ON MEYER-KÖNIG AND ZELLER OPERATORS

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ABSTRACT

ON MEYER-KÖNIG AND ZELLER OPERATORS

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This thesis is a survey on Meyer-König and Zeller operators which are well-known positive linear operators in the approximation theory.

This thesis consists of four chapters. In the first chapter, a short history of the studies on the approximation of the linear positive operators, and some basic definitions and theorems are given. In the second chapter, some approximation properties of a generalization of Meyer-König and Zeller operators via generating functions and a Kantarovich type generalization of Meyer-König and Zeller operators are obtained. Also, the rate of convergence of these operators, with the help of usual modulus of continuity and the elements of Lipschitz class, is computed and an application to functional differential equations is presented. In the third chapter, explicit formulas and some estimates for the moments of Meyer-König and Zeller operators are given. In the final chapter, it is shown that Meyer-König and Zeller operators preserve the Lipschitz constants and satisfy an initial value problem.

Keywords: Linear positive operators, Meyer-König and Zeller operators, Korovkin theorem, Lipschitz class, modulus of continuity, functional differential equation.

ÖZET

MEYER-KÖNIG VE ZELLER OPERATÖRLERİ ÜZERİNE

Doğru Akgöl, Sibel

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Bu tez, yaklaşım teorisinde tanınmış lineer pozitif operatörler olan Meyer-König ve Zeller operatörleri üzerine bir incelemedir.

Bu tez dört bölümden oluşmaktadır. Birinci bölümde, lineer pozitif operatörlerin yaklaşımı üzerine yapılan çalışmaların kısa bir tarihçesi ve bazı temel tanım ve teoremler verilmiştir. İkinci bölümde, Meyer-König ve Zeller operatörlerinin doğurucu fonksiyonlar aracılığıyla tanımlanan bir genelleştirmesi ve Kantarovich tipli bir genelleştirmesi sunularak bu genelleştirmelerin yaklaşım özellikleri elde edilmiştir. Ayrıca süreklilik modülü ve Lipschitz sınıfının elemanları yardımıyla bu operatörlerin yaklaşım hızları hesaplanmış ve fonksiyonel diferansiyel denklemlere bir uygulamaları verilmiştir. Üçüncü bölümde, Meyer-König ve Zeller operatörlerinin momentleri için açık formüller ve bazı hesaplamalar elde edilmiştir. Son bölümde ise Meyer-König ve Zeller operatörlerinin Lipschitz sabitini koruduğu ve bir başlangıç değer problemini sağladığı gösterilmiştir.

Anahtar Kelimeler: Lineer pozitif operatörler, Meyer-König ve Zeller operatörleri, Korovkin teoremi, Lipschitz sınıfı, süreklilik modülü, fonksiyonel diferansiyel denklem.

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CHAPTER 1

INTRODUCTION

The approximation theory has a close relationship with other branches of mathematics, so its development has assisted the development in other mathematical domains. It plays an increasingly important role in applications to many subjects of applied sciences and engineering. Moreover, it has significant intersections with every other topics of analysis. In particular, this theory has been used in the theory of approximation of continuous functions by means of sequences of linear positive operators. The main aim of this theory is to find representations of a function space's elements as limits of elements of another space and in this way, to solve the approximation problem.

1.1 Basic Definitions and Lemmas

In this section, we give some basic definitions and useful theorems for the linear positive operators which will be used throughout this thesis.

Definition 1.1 *A nonempty set X is said to be a "linear space" (or a vector space) over a field $\mathbb{K}$, if it satisfies the following conditions:*

(i) $\forall x, y \in X,\ \ x + y \in X$

(ii) $\forall x, y \in X,\ \ x + y = y + x$

(iii) $\forall x, y, z \in X,\ \ x + (y + z) = (x + y) + z$

(iv) $\forall x \in X,\ \exists \theta \in X$ *such that* $x + \theta = \theta + x = x$

(v) $\forall x \in X,\ \exists \tilde{x} \in X$ *such that* $x + \tilde{x} = \theta$

(vi) $\forall x \in X$ *and* $\forall \alpha \in \mathbb{K}, \;\; \alpha x \in X$

(vii) $\forall x \in X$ *and* $\forall \alpha, \beta \in \mathbb{K}, \;\; \alpha(\beta x) = (\alpha\beta)x$

(viii) $\forall x \in X, \;\; 1x = x$

(ix) $\forall x \in X$ *and* $\forall \alpha \in \mathbb{K}, \;\; \alpha(x + y) = \alpha x + \alpha y$

(x) $\forall x \in X$ *and* $\forall \alpha, \beta \in \mathbb{K}, \;\; (\alpha + \beta)x = \alpha x + \beta x.$

Definition 1.2 *Let X be a linear space over a field $\mathbb{K}$. A function $||.|| : X \longrightarrow \mathbb{R}^+$ satisfying the following conditions is said to be a "norm".*

(i) $\forall x \in X, \;\; ||x|| \geq 0$

(ii) $\forall x \in X, \;\; ||x|| = 0 \iff x = 0$

(iii) $\forall x \in X$ *and* $\forall \alpha \in \mathbb{K}, \;\; ||\alpha x|| = |\alpha| \; ||x||$

(iv) $\forall x, y \in X, \;\; ||x + y|| \leq ||x|| + ||y||.$

A linear space on which a norm is defined is then called a "linear normed space".

Definition 1.3 *Let X and Y be two linear normed function spaces. An operator $L : X \to Y$ is a rule which assigns to each function of X a function of Y.*
We denote the operators by $L(f; x)$ or $L(f(s); x)$.

Definition 1.4 *Let X and Y be two linear normed function spaces. Also, let $L : X \to Y$ be an operator. $L : X \to Y$ is said to be a "linear operator" if it satisfies the two conditions:*

(i) $L(f + g; x) \;=\; L(f; x) + L(g; x),$

(ii) $L(\alpha f; x) \;=\; \alpha L(f; x),$

for every $f, g \in X$ and for every scalar α.

By the definition of the linear operator, it is easily seen that $L(0; x) = 0$.

Definition 1.5 *Let X and Y be two linear normed function spaces, and also, let $f \in X$ such that $f \geq 0$. If $L(f;x) \geq 0$, then the operator $L : X \to Y$ is said to be a "positive operator".*

If $L : X \to Y$ is positive and linear operator, then it is called as "linear positive operator."

Monotonicity is one of the crucial properties of the linear positive operators. The following lemma states that every linear positive operator is monotone increasing.

Lemma 1.6 *Let L be a linear positive operator. If $f \leq g$, then we have*

$$L(f;x) \leq L(g;x). \tag{1.1}$$

Proof. If $f \leq g$, then we can write $g - f \geq 0$. By positivity and linearity of the operator L, one gets

$$L(g-f;x) = L(g;x) - L(f;x) \geq 0$$

which gives the desired result. □

As a result of Lemma 1.6, we can give the following lemma.

Lemma 1.7 *Let L be a linear positive operator. Then, we have*

$$|L(f;x)| \leq L(|f|;x). \tag{1.2}$$

Proof. Since $-|f| \leq f \leq |f|$, by the Lemma 1.6 we have

$$L(-|f|;x) \leq L(f;x) \leq L(|f|;x). \tag{1.3}$$

The linearity of the operator L implies that $L(-|f|;x) = -L(|f|;x)$. Thus, from inequality (1.3) we obtain the inequality (1.2). □

We now give the definition of the function space $C[a,b]$ which frequently appears in this thesis.

Definition 1.8 *$C[a,b]$ is the space of functions defined on $[a,b]$ such that they are continuous in that interval, continuous on the right at the point a and on the left at the point b. The space $C[a,b]$ is normed by*

$$\|f\|_{C[a,b]} = \max_{a\le x\le b} |f(x)|.$$

Definition 1.9 *Let $n \in \mathbb{N}$ and $f_n(x) \in C[a,b]$ be a sequence of functions. If*

$$\lim_{n\to\infty} \|f_n(x) - f(x)\|_{C[a,b]} = 0,$$

then $f_n(x)$ is said to "converge uniformly to the function $f(x)$ in $C[a,b]$", and denoted by

$$f_n(x) \rightrightarrows f(x).$$

1.2 Fundamental Theorems

Positive approximation processes have a fundamental role in approximation theory since the linear positive operators, being the main elements of these processes, are the simplest structures providing functions to converge.

In 1895 Weierstrass proved that if $f(x)$ is a continuous function on a closed interval $[a,b]$, then for each $\epsilon > 0$, there exists a polynomial $p(x)$ such that

$$\max_{a\le x\le b} |f(x) - p(x)| < \epsilon.$$

Weierstrass' theorem includes the existence of a polynomial converging to a continuous function uniformly on a closed interval.

Bernstein [4], to give a simple proof of the Weierstrass theorem, introduced the polynomials

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \tag{1.4}$$

which are called as Bernstein polynomials. The author proved that if $f(x)$ is continuous on $[0,1]$, then $B_n(f;x)$ converges uniformly to $f(x)$ on $[0,1]$. It is clear that the Bernstein polynomials are linear positive operators. So, discussing further properties of the Bernstein polynomials, theory of linear positive operators occurred in the approximation theory. This theory was first investigated by Bohman and Korovkin (see [2]-[4], [12], [13]).

Bohman stated the following theorem.

Theorem 1.10 (Bohman): *Let f be a continuous function on $[0,1]$. Then, the necessary and sufficient conditions that the polynomials*

$$P_n(f;x) = \sum_{k=0}^{n} f(\alpha_{k,n})Q_{k,n}(x), \quad 0 \le \alpha_{k,n} \le 1,\ Q_{k,n}(x) \ge 0$$

converge uniformly to f are:

(i) $\lim\limits_{n\to\infty} \sum_{k=0}^{n} Q_{k,n}(x) = 1,$

(ii) $\lim\limits_{n\to\infty} \sum_{k=0}^{n} \alpha_{k,n} Q_{k,n}(x) = x,$

(iii) $\lim\limits_{n\to\infty} \sum_{k=0}^{n} \alpha_{k,n}^2 Q_{k,n}(x) = x^2.$

Later, P.P. Korovkin proved a more general theorem known as Korovkin theorem.

Theorem 1.11 (P.P. Korovkin): *Let $L_n : C[a,b] \to C[a,b]$ be a sequence of linear positive operators satisfying the conditions*

(i) $L_n(1;x) \rightrightarrows 1,$

(ii) $L_n(s;x) \rightrightarrows x,$

(iii) $L_n(s^2;x) \rightrightarrows x^2.$

Then, for every $f \in C[a,b]$, $L_n(f;x)$ converge uniformly to f on [a,b], i.e.,

$$L_n(f;x) \rightrightarrows f(x).$$

CHAPTER 2

APPROXIMATION BY SOME GENERALIZATIONS OF THE MEYER-KÖNIG AND ZELLER OPERATORS

The classical Meyer-König and Zeller operators [5] are defined by

$$\tilde{M}_n(f;x) = \begin{cases} (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k}{k}x^k, & \text{if } 0 \leq x < 1; \\ f(1), & \text{if } x = 1. \end{cases} \tag{2.1}$$

In order to give the monotonicity properties, Cheney and Sharma [6] modified these operators as follows:

$$M_n(f;x) = \begin{cases} (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k, & \text{if } 0 \leq x < 1; \\ f(1), & \text{if } x = 1. \end{cases} \tag{2.2}$$

These operators are called as Bernstein power series. Throughout this thesis, we refer both of the operators (2.1) and (2.2) as Meyer-König and Zeller operators.

In this chapter, we first introduce a generalization by means of generating functions [17] and a Kantarovich type [15] generalization of the Meyer-König and Zeller operators defined by (2.1) and (2.2). We compute the order of approximation by means of modulus of continuity and the elements of Lipschitz class. Finally, an r-th order generalization and an application to functional differential equations are presented.

2.1 A Generalization of the Meyer-König and Zeller Operators by a Class of Generating Functions

In this section, we consider the sequence of linear positive operators,

$$L_n(f,x) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} f\left(\frac{k}{a_n(k)}\right) C_k^{(n)}(t) x^k, \quad x \in [0,1),\ t \in (-\infty, 0], \tag{2.3}$$

where $\{F_n(x,t)\}$, $n \in \mathbb{N}$ are the generating functions for the sequence of functions $\left\{C_k^{(n)}(t)\right\}_{k\in\mathbb{N}_0}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in the form

$$F_n(x,t) = \sum_{k=0}^{\infty} C_k^{(n)}(t) x^k \tag{2.4}$$

and $C_k^{(n)}(t) \geq 0$ for $t \in (-\infty, 0]$.

We assume that the following conditions are valid:

(a) $F_{n+1}(x,t) = p(x) F_n(x,t),\ p(x) < M < \infty,\ x \in [0,1)$,

(b) $AtC_{k-1}^{(n+1)}(t) = a_n(k) C_{k-1}^{(n)}(t) - kC_k^{(n)}(t),\ A \in [0,a],\ C_k^{(n)}(t) = 0$ for $k \in \mathbb{Z}^-$,

(c) $\max\{k,n\} \leq a_n(k) \leq a_n(k+1)$.

Remark 2.1 *The following choices show that the operators L_n given in (2.3) are the generalizations of some well known operators:*

(i) If we take $a_n(k) = n+k$, $C_k^{(n)}(t) = L_k^{(n)}(t)$, where $L_k^{(n)}(t)$ is the Laguerre polynomial [19] and

$$F_n(x,t) = (1-x)^{-n-1} \exp\left(\frac{tx}{x-1}\right),$$

then the operators (2.3) become

$$P_n(f;x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) L_k^{(n)}(t) x^k$$

which is given in [6].

(ii) Taking $C_k^{(n)}(t) = L_k^{(n)}(t)$ again, $F_n(x,t)$ as in part (i) and choosing $a_n(k) = n+k+1$ for the operators (2.3), we get

$$Z_n(f;x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right) L_k^{(n)}(t) x^k,$$

defined in [10].

(iii) Since $L_k^{(n)}(0) = \binom{n+k}{k}$, by choosing $t = 0$ in $P_n(f;x)$ defined in (i), one obtains the Meyer-König and Zeller operators defined by (2.1):

$$M_n(f;x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k.$$

(iv) Now, if we insert $t = 0$ in the operators $Z_n(f;x)$ defined in (ii), then the operators $Z_n(f;x)$ turn out to the Bernstein power series defined by (2.2):

$$\tilde{M}_n(f;x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right) \binom{n+k}{k} x^k.$$

(v) If we put $F_n(x,t) = e^{nx}, a_n(k) = n$ and $C_k^{(n)}(t) = \frac{n^k}{k!}$ in the operators (2.3), we obtain the well known Szasz-Mirakjan operators [1]:

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}.$$

(vi) For the choices $F_n(x,t) = e^{h(n)x},\ a_n(k) = h(n)$ and $C_k^{(n)}(t) = \frac{(h(n))^k}{k!}$, the operators (2.3) turn out to be

$$S_n(f;x) = e^{-h(n)x} \sum_{k=0}^{\infty} f\left(\frac{k}{h(n)}\right) \frac{(h(n)x)^k}{k!}$$

which were introduced in [17].

2.1.1 Approximation Properties of L_n

We now give Korovkin type approximation properties of the operators L_n defined by (2.3). The following theorem states the convergence of the operators L_n.

Theorem 2.2 *Let* $x \in [0,1)$, $t \in (-\infty, 0]$ *and* b *be a real number in the interval* $(0,1)$. *If* f *is continuous on* $[0,b]$ *and* $\frac{|t|}{n} \to 0$, *then* $L_n(f;x)$ *converges to* $f(x)$ *uniformly on* $[0,b]$.

Proof. By the Korovkin theorem 1.11, it is sufficient to show that the conditions

$$L_n(f(s);x) \rightrightarrows x^i, \text{ for } f(s) \equiv s^i, \; i = 0,1,2$$

are satisfied.

By the definition of the operators L_n, for the function $f(s) \equiv 1$ it is easily seen that

$$L_n(1;x) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} C_k^{(n)}(t) x^k = 1$$

which gives

$$\lim_{n\to\infty} \|L_n(1;x) - 1\|_{C[0,b]} = 0. \tag{2.5}$$

We now consider the function $f(s) \equiv s$. By using the condition (b), equation (2.4) and the definition of the operators L_n, one gets

$$\begin{aligned} L_n(s;x) &= \frac{1}{F_n(x,t)} \sum_{k=1}^{\infty} \frac{k}{a_n(k)} C_k^{(n)}(t) x^k \\ &= \frac{1}{F_n(x,t)} \sum_{k=1}^{\infty} C_{k-1}^{(n)}(t) x^k - \frac{At}{F_n(x,t)} \sum_{k=1}^{\infty} \frac{C_{k-1}^{(n+1)}(t)}{a_n(k)} x^k \\ &= x \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} C_k^{(n)}(t) x^k - \frac{Atx}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{C_k^{(n+1)}(t)}{a_n(k+1)} x^k \\ &= x - \frac{Atx}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{C_k^{(n+1)}(t)}{a_n(k+1)} x^k. \end{aligned} \tag{2.6}$$

Since $t \in (-\infty, 0]$, we can write

$$\frac{Atx}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{1}{a_n(k+1)} C_k^{(n+1)}(t) x^k \leq 0,$$

thus, from the equation (2.6), it follows that

$$L_n(s;x) - x \geq 0. \tag{2.7}$$

By the condition (c), we have $\max\{k,n\} \leq a_n(k+1)$. If $\max\{k,n\} = n$, then $k \leq n \leq a_n(k+1)$, for the other case, if $\max\{k,n\} = k$, then $n \leq k \leq a_n(k+1)$. This implies that

$$\frac{1}{a_n(k+1)} \leq \frac{1}{n}. \tag{2.8}$$

Thus, using the equations (2.4) and (2.6), and the conditions (c) and (a), we obtain

$$\begin{aligned} L_n(s;x) &\leq x - \frac{1}{n} \frac{Atx}{F_n(x,t)} \sum_{k=0}^{\infty} C_k^{(n+1)}(t) x^k \\ &= x - \frac{Atx}{nF_n(x,t)} F_{n+1}(x,t) \\ &= x - \frac{xp(x)At}{n}. \end{aligned} \tag{2.9}$$

From inequalities (2.7) and (2.9), it is obvious that

$$|L_n(s;x) - x| \leq -\frac{xp(x)At}{n}. \tag{2.10}$$

Taking maximum of both sides of the inequality (2.10) over $[0,b]$, we find that

$$\|L_n(s;x) - x\|_{C[0,b]} \leq \frac{bMa|t|}{n}. \tag{2.11}$$

Since $\frac{|t|}{n} \to 0$, this gives

$$\lim_{n\to\infty} \|L_n(s;x) - x\|_{C[0,b]} = 0. \tag{2.12}$$

Now we consider the function $f(s) \equiv s^2$. By using the condition (b), we have

$$
\begin{aligned}
L_n(s^2;x) &= \frac{1}{F_n(x,t)}\sum_{k=1}^{\infty}\frac{k^2}{[a_n(k)]^2}C_k^{(n)}(t)x^k \\
&= \frac{1}{F_n(x,t)}\sum_{k=1}^{\infty}\frac{k}{[a_n(k)]^2}\left[a_n(k)C_{k-1}^{(n)}(t)-AtC_{k-1}^{(n+1)}(t)\right]x^k \\
&= \frac{1}{F_n(x,t)}\sum_{k=1}^{\infty}\left[\frac{(k-1)C_{k-1}^{(n)}(t)}{a_n(k)}+\frac{1}{a_n(k)}C_{k-1}^{(n)}(t)-\frac{k}{[a_n(k)]^2}AtC_{k-1}^{(n+1)}(t)\right]x^k \\
&= \frac{1}{F_n(x,t)}\sum_{k=2}^{\infty}\left[\frac{a_n(k-1)}{a_n(k)}C_{k-2}^{(n)}(t)-\frac{At}{a_n(k)}C_{k-2}^{(n+1)}(t)\right]x^k \\
&\quad +\sum_{k=1}^{\infty}\left[\frac{1}{a_n(k)}C_{k-1}^{(n)}(t)-\frac{k}{[a_n(k)]^2}AtC_{k-1}^{(n+1)}(t)\right]x^k
\end{aligned}
$$

or

$$
\begin{aligned}
L_n(s^2;x)-x^2 &\le \left[\frac{1}{F_n(x,t)}\sum_{k=2}^{\infty}\frac{a_n(k-1)}{a_n(k)}C_{k-2}^{(n)}(t)x^k-x^2\right]+\left|\frac{At}{F_n(x,t)}\sum_{k=2}^{\infty}\frac{1}{a_n(k)}C_{k-2}^{(n+1)}(t)x^k\right| \\
&\quad +\left|\frac{1}{F_n(x,t)}\sum_{k=1}^{\infty}\frac{1}{a_n(k)}C_{k-1}^{(n)}(t)x^k\right|+\left|\frac{At}{F_n(x,t)}\sum_{k=1}^{\infty}\frac{k}{[a_n(k)]^2}C_{k-1}^{(n+1)}(t)x^k\right| \\
&=: I_1+|I_2|+|I_3|+|I_4|. \qquad (2.13)
\end{aligned}
$$

From the condition (c), it is easily seen that $\frac{a_n(k-1)}{a_n(k)} \le 1$. By using this fact, we can write

$$
I_1 = \frac{1}{F_n(x,t)}\sum_{k=2}^{\infty}\frac{a_n(k-1)}{a_n(k)}C_{k-2}^{(n)}(t)x^k - x^2 \le 0. \qquad (2.14)
$$

If we use the equation (2.4) and the conditions (a) and (c), then we find that

$$
\begin{aligned}
|I_2| = \left|\frac{At}{F_n(x,t)}\sum_{k=2}^{\infty}\frac{1}{a_n(k)}C_{k-2}^{(n+1)}(t)x^k\right| &= \left|\frac{Atx^2}{F_n(x,t)}\sum_{k=0}^{\infty}\frac{1}{a_n(k+2)}C_k^{(n+1)}(t)x^k\right| \\
&\le \frac{a|t|x^2}{F_n(x,t)}\sum_{k=0}^{\infty}\frac{1}{a_n(k+2)}C_k^{(n+1)}(t)x^k \\
&\le \frac{a|t|x^2}{nF_n(x,t)}\sum_{k=0}^{\infty}C_k^{(n+1)}(t)x^k \\
&\le \frac{a|t|x^2}{n}p(x). \qquad (2.15)
\end{aligned}
$$

Similarly, in terms of the equation (2.8) and the condition (c), one gets

$$|I_3| = \left|\frac{1}{F_n(x,t)}\sum_{k=1}^{\infty}\frac{1}{a_n(k)}C_{k-1}^{(n)}(t)x^k\right| = \frac{x}{F_n(x,t)}\sum_{k=0}^{\infty}\frac{1}{a_n(k+1)}C_k^{(n)}(t)x^k \le \frac{x}{n}. \tag{2.16}$$

To calculate I_4, by using the condition (c), equation (2.8) and the condition (a), we can write

$$\begin{aligned}|I_4| = \left|\frac{At}{F_n(x,t)}\sum_{k=1}^{\infty}\frac{k}{[a_n(k)]^2}C_{k-1}^{(n+1)}(t)x^k\right| &= \frac{A|t|x}{F_n(x,t)}\sum_{k=0}^{\infty}\frac{k+1}{[a_n(k+1)]^2}C_k^{(n+1)}(t)x^k \\ &\le \frac{A|t|x}{F_n(x,t)}\sum_{k=0}^{\infty}\frac{1}{a_n(k+1)}C_k^{(n+1)}(t)x^k \\ &\le \frac{a|t|x}{n}p(x). \end{aligned}\tag{2.17}$$

On the other hand, by the linearity of the operators (2.3) and the equation (2.7), we find

$$L_n(s^2;x) - x^2 = L_n((s-x)^2;x) + 2xL_n(s-x;x), \tag{2.18}$$

and

$$L_n(s-x;x) \ge 0. \tag{2.19}$$

The inequalities (2.18), (2.19) and the positivity of L_n implies that

$$L_n(s^2;x) - x^2 \ge 0. \tag{2.20}$$

Finally, inserting the equations (2.14) - (2.17) into the equation (2.13) we obtain

$$L_n(s^2;x) - x^2 \le \frac{a|t|x^2}{n}p(x) + \frac{x}{n} + \frac{a|t|x}{n}p(x). \tag{2.21}$$

In addition, from inequalities (2.20) and (2.21), it is easily seen that

$$0 \le L_n(s^2;x) - x^2 \le \frac{a|t|x^2}{n}p(x) + \frac{x}{n} + \frac{a|t|x}{n}p(x).$$

Hence, taking the maximum of both sides over $[0,b]$, we have

$$0 \leq \left\|L_n(s^2;x) - x^2\right\|_{C[0,b]} \leq \frac{a|t|b^2}{n}M + \frac{b}{n} + \frac{a|t|b}{n}M$$
$$= \frac{1}{n}\Big(b + a|t|bM(1+b)\Big) \tag{2.22}$$

which obviously implies that

$$\lim_{n\to\infty} \left\|L_n(s^2;x) - x^2\right\|_{C[0,b]} = 0. \tag{2.23}$$

The equations (2.5), (2.12) and (2.23) gives the desired result. □

2.1.2 Rate of Convergence of L_n

In this section, we compute the rate of convergence of the linear positive operators L_n defined by (2.3) which converge to $f(x)$ on $[0,b]$ uniformly. For this purpose, we find an inequality of the form

$$\|L_n(f;x) - f(x)\|_{C[a,b]} \leq C\alpha_n, \quad 0 < C \in \mathbb{R}, \tag{2.24}$$

where α_n is a sequence of positive numbers such that

$$\lim_{n\to\infty} \alpha_n = 0.$$

From (2.24), it follows that the rate of convergence of the operators L_n to f depends on how fast the sequence α_n converges to zero.

Firstly we give the rate of convergence of the operators L_n by means of the modulus of continuity, defined as follows:

Definition 2.3 *Let $f \in C[0,b]$. The modulus of continuity of f, denoted by $\omega(f,\delta)$, is defined by*

$$\omega(f,\delta) = \sup_{\substack{|s-x|<\delta \\ s,x\in[0,b]}} |f(s) - f(x)|\,. \tag{2.25}$$

For any $\delta > 0$, the well known properties of modulus of continuity (see [12]- [17], etc.) are:

$$\text{i) If } \delta_1 \leq \delta_2, \text{ then } \omega(f, \delta_1) \leq \omega(f, \delta_2) \tag{2.26}$$

$$\text{ii) } \lim_{\delta \to 0} \omega(f, \delta) = 0 \tag{2.27}$$

$$\text{iii) } |f(s) - f(x)| \leq \omega(f, \delta)\left(1 + \frac{|s-x|}{\delta}\right) \tag{2.28}$$

$$\text{iv) } |f(s) - f(x)| \leq \omega(f, |s-x|) \leq \left(1 + \frac{|s-x|^2}{\delta^2}\right)\omega(f, \delta). \tag{2.29}$$

Theorem 2.4 *Let L_n defined by (2.3). Then, for all $f \in C[0, b]$, we have*

$$\|L_n(f; x) - f(x)\|_{C[0,b]} \leq \left(1 + (3B)^{\frac{1}{2}}\right)\omega(f, \delta_n),$$

where $\delta_n = \dfrac{1}{\sqrt{n}}$ *and* $B = \max\left\{b, bMa|t|, 3b^2Ma|t|\right\}$.

Proof. Let $f \in C[0, b]$. By using linearity and monotonicity of L_n and the property (2.28), we can write

$$\begin{aligned} |L_n(f; x) - f(x)| &\leq L_n\left(|f(s) - f(x)|\,; x\right) \\ &\leq \omega(f, \delta_n)L_n\left(1 + \frac{|s-x|}{\delta_n}; x\right) \\ &= \omega(f, \delta_n)\left(L_n(1; x) + \frac{1}{\delta_n}L_n(|s-x|; x)\right) \\ &= \omega(f, \delta_n)\left(1 + \frac{1}{\delta_n}\left[\frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}\left|\frac{k}{a_n(k)} - x\right| C_k^{(n)}(t)x^k\right]\right). \end{aligned} \tag{2.30}$$

Then, by means of the Cauchy-Schwarz inequality and letting

$$\frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}\left(\frac{k}{a_n(k)} - x\right)^2 C_k^{(n)}(t)x^k \;=:\; A_n(x,t), \tag{2.31}$$

one gets

$$
\begin{aligned}
\frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}\left|\frac{k}{a_n(k)}-x\right|C_k^{(n)}(t)x^k &= \sum_{k=0}^{\infty}\left\{\left[\frac{1}{F_n(x,t)}\left(\frac{k}{a_n(k)}-x\right)^2 C_k^{(n)}(t)x^k\right]^{\frac{1}{2}}\right.\\
&\quad \left.\times\left[\frac{1}{F_n(x,t)}C_k^{(n)}(t)x^k\right]^{\frac{1}{2}}\right\}\\
&\leq \left[\frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}\left(\frac{k}{a_n(k)}-x\right)^2 C_k^{(n)}(t)x^k\right]^{\frac{1}{2}}\\
&\quad \times\left[\frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}C_k^{(n)}(t)x^k\right]^{\frac{1}{2}}\\
&= \left[\frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}\left(\frac{k}{a_n(k)}-x\right)^2 C_k^{(n)}(t)x^k\right]^{\frac{1}{2}}\\
&= [A_n(x,t)]^{\frac{1}{2}}. \qquad (2.32)
\end{aligned}
$$

Hence, from (2.30), it follows that

$$|L_n(f;x)-f(x)| \leq \omega(f;\delta_n)\left(1+\frac{1}{\delta_n}\left(A_n(x,t)\right)^{\frac{1}{2}}\right).$$

Taking maximum of both sides over $[0,b]$, we find

$$\|L_n(f;x)-f(x)\|_{C[0,b]} \leq \omega(f;\delta_n)\left(1+\frac{1}{\delta_n}\max_{x\in[0,b]}\left(A_n(x,t)^{\frac{1}{2}}\right)\right).$$

Since

$$
\begin{aligned}
A_n(x,t) &= \frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}\left(\frac{k}{a_n(k)}-x\right)^2 C_k^{(n)}(t)x^k\\
&= L_n((s-x)^2;x)\\
&\leq \left|L_n(s^2;x)-x^2\right| + 2x\left|L_n(s;x)-x\right|,
\end{aligned}
$$

by using the inequalities (2.11) and (2.22), one can write

$$\begin{aligned}\max_{x\in[0,b]} A_n(x,t) &\leq \left\|L_n(s^2;x)-x^2)\right\|_{C[0,b]} + 2b\,\|L_n(s;x)-x\|_{C[0,b]} \\ &\leq \frac{1}{n}\Big(b+a|t|bM(1+b)\Big) + \frac{2b^2Ma|t|}{n} \\ &= \frac{1}{n}\Big[b+a|t|bM+3a|t|b^2M\Big] \\ &\leq \frac{3B}{n}. \end{aligned} \tag{2.33}$$

This implies that

$$\|L_n(f;x)-f(x)\|_{C[0,b]} \leq \omega(f,\delta_n)\left(1+\frac{1}{\delta_n}\left(\frac{3B}{n}\right)^{\frac{1}{2}}\right).$$

Finally, choosing $\delta_n = \dfrac{1}{\sqrt{n}}$, we obtain

$$\|L_n(f;x)-f(x)\|_{C[0,b]} \leq \omega(f,\delta_n)\left(1+(3B)^{\frac{1}{2}}\right),$$

which is the desired result. □

Let us recall the definition of the Lipschitz class denoted by $\mathrm{Lip}_M(\alpha)$.

Definition 2.5 *Let $M > 0$ and $0 < \alpha \leq 1$. Then, a function $f \in C[0,b]$ belongs to $Lip_M(\alpha)$, if the inequality*

$$|f(s)-f(x)| \leq M|s-x|^{\alpha}, \quad s,x\in[0,b] \tag{2.34}$$

is satisfied.

We now give the rate of convergence by means of the elements of Lipschitz class.

Theorem 2.6 *Let $f \in Lip_M(\alpha)$. Then, we have*

$$\|L_n(f;x)-f(x)\|_{C[0,b]} \leq M(3B)^{\frac{\alpha}{2}}\delta_n^{\alpha}, \tag{2.35}$$

where B and δ_n are the same as in the Theorem 2.4.

Proof. Let $f \in \mathrm{Lip}_M(\alpha)$. By using the linearity and monotonicity of the operators L_n defined by (2.3) and using (2.34), we have

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq L_n(|f(s) - f(x)|;x) \\ &= \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \left| f\left(\frac{k}{a_n(k)}\right) - f(x) \right| C_k^n(t) x^k \\ &\leq \frac{M}{F_n(x,t)} \sum_{k=0}^{\infty} \left| \frac{k}{a_n(k)} - x \right|^{\alpha} C_k^n(t) x^k. \end{aligned} \tag{2.36}$$

Applying the Hölder inequality with $p = \dfrac{2}{\alpha},\ q = \dfrac{2}{2-\alpha}$, we can write

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq M \left[\frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \left(\frac{k}{a_n(k)} - x \right)^2 C_k^n(t) x^k \right]^{\frac{\alpha}{2}} \left[\frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} C_k^n(t) x^k \right]^{\frac{2-\alpha}{2}} \\ &= M \left[A_n(x,t) \right]^{\frac{\alpha}{2}}, \end{aligned} \tag{2.37}$$

where $A_n(x,t)$ is defined by (2.31). Taking into account the inequality (2.33), we arrive at the required result. □

Finally, we compute the rate of convergence of the operators L_n defined by (2.3) by using the Peetre's K-functional (see [7], [17]) which is defined as follows.

Definition 2.7 *The Peetre's K-functional $K(f, \delta_n)$ is defined by*

$$K(f, \delta_n) = \inf_{g \in C^2[0,b]} \left\{ \|f - g\|_{C[0,b]} + \delta_n \|g\|_{C^2[0,b]} \right\}, \tag{2.38}$$

where the space $C^2[0,b]$ is the space of functions f for which $f, f', f'' \in C[0,b]$. The norm in the space $C^2[0,b]$ is defined by

$$\|f\|_{C^2[0,b]} := \|f\|_{C[0,b]} + \|f'\|_{C[0,b]} + \|f''\|_{C[0,b]}. \tag{2.39}$$

Theorem 2.8 *Let L_n defined by (2.3). If $f \in C[0,b]$, then we have*

$$\|L_n(f;x) - f(x)\|_{C[0,b]} \leq 2K(f, \delta_n)$$

where

$$\delta_n = \frac{b + a|t|bM(2+3b)}{4n}.$$

Proof. Let $g \in C^2[0,b]$. Then, by the Taylor formula, we can write

$$g(s) - g(x) = g'(x)(s-x) + \frac{1}{2}g''(x)(s-x)^2$$

which implies

$$|L_n(g;x) - g(x)| \leq |g'(x)|\,|L_n(s-x;x)| + \frac{1}{2}\,|g''(x)|\,\left|L_n\left((s-x)^2;x\right)\right|.$$

Using the expression

$$L_n((s-x)^2;x) = \left(L_n(s^2;x) - x^2\right) - 2x\left(L_n(s;x) - x\right),$$

we can write

$$|L_n(g;x) - g(x)| \leq |g'(x)|\,|L_n(s-x;x)| + \frac{1}{2}\,|g''(x)|\left[\left|L_n(s^2;x) - x^2\right| + 2x\left|L_n(s;x) - x\right|\right].$$

If we take the maximum of both sides of this inequality over $[0,b]$ and use the inequalities (2.11) and (2.22), then we find

$$\|L_n(g;x) - g(x)\|_{C[0,b]} \leq \|g'\|_{C[0,b]}\,\frac{bMa|t|}{n} + \frac{1}{2n}\left[b + a|t|bM(1+3b)\right]\|g''\|_{C[0,b]}.$$

Then, for each $t \in (-\infty, 0]$ and each $b \in (0,1)$ it follows that

$$\|L_n(g;x) - g(x)\|_{C[0,b]} \leq \left(\|g'\|_{C[0,b]} + \|g''\|_{C[0,b]}\right)\frac{1}{2n}\left[b + a|t|bM(2+3b)\right]$$

and so

$$\|L_n(g;x) - g(x)\|_{C[0,b]} \leq \|g\|_{C^2[0,b]}\,\frac{1}{2n}\left[b + a|t|bM(2+3b)\right]. \qquad (2.40)$$

On the other hand, by using the linearity of the operators (2.3), we have

$$|L_n(f;x) - f(x)| \leq |L_n(f-g;x)| + |f(x) - g(x)| + |L_n(g;x) - g(x)|.$$

Hence, noting that $L_n(1;x) = 1$ and taking maximum over $[0,b]$ of both sides of the final equation, we obtain

$$||L_n(f;x) - f(x)||_{C[0,b]} \leq 2||f-g||_{C[0,b]} + ||L_n(g;x) - g(x)||_{C[0,b]}. \tag{2.41}$$

Finally, with the help of (2.40), the inequality (2.41) takes the form

$$||L_n(f;x) - f(x)||_{C[0,b]} \leq 2\left(||f-g||_{C[0,b]} + \frac{1}{4n}\Big[b + a|t|bM(2+3b)\Big]||g||_{C^2[0,b]}\right)$$

or

$$||L_n(f;x) - f(x)||_{C[0,b]} \leq 2\left(||f-g||_{C[0,b]} + \delta_n||g||_{C^2[0,b]}\right).$$

Therefore, this implies that

$$\begin{aligned} ||L_n(f;x) - f(x)||_{C[0,b]} &\leq \inf_{g\in C^2[0,b]} (2||f-g||_{C[0,b]} + \delta_n||g||_{C^2[0,b]}) \\ &= 2K(f,\delta_n). \end{aligned}$$

So the proof is completed. □

2.1.3 A Generalization of r-th Order of L_n

In this section, we introduce the r-th order generalization of the operators L_n defined by (2.3). Let us recall the definition of the function space $C^r[0,b]$.

Definition 2.9 *We denote by $C^r[0,b]$, $r = 0,1,2\ldots$, the set of functions f having continuous r-th derivatives $f^{(r)}\left(f^{(0)}(x) = f(x)\right)$ on the interval $[0,b]$.*

We now consider the following generalization of the linear positive operators L_n which are introduced in [11] and [17].

$$L_n^{[r]}(f;x) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \sum_{i=0}^{r} f^{(i)}\left(\frac{k}{a_n(k)}\right) \frac{\left(x - \frac{k}{a_n(k)}\right)^i}{i!} C_k^{(n)}(t) x^k, \tag{2.42}$$

where $f \in C^r[0,b]$, $r = 0, 1, 2\ldots$ and $n \in \mathbb{N}$. These operators are called as the r-th order generalization of the operators L_n.

Note that for $r = 0$, we have the sequence of the operators (2.3).

Theorem 2.10 *If $f^{(r)} \in Lip_M(\alpha)$ and $f \in C^r[0,b]$, then we have*

$$\left\| L_n^{[r]}(f;x) - f(x) \right\|_{C[0,b]} \leq \frac{M}{(r-1)!} \frac{\alpha}{\alpha + r} B(\alpha, r) \left\| L_n(|s-x|^{\alpha+r}; x) \right\|_{C[0,b]},$$

where $B(\alpha, r)$ is the Beta function and $r, n \in \mathbb{N}$.

Proof. Let $f \in C^r[0,b]$. By (2.42), we have

$$\begin{aligned} f(x) - L_n^{[r]}(f;x) &= f(x) - \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \sum_{i=0}^{r} f^{(i)}\left(\frac{k}{a_n(k)}\right) \frac{\left(x - \frac{k}{a_n(k)}\right)^i}{i!} C_k^{(n)}(t) x^k \\ &= \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} \left[f(x) - \sum_{i=0}^{r} f^{(i)}\left(\frac{k}{a_n(k)}\right) \frac{\left(x - \frac{k}{a_n(k)}\right)^i}{i!} \right] C_k^{(n)}(t) x^k. \end{aligned} \tag{2.43}$$

From the Taylor's theorem, it follows that

$$\begin{aligned} f(x) - \sum_{i=0}^{r} f^{(i)}\left(\frac{k}{a_n(k)}\right) \frac{\left(x - \frac{k}{a_n(k)}\right)^i}{i!} &= \frac{\left(x - \frac{k}{a_n(k)}\right)^r}{(r-1!)} \int_0^1 (1-t)^{r-1} \\ &\times \left[f^{(r)}\left(\frac{k}{a_n(k)} + t\left(x - \frac{k}{a_n(k)}\right)\right) - f^{(r)}\left(\frac{k}{a_n(k)}\right) \right] dt. \end{aligned} \tag{2.44}$$

Since $f^{(r)} \in \mathrm{Lip}_M(\alpha)$, we can write

$$\left| f^{(r)}\left(\frac{k}{a_n(k)} + t\left(x - \frac{k}{a_n(k)}\right)\right) - f^{(r)}\left(\frac{k}{a_n(k)}\right) \right| \leq M t^\alpha \left| x - \frac{k}{a_n(k)} \right|^\alpha.$$

Now, using this and the following relation

$$B(1+\alpha,r)=\int_0^1 t^{\alpha}(1-t)^{r-1}dt=\frac{\alpha}{\alpha+r}B(\alpha,r),$$

we have

$$\begin{aligned}
&\left|f(x)-\sum_{i=0}^{r} f^{(i)}\left(\frac{k}{a_n(k)}\right)\frac{\left(x-\frac{k}{a_n(k)}\right)^i}{i!}\right| \\
&\quad=\left|\frac{\left(x-\frac{k}{a_n(k)}\right)^r}{(r-1)!}\int_0^1(1-t)^{r-1}\left[f^{(r)}\left(\frac{k}{a_n(k)}+t\left(x-\frac{k}{a_n(k)}\right)\right)-f^{(r)}\left(\frac{k}{a_n(k)}\right)\right]dt\right| \\
&\quad\le \frac{\left|\left(x-\frac{k}{a_n(k)}\right)^r\right|}{(r-1)!}\int_0^1\left|(1-t)^{r-1}\right|\left|f^{(r)}\left(\frac{k}{a_n(k)}+t\left(x-\frac{k}{a_n(k)}\right)\right)-f^{(r)}\left(\frac{k}{a_n(k)}\right)\right|dt \\
&\quad\le \frac{\left|\left(x-\frac{k}{a_n(k)}\right)^r\right|}{(r-1)!}M\left|x-\frac{k}{a_n(k)}\right|^{\alpha}\int_0^1(1-t)^{r-1}t^{\alpha}dt \\
&\quad= \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha;r)\left|x-\frac{k}{a_n(k)}\right|^{\alpha+r}. \qquad (2.45)
\end{aligned}$$

Thus, by using the equation (2.43) and the inequality (2.45), we may conclude that

$$\left|f(x)-L_n^{[r]}(f;x)\right| \le \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha;r)\frac{1}{F_n(x,t)}\sum_{k=0}^{\infty}\left|x-\frac{k}{a_n(k)}\right|^{\alpha+r}C_k^{(n)}(t)x^k.$$

Taking maximum of both sides of this inequality over $[0,b]$, we obtain

$$\left\|f(x)-L_n^{[r]}(f;x)\right\|_{C[0,b]} \le \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha;r)\left\|L_n\left(|x-s|^{\alpha+r};x\right)\right\|_{C[0,b]}$$

which is the desired result. □

Considering now the function $g\in C[0,b]$ defined by $g(s)=|s-x|^{\alpha+r}$, it is seen that $g(x)=0$. Thus, by Theorem 2.2, this follows that

$$\lim_{n\to\infty}\|L_n(g;x)\|_{C[0,b]}=0.$$

Then, for all $f^{(r)} \in \mathrm{Lip}_M(\alpha)$ and $f \in C^r[0,b]$, Theorem 2.10 implies that

$$\lim_{n\to\infty} \left\| L_n^{[r]}(f;x) - f(x) \right\|_{C[0,b]} = 0.$$

2.1.4 An Application to Differential Equations

Consider the linear positive operators

$$L_n^*(f;x) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} f\left(\frac{k}{b_n+k}\right) C_k^{(n)}(t) x^k, \tag{2.46}$$

where $b_n \le b_{n+1}$. We observe that the operators L_n defined by (2.3) can be reduced to the operators (2.46) for $a_n(k) = b_n + k$. In this part, we give an application to differential equations by the next theorem.

Theorem 2.11 *Let*

$$\frac{\partial}{\partial x}(F_n(x,t)) = K_n(x) F_n(x,t) \tag{2.47}$$

and $g(s) = \dfrac{s}{1-s}$. *For each* $x \in [0,b]$, $b \in (0,1)$ *and* $f \in C[0,b]$, *the operators* $L_n^*(f;x)$ *defined by (2.46) satisfy the differential equation*

$$x \frac{d}{dx} L_n^*(f;x) = -x K_n(x) L_n^*(f;x) + b_n L_n^*(fg;x). \tag{2.48}$$

Note that the equation (2.48) is indeed not a differential equation for $L_n^*(f;x)$ but rather a functional differential equation.

Proof. Let $f \in C[0,b]$. In section 2.1.1, we have shown that the operators L_n defined by (2.3) converge uniformly on the interval $[0,b]$. Hence, the power series on the right hand side of (2.46) can be differentiated term by term in $[0,b]$. Doing this, we have

$$\frac{d}{dx} L_n^*(f;x) = \frac{-\frac{\partial}{\partial x} F_n(x,t)}{F_n^2(x,t)} \sum_{k=0}^{\infty} f\left(\frac{k}{b_n+k}\right) C_k^{(n)}(t) x^k + \frac{1}{F_n(x,t)} \sum_{k=1}^{\infty} f\left(\frac{k}{b_n+k}\right) k C_k^{(n)}(t) x^{k-1}.$$

By using the equation (2.47) and keeping in mind that $g\left(\frac{k}{b_n+k}\right)=\frac{k}{b_n}$, we can write

$$\begin{aligned} x\frac{d}{dx}L_n^*(f;x) &= \frac{-xK_n(x)}{F_n(x,t)}\sum_{k=0}^{\infty} f\left(\frac{k}{b_n+k}\right)C_k^{(n)}(t)x^k \\ &\quad + \frac{xb_n}{F_n(x,t)}\sum_{k=0}^{\infty} f\left(\frac{k}{b_n+k}\right)\frac{k}{b_n}C_k^{(n)}(t)x^{k-1} \\ &= \frac{-xK_n(x)}{F_n(x,t)}\sum_{k=0}^{\infty} f\left(\frac{k}{b_n+k}\right)C_k^{(n)}(t)x^k \\ &\quad + \frac{b_n}{F_n(x,t)}\sum_{k=0}^{\infty} f\left(\frac{k}{b_n+k}\right)g\left(\frac{k}{b_n+k}\right)C_k^{(n)}(t)x^k. \end{aligned}$$

Therefore, by using the definition of the operators L_n^*, we arrive at the required result.

□

2.2 A Kantarovich Type Generalization of the Meyer-König and Zeller Operators

In this section, we consider a Kantarovich type generalization of the Meyer-König and Zeller operators [15]

$$M_n^*(f;x)=\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{1}{\alpha_{n,k}}\int_k^{k+\alpha_{n,k}} f\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)\frac{x^k}{k!}, \tag{2.49}$$

where $0<\alpha_{n,k}\leq 1$, and f is an integrable function on $(0,1)$; under the conditions given below.

(i) Let A be a real number in the interval $(0,1)$ and also, let $\{\varphi_n\}$ be a sequence of functions. Every element of the sequence $\{\varphi_n\}$ is analytic on a domain D which contains the disk $B=\{z\in\mathbb{C}:|z|\leq A\}$,

(ii) $\varphi_n^{(0)}(x)=\varphi_n(x)>0$,

(iii) $\varphi_n^{(k)}(x)=\gamma_n(n+k)(1+l_{n,k})\varphi_n^{(k-1)}(x),\ k=1,2,\ldots,$

(iv) $\varphi_n^{(k)}(0)=\gamma_n(n+k)(1+l_{n,k})\varphi_n^{(k-1)}(0),\ k=1,2,\ldots,$

where $\varphi_n^{(k)}(x) = \dfrac{d^k}{dx^k}\varphi_n(x)$, and $l_{n,k}$ and γ_n are sequences of positive numbers having the properties

$$l_{n,k} = O\left(\frac{1}{n}\right),\ l_{n,k} \geq 0,\ \ \gamma_n = 1 + O\left(\frac{1}{n}\right),\ \gamma_n \geq 1.$$

Note that, if we choose $\varphi_n^{(k)}(x) = (1-x)^{-n-1}$, $\gamma_n = 1$ and $l_{n,k} = 0$ for the operators M_n^*, we obtain the Meyer-König and Zeller operators defined by (2.2).

2.2.1 Approximation Properties of M_n^*

We now investigate the approximation properties of the operators $M_n^*(f;x)$ with the help of the Korovkin theorem 1.11. Before giving our main result we need the following lemma which is given by O. Doğru in [14].

Lemma 2.12 *The sequence of linear positive operators T_n given by*

$$T_n(f;x) = \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \varphi_n^{(k)}(0) \frac{x^k}{k!} \tag{2.50}$$

converges uniformly to the function $f \in C[0,A]$ in $[0,A]$ under the conditions (i)-(iv).

Proof. By the Korovkin theorem 1.11, it is sufficient to show that the conditions

$$T_n(f(s);x) \rightrightarrows x^i, \quad \text{for } f(s) \equiv s^i,\ i = 0,1,2$$

are satisfied.

For $f(s) \equiv 1$, (2.50) reduces to

$$T_n(1;x) = \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \varphi_n^{(k)}(0) \frac{x^k}{k!}.$$

By means of the Maclaurin series expansion of the function $\varphi_n(x)$, it follows that $T_n(1;x) = 1$ which gives

$$\lim_{n\to\infty} \|T_n(1;x) - 1\|_{C[0,A]} = 0.$$

For $f(s) = s$, by using the condition (iv) and the fact that $T_n(1; x) = 1$, we have

$$\begin{aligned} T_n(s;x) &= \frac{1}{\varphi_n(x)} \sum_{k=1}^{\infty} \frac{\varphi_n^{(k)}(0)}{n+k} \frac{x^k}{(k-1)!} \\ &= \frac{1}{\varphi_n(x)} \sum_{k=1}^{\infty} \gamma_n (1 + l_{n,k}) \varphi_n^{(k-1)}(0) \frac{x^k}{(k-1)!} \\ &= \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=1}^{\infty} (1 + l_{n,k}) \varphi_n^{(k-1)}(0) \frac{x^{k-1}}{(k-1)!} \end{aligned} \tag{2.51}$$

and so

$$\begin{aligned} T_n(s;x) &= \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=1}^{\infty} \varphi_n^{(k-1)}(0) \frac{x^{k-1}}{(k-1)!} + \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=1}^{\infty} l_{n,k} \varphi_n^{(k-1)}(0) \frac{x^{k-1}}{(k-1)!} \\ &= x\gamma_n + \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=0}^{\infty} l_{n,k+1} \varphi_n^{(k)}(0) \frac{x^k}{k!}. \end{aligned}$$

Since $0 \le l_{n,k} = O\left(\frac{1}{n}\right)$, there exists a positive number d such that $l_{n,k} \le \frac{d}{n}$ for every $k \in \mathbb{N}_0$. Thus, we obtain

$$T_n(s;x) \le x\gamma_n + \frac{x\gamma_n d}{n}$$

or

$$T_n(s;x) - x \le (\gamma_n - 1)x + \frac{x\gamma_n d}{n}. \tag{2.52}$$

Now, by equation (2.51), one gets

$$\begin{aligned} T_n(s;x) &= \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=0}^{\infty} (1 + l_{n,k+1}) \varphi_n^{(k)}(0) \frac{x^k}{k!} \\ &= \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=0}^{\infty} \varphi_n^{(k)}(0) \frac{x^k}{k!} + \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=0}^{\infty} l_{n,k+1} \varphi_n^{(k)}(0) \frac{x^k}{k!} \\ &= x\gamma_n + \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=0}^{\infty} l_{n,k+1} \varphi_n^{(k)}(0) \frac{x^k}{k!}, \end{aligned}$$

which yields

$$T_n(s;x) - x = (\gamma_n - 1)x + \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=0}^{\infty} l_{n,k+1} \varphi_n^{(k)}(0) \frac{x^k}{k!} .$$

Taking into consideration the facts $\gamma_n \geq 1$, $l_{n,k+1} \geq 0$, and $x \in [0, A]$, it immediately follows that

$$T_n(s; x) - x \geq 0. \tag{2.53}$$

From inequalities (2.52) and (2.53), we get

$$0 \leq T_n(s; x) - x \leq (\gamma_n - 1)x + \frac{x\gamma_n d}{n} \tag{2.54}$$

which gives

$$\lim_{n\to\infty} \|T_n(s; x) - x\|_{C[0,A]} = 0.$$

For $f(s) \equiv s^2$, by using the condition (iv), one has

$$\begin{aligned}
T_n(s^2; x) &= \frac{1}{\varphi_n(x)} \sum_{k=1}^{\infty} \frac{k}{(n+k)^2} \varphi_n^{(k)}(0) \frac{x^k}{(k-1)!} \\
&= \frac{x}{\varphi_n(x)} \sum_{k=1}^{\infty} \frac{k}{(n+k)^2} \gamma_n (n+k)(1 + l_{n,k}) \varphi_n^{(k-1)}(0) \frac{x^{k-1}}{(k-1)!} \\
&= \frac{x^2\gamma_n}{\varphi_n(x)} \sum_{k=2}^{\infty} \frac{1}{n+k} (1 + l_{n,k}) \varphi_n^{(k-1)}(0) \frac{x^{k-2}}{(k-2)!} \\
&\quad + \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=1}^{\infty} \frac{1}{n+k} (1 + l_{n,k}) \varphi_n^{(k-1)}(0) \frac{x^{k-1}}{(k-1)!} \\
&= \frac{x^2\gamma_n^2}{\varphi_n(x)} \sum_{k=2}^{\infty} (1 + l_{n,k})(1 + l_{n,k-1}) \frac{n+k-1}{n+k} \varphi_n^{(k-2)}(0) \frac{x^{k-2}}{(k-2)!} \\
&\quad + \frac{x\gamma_n}{\varphi_n(x)} \sum_{k=1}^{\infty} \frac{1}{n+k} (1 + l_{n,k}) \varphi_n^{(k-1)}(0) \frac{x^{k-1}}{(k-1)!}.
\end{aligned} \tag{2.55}$$

Noting that $1 + l_{n,k} \leq 1 + \frac{d}{n}$, $\frac{n+k-1}{n+k} \leq 1$ and $\frac{1}{n+k} \leq \frac{1}{n}$, we can write

$$\begin{aligned}
T_n(s^2; x) &\leq x^2\gamma_n^2 \left(1 + \frac{d}{n}\right)^2 \frac{1}{\varphi_n(x)} \sum_{k=2}^{\infty} \varphi_n^{(k-2)}(0) \frac{x^{k-2}}{(k-2)!} \\
&\quad + \frac{x\gamma_n}{n} \left(1 + \frac{d}{n}\right) \frac{1}{\varphi_n(x)} \sum_{k=1}^{\infty} \varphi_n^{(k-1)}(0) \frac{x^{k-1}}{(k-1)!}
\end{aligned}$$

and thus

$$T_n(s^2; x) - x^2 \leq x^2(\gamma_n^2 - 1) + \frac{2dx^2\gamma_n^2 + x\gamma_n}{n} + \frac{x^2\gamma_n^2 d^2 + x\gamma_n d}{n^2}. \tag{2.56}$$

On the other hand, the inequality (2.53) implies that

$$T_n(s - x; x) \geq 0. \tag{2.57}$$

With the help of the positivity of T_n and the inequality (2.57), we have

$$T_n(s^2; x) - x^2 = T_n((s - x)^2; x) + 2xT_n(s - x; x) \geq 0. \tag{2.58}$$

If we now use the inequalities (2.56) and (2.58), then we obtain

$$0 \leq T_n(s^2; x) - x^2 \leq x^2(\gamma_n^2 - 1) + \frac{2dx^2\gamma_n^2 + x\gamma_n}{n} + \frac{x^2\gamma_n^2 d^2 + x\gamma_n d}{n^2}. \tag{2.59}$$

This leads to

$$\lim_{n\to\infty} \|T_n(s^2; x) - x^2\|_{C[0,A]} = 0.$$

Thus the proof is completed. □

Now we can give our main result.

Theorem 2.13 *The sequence of linear positive operators defined by (2.49) with conditions (i)-(iv) converge uniformly to the function* $f \in C[0, A]$ *in* $[0, A]$.

Proof. As in Lemma 2.12, it is enough to prove that the conditions

$$M_n^*(f(s); x) \rightrightarrows x^i, \text{ for } f(s) \equiv s^i, \ i = 0, 1, 2$$

are satisfied.

For $f(s) \equiv 1$, by using the Maclaurin series expansion of $\varphi_n(x)$, we have

$$M_n^*(1;x) \;=\; \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{1}{\alpha_{n,k}}\int_{k}^{k+\alpha_{n,k}} d\xi \;\varphi_n^{(k)}(0)\frac{x^k}{k!} \;=\; 1.$$

This implies that

$$\lim_{n\to\infty}\|M_n^*(1;x)-1\|_{C[0,A]} = 0. \tag{2.60}$$

By using the operators T_n defined by (2.50), for $f(s) \equiv s$ we can write

$$\begin{aligned}
M_n^*(s;x)-x &= \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{1}{\alpha_{n,k}}\int_{k}^{k+\alpha_{n,k}}\frac{\xi}{n+k}d\xi\;\varphi_n^{(k)}(0)\frac{x^k}{k!}-x\\
&= \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{1}{2\alpha_{n,k}(n+k)}\left[(k+\alpha_{n,k})^2-k^2\right]\varphi_n^{(k)}(0)\frac{x^k}{k!}-x\\
&= \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{k}{n+k}\varphi_n^{(k)}(0)\frac{x^k}{k!}+\frac{1}{2\varphi_n(x)}\sum_{k=0}^{\infty}\frac{\alpha_{n,k}}{n+k}\varphi_n^{(k)}(0)\frac{x^k}{k!}-x\\
&= T_n(s;x)-x+\frac{1}{2\varphi_n(x)}\sum_{k=0}^{\infty}\frac{\alpha_{n,k}}{n+k}\varphi_n^{(k)}(0)\frac{x^k}{k!}.
\end{aligned} \tag{2.61}$$

Since $T_n(s;x)-x \geq 0$, it follows that

$$M_n^*(s;x)-x \geq 0. \tag{2.62}$$

On the other hand, since

$$\frac{\alpha_{n,k}}{n+k} \leq \frac{1}{n}, \tag{2.63}$$

(2.61) implies that

$$\begin{aligned}
M_n^*(s;x)-x &\leq T_n(s;x)-x+\frac{1}{2n\varphi_n(x)}\sum_{k=0}^{\infty}\varphi_n^{(k)}(0)\frac{x^k}{k!}\\
&= T_n(s;x)-x+\frac{1}{2n}.
\end{aligned} \tag{2.64}$$

Thus, from the inequalities (2.62), (2.64) and by the Lemma 2.12, we find

$$\lim_{n\to\infty}\left\|M_n^*(s;x)-x\right\|_{C[0,A]} = 0. \tag{2.65}$$

Finally, for $f(s) \equiv s^2$ we have

$$\begin{aligned} M_n^*(s^2;x) &= \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{1}{\alpha_{n,k}}\int_k^{k+\alpha_{n,k}}\left(\frac{\xi}{n+k}\right)^2 d\xi\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\ &= \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{1}{\alpha_{n,k}}\frac{(k+\alpha_{n,k})^3-k^3}{3(n+k)^2}\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\ &= \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\left[\frac{k^2}{(n+k)^2}+\frac{k\alpha_{n,k}}{(n+k)^2}+\frac{\alpha_{n,k}^2}{3(n+k)^2}\right]\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\ &= T_n(s^2;x)+\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{k\alpha_{n,k}}{(n+k)^2}\ \varphi_n^{(k)}(0)\frac{x^k}{k!}+\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\frac{\alpha_{n,k}^2}{3(n+k)^2}\ \varphi_n^{(k)}(0)\frac{x^k}{k!}. \end{aligned}$$

Using now the inequalities (2.58) and (2.63), we obtain

$$\begin{aligned} 0 \le M_n^*(s^2;x)-x^2 &\le \left(T_n(s^2;x)-x^2\right)+\frac{1}{n\varphi_n(x)}\sum_{k=0}^{\infty}\frac{k}{n+k}\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\ &\quad +\frac{1}{3n^2}\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\ &= T_n(s^2;x)-x^2+\frac{1}{n}T_n(s;x)+\frac{1}{3n^2}. \end{aligned} \tag{2.66}$$

Hence, from Lemma 2.12, it follows that

$$\lim_{n\to\infty}\left\|M_n^*(s^2;x)-x^2\right\|_{C[0,A]}=0. \tag{2.67}$$

By means of the Korovkin theorem, the statements (2.60), (2.65) and (2.67) give

$$M_n^*(f;x) \rightrightarrows f(x), \quad x\in[0,A]$$

which completes the proof. □

2.2.2 Rate of Convergence of M_n^*

Now, we compute the rate of convergence of the operators $M_n^*(f;x)$ given by (2.49) with the help of the modulus of continuity defined in section 2.1.2.

Theorem 2.14 *Let f be a continuous function on* $[0, A]$. *Then, the sequence of linear positive operators defined by (2.49) under the conditions (i)-(iv), satisfies the inequality*

$$|M_n^*(f;x) - f(x)| \le C\omega\left(f;\frac{1}{\sqrt{n}}\right), \tag{2.68}$$

where C is a positive number.

Proof. Using the linearity of the operators M_n^* and the triangle inequality, we obtain

$$\begin{aligned} |M_n^*(f;x) - f(x)| &= \left| \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} \left[f\left(\frac{\xi}{n+k}\right) - f(x) \right] d\xi \; \varphi_n^{(k)}(0) \frac{x^k}{k!} \right| \\ &\le \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} \left| f\left(\frac{\xi}{n+k}\right) - f(x) \right| d\xi \; \varphi_n^{(k)}(0) \frac{x^k}{k!}. \end{aligned}$$

Then, the inequality (2.29) leads to

$$\begin{aligned} |M_n^*(f;x) - f(x)| &\le \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} \omega(f;\delta) \left[1 + \frac{1}{\delta^2} \left(\frac{\xi}{n+k} - x \right)^2 \right] d\xi \; \varphi_n^{(k)}(0) \frac{x^k}{k!} \\ &\le \omega(f;\delta) \left[\frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} d\xi \; \varphi_n^{(k)}(0) \frac{x^k}{k!} \right. \\ &\qquad \left. + \frac{1}{\delta^2} \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} \left(\frac{\xi}{n+k} - x \right)^2 d\xi \; \varphi_n^{(k)}(0) \frac{x^k}{k!} \right]. \end{aligned}$$

By the fact that $M_n^*(1;x) = 1$, we have

$$|M_n^*(f;x) - f(x)| \le \omega(f;\delta) \left[1 + \frac{1}{\delta^2} M_n^*((s-x)^2;x) \right]. \tag{2.69}$$

Then, with the help of the inequalities (2.64) and (2.66), one gets

$$\begin{aligned} M_n^*((s-x)^2;x) &= [M_n^*(s^2;x) - x^2] - 2x\,[M_n^*(s;x) - x] \\ &\le \left[T_n(s^2;x) - x^2 \right] + \frac{1}{n} T_n(s;x) + \frac{1}{3n^2} + 2x \left[T_n(s;x) - x + \frac{1}{2n} \right]. \end{aligned}$$

Using (2.54) and (2.56), by direct computation, we obtain

$$M_n^*((s-x)^2;x) \;=\; O\left(\frac{1}{n}\right).$$

Thus, for the choose of $\delta = \dfrac{1}{\sqrt{n}}$, from (2.69) we find the desired result. □

2.2.3 An Application to Differential Equations

In this part, we give an application of the linear positive operators (2.49) to functional differential equations by the following theorem.

Theorem 2.15 *Let*

$$g(s) = \frac{as}{b(1-s)} \quad s \in [0, A],\;\; a, b \neq 0.$$

For each $x \in [0, A]$, $f \in C[0, A]$ *and* $\alpha_{n,k} = 1$; $M_n^*(f;x)$ *satisfies the following functional differential equation for* $n = 2, 3, \ldots$:

$$\begin{aligned} x\frac{d}{dx}M_n^*(f;x) \;&=\; \left[-\gamma_n(1+n)(1+l_{n,1})x + \frac{1}{\ln\left(\frac{n-1}{n}\right)} - n\right]M_n^*(f;x) \\ &\quad - \frac{b}{a\ln\left(\frac{n-1}{n}\right)}M_n^*((f,g);x), \end{aligned} \tag{2.70}$$

where

$$M_n^*((f,g);x) = \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\int_k^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi \int_k^{k+1} g\left(\frac{\xi}{n+k}\right)d\xi\;\, \varphi_n^{(k)}(0)\frac{x^k}{k!}.$$

Proof. Let $f \in C[0, A]$. We have shown that $M_n^*(f;x)$ is uniform convergent on $[0, A]$, therefore we can differentiate this series term by term in this interval. For $k = 1$, the condition (iii) turns out to be

$$\varphi_n'(x) = \gamma_n(1+n)(1+l_{n,1})\varphi_n(x).$$

By taking this into consideration, we have

$$
\begin{aligned}
x\frac{d}{dx}M_n^*(f;x) &= x\Bigg[\frac{-\varphi_n'(x)}{\varphi_n^2(x)}\sum_{k=0}^{\infty}\int\limits_{k}^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\
&\quad + \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\int\limits_{k}^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)\frac{kx^{k-1}}{k!}\Bigg] \\
&= \frac{-\gamma_n(1+n)(1+l_{n,1})x}{\varphi_n(x)}\sum_{k=0}^{\infty}\int\limits_{k}^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\
&\quad + \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\int\limits_{k}^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)k\frac{x^k}{k!}. \qquad (2.71)
\end{aligned}
$$

By using the definition of the function g, it is seen that $g\left(\frac{\xi}{n+k}\right) = \frac{a\xi}{b(n+k-\xi)}$.

Then we have

$$
\begin{aligned}
\int_k^{k+1} g\left(\frac{\xi}{n+k}\right)d\xi &= \frac{a}{b}\int_k^{k+1}\left(-1+\frac{n+k}{n+k-\xi}\right)d\xi \\
&= -\frac{a}{b}(n+k)\ln\left(\frac{n-1}{n}\right) - \frac{a}{b}.
\end{aligned}
$$

This gives

$$
k = \frac{-\frac{a}{b} - \int\limits_{k}^{k+1} g\left(\frac{\xi}{n+k}\right)d\xi}{\frac{a}{b}\ln\left(\frac{n-1}{n}\right)} - n.
$$

Substituting this result in the equation (2.71), we obtain

$$
\begin{aligned}
x\frac{d}{dx}M_n^*(f;x) = \frac{-\gamma_n(1+n)(1+l_{n,1})x}{\varphi_n(x)}\sum_{k=0}^{\infty}\int\limits_{k}^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\
+ \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\int\limits_{k}^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)\left(\frac{-\frac{a}{b} - \int\limits_{k}^{k+1} g\left(\frac{\xi}{n+k}\right)d\xi}{\frac{a}{b}\ln\left(\frac{n-1}{n}\right)} - n\right)\frac{x^k}{k!}.
\end{aligned}
$$

This, by the linearity of the operators M_n^*, implies that

$$\begin{aligned} x\frac{d}{dx}M_n^*(f;x) &= \left(-\gamma_n(1+n)(1+l_{n,1})x - \frac{1}{\ln\left(\frac{n-1}{n}\right)} - n\right)\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\int_k^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\ &\quad - \frac{1}{\frac{a}{b}\ln(\frac{n-1}{n})}\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\int_k^{k+1} f\left(\frac{\xi}{n+k}\right)d\xi\int_k^{k+1} g\left(\frac{\xi}{n+k}\right)d\xi\ \varphi_n^{(k)}(0)\frac{x^k}{k!} \\ &= \left[-\gamma_n(1+n)(1+l_{n,1})x - \frac{1}{\ln\left(\frac{n-1}{n}\right)} - n\right]M_n^*(f;x) - \frac{b}{a\ln\left(\frac{n-1}{n}\right)}M_n^*((f,g);x) \end{aligned}$$

which completes the proof. □

CHAPTER 3

THE MOMENTS OF THE MEYER-KÖNIG AND ZELLER OPERATORS

In this chapter, we consider the Meyer-König and Zeller operators M_n (see [6],[8], [16],[18], etc.) defined by

$$M_n(f;x) = \begin{cases} (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k, & \text{if } 0 \le x < 1; \\ f(1), & \text{if } x = 1, \end{cases} \tag{3.1}$$

and give explicit formulas [8] and some estimates for the moments of these operators [18].

3.1 Explicit Formulas for Central Moments of M_n

We begin with the following theorem.

Theorem 3.1 *Let M_n be the positive linear operators defined by (3.1). Then, we have*

$$M_n(1;x) = 1$$

and

$$M_n(s;x) = x.$$

Proof. We now consider the Taylor series expansion of $\frac{1}{(1-x)^\alpha}$:

$$\frac{1}{(1-x)^\alpha} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} x^k.$$

If we take $\alpha = n + 1$, then we see that

$$(1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k = 1. \tag{3.2}$$

Thus, we may conclude that for $f(s) = 1$, $M_n(1; x) = 1$.

For $f(s) = s$, by (3.1) we can write

$$\begin{aligned} M_n(s;x) &= (1-x)^{n+1} \sum_{k=1}^{\infty} \frac{(n+k-1)!}{n!(k-1)!} x^k \\ &= x(1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k. \end{aligned}$$

By using the fact (3.2), we arrive at the desired result.□

We now give the following theorem and lemma [8] which we will use to find an explicit formula for the second moment of the operators $M_n(f; x)$ in terms of hypergeometric series.

Theorem 3.2 *Let* $g(s) = \dfrac{s}{1-s}$, $s \in [0, 1)$. *For each* $n \in \mathbb{N}$, $x \in [0, 1)$ *and* $f \in C[0, 1)$ *the linear positive operators* $M_n(f; x)$ *defined in (3.1) satisfy the differential equation*

$$x(1-x)\frac{d}{dx} M_n(f;x) = -(n+1)x M_n(f;x) + n(1-x) M_n(fg;x). \tag{3.3}$$

Proof. Let $n \in \mathbb{N}$. In Chapter 2, we showed that the operators L_n defined by (2.3), which is a generalization of the operators (3.1) converges uniformly on $[0, 1)$. Thus we can differentiate the series in (3.1) term by term in the interval $[0, 1)$. Hence, we have

$$\begin{aligned} \frac{d}{dx} M_n(f;x) &= -(n+1)(1-x)^n \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k \\ &\quad + (1-x)^{n+1} \sum_{k=1}^{\infty} f\left(\frac{k}{n+k}\right) k \binom{n+k}{k} x^{k-1}. \end{aligned}$$

Then, it follows that

$$\begin{aligned} x(1-x)\frac{d}{dx}M_n(f;x) &= -(n+1)x(1-x)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k \\ &\quad + n(1-x)^{n+2}\sum_{k=1}^{\infty} f\left(\frac{k}{n+k}\right)\frac{k}{n}\binom{n+k}{k}x^k. \end{aligned}$$

By using the definition of the operators M_n and the fact $g\left(\frac{k}{n+k}\right) = \frac{k}{n}$, we can write

$$\begin{aligned} x(1-x)\frac{d}{dx}M_n(f;x) &= -(n+1)xM_n(f;x) + n(1-x)^{n+2}\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)g\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k \\ &= -(n+1)xM_n(f;x) + n(1-x)M_n(fg;x), \end{aligned}$$

which is the desired result. □

By means of this theorem, the following lemma can be proven.

Lemma 3.3 *For each $n \in \mathbb{N}$, $M_n(s^2;x)$ is a solution of the differential equation*

$$x(1-x)y'(x) + (n+x)y(x) = nx^2 + x, \qquad x \in [0,1) \tag{3.4}$$

satisfying the condition y(0)=0.

Proof. Let $n \in \mathbb{N}$ and $x \in [0,1)$. Then, for $f(s) = s^2$, by definition of the operators M_n, it is easily seen that

$$M_n(s^2;0) = 0.$$

We now set $f = s - s^2$ in equation (3.3) to obtain

$$x(1-x)\frac{d}{dx}M_n(s-s^2;x) = -(n+1)xM_n(s-s^2;x) + n(1-x)M_n((s-s^2)g;x). \tag{3.5}$$

By the linearity properties of the operators M_n, we can rewrite the last term in (3.5) as

$$\begin{aligned} M_n((s-s^2)g;x) &= M_n(sg;x) - M_n(s^2g;x) \\ &= (1-x)^{n+1}\sum_{k=0}^{\infty}\frac{k}{n}\frac{k}{n+k}\binom{n+k}{k}x^k - (1-x)^{n+1}\sum_{k=0}^{\infty}\left(\frac{k}{n+k}\right)^2\frac{k}{n}\binom{n+k}{k}x^k \\ &= (1-x)^{n+1}\sum_{k=0}^{\infty}\frac{k}{n}\left[\frac{k}{n+k}-\left(\frac{k}{n+k}\right)^2\right]\binom{n+k}{k}x^k \\ &= (1-x)^{n+1}\sum_{k=0}^{\infty}\left(\frac{k}{n+k}\right)^2\binom{n+k}{k}x^k \\ &= M_n(s^2;x). \end{aligned}$$

Substitution of this result into (3.5) gives

$$x(1-x)\frac{d}{dx}M_n(s-s^2;x) = -(n+1)xM_n(s-s^2;x) + n(1-x)M_n(s^2;x).$$

Again by using the linearity properties of M_n and Theorem 3.1, we find

$$x(1-x)\frac{d}{dx}M_n(s^2;x) + (n+x)M_n(s^2;x) = nx^2 + x. \tag{3.6}$$

Thus the proof is completed. □

Now using the equation (3.6), we can give an explicit expression for $M_n(s^2;x)$ by means of the hypergeometric series. The hypergeometric series [9] is defined by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}, \tag{3.7}$$

where a, b and $c \neq 0, -1, -2, \ldots$ are constants and

$$(\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1) \tag{3.8}$$

is the Pochhammer symbol. It is easily seen that, by using the well known properties of the Gamma function

$$\Gamma(z+1) = z!, \quad z \geq -1 \tag{3.9}$$

we have

$$(\alpha)_k = \frac{(\alpha+k-1)!}{(\alpha-1)!} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad k \in \mathbb{N},\ \alpha \in \mathbb{R}. \tag{3.10}$$

The hypergeometric series defined by (3.7) is convergent for $|x| < 1$, if $c - a - b > 0$ so is for $x = 1$. Indeed, using the ratio test we have

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(a+k)(b+k)}{(c+k)(k+1)} x = x,$$

which implies that $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = |x|$. Thus, the series (3.7) is convergent for $|x| < 1$. For the case $x = 1$, we have ${}_2F_1(a, b; c; 1) = \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}$. By means of the Raabe's test, we get

$$\begin{aligned}
\lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right) &= \lim_{k\to\infty} k\left(\frac{(k+1)(c+k)}{(a+k)(b+k)} - 1\right) \\
&= \lim_{k\to\infty} \frac{k(c-ab)}{(a+k)(b+k)} + \lim_{k\to\infty} \frac{k^2(c+1-a-b)}{k^2+k(a+b)+ab} \\
&= c + 1 - a - b.
\end{aligned}$$

Therefore, the series ${}_2F_1(a, b; c; 1)$ is convergent where $c+1-a-b > 1$ or $c-a-b > 0$. Furthermore, as it is given in [9], the following expression holds

$$\begin{aligned}
{}_2F_1(a, b; c; x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b-c+1, 1-x) \\
&+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} {}_2F_1(c-a, c-b, c-a-b+1, 1-x).
\end{aligned}$$

For $x = 1$, this gives

$${}_2F_1(a, b; c; 1) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{1}{k!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (c - a - b > 0). \tag{3.11}$$

We now give an explicit expression for $M_n(s^2; x)$ by the following theorem presented in [8].

Theorem 3.4 *For $n \in \mathbb{N}$ and for each $x \in [0,1)$,*

$$M_n(s^2;x) = x^2 + \frac{x(1-x)^2}{n+1}{}_2F_1(1,2;n+2;x) \tag{3.12}$$

holds and for $n \geq 2$ it also holds at $x = 1$.

Proof. Let $n \in \mathbb{N}$ and $x \in [0,1)$. Substituting

$$y(x) = x^2 + x(1-x)^2 z(x), \tag{3.13}$$

where z is to be determined, into the equation (3.4) we have

$$\begin{aligned} x(1-x)y'(x) + (n+x)y(x) &= x(1-x)\Big(2x + [(1-x)^2 - 2x(1-x)]z(x) + x(1-x)^2 z'(x)\Big) \\ &\quad + (n+x)[x^2 + x(1-x)^2 z(x)] \\ &= nx^2 + x \end{aligned}$$

or

$$x(1-x)z'(x) + (n+1-2x)z(x) = 1\,, \quad x \in [0,1). \tag{3.14}$$

Now we seek a particular solution z_p of the equation (3.14) of the form

$$z(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{3.15}$$

where the coefficients a_k to be determined. Substitution of (3.15) into the equation (3.14) gives

$$x(1-x)\sum_{k=1}^{\infty} k a_k x^{k-1} + (n+1-2x)\sum_{k=0}^{\infty} a_k x^k = 1$$

or

$$\sum_{k=1}^{\infty} k a_k x^k - \sum_{k=2}^{\infty}(k-1)a_{k-1}x^k + (n+1)\sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 2a_{k-1}x^k = 1.$$

This follows that

$$(n+1)a_0 + [(n+2)a_1 - 2a_0]x + \sum_{k=2}^{\infty}[(n+k+1)a_k - (k+1)a_{k-1}]x^k = 1.$$

To hold this equation, we must have

$$(n+1)a_0 = 1,\ \ (n+2)a_1 - 2a_0 = 0$$

and

$$(n+k+1)a_k - (k+1)a_{k-1} = 0, \quad k \geq 2.$$

From these one gets

$$a_0 = \frac{1}{n+1},$$
$$a_1 = \frac{2}{n+2}a_0$$

and the recurrence relation

$$a_k = \frac{k+1}{n+k+1}a_{k-1}, \qquad k \geq 2.$$

By means of this recurrence relation, we obtain the following formula for the coefficients a_k:

$$a_k = \frac{2.3\ldots k(k+1)}{(n+1)(n+2)\ldots(n+k+1)}, \qquad k \geq 2.$$

If we substitute these coefficients into (3.15), then we obtain

$$z(x) \quad = \quad \frac{1}{n+1} + \frac{2x}{(n+1)(n+2)} + \sum_{k=2}^{\infty} \frac{2.3\ldots k(k+1)}{(n+1)(n+2)\ldots(n+k+1)}x^k.$$

Since $(2)_0 = 1$ and $(2)_1 = 2$, we can rewrite this solution in the form

$$z(x) \quad = \quad \frac{1}{n+1}\sum_{k=0}^{\infty}\frac{(2)_k}{(n+2)_k}x^k. \tag{3.16}$$

By means of the hypergeometric functions defined by (3.7), the solution (3.16) can be expressed as

$$z(x) = \frac{1}{n+1}{}_2F_1(1,2;n+2;x).$$

Now, we find the general solution z_h of the corresponding homogeneous differential equation

$$x(1-x)z'(x)+(n+1-2x)z(x)=0.$$

It is easily seen that this equation is separable in x and z. Thus the general solution of (3.14) is given by

$$z(x)=\frac{1}{n+1}{}_2F_1(1,2;n+2;x)+Cx^{-n-1}(1-x)^{n-1},\quad x\in[0,1).$$

Finally, setting this result into equation (3.13), we find

$$y(x)=x^2+\frac{x(1-x)^2}{n+1}{}_2F_1(1,2;n+2;x)+Cx^{-n}(1-x)^{n+1}\,,\ C\in\mathbb{R},\ x\in[0,1).$$

To satisfy the condition $y(0)=0$, we must have $C=0$. So we have

$$y(x)=x^2+\frac{x(1-x)^2}{n+1}{}_2F_1(1,2;n+2;x). \tag{3.17}$$

In the Lemma 3.3, we have shown that $M_n(s^2;x)$ is a solution of the differential equation (3.4), then by (3.17), we can write

$$M_n(s^2;x)=x^2+\frac{x(1-x)^2}{n+1}{}_2F_1(1,2;n+2;x) \tag{3.18}$$

which is the desired result (3.12).

For $x=1$, with the help of (3.11) and the property (3.9) we obtain

$${}_2F_1(1,2;n+2;1)=\frac{\Gamma(n+2)\Gamma(n-1)}{\Gamma(n+1)\Gamma(n)}=\frac{(n+1)!(n-2)!}{n!(n-1)!}=\frac{n+1}{n-1}.$$

Thus, for $x=1$ the left side of (3.12) is equal to 1. On the other hand, by the definition of the operators M_n given by (3.1), we have

$$M_n(s^2;1)=1.$$

Thus, we may conclude that (3.12) is also valid for $x=1$. Therefore, the proof is completed. □

3.2 Some Estimates for $M_n((s-x)^{2p};x)$

Let us first state the following lemma.

Lemma 3.5 *For $n \geq 2$ and $x \in [0,1]$ there holds for any $m \in \mathbb{N}$,*

$$_2F_1(1,2;n+2;x) \leq \sum_{k=0}^{m-1} \frac{(2)_k}{(n+2)_k} x^k + \frac{(m+1)!}{(n-1)(n+2)_{m-1}} x^m. \tag{3.19}$$

Proof. By (3.8), one gets

$$_2F_1(1,2;n+2;x) = \sum_{k=0}^{\infty} \frac{(1)_k(2)_k}{(n+2)_k} \frac{x^k}{k!}.$$

Using the fact $(1)_k = 1.2\ldots k = k!$, we can write

$$_2F_1(1,2;n+2;x) = \sum_{k=0}^{\infty} \frac{(2)_k}{(n+2)_k} x^k. \tag{3.20}$$

If we let

$$\Phi_m(x) = \sum_{k=m}^{\infty} \frac{(2)_k}{(n+2)_k} x^k, \tag{3.21}$$

then (3.20) can be expressed as follows:

$$_2F_1(1,2;n+2;x) = \sum_{k=0}^{m-1} \frac{(2)_k}{(n+2)_k} x^k + \Phi_m(x). \tag{3.22}$$

Now consider (3.21). By means of the definition of the Pochhammer symbol (3.8), it is clear that

$$(2)_k = (k+1)!, \ (2)_k = (2)_m(m+2)_{k-m}$$

and

$$(n+2)_k = (n+2)_m(n+m+2)_{k-m}.$$

Hence, (3.21) can be written as

$$\begin{aligned}\Phi_m(x) &= \sum_{k=m}^{\infty} \frac{(2)_m (m+2)_{k-m}}{(n+2)_m (n+2+m)_{k-m}} x^m x^{k-m} \\ &= \frac{(2)_m}{(n+2)_m} x^m \sum_{k=0}^{\infty} \frac{(m+2)_k}{(n+m+2)_k} x^k \\ &= \frac{(m+1)!}{(n+2)_m} x^m \sum_{k=0}^{\infty} \frac{(m+2)_k}{(n+m+2)_k} x^k.\end{aligned}$$

Now, with the help of the hypergeometric series (3.7) and (3.11), we can write the above equation as follows:

$$\begin{aligned}\Phi_m(x) &= \frac{(m+1)!}{(n+2)_m} x^m \,{}_2F_1(1, m+2; n+m+2; x) \\ &\leq \frac{(m+1)!}{(n+2)_m} x^m \,{}_2F_1(1, m+2; n+m+2; 1) \\ &= \frac{(m+1)!}{(n+2)_m} x^m \frac{\Gamma(n+m+2)\Gamma(n-1)}{\Gamma(n+m+1)\Gamma(n)} \\ &\leq \frac{(m+1)!}{(n-1)(n+2)_{m-1}} x^m.\end{aligned}$$

Thus, by (3.22) we arrive at the required result. □

By using Lemma 3.5, we can prove the following theorem.

Theorem 3.6 *Let $n \in \mathbb{N}$ and $x \in [0,1]$. Then, for the linear positive operators M_n given by (3.1), we have*

$$\frac{x(1-x)^2}{n+1}\left(1+\frac{2x}{n+2}\right) \leq M_n(s^2; x) - x^2 \leq \frac{x(1-x)^2}{n+1}\left(1+\frac{2x}{n-1}\right), \tag{3.23}$$

for $n \geq 2$.

Proof. For $m = 1$, the inequality (3.19) turns out to be

$${}_2F_1(1, 2; n+2; x) \leq \frac{(2)_0}{(n+2)_0} + \frac{2!}{(n-1)(n+2)_0} x = 1 + \frac{2x}{n-1}$$

where $n \geq 2$ and $x \in [0,1]$. Thus, by (3.12) we may write

$$M_n(s^2;x) \leq x^2 + \frac{x(1-x)^2}{n+1}\left(1 + \frac{2x}{n-1}\right)$$

and this implies that

$$M_n(s^2;x) - x^2 \leq \frac{x(1-x)^2}{n+1}\left(1 + \frac{2x}{n-1}\right), \qquad n \geq 2, \quad x \in [0,1]. \tag{3.24}$$

On the other hand, by (3.20) and the fact that $\sum_{k=2}^{\infty} \frac{(2)_k}{(n+2)_k} x^k \geq 0$, we can write

$$_2F_1(1,2;n+2;x) = \frac{(2)_0}{(n+2)_0} + \frac{(2)_1}{(n+2)_1}x + \sum_{k=2}^{\infty} \frac{(2)_k}{(n+2)_k} x^k \geq 1 + \frac{2x}{n+2}.$$

Setting this into (3.12), we can conclude that

$$M_n(s^2;x) - x^2 \geq \frac{x(1-x)^2}{n+1}\left(1 + \frac{2x}{n+2}\right). \tag{3.25}$$

If we combine (3.24) and (3.25), then we obtain the desired result. □

After this result, we are now ready to give the estimates for $M_n((s-x)^{2p};x)$ referring to [18].

Theorem 3.7 *Let* $\varphi(x) = \sqrt{x}(1-x)$, $A_{n,2p}(x) = M_n((s-x)^{2p};x)$, $p \in \mathbb{N}$. *For* $n > 2p$ *and each* $x \in [0,1)$, *we have the estimates;*

$$A_{n,2p}(x) \leq C \begin{cases} \dfrac{\varphi^{2p}(x)}{n^p}, & \text{for } x \geq \dfrac{1}{n}; \\ \dfrac{\varphi^2(x)(1-x)^{2p-2}}{n^{2p-1}}, & \text{for } x < \dfrac{1}{n}. \end{cases} \tag{3.26}$$

Here, C is a constant which is independent of n and x and not necessarily the same at each occurrence.

Proof. We prove this theorem by induction on p. For $p = 1$, we must show that

$$A_{n,2}(x) = M_n((s-x)^2;x) \leq C\frac{\varphi^2(x)}{n}, \quad x \in [0,1). \tag{3.27}$$

With the help of the linearity of the operators M_n and by using (3.2), (3.3) and (3.23), we can write

$$\begin{aligned} A_{n,2}(x) = M_n((s-x)^2;x) &= M_n(s^2;x) - 2xM_n(s;x) + x^2M_n(1;x) \\ &= M_n(s^2;x) - x^2 \\ &\leq \frac{\varphi^2(x)}{n+1}\left(1+\frac{2x}{n-1}\right). \end{aligned} \tag{3.28}$$

Since $n > 2$ and $x \in [0,1)$, it is clearly seen that $\left(1+\frac{2x}{n-1}\right) < 3$. Thus, from (3.28) it follows that

$$A_{n,2}(x) \leq 3\frac{\varphi^2(x)}{n+1} < 3\frac{\varphi^2(x)}{n}$$

which shows that (3.26) is valid for $p = 1$ with $C = 3$.

We now assume that (3.26) is true for $p = r > 1$ and $n > 2r$. That is, we have

$$A_{n,2r}(x) \leq C \begin{cases} \dfrac{\varphi^{2r}(x)}{n^r}, & \text{for } x \geq \dfrac{1}{n}; \\ \dfrac{\varphi^2(x)(1-x)^{2r-2}}{n^{2r-1}}, & \text{for } x < \dfrac{1}{n} \end{cases} \tag{3.29}$$

and show that (3.26) holds for $p = r+1$. Hence, for $n > 2(r+1)$, by the definition of the operators M_n, letting $m_{n,k}(x) = \binom{n+k}{k}x^k(1-x)^{n+1}$, we can write

$$\begin{aligned} A_{n,2(r+1)}(x) &= M_n((s-x)^{2(r+1)};x) \\ &= \sum_{k=0}^{\infty}\left(\frac{k}{n+k}-x\right)^{2(r+1)} m_{n,k}(x) \\ &= x^{2(r+1)}(1-x)^{n+1} + \sum_{k=1}^{\infty}\frac{k}{n+k}\left(\frac{k}{n+k}-x\right)^{2r+1} m_{n,k}(x) \\ &\quad - \sum_{k=1}^{\infty} x\left(\frac{k}{n+k}-x\right)^{2r+1} m_{n,k}(x) \\ &= x^{2(r+1)}(1-x)^{n+1} + \frac{1}{n+1}\left(\frac{1}{n+1}-x\right)^{2r+1} m_{n,1}(x) \\ &\quad + \sum_{k=2}^{\infty}\frac{k}{n+k}\left(\frac{k}{n+k}-x\right)^{2r+1} m_{n,k}(x) - \sum_{k=1}^{\infty} x\left(\frac{k}{n+k}-x\right)^{2r+1} m_{n,k}(x). \end{aligned}$$

Since

$$\frac{1}{n+1} m_{n,1}(x) = x(1-x)^{n+1}$$

and

$$\frac{k}{n+k} m_{n,k}(x) = x\, m_{n,k-1}(x),$$

one gets

$$\begin{aligned} A_{n,2(r+1)}(x) &= x^{2(r+1)}(1-x)^{n+1} + x\left(\frac{1}{n+1} - x\right)^{2r+1} (1-x)^{n+1} \\ &\quad + x \sum_{k=1}^{\infty} \left[\left(\frac{k+1}{n+k+1} - x\right)^{2r+1} - \left(\frac{k}{n+k} - x\right)^{2r+1}\right] m_{n,k}(x) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Firstly, we compute I_1.

$$\begin{aligned} I_1 = x^{2(r+1)}(1-x)^{n+1} &= x^{r+1}(1-x)^{2(r+1)} x^{r+1}(1-x)^{n-2r-1} \\ &= \varphi^{2(r+1)}(x) x^{r+1}(1-x)^{n-2r-1}. \end{aligned} \tag{3.30}$$

Since

$$\sup_{x\in[0,1)} x^{r+1}(1-x)^{n-2r-1} = \left(\frac{r+1}{n-r}\right)^{r+1} \left(\frac{n-2r-1}{n-r}\right)^{n-2r-1},$$

taking into consideration the fact $n > 2(r+1)$, by (3.30), we can write

$$\begin{aligned} I_1 &\le \varphi^{2(r+1)}(x) \left(\frac{r+1}{n-r}\right)^{r+1} \left(\frac{n-2r-1}{n-r}\right)^{n-2r-1} \\ &\le \varphi^{2(r+1)}(x)(r+1)^{r+1} \left(\frac{1}{n-r}\right)^{r+1} \\ &= \varphi^{2(r+1)}(x)(r+1)^{r+1} \left(\frac{1}{n} + \frac{r}{n(n-r)}\right)^{r+1} \\ &\le \varphi^{2(r+1)}(x)(r+1)^{r+1} \left(\frac{1}{n} + \frac{r}{n(r+2)}\right)^{r+1}. \end{aligned}$$

Since $\dfrac{r}{r+2} < 1$ for each $r \geq 1$, it follows that

$$\begin{aligned} I_1 &\leq \varphi^{2(r+1)}(x)(r+1)^{r+1}\left(\frac{1}{n}+\frac{1}{n}\right)^{r+1} \\ &= \varphi^{2(r+1)}(x)\left(\frac{2(r+1)}{n}\right)^{r+1} \\ &\leq C\begin{cases} \dfrac{\varphi^{2(r+1)}(x)}{n^{r+1}}, & \text{for } x \geq \dfrac{1}{n}; \\ \dfrac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}, & \text{for } x < \dfrac{1}{n}, \end{cases} \end{aligned} \tag{3.31}$$

where $C = (2(r+1))^{r+1}$.

Indeed, for $x < \dfrac{1}{n}$ we have

$$\frac{\varphi^{2(r+1)}(x)}{n^{r+1}} = \frac{\varphi^2(x)(1-x)^{2r}x^r}{n^{r+1}} \leq \frac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}.$$

For I_2 there are two cases; either $x \geq \dfrac{1}{n}$ or $0 < x < \dfrac{1}{n}$.

If $x \geq \dfrac{1}{n}$, it is easily seen that $\left(\dfrac{1}{n+1} - x\right) < 0$. Therefore $I_2 \leq 0$ for $x \geq \dfrac{1}{n}$, and so we omit this case.

If $0 < x < \dfrac{1}{n}$, then we have $\left|\dfrac{1}{n+1} - x\right| < \dfrac{1}{n}$. Thus, one has

$$I_2 \leq |I_2| \leq x\frac{1}{n^{2r+1}}(1-x)^{n+1} \leq \frac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}. \tag{3.32}$$

For I_3, firstly we investigate the expression

$$\left|\left(\frac{k+1}{n+k+1} - x\right)^{2r+1} - \left(\frac{k}{n+k} - x\right)^{2r+1}\right|.$$

Let $f(s) = (s-x)^{2r+1}$. Then, this expression takes the form

$$\left|f\left(\frac{k+1}{n+k+1}\right) - f\left(\frac{k}{n+k}\right)\right|.$$

Now, applying the mean value theorem on the interval (a,b) with $a = \dfrac{k}{n+k}$ and $b = \dfrac{k+1}{n+k+1}$, we get

$$\left|\left(\frac{k+1}{n+k+1}-x\right)^{2r+1}-\left(\frac{k}{n+k}-x\right)^{2r+1}\right| = \frac{n(2r+1)}{(n+k)(n+k+1)}(c-x)^{2r}, \quad a<c<b.$$

Since $a-x<c-x<b-x$, choosing $H=\max\{|a-x|,|b-x|\}$ we obtain $|c-x|<H$ and this implies that $(c-x)^{2r}<H^{2r}$. Therefore

$$(c-x)^{2r} < (a-x)^{2r}+(b-x)^{2r} = \left(\frac{k+1}{n+k+1}-x\right)^{2r}+\left(\frac{k}{n+k}-x\right)^{2r}, \tag{3.33}$$

and so

$$\begin{aligned}\left|\left(\frac{k+1}{n+k+1}-x\right)^{2r+1}-\left(\frac{k}{n+k}-x\right)^{2r+1}\right| \;\le\; & \frac{n(2r+1)}{(n+k)(n+k+1)} \\ & \times\left[\left(\frac{k+1}{n+k+1}-x\right)^{2r}+\left(\frac{k}{n+k}-x\right)^{2r}\right].\end{aligned}$$

Thus, from this it follows that

$$\begin{aligned} I_3 &= x\sum_{k=1}^{\infty}\left[\left(\frac{k+1}{n+k+1}-x\right)^{2r+1}-\left(\frac{k}{n+k}-x\right)^{2r+1}\right]m_{n,k}(x) \\ &\le x(2r+1)\frac{n}{(n+k)(n+k+1)}\left[\left(\frac{k+1}{n+k+1}-x\right)^{2r}+\left(\frac{k}{n+k}-x\right)^{2r}\right]m_{n,k}(x).\end{aligned}$$

Now, by using the fact

$$\begin{aligned}\frac{n}{(n+k)(n+k+1)}m_{n,k}(x) &= \frac{(1-x)^2}{n-1}\frac{n+k-1}{n+k+1}m_{n-2,k}(x) \\ &\le \frac{2(1-x)^2}{n}m_{n-2,k}(x),\end{aligned} \tag{3.34}$$

we can write

$$|I_3| \le (4r+2)x(1-x)^2\sum_{k=1}^{\infty}\frac{1}{n}\left[\left(\frac{k+1}{n+k+1}-x\right)^{2r}+\left(\frac{k}{n+k}-x\right)^{2r}\right]m_{n-2,k}(x).$$

Let

$$I_{31} = \sum_{k=1}^{\infty} \left(\frac{k+1}{n+k+1} - x \right)^{2r} m_{n-2,k}(x)$$

and

$$I_{32} = \sum_{k=1}^{\infty} \left(\frac{k}{n+k} - x \right)^{2r} m_{n-2,k}(x),$$

and let us recall the following inequality;

$$(a+b)^n \leq 2^{n-1}(a^n + b^n), \quad \text{where } a, b \geq 0, \; n = 1, 2, 3 \ldots \tag{3.35}$$

For $n > 2(r+1)$, the inequalities (3.29) and (3.35) imply that

$$\begin{aligned}
I_{31} &= \sum_{k=1}^{\infty} \left(\frac{k+1}{n+k+1} - x \right)^{2r} m_{n-2,k}(x) \\
&= \sum_{k=1}^{\infty} \left[\left(\frac{k+1}{n+k+1} - \frac{k}{n+k-2} \right) + \left(\frac{k}{n+k-2} - x \right) \right]^{2r} m_{n-2,k}(x) \\
&\leq 2^{2r-1} \sum_{k=1}^{\infty} \left[\left(\frac{k+1}{n+k+1} - \frac{k}{n+k-2} \right)^{2r} + \left(\frac{k}{n+k-2} - x \right)^{2r} \right] m_{n-2,k}(x) \\
&= 2^{2r-1} \sum_{k=1}^{\infty} \left(\frac{n-2k-2}{(n+k+1)(n+k-2)} \right)^{2r} m_{n-2,k}(x) \\
&\quad + 2^{2r-1} \sum_{k=1}^{\infty} \left(\frac{k}{n+k-2} - x \right)^{2r} m_{n-2,k}(x) \\
&\leq 2^{2r-1} \sum_{k=1}^{\infty} \frac{1}{(n+k+1)^{2r}} m_{n-2,k}(x) + 2^{2r-1} A_{n-2,2r}(x) \\
&\leq 2^{2r-1} \sum_{k=1}^{\infty} \frac{1}{(n+k+1)^{2r}} m_{n-2,k}(x) + C \begin{cases} \dfrac{\varphi^{2r}(x)}{(n-2)^r}, & \text{for } x \geq \dfrac{1}{n-2}; \\ \dfrac{\varphi^2(x)(1-x)^{2r-2}}{(n-2)^{2r-1}}, & \text{for } x < \dfrac{1}{n-2}. \end{cases} \\
&\leq 2^{2r-1} \sum_{k=1}^{\infty} \frac{1}{(n+k+1)^{2r}} m_{n-2,k}(x) + C \begin{cases} \dfrac{\varphi^{2r}(x)}{n^r}, & \text{for } x \geq \dfrac{1}{n-2}; \\ \dfrac{\varphi^2(x)(1-x)^{2r-2}}{n^{2r-1}}, & \text{for } x < \dfrac{1}{n-2}. \end{cases}
\end{aligned} \tag{3.36}$$

By some computations, we have

$$\begin{aligned}
\frac{1}{(n+k+1)^{2r}} m_{n-2,k}(x) &= \frac{1}{(n+k+1)^{2r}} \binom{n+k-2}{k} x^k (1-x)^{n-1} \\
&= \frac{(1-x)^{2r}}{(n+k+1)^{2r}} \frac{(n+k-2)!(n-2r-2)!k!}{(n-2)!k!(n+k-2r-2)!} m_{n-2r-2,k}(x) \\
&\leq (1-x)^{2r} \frac{1}{(n-2)(n-3)\dots(n-2r-1)} m_{n-2r-2,k}(x) \\
&\leq C \frac{(1-x)^{2r}}{n^{2r}} m_{n-2r-2,k}(x). \qquad (3.37)
\end{aligned}$$

We now need to show that $\dfrac{\varphi^{2r}(x)}{n^r} \sim \dfrac{\varphi^2(x)(1-x)^{2r-2}}{n^{2r-1}}$ for $\dfrac{1}{n} \leq x \leq \dfrac{1}{n-2}$. This means that we must prove that the inequality

$$C^{-1} \frac{x^r(1-x)^{2r}}{n^r} \leq \frac{x(1-x)^{2r}}{n^{2r-1}} \leq C \frac{x^r(1-x)^{2r}}{n^r} \qquad (3.38)$$

holds for a positive constant C.

Since $\dfrac{1}{n} \leq x \leq \dfrac{1}{n-2}$ one has $\dfrac{1}{nx} \leq \dfrac{1}{(n-2)x}$ and $\dfrac{1}{x^{r-1}} \leq n^{r-1}$.
Moreover, for $n > 2(r+1)$ we have $\dfrac{2}{n-2} \leq \dfrac{1}{r}$ which implies that

$$\left(\frac{n}{n-2}\right)^{r-1} = \left(1 + \frac{2}{n-2}\right)^{r-1} \leq \left(1 + \frac{1}{r}\right)^{r-1} \leq 2^{r-1}.$$

Using these inequalities, we may conclude that

$$\frac{x(1-x)^{2r}}{n^{2r-1}} \leq \frac{x(1-x)^{2r}}{n^r} \left(\frac{n}{n-2}\right)^{r-1} \leq \frac{x(1-x)^{2r}}{n^r} 2^{r-1}. \qquad (3.39)$$

On the other hand, the inequality $x \leq \dfrac{1}{n-2}$ gives that $\left(\dfrac{1}{x}\right)^{r-1} \geq (n-2)^{r-1}$. Then, since $n > 2(r+1)$ and $r \geq 1$, we have

$$\left(\frac{n-2}{n}\right)^{r-1} = \left(1 - \frac{2}{n}\right)^{r-1} \geq \left(1 - \frac{1}{r+1}\right)^{r-1} \geq \frac{1}{2^{r-1}},$$

and

$$\left(\frac{n-2}{n}\right) \leq \frac{1}{(nx)^{r-1}}.$$

Thus, one has

$$\frac{x(1-x)^{2r}}{n^{2r-1}} \geq \frac{x^r(1-x)^{2r}}{n^r(nx)^{r-1}} \geq \frac{x^r(1-x)^{2r}}{n^r}\left(\frac{n-2}{n}\right)^{r-1} \geq \frac{x^r(1-x)^{2r}}{n^r}\frac{1}{2^{r-1}}. \tag{3.40}$$

Combining inequalities (3.39) and (3.40), we obtain that (3.38) holds for $C = 2^{r-1}$. Substitution of (3.37) and (3.38) into (3.36) implies that

$$I_{31} \leq C\frac{(1-x)^{2r}}{n^{2r}}\sum_{k=1}^{\infty} m_{n-2r-2,k}(x) + C\begin{cases} \dfrac{\varphi^{2r}(x)}{n^r}, & \text{for } x \geq \dfrac{1}{n}; \\ \dfrac{\varphi^2(x)(1-x)^{2r-2}}{n^{2r-1}}, & \text{for } x < \dfrac{1}{n}. \end{cases}$$

Since the equality $\sum_{k=0}^{\infty} m_{n,k}(x) = 1$ is valid for all $n \in \mathbb{N}$ and $m_{n-2r-2,0}(x) = (1-x)^{n-2r-1}$, we may conclude that

$$\begin{aligned}\sum_{k=1}^{\infty} m_{n-2r-2,k}(x) &= \sum_{k=0}^{\infty} m_{n-2r-2,k}(x) - m_{n-2r-2,0}(x) \\ &\leq 1-(1-x)^{n-2r-1} \\ &\leq 1.\end{aligned}$$

On the other hand, it is easy to see that for $x \geq \frac{1}{n}$ the inequality

$$\frac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}} = \frac{\varphi^{2r+2}(x)}{n^{r+1}}\frac{1}{(nx)^r} \leq \frac{\varphi^{2r+2}(x)}{n^{r+1}}$$

and for $x < \frac{1}{n}$ the inequality

$$\frac{\varphi^4(x)(1-x)^{2r-2}}{n^{2r}} = \frac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}nx \leq \frac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}$$

hold. Hence, with the help of these inequalities one gets

$$\frac{\varphi^2(x)}{n}I_{31} \leq C\frac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}} + C\frac{\varphi^2(x)}{n}\begin{cases} \dfrac{\varphi^{2r}(x)}{n^r}, & \text{for } x \geq \dfrac{1}{n}; \\ \dfrac{\varphi^2(x)(1-x)^{2r-2}}{n^{2r-1}}, & \text{for } x < \dfrac{1}{n}, \end{cases}$$

which obviously implies that

$$\frac{\varphi^2(x)}{n}I_{31} \leq C\begin{cases}\dfrac{\varphi^{2(r+1)}(x)}{n^{r+1}}, & \text{for } x \geq \dfrac{1}{n};\\ \dfrac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}, & \text{for } x < \dfrac{1}{n}.\end{cases} \tag{3.41}$$

We now consider I_{32}. Using the inequality (3.35) for $n > 2(r+1)$, we can write

$$\begin{aligned} I_{32} &= \sum_{k=1}^{\infty}\left[\left(\frac{k}{n+k}-\frac{k}{n+k-2}\right)+\left(\frac{k}{n+k-2}-x\right)\right]^{2r} m_{n-2,k}(x) \\ &\leq 2^{2r-1}\sum_{k=1}^{\infty}\left[\left(\frac{k}{n+k}-\frac{k}{n+k-2}\right)^{2r}+\left(\frac{k}{n+k-2}-x\right)^{2r}\right] m_{n-2,k}(x) \\ &= 2^{2r-1}2^{2r}\sum_{k=1}^{\infty}\frac{k^{2r}}{(n+k)^{2r}(n+k-2)^{2r}} m_{n-2,k}(x) + 2^{2r-1}A_{n-2,2r}(x) \\ &\leq 2^{4r-1}\sum_{k=1}^{\infty}\frac{1}{(n+k)^{2r}} m_{n-2,k}(x) + 2^{2r-1}A_{n-2,2r}(x). \end{aligned}$$

In a similar way that of (3.37), we find that

$$\frac{1}{(n+k)^{2r}} m_{n-2,k}(x) \leq C\frac{(1-x)^{2r}}{n^{2r}} m_{n-2r-2,k}(x).$$

Hence, we may write

$$I_{32} \leq C\sum_{k=1}^{\infty}\frac{(1-x)^{2r}}{n^{2r}} m_{n-2r-2,k}(x) + CA_{n-2,2r}(x).$$

Multiplying both sides of this inequality by $\dfrac{\varphi^2(x)}{n}$ and taking (3.29) into account, we have

$$\begin{aligned} \frac{\varphi^2(x)}{n}I_{32} &\leq C\frac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}} + C\frac{\varphi^2(x)}{n}\begin{cases}\dfrac{\varphi^{2r}(x)}{n^{r}}, & \text{for } x \geq \dfrac{1}{n};\\ \dfrac{\varphi^2(x)(1-x)^{2r-2}}{n^{2r-1}}, & \text{for } x < \dfrac{1}{n}\end{cases} \\ &\leq C\begin{cases}\dfrac{\varphi^{2(r+1)}(x)}{n^{r+1}}, & \text{for } x \geq \dfrac{1}{n};\\ \dfrac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}, & \text{for } x < \dfrac{1}{n}.\end{cases} \end{aligned} \tag{3.42}$$

For some constant C, the inequalities (3.41) and (3.42) imply that

$$|I_3| \le C \begin{cases} \dfrac{\varphi^{2(r+1)}(x)}{n^{r+1}}, & \text{for } x \ge \dfrac{1}{n}; \\ \dfrac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}, & \text{for } x < \dfrac{1}{n}. \end{cases} \tag{3.43}$$

Thus, for $n > 2(r+1)$, by the inequalities (3.31), (3.32) and (3.43), we conclude that

$$A_{n,2(r+1)}(x) \le I_1 + I_2 + |I_3| \le C \begin{cases} \dfrac{\varphi^{2(r+1)}(x)}{n^{r+1}}, & \text{for } x \ge \dfrac{1}{n}; \\ \dfrac{\varphi^2(x)(1-x)^{2r}}{n^{2r+1}}, & \text{for } x < \dfrac{1}{n}. \end{cases}$$

Therefore, by induction on p the theorem has been proved. □

CHAPTER 4

ON SOME PROPERTIES OF THE MEYER-KÖNIG AND ZELLER OPERATORS

The aim of this chapter is to give some basic properties of the Meyer-König and Zeller operators

$$M_n(f;x) = \begin{cases} (1-x)^{n+1} \displaystyle\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} x^k, & \text{if } 0 \le x < 1; \\ f(1), & \text{if } x = 1. \end{cases} \tag{4.1}$$

Firstly, we show that these operators preserve the Lipschitz constants [16] and then we give the monotonicity properties [6] of these operators. Finally, we prove that these linear positive operators satisfy an initial value problem.

4.1 Preservation of Lipschitz Constants

Let us recall the definition of concave function which is given in [6].

Definition 4.1 *A function f is said to be concave on an interval $[a,b]$ if for any points $t_j,\ j = 1,2,\ldots n$ in $[a,b]$ and arbitrary constants $\alpha_i,\ i = 1,2,\ldots n,$ f satisfies the inequality*

$$\alpha_1 f(t_1) + \alpha_2 f(t_2) + \ldots \alpha_n f(t_n) \le f(\alpha_1 t_1 + \alpha_2 t_2 + \ldots \alpha_n t_n) \tag{4.2}$$

where $0 < \alpha_i < 1$ *and* $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1.$

If the reverse of the inequality (4.2) holds, then f is said to be convex.

Theorem 4.2 *Let M_n be the linear positive operators defined by (4.1). Then, for all $f \in Lip_M(\alpha)$ and $n \in \mathbb{N}$, we have*

$$M_n(f;x) \in Lip_M(\alpha). \tag{4.3}$$

Proof. Let $f \in \mathrm{Lip}_M(\alpha)$ and $n \in \mathbb{N}$. Now, consider the linear positive operators M_n defined by (4.1) and assume that $0 \le x_1 < x_2 < 1$. Since $M_n(f;1) = f(1)$, we will consider only the case $x_2 < 1$. Then, with the help of the binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

we have

$$\begin{aligned}
M_n(f;x_2) &= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j} x_2^j (1-x_2)^{n+1} \\
&= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j} (1-x_2)^{n+1}\left(\frac{x_2 - x_1 + x_1 - x_1 x_2}{1-x_1}\right)^j \\
&= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j} \frac{(1-x_2)^{n+1}}{(1-x_1)^j} \sum_{k=0}^{j} \binom{j}{k} {x_1}^k (1-x_2)^k (x_2-x_1)^{j-k} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{j} f\left(\frac{j}{n+j}\right) \frac{(n+j)!}{n!k!(j-k)!} \frac{x_1^k (x_2-x_1)^{j-k}(1-x_2)^{n+k+1}}{(1-x_1)^j} \\
&= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} f\left(\frac{j}{n+j}\right) \frac{(n+j)!}{n!k!(j-k)!} \frac{x_1^k (x_2-x_1)^{j-k}(1-x_2)^{n+k+1}}{(1-x_1)^j}.
\end{aligned}$$

By the change of index $j - k = \ell$, we can write this as follows:

$$M_n(f;x_2) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} f\left(\frac{k+\ell}{n+k+\ell}\right) \frac{(n+k+\ell)!}{n!k!\ell!} \frac{x_1^k (x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}}. \tag{4.4}$$

On the other hand, by the Taylor expansion

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n} x^k$$

we have

$$\begin{aligned} M_n(f;x_1) &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} x_1^k (1-x_1)^{n+1} \\ &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} x_1^k \frac{(1-x_2)^{n+k+1}}{(1-x_1)^k} \frac{1}{\left(1-\frac{x_2-x_1}{1-x_1}\right)^{n+k+1}} \\ &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} \frac{x_1^k(1-x_2)^{n+k+1}}{(1-x_1)^k} \sum_{\ell=0}^{\infty}\binom{n+k+\ell}{\ell}\left(\frac{x_2-x_1}{1-x_1}\right)^{\ell} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} f\left(\frac{k}{n+k}\right)\frac{(n+k+\ell)!}{n!k!\ell!}\frac{x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}}. \qquad (4.5) \end{aligned}$$

By means of the facts $M_n(1;x)=1,\ \ M_n(s;x)=x$ and the equalities (4.4), (4.5), it is seen that

$$\sum_{k,\ell=0}^{\infty}\frac{(n+k+\ell)!}{n!k!\ell!}\frac{x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = 1, \qquad (4.6)$$

$$\sum_{k,\ell=0}^{\infty}\frac{k+\ell}{n+k+\ell}\frac{(n+k+\ell)!}{n!k!\ell!}\frac{x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = x_2, \qquad (4.7)$$

$$\sum_{k,\ell=0}^{\infty}\frac{k}{n+k}\frac{(n+k+\ell)!}{n!k!\ell!}\frac{x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = x_1. \qquad (4.8)$$

Since $f \in \mathrm{Lip}_M(\alpha)$, it follows that

$$\begin{aligned} |M_n(f;x_2)-M_n(f;x_1)| &\le \sum_{k,\ell=0}^{\infty}\frac{(n+k+\ell)!}{n!k!\ell!}\frac{x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \\ &\quad \times\left|f\left(\frac{k+\ell}{n+k+\ell}\right)-f\left(\frac{k}{n+k}\right)\right| \\ &\le M\sum_{k,\ell=0}^{\infty}\frac{(n+k+\ell)!}{n!k!\ell!}\frac{x_1^k(x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \\ &\quad \times\left|\frac{k+\ell}{n+k+\ell}-\frac{k}{n+k}\right|^{\alpha}. \end{aligned}$$

Since the function

$$g(t)=t^{\alpha},\ \ 0<\alpha\le 1,\ \ t\in[0,\infty)$$

is concave, by using (4.6), we can deduce that

$$|M_n(f;x_2) - M_n(f;x_1)| \leq M\Bigg[\sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \frac{x_1^k (x_2 - x_1)^{\ell} (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \times \left(\frac{k+\ell}{n+k+\ell} - \frac{k}{n+k}\right)\Bigg]^{\alpha}.$$

Using now (4.7) and (4.8), we obtain

$$|M_n(f;x_2) - M_n(f;x_1)| \leq M(x_2 - x_1)^{\alpha},$$

which shows that $M_n(f;x) \in \mathrm{Lip}_M(\alpha)$. Thus the proof is completed. □

We now give the monotonicity properties of the linear positive operators M_n defined by (4.1).

Theorem 4.3 *Let $n \in \mathbb{N}$. If f is convex, then $M_n(f;x)$ is decreasing in n.*

Proof. Let f be a convex function. Then, by the definition of the linear positive operators M_n, one gets

$$\begin{aligned} M_n(f;x) - M_{n+1}(f;x) &= (1-x)^{n+1}\Bigg\{\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} \\ &\quad - (1-x)\sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k+1}{k}\Bigg\}x^k \\ &= (1-x)^{n+1}\Bigg\{\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k \\ &\quad - \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k+1}{k}x^k \\ &\quad + \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k+1}{k}x^{k+1}\Bigg\} \\ &= (1-x)^{n+1}\Bigg\{f(0) + \sum_{k=1}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k \\ &\quad - f(0) - \sum_{k=1}^{\infty} f\left(\frac{k}{n+k+1}\right)\binom{n+k+1}{k}x^k \\ &\quad + \sum_{k=1}^{\infty} f\left(\frac{k-1}{n+k}\right)\binom{n+k}{k-1}x^k\Bigg\}. \end{aligned}$$

Since,

$$\binom{n+k}{k} = \frac{n+1}{n+k+1}\binom{n+k+1}{k} \text{ and } \binom{n+k}{k-1} = \frac{k}{n+k+1}\binom{n+k+1}{k}$$

we have

$$\begin{aligned} M_n(f;x) - M_{n+1}(f;x) = \ & (1-x)^{n+1} \sum_{k=1}^{\infty} \left\{ \frac{n+1}{n+k+1} f\left(\frac{k}{n+k}\right) - f\left(\frac{k}{n+k+1}\right) \right. \\ & \left. + \frac{k}{n+k+1} f\left(\frac{k-1}{n+k}\right) \right\} \binom{n+k+1}{k} x^k. \end{aligned}$$

Let

$$A_{n,k} := \frac{n+1}{n+k+1} f\left(\frac{k}{n+k}\right) - f\left(\frac{k}{n+k+1}\right) + \frac{k}{n+k+1} f\left(\frac{k-1}{n+k}\right).$$

Then, one has

$$M_n(f;x) - M_{n+1}(f;x) = (1-x)^{n+1} \sum_{k=1}^{\infty} A_{n,k} \binom{n+k+1}{k} x^k.$$

If we choose

$$\alpha_1 = \frac{n+1}{n+k+1}, \quad \alpha_2 = \frac{k}{n+k+1}, \quad x_1 = \frac{k}{n+k} \quad \text{and} \quad x_2 = \frac{k-1}{n+k},$$

then, it is clear that

$$\alpha_1 + \alpha_2 = 1$$

and

$$\alpha_1 x_1 + \alpha_2 x_2 = \frac{(n+1)k}{(n+k+1)(n+k)} + \frac{k(k-1)}{(n+k+1)(n+k)} = \frac{k}{n+k+1}.$$

Hence, we can write

$$A_{n,k} = \alpha_1 f(x_1) + \alpha_2 f(x_2) - f(\alpha_1 x_1 + \alpha_2 x_2).$$

Using the fact that f is convex, we find

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) - f(\alpha_1 x_1 + \alpha_2 x_2) \geq 0.$$

This gives $A_{n,k} \geq 0$, and so we have

$$M_n(f;x) - M_{n+1}(f;x) \geq 0.$$

Thus, the proof is completed. □

4.2 An Application to Initial Value Problems

In this section, we show that the linear positive operators M_n given by (4.1) is a solution of an initial value problem for a first order ordinary differential equation. For this purpose, let us first prove the following lemmas.

Lemma 4.4 *If* $|f''| \leq \nu$, $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha + \beta = 1$, *then we have*

$$\left|\alpha f(x) + \beta f(y) - f(\alpha x + \beta y)\right| \leq \frac{\nu}{8}(x-y)^2.$$

Proof. Let $t = \alpha x + \beta y$, then $\alpha = \dfrac{t-y}{x-y}$ and $\beta = 1 - \alpha = \dfrac{x-t}{x-y}$. By the Taylor formula $f(x) = f(a) + (x-a)f'(a) + \dfrac{(x-a)^2}{2} f''(a)$, we may write

$$\begin{aligned}
\left|\alpha f(x) + \beta f(y) - f(\alpha x + \beta y)\right| &= \left|\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + \beta y)\right| \\
&= \left|\frac{1}{x-y}\left[(t-y) f(x) + (x-t) f(y) - (x-y) f(t)\right]\right| \\
&= \left|\frac{1}{x-y}\left[(t-y)\left\{f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a)\right\}\right.\right. \\
&\quad + (x-t)\left\{f(a) + (y-a)f'(a) + \frac{(y-a)^2}{2} f''(a)\right\} \\
&\quad \left.\left. - (x-y)\left\{f(a) + (t-a)f'(a) + \frac{(t-a)^2}{2} f''(a)\right\}\right]\right|.
\end{aligned}$$

Using the assumption $|f''| \le \nu$, by simple computation we find

$$
\begin{aligned}
\left|\alpha f(x) + \beta f(y) - f(\alpha x + \beta y)\right| &= \left|\frac{1}{2}\left[t(x+y) - t^2 - xy\right]\right| |f''(a)| \\
&\le \frac{\nu}{2}\left|t(x+y) - t^2 - xy\right| \\
&= \frac{\nu}{2}\left|\left[\alpha x + (1-\alpha)y\right](x+y) - \left[\alpha x + (1-\alpha)y\right]^2 - xy\right| \\
&= \frac{\nu}{2}\left|\alpha - \alpha^2\right|(x-y)^2.
\end{aligned}
$$

Now, let $g(\alpha) = \alpha - \alpha^2$. Since $g(\alpha) \le \frac{1}{4}$, it follows that

$$\left|\alpha f(x) + \beta f(y) - f(\alpha x + \beta y)\right| \le \frac{\nu}{8}(x-y)^2.$$

So, the proof is completed. □

Lemma 4.5 *If $|f''| \le \nu$, then we have*

(i) $\left|M_n(f;x) - M_{n+1}(f;x)\right| \le \dfrac{\nu}{8n^2}(1-x)$

(ii) $\left|M_n(f;x) - f(x)\right| \le \dfrac{\nu}{3n}(1-x).$

Proof. By the Theorem 4.3, we have

$$
\begin{aligned}
M_n(f;x) - M_{n+1}(f;x) = (1-x)^{n+1} \sum_{k=1}^{\infty} \binom{n+k+1}{k} \\
\times \left[\alpha_1 f(x_1) + \alpha_2 f(x_2) - f(\alpha_1 x_1 + \alpha_2 x_2)\right] x^k,
\end{aligned}
$$

where

$$\alpha_1 = \frac{n+1}{n+k+1}, \quad \alpha_2 = \frac{k}{n+k+1}; \quad x_1 = \frac{k}{n+k} \quad \text{and} \quad x_2 = \frac{k-1}{n+k}.$$

Taking into consideration the Lemma 4.4, and using the expressions

$$x_1 - x_2 = \frac{1}{n+k} \quad \text{and} \quad \frac{n+k+1}{n(n+1)(n+k)} \le \frac{1}{n^2},$$

one has

$$\begin{aligned}
\left|M_n(f;x) - M_{n+1}(f;x)\right| &\leq (1-x)^{n+1}\sum_{k=1}^{\infty}\binom{n+k+1}{k}\frac{\nu}{8}(x_1-x_2)^2x^k \\
&= \frac{\nu}{8}(1-x)^{n+1}\sum_{k=1}^{\infty}\frac{n+k+1}{n(n+1)(n+k)}\binom{n+k-1}{k}x^k \\
&\leq \frac{\nu}{8n^2}(1-x)^{n+1}\sum_{k=1}^{\infty}\binom{n+k-1}{k}x^k.
\end{aligned}$$

Now, we use the expansion

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k \tag{4.9}$$

in order to obtain

$$\begin{aligned}
\left|M_n(f;x) - M_{n+1}(f;x)\right| &\leq \frac{\nu}{8n^2}(1-x)^{n+1}\left[\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k - 1\right] \\
&= \frac{\nu}{8n^2}(1-x)^{n+1}\left[\frac{1}{(1-x)^n} - 1\right] \\
&= \frac{\nu}{8n^2}\left[(1-x) - (1-x)^{n+1}\right] \\
&\leq \frac{\nu}{8n^2}(1-x).
\end{aligned}$$

Thus, the desired result (i) is obtained. The statement (ii) can be proven similarly . □

We shall now give the main theorem.

Theorem 4.6 *The functions $y_n(x)$ defined recursively by*

$$y_0(x) = y_0, \qquad y_n(x) = y_0 + \int_0^x M_n\{f(t, y_{n-1}(t)); s\}\,ds, \ \ n = 1, 2, \ldots, \tag{4.10}$$

converge uniformly to a solution of the initial value problem

$$y' = f(x, y), \quad y(0) = y_0 \quad \textit{for } x \in [0, 1)$$

provided that f and its first two partial derivatives are bounded in the strip $0 \leq x < 1$,

$-\infty < y < \infty$ *and that* f *satisfies the Lipschitz condition*

$$|f(x, y_1) - f(x, y_2)| \leq \lambda |y_1 - y_2|, \quad \textit{with } \lambda < 1. \tag{4.11}$$

Proof. Let $y_n(x)$ be defined as in (4.10). We consider the series

$$y_0 + \sum_{k=0}^{\infty} \Big[y_{k+1}(x) - y_k(x) \Big].$$

This series converges uniformly on $[0, 1)$ if and only if its partial sum converges uniformly on $[0, 1)$. Let $\epsilon_n(x) = y_{n+1}(x) - y_n(x)$. Then, by means of (4.10) one gets

$$\begin{aligned}
|\epsilon_n(x)| &= \left| y_0 + \int_0^x M_{n+1}\Big\{ f(t, y_n(t)); s \Big\} ds - y_0 - \int_0^x M_n \Big\{ f(t, y_{n-1}(t)); s \Big\} ds \right| \\
&\leq \int_0^x \left| M_{n+1}\Big\{ f(t, y_n(t)); s \Big\} - M_n \Big\{ f(t, y_{n-1}(t)); s \Big\} \right| ds \\
&\leq \int_0^x \left| M_{n+1}\Big\{ f(t, y_n(t)); s \Big\} - M_n \Big\{ f(t, y_n(t)); s \Big\} \right| ds \\
&+ \int_0^x \left| M_n \Big\{ f(t, y_n(t)); s \Big\} - M_n \Big\{ f(t, y_{n-1}(t)); s \Big\} \right| ds \\
&=: I_1 + I_2.
\end{aligned}$$

By the Lemma 4.5, we can write

$$\begin{aligned}
I_1 &= \int_0^x \left| M_{n+1}\Big\{ f(t, y_n(t)); s \Big\} - M_n \Big\{ f(t, y_n(t)); s \Big\} \right| ds \\
&\leq \frac{\nu}{8n^2} \int_0^x (1 - s) ds \\
&= \frac{\nu}{8n^2} \left(x - \frac{x^2}{2} \right).
\end{aligned}$$

Since $x - \frac{x^2}{2} \leq \frac{1}{2}$, $x \in [0, 1)$, from this it follows that

$$I_1 \leq \frac{\nu}{16n^2}, \tag{4.12}$$

where

$$\nu = \sup_{0\le x<1} \left| \frac{d^2}{dx^2} f(x, y_n(x)) \right|.$$

We now prove that our hypotheses guarantee that $\nu < \infty$. Since f and its first two partial derivatives are bounded in the strip $x \in [0, 1)$, $-\infty < y < \infty$, we let

$$K = \sup_{\substack{0\le x<1 \\ -\infty<y<\infty}} \{|f|, |f_1|, |f_2|, |f_{11}, |f_{12}|, |f_{22}|\},$$

where f_1, f_2 denote the first partial derivatives of f with respect to the first and second variable of f respectively, and similarly f_{11}, f_{12}, f_{22} denote the second partial derivatives. On the other hand, since $M_n(1; x) = 1$, for a function g we have

$$\begin{aligned} \left|M_n(g; x)\right| &\le \sup_{0\le s<1} \left|g(s)\right| \left|(1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k \right| \\ &\le \sup_{0\le s<1} \left|g(s)\right|. \end{aligned} \tag{4.13}$$

Thus, by the definition of $y_n(x)$, one has

$$\left|y_n'(x)\right| = \left|M_n\{f(t, y_{n-1}(t)); x\}\right| \le \sup_{0\le t<1} \left|f(t, y_{n-1}(t))\right| = K$$

or

$$|y_n'(x)| \le K.$$

Now, let $F(x) = f(x, y_{n-1}(x))$. Then, one can write

$$\begin{aligned} y_n''(x) &= \frac{d}{dx} \left(M_n\{f(t, y_{n-1}(t)); x\}\right) \\ &= \frac{d}{dx} \left\{(1-x)^{n+1} \sum_{k=0}^{\infty} F\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k \right\} \\ &= (1-x)^n \Bigg[-(n+1) \sum_{k=0}^{\infty} F\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k \\ &\quad + \sum_{k=1}^{\infty} F\left(\frac{k}{n+k}\right) \binom{n+k}{k} k x^{k-1} - \sum_{k=1}^{\infty} F\left(\frac{k}{n+k}\right) \binom{n+k}{k} k x^k \Bigg]. \end{aligned}$$

Using the relation $\binom{n+k}{k} = \frac{n+k}{k}\binom{n+k-1}{k-1}$, we find

$$\begin{aligned}
y_n''(x) &= (1-x)^n\Bigg[-(n+1)\sum_{k=0}^{\infty} F\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k \\
&\quad + \sum_{k=1}^{\infty} F\left(\frac{k}{n+k}\right)(n+k)\binom{n+k-1}{k-1}x^{k-1} - \sum_{k=1}^{\infty} F\left(\frac{k}{n+k}\right)\binom{n+k}{k}kx^k\Bigg] \\
&= (1-x)^n\left[-(n+k+1)\sum_{k=0}^{\infty} F\left(\frac{k}{n+k}\right) + \sum_{k=0}^{\infty}(n+k+1)F\left(\frac{k+1}{n+k+1}\right)\right]\binom{n+k}{k}x^k \\
&= (1-x)^n\sum_{k=0}^{\infty}(n+k+1)\left[F\left(\frac{k+1}{n+k+1}\right) - F\left(\frac{k}{n+k}\right)\right]\binom{n+k}{k}x^k.
\end{aligned}$$

By using the mean value theorem on the interval $\left(\frac{k}{n+k}, \frac{k+1}{n+k+1}\right)$ for the function F, we obtain

$$\begin{aligned}
\left|F\left(\frac{k+1}{n+k+1}\right) - F\left(\frac{k}{n+k}\right)\right| &= \left|F'(\xi)\right|\left|\frac{k+1}{n+k+1} - \frac{k}{n+k}\right| \\
&= |F'(\xi)|\frac{n}{(n+k)(n+k+1)}, \quad \xi \in \left(\frac{k}{n+k}, \frac{k+1}{n+k+1}\right),
\end{aligned}$$

and so

$$y_n''(x) \le \nu(1-x)^n\sum_{k=0}^{\infty}|F'(\xi)|\,\frac{n}{n+k}\binom{n+k}{k}x^k. \tag{4.14}$$

Then, differentiation of F with respect to x gives that

$$F'(x) = \frac{d}{dx}\left[f(x, y_{n-1}(x))\right] = f_1(x, y_{n-1}(x)) + f_2(x, y_{n-1}(x))y_{n-1}'(x)$$

which implies

$$\left|F'(x)\right| \le |f_1| + |f_2||y_{n-1}'| \le K + K^2.$$

Using (4.9) and this inequality in (4.14), we find

$$|y_n''(x)| \le K + K^2.$$

On the other hand, since

$$\frac{d^2}{dx^2}\left[f(x, y_n(x))\right] = f_{11} + (2f_{12} + f_{22}y_n')y_n' + f_2 y_n'',$$

taking supremum of both sides of this over $x \in [0, 1)$, we find

$$\nu \leq K + (2K + K^2)K + K(K + K^2) = C.$$

Setting this into (4.12), we may conclude that $I_1 \leq \dfrac{C}{n^2}$.

Now, we turn to I_2. Since the operators M_n are linear and the function f satisfies the Lipschitz condition (4.11), using (4.13), for $0 \leq x < 1$, we can write

$$\begin{aligned}
I_2 &= \int_0^x \left| M_n\left\{ f(t, y_n(t)); s\right\} - M_n\left\{ f(t, y_{n-1}(t)); s\right\}\right| ds \\
&\leq \int_0^x M_n\left\{ \left| f(t, y_n(t)) - f(t, y_{n-1}(t))\right|; s\right\} ds \\
&\leq \lambda \int_0^x M_n\left\{ \left| y_n(t) - y_{n-1}(t)\right|; s\right\} ds \\
&= \lambda \int_0^x M_n\left\{ \left| \epsilon_{n-1}(t)\right|; s\right\} ds \\
&\leq \sup_{0\leq t<1} \left| \epsilon_{n-1}(t)\right| \int_0^x ds \\
&\leq \lambda \sup_{0\leq t<1} \left| \epsilon_{n-1}(t)\right|.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sup_{0\leq t<1} \left|\epsilon_n(t)\right| &\leq \sup_{0\leq t<1} \left| y_0 + \int_0^x M_{n+1}\{ f(t, y_n(t)); s\} ds \right| + \sup_{0\leq t<1} \left| y_0 + \int_0^x M_n\{ f(t, y_{n-1}(t)); s\} ds \right| \\
&\leq 2|y_0| + \sup_{0\leq t<1} \int_0^x \left| M_{n+1}\{ f(t, y_n(t)); s\}\right| ds + \sup_{0\leq t<1} \int_0^x \left| M_n\{ f(t, y_{n-1}(t)); s\}\right| ds.
\end{aligned}$$

Using again the inequality (4.13), it implies that

$$\begin{aligned}\sup_{0\le t<1}\left|\epsilon_n(t)\right| &\le 2|y_0| + \sup_{0\le t<1}|f(t,y_n(t))|\int_0^x ds + \sup_{0\le t<1}|f(t,y_{n-1}(t))|\int_0^x ds \\ &= 2(|y_0| + K).\end{aligned}$$

Therefore, we find

$$\left|\epsilon_n(t)\right| \le I_1 + I_2 \le \frac{C}{n^2} + \lambda \sup_{0\le t<1}\left|\epsilon_{n-1}(t)\right|. \tag{4.15}$$

By using the Weierstrass-M test, it is easily seen that the series $\sum_{n=1}^{\infty}\left|\epsilon_n(t)\right|$ converges uniformly. In this case, we may conclude that its partial sum converges uniformly. That is, we have

$$\lim_{n\to\infty} y_n(x) = y(x).$$

From this, it follows that

$$\lim_{n\to\infty} y_n'(x) = \lim_{n\to\infty} M_n\{f(t,y_{n-1}(t));x\} = y'(x).$$

Finally, by using the fact that f satisfies the Lipschitz condition and taking into consideration Lemma 4.5, we have

$$\begin{aligned}\left|M_n\{f(t,y_n(t));x\} - f(x,y(x))\right| &\le \left|M_n\{f(t,y_n(t));x\} - M_n\{f(t,y(t));x\}\right| \\ &\quad + \left|M_n\{f(t,y(t));x\} - f(x,y(x))\right| \\ &\le \lambda M_n\left\{\left|y_n(t) - y_{n-1}(t)\right|;x\right\} \\ &\quad + \left|M_n\{f(t,y(t));x\} - f(x,y(x))\right| \\ &\le \lambda \sup_{0\le t<1}\left|y_n(t) - y(t)\right| + \frac{\nu}{3n}(1-x)\end{aligned}$$

or

$$0 \le \lim_{n\to\infty} \left| M_n\{f(t, y_n(t)); x\} - f(x, y(x)) \right| \le \lim_{n\to\infty} \left(\lambda \sup_{0\le t<1} \left| y_n(t) - y(t) \right| + \frac{\nu}{3n}(1 - x) \right). \tag{4.16}$$

Since $y_n(t) \rightrightarrows y(t)$, the right hand side of the above equation is equal to zero and this leads to

$$M_n\{f(t, y_n(t)); x\} \rightrightarrows f(x, y(x)), \quad x \in [0, 1).$$

Thus, we complete the proof. □

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