.28 *Let R be a ring and let M be a right R-module. If every complement submodule K of M is a direct summand of M then M is called CS-module ((C_1) condition holds). Equivalently, for every submodule K of M there exists a direct summand N of M such that K is essential in N.*

The ring R is called right CS-ring if R_R is CS-module. For every $I \leq_c R_R$ there exists idempotent $e \in R$ such that $I = eR$. For example, semi-simple modules, uniform modules and injective modules are CS-modules.

Every complement of a CS-module is CS-module. But any submodule of a CS-module may not be CS-module. For example, let M be not a CS-module. Since $E(M)$ is injective module, $E(M)$ is CS-module. M is essential in $E(M)$ but M is not CS module. Also the direct sum of two CS-modules may not be CS-module.

Example 1.29 *Let Z denote the integers, let p be any prime, let $M_1 = Z/Z_p$ and let $M_2 = Z/Z_{p^3}$. M_1 and M_2 are CS-Z-modules. But $M = M_1 \oplus M_2$ is not CS-module.*

Definition 1.30 *A right R module M is called indecomposable module if M has no non-zero proper direct summand. Equivalently, M is indecomposable if and only if for any $K \leq_d M$, $K = 0$ or $K = M$.*

Proposition 1.31 *Let M be an indecomposable right R-module. If M is CS-module then M is uniform module.*

Definition 1.32 *Let M be a right R-module.*

(C_2): *Every submodule of M which isomorphic to a direct summand of M is a direct summand of M.*

(C_3): *If N_1, N_2 be two direct summands of M such that $N_1 \cap N_2 = 0$, then $N_1 \oplus N_2$ is a direct summand of M.*

Lemma 1.33 *Every direct summand of M satisfying $(C_i)(i = 1, 2)$ satisfies $(C_i)(i = 1, 2)$.*

Definition 1.34 *A right R-module M is called continuous (quasi-continuous) if M is CS-module satisfying the condition (C_2) $((C_3))$.*

Lemma 1.35 *Every module M satisfying the condition (C_2) satisfies the condition (C_3).*

Proof. Let K, L be direct summands of M with $K \cap L = 0$, $M = K \oplus K'$ for a submodule K' of M. Let $\pi : M \to K'$ be the projection map. $K \cap L = 0$ then $\pi(L) \cong L$ and $\pi(L) \leq K'$. By the condition (C_2), $\pi(L) \leq M$ and hence $M = \pi(L) \oplus L'$ for a submodule L' of M. Then $K' = \pi(L) \oplus (K' \cap L')$ and $M = K \oplus \pi(L) \oplus (K' \cap L')$. Hence $K \oplus \pi(L) \leq_d M$. $K \oplus \pi(L) = K \oplus L$ then $K \oplus L \leq_d M$. □

CHAPTER 2

FINITE DIRECT SUMS OF CS-MODULES

In this chapter, all rings are associative with identity element and all modules are unital right modules. We concern with when a direct sum of CS-modules is CS-module. In [45], it is proved that for any ring R, the direct sum $M = \bigoplus_{i \in I} M_i$ is CS if and only if there exists $i \neq j$ in I such that every closed submodule K of M with $K \cap M_i = 0$ or $K \cap M_j = 0$ is direct summand. In addition, if R is any ring, M_1 is a uniform R-module of finite composition length and M_2 is a simple R-module, then $M_1 \oplus M_2$ is CS if and only if M_2 is M_1/N-injective for every non-zero submodule N of M_1. In [18], it is proved that if M_1 and M_2 are relatively injective CS-modules then $M = M_1 \oplus M_2$ is CS-module.

Lemma 2.1 *Let M be any module and $K \subseteq L$ submodules of M such that K is a complement in L and L is a complement in M. Then K is a complement in M.*

Proof. Let K_1 be a complement of K in L. Then $K \cap K_1 = 0$ and $K \oplus K_1$ is essential in L. Let L_1 be a complement of L in M. Then $L \cap L_1 = 0$ and $L \oplus L_1$ is essential in M.

$$\frac{K \oplus K_1}{K} \subseteq^{ess} \frac{L}{K} \text{ and } \frac{L \oplus L_1}{L} \subseteq^{ess} \frac{M}{L}$$

Claim: $\frac{K+K_1+L_1}{K} \subseteq^{ess} \frac{M}{K}$

proof. Observe first that

$$(K + K_1) \cap (K + L_1) = K + ((K + K_1) \cap L_1) \subseteq K + (L \cap L_1) = K.$$

We have

$$\frac{K+K_1+L_1}{K} = \frac{K+K_1}{K} \oplus \frac{K+L_1}{K} \subseteq^{ess} \frac{L}{K} \oplus \frac{K+L_1}{K} = \frac{L+L_1}{K}$$

So it suffices to show that $\frac{L+L_1}{K} \subseteq^{ess} \frac{M}{K}$. Let $\alpha : \frac{M}{K} \to \frac{M}{L}$ given by $\alpha(m+K) = m+L$. Since $\frac{L \oplus L_1}{L} \subseteq^{ess} \frac{M}{L}$ and $\alpha^{-1}(\frac{L \oplus L_1}{L}) = \frac{L+L_1}{K}$, $\frac{L+L_1}{K} \subseteq^{ess} \frac{M}{K}$. This proves the claim.

Now suppose that $K \subseteq^{ess} N \subseteq M$. We must show that $K = N$. $K \cap (K_1 + L_1) = 0$ (in fact, if $k \in K \cap (K_1 + L_1)$, then $k = k_1 + l_1$ where $k_1 \in K_1$, $l_1 \in L_1$. Then $k - k_1 = l_1 \in L \cap L_1 = 0$). Since $K \subseteq^{ess} N$, $N \cap (K_1 + L_1) = 0$. Hence $\frac{N}{K} \cap \frac{L+L_1+K_1}{K} = 0$ implies that $\frac{N}{K} = 0$ and so $N = K$. □

Lemma 2.2 *Any direct summand of a CS-module is a CS-module.*

Proof. Let M be a CS-module and M_1 be a direct summand of M. Let K be a complement submodule of M_1. By Lemma 2.1, K is a complement in M. Since M is CS-module, K is a direct summand of M. Then there exists a direct summand K_1 of M such that $M = K \oplus K_1$. By modularity $M_1 = M \cap M_1 = M_1 \cap (K \oplus K_1) = K \oplus (M_1 \cap K_1)$. Hence K is a direct summand of M_1 and so M_1 is a CS-module. □

Proposition 2.3 *Any indecomposable module M is a CS-module if and only if M is uniform.*

Proof. Let M be an indecomposable CS-module. Let N be a submodule of M such that it is not essential in M. Since M is CS-module, there exists a direct summand K of M such that $N \subseteq^{ess} K \subseteq^{d} M$. Since M is indecomposable, $K = M$. This is a contradiction. Thus, M is uniform.

Conversely, suppose that M is indecomposable uniform module. Let K be a non-zero complement submodule of M. Then there exists a submodule L of M such that $K \cap L = 0$ and $K \oplus L \subseteq^{ess} M$. Since M is uniform, $L = 0$ and also $K = M$. □

Proposition 2.4 *Any (quasi-)injective module M is a CS-module.*

Proof. Let N be a submodule of M. Then $E(M) = E_1 \oplus E_2$ where $E_1 = E(N)$. The quasi-injectivity of M implies that $M = (M \cap E_1) \oplus (M \cap E_2)$. Since $N \subseteq^{ess} E_1$, $N \subseteq^{ess} M \cap E_1 \subseteq^{d} M$. □

In general, it is not true that the direct sum of two CS-module is CS-module.

Lemma 2.5 *Let K be a complement in M. Then K is a direct summand of M if and only if there exists a complement L of K in M such that every homomorphism $\varphi : K \oplus L \to M$ can be lifted to a homomorphism $\theta : M \to M$.*

Proof. Suppose first that K is a direct summand of M. Then $M = K \oplus K'$ for some module K' of M. Clearly, $L = K'$ will do.

Conversely, suppose that there exists a complement L of K in M with the stated property. Let $\varphi : K \oplus L \to M$ be the homomorphism defined by

$$\varphi(x + y) = x(x \in K, y \in L).$$

By hypothesis, there exists a homomorphism $\theta : M \to M$ such that

$$\theta(x + y) = x(x \in K, y \in L).$$

Note that $K \subseteq im\theta$ and $L \subseteq ker\theta$.

Let $0 \neq \upsilon \in im\theta$. Then there exists $u \in M$ such that $\upsilon = \theta(u)$. Note that $u \notin L$. Thus $K \cap (L + uR) \neq 0$. There exists $x \in K, y \in L$ and $r \in R$ such that $0 \neq x = y + ur$. Then $x = \theta(x) = \theta(y + ur) = \upsilon r$. It follows that $\upsilon R \cap K \neq 0$ for all non-zero $\upsilon \in im\theta$. Thus K is an essential submodule of $im\theta$. But K is a complement in M. Hence $K = im\theta$. □

Corollary 2.6 *A module satisfies (C_1) if and only if for every complement K in M there exists a complement L of K in M such that every homomorphism $\varphi : K \oplus L \to M$ can be lifted to a homomorphism $\theta : M \to M$.*

Proof. Immediate by Lemma 2.5. □

Let n be a positive integer. We consider the following condition for a module M:

(P_n) For every submodule K of M such that K is a direct sum $K_1 \oplus \ldots \oplus K_n$ of complements $K_i (1 \leq i \leq n)$ in M, every homomorphism $\varphi : K \to M$ can be lifted to a homomorphism $\theta : M \to M$.

It is clear that if M satisfies (P_n) then M satisfies (P_{n-1}) for all $n \geq 2$. Modules satisfying (P_1) have been considered in [44].

Example 2.7 *Let Z denote the integers, let p be any prime, let $M_1 = Z/Z_p$ and let $M_2 = Z/Z_{p^3}$. M_1 and M_2 are CS-Z-modules. But $M = M_1 \oplus M_2$ is not CS-module.*

Theorem 2.8 *Let M be any module, and let $Z_2(M)$ denote its second singular submodule. Then M is a CS-module if and only if $M = Z_2(M) \oplus N$, where $Z_2(M)$ and N are CS-modules and $Z_2(M)$ is N-injective.*

Proof. Suppose that M is a CS-module. Since $Z_2(M)$ is closed in M and M is a CS-module, we have $M = Z_2(M) \oplus N$, where N is non-singular. By Lemma 2.2, $Z_2(M)$ and N are CS-modules.

To show that $Z_2(M)$ is N-injective, let $\phi : X \to Z_2(M)$ be a homomorphism from a submodule X of N to $Z_2(M)$. Consider

$$X_1 = \{x - \phi(x) \mid x \in X\}.$$

Since M is CS-module, there exists $X_1 \leq_e X^* \leq_d M$. Write $M = X^* \oplus Y$ where Y is a submodule of M. Let $x \in X_1 \cap Z_2(M)$. Then $x = z - \phi(z)$ where $z \in X$. It follows that $x + \phi(z) = z \in X \cap Z_2(M) = 0$. So $X_1 \cap Z_2(M) = 0$ and also $X^* \cap Z_2(M) = 0$. Thus X^* is non-singular and that $Z_2(M) = Z_2(Y) \leq_d Y$, say $Y = Y_1 \oplus Z_2(M)$. Let $\pi : X^* \oplus Y_1 \oplus Z_2(M) \to Z_2(M)$ be the projection. $\alpha = \pi \mid_N$ extends ϕ. In fact, for any $x \in X$, $x = (x - \phi(x)) + \phi(x)$.

$$\pi(x) = \pi((x - \phi(x)) + \phi(x)) = \pi(x - \phi(x)) + \pi(\phi(x)) = \phi(x).$$

Conversely, let $M = Z_2(M) \oplus N$, where $Z_2(M)$ and N are CS-modules and $Z_2(M)$ is N-injective. Let A be a complement submodule of M. Since $Z_2(M)$ is CS-module, we have $Z_2(A) \subseteq_d Z_2(M)$, and hence $Z_2(A) \subseteq_d A$. Write $A = Z_2(A) \oplus B$, where B is a non-singular submodule of A. Since $B \cap Z_2(M) = 0$ and $Z_2(M)$ is N-injective, there exists a homomorphism $\psi : N \to Z_2(M)$ such that $\psi\pi_2 |_B = \pi_1 |_B$, where $\pi_1 : M \to Z_2(M)$ and $\pi_2 : M \to N$ are projections. Consider

$$N^* = \{n + \psi(n) \mid n \in N\}.$$

For $x \in B$, $x = m_1 + m_2$, where $m_1 \in Z_2(M)$, $m_2 \in N$.

$$x = m_1 + m_2 = \pi_1(x) + \pi_2(x) = \pi_2(x) + \psi(\pi_2(x)) \in N^*.$$

Hence $B \subseteq N^*$. It follows that B is closed in N^*. Let $x \in N^* \cap Z_2(M)$. Then there exists $n \in N$ such that $x = n + \psi(n)$ and $x - \psi(n) = n \in N \cap Z_2(M) = 0$ and so $x = 0$. This implies that $N^* \cap Z_2(M) = 0$. For any $m \in M$, $m = m_1 + m_2$; where $m_1 \in Z_2(M)$, $m_2 \in N$. $m = m_1 + m_2 = (m_1 + \psi(m_2)) + (m_2 - \psi(m_2)) \in Z_2(M) + N^*$. Hence $M = Z_2(M) \oplus N^* = Z_2(M) \oplus N$, implies $N^* \cong N$. Since $N^* \cong N$, N^* is a CS-module, we have $B \leq_d N^*$. It is clear that $M = Z_2(M) \oplus N^*$; therefore $A \leq_d M$. □

Lemma 2.9 *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. Then the following statements are equivalent.*

(i) M_2 is M_1-injective.

(ii) For each submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \subseteq M'$.

Proof. (i) $\Rightarrow$ (ii). For $i = 1,2$, let $\pi_i : M \to M_i$ denote the projection mapping. Let $\alpha = \pi_1 |_N$ and $\beta = \pi_2 |_N$. Then α is a monomorphism. By (i), there exists a homomorphism $\phi : M_1 \to M_2$ such that $\phi\alpha = \beta$. Let

$$M' = \{x + \phi(x) : x \in M_1\}.$$

Since $M' \cap M_2 = 0$ and $M = M' + M_2$, $M = M' \oplus M_2$. For $x \in N$, $x = m_1 + m_2$, where $m_1 \in M_1$, $m_2 \in M_2$.

$$x = m_1 + m_2 = \pi_1(x) + \pi_2(x) = \pi_1(x) + \phi(\pi_1(x)) \in M'.$$

Hence $N \subseteq M'$.

(ii) $\Rightarrow$ (i). Let K be a submodule of M_1, and $\alpha : K \to M_2$ be a homomorphism. Let

$$L = \{y - \alpha(y) : y \in K\}.$$

Then L is a submodule of M and $L \cap M_2 = 0$. By (ii), $M = L' \oplus M_2$ for some submodule L' such that $L \leq L'$. Let $\pi : L' \oplus M_2 \to M_2$ denote the canonical projection. Then $\beta = \pi |_{M_1}: M_1 \to M_2$ and, for any $y \in K$,

$$\beta(y) = \beta((y - \alpha(y)) + \alpha(y)) = \alpha(y).$$

It follows that β lifts α to M_1. Thus M_2 is M_1-injective. □

Theorem 2.10 *Let M be a module such that $M = M_1 \oplus M_2$, where M_1 and M_2 are CS-modules. Suppose that M_1 is nonsingular and M_2 is M_1-injective. Then M is a CS-module.*

Proof. Because M_2 is a CS-module, then by Theorem 2.8, $M_2 = Z_2(M_2) \oplus M'$ for some nonsingular submodule M' of M_2 such that M' and $Z_2(M_2)$ are CS-modules and $Z_2(M_2)$ is
M'-injective. Since $Z(M_1) = 0$, $Z_2(M) = Z_2(M_2)$ and $Z_2(M)$ is M_1-injective. Thus $M = Z_2(M) \oplus (M_1 \oplus M')$, where $Z_2(M)$ is a CS-module, $Z_2(M)$ is $(M_1 \oplus M')$-injective, M_1 and M' are CS-modules and M' is M_1-injective. By [7, Theorem 1], M is a CS module if $M_1 \oplus M'$ is a CS-module. Thus we can suppose without loss of generality that M_2 is nonsingular, and hence M is nonsingular.

Let K be a complement in M. Because M_2 is a CS-module, there exist submodules L_1, L_2 of M_2 such that $M_2 = L_1 \oplus L_2$ and $K \cap M_2$ is essential in L_1. Let $0 \neq x \in K + L_1$.

Then $x = y + z$ for some $y \in K$, $z \in L_1$. Because $K \cap M_2$ is essential in L_1, there exists an essential right ideal E of R such that $zE \subseteq K$. Then M nonsingular gives

$$0 \neq xE = (y + z)E \subseteq xR \cap K \subseteq K.$$

It follows that K is essential in $K + L_1$.

Now $M = M_1 \oplus M_2 = M_1 \oplus L_1 \oplus L_2$ and, by the Modular Law,

$$K = K \cap M = K \cap (M_1 \oplus L_1 \oplus L_2) = L_1 \oplus (K \cap (M_1 \oplus L_2))$$

Note that

$$(K \cap (M_1 \oplus L_2)) \cap L_2 \subseteq K \cap M_2 \cap L_2 \subseteq L_1 \cap L_2 = 0.$$

By Lemma 2.9, $M_1 \oplus L_2 = M'' \oplus L_2$ for some submodule M'' with $K \cap (M_1 \oplus L_2) \subseteq M''$. Clearly $M'' \cong M_1$, so that M'' is a CS-module and $K \cap (M_1 \oplus L_2)$ is a complement in M''. Thus $K \cap (M_1 \oplus L_2)$ is a direct summand of M'', and $K = L_1 \oplus (K \cap (M_1 \oplus L_2))$ is a direct summand of M. It follows that M is a CS-module. □

Theorem 2.11 *A module M is a CS-module with finite Goldie dimension if and only if*

(i) M is a finite direct sum of uniform submodules, and

(ii) every direct summand of M of uniform dimension 2 is a CS-module.

Proof. Suppose M is a CS-module with finite non-zero Goldie dimension. Let U be a maximal uniform submodule of M. Then U is a complement in M. By hypothesis, $M = U \oplus U'$ for some submodule U' of M. By induction on Goldie dimension and Lemma 2.2, U' is a finite direct sum of uniform submodules. This proves (i). Also Lemma 2.2 proves (ii).

Conversely, suppose M satisfies (i), (ii). Let $M = U_1 \oplus \ldots \oplus U_n$, where n is a positive integer and U_i is uniform submodule of M for each $1 \leq i \leq n$. Let V be a maximal uniform submodule of M. Suppose $V \neq M$. Then $V \cap U_i = 0$ for some $1 \leq i \leq n$.

Without loss of generality, i = 1. Let $U' = U_2 \oplus \ldots \oplus U_n$. There exists a complement K in M such that $V \oplus U_1$ is essential in K. By the Modular Law

$$K = U_1 \oplus (K \cap U')$$

Clearly $K \cap U'$ is a complement in K, and hence also in M by Lemma 2.1. Thus $K \cap U'$ is a complement in U'. By induction on Goldie dimension, $K \cap U'$ is a direct summand of U'. This implies at once that K is a direct summand of M. Clearly K has Goldie dimension 2, so that, by hypothesis, K is a CS-module. Hence V is a direct summand of K, and hence also of M.

Now let L be any complement in M. Let W be a maximal uniform submodule of L. Then $W \leq_c L$ and by Lemma 2.1 W is a complement in M. By above argument W is a direct summand of M. Thus $M = W \oplus W'$ for some submodule W' of M. Thus $L = W \oplus (L \cap W')$ and $L \cap W'$ is a complement in M by Lemma 2.1. By induction on the Goldie dimension of L, $L \cap W'$ is a direct summand of M, and hence also of W'. Thus L is a direct summand of M. It follows that M is a CS-module. □

For any set I, $|I|$ will denote its cardinality.

Theorem 2.12 *Let M be a module such that $M = \bigoplus_{i\in I} M_i$ be the direct sum of R-modules $M_i (i \in I)$, for some index set I with $|I| \geq 2$. Then the following statements are equivalent.*

(i) M is CS.

(ii) There exist $i \neq j$ in I such that every closed submodule K of M with $K \cap M_i = 0$ or $K \cap M_j = 0$ is a direct summand.

(iii) There exist $i \neq j$ in I such that every complement of M_i or of M_j in M is a CS-module and a direct summand of M.

Proof. (i) ⇒ (ii). Suppose that M is a CS-module.Then every complement of M is direct summand.

(ii) $\Rightarrow$ (iii). Let K be a complement of M_i in M. By (ii), K is a direct summand of M. Let L be a closed submodule of K. By Lemma 2.1, L is a closed submodule of M, and clearly $L \cap M_i = 0$. By (ii), L is a direct summand of M, and hence also of K. Thus K is CS.

(iii) $\Rightarrow$ (i). Let N be a closed submodule of M. There exists a closed submodule H of N such that $N \cap M_i$ is essential in H. Clearly $H \cap M_j = 0$. By Zorn's Lemma there exists a complement P of M_j in M such that $H \leq P$. Now Lemma 2.1 gives H closed in M and hence H is closed in P. Applying (iii) we see that H is a direct summand of the CS-module P and P is a direct summand of M. Hence H is a direct summand of M.

There exists a submodule H' of M such that $M = H \oplus H'$. The Modular Law gives $N = H \oplus (N \cap H')$. By Lemma 2.1, $N \cap H'$ is a closed submodule of M and clearly $(N \cap H') \cap M_i = 0$. By the above argument, (iii) gives that $N \cap H'$ is a direct summand of M, and hence also of H'. It follows that N is a direct summand of M. Thus M is CS. □

Definition 2.13 *Let M be a module and K, L are direct summands of M with $K \cap L = 0$. M satisfies condition (C_3) if $K \oplus L$ is a direct summand of M.*

Lemma 2.14 *The following statements are equivalent for a module M.*

(i) M satisfies (C_3).

(ii) For all direct summands P, Q of M with $P \cap Q = 0$, there exists a submodule P' of M such that $M = P \oplus P'$ and $Q \subseteq P'$.

Proof. $(i) \Rightarrow (ii)$. Let P and Q be direct summands of M with $P \cap Q = 0$. By (i), $P \oplus Q$ is a direct summand of M and hence $M = P \oplus Q \oplus Q''$ for some submodule Q'' of M. Thus $P' = Q \oplus Q''$ has the required properties.

$(ii) \Rightarrow (i)$. Let K, L be direct summands of M such that $K \cap L = 0$. By (ii), $M = K \oplus K'$ for some submodule K' of M such that $L \subseteq K'$. But $M = L \oplus L'$ for some submodule L' of M, and hence

$$K' = K' \cap M = K' \cap (L \oplus L') = L \oplus (K' \cap L').$$

Thus $M = K \oplus K' = K \oplus L \oplus (K' \cap L')$ and $K \oplus L$ is a direct summand of M. Therefore M satisfies (C_3). □

Definition 2.15 *A module M is called quasi-continuous if M is CS-module satisfying (C_3).*

Proposition 2.16 *A CS-module M is quasi-continuous if and only if whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2, then M_2 is M_1-injective.*

Proof. Suppose that M is quasi-continuous. Suppose $M = M_1 \oplus M_2$. Let N be a submodule of M with $N \cap M_2 = 0$. Because M is a CS-module, there exists a direct summand N' of M such that N is essential in N'. Clearly $N' \cap M_2 = 0$. By Lemma 2.14, $M = M' \oplus M_2$ for some submodule M' of M such that $N' \subseteq M'$. By Lemma 2.9, M_2 is M_1-injective.

Conversely, suppose M_2 is M_1-injective whenever $M = M_1 \oplus M_2$. By Lemma 2.9 and Lemma 2.14, M satisfies (C_3). Thus M is quasi-continuous. □

Definition 2.17 *Let n be a positive integer. Modules $M_1, M_2, \ldots, M_n$ are called relatively injective if M_i is M_j-injective for all $1 \leq i \neq j \leq n$.*

Theorem 2.18 *Let M be a CS-module such that $M = M_1 \oplus \ldots \oplus M_n$ is a finite direct sum of relatively injective modules M_i $(1 \leq i \leq n)$. Then M is a CS-module if and only if M_i is a CS-module for each $1 \leq i \leq n$.*

Proof. Suppose that $M = M_1 \oplus \ldots \oplus M_n$ is a CS-module. By Lemma 2.2, M_i is CS-module for each $1 \leq i \leq n$.

Conversely suppose that M_i is a CS-module ($1 \leq i \leq n$). We prove that M is a CS-module by induction on n. It is clearly sufficient to prove the case $n = 2$. Suppose $M = M_1 \oplus M_2$. Let K be a complement in M. By Zorn's Lemma there exists a submodule L of K maximal with respect to the property $L \cap M_1 = L \cap (K \cap M_1) = 0$. This implies that $L \oplus (K \cap M_1)$ is essential in K. Clearly L is a complement in K, and hence also in M. Because M_1 is M_2-injective, there exists a submodule M' of M such that $M = M_1 \oplus M'$ and $L \subseteq M'$. Note that $M' \cong M_2$, so that without loss of generality $M' = M_2$, and hence $L \subseteq M_2$. Now L is a complement in M_2 which is a CS-module, so that $M_2 = L \oplus L'$ for some submodule L' of M_2.

Note that $M = M_1 \oplus M_2 = M_1 \oplus L \oplus L'$ and $K = L \oplus K'$, where $K' = K \cap (M_1 \oplus L')$ is a complement in $M_1 \oplus L'$. We now claim that $K' \cap M_1$ is essential in K'. In fact, $L \oplus (K \cap M_1)$ is essential in K. Hence $[L \oplus (K \cap M_1)] \cap K'$ is essential in $K' \subseteq K$. But clearly $K' \cap M_1 = K \cap M_1$, and hence

$$[L \oplus (K \cap M_1)] \cap K' = [L \oplus (K' \cap M_1)] \cap K' = (L \cap K') \oplus (K' \cap M_1) = K' \cap M_1.$$

Thus $K' \cap M_1$ is essential is K'. But clearly

$$(K' \cap M_1) \cap (K' \cap L') \subseteq M_1 \cap L' = 0,$$

so that $K' \cap L' = 0$. By hypothesis, L' is M_1-injective and hence, by Lemma 2.9, $M_1 \oplus L' = M'' \oplus L'$ for some submodule M'' with $K' \subseteq M''$. Clearly $M'' \cong M_1$ and K' is a complement in M''. Thus K' is a direct summand of $M_1 \oplus L'$, and K is a direct summand of M. It follows that M is a CS-module. □

Example 2.19 *Let p be any prime integer and let R denote the local ring Z_p. Let M denote the Z-module $(Z/Zp) \oplus Q$. Then*

(i) M is an R-module.

(ii) K is a complement in M if and only if K is a direct summand of M or $K = R(1 + Zp, q)$ for some non-zero element q in Q.

(iii) M is not a CS-module.

Proof. (i) Let $M_1 = (Z/Zp) \oplus 0$ and $M_2 = 0 \oplus Q$, so that $M = M_1 \oplus M_2$. The ring R is the subring of Q consisting of all rational numbers s/t such that $s, t \in Z, t \neq 0$ and t is coprime to p. Note first that for any element m in M and any $s, t \in Z$ such that p does not divide t, there exists a unique element $m' \in M$ such that $tm' = sm$, and we shall denote m' by $(s/t)m$. In this way M is an R-module.

(ii) Let $q \in Q$ and $K = R(1 + Zp, q)$. We show first that K is a complement in the Z-module M. Note that K is a uniform submodule of M. Suppose that N is a submodule of M such that K is an essential submodule of N. Let $x \in N$. Then $U = Zx + Z(1 + Zp, q)$ is a finitely generated uniform Z-module, and hence U is cyclic. Suppose that $U = Z(a + Zp, b)$, where $a \in Z, b \in Q$. There exists $n \in Z$ such that $(1 + Zp, q) = n(a + Zp, b)$. Note that $1 - na \in Zp$ and hence n is coprime to p, and $(a + Zp, b) \in R(1 + Zp, q) = K$. Thus $x \in K$. It follows that $K = N$. Hence K is a complement in M.

Let L be a complement in the Z-module M. Suppose that $L \neq 0, M$. Note that M has uniform dimension 2 and hence L is uniform [8, Lemma 1.9]. We shall show first that L is an R-submodule of M. Let

$$L' = \{m \in M : tm \in L \text{ for some } t \in Z, t \text{ coprime to } p\}.$$

Then L' is a submodule of M, in fact $L' = RL$. If $0 \neq m \in L'$ then $tm \in L$ for some $t \in Z$, coprime to p, and hence $tm \neq 0$. It follows that L is an essential submodule of L'. Thus $L = L'$, and L is an R-submodule of M.

Next we show that $L = 0, M, M_1, M_2$ or $R(1 + Zp, q)$ for some $q \in Q$. Suppose that $L \neq 0, M, M_1$ or M_2. Note that M_1 and M_2 are both uniform, so that L is not contained in either M_1 or M_2. Thus $(c + Zp, d) \in L$ for some $c \in Z$, coprime to p and $0 \neq d \in Q$. Without loss of generality we can suppose that $c = 1$. Because L is an R-submodule of M, $R(1 + Zp, d) \subseteq L$. But $R(1 + Zp, d)$ is a complement in M, and hence $L = R(1 + Zp, d)$. This completes the proof of (ii).

(iii) Let $N = R(1 + Zp, 1)$ is a complement submodule of M by (ii). Since N is not a direct summand of M, M is not a CS-module. □

Lemma 2.20 *Let module $M = M_1 \oplus M_2$ be a direct sum of relatively injective submodules M_1, M_2 such that M_2 is quasi-continuous. Let K, L be a direct summands of M such that $K \cap L = 0$. Suppose further that $K \cap M_1 = 0$. Then $K \oplus L$ is a direct summand of M.*

Proof. By Lemma 2.9, we can suppose without loss of generality that $K \subseteq M_2$. Then $M_2 = K \oplus K'$ for some submodule K' of M_2. Note that K is K'-injective (Proposition 2.16). Therefore K is $(M_1 \oplus K')$-injective. Now $M = K \oplus (M_1 \oplus K')$ and $L \cap K = 0$ so that, again using Lemma 2.9, $M = K \oplus K''$ for some submodule K'' with $L \subseteq K''$. Now L is a direct summand of M, hence also of K''. Thus $K \oplus L$ is a direct summand of M. □

Theorem 2.21 *Let R be a ring and M an R-module such that $M = M_1 \oplus \ldots \oplus M_n$ is a finite direct sum of submodules M_i $(1 \leq i \leq n)$. Then M is quasi-continuous if and only if $M_1, \ldots M_n$ are relatively injective quasi-continuous modules.*

Proof. Suppose that M is quasi-continuous. By Proposition 2.16 and [2, Proposition 2.7] M_i is quasi-continuous for each $1 \leq i \leq n$.

Conversely, suppose that M_i $(1 \leq i \leq n)$ are relatively injective and quasi-continuous. By induction on n, it is sufficient to prove the case $n = 2$. Thus suppose $M = M_1 \oplus M_2$. By Theorem 2.18, M is a CS-module. Let K, L be direct summands of M with $K \cap L = 0$. Then K is a CS-module, by Lemma 2.1, and hence $K = K_1 \oplus K_2$ for some submodules K_1, K_2 with $K \cap M_1$ essential in K_1.

Note that $K_2 \cap M_1 = K_2 \cap (K \cap M_1) = 0$. By Lemma 2.20, $K_2 \oplus L$ is a direct summand of M. On the other hand, $(K_1 \cap M_2) \cap (K \cap M_1) = 0$ implies that $K_1 \cap M_2 = 0$.

Again using Lemma 2.20, $K \oplus L = K_1 \oplus (K_2 \oplus L)$ is a direct summand of M. It follows that M is quasi-continuous. □

Lemma 2.22 *Let $M = M_1 \oplus M_2$ be a module and let K be a submodule of M. Then K is a complement of M_2 in M if and only if there exists a homomorphism $\varphi : M_1 \to E(M_2)$ such that $K = \{x + \varphi(x) : x \in \varphi^{-1}(M_2)\}$.*

Proof. Suppose that K is a complement of M_2 in M. Let $\pi_i : M \to M_i (i = 1, 2)$ denote the canonical projections. Note that $\pi_1|_K : K \to M_1$ is a monomorphism. If $\epsilon : M_2 \to E(M_2)$ is the inclusion mapping then there exists a homomorphism $\varphi : M_1 \to E(M_2)$ such that $\varphi(\pi_1|_K) = \epsilon(\pi_2|_K)$. For any $x \in K, \varphi\pi_1(x) = \pi_2(x) \in M_2$ so that $\pi(x) \in \varphi^{-1}(M_2)$, and

$$x = \pi_1(x) + \pi_2(x) = \pi_1(x) + \varphi(\pi_1(x)).$$

Thus $K \subseteq \{y+\varphi(y) : y \in \varphi^{-1}(M_2)\} = K_1$. But K_1 is a sub module of M and $K_1 \cap M_2 = 0$, so that $K = K_1$, as required.

Conversely, suppose that $\theta : M_1 \to E(M_2)$ is a homomorphism and $K = \{x + \theta(x) : x \in \theta^{-1}(M_2)\}$. Clearly K is a submodule of M and $K \cap M_2 = 0$. Suppose that L is a submodule of M such that $L \cap M_2 = 0$. Now suppose there exists $u \in L$ such that $\pi_2(u) \neq \theta\pi_1(u)$. Because $0 \neq \pi_2(u) - \theta\pi_1(u) \in E(M_2)$, there exists $r \in R$ such that $0 \neq \{\pi_2(u) - \theta\pi_1(u)\}r \in M_2$. But, in this case, $\theta\pi_1(u)r \in M_2$ and

$$\{\pi_2(u)-\theta\pi_1(u)\}r = \pi_2(ur)-\theta\pi_1(ur) = ur-\{\pi_1(ur)+\theta\pi_1(ur)\} \in (L+K)\cap M_2 = L\cap M_2 = 0,$$

a contradiction.

Let $v \in L$. Then $\theta\pi_1(v) = \pi_2(v) \in M_2$, so that $\pi_1(v) \in \theta^{-1}(M_2)$ and

$$v = \pi_1(v) + \pi_2(v) = \pi_1(v) + \theta(\pi_1(v)) \in K.$$

It follows that $L = K$. Thus K is a complement of M_2 in M. □

2.1 Arbitrary Direct Sums

Theorem 2.23 *Let R be any ring and let $M = \bigoplus_{i \in I} M_i$ be the direct sum of R-modules $M_i (i \in I)$, for some index set with $|I| \geq 2$. Then the following statements are equivalent:*

(i) M is CS.

(ii) For each $i \in I$ and each homomorphism $\varphi : M_{-i} = \bigoplus_{j \neq i} M_j \to E(M_i)$, the submodule $\{x + \varphi(x) : x \in \varphi^{-1}(M_i)\}$ is a CS-module and a direct summand of M.

(iii) There exist $i \neq j$ in I such that for each $k \in \{i, j\}$ and each homomorphism $\varphi : M_{-k} \to E(M_k)$, the submodule $\{x + \varphi(x) : x \in \varphi^{-1}(M_k)\}$ is a CS-module and a direct summand of M.

Proof. By Theorem 2.12, and Lemma 2.5. □

2.2 UC-modules

Definition 2.24 *A module M is called a UC-module if every submodule has a unique closure.*

Semisimple modules, uniform modules and nonsingular modules are all examples of *UC*-modules.

Theorem 2.25 *Let M be a UC-module such that $M = \bigoplus_{i \in I} M_i$ is the direct sum of R-modules $M_i (i \in I)$, for some non-empty index set I. Then the following statements are equivalent.*

(i) M is CS.

(ii) There exists $i \in I$ such that M_i is CS and every closed submodule K of M with $K \cap M_i = 0$ is a direct summand.

(iii) There exists $i \in I$ such that M_i is CS and every complement of M_i in M is a CS-module and a direct summand of M.

(iv) The module M_i is CS for each $i \in I$ and every closed submodule L of M with $L \cap M_i = 0 (i \in I)$ is a direct summand of M.

Proof. (i) ⇒ (ii). By Lemma 2.2.

(ii) ⇒ (iii). Let L be a complement of M_i in M. Then $L \cap M_i = 0$ and by (ii) L is a direct summand of M. Let N be a closed submodule of L. By Lemma 2.1 and (ii), N is a direct summand of M, and hence also of L. Thus L is a CS-module.

(iii) ⇒ (i). Let H be a closed submodule of M. By [8, Theorem 1], $H \cap M_i$ is a closed submodule of M_i and hence, by (iii), $H \cap M_i$ is a direct summand of M. Thus $M = (H \cap M_i) \oplus H'$ for some submodule H' of M. Now $H = (H \cap M_i) \oplus (H \cap H')$ and $H \cap H'$ is a closed submodule of M. Moreover $(H \cap H') \cap M_i = 0$. By the proof of Theorem 2.12 (iii) ⇒ (i), it follows that $H \cap H'$ is a direct summand of M and hence H is a direct summand of M.

(i) ⇒ (iv). By Lemma 2.2.

(iv) ⇒ (i). Let P be a closed submodule of M. For each $i \in I$, $P \cap M_i$ is closed in M_i and hence $M_i = (P \cap M_i) \oplus M_i'$ for some submodule M_i' of M. Let $M' = \oplus_{i \in I} M_i'$, $P' = \oplus_{i \in I}(P \cap M_i)$. Then $M = P' \oplus M'$ and $P' \leq P$. It follows that $P = P' \oplus (P \cap M')$. By Lemma 2.1, $P \cap M'$ is closed in M and $(P \cap M') \cap M_i = 0 (i \in I)$. By (iv) $P \cap M'$ is a direct summand of M. Thus P is a direct summand of M. We conclude that M is CS. □

2.3 Modules with Semisimple Summands

Example 2.26 *Let p be any prime and M the Z-module $M = (Z/Z_p) \oplus (Z/Z_{p^3})$. Let $M_1 = (Z/Z_p) \oplus 0$ and $M_2 = 0 \oplus (Z/Z_{p^3})$. M_1 and M_2 are CS-modules. But M is neither CS nor UC. In fact, the submodule $K = (1 + Z_p, p + Z_{p^3})$ is a complement submodule of M of order p^2. If K were a direct summand of M then $M = K \oplus K'$, for some submodule K' of M, and hence K' has order p^2 also, giving $p^2M = 0$, a contradiction. Thus Theorem 2.25 (iv) $\Rightarrow$ (i) fails if M is not UC.*

Theorem 2.27 *Let M be a UC-module such that $M = \bigoplus_{i \in I} M_i$ is the direct sum of R-modules $M_i (i \in I)$, for some non-empty index set I. Then the following statements are equivalent.*

(i) M is CS.

(ii) There exists $i \in I$ such that M_i is CS and for each homomorphism $\varphi : M_{-i} \to E(M_i)$ the submodule $\{x + \varphi(x) : x \in \varphi^{-1}(M_i)\}$ is a CS-module and a direct summand of M.

Proof. Follows from Lemma 2.22 and Theorem 2.25. □

Proposition 2.28 *Let M be a UC R-module such that $M = M_1 \oplus M_2$ is the direct sum of a module M_1 and a semisimple module M_2. Then M is CS if and only if M_1 is CS.*

Proof. The necessity is clear by Lemma 2.2.

Conversely, suppose that M_1 is CS. Let K be a complement of M_1 in M. Then $M_1 \oplus K$ is essential in M and hence $M_2 \leq \text{Soc}\, M \leq M_1 \oplus K$. Thus $M = M_1 \oplus K$. It follows that $K \cong M/M_1 \cong M_2$, so that K is CS. By Theorem 2.25, M is CS. □

Proposition 2.29 *Let M_1 be an R-module with zero socle and let M_2 be a semisimple R-module. Then the module $M = M_1 \oplus M_2$ is CS if and only if M_1 is CS and M_2 is M_1-injective.*

Proof. The necessity follows by Lemma 2.2 and [6, Lemma 11] Conversely, suppose that M_1 is CS and M_2 is M_1-injective. Clearly M_1 is M_2-injective. By Theorem 2.21, M is CS. □

Lemma 2.30 *Let M_1 and M_2 be modules with M_2 semisimple. Then the module $M_1 \oplus M_2$ is CS if and only if every complement K of M_2 in M is a CS-module and a direct summand of M.*

Proof. Suppose that every complement of M_2 in M is a CS-module and direct summand of M. Let K be a complement in M such that $K \cap M_2 = 0$. By Zorn's Lemma there exists a complement L of M_2 in M such that $K \leq L$. By assumption L is a CS-module and direct summand of M. Since K is a complement submodule in L then $K \leq_d L \leq_d M$ this implies $K \leq_d M$.

Conversely, it is clear. □

Theorem 2.31 *Let M_1 be a CS module and let M_2 be a semisimple module such that M_2 is (M_1/N)-injective for every non-zero submodule N of M_1. Then the module $M = M_1 \oplus M_2$ is CS.*

Proof. Let K be a complement of M_2 in M. There exists a homomorphism $\varphi : M_1 \to E(M_2)$ such that $K = \{x + \varphi(x) : x \in \varphi^{-1}(M_2)\}$ by lemma 2.22. Let $Q = \varphi^{-1}(M_2)$ and let $P = Ker\varphi$. Then $P \leq Q$ are submodules of M_1.

Suppose that $P = 0$. Then $K \cap M_1 = 0$, and hence $M_1 \oplus K = M_1 \oplus \varphi(Q)$, which is a direct summand of M, because $\varphi(Q)$ is a direct summand of M_2. Thus K is a direct summand of M and, because K embeds in $M/M_1 \cong M_2$, K is semisimple and thus CS.

Now suppose that $P \neq 0$. By hypothesis, M is (M_1/P)-injective. Now $Q/P \cong \varphi(Q)$, which is a direct summand of M_2. Thus Q/P is (M_1/P)-injective. There exists a submodule Q' of M_1 such that $P \subseteq Q'$ and $M_1/P = (Q/P) \oplus (Q'/P)$. Define $\theta : M_1 \to E(M_2)$ by

$$\theta(q + q') = \varphi(q)(q \in Q, q' \in Q').$$

It can easily be checked that θ is well-defined and a homomorphism. Moreover $\theta|_Q = \varphi$. Let

$$K' = \{x + \theta(x) : x \in \theta^{-1}(M_2)\} = \{x + \theta(x) : x \in M_1\},$$

noting that $\theta(M_1) = \varphi(Q) \leq M_2$. Lemma 2.22 gives that K' is a complement of M_2 in M. But $K \leq K'$ so that $K = K'$. Clearly $M = K \oplus M_2$. Thus K is a CS-module and a direct summand of M. By Lemma 2.30 M is CS. □

Lemma 2.32 *Let M_1 be a uniform module of finite composition length and let M_2 be a semisimple module such that $M = M_1 \oplus M_2$ is CS. Let $\varphi : M_1 \to E(M_2)$ be a homomorphism such that $\varphi(M_1) \nleq M_2$. Then $\varphi^{-1}(M_2) = 0$ or $\varphi^{-1}(M_2)$ is isomorphic to a simple submodule of M_2.*

Proof. Let $U = \varphi^{-1}(M_2)$. Let $K = \{x + \varphi(x) : x \in U\}$. By Lemma 2.22, K is a closed submodule and hence K is a direct summand. Note that $K \cong U \subseteq M_1$. Thus $K = 0$ or K is uniform. Suppose that $K \neq 0$.By the Krull-Schmidt Theorem, $K \cong M_1$ or K is isomorphic to a simple submodule of M_2. Suppose that $K \cong M_1$. Comparing composition lengths, $U = M_1$ and hence $\varphi(M_1) \leq M_2$, a contradiction. Thus $U = 0$ or U is isomorphic to a simple submodule of M_2. □

Theorem 2.33 *Let M_1 be a uniform module of finite composition length and let M_2 be semisimple module. Then $M = M_1 \oplus M_2$ is a CS-module if and only if M_2 is (M_1/N)-injective for every non-zero submodule N of M_1.*

Proof. The sufficiency is proved in Theorem 2.31. Conversely, suppose that M is CS. Suppose that N is a non-zero submodule of M_1, L is is a submodule containing N and there exists a monomorphism $\alpha : L/N \to M_2$. Note that $\alpha(L/N)$ is a direct summand of M_2 and hence $M_1 \oplus \alpha(L/N)$ is CS by Lemma 2.2. Thus without loss of generality, $\alpha : L/N \to M_2$ is an isomorphism.

Let $\pi : L \to L/N$ denote the canonical epimorphism. Let $\theta = \alpha\pi : L \to M_2$. Then θ can be lifted to a homomorphism $\varphi : M_1 \to E(M_2)$. Let $Q = \varphi^{-1}(M_2)$. Clearly $L \leq Q$. For any q in Q there exists $x \in L$ such that $\varphi(q) = \theta(x) = \varphi(x)$, so that $Q = L + ker\varphi$. Moreover, $L \cap ker\varphi = L \cap ker\theta = N$. Thus $Q/N = (L/N) \oplus ((ker\varphi)/N)$.

But $N \neq 0$ implies that the composition length of Q is at least 2. By Lemma 2.32, $\varphi(M_1) \leq M_2$, i.e. $Q = M_1$. Thus $M_1/N = (L/N) \oplus ((ker\varphi)/N)$. It follows that M_2 is (M_1/N)-injective. □

Corollary 2.34 *Let M_1 be a module with unique composition series $M_1 > L > N > 0$.Then $M_1 \oplus (L/N)$ is not CS.*

CHAPTER 3

ON P-EXTENDING AND EF-EXTENDING MODULES

In this chapter, it is given some characterizations and properties of principally injective modules.

Definition 3.1 *1. A right module M over a ring R is called principally injective (P-injective) if for every R-homomorphism for a principal right ideal of R to M can be extended to R.*

2. M is called P-extending (PC1) module if every cyclic submodule of M is essential in a direct summand of M.

3. M is called FP-extending module if every finite uniform dimension closed submodule which contains essentially a cyclic submodule (EC-closed) is a direct summand of M.

4. A module M satisfies the condition (PC2) if for each $a, b \in M$ such that $aR \cong bR$ and $bR \leq_d M$ then $aR \leq_d M$.

5. A module M satisfies the condition (PC3) if for each $a, b \in M$ such that aR and bR are direct summands of M and $aR \cap bR = 0$ then $aR \oplus bR \leq_d M$.

Definition 3.2 *1. A module M is called P-quasi-continuous module if the conditions (PC1) and (PC3) hold.*

2. A module M is called P-continuous module if the conditions (PC1) and (PC2) hold.

It is clear that

$$(C1) \Rightarrow (PC1),\ (C2) \Rightarrow (PC2),\ (C3) \Rightarrow (PC3).$$

Hence

$$continuous \Rightarrow P\text{-}continuous \text{ and } quasi\text{-}continuous \Rightarrow P\text{-}quasi\text{-}continuous.$$

Definition 3.3 *Let M and N be R-modules and $f : N \to M$ be a R-homomorphism. The set*

$$< f >= \{n - f(n) \mid n \in N\} \subseteq N \oplus M$$

is called graph of f.

Definition 3.4 *Let M and N be R-modules. M is called N-principally-injective (N-P-injective) if every R-homomorphism from a cyclic submodule of N to M can be extended to N.*

A module M is *extending*(n – *extending*) if every closed submodule A (with U-dim(A) $\leq n$) is a direct summand of M, or equivalently to the requirement that every submodule A (with U-dim(A) $\leq n$) is essential in a direct summand of M.

Lemma 3.5 *Let M and N be R-modules. The followings are equivalent*

(i) M is N-P-injective

(ii) For each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$ there exists $f \in Hom_R(N, M)$ such that $m = f(n)$.

Proof. (i) ⇒ (ii) Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$. nR is a cyclic submodule of N. $\alpha : nR \to M$; $\alpha(nr) = mr$ is a homomorphism. By (i) there exists a homomorphism $f : N \to M$ such that $f\mid_{nR} = \alpha$.

$$f(n) = f(n1_R) = \alpha(n1_R) = m1_R = m.$$

(ii) ⇒ (i) Let X be a cyclic submodule of N. Then there exists $n \in N$ such that $X = nR$. Let $\alpha : X \to M$ be a homomorphism. $\alpha(n) \in M$, say $\alpha(n) = m$. Let $k \in r_R(n)$.

$$mk = \alpha(n)k = \alpha(nk) = \alpha(0) = 0.$$

Hence $k \in r_R(m)$ and so $r_R(n) \subseteq r_R(m)$. By assumption, there exists a homomorphism $f : N \to M$; $f(n) = m$.

$$f(nr) = f(n)r = mr = \alpha(n)r = \alpha(nr).$$

Hence $f|_{nR} = \alpha$. So M is N-P-injective. □

Proposition 3.6 *Let M and N be R-modules, and $S = End(M)$. Then the following are equivalent :*

(i) M is N P-injective ;

(ii) For each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, we have $Sm \subseteq Hom_R(N, M)n$;

(iii) For each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, there is a complement C of M in $N \oplus M$ with $n - m \in C$ and $N \oplus M = C \oplus M$;

(iv) For each $n \in N$, $l_M r_R(n) = Hom_R(N, M)n$;

(v) For each $n \in N$ and $a \in R$, $l_M[aR \cap r_R(n)] = l_M(a) + Hom_R(N, M)n$.

Proof. (i) ⇒ (ii) : Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$. Since M is $N-P$-injective, then there exists a homomorphism $f : N \to M$ such that $m = f(n)$. Let $\phi \in S$, then $\phi(m) \in \mathrm{Hom}_R(N, M)n$. Therefore, $Sm \subseteq \mathrm{Hom}_R(N, M)n$.

(ii) ⇒ (iii). : Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, then by (ii), there exists a homomorphism $f : N \to M$ such that $m = f(n)$. Hence $N \oplus M = \langle f \rangle \oplus M$, where $\langle f \rangle$ is the graph of a homomorphism $f : N \to M$. Therefore, $C = \langle f \rangle$ is a complement of M in $N \oplus M$ with $N \oplus M = C \oplus M$ and $n - m \in C$.

(iii) ⇒ (iv) : Let $n \in N$ and $x \in l_M r_R(n)$, then $r_R(n) \subseteq r_R(x)$. By (iii), there is a complement C of M in $N \oplus M$ with $n - x \in C$ and $N \oplus M = C \oplus M$. So, there exists a homomorphism $f : N \to M$ such that $C = \langle f \rangle$. Since $n - x \in C$, then $n - x = n' - f(n')$, for some $n' \in N$. So, $n = n'$ and $x = f(n') = f(n)$. Hence $x \in Hom_R(N, M)n$, and $l_M r_R(n) \subseteq Hom_R(N, M)n$. The other conclusion is obvious.

(iv) ⇒ (v) : Let $n \in N$, $a \in R$, and $x \in l_M[aR \cap r_R(n)]$, then $x(aR \cap r_R(n)) = 0$ and so $r_R(na) \subseteq r_R(xa)$. Hence $l_M r_R(xa) \subseteq l_M r_R(na) = Hom_R(N, M)na$, by (iv). Therefore,

$xa = f(na) = f(n)a$, for some $f \in Hom_R(N, M)$. So $(x - f(n))a = 0$ and $x - f(n) \in l_M(a)$. Thus $x \in l_M(a) + Hom_R(N, M)n$, and so $l_M[aR \cap r_R(n)] \subseteq l_M(a) + Hom_R(N, M)n$. On the other hand, let $x \in l_M(a) + Hom_R(N, M)n$, then $x = m + f(n)$ for some $m \in l_M(a)$ and $f \in Hom_R(N, M)$. So $xa = ma + f(n)a = f(na)$. Let $ar \in aR \cap r_R(n)$, then $x(ar) = f(na)r = f(nar) = 0$, and so $x \in l_M[aR \cap r_R(n)]$. Thus $l_M(a) + Hom_R(N, M)n \subseteq l_M[aR \cap r_R(n)]$.

(v) $\Rightarrow$ (i) : Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, then $l_M r_R(m) \subseteq l_M r_R(n)$. By (v), we get $l_M r_R(n) = Hom_R(N, M)n$, and so there is a homomorphism $f : N \to M$ such that $f(n) = m$. Thus M is N-P-injective. □

Proposition 3.7 *Let M be N-P-injective, then M is X-P-injective, for every submodule X of N. If, in addition, X is a direct summand of N, then M is N/X-P-injective.*

Proof. Let $N = X \oplus Y$ for some submodule Y of N. Then $\frac{N}{X} \cong Y$ and M is N/X-P-injective. □

Lemma 3.8 *Let M be N-P-injective and $K \leq^{\oplus} M$, then K is N-P-injective.*

Proof. Let $X = nR$ be a cyclic submodule of N and $\alpha : nR \to K$ be a homomorphism. Since $K \leq^{\oplus} M$, there exists a direct summand L of M such that $M = K \oplus L$. Let $\pi : M \to K$ be projection map and $i : K \to M$ be inclusion map. Since M is N-P-injective there exists $\beta : N \to M$ a homomorphism such that $\beta |_{nR} = i\alpha$. Let $\overline{\beta} : N \to K$; $\overline{\beta} = \pi\beta$ is a homomorphism and $\overline{\beta} |_{nR} = \alpha$. Hence K is N-P-injective. □

Lemma 3.9 *Let $\{M_i\}_{i \in I}$ be a family of modules. Then the direct product $\prod_{i \in I} M_i$ is N-P-injective if and only if M_i is N-P-injective, for every $i \in I$.*

Proof. It is obvious. □

Proposition 3.10 *If M is a quasi-principally injective module, and $S = End(M)$, then $SH = SK$, for any isomorphic R-submodules H, K of M.*

Proof. Since $H \cong K$, then there is a right R-isomorphism $\sigma : H \to K$. For each $k \in K$, $k = \sigma(h)$ for some $h \in H$ and $r_R(h) = r_R(k)$. Since M is quasi-principally injective, then $Sh = Sk$ by Proposition 3.6, and so $Sk \subseteq SH$, for each $k \in K$. Then $SK \subseteq SH$. Similarly, we get $SH \subseteq SK$, and so the result. □

Lemma 3.11 *The following conditions are equivalent for a ring R.*

(i) R is right P-injective.

(ii) $lr(a) = Ra$ for all $a \in R$.

(iii) $r(a) \subseteq r(b)$, where $a, b \in R$, implies that $Rb \subseteq Ra$.

(iv) $l[bR \cap r(a)] = l(b) + R(a)$ for all $a, b \in R$.

(v) If $\gamma : aR \to R$, $a \in R$, is R-linear, then $\gamma(a) \in Ra$.

Proof. (i) ⇒ (ii) : Always $Ra \subseteq lr(a)$. If $b \in lr(a)$ then $r(a) \subseteq r(b)$, so $\gamma : aR \to R$ is well defined by $\gamma(ar) = br$. Thus $\gamma = c.$ for some $c \in R$ by (i), whence $b = \gamma(a) = ca \in Ra$. This implies $lr(a) = Ra$.

(ii) ⇒ (iii) : If $r(a) \subseteq r(b)$ then $b \in lr(a) = Ra$ and $b = ra$ for some $r \in R$. Then $Rb \subseteq Ra$.

(iii) ⇒ (iv) : Let $x \in l[bR \cap r(a)]$. Then $r(ab) \subseteq r(xb)$, so $xb = rab$ for some $r \in R$. Hence $x - ra \in l(b)$, proving that $l[bR \cap r(a)] \subseteq l(b) + R(a)$. The other inclusion always holds.

(iv) ⇒ (v) : Let $\gamma : aR \to R$, be R-linear, and write $\gamma(a) = d$. Then $r(a) \subseteq r(d)$, so $d \in lr(a)$. But $lr(a) = Ra$. Then $d = \gamma(a) \in Ra$.

(v) ⇒ (i) : Let $\gamma : aR \to R_R$. By (v) write $\gamma(a) = ca$, $c \in R$. Then $\gamma = c.$. Hence R is right P-injective. □

Corollary 3.12 *Let R be a P-injective ring and H, K be two-sided ideals of R. If $H \cong K$, as right ideals of R, then $H = K$.*

Proof. By Lemma 3.11. □

Theorem 3.13 *Let M be a quasi-principally injective module, then M has (PC_2).*

Proof. Let $a, b \in M$ with $aR \cong bR$ and $bR \leq^{\oplus} M$. Then $bR = eM$ for some idempotent $e \in End(M)$. Since $aR \cong bR$, then there is an isomorphism $\sigma : bR \to aR$. Let $\sigma e = h$, then $aR = hM$ and $\sigma^{-1}h = e$. Since $bR \leq^{\oplus} M$, then by Lemma 3.8, bR is M-P-injective, and so there exists a homomorphism $\phi : M \to bR$ such that $\phi(a) = \sigma^{-1}(a)$. Then ϕ is an epimorphism, $\phi h = e$, and so $f = h\phi$ is an idempotent endomorphism of M. Hence $fM = h\phi M = h(bR) = heM = hM$, and so $aR \leq^{\oplus} M$. □

Corollary 3.14 *If R is a P-injective ring, then R has (C_2).*

Lemma 3.15 *Let M be an R-module. If M has (PC_2), then M has (PC_3).*

Proof. Let $aR \leq^{\oplus} M$ and $bR \leq^{\oplus} M$ with $aR \cap bR = 0$, then $aR = eM = \text{Im } e$, for some $e^2 = e \in End(M)$, and so $aR \oplus bR = eM \oplus (1 - e)bR$. Since $(1 - e)bR \cong bR \leq^{\oplus} M$ and M has (PC_2), then $(1 - e)bR = fM$ for some $f^2 = f \in End(M)$. Then $ef = 0$, and $h = e + f - fe$ is an idempotent in $End(M)$. Therefore, $aR \oplus bR = eM \oplus fM = (e + f - fe)M = hM \leq^{\oplus} M$. □

Corollary 3.16 *If M is a quasi-principally injective module, then M has (PC_3).*

Definition 3.17 *By an EC-(closed) submodule C of a module M, we mean a (closed) submodule C which contains essentially a cyclic submodule; i.e. there exists $c \in C$ such that $cR \leq_e C$.*

Lemma 3.18 *Every summand of an EC-submodule of M is EC-submodule.*

Proof. Let $cR \leq_e C$ be an EC-submodule of M, and $C_1 \leq^{\oplus} C$, then $C = C_1 \oplus C_2$, for some submodule C_2 in C. Let $c = c_1 + c_2$, where $c_1 \in C_1$ and $c_2 \in C_2$. It is easy to see that $c_1R \leq_e C_1$. Therefore, C_1 is an EC-submodule of M. □

Corollary 3.19 *Every summand of an EC-closed submodule of M is EC-closed.*

Lemma 3.20 *Every summand of a P-(quasi-)continuous module is P-(quasi-)continuous.*

Proof. It is obvious by Corollary 3.19. □

Lemma 3.21 *For an indecomposable module M, the following are equivalent:*

(i) M is extending;

(ii) M is P-extending;

(iii) M is uniform.

Proof. (i) ⇒ (ii) It is obvious.

(ii) ⇒ (iii) Suppose that M is not uniform. Then there exists $m \in M$ such that mR is not essential in M and also there exists a complement submodule K in M such that mR is essential submodule of K. Since M is P-extending, K is direct summand of M and $K \neq M$. This contradicts with the indecomposability of M.

(iii) ⇒ (ii) It is obvious. □

Lemma 3.22 *Let M be a 1-extending-module. Then every closed submodule of M of the form $\bigoplus_{i=1}^{n} A_i$ with all A_i uniform, is a direct summand.*

Proof. By induction. Assume that the claim is true for n,and let $A = \bigoplus_{i=0}^{n} A_i$ be closed submodule of M. By assumption, $A^* = \bigoplus_{i=1}^{n} A_i$ is direct summand of M. Write $M = A^* \oplus M^*$ for $M^* \leq^{\oplus} M$. It follows that $A = A^* \oplus (A \cap M^*)$. It is clear that $A \cap M^*$ is closed uniform submodule of M. Since direct summand of 1-extending modules are 1-extending, we have $A \cap M^* \leq^{\oplus} M$. Hence $A \leq^{\oplus} M$. □

Lemma 3.23 *Let M be a 1-extending module. Then every non-zero closed submodule of M, of finite uniform dimension contains a uniform summand.*

Proof. Let $A \neq 0$ be a closed submodule of M, with U-dimension(A)$< \infty$. Let A_1 be a uniform submodule in A, and let U be a maximal essential extension of A_1 in A. Since U is complement in A and A is complement in M, U is complement in M. Since M is 1-extending, U is a direct summand in M and therefore U is a direct summand in A. □

Lemma 3.24 *A module M over a noetherian ring R, is 1-extending if and only if it is P-extending.*

Proof. Let M be a 1-extending module, and $cR \leq^e C$ be an EC-closed submodule of M. Since R is a noetherian ring, then C has a finite uniform dimension. Since M is 1-extending, then by Lemma 3.22 and Lemma 3.23, M is n-extending. Hence C is a summand, and so M is P-extending. For the converse, it is obvious. □

Corollary 3.25 *Let M be a module with finite uniform dimension, then the following are equivalent:*

(i) M is extending;

(ii) M is 1-extending;

(iii) M is P-extending.

Proposition 3.26 *Let $M = M_1 \oplus M_2$, and let $C \cap M_1$ be an EC-submodule of M, for every EC-closed submodule C of M. Then M is P-extending if and only if every EC-closed submodule C, with $C \cap M_1 = 0$, or $C \cap M_2 = 0$, is a summand.*

Proof. The necessary condition is obvious. For the sufficient condition, let $cR \leq^e C$ be an EC-closed submodule of M. If $C \cap M_1 = 0$, then we are done. Otherwise, $C \cap M_1$ is an EC-submodule of M, by assumption. Let C_1 be a maximal essential extension of $C \cap M_1$ in C, then C_1 is an EC-closed submodule of M, with $C \cap M_2 = 0$. Hence by the assumption, C_1 is a summand of M. Write $M = C_1 \oplus C_2$, by the modular law, $C = C_1 \oplus (C \cap C_2)$ by Corollary 3.19, $C \cap C_2$ is an EC-closed submodule of M with $(C \cap C_2) \cap M_1 = 0$, and therefore, $C \cap C_2$ is an summand of M. Thus C is a summand of M, and therefore, M is P-extending. □

Proposition 3.27 *Let $M = M_1 \oplus M_2$, where M_1 is of finite uniform dimension. Then M is P-extending if and only if every EC-closed submodule C of M with $C \cap M_1 = 0$, or C is of finite uniform dimension, is a summand.*

Proof. The necessary condition is obvious. For the sufficient condition, let $mR \leq^e C$ be an EC-closed submodule of M. If $C \cap M_1 = 0$, then we are done. Now let $0 \neq c \in C \cap M_1$, and C_1 be a maximal essential extension of cR in C. Since M_1 is of finite uniform dimension, so is C_1. By the given assumption, C_1 is a summand of M. Write $M = C_1 \oplus K$. Hence $C = C_1 \oplus C^*$, where $C^* = K \cap C$ is closed in M. Let $m = c_1 + c^*$, where $c_1 \in C_1$ and $c^* \in C^*$. Since C^* is a summand of an EC-closed submodule C, then by Corollary 3.19, C^* is EC-closed. If $C^* \cap M_1 = 0$, then by assumption C^* is a summand, and hence C is a summand of M. On the other hand, if $C^* \cap M_1 \neq 0$, then by repeating the previous steps, we have $C^* = C_2 \oplus C_3$, where C_2 is a summand and has a non-zero intersection with M_1. Continuing in this manner, we should stop after a finite steps (due to M_1 a finite uniform dimensional module) and end with $C = C_1 \oplus C_2 \oplus \ldots \oplus C_n$, where C_i is a summand of M ($i = 1, 2, \ldots, n-1$), and C_n contains an essential cyclic submodule with $C_n \cap M_1 = 0$. Hence C_n is a summand of M, by assumption, and therefore C is a summand of M. □

Corollary 3.28 *Let $M = M_1 \oplus M_2$, where M_1 is of finite uniform dimension. Then M is P-extending if and only if every EC-closed submodule of M, with $C \cap M_1 = 0$, or $C \cap M_2 = 0$, is a summand.*

Proposition 3.29 *Let $M = M_1 \oplus M_2$. Then M is FP-extending if and only if every EC-closed submodule C of M with finite uniform dimensional such that $C \cap M_1 = 0$, or $C \cap M_2 = 0$, is a summand.*

Proof. It is similar to the proof of Proposition 3.27. □

Proposition 3.30 *Let $M = M_1 \oplus M_2$, where M_1 is a semisimple module. Then M is P-extending if and only if every EC-closed submodule C of M with $C \cap M_1 = 0$, is a summand.*

Proof. The necessary condition is obvious. For the sufficient condition, let C be an EC-closed submodule of M. If $C \cap M_1 = 0$, then we are done. On the other hand, since M_1 is a semisimple, we get $C \cap M_1 \leq^{\oplus} M_1$ and so $C = C \cap M_1 \oplus C^*$. Since C^* is an EC-closed submodule of M and $C^* \cap M1 = 0$, then C^* is a summand of M. Therefore C is a summand of M. □

Proposition 3.31 *Let $M = M_1 \oplus M_2$, where M_1 is P-extending and M_2 is M_1-P-injective. If M_2 is nonsingular, then every EC-closed submodule C of M, with $C \cap M_2 = 0$, is a summand of M.*

Proof. Let $cR \leq_e C$ be an EC-closed submodule of M with $C \cap M_2 = 0$, and write $c = c_1 + c_2$, where $c_1 \in M_1$ and $c_2 \in M_2$. Since M_2 is M_1-P-injective, then the homomorphism $\alpha : c_1R \to M_2$; $\alpha(c_1) = c_2$, there exists a homomorphism $\phi : M_1 \to M_2$ such that $\phi |_{c_1R}= \alpha$. Let

$$(c_1R)^* = \{c_1r + \phi(c_1)r \mid r \in R\}.$$

$(c_1R)^*$ is a submodule of

$$M_1^* = \{m_1 + \phi(m_1) \mid m_1 \in M_1\}$$

Let $cr \in cR$. $cr = c_1r + c_2r = c_1r + \phi(c_1)r$. Then $cR = (c_1R)^*$. Let $y \in M_1^* \cap M_2 = 0$. Let $m \in M$. $m = (m_1 + \phi(m_1)) + (m_2 - \phi(m_1)) \in M_1^* + M_2$ where $m_1 \in M_1$ and $m_2 \in M_2$. Then $M = M_1^* \oplus M_2$. Therefore $M_1^* \cong M_1$. Let $x \in C$ and write $x = y + m_2$, where $y \in (M_1)^*$ and $m_2 \in M_2$. Since $cR \leq_e C$, then there exists an essential right ideal I of R such that $m_2I = 0$. Since M_2 is nonsingular, then $m_2 = 0$. Let $c \in C$. Then $c = m_1 + \phi(m_1) + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$. Since $cR \leq_e C$, there exists $0 \neq r \in R$ such that $cr = (m_1 + \phi(m_1) + m_2)r \in cR \leq M_1^*$. $m_1r + \phi(m_1)r + m_2r \in M_1^*$. Then there exists $z \in M_1^*$ such that $m_1r + \phi(m_1)r - z = -m_2r \in M_1^* \cap M_2 = 0$. It follows that $m_2 = 0$ and also $c = m_1 + \phi(m_1) \in M_1^*$. It follows that $C \subseteq (M_1)^*$. Since $(M_1)^*$ is P-extending, we have $C \leq^{\oplus} (M_1)^* \leq^{\oplus} M$. □

Definition 3.32 *Let $M = M_1 \oplus M_2$ be a module. The module M_2 is called M_1-EC-injective, if for every EC-(closed) submodule N of M_1, and every homomorphism from N to M_2 can be extended to M_1.*

This is equivalent to for every EC-(closed) submodule N of M such that $N \cap M_2 = 0$, there exists $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$.

Observe that every module over a regular ring R is R-EC-injective.

Lemma 3.33 *Let $M = M_1 \oplus M_2$ and M_2 be M_1-EC-injective. Then:*

(i) M_2 is K-EC-injective, for all $K \leq M_1$.

(ii) H is M_1-EC-injective, for all $H \leq^{\oplus} M_2$.

(iii) H is K-EC-injective, for all $K \leq^{\oplus} M_1$, and $H \leq^{\oplus} M_2$.

Proof. (i) Let K be a submodule of M_1, and N be an EC-submodule $K \oplus M_2$ with $N \cap M_2 = 0$. Then N is an EC-submodule of M. Since M_2 is M_1-EC-injective, then there is $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$. Then $K \oplus M_2 = (K \oplus M_2) \cap (N' \oplus M_2) = (N' \cap (K \oplus M_2)) \oplus M_2$ and $N \leq N' \oplus (K \oplus M_2)$. Hence M_2 is K-EC-injective.

(ii) Let H be a summand of M_2, and N be an EC-submodule of $M_1 \oplus H$ with $N \cap H = 0$. Then N is an EC-submodule of M and $N \cap M_2 = 0$ since M_2 is M_1-EC-injective, then there is $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$. Since $H \leq^{\oplus} M_2$, then $M_2 = H \oplus H'$, and so $M_1 \oplus H = (M_1 \oplus H) \cap (N' \oplus H \oplus H') = H \oplus (M_1 \oplus H) \cap (N' \oplus H')$. Since $N \leq N'$, then $N \leq (M_1 \oplus H) \cap (N' \oplus H)$. Therefore H is M_1-EC-injective.

(iii) Follows from (i) and (ii). □

Proposition 3.34 *Let $M = M_1 \oplus M_2$ where M_1 is P-extending and M_2 is M_1-EC-injective. Then $M = C \oplus M_1' \oplus M_2$; where $M_1' \leq M_1$, for every EC-closed submodule C of M, with $C \cap M_2 = 0$.*

Proof. Let $cR \leq^e C$ be an EC-closed submodule of M with $C \cap M_2 = 0$. Define $X = M_1 \cap (C \oplus M_2)$. Then $c_1 R \leq_e X$, where $c = c_1 + c_2$, where $c_1 \in M_1$ and $c_2 \in M_2$. Let N_1 be a maximal essential extension of X in M_1. Then N_1 is an EC-closed submodule of M_1. Since M_1 is P-extending, we have $N_1 \leq^{\oplus} M_1$. Write $M_1 = N_1 \oplus M_1'$, where $M_1' \leq M_1$. Now $C \oplus M_2 = X \oplus M_2 \leq_e N_1 \oplus M_2$; i.e. $C \leq N_1 \oplus M_2$, and $C \leq_c N_1 \oplus M_2$. Then C is complement of M_2 in $N_1 \oplus M_2$. Since M_2 is M_1-EC-injective, and N_1 is a summand of M_1, then by Lemma 3.33 (i), M_2 is N_1-EC-injective, and so there exists $N' \leq N_1 \oplus M_2$ such that $C \leq N'$, and $N_1 \oplus M_2 = N' \oplus M_2$. Hence N' is a complement of M_2 in $N_1 \oplus M_2$, but C is a complement of M_2 in $N_1 \oplus M_2$. Therefore, $N' = C$ and $M = M_1 \oplus M_2 = N_1 \oplus M_1' \oplus M_2 = C \oplus M_1' \oplus M_2$. □

Corollary 3.35 *Let $M = M_1 \oplus M_2$, where M_i is P-extending and is M_j-EC-injective ($i \neq j = 1, 2$) if and only if $M = C \oplus M_i' \oplus M_j$; where $M_i' \leq M_i$, for every EC-closed submodule C of M, with $C \cap M_j = 0$ ($i \neq j = 1, 2$).*

Proposition 3.36 *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are relatively EC-injective, and either M_1 or M_2 is of finite uniform dimension. Then M is P-extending if and only if M_1 and M_2 are P-extending.*

Proof. It is follows by Corollaries 3.35, and 3.28. □

Proposition 3.37 *Let $M = \bigoplus_{i \in I} M_i$ be an R-module, where $M(F)$ is P-extending and $M(I \backslash F)$ is $M(F)$-EC-injective, for all finite subset F of I. Then M is P-extending.*

Proof. Let $c \in M$ and C be a maximal essential extension of cR in M. Then $cR \leq M(F)$ and $cR \cap M(I \backslash F) = 0$, for a finite subset F of I. Since $cR \leq_e C$, then $C \cap M(I \backslash F) = 0$. Since $M(I \backslash F)$ is $M(F)$-EC-injective and C is EC-closed submodule of M, then by Proposition 3.34, C is a summand of M. Hence M is P-extending. □

Definition 3.38 *A module M is called ef-extending if every closed submodule which contains essentially a finitely generated submodule is a direct summand.(Equivalently, A module M is called ef-extending if every submodule N of M such that N is finitely generated there exists a direct summand L of M such that N is essential in L.*

Definition 3.39 *A module M is called $uniform-extending$ (u-extending) if every uniform submodule is essential in a direct summand of M.*

The following implications are obvious

extending ⇒ ef-extending ⇒ p-extending⇒ uniform-extending

The following example shows that the implication ef-extending ⇒ extending is not true.

Example 3.40 *The Z-module $M = \prod_{i=1}^{\infty} Z_2$ is ef-extending but it is not extending.*

Proof. It is easy to see that $N = \bigoplus_{i=1}^{\infty} Z_2$ is local direct summand of M. Since Z is a Noetherian ring, N is closed submodule of M [10, 8.1]. But N is not a direct summand of M. In fact, suppose that $M = N \oplus K$. Set $x = (0, 1, 1, \ldots, 1, \ldots) \in K$, $x' = (0, 0, 0, 1, \ldots, 1, \ldots) \in K$. Then $x - x' = (0, 1, 1, 0, \ldots, 0, \ldots) \in K \cap N$, a contradiction. Thus M is not extending. We now show that M is ef-extending. $Z/2Z = \{0, 1\}$, M has some of the following properties:

($*$) Since $x = (x_i) \in M$, $x_i = 0$ or $x_i = 1$. This implies that $xk = 0$ if k is even and $xk = x$ if k is odd. Hence $xZ = \{0, x\}$. This means that xZ is a simple submodule of M.

($**$) For every $x \in M$, xZ is a direct summand of M. In fact, we can suppose that $x \neq 0$, $x = x_i$. Then there exists an integer i such that $x_i = 1, x_1 = 1$ says, i.e., $x = (1, x_2, x_3, \ldots)$. Take $N'' = \{(0, y_2, y_3, \ldots) \mid y_i \in Z_2, i > 1\} \leq M$. We can easily see that $N'' \cap xZ = 0$ and $M = xZ \oplus N''$.

Thus, every cyclic submodule of M is a simple submodule and a direct summand of M. So if K is an essentially finitely generated submodule, then we can easily see that K is direct summand of M. Hence M is ef-extending. □

Proposition 3.41 *Let M be an ef-extending module such that every local direct summand is a direct summand of M. Then M is an extending module.*

Proof. Let K be a non-zero closed submodule of M. For any $0 \neq x \in K$, xR is essential in a submodule A of K which is closed in K. Since K is closed in M, A is closed in M and therefore A is a direct summand of M. By Zorn's lemma, there exists a maximal local direct summand $N = \bigoplus_I A_i$ where each $A_i \subset K$. By hypothesis, N is a direct summand of M, i.e., $M = N \oplus N'$ for some submodule N' of M, so $K = N \oplus (K \cap N')$. Assume that $K \cap N' \neq 0$. Then there exists $A \neq 0$ A is a direct summand of M. This implies that A is also a direct summand of $K \cap N'$. So $N \oplus A$ is a local direct summand of M, contradicting the choice of N. Thus $K \cap N' = 0$. This means that $K = N$. This shows that M is an extending module.

By the example above, we see that the Z-module $M = \prod_{i=1}^{\infty} Z_2$ is ef-extending but not extending. Note that $N = \bigoplus_{i=1}^{\infty} Z_2$ is local direct summand of M but it is not a direct summand of M. □

Lemma 3.42 *A module M is uniform-extending if and only if every closed submodule K of M that has finite uniform dimension is a direct summand of M.*

Proof. Suppose M is u-extending. Let K be a closed submodule of M that has finite uniform dimension. Without loss of generality, we can assume uniform dimension of K is 2. Then we have a uniform closed submodule K_1 of K. Since K is closed submodule of M, K_1 is closed in M and M is u-extending K_1 is direct summand of M. $M = K_1 \oplus L$ for some direct summand L of M. By modularity $K = K_1 \oplus (K \cap L)$. Since $ud(K) = 2$, $K \cap L$ is a uniform closed submodule and so it is a direct summand of M and also of L. Hence K is a direct summand of M. Conversely, it is obvious. □

Proposition 3.43 *For a module M over a noetherian ring, the following conditions are equivalent:*

(i) M is ef-extending.

(ii) M is uniform-extending.

Proof. Since a finitely generated module over a noetherian ring is noetherian, every finitely generated module has finite uniform dimension. By Lemma 3.42, the proposition follows. □

Definition 3.44 *A module M is said to satisfy (C_{11}) if and only if for every submodule A of M, there exists a direct summand K of M such that $A \cap K = 0$ and $A \oplus K \trianglelefteq M$*

Lemma 3.45 *Any direct sum of modules (C_{11}) satisfies (C_{11}).*

Proof. Let $M_\lambda(\lambda \in \Lambda)$ be a non-empty collections of modules, each satisfying (C_{11}). Let $M = \oplus_{\lambda\in\Lambda} M_\lambda$. Let N be any submodule of M. Let $\lambda \in \Lambda$. Note that $N \cap M_\lambda$ is a submodule of M_λ and M_λ satisfies (C_{11}). By [13 Proposition 2.3], there exists a direct summand K_λ of M_λ such that $(N \cap M_\lambda) \cap K_\lambda = 0$ and $(N \cap M_\lambda) \oplus K_\lambda$ is an essential submodule of M_λ. Note that $N \cap K_\lambda = 0$, $(N \oplus K_\lambda) \cap M_\lambda = (N \cap M_\lambda) \cap K_\lambda$ and $(N \oplus K_\lambda) \cap M_\lambda$ is an essential submodule of M_λ. Let Λ' be a non-empty subset of Λ containing λ such that there exists a direct summand K' of $M' = \oplus_{\lambda\in\Lambda'} M_{\lambda'}$, with $N \cap K' = 0$ and with $(N \oplus K') \cap M'$ an essential submodule of M'. Suppose $\Lambda' \neq \Lambda$. Let $\mu \in \Lambda$, μ is not in Λ'. Now $L = (N \oplus K') \cap M_\mu$ is a submodule of M_μ, so there exists a direct summand K_μ of M_μ such that $L \cap K_\mu = 0$ and $L \oplus K$ is an essential submodule of M_μ. Let $\Lambda'' = \Lambda' \cup \{\mu\}$ and $M'' = \oplus_{\lambda\in\Lambda''} M_\lambda = M' \oplus M_\lambda$. Note that $K' \cap K_\mu = 0$. Let $K'' = K' \oplus K_\mu$. Note that K'' is a direct summand of M'' and moreover $N \cap K'' = 0$. Consider the submodule $N \oplus K''$. Note that $(N \oplus K'') \cap M'$ contains $(N \oplus K') \cap M'$, so that $(N \oplus K'') \cap M'$ is an essential submodule of M'. Moreover

$$(N \oplus K'') \cap M_\mu = (N \oplus K' \oplus K_\mu) \cap M_\mu = [(N \oplus K') \cap M_\mu] \oplus K_\mu = L \oplus K_\mu,$$

which is an essential submodule of M_μ. It follows that $(N \oplus K'') \cap M''$ is an essential submodule of M''. Repeating this argument, there exists a direct summand K of M such that $N \cap K = 0$ and $N \oplus K$ is an essential submodule of M. By [13 Proposition 2.3] M satisfies (C_{11}). □

Lemma 3.46 *Let $M = \bigoplus_I M_i$ be a decomposition with all M_i uniform and End(M_i) local. If the family $\{M_i \mid i \in I\}$ is relatively injective, then there does not exist an infinite sequence of non-isomorphic monomorphism $\{f_k : M_{i_k} \to M_{i_{k+1}}\}_N$ with all $i_k \in I$ distinct.*

Proof. Suppose that there exists an infinite sequence of non-isomorphic monomorphisms (f_i) where $f_i : M_i \to M_{i+1}, i \geq 1$.

Let $N_i = \{x - f_i(x) \mid x \in M_i\}$. Then we can easily see that the family
$\{N_i \mid i = 1, 2, \ldots\}$ independent, so the sum $\sum_{i=1}^{\infty} N_i$ is direct. Since each M_i is a uniform module, it satisfies (C_{11}), so thus $\bigoplus_{i=1}^{\infty} M_i$. Therefore, there exists a direct summand K of $\bigoplus_{i=1}^{\infty} M_i$ such that $(\bigoplus_{i=1}^{\infty} N_i) \cap K = 0$ and $(\bigoplus_{i=1}^{\infty} N_i) \oplus K$ is essential in $\bigoplus_{i=1}^{\infty} M_i$. Assume that $K \neq 0$. Then by [1, 12.6] there exists a $k \in N$ such that M_k is direct summand of K. The relative injectivity of the family $\{M_i \mid i = 1, 2, \ldots\}$ implies that M_k is $\bigoplus_{i \neq k} M_i$-injective [2, 1.5]. Hence, there exists M' such that $\bigoplus_{i=1}^{\infty} N_i \leq M'$ and $\bigoplus_{i=1}^{\infty} M_i = M' \oplus M_k$. This implies that N_k is a direct summand of M' so that $M_k \oplus N_k$ is a direct summand of M or $M_k \oplus N_k$ is a closed submodule of M. Moreover, $M_k \oplus N_k$ is essential in $M_k \oplus M_{k+1}$. Hence $M_k \oplus N_k = M_k \oplus M_{k+1}$. This implies that f_k is epimorphic, a contradiction. Therefore $K = 0$ and hence $\bigoplus_{i=1}^{\infty} N_i$ is essential in $\bigoplus_{i=1}^{\infty} M_i$. Thus $M_1 \cap (\bigoplus_{i=1}^{\infty} N_i) \neq 0$, so there exists $x_1 \neq 0, x_1 = y_1 - f_1(y_1) + \ldots + y_n - f_n(y_n)$, where $y_i \in M_i$ $(i = 1, \ldots, n)$. This would imply that $f_n f_{n-1} \ldots f_2 f_1(x) = 0$, which contradicts to the fact that all f_i are monomorphic, proving our lemma. □

Theorem 3.47 *Let $M = \bigoplus_I M_i$ be a decomposition with M_i uniform and End(M_i) local. Assume the family $\{M_i \mid i \in I\}$ is relatively injective. Then the following conditions are equivalent:*

(i) M is extending.

(ii) M is ef-extending.

(iii) M is uniform-extending.

Proof. The proof follows by Lemma 3.46. and [14 Theorem 3.4] □

Lemma 3.48 *Let $M = M_1 \oplus M_2$ having the following property: either every closed submodule K in M with $K \cap M_1 = 0$ is a direct summand of M, or every closed submodule K in M which is essentially finitely generated such that $K \cap M_2 = 0$ is a direct summand of M. Then M is an ef-extending module.*

Proof. Let K be a closed submodule of M that contains essentially a finitely generated submodule $N = x_1R + \ldots + x_nR$. Then there exists a closed submodule H in K such that $K \cap M_2$ is essential in H. From this, H is a closed submodule of M, $H \cap M_1 = 0$ and then H is a direct summand of M, $M = H \oplus H'$ says. This implies that $K = H \oplus (K \cap H')$. So $K \cap H'$ is closed submodule in M and $(K \cap H') \cap M_2 = 0$. We now prove that $K \cap H'$ is essentially finitely generated. In fact, since $N = x_1R + \ldots + x_nR$ is essential in $K = H \oplus (H' \cap K)$, we have $x_1 = h_1 + k_1, \ldots, x_n = h_n + k_n$, where $h_i \in H$, $k_i \in H' \cap K$ $(i = 1, \ldots, n)$. Let $B = k_1R + \ldots + k_nR$. Since N is essential in K, B is essential in $K \cap H'$. By hypothesis, we have $H' \cap K$ is a direct summand of M and hence of H', i.e., $H' = (H' \cap K) \oplus P$ for some P. It follows that $M = H \oplus (H' \cap K) \oplus P = K \oplus P$, proving our lemma. □

Proposition 3.49 *A direct sum of an extending module and an ef-extending module which are relatively injective is also an ef-extending module.*

Proof. By Lemma 3.48 and [10 Theorem 7.5]. □

Lemma 3.50 *Let $M = M_1 \oplus M_2$ with each M_i uniform and End(M_i) local $(i = 1, 2)$. Assume M is uniform-extending. Then for any $A \leq M_i$ every homomorphism $f : A \to M_j$ can be extended to a homomorphism*
$f' : B \to M_j$, where B is a submodule of M_i such that either $B = M_i$ or $B \neq M_i$ and f' is an isomorphism.

Proof. Assume that $A \leq M_1$ and $f : A \to M_2$ is a homomorphism. Let

$$A'' = \{a - f(a) \mid a \in A\}.$$

Then $A'' \simeq A$ is a uniform submodule of M. Since M is uniform extending, A'' is essential in a direct summand D of M. By [1, 12.7], either $M = M_1 \oplus D$ or $M = D \oplus M_2$. Assume first that $M = D \oplus M_2$. Let $p : D \oplus M_2 \to M_2$ be the projection. Then it is easy to check the restriction of p on M_1 is an extension of f. So p is the desired homomorphism. Now assume that $M = M_1 \oplus D$. Then $D \cap M_1 = 0$ and clearly $\ker f = 0$, therefore there exists $f^{-1} : f(A) \to A$. We can easily see that the projection $q : M_1 \oplus D \to M_1$ which restricts on M_2 is an extension of f^{-1} and we call this extension j. Since f^{-1} is a monomorphism and M_2 is a uniform module, j is also a monomorphism. We can easily see that $A \leq j(M_2)$. Set $B = j(M_2)$. Then we see that $j^{-1} : B \to M_2$ is an extension of f. So j^{-1} is the desired isomorphism. □

Definition 3.51 *A module A is called nearly B-injective if for each $C \leq B$ and for each homomorphism $f : C \to A$ with $\ker f \leq 0$, then there exists a homomorphism $f' : B \to A$ such that it is extension of f.*

The family $\{M_i \mid i \in I\}$ of right R-modules is said to satisfy A_2) if for any choice of $x_n, x_n \in M_{i_n}$ with distinct $i_n \in I$ such that $r_R(y) \subseteq \bigcap_{i=1}^{\infty} r_R(x_n)$ for some $y \in M_j$, the ascending sequence :

$$\bigcap_{n=1}^{\infty} r_R(x_n) \subseteq \bigcap_{n=2}^{\infty} r_R(x_n) \ldots$$

becomes stationary.

Lemma 3.52 *A module A is nearly B-injective if and only if A is nearly xR-injective for each $x \in B$.*

Proof. We use the same argument as that given in [2, 1.4]. □

Lemma 3.53 *Let $M = \bigoplus_I M_i$ be a decomposition with all M_i-uniform and $End(M_i)$ local. Assume $M_i \oplus M_j$ is uniform-extending for each pair $i \neq j$ in I and the family $\{M_i \mid i \in I\}$ satisfies (A_2). Then for each $k \in I$, $\bigoplus_{i \neq k} M_i$ is nearly M_k-injective.*

Proof. By Lemma 3.52, it suffices to prove that $\bigoplus_{i \neq k} M_i$ is nearly xR-injective for each $x \in M_k$. Assume that $A \leq xR$ and $f : A \to \bigoplus_{i \neq k} M_i$ is a homomorphism such that $\ker f \neq 0$. Define $S = \{r \in R \mid xr \in A\}$. Then it is easy to check that S is an ideal of R and $A = xS$. For each $i \in I \setminus \{k\}$, put $f_i = p_i f : xS \to M_i$, where each $p_i : \bigoplus_{i \neq k} M_i \to M_i$ is the projection. Since $M_k \oplus M_i$ is uniform-extending, $\ker f \neq 0$ and by Lemma 3.50, f_i can be extended to a homomorphism $h_i : xR \to M_i$. So we can easily see that $h : xR \to \prod_{i \neq k} M_i$

$$xr \mapsto (h_i(xr))_{I \setminus \{k\}}$$

is an extension of f on A. Put $a = (a_i)_{I \setminus \{k\}} = h(x) \in \prod_{i \neq k} M_i$. Clearly

$$r_R(x) \subseteq r_R(a) = \bigcap_{i \neq k} r_R(a_i).$$

For each element $s \in S$, let $I_s = \{i \in I \setminus \{k\}$ such that $a_i s \neq 0\}$. Then I_s is a finite subset of $I \setminus \{k\}$. If $\bigcup_{s \in S} I_s$ such that $\bigcup_{n=1}^{\infty} I_{s_n}$ is countable. Since I_s is finite for each $s \in S$, we can choose a sequence $(s_n)_n$ satisfying

$$I_{s_1} \subsetneqq I_{s_1} \subsetneqq \ldots$$

and $i_1 \in I_{s_1}, i_2 \in I_{s_2} \setminus I_{s_1}, \ldots, i_n \in I \setminus (\bigcup_{j=1}^{n-1} I_{s_j})$. Since $i_1 \in I_{s_1}$, it follows that $a_{i_1} s_1 \neq 0$, $a_j s_1 = 0$ for each $j \in I \setminus I_{s_1}$. Similarly, for $i_2 \in I_{s_2} \setminus I_{s_1}$, we have

$$a_{i_2} s_1 = 0, a_{i_2} s_2 \neq 0, \ldots$$

and finally, $i_n \in I \setminus (\bigcup_{j=1}^{n-1} I_{s_j})$, we have $a_{i_n} s_1 = \ldots = a_{i_n} s_{n-1} = 0$, $a_{i_n} s_n \neq 0$.

Thus the sequence $(\bigcap_{k=n}^{\infty} r_R(a_{i_k}))_{n \in N}$ is strictly increasing, contradicting to the assumption that $\{M_i\}_{i \in I}$ satisfies (A_2). We now assume that $\bigcup_{s \in S} I_s = \{i_1, \ldots, i_n\}$. For each $t \in I \setminus \{i_1, \ldots, i_n\}$, $a_t s = 0$. This would imply $f(xs) = (a_i s)_{i \in I \setminus \{k\}} \in \bigoplus_{t=1}^{n} M_{i_t}$ for each $s \in S$. Hence $f(A) \subseteq \bigoplus_{t=1}^{n} M_{i_t}$. Since each M_{i_t} is nearly M_k-injective, $\bigoplus_{t=1}^{n} M_{i_t}$ is nearly M_k-injective. So there exists a homomorphism $h' : M_k \to \bigoplus_{t=1}^{n} M_{i_t}$ such that h' is an extension of f. The proof of our lemma is completed. □

Theorem 3.54 *Let $M = \bigoplus_I M_i$ be a decomposition with all M_i-uniform and $End(M_i)$ local. Then the following conditions are equivalent:*

(i) M_i is uniform-extending.

(ii) $M_i \oplus M_k$ is extending for each pair $k \neq i$ in I and the family $\{M_i \mid i \in I\}$ satisfies (A_2).

(iii) $M_i \oplus M_k$ is ef-extending for each pair $k \neq i$ in I and the family $\{M_i \mid i \in I\}$ satisfies (A_2).

(iv) $M_i \oplus M_k$ is uniform-extending for each pair $k \neq i$ in I and the family $\{M_i \mid i \in I\}$ satisfies (A_2).

Proof. (i) $\Rightarrow$ (ii). By [14,Lemma 2.3]

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i). Suppose that $\{M_i \mid i \in I\}$ satisfies (A_2) and U is a uniform submodule of M. By Zorn's lemma, there exists $k \in I$ such that $U \cap \bigoplus_{i \neq k} M_i = 0$. Thus, the projection $p_k : M = (\oplus_{i \neq k} M_i) \oplus M_k \to M_k$ restricts on U is a monomorphism. Let $A = p_k(U)$ and $p : (\oplus_{i \neq k} M_i) \oplus M_k \to \oplus_{i \neq k} M_i$ be the projection. Consider the homomorphism $h : A \to \oplus_{i \neq k} M_i$, defined by $h(p_k(u)) = p(u)$ for each $u \in U$. If $h = 0$ then $U \leq M_k$ and since A is closed in M_k, it follows that $U = M_k$. So U is a direct summand of M. Now assume that $h \neq 0$. Then there exists $u \in U$ such that $h(p_k(u)) \neq 0$. Thus, we can choose $i_1, i_2, \ldots i_n$ in $I \setminus \{k\}$ such that $h(p_k(u)) \in M_{i_1} \oplus \ldots \oplus M_{i_n}$. Put $N_1 = M_{i_1} \oplus \ldots \oplus M_{i_n}$ and $N_2 = \bigoplus_{i \neq k} M_i \setminus N_1$. By Lemma 3.52, N_2 is nearly M_k-injective and $p_2 h$ is not a monomorphism (where $p_2 : \bigoplus_{i \neq k} M_i = N_1 \oplus N_2 \to N_2$ is the projection), it would implies that $p_2 h$ can be extended to a homomorphism $h_2 : M_k \to N_2$. If for each $t = 1, 2, \ldots, n$, $p_t h : A \to M_{i_t}$ is not a monomorphism, then $p_t h$ can be extended to a homomorphism $h_t : M_k \to M_{i_t}$. Therefore h can be extended to a homomorphism $h' : M_k \to \oplus_{i\neq k} M_i$. Set $M_k^* = \{x - h'(x) \mid x \in M_k\}$. It is easy to see that $M = M_k^* \oplus (\oplus_{i\neq k} M_i)$ and $U \trianglelefteq M_k^*$. Hence $U = M_k^*$, i.e., U is a direct summand of M. If there exists some t such that $p_t h$ is isomorphic then, without loss of generality, we suppose that $p_1 h, \ldots, p_m h$ are monomorphic for some $m \leq n$. By Lemma 3.50, $p_t h$ can be extended to a homomorphism $f_t : B_t \to M_{i_t}$ and f_t is isomorphic for each $t = 1, 2, \ldots, m$. We can easily see that :

($*$) $A = \bigcap_{t=1}^{m} B_t$.

($**$) The family $\{B_t \mid t = 1, \ldots, m\}$ is total ordered.

Thus there exists $t \in \{1, \ldots, m\}$ such that $A = B_t$, i.e., $f_t = p_{i_t} : A \to M_{i_t}$ is isomorphic. It follows that $p_t : U \to M_{i_t}$ is isomorphic. Hence U is a direct summand of M and hence M is uniform-extending. □

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