BAER AND QUASI-BAER MODULES

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ABSTRACT

BAER AND QUASI-BAER MODULES

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This study contains the endomorphism rings of retractable modules, Baer, quasi-Baer modules and rings. A module M_R is said to be retractable if $Hom_R(M, N) \neq 0$ for each nonzero submodule N of M. It is shown that if M_R is nonsingular and retractable, then $End_R(M) = S$ is a right CS ring if and only if M is CS module. A module M_R is called (quasi-) Baer if the right annihilator of a (two-sided) left ideal of $S = End_R(M)$ is a direct summand of M. After these definitions, it is shown that a direct summand of a (quasi-) Baer module is also a (quasi-) Baer module and a finitely generated Z-module M is a Baer module if and only if M is semisimple or torsion free. Beside these it is shown that direct sums of (quasi-) Baer modules are not (quasi-) Baer module. Furthermore, it is shown every free (projective) module over a (quasi-) Baer ring is always a (quasi-) Baer module. The relation between CS-modules and FI-extending modules are exhibited and it is shown that a module MR is (quasi-) Baer and (FI-) K-cononsingular if and only if M_R is (FI-) extending and (FI-) K-nonsingular. It is also shown that if R is semisimple and artinian if and only if every (right) R-module is Baer. Among other results, the endomorphism ring of a (quasi-) Baer module is a (quasi-) Baer ring, while the converse is not true in general.

Key words: retractable modules, CS-modules, (FI-) extending modules, (FI-) K-nonsingular modules, injective modules, fully invariant modules, endomorphism rings, annihilator.

ÖZET

BAER VE QUASI-BAER MODULLER

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Bu çalışma retractable modüllerin endomorphizma halkalarını, Baer, quasi-Baer modülleri ve halkaları içermektedir. M'nin her sıfırdan farklı alt modülü için $Hom_R(M, N) \neq 0$ ise bu M modülüne retractable module denir. Gösterildi ki, M_R nonsingular ve retractable olsun, M_R CS modüldür ancak ve ancak endomorfizma halkası CS halkadır ($End_R(M) = S$). M'nin endomorfizma halkasının sol idealinin sağ sıfırlayıcısı M'nin direk toplamıysa M'ye Baer modül denir. Bu tanımlardan sonra (quasi-) Baer modülün direk toplamının (quasi-) Baer modül olduğu gösterildi. Bunların yanında (quasi-) Baer modülün dik toplamlarının (quasi-) Baer modül olmadığı gösterildi. (Quasi-) Baer halkasındaki her serbest modülün her zaman (quasi-) Baer modül olduğu gösterildi. CS-modüller ve FI-extending modüller arasındaki ilişki gösterildi. Ayrıca kanıtlandı ki, M_R (quasi-) Baer ve (FI-) K-cononsingulardir ancak ve ancak M_R (FI-) extending ve (FI-) K-nonsingulardir. Ve her (sağ) R-modulü Baerdir ancak ve ancak R semisimple ve artiniandır. Bu sonuçların yanında, (quasi-) Baer modulün endomorfizma halkasının (quasi-) Baer halka olduğu ama tersinin genellikle doğru olmadığı gösterildi.

Anahtar Kelimeler: retractable modüller, CS-modüller, (FI-) extending modüller, (FI-) K- nonsingular modüller, injektif modüller, fully invariant moduller, endomorphizma halkası, annihilatör.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Some Basic Concepts in Modules and Rings

Definition 1.1 *Let M be a module. If there does not exist a properly descending (ascending) infinite chain $M_1 \geq M_2 \geq ...$ ($M_1 \leq M_2 \leq ...$) of submodules of M then M satisfies the descending (ascending) chain condition DCC (ACC). A module M is artinian (noetherian) if it satisfies DCC (ACC).*
A ring R is called right artinian if R_R is artinian (noetherian) module. A ring R is called left artinian if ${}_RR$ is artinian (noetherian) module. If R is both left and right artinian (noetherian) then it is called artinian (noetherian) ring.

Proposition 1.2 *Let N be a submodule of M. Then M is artinian if and only if N and $\frac{M}{N}$ are both artinian.*

Corollary 1.3 *Any finite direct sum of artinian modules is artinian.*

Corollary 1.4 *If R is right artinian ring then all finitely generated R modules are artinian.*

Definition 1.5 *A right R-module M is called free if it has a basis, $\{m_i \mid i \in I\}$, $m_i \in M$ such that every element of M can be written uniquely in the form;*

$$m = \sum_{i \in I} m_i r_i$$

where $r_i \in R$ and all but a finite number of r_i are 0.

Proposition 1.6 *Let M be a right R-module.*

(i) *A right R module M is free if and only if it is isomorphic to a direct sum of copies of R_R.*

(ii) *Every module M is homomorphic image of a free module.*

Definition 1.7 *A right R-module M is said to be finitely generated if there exist elements $m_1, m_2, ...m_n \in M$ such that $M = \sum_{j=1}^{n} m_j$. In this case, we say that $\{m_1, m_2, ..., m_n\}$ is a set of generators of M.*

Definition 1.8 *A right R-module M is said to be cyclic if there is an element $m_0 \in M$ such that every $m \in M$ is of the form $m = m_o r$, where $r \in R$. Also m_0 is called the generator of M and we write $M =< m_o >$.*

Definition 1.9 *Let M be an R-module and N be a submodule of M. N is called an essential submodule of M if $K \cap N \neq 0$ for all nonzero submodules K of M, denoted by $N \leq_e M$.*

Proposition 1.10 *Let M be an R module. Then;*

(i) *$N \leq_e M$ if and only if $N \cap mR \neq 0$ for all nonzero $m \in M$.*

(ii) *Let $K \leq N \leq M$. Then $K \leq_e M$if and only if $K \leq_e N$ and $K \leq_e M$.*

(iii) *Let $N \leq_e M$ and $K \leq M$. Then $N \cap K \leq_e K$.*

(iv) *Let $N_i \leq_e K_i$, $1 \leq i \leq t, t \geq 1$. Then; $(N_1 \cap N_2 \cap ...N_t) \leq_e (K_1 \cap K_2 \cap ...K_t)$.*

(v) *Let $K \leq N \leq M$. If $(\frac{N}{K}) \leq_e (\frac{M}{K})$ then $N \leq_e M$.*

(vi) *Let $N \leq_e M$ and $m \in M$ then $(N : m) = \{r \in R \mid mr \in N\} \leq_e R_R$.*

(vii) *For any nonempty index set I, let $N_i \leq_e M_i (i \in I)$. Then;*

$$\bigoplus_{i \in I} N_i \leq_e \bigoplus_{i \in I} M_i$$

.

Definition 1.11 *Let M be a right R-module and $K, N \leq M$. K is called complement of N in M if K is maximal with respect to the property $K \cap N = 0$. If a submodule K of M is complement submodule in M, then it is denoted by $K \leq_c M$.*

Lemma 1.12 *Let M be a right R-module and $K, N \leq M$. If $K \cap N = 0$,there exists a complement L of N such that $K \leq_e L$ and $(L \oplus N) \leq_e M$.*

Definition 1.13 *Let M be an R-module. For any nonzero $x \in M$ define $ann_R(x) = \{r \in R \mid rx = 0\}$ that is a left ideal of R and is called the annihilator ideal of x. Also it follows that $Rx \cong \frac{R}{ann_R(x)}$.*

Definition 1.14 *Let M be an R-module. $Z(M) = \{x \in M \mid ann_R(x) \leq_e R\}$ is called singular submodule of M. If $Z(M) = M$ then M is called singular module. And if $Z(M) = 0$ then M is called nonsingular module.*

Definition 1.15 *An R-module M is said to be torsion module if $ann(x) \neq 0$ for all $x \in M$.*

Definition 1.16 *An R-module M is called torsion-free module if $ann(x) = 0$ for all $x \in M$.*

Example 1.17 $Z_2(M) = \{m \in M \mid m + Z(M) \in Z(\frac{M}{Z(M)})\}$ *is a submodule of M and it is the largest singular submodule of M. Also $Z(M) \leq_e Z_2(M)$. In fact, let $m \in Z_2(M)$. Then $m + Z(M) \in Z(\frac{M}{Z(M)})$. This implies that there exists an essential ideal I in R such that $mI \leq Z(M)$. Hence, $Z(M) \leq_e Z_2(M)$.*

Lemma 1.18 *Let M be a nonsingular right R-module and let N be a submodule of M. Then;*

(i) $N \leq_e M$ *if and only if* $Z(\frac{M}{N}) = (\frac{M}{N})$.

(ii) $Z_2(M) \leq_c M$.

Definition 1.19 *Let M be a right R-module and $N \leq M$. K is called essential closed(closure) of N in M such that $N \leq_e K \leq_c M$.*

Proposition 1.20 *Let M be a right R-module and $N \leq K \leq M$. Then;*

(i) $N \leq_c M$ *if and only if the essential closed(closure) of N in M is itself.*

(ii) $N \leq_c K \leq_c M$ *then* $N \leq_c M$ *and if* $N \leq_e M$ *then* $N \leq_c K$.

(iii) *If L is the complement of N in M and U is the complement of L in M with $N \leq U$, then $N \leq_e U$.*

(iv) *L is essential closed of N in M if and only if L is maximal submodule with respect to the property $N \leq_e L$ if and only if L is the minimal submodule of the complement submodules which contain N in M.*

Definition 1.21 *Let M be a right R-module. The submodule*
$soc(M) = \bigcap\{N \leq M \mid N \text{ is essential submodule }\} = \sum\{N \leq M \mid N \text{ is simple submodule}\}$
is called socle of M.

Theorem 1.22 *Let M be a right R-module. The followings are equivalent.*

(i) *Every submodule of M is a sum of the simple submodules of M.*

(ii) *M is a sum of simple submodules of M.*

(iii) *M is a direct summand of simple submodules of M.*

(iv) *Every submodule of M is a direct summand of M.*

Definition 1.23 *Let M be a right R-module. If it satisfies one of the conditions of the theorem above then it is called semisimple module.*

Corollary 1.24 **(i)** *Every submodule of semisimple module is semisimple.*

(ii) *Homomorphic image of every semisimple module is semisimple.*

(iii) *Every sum of semisimple modules is semisimple.*

Lemma 1.25 *Let* $\{M_i \mid i \in I\}$ *be a family of modules.Then;*

$$\bigoplus_{i\in I} Soc(M_i) = Soc(\bigoplus_{i\in I} M_i)$$

.

Definition 1.26 *Let M be a right R-module. M is called uniform module if every submodule of M is essential in M.*

Definition 1.27 *Let M be a right R-module. Then we call M has a finite uniform dimension (finite Goldie dimension) if there exists an independent sequence* $H_1, H_2, ..., H_n$ $(n < \infty)$ *of uniform submodules of M with* $(H_1 \oplus H_2 \oplus ... \oplus H_n) \leq_e M$*. Also it is denoted by* $ud(M) = n < \infty$.

Proposition 1.28 *Let M be a right R-module and* $A \leq M$.

- **(i)** *M has a finite uniform dimension if and only if every submodule of M has a finite uniform dimension.*
- **(ii)** *If* $A \leq_c M$ *has a finite uniform dimension then* $\frac{M}{A}$ *has a finite uniform dimension.*
- **(iii)** *If* $A_1, A_2, ..., A_n \leq M$ *and for each i,* A_i *has a finite uniform dimension then* $(A_1 \oplus A_2 \oplus ... \oplus A_n)$ *has a finite uniform dimension.*
- **(iv)** *If* $A \leq_e M$ *and A has finite uniform dimension then M has finite uniform dimension.*

Lemma 1.29 *Let M be a right R-module.*

- **(i)** *If* $A_1, A_2, ..., A_n \leq M$ *then;* $ud(A_1 \oplus A_2 \oplus ... \oplus A_n) = ud(A_1) + ud(A_2) + ... + ud(A_n)$.
- **(ii)** *Let* $A \leq M$ *and A has finite uniform dimension. Then* $A \leq_e M$ *if and only if* $ud(M) = ud(A)$.

Proposition 1.30 *Let M be a right R-module and* $A \leq M$.

- **(i)** *If* $A \leq_c M$ *then* $ud(M) = ud(A) + ud(\frac{M}{A})$.

(ii) *Let M has a uniform finite dimension. If* $ud(M) = ud(A) + ud(\frac{M}{A})$ *then* $A \leq_c M$.

Definition 1.31 *Let R be a ring, M and N be R-modules with identity. If every homomorphism from a submodule X of N to M extend from N to M then M is said to be N-injective. For every R-module N if M is N-injective then M is called injective module. If M is M-injective then M is called quasi-injective module. M and N are relatively injective if M is N-injective and N is M-injective. Also if M is* R_R *injective then M is injective.*

Proposition 1.32 *Let* $\{M_i \mid i \in I\}$ *be a family of R-modules.* $\prod_{i\in I} M_i$ *is injective if and only if each* $i \in I$, M_i *is injective.*

Proposition 1.33 *Let M be a right R-module.*

(i) *M is injective if and only if M is direct summand of every R-module which contains M.*

(ii) *Let A be an R-module and B be a submodule of A. If M is A-injective then M is* $\frac{A}{B}$ *and B-injective.*

Proposition 1.34 *A module M is* $(\bigoplus_{i\in I} A_i)-$ *injective if and only if M is* A_i- *injective for every* $i \in I$.

Definition 1.35 *Let M be a right R-module. The injective module which contains M as essential is called the injective hull of M and it is denoted by* $E(M)$.

Proposition 1.36 *Let M be a right R-module. The followings are equivalent.*

(i) *The injective hull of M is* $E(M)$.

(ii) $E(M)$ *is the maximal module of the modules which contains M as essential.*

(iii) $E(M)$ *is the minimal module of the injective modules which contains M.*

Definition 1.37 *Let R be a ring and let M be a right R-module. If every complement submodule K of M is a direct summand of M then M is called CS-module* $((C_1) holds)$.

Equivalently, for every submodule K of M there exists a direct summand N of M such that K is essential in N. The ring R is called right CS-ring if R_R is CS-module. For every $I \leq_c R$ there exists idempotent $e \in R$ such that $I = eR$. For example, semisimple modules, uniform modules and injective modules are CS-modules. Every complement of a CS-module is CS-module. But any submodule of a CS-module may not be CS-module.

For example, let M be not a CS-module then since $E(M)$ is injective module then it is CS-module. Even M is essential in $E(M)$ it is not CS-module. Also a direct sum of two CS-modules may not be CS-module.

Example 1.38 *Let $\mathbb{Z}$ denote the integers, let p be any prime, let $M_1 = \mathbb{Z}_p$ and let $M_2 = \mathbb{Z}_{p^3}$. M_1 and M_2 are CS-$\mathbb{Z}$-modules. But $M = (M_1 + M_2)$ is not a CS-module.*

Definition 1.39 *A right R-module M is called indecomposable module if M has no nonzero proper direct summand. Equivalently, M is indecomposable if and only if for any $K \leq_d M$,$K = 0$ or $K = M$.*

Proposition 1.40 *Let M be an indecomposable right R-module. If M is CS-module then M is uniform module.*

Definition 1.41 *Let M be a right R-module.*

(C_2)*: Every submodule of M which is isomorphic to a direct summand of M is a direct summand of M.*

(C_3)*: If N_1, N_2 be two direct summand of M such that $N_1 \cap N_2 = 0$ then $(N_1 \oplus N_2)$ is a direct summand of M.*

Lemma 1.42 *Every direct summand of M satisfying C_i $(i = 1, 2)$ satisfies C_i $(i = 1, 2)$.*

Definition 1.43 *A right R-module M is called continuous (quasi-continuous) if M is CS-module satisfying the condition (C_2) $((C_3))$.*

Lemma 1.44 *Every module M satisfying the condition (C_2) satisfies the condition (C_3).*

Proof. Let K, L be direct summand of M with $K \cap L = 0$, $M = K \oplus K'$ for a submodule K' of M. Let $\pi : M \to K'$ be a projection map. $K \cap L = 0$ then $\pi(L) \cong L$ and $\pi(L) \leq K'$. By the condition (C_2), $\pi(L) \leq_d M$ and hence $M = \pi(L) \oplus L_{'}$ for a submodule L' of M. Then $K' = \pi(L) \oplus (K' \cap L')$ and $M = K \oplus \pi(L) \oplus (K' \cap L')$. Hence, $K \oplus \pi(L) \leq_d M.K \oplus \pi(L) \cong K \oplus L$ then $K \oplus L \leq_d M$.□

Proposition 1.45 *In any ring R, the following sets coincide:*

(i) *The intersection of all maximal right ideals of R.*

(ii) *The intersection of all maximal left ideals of R.*

(iii) *The intersection of all right primitive ideals of R.*

(iv) *The intersection of all left primitive ideals of R.*

Definition 1.46 *In any ring R, the ideal defined by the intersections given in Proposition 1.45 is called the Jacobson radical of R, denoted by $J(R)$.*

CHAPTER 2

NONSINGULAR RETRACTABLE MODULES AND THEIR ENDOMORPHISM RINGS

2.1 Preliminaries

In this chapter [26] was taken as a reference basically. Also in this chapter all the rings are assumed to be associative with unit but not necessarily commutative. The modules are unital right modules. The base ring, the right R-module and the endomorphism ring are denoted by, R, M_R, $S = End_R(M)$ respectively. The notation $End(M)$ is used instead of $End_R(M)$. The $N \leq_e M$ means N is essential in M; $N \leq_c M$ means N is closed in M; $N \leq_d M$ means N is direct summand of M.

Definition 2.1 *A module M is said to be retractable if for every nonzero submodule N of M,* $Hom_R(M, N) \neq 0$.

Definition 2.2 *A module M is said to be e-retractable if for every nonzero complement submodule C of M,* $Hom_R(M, C) \neq 0$.

Definition 2.3 *Let A be a right R-module and B be a left R-module. Let F be a free abelian group that generated by* $A \times B$ *and K be a subgroup of F whose elements are generated by the following elements of F;*

(i) $(a + a', b) - (a, b) - (a', b)$

(ii) $(a, b + b') - (a, b) - (a, b')$

(iii) $(ar, b) - (a, rb)$

Then F/K *is an abelian factor group which is called the tensor product of A and B and denoted by* $A \otimes_R B$.

Before defining a nondegenerate module, we define some notations;

$$I_S(N) = \{s \in S \mid sM \subseteq N\}$$

and let

$$A_M(H) = HM = \sum_{h \in H} hM$$

where $N \leq M_R$, $H \leq S_S$ and $S = End(M)$.

Clearly, $I_S(N)$ is a left ideal of S and $A_M(H) = HM$ is a submodule of M. The notations $A_M(H)$ and HM will be used interchangeably, and we will identify $I_S(N)$ and $Hom_R(M, N)$ for $N \leq M_R$; in particular, M is retractable if $I_S(N) \neq 0$ for $0 \neq N \leq M_R$.

Definition 2.4 *Let $M^* = Hom_R(M, R)$ and let $T = (M^*, M) = \{\sum_{i=1}^{n} f_i m_i \mid f_i \in M^*, m_i \in M\} = \sum_{f \in M^*} Im(f)$ be the trace of M in R, M is said to be nondegenerate if $mT \neq 0$ for all $0 \neq m \in M$.*

Also, define $(,) : M^* \otimes_S M \to R$ by $(f, m) = f(m)$ for $m \in M$ and $f \in M^*$ is R-module homomorphism. And $[,] : M \otimes_R M^* \to S$ by $[m, f]m_1 = m(f, m_1) = mf(m_1)$ for $m, m_1 \in M$ and $f \in M^*$.

Proposition 2.5 *Let M_R be nondegenerate. Then, for any nonzero submodule, N of M $I_S(N) \neq 0$ but not conversely (for example, let M be the $\mathbb{Z}$-module $\frac{\mathbb{Z}}{p^n\mathbb{Z}}$).*

Proof. Let N be a nonzero submodule of M, and let $0 \neq n \in N$. Then, since M is nondegenerate, $[n, M^*] \neq 0$ by [25]. From $[n, M^*]M = n(M, M^*) \subseteq nR$, so $[n, M^*] \subseteq I_S(N)$, hence $I_S(N) \neq 0$.□

Proposition 2.6 *For any right R-module M, right M-module N and right S-module H;*

- **(i)** $N_1 \leq N_2 \leq M_R$ *then* $I_S(N_1) \leq I_S(N_2)$.
- **(ii)** $H_1 \leq H_2 \leq S_S$ *then* $A_M(H_1) \leq A_M(H_2)$.
- **(iii)** $A_M I_S(N) \leq N$ *and* $H \leq I_S A_M(H)$.
- **(iv)** $I_S(N) = I_S A_M I_S(N)$ *and* $A_M(H) = A_M I_S A_M(H)$.

Proof.

(i) Let $s \in I_S(N_1)$ then $sM \subseteq N_1 \leq N_2$ so $s \in I_S(N_2)$ which implies $I_S(N_1) \leq I_S(N_2)$.

(ii) Let $H_1 \leq H_2 \leq S_S$ then $A_M(H_1) = H_1M \leq H_2M = A_M(H_2)$ which implies $A_M(H_1) \leq A_M(H_2)$.

(iii) Let $N \leq M_R$ and $H \leq S_S$ then $A_M I_S(N) = I_S(N)M \leq N$ since $I_S(N) = \{s \in S \mid sM \subseteq N\}$. Also since $I_S A_M(H) = I_S(HM) = \{s \in S \mid sM \subseteq HM\}$, for all $h \in H$ then $h \in I_S A_M(H)$ which implies $H \leq I_S A_M(H)$.

(iv) Let $N \leq M_R$ and $H \leq S_S$ then by (iii) $A_M I_S(N) \leq N$ if we apply I_S to both side we found $I_S A_M I_S(N) \leq I_S(N)$. And let $I_S(N) = H_1 = \{s \in S \mid sM \subseteq N\}$ is a right ideal of S. Then again by (iii) $I_S(N) \leq I_S A_M I_S(N)$. Consequently, $I_S(N) = I_S A_M I_S(N)$. Similarly, we can see that $A_M(H) = A_M I_S A_M(H)$.

□

2.2 Nondegenerate, Retractable and e-Retractable Modules

Proposition 2.7 *When M is nondegenerate then M is retractable and M has the following two properties.*
(I) For $N_1 \leq N_2 \leq M_R$, $N_1 \leq_e N_2$ if and only if $I_S(N_1) \leq_e I_S(N_2)$.
(II) For $H_1 \leq H_2 \leq S_S$, $H_1 \leq_e H_2$ if and only if $A_M(H_1) \leq_e A_M(H_2)$.

Proof. (II) First assume $H_1 \leq H_2 \leq S_S$ and let $0 \neq m = \Sigma_{i=1}^{n} h_i m_i \in A_M(H_2)$, with $m_i \in M$ and $h_i \in H_2$ for $i = 1, ..., n$. Then $0 \neq [m, M^*] = \Sigma_{i=1}^{n} h_i[m_i, M^*] \subseteq H_2$, hence $[m, M^*] \cap H_1 \neq 0$. We have $0 \neq [m, M^*] \cap H_1M \subseteq H_1M \cap m(M, M^*) \subseteq H_1M \cap mR$, therefore, $H_1M = A_M(H_1) \leq_e A_M(H_2)$.

Conversely, assume that $A_M(H_1) \leq_e A_M(H_2)$, for $H_1 \leq H_2$, and let $0 \neq h \in H_2$ then $hM \neq 0$ implies $hM \cap A_M(H_1) \neq 0$ and this implies that $0 \neq [hM \cap A_M(H_1), M^*] \subseteq [hM, M^*] \cap [A_M(H_1), M^*] \subseteq hS \cap H_1$, hence $H_1 \leq_e H_2$.

(I) Let $I_S(N_1) \leq_e I_S(N_2)$. Since M is retractable then it follows from Proposition 2.5 that, for every nonzero submodule N_1 of M, there is $0 \neq s \in I_S(n_1R)$ where $0 \neq n_1 \in N_1$. Hence, since $I_S(n_1R) \subseteq I_S(N_1)$, we have $0 \neq sM \subseteq n_1R \cap A_M I_S(N_1)$, so that $A_M I_S(N_1) \leq_e N_1$. Similarly, $A_M I_S(N_2) \leq_e N_2$. And by hypothesis $I_S(N_1) \leq_e I_S(N_2)$ so $A_M I_S(N_1) \leq_e A_M I_S(N_2)$ which implies $N_1 \leq_e N_2$.

Now let N_1 and N_2 are nonzero submodules of M such that $N_1 \leq N_2$ and assume first that $N_1 \leq_e N_2$. Then we have $I_S(N_1) \leq I_S(N_2)$, $A_M I_S(N_1) \leq A_M I_S(N_2) \leq_e N_2$, and $A_M I_S(N_1) \leq_e N_1 \leq N_2$, therefore $A_M I_S(N_1) \leq_e N_2$ and hence $A_M I_S(N_1) \leq_e A_M I_S(N_2)$, and this implies by (II), that $I_S(N_1) \leq_e I_S(N_2)$. □

Proposition 2.8 *For any M_R, the followings are equivalent.*

(i) *M_R is retractable.*

(ii) *For any $N \leq M_R$, $A_M I_S(N) \leq_e N$.*

(iii) *For $N_1 \leq N_2 \leq M_R$, $I_S(N_1) \leq_e I_S(N_2)$ then $N_1 \leq_e N_2$.*

Proof.

$(i \Rightarrow ii)$ If M is retractable and $0 \neq N \leq M_R$ then for any $0 \neq n \in N$, there exists $0 \neq s \in I_S(nR)$, since $I_S(nR) \subseteq I_S(N)$, we have that, $0 \neq sM \subseteq nR \cap A_M I_S(N)$ so that $A_M I_S(N) \leq_e N$.

$(ii \Rightarrow i)$ If $A_M I_S(N) \leq_e N$ for any nonzero N this implies $I_S(N) = \{s \in S \mid sM \subseteq N\} \neq 0$. Then M_R is retractable.

$(i \Rightarrow iii)$ If M is retractable and $I_S(N_1) \leq_e I_S(N_2)$ for $N_1 \leq N_2 \leq M_R$ then for any $0 \neq n_2 \in N_2$, there is $0 \neq s \in I_S(n_2R) \cap I_S(N_1)$ we have, $0 \neq sM \subseteq n_2R \cap N_1$, which shows $N_1 \leq_e N_2$.

$(iii \Rightarrow ii)$ Assume (iii) holds, and let $0 \neq N \leq M_R$. Then since $I_S(N) = I_S A_M I_S(N)$ implies in particular, that $I_S A_M I_S(N) \leq_e I_S(N)$ we have by (iii) that $A_M I_S(N) \leq_e N$.□

It can be seen from Proposition 2.8 that whereas nondegenerate modules are satisfied property (I), retractable modules are satisfied only one direction of (I). However, if M is nonsingular as well as retractable, then the other direction is also satisfied.

Theorem 2.9 *If M_R is nonsingular and retractable then we have.*

(I) For $N_1 \leq N_2 \leq M_R$, $N_1 \leq_e N_2$ if and only if $I_S(N_1) \leq_e I_S(N_2)$.

Proof.

($\Leftarrow$:) Already proved.(Proposition 2.8)

($\Rightarrow$:) Assume that $N_1 \leq_e N_2$ and let $0 \neq s \in I_S(N_2)$. Choose $m \in M$ such that $0 \neq sm = n_2 \in N_2$, then since $N_1 \leq_e N_2$ there is $r \in R$ such that $0 \neq n_1 = n_2 r = smr$, and also $0 \neq smrR = n_1 R \subseteq N_1$. By Proposition 2.8 ($ii$) $A_M I_S(mrR) \leq_e mrR$, if x is any nonzero element in mrR, then it is known [18, p.46, Lemma 3]that the right ideal $J_x = \{r \in R \mid xr \in A_M I_S(mrR)\}$ is an essential right ideal of R. Assume $s[A_M I_S(mrR)] = 0$, then for any nonzero $x \in mrR$, since $xJ_x \subseteq A_M I_S(mrR)$, we will have $sxJ_x = 0$ and consequently $sx = 0$ since M is nonsingular and $J_x \leq_e R_R$, but this contradicts the fact that $sx = smrR \neq 0$. Hence $s[A_M I_S(mrR)] \neq 0$ and there is $c \in I_S(mrR)$ such that $sc \neq 0$. Then $0 \neq sc \in sS \cap I_S(N_1)$, proving that $I_S(N_1) \leq_e I_S(N_2)$.□

Corollary 2.10 *Let M_R be nonsingular. Then M is retractable if and only if (I) holds.*

Proof. By Proposition 2.8 and Theorem 2.9. □

Recall that a submodule C of M is complement submodule of M if C has no proper essential in M. When M_R is nonsingular, then for any submodule N of M, there is a unique complement, C in M such that $N \leq_e C$.

Theorem 2.11 *Let M_R be nonsingular. Then the followings are equivalent;*

(i) *M is e-retractable.*

(ii) *For any nonzero complement C in M, $A_M I_S(C) \leq_e C$.*

(iii) *If $N_1 \leq N_2 \leq M_R$ and N_2 is a complement in M, if $I_S(N_1) \leq_e I_S(N_2)$ then $N_1 \leq_e N_2$.*

Proof.

($i \Rightarrow ii$) Let assume M is e-retractable and let $0 \neq x \in C$ then there exists a complement submodule Y of C such that $xR \leq_e Y$. Since $Y \subseteq C$ we have, $I_S(Y) \subseteq$

$I_S(C)$, and since $0 \neq Y$ is a complement in M and M is e-retractable, there is $0 \neq s \in I_S(Y)$. Then $sM \subseteq Y$ and since $xR \leq_e Y$, there is $0 \neq z \in sM \cap xR$. Therefore, $0 \neq z \in I_S(C)M \cap xR$ proving that $I_S(C)M \leq_e C$ which implies $A_M I_S(C) \leq_e C$.

($ii \Rightarrow i$) Let $0 \neq C \leq_c M$ by (ii) $A_M I_S(C) \leq_e C$. Then there exists $0 \neq z \in I_S(C)M \cap xR$ where $x \in C$ so $I_S(C) = \{s \in S \mid sM \subseteq C\} \neq 0$ which provides $Hom(M, C) \neq 0$.

($i \Rightarrow iii$) Assume M is e-retractable and $N_1 \leq N_2 \leq M_R$ where $N_2 \leq_c M$, suppose that $I_S(N_1) \leq_e I_S(N_2)$. Let $0 \neq n_2 \in N_2$. Then there exists $C \leq_c N_2$ such that $n_2R \leq_e C \leq_c N_2$ and so by e-retractability $I_S(C) \neq 0$ hence there is $0 \neq b \in I_S(C) \cap I_S(N_1)$. Then $0 \neq bM \subseteq C \cap N_1$, and this implies that $C \cap N_1 \neq 0$ and so $n_2R \cap N_1 \neq 0$. This implies $N_1 \leq_e N_2$.

($iii \Rightarrow ii$) Assume (iii) holds. And let N_2 be nonzero complement in M. From $I_S(N_2) = I_S A_M I_S(N_2)$ we have in particular, $I_S A_M I_S(N_2) \leq_e I_S(N_2)$ by (iii) $A_M I_S(N_2) \leq_e (N_2)$.□

As regards property (II), here again it holds when M is nondegenerate, whereas for M is nonsingular and retractable, (II) holds if and only if $H \leq_e I_S(HM)$ for each $H \leq S_S$. Next theorem gives the relationship between properties (I) and (II).

Theorem 2.12 **(i)** *Given (I), then (II) holds if and only if* $H_1 \leq_e I_S A_M(H_1)$ *for each* $H_1 \leq S_S$.

(ii) *Given (II), then (I) holds if and only if* $A_M I_S(N_1) \leq_e N_1$ *for each* $N_1 \leq M_R$

Proof.

(i) (⇒:) Let (I) be given. Suppose (II) holds and let $H_1 \leq S_S$. Then $A_M I_S A_M(H_1) = A_M(H_1)$ implies in particular that $A_M(H_1) \leq_e A_M I_S A_M(H_1)$ and by proposition 2.6 (iii) $H_1 \leq I_S A_M(H_1)$ these implies that by (II) $H_1 \leq_e I_S A_M(H_1)$.

(⇐:) Conversely, assume that $H_1 \leq_e I_S A_M(H_1)$ for each $H_1 \leq S_S$. To prove (II) assume first $H_1 \leq_e H_2$. Then we have $A_M(H_1) \leq A_M(H_2)$, $H_1 \leq_e I_S A_M(H_1) \leq I_S A_M(H_2)$ and $H_1 \leq_e H_2 \leq_e I_S A_M(H_2)$, therefore $H_1 \leq_e I_S A_M(H_2)$, which implies that $I_S A_M(H_1) \leq_e I_S A_M(H_2)$ and by (I) $A_M(H_1) \leq_e A_M(H_2)$. For the other direction of (II) assume $A_M(H_1) \leq_e A_M(H_2)$, where $H_1 \leq H_2 \leq S_S$ using (I) we have, $H_1 \leq_e I_S A_M(H_1) \leq_e I_S A_M(H_2)$, hence $H_1 \leq_e I_S A_M(H_2)$. But $H_1 \leq_e H_2 \leq_e I_S A_M(H_2)$, hence $H_1 \leq_e H_2$.

(ii) ($\Rightarrow$:) Let (II) be given. Suppose (I) holds, then we know by Proposition 2.6 $A_M I_S(N_1) \leq N_1$ and $I_S A_M I_S(N_1) = I_S(N_1)$ implies in particular $I_S A_M I_S(N_1) \leq_e I_S(N_1)$ then by (I) $A_M I_S(N_1) \leq_e (N_1)$ for each $N_1 \leq M_R$.

($\Leftarrow$:) Conversely, assume that $A_M I_S(N_1) \leq_e N_1$ for each $N_1 \leq M_R$. To prove (I) assume first $N_1 \leq_e N_2$. Then we have $I_S(N_1) \leq I_S(N_2)$, $A_M I_S(N_1) \leq A_M I_S(N_2) \leq_e N_2$ and $A_M I_S(N_1) \leq_e N_1 \leq_e N_2$ therefore $A_M I_S(N_1) \leq_e N_2$. Hence $A_M I_S(N_1) \leq_e A_M I_S(N_2)$ which by (II) implies that $I_S(N_1) \leq_e I_S(N_2)$. For the other direction of (I) assume that $I_S(N_1) \leq_e I_S(N_2)$ with $N_1 \leq N_2 \leq M_R$. Then again by using (II) $A_M I_S(N_1) \leq_e A_M I_S(N_2) \leq_e N_2$, hence $A_M I_S(N_1) \leq_e (N_2)$. But $A_M I_S(N_1) \leq_e N_1 \leq N_2$, hence $N_1 \leq_e N_2$.

□

Corollary 2.13 *Let M_R be a nonsingular and retractable. Then (II) holds if and only if $H \leq_e I_S A_M(H)$ for each $H \leq S_S$.*

Proof. By Theorem 2.9 since M_R is nonsingular and retractable then (I) holds so by Theorem 2.12(*i*) (II) holds if and only if $H \leq_e I_S A_M(H)$ for each $H \leq S_S$. □

Theorem 2.14 **(i)** *For any M_R and $T = (M^*, M) = \{\sum_{i=1}^{n} f_i m_i \mid f_i \in M^*,\ m_i \in M\}$, $MT \leq_e M$ then $NT \leq_e Hom_R(M, N)M$ for every nonsingular N_R, if M is nonsingular, the converse also holds.*

(ii) *$T \leq_e R$ then $NT \leq_e N$ for every nonsingular N_R, if R is right nonsingular, the converse also holds.*

Proof.

(i) Assume that $MT \leq_e M$, and let N_R be nonsingular. Let $0 \neq n \in Hom_R(M,N)M$ and $n = \sum_{i=1}^{k} f_i m_i$ with $0 \neq f_i \in Hom_R(M,N)$ and $0 \neq m_i \in M$, (assume also $f_i m_i \neq 0$), for $i = 1, ..., k$. Set $K_i = \{r \in R \mid m_i r \in MT\}$, then $K_i \leq_e R$ (Let $r \in R \setminus K_i$ then $m_i r \notin MT$ since $MT \leq_e M$, there exists $0 \neq r' \in R$ such that $0 \neq (m_i r)r' \in MT$ so $0 \neq m_i(r' r) \in MT$ then $r' r \in K_i$ implies that $K_i \leq_e R$). Hence, since $0 \neq f_i m_i \in N$ and N is nonsingular, we have $0 \neq f_i m_i K_i \subseteq f_i MT \cap f_i m_i R$, for $i = 1, ..., k$. Let $J = r_R(n) = \{r \in R \mid nr = 0\}$. Then, since N_R is nonsingular and $n \neq 0$, J is not an essential right ideal of R. Let $0 \neq P$ be right ideal of R such that $P \cap J = 0$. Since $\bigcap_{i=1}^{k} K_i \leq_e R$, there is $0 \neq s \in P \cap (\bigcap_{i=1}^{k} K_i)$. Then we have $0 \neq ns = (\sum_{i=1}^{k} f_i m_i)s = \sum_{i=1}^{k} f_i m_i s \in \sum_{i=1}^{k} f_i MT$, since $s \in \bigcap_{i=1}^{k} K_i$. Therefore, since $f_i M \subseteq N$, $0 \neq ns \in NT \cap nR$, which shows that $NT \leq_e Hom_R(M,N)M$. The second statement is clear.

(ii) Assume that $T \leq_e R$ and let N_R be nonsingular. Let $0 \neq n \in N$, then $r_R(n) = \{r \in R \mid nr = 0\}$ is not essential as a right ideal of R, so there is a nonzero right ideal J in R such that $J \cap r_R(n) = 0$. Then $J \cap T \neq 0$ and $0 \neq n(J \cap T) \subseteq nJ \cap NT$, so $NT \leq_e N$. Again, the second statement is clear.

□

Corollary 2.15 *Let M_R be nonsingular. Then $MT \leq_e M$ if and only if $NT \leq_e A_M I_S(N)$ for every $N \leq M_R$, and M is nondegenerate if and only if $MT \leq_e M_R$ and M is retractable.*

Proof. By Proposition 2.7 and Theorem 2.14.□

2.3 Endomorphism Rings of Nonsingular Retractable Modules

Definition 2.16 *The set $C^e = \{N \leq M_R \mid N \text{ is a complement submodule of } M\}$, and the set $C^1(S) = \{H \leq S_S \mid H \text{ is a complement right ideal of } S\}$ for $S = End(M)$.*

Theorem 2.17 *Let M_R be nonsingular and nondegenerate. Then $S = End(M)$ is a right CS ring if and only if M is a CS module.*

Proof. Assume that M is CS module and let H be a right complement in S. Then $[A_M(H)]^e = eM$, for $e = e^2 \in S$, and $H = \{I_S[A_M(H)]^e\} = I_S(eM) = eS$ where $[A_M(H)]^e$ is a complement of $[A_M(H)]$ in M_R. Hence H is a direct summand in S, proving that S is a right CS ring.

Conversely, assume that S is a right CS ring, and let N be a complement in M. Then $I_S(N) = eS$, for $e = e^2 \in S$, and $N = [A_M I_S(N)]^e = [A_M(eS)]^e = [eM]^e = eM$, since every direct summand is e-closed. Hence N is a direct summand in M then M is a CS module.□

Theorem 2.18 *Let M_R be nonsingular and retractable. Then the maps $N \to I_S(N)$ and $H \to [A_M(H)]^e$ determine a projectivity between C^e and $C^1(S)$ if and only if $K \leq_e I_S A_M(K)$ for every $K \leq S_S$.*

Proof. Assume that $K \leq_e I_S A_M(K)$ for every $K \leq S_S$. Then, by Corollary 2.13, property (II) holds, also, by Theorem 2.9, property (I) holds. Let $N \in C^e$ and suppose that $I_S(N) \leq_e J$. By Zorn's Lemma, we may assume that $J \in C^1(S)$. Since (II) holds, $I_S(N) \leq_e J$ implies that $A_M I_S(N) \leq_e A_M(J)$, since M is retractable, $A_M I_S(N) \leq_e N$, by Proposition 2.8. Therefore, $N = [A_M I_S(N)]^e = [A_M(J)]^e$, so that $A_M(J) \subseteq N$ and hence $J \subseteq I_S(N)$, then $I_S(N) = J$, that is I_S maps $N \in C^e$ to $I_S(N) \in C^1(S)$. Clearly, for any $H \leq S_S$, $[A_M(H)]^e \in C^e$.

Let $H \in C^1(S)$, by (I), $A_M(H) \leq_e [A_M(H)]^e$ implies that $I_S A_M(H) \leq_e I_S\{[A_M(H)]^e\}$. Then $H \leq_e I_S A_M(H) \leq_e I_S\{[A_M(H)]^e\}$ implies that $H = I_S A_M(H) = I_S\{[A_M(H)]^e\}$.

We have $N \in C^e \to I_S(N) \in C^1(S) \to [A_M I_S(N)]^e = N$, and $H \in C^1(S) \to [A_M(H)]^e \in C^e \to I_S\{[A_M(H)]^e\} = H$.

Hence the two order-preserving maps are inverses of each other and so determine a projectivity between C^e and $C^1(S)$.

Conversely, assume that the maps $N \to I_S(N)$ and $H \to [A_M(H)]^e$ determine a projectivity between C^e and $C^1(S)$. Then, if $H \in C^1(S)$, $H = I_S\{[A_M(H)]^e\}$, since $H \leq$

$I_SA_M(H) \leq I_S\{[A_M(H)]^e\}$, it follows that $H = I_SA_M(H)$. Let K be any right ideal in S, there is $J \in C^1(S)$ such that $K \leq_e J$. We have $J = I_SA_M(J)$, $A_M(K) \leq A_M(J)$, and $K \leq I_SA_M(K) \leq I_SA_M(J) = J$, so $K \leq_e J$ implies that $K \leq_e I_SA_M(K)$.□

The next theorem uses injective hull, $\tilde{M}$, of M and its endomorphism ring, $A = End(\tilde{M})$.

Theorem 2.19 *Let M_R be nonsingular and retractable. Then $S = End(M)$ is a right CS ring if and only if M is a CS module.*

Proof. Let M be nonsingular, retractable and CS. Since M is nonsingular and retractable, we have, by [27, Theorem 3.1] that S is right nonsingular, $S \leq_e A$ and A is maximal right quotient ring of S. Since M is CS module, if $N \in C^e$ and $N \neq M$, then we have $N = eM$, for $1 \neq e = e^2 \in S$, and therefore $0 \neq 1 - e \in l_S(N) = \{s \in S \mid sN = 0\}$ for $N \leq M_R$. Since M is nonsingular, we know by [24, Theorem 3.5] that, if for every complement N in M such that $N \neq M$ we have $l_S(N) \neq 0$, then S has nonzero intersection with every nonzero left ideal of $A = End(\tilde{M})$. Hence, the right nonsingular ring S has nonzero intersection with every nonzero left ideal of its maximal right quotient ring A, therefore, it follows by Utumi's Theorem [42, Theorem 3.13] that every complement right ideal in S is a right annihilator in S. But, by [25, Theorem 3.13], since M is nonsingular and CS, every right annihilator in S is a direct summand in S (that is, S is a Baer ring). Hence every complement right ideal in S is a direct summand in S, and S is a right CS ring.

Conversely, if M_R is nonsingular, retractable and S is right CS ring then M is CS module [16, Theorem 3.1]. However, for completeness slightly shorter version of Theorem 2.11 will be used. Let M be nonsingular and e-retractable and assume that S is a right CS ring. Let $N \in C^e$ and set $H = I_S(N)$. There is $K \in C^1(S)$ such that $H \leq_e K$, and, since S is a right CS ring, $K = eS$ for $e = e^2 \in S$. We have $A_M(H) \leq A_M(K) = A_M(eS) = eM$, and $I_SA_M(K) = I_S(eM)$. Clearly, $e \in I_S(eM)$, so that $eS \subseteq I_S(eM)$, on the other hand, if $s \in I_S(eM)$, then, for any $m \in M$, $sm = em_1$ some $m_1 \in M$, hence $esm = e^2m_1 = em_1 = sm$, that is, $s = es \in eS$. Therefore $I_S(eM) = eS = K$, that is, $I_S A_M(K) = K$, hence, since $H \leq_e K$ we have $H \leq_e I_S A_M(H) \leq_e I_S A_M(K) = K$. Thus, we have $A_M(H) \leq A_M(K)$ and $I_S A_M(H) \leq_e I_S A_M(K)$, with $A_M(K) = eM$ a direct summand and hence a complement in M. Therefore, by Theorem 2.11, it follows that $A_M(H) \leq_e A_M(K)$. Since M is e-retractable and $N \in C^e$, we have, again by Theorem 2.11, $A_M I_S(N) \leq_e N$. Therefore, we have $N = [A_M I_S(N)]^e = [A_M(H)]^e = A_M(K)$, that is, $N = A_M(K) = eM$, and N is a direct summand in M, proving that M is a CS module.□

CHAPTER 3

BAER AND QUASI-BAER MODULES

3.1 Preliminaries

In this chapter [38] was taken as a reference basically. And in this chapter all the ring are assumed to be with unit, and not necessarily commutative. The modules are unital right modules. The base ring, the right R-module and the endomorphism ring are denoted by, R, M_R, $S = End_R(M)$ respectively. The notation $End(M)$ is used instead of $End_R(M)$. The right annihilator of $X \subseteq M$ in R (i.e. all elements $r \in R$ so that $Xr = 0$) is denoted by $r_R(X)$, the left annihilator of $X \subseteq M$ in S (i.e. all elements $\varphi \in S$ so that $\varphi X = 0$) is denoted by $l_S(X)$; the right annihilator of $T \subseteq S$ in M (i.e. all elements $m \in M$ so that $Tm = 0$) is denoted by $r_M(T)$ and the left annihilator of $P \subseteq R$ in M (i.e. all elements $m \in M$ so that $mP = 0$) is denoted by $l_M(P)$. And $N \trianglelefteq M$ means N is fully invariant in M.

Definition 3.1 *A module M is called an extending (CS-) module if, for all $N \leq M$, there exists a direct summand $N' \leq_d M$ such that $N \leq_e N'$.*

Definition 3.2 *A submodule N of a module M is called fully invariant if $\varphi(N) \subseteq N$ for all $\varphi \in End_R(M)$.*

Definition 3.3 *A module M is called an FI-extending module if, for all $N \trianglelefteq M$, there exists a direct summand $N' \leq_d M$ such that $N \leq_e N'$.*

Definition 3.4 *A ring R is called a Baer ring if the right annihilator in R of any left ideal is generated, as a right ideal, by an idempotent element of R (in other words, for all $I \leq {}_R R$, $r_R(I) = eR$ where $e^2 = e \in R$).*

Definition 3.5 *A ring R is called a quasi-Baer ring if the right annihilator in R of any two-sided ideal is generated, as a right ideal, by an idempotent element of R (for all* $I \trianglelefteq R$*,* $r_R(I) = eR$ *where* $e^2 = e \in R$*).*

Remark 3.6 *The Baer and quasi-Baer properties for rings are left-right symmetric: a ring R is a (quasi-) Baer ring if and only if the left annihilator in R of any (two-sided) right ideal is generated, and a left ideal, by an idempotent element of R.*

Definition 3.7 *A ring R is called right nonsingular if no nonzero element has an essential right annihilator in* R_R*.*

Definition 3.8 *A ring R is called right cononsingular if any right ideal, with zero left annihilator, is essential in* R_R*.*

Definition 3.9 *An idempotent* $e^2 = e \in R$ *is called a left (respectively, right) semi-central idempotent if eR (respectively, Re) is a two-sided ideal of R.*

The next lemma will be useful.

Lemma 3.10 *For* $N \leq M$*,* $I \leq R_R$*,* $K \leq {}_SS$*,* $P \trianglelefteq M$*,* $J \trianglelefteq R$*,* $L \trianglelefteq S$*, the followings hold:*

(i) $l_M(r_R(l_M(I))) = l_M(I)$

(ii) $r_R(l_M(r_R(N))) = r_R(N)$

(iii) $l_S(r_M(l_S(N))) = l_S(N)$

(iv) $r_M(l_S(r_M(K))) = r_M(K)$

(v) $l_M(J) \trianglelefteq M$

(vi) $r_R(P) \trianglelefteq R$

(vii) $l_S(P) \trianglelefteq S$

(viii) $r_M(L) \trianglelefteq M$.

Proof. It is well-known that the pairs $r_R()$-$l_M()$, respectively $l_S()$-$r_M()$ are Galois pairs, hence equalities (i) through (iv) hold true. Let show (i);

Let $m \in l_M(r_R(l_M(I)))$ then $mr_R(l_M(I)) = 0$ which implies $r_R(l_M(I)) \subseteq r_R(m)$, so $m \in l_M(I)$. Also if we take an element m in $l_M(I)$ then $r_R(l_M(I)) \subseteq r_R(m)$ which implies $mr_R(l_M(I)) = 0$, so $m \in l_M(r_R(l_M(I)))$.

For assertion (v) we observe that, in general, $l_M(J) \leq_S M$. On the other hand, if $J \trianglelefteq R$ then $rJ \subseteq J$, and so, if $m \in l_M(J)$, $mr \in l_M(J)$ which implies that $mrJ \subseteq mJ = 0$. Hence $l_M(J) \trianglelefteq M$. The last three statements follow similarly. □

Lemma 3.11 *Let M be a module, and let $M = M_1 \oplus M_2$ be a direct sum decomposition. If $N \trianglelefteq M$ then $N = N_1 \oplus N_2$, where $N_i = N \cap M_i \trianglelefteq M_i$, for $i = 1, 2$.*

Proof. Let π_i be the canonical projection of M onto M_i, for $i = 1, 2$. Since $N \trianglelefteq M$, $\pi_i(N) \subseteq N$, and so $\pi_i(N) = N \cap M_i = N_i$, for $i = 1, 2$. Hence $N \subseteq \pi_1(N) + \pi_2(N) = N_1 + N_2$. But since $N_i \subseteq N$ $(i = 1, 2)$, $N_1 + N_2 \subseteq N$. As $N_1 \cap N_2 = N \cap M_1 \cap M_2 = 0$ we get that $N = N_1 \oplus N_2$.□

Lemma 3.12 *Let M be a module, with $M = N_1 \oplus N_2$ and let $F_1 \trianglelefteq N_1$. Then there exists $F_2 \trianglelefteq N_2$ so that $F_1 \oplus F_2 \trianglelefteq M$.*

Proof. Let

$$F_2 = \sum_{\varphi \in Hom(N_1, N_2)} \varphi(F_1)$$

Then $F_2 \leq N_2$. Take any $\psi \in End(N_2)$. Since $\psi\varphi \in Hom(N_1, N_2)$ $\forall \varphi \in Hom(N_1, N_2)$, we obtain $\psi(F_2) = \psi(\sum \varphi(F_1)) = \sum \psi\varphi(F_1) \subseteq F_2$. Hence $F_2 \trianglelefteq N_2$. Consider $\chi \in End(M)$; then $\chi = (\chi_{ij})_{i,j=1,2}, \chi_{ij}\colon N_j \to N_i$, with $i, j = 1, 2$. Note that $\chi_{ii}(F_i) \subseteq F_i$, since $F_i \trianglelefteq N_i$, $i = 1, 2$, and $\chi_{21}(F_1) \subseteq F_2$, from the definition of F_2. For $\varphi \in Hom(N_1, N_2)$, $\chi_{12}\varphi \in End(N_1)$; it follows that $\chi_{12}(F_2) = \chi_{12}(\sum \varphi(F_1)) = \sum_{12} \varphi(F_1) \subseteq F_1$. Since each component of χ maps $F_1 \oplus F_2$ back into $F_1 \oplus F_2$, $F_1 \oplus F_2 \trianglelefteq M$.□

3.2 Baer Modules

Definition 3.13 *A right R-module M is called a Baer module if for all $N \leq M$, $l_S(N) \leq_d {}_S S$ (or, equivalently, $l_S(N) = Se$, with $e^2 = e \in S = End(M)$).*

Remark 3.14 *By lemma 3.10, one can easily prove that a module M is Baer if and only if $\forall\ I \leq {}_S S$, $r_M(I) = eM$ where $e^2 = e \in S = End(M)$.*

The following theorem given by Chatters and Khuri (1980) is generalized by Rizvi and Roman (2004) as Theorem 3.29.

Theorem 3.15 $[16, Theorem 2.1]$ *Let R be a ring. Then R is a right nonsingular, right extending if and only if R is a right cononsingular, Baer ring.*

Definition 3.16 *We call the module M is K-nonsingular if, for all $\varphi \in S = End(M)$, $r_M(\varphi) = Ker\varphi \leq_e M$ implies $\varphi = 0$.*

Example 3.17 *All semisimple modules are obviously Baer and so K-nonsingular modules, as are all Baer rings viewed as right modules over themselves. $\mathbb{Z}^n$ is a Baer $\mathbb{Z}$-module, for all $n \in N$.*

Lemma 3.18 *A module M is K-nonsingular if and only if for all $I \leq {}_S S$, $r_M(I) \leq_e M$ implies $I = 0$.*

Proof. For the necessity, assume the module M is K-nonsingular. Take $I \leq {}_S S$ so that $r_M(I) \leq_e M$. Let $\varphi \in I$. Then $r_M(I) = \bigcap_{\psi \in I} Ker(\psi) \subseteq Ker(\varphi)$, hence $Ker(\varphi) \leq_e M$ and so $\varphi = 0$. Since φ was arbitrarily chosen, it implies that $I = 0$. Conversely, let $\varphi \in S = End(M)$, with $Ker(\varphi) \leq_e M$. But $Ker(\varphi) = r_M(S\varphi)$, hence $S\varphi = 0$. This implies $\varphi = 0$.□

Recall that a ring R is said to be cononsingular [16] if $\forall\ I \leq R_R$, $rI \neq 0$, $\forall 0 \neq r \in R \Rightarrow I \leq_e R_R$. A module theoretic version for this concept is like as follows.

Definition 3.19 *A module M is called K-cononsingular if, for all $N \leq M, l_S(N) = 0$ implies $N \leq_e M$ (equivalently, $\varphi(N) \neq 0$ for all $0 \neq \varphi \in S = End(M)$ implies $N \leq_e M$).*

Example 3.20 *Uniform modules are K-cononsingular.*

Proposition 3.21 *Let M be an R-module.*

(i) *M is K-nonsingular if and only if, for all* $I \leq_S S$, $r_M(I) \leq_e eM$ *for* $e^2 = e \in S = End(M)$*, implies* $I \cap Se = 0$*;*

(ii) *M is K-cononsingular if and only if, for all* $N \leq M, r_M(l_S(N)) \leq_d M$ *implies* $N \leq_e r_M(l_S(N))$.

Proof.

(i) Let $I \leq S$ so that $r_M(I) \leq_e eM$. Then $r_M(I \cap Se) = r_M(I) \oplus (1-e)M \leq_e M$. By K-nonsingularity of M; $I \cap Se = 0$. Conversely, let $I \leq_S S$ such that $r_M(I) \leq_e M$. Then, by hypothesis, we have that $I \cap S = 0$, thus $I = 0$.

(ii) Let $r_M(l_S(N)) = eM$ for $e^2 = e \in S = End(M)$ implies $l_S(N) \subseteq S(1-e)$. Since, by Lemma 3.10 $N \leq r_M(l_S(N)) = eM$ we obtain that $l_S(N \oplus (1-e)M) = 0$. By K-cononsingularity, $N \oplus (1-e)M \leq_e M = eM \oplus (1-e)M$ implies $N \leq_e eM = r_M(l_S(N))$. Conversely, let $N \leq M$ with $l_S(N) = 0$ implies $r_M(l_S(N)) = M$. Then $N \leq_e r_M(l_S(N)) = M$.

□

Remark 3.22 *When* $M = R$ *the Definitions 3.16 and 3.19 coincide with the usual concepts of nonsingularity and cononsingularity, respectively. On the other hand, in the general case a K-nonsingular module is not nonsingular, as the following example shows. It can be seen that every Baer module is K-nonsingular and every extending module is K-cononsingular. K-nonsingularity is a weaker form of nonsingularity, as shown below.*

Proposition 3.23 *Every nonsingular module is K-nonsingular.*

Proof. Assume M is not K-nonsingular; hence $\exists\, 0 \neq \varphi \in S = End(M)$ so that $Ker(\varphi) \leq_e M$. Since $0 \neq \varphi$, $\exists\, 0 \neq m \in M \backslash Ker(\varphi)$. The set $I = \{r \in R \mid mr \in Ker(\varphi)\}$

is a right ideal in R. In fact, $I \leq_e R : r \notin I \Rightarrow mr \notin Ker(\varphi) \Rightarrow \exists\, r'$ so that $0 \neq mrr' \in Ker(\varphi) \Rightarrow 0 \neq rr' \in I$. But for $0 \neq \varphi(m), \varphi(m)I = 0$, that contradictions with the nonsingularity of M.□

Example 3.24 *The $\mathbb{Z}$-module $\mathbb{Z}_p$, where p is prime, is K-nonsingular (it is a simple module, hence all nonzero endomorphisms are automorphisms); however, the module $\mathbb{Z}_p$ is not nonsingular in fact, for all $\overline{x} \in \mathbb{Z}_p$, $\overline{x}.p\mathbb{Z} = 0$ ($p\mathbb{Z} \in Ker(\overline{x}) \neq 0$), and $p\mathbb{Z} \leq_e \mathbb{Z}$.*

Lemma 3.25 *An extending module M is K-cononsingular.*

Proof. Let $N \leq M$ so that $\varphi(N) \neq 0, \forall 0 \neq \varphi \in S = End(M)$. If $N \nleq_e M$, by extending property we have $N \leq_e eM$, for some idempotent $e \in S = End(M)$, such that $e \neq 1$. Hence $(1-e) \neq 0$; but $(1-e)N \subseteq (1-e)eM = 0$, thus getting a contradiction. Hence, M is K-cononsingular.□

Lemma 3.26 *A K-nonsingular extending module M is a Baer module.*

Proof. Assume that M is a K-nonsingular extending module. Let $N \leq M$. By the extending property, there exists $e^2 = e \in S = End(M)$ so that $N \leq_e eM$. Hence $l_S(N) \supseteq l_S(eM) = S(1-e)$. Assume that the inclusion is strict; then there exists $\varphi \in l_S(N) \backslash S(1-e)$. Since $S = Se \oplus S(1-e)$ (as a left S-module) we have that $\varphi = s_1 e + s_2(1-e)$ for some $s_1, s_2 \in S = End(M)$ with $s_1 \neq 0$; replacing φ with $\varphi - s_2(1-e) \in l_S(N)$,we can safely assume φ is in Se. We obtain that $\varphi(N) = 0$ and $\varphi((1-e)M) = 0$ and so $\varphi(N \oplus (1-e)M) = 0$. But $N \oplus (1-e)M \leq_e M$, hence K-nonsingularity of M yields that $\varphi = 0$ which contradicts our hypothesis. Therefore $l_S(N) = S(1-e)$, and so M is Baer.□

Lemma 3.27 *A Baer module M is K-nonsingular.*

Proof. Let M be Baer. Let $\varphi \in S = End(M)$ be any endomorphism of M with $Ker\varphi \leq_e M$. Since M is Baer, $Ker\varphi = r_M(S\varphi) = fM$ for some $f^2 = f \in S = End(M)$. Being a summand and an essential submodule in M implies that $Ker\varphi = M$. Thus $\varphi = 0$. This proves that M is K-nonsingular.□

Lemma 3.28 *A K-cononsingular Baer module M is an extending module.*

Proof. Assume M be K-cononsingular and Baer. From Lemma 3.27 it follows that M is also K-nonsingular. To prove that M is extending, let $N \leq M$ then $l_S(N) = Sf$ for $f^2 = f \in S = End(M)$. Hence $N \subseteq r_M(l_S(N)) = (1-f)M$. Assume that $N \nleq_e (1-f)M$. Hence there exists $P \leq (1-f)M$ so that $N \cap P = 0$. Take $\overline{N} \supset N$ a complement of P in M. Note that $l_S(\overline{N}) \neq 0$ by K-cononsingularity, clearly, $\overline{N} \nleq_e M$. Let $0 \neq s \in S = End(M), s\overline{N} = 0$.Then $sN = 0$ and since $l_S(N) = Sf$, hence $s(1-f) = 0$, $s((1-f)M) = 0$. It follows that $sP = 0$, and so $s(\overline{N} \oplus P) = 0$. But $P \oplus \overline{N} \leq_e M$, hence, by K-nonsingular, $s = 0$, it is a contradiction. Thus M is extending module.□

Theorem 3.29 *A module M is extending and K-nonsingular if and only if M is Baer and K-cononsingular.*

Proof.

(⇒:) Since M is extending and K-nonsingular by Lemma 3.25 and Lemma 3.26 it is K-cononsingular and Baer.

(⇐:) Since M is M is Baer and K-cononsingular by Lemma 3.27 and Lemma 3.28 it is K-nonsingular and extending.

□

Theorem 3.30 *Let M be a Baer module. Then every direct summand N of M is also a Baer module.*

Proof. Let $M = N \oplus P$. Let $S' = End_R(N)$. Then, for any $\varphi' \in S'$ there exists a $\varphi \in S$, defined as $\varphi = \varphi' \oplus 0_{|P}$. Take $I' \leq {}_{S'}S'$; let $I = \{\varphi \mid \varphi = \varphi' \oplus 0_{|P}, \varphi' \in I'\}$. I is not necessarily a left ideal of S=End(M), so consider $\overline{I} = SI$, the left ideal of S=End(M) generated by the set I. We observe that $\forall\ \varphi \in \overline{I}$, $\varphi(P) = 0$, since $\varphi = \sum_{i \in F} s_i(\varphi'_i \oplus 0_{|P})$ and $s_i(\varphi'_i \oplus 0_{|P})(P) = s_i(0) = 0$, $s_i \in S = End(M)$, $\varphi'_i \in I'$(where F is a finite index set). Since M is Baer module, $r_M(\overline{I}) \leq_d M$, and so there exists $Q \leq_d M$ so that $r_M(\overline{I}) \oplus Q =$

M. Also, since $P \subseteq r_M(\bar{I})$, there exists $L \subseteq r_M(\bar{I})$ so that $r_M(\bar{I}) = P \oplus L$, thus $L \leq_d M$. So $M = r_M(\bar{I}) \oplus Q = Q \oplus L \oplus P$. Set π_N to be the projection of M onto N; then we can see that $\pi_N|_{Q\oplus L}; Q \oplus L \to N$ is an isomorphism (its kernel is $P \cap (Q \oplus L) = 0$) and we obtain a decomposition $N = \pi_N(Q) \oplus \pi_N(L)$. It will be shown that $r_N(I') = \pi_N(L)$.
Let $\varphi' \in I'$ then $\varphi' \oplus 0_{|P} \in I$, $r_M(I) \subseteq r_M(\bar{I})$ and so $(\varphi' \oplus 0_{|P})(P \oplus L) = 0$. It implies that $(\varphi' \oplus 0_{|P})(L) = 0$. But every element $l \in L$ can be written as $l = \pi_N(l) + \pi_P(l)$, since $\pi_P(l)$ is annihilated by $\varphi' \oplus 0_{|P}$, so $\varphi' \oplus 0_{|P}(\pi_N(l)) = 0 \Rightarrow \varphi' \oplus 0_{|P}(\pi_N(L)) = 0$. Hence $\varphi'(\pi_N(l)) = 0$, and since $\varphi' \in I'$ was arbitrarily chosen, $\pi_N(L) \subseteq r_N(I')$.
Next, let $n \in N \setminus \pi_N(L)$. Then $n = n_1 + n_2$ for some $n_1 \in \pi_N(L)$ and some $0 \neq n_2 \in \pi_N(Q)$. Since $\pi_N|_{Q\oplus L}$ is an isomorphism, there exists $\overline{n_2} \in Q$ so that $\pi_N(\overline{n_2}) = n_2$. Since $r_M(I) \subseteq r_M(\bar{I})$ then $Q \cap r_M(\bar{I}) = 0$, hence there exists $\varphi \in \bar{I}$ so that $\varphi(\overline{n_2}) \neq 0$. But as $\varphi = \sum_{i\in F} s_i(\varphi'_i \oplus 0_{|P})$ there exists $s_{i_0}(\varphi'_{i_0} \oplus 0_{|P})(\overline{n_2}) \neq 0$, and hence $(\varphi'_{i_0} \oplus 0_{|P})(\overline{n_2}) \neq 0$. Decomposing $\bar{n}_2$ into $\pi_N(\overline{n_2}) + \pi_P(\overline{n_2})$ we get that $\varphi'_{i_0}(\pi_N(\overline{n_2})) \neq 0$ (as $\pi_P(\overline{n_2})$ gets mapped into 0) $\Leftrightarrow \varphi'_{i_0}(n_2) \neq 0$. Hence $\pi_N(L) = r_N(I')$. Since $r_N(I')$ is clearly a summand of N, N is a Baer module.□

Example 3.31 *Let R be a Baer ring, and let $e^2 = e \in R$ be any idempotent of R. Then $M = eR$ is an R-module which is Baer.*

As an application of above results, the Baer modules can be characterized in the class of finitely generated $\mathbb{Z}$-modules.

Theorem 3.32 [19, $Theorem 8.4$] *A torsion group A is the direct sum of p-groups A_p belonging to different primes p. The A_p are uniquely determined by A.*

Proof. Let A_p consist of all $a \in A$ whose order is a power of the prime p. In view of $0 \in A_p$, A_p is not empty. If $a, b \in A_p$, that is, if $p^m a = p^n b = 0$ for some integers $m, n \geq 0$, then $p^{max(m,n)}(a - b) = 0$, $(a - b) \in A_p$, and A_p is a subgroup. Every element in $A_{p_1} + ... + A_{p_k}$ is annihilated by a product of a power of $p_1, ..., p_k$, therefore,

$$A_p \cap (A_{p_1} + ... + A_{p_k}) = 0$$

whenever $p \neq p_1, ..., p_k$. Thus the A_p generate their direct sum $\bigoplus_p A_p \in A$. In order to show that every $a \in A$ lies in the direct sum, let $\circ(a) = m = p_1^{r_1}...p_n^{r_n}$ with different primes p_i. The numbers $m_i = mp_i^{-r_i}$ ($i = 1, ..., n$) are relatively prime, hence there are integers, $s_1, ..., s_n$ such that $s_1 m_1 + ... + s_n m_n = 1$. Thus $a = s_1 m_1 a + ... + s_n m_n a$ where $m_i a \in A_{p_i}$ (in view of $p_i^{r_i} m_i a = ma = 0$), and so $a \in A_{p_1} + ... + A_{p_n} \leq \bigoplus_p A_p$. If $A = \bigoplus_p A_p$ is any direct decomposition of A into p-groups B_p with different primes p, then by the definition of A_p, we have $B_p \leq A_p$ for every p. Since the B_p and A_p generate direct sums which are both equal to A we must have $B_p = A_p$ for every p. □

Proposition 3.33 *A finitely generated $\mathbb{Z}$-module M is Baer if and only if M is semisimple or torsion-free.*

Proof.

(⇐:)If M is semisimple then M is obviously Baer. If M is finitely generated and torsion-free, $M \cong \mathbb{Z}^n$, where $n \in \mathbb{N}$; $\mathbb{Z}^n$ is extending and nonsingular, hence by Theorem 3.29 it is Baer.

(⇒:) Assume now M is finitely generated Baer module. We can always decompose $M = t(M) \oplus f(M)$, where $t(M)$ is torsion submodule of M and $f(M)$ is torsion-free submodule of M. Assume $t(M) \neq 0$ and $f(M) \neq 0$; by structure [19, Theorem 8.4], $t(M) \cong \oplus_{p \in P} \mathbb{Z}_{p^{n(p)}}$, where $P \subseteq \mathbb{Z}$ is a finite set of primes; $n(p) \in \mathbb{N}$, for all $p \in P$. Also, $f(M) \cong \mathbb{Z}^n$, $0 \neq n \in \mathbb{N}$. Let p_0 be a prime so that $n(p_0) \neq 0$ (such a prime must exist), and let $\varphi : \mathbb{Z} \to \mathbb{Z}_{p_0^{n(p_0)}}$ be the morphism defined by $\varphi(x) = \overline{x}$, for $x \in \mathbb{Z}$. $Ker(\varphi)$ is a proper submodule of $\mathbb{Z}$, hence it is essential in $\mathbb{Z}$. Extend φ to $\overline{\varphi}$, an endomorphism of M, where $\overline{\varphi} = \varphi(\pi_{p_0})$, π_{p_0} being the canonical projection of M onto $\mathbb{Z}$. The kernel $Ker(\overline{\varphi}) \leq_e M$, but$Ker(\overline{\varphi}) \neq M$, hence M is not Baer, a contradiction. Hence either $t(M) = 0$ of $f(M) = 0$.

Suppose $f(M) = 0$. Then $M = t(M)$; it is finite direct sum of $\mathbb{Z}_{p^{n(p)}}$, where p is prime and $n(p) \in \mathbb{N}$. Therefore $\mathbb{Z}_{p^{n(p)}}$ must be Baer module, by Theorem 3.30. Assume there exists prime p such that $n(p) > 1$; for such $\mathbb{Z}_{p^{n(p)}}$, we set $\varphi(x) = px : \mathbb{Z}_{p^{n(p)}} \to \mathbb{Z}_{p^{n(p)}}$. The morphism φ is not 0 ($p.1 = p \neq 0$ modulo p^{n_p}, where $n(p) > 1$); $Ker(\varphi) \neq 0$

$(p.p^{n(p)-1} = p^{n(p)} = 0)$, and since $\mathbb{Z}_{p^{n(p)}}$ is uniform, $Ker(\varphi)$ cannot be a summand. Thus $\mathbb{Z}_{p^{n(p)}}$ is not a Baer module, a contradiction. Hence $t(M) = \oplus_P \mathbb{Z}_p$, with $P \subseteq \mathbb{Z}$ a finite collection of primes (possibly in multiple instances).

Finally suppose that $t(M) = 0$; then $M = f(M) \cong \mathbb{Z}^n$ which we already know is a torsion-free module.□

Remark 3.34 *The statement of Proposition 3.33 holds true for any finitely generated module over a Principal Integral Domain.*

Definition 3.35 *A module M is said to have the summand intersection property (SIP) if the intersection of any two direct summands of M is a direct summand. A module is said to have generalized summand intersection property (GSIP) if the intersection of any family of direct summands of M is a direct summand.*

Theorem 3.36 [22] *M has the SIP if and only if for every decomposition $M = A \oplus B$ and every R-homomorphism f from A to B, the kernel of f is a direct summand of M.*

Proof. Assume M is a module with SIP. Let $M = A \oplus B$ and f an R-homomorphism from A to B. Let $T = \{a + f(a) \mid a \in A\}$. To show that $M = T \oplus B$, let $x \in M$, then $x = a + b$ where $a \in A$ and $b \in B$. Now, $x = a + f(a) - f(a) + b$. But $a + f(a) \in T$ and $-f(a) + b \in B$, so $M = T + B$. Now, let $x \in T \cap B$. Hence $x = a + f(a)$, where $a \in A$ and so $a = x - f(a) \in A \cap B = 0$. Therefore, $f(a) = 0$. Thus, $x = 0$. Since M has SIP, then $T \cap A$ is a direct summand in M. It is easy to show that $T \cap A = Ker(f)$, so $Ker(f)$ is a direct summand in M.

The converse, assume that for every decomposition $M = A \oplus B$ and every R-homomorphism from A to B, the kernel of f is a direct summand of M. Let $M = N \oplus N_1$, $M = K \oplus K_1$ and let $\pi_{N_1} : M \to N_1$ and $\pi_K : M \to K$ be the natural epimorphisms. Now, define $h = (\pi_{N_1} \circ \pi_K) |_N$. Note that h is defined from $N \to N_1$. Thus, $Ker(h)$ is a direct summand of M. It is easy to check that $Ker(h) = (N \cap K) \oplus (N \cap K_1)$. Since $N \cap K$ is a direct summand of $Ker(h)$ and $Ker(h)$ is a direct summand of M, then $N \cap K$ is a direct summand of M.□

Example 3.37 *There are modules with the SIP such that their direct sum need not have the SIP.*

Proof. Consider $\mathbb{Z}_4$ and $\mathbb{Z}_2$ as $\mathbb{Z}$- modules. It is clear that each of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ is indecomposable and hence, has SIP. Define f from $\mathbb{Z}_4$ to $\mathbb{Z}_2$ by $f(\overline{x}) = \overline{x}$. Then $Ker(f) = \{\overline{0}, \overline{2}\}$ is not a direct summand of $\mathbb{Z}_4$. By theorem 3.36, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ does not have SIP. □

The characterization of Baer modules in terms of GSIP is like as follows.

Proposition 3.38 *A module M is Baer if and only if M has the generalized summand intersection property and $Ker(\varphi) \leq_d M, \forall \varphi \in S = End(M)$.*

Proof. Let M be a Baer module. Then it is clear that $Ker(\varphi) \leq_d M$ for all $\varphi \in S$. Take $e_i^2 = e_i \in S = End(M)$, $i \in F$ (for an index set F) and let $I = \sum_{i \in F} S(1 - e_i)$. Then $Ker((1 - e_i)) \supseteq r_M(I)$ $\forall i \in F$ (elements of M that annihilate all morphisms in I must annihilate, in particular, $(1 - e_i)$). Let $N = \bigcap_{i \in F} Ker((1 - e_i))$; then $r_M(I) \subseteq N$. Then for any morphism $\sum_{i \in F} s_i(1 - e_i) \in I$, we have that $(\sum_{i \in F} s_i(1 - e_i))(N) = 0$ (where $s_i = 0$ for all but finite number of $i \in F$), and so we obtain $r_M(I) = N$. This yields $\bigcap_{i \in F} e_i M = N = r_M(I) \leq_d M$, since M is Baer. Therefore M satisfies GSIP.
Conversely, for each $\varphi \in I$, where $I \leq_S S$, we get that $Ker(\varphi) \leq_d M$. Also, $r_M(I) = \bigcap_{\varphi \in I} Ker(\varphi) \leq_d M$, by the GSIP. Hence we get that M is Baer.□

Theorem 3.39 *Alternative proof of Theorem 3.30 can be done by using the property that M is Baer if and only if M has the generalized summand intersection property and $Ker(\varphi) \leq_d M, \forall \varphi \in S = End(M)$.*

Proof. By previous proposition M has GSIP and $Ker(\varphi) \leq_d M$, $\forall \varphi \in S = End(M)$. Since for all $P \leq_d N$, where $N \leq_d M$, $P \leq_d M, N$ obviously has the GSIP. Taking now $\psi \in End(N)$, we can extend ψ to an endomorphism of M, by taking $\overline{\psi} = \psi \pi_N : M \to N \subseteq M$, where π_N is the canonical projection onto N. $Ker\overline{\psi} \leq_d M$, but $Ker\overline{\psi} = N' \oplus Ker\psi$ as it is easily checked (for $M = N \oplus N'$). This implies that $Ker\psi \leq_d N$ (by using GSIP). In conclusion, N has GSIP and since $\psi \in End(N)$ was arbitrarily chosen then $Ker\psi \leq_d N$, $\forall \psi \in End(N)$ then N is Baer module. □

Theorem 3.40 *M is an indecomposable Baer module if and only if $\forall 0 \neq \varphi \in End(M)$, φ is a monomorphism.*

Proof. Let M be indecomposable and $0 \neq \varphi \in End(M)$. M being Baer, $Ker(\varphi) \leq_d M$, hence $Ker(\varphi) = 0$ or $Ker(\varphi) = M$. As $\varphi \neq 0$ it follows that φ is a monomorphism. Conversely, assume that M is not indecomposable, hence $M = M_1 \oplus M_2$ with $M_1, M_2 \neq 0$. Take $\varphi = \pi_1$ the canonical projection of M onto M_1; $Ker(\varphi) = M_2 \neq 0$, a contradiction. Baer condition for M follows obviously.□

In general, a direct sum of Baer modules is not a Baer module, as the following example shows.

Example 3.41 *$\mathbb{Z}$ and $\mathbb{Z}_2$ are Baer $\mathbb{Z}$-modules ($\mathbb{Z}$ is domain; $\mathbb{Z}_2$ is simple). By Theorem 3.38, $\mathbb{Z} \oplus \mathbb{Z}_2$ is not a Baer module (in fact, for the endomorphism $\varphi(n, \hat{m}) = \hat{n}$ we obtain that $Ker(\varphi) = 2\mathbb{Z} \oplus \mathbb{Z}_2$, which is not a direct summand of $\mathbb{Z} \oplus Z_2$).*

Proposition 3.42 *Let $\{M_i \mid i \in F\}$ be a class of Baer modules, for an index set F. If $\bigoplus_{i \in F} M_i$ is a Baer module, then the class $\{M_i\}$ satisfies the following:*

- **(i)** *Relatively Baer Condition: for any pair (i_0, j_0), $i_0 \neq j_0 \in F$, and any $\psi \in Hom(M_{j_0}, M_{i_0})$, $Ker(\psi) \leq_d M_{j_0}$.*

- **(ii)** *$\forall i_0 \neq j_0 \in F$, for all monomorphisms $\varphi : M'_{i_0} \leq_d M_{i_0} \to M_{i_0}$ and $\psi : M'_{j_0} \leq_d M_{j_0} \to M_{i_0}$ the set $A = \{(\varphi^{-1}(a), -\psi^{-1}(a)) \mid a \in Im(\varphi) \cap Im(\psi)\}$ is a direct summand of $M'_{i_0} \oplus M'_{j_0}$.*

Proof. The elements of the endomorphism ring of $\bigoplus M_i$ are matrices, for which the (i, j) entries are morphisms $M_j \to M_i$. Since $\bigoplus M_i$ is Baer, the kernel of every endomorphism is a direct summand.

To show (i), take the endomorphism $(\varphi_{ij})_{i,j \in F}$, with

(1) $\varphi_{ij} = 0$, $\forall i \neq i_0$ and $j \neq j_0$;

(2) $\varphi_{i_0, j_0} = \psi$.

$Ker((\varphi_{ij})) = (\bigoplus_{j \in (F \setminus j_0)} M_k) \oplus Ker(\psi)$, as it is easily checked. This must be summand then $Ker(\psi) \leq_d M_{j_0}$.

To prove (*ii*), observe that as φ is defined on a summand of M_{i_0}, it can be extended to the whole M_{i_0}, by setting it equal to 0 on the other component; similarly with ψ. To simplify notation, we use the same symbols for these new morphism. Take the endomorphism $(\alpha_{ij})_{i,j\in F}$, with:

(1) $\alpha_{ij} = 0, \forall (i, j) \neq (i_0, j_0), (i_0, i_0)$;

(2) $\alpha_{i_0 i_0} = \varphi$;

(3) $\alpha_{i_0 j_0} = \psi$.

$Ker((\alpha_{ij})) = K = \{(b, c) \mid \varphi(b) + \psi(c) = 0\}$. Notice that $Ker(\varphi) \oplus Ker(\psi) \subseteq K$. Moreover, since both the kernels of φ and ψ are direct summands, we have $M_{i_0} = Ker(\varphi) \oplus M'_{i_0}$ and $M_{j_0} = Ker(\psi) \oplus M'_{j_0}$. Note that φ is mono on M'_{i_0} and ψ is mono on M'_{j_0}. We have $\varphi(b) + \psi(c) = 0$ only if $\varphi(b) = -\psi(c) \in Im(\varphi) \cap Im(\psi)$. For $(b, c) \in (M'_{i_0} \oplus M'_{j_0}) \cap K$, we get $(b, c) \in \{(\varphi|^{-1}_{M'_{i_0}}(a), -\psi|^{-1}_{M'_{j_0}}(a)), a \in Im(\varphi) \cap Im(\psi)\} = A$. $(Ker(\varphi) \oplus Ker(\psi)) \cap A = \{(0, 0)\}$, obviously. Given the fact that any pair $(b, c) \in K$ can be written uniquely as $(b, c) = (b', c') + (b'', c'')$ with $(b', c') \in Ker(\varphi) \oplus Ker(\psi)$ and $(b'', c'') \in (M'_{i_0} \oplus M'_{j_0})$, we have that $K = Ker(\varphi) \oplus Ker(\psi) \oplus A$. Now, K must be a summand of $M_{i_0} \oplus M_{j_0}$; hence $A \leq_d M'_{i_0} \oplus M'_{j_0}$.□

Proposition 3.43 *Let M be a K-nonsingular module, and let $N \leq_d M$. Then N is K-nonsingular.*

Proof. Let $\varphi : N \to N$ so that $Ker(\varphi) \leq_e N$. Extend this morphism to $\overline{\varphi} = \varphi \oplus 0_{N'}$ on the module $M = N \oplus N'$. $Ker(\overline{\varphi}) = Ker(\varphi) \oplus N' \leq_e N \oplus N' = M$, and since M is K-nonsingular, $\overline{\varphi} = 0$ hence $\varphi = 0$. Since φ was arbitrarily chosen, it implies that N is K-nonsingular.□

It is well-known that if R is right nonsingular, the essential closure of a submodule is unique.

A similar result for K-nonsingularity is like as follows.

Proposition 3.44 *Let M be a K-nonsingular module. If X is essential in a summand N of M, then N is unique.*

Proof. Assume $X \leq_e N_i \leq_d M$, with $N_i \oplus P_i = M$, $i = 1, 2$; assume $N_1 \neq N_2$ (which implies $N_1 \nsubseteq N_2$ and $N_2 \nsubseteq N_1$). Take $\varphi = \pi_{P_2}(\pi_{N_1})$; $\varphi \neq 0$, since there exists $x \in N_1 \setminus N_2$ which will have a nonzero projection onto P_2. Take $y \in M$; $y = n_1 + p_1$; since $X \leq_e N_1$ there exists $r \in R$ so that $0 \neq n_1 r \in X$; $yr \neq 0$, since $n_1 r \neq 0$; $\varphi(yr) = \pi_{P_2}(\pi_{N_1}(n_1 r + p_1 r)) = \pi_{P_2}(n_1 r) = 0$, since $n_1 r \in X \subseteq N_2$. Hence, since y was arbitrarily chosen, we get that $Ker(\varphi) \leq_e M$. But M is K-nonsingular, so $\varphi = 0$, a contradiction. Hence, the summand in which X is essential is unique.□

Proposition 3.45 *Let M be a K-nonsingular module and $X \trianglelefteq M$. Let $N \leq_d M$ with $X \leq_e N$. Then $N \trianglelefteq M$.*

Proof. We have $M = N \oplus P$. Let $\varphi \in S = End(M)$. Assume $\varphi(N) \nsubseteq N$. Take $\psi = \pi_P(\varphi(\pi_N))$, with π_N and π_P the respective canonical projections. Let $x \in N$ so that $\varphi(x) \notin N$, hence $\psi(x) \neq 0$. But $X \oplus P \subseteq Ker(\psi)$ (since all elements from P are sent to 0 through π_N, while elements from $X \subseteq N$ are sent into $X \subseteq N$ through $\varphi(\pi_N)$) and $X \oplus P \leq_e M$. This is a contradiction, since M is K-nonsingular. Hence $\varphi(N) \subseteq N$, and since φ was arbitrarily chosen, $N \trianglelefteq M$. □

Recall that a module M is called an FI-extending module if, for every $N \trianglelefteq M$, there exists $e^2 = e \in S = End(M)$ so that $N \leq_e eM$. A large class of modules and rings are FI-extending, but not necessarily extending (for example, direct sums of uniform modules, ring of upper-triangular matrices over $\mathbb{Z}$).

Proposition 3.46 [4] *Let M be a module and X is a fully invariant submodule of M. If M is FI-extending, then X is FI-extending.*

Proof. Assume M is a FI-extending module. Let S be fully invariant submodule of X. Since $S \trianglelefteq X \trianglelefteq M$, then $S \trianglelefteq M$ (In fact, let $\varphi \in End(M)$. Then $\varphi(X) \subseteq X$, and also $\varphi|_X : X \to X$ since $S \trianglelefteq X$, $(\varphi|_X)(S) \subseteq S$. Thus, $\varphi(S) \subseteq S$). Since M is FI-extending and $S \trianglelefteq M$, there is a direct summand D of M such that $S \leq_e D$. Let $\pi : M \to D$ be the projection endomorphism. Then

$$S = \pi(S) \leq \pi(X) \cap D = \pi(X)$$

Hence, $S \leq_e \pi(X)$ and $\pi(X)$ is a direct summand of X ($X \trianglelefteq M = D \oplus P$ for some $P \leq M$ implies $X = (D \cap X) \oplus (P \cap X) = (D \cap \pi(X)) \oplus (P \cap X) = \pi(X) \oplus (P \cap X)$).□

Theorem 3.47 *Let $M = \bigoplus_{i \in I} X_i$ if each X_i is an FI-extending module, then M is FI-extending module.*

Proof. Assume each X_i is an FI-extending module, and S is a fully invariant submodule of M. Since, for each i such that $\pi_i(S) \neq 0$, $\pi_i(S)$ is a fully invariant submodule of X_i, there exists D_i, a direct summand of X_i, such that $\pi_i(S) \leq_e D_i$. $S = \bigoplus \pi_i(S) \leq_e \bigoplus D_i$. Since $\bigoplus D_i$ is a direct summand of M, we have that M is an FI-extending module.□

Corollary 3.48 *If M is direct sum of extending modules, then M is FI-extending.*

Proposition 3.49 *Let M be K-nonsingular module, then the following conditions are equivalent:*

(i) *M is FI-extending;*

(ii) *M is strongly FI-extending (M is called a strongly FI-extending module if every fully invariant submodule of M is essential in a fully invariant summand of M [4])*

Proof.
($i \Rightarrow ii$) Let $X \trianglelefteq M$. Since M is FI-extending, $X \leq_e N \leq_d M$. But by the proposition 3.45, this summand is fully invariant (and also it is unique). Hence M is strongly FI-extending.
($ii \Rightarrow i$) is obvious.□

3.3 Quasi-Baer Modules

Definition 3.50 *A right R-module M is called a quasi-Baer module if for all $N \trianglelefteq M$, $l_S(N) = Se$, with $e^2 = e \in S = End(M)$ (or, equivalently, $\forall J \trianglelefteq S, r_M(J) = fM$ for $f^2 = f \in S = End(M)$).*

Example 3.51 *All semisimple modules are quasi-Baer; all Baer and quasi-Baer rings are quasi-Baer modules, viewed as modules over themselves. The Baer modules are obviously quasi-Baer modules. The finitely generated abelian groups are also quasi-Baer.*

Theorem 3.52 $[2, Proposition 4.4]$ *Let R be right nonsingular. Then R is right FI-extending if and only if R is quasi-Baer and $A \leq_e r(l(A))$, for $A \trianglelefteq R$.*

Proof. Assume R is right FI-extending and let $A \trianglelefteq R$. Then there exists $e = e^2$ such that $A_R \leq_e eR$. Since R is right nonsingular, $l(A) = l(eR) = R(1 - e)$ hence R is quasi-Baer. Moreover, $A_R \leq_e eR = r(l(eR)) = r(l(A))$ so $A_R \leq_e r(l(A))$.
The converse is obvious.□

As in the Baer case, we need to introduce a concept of nonsingularity, in this case taking in account not only the endomorphisms ring but also the fully invariant submodules of M.

Definition 3.53 *A module M is called FI-K-nonsingular if, for any $I \trianglelefteq S$ so that $r_M(I) \leq_e eM$ for $e^2 = e \in S = End(M)$,$r_M(I) = eM$.*

Definition 3.54 *A module M is called FI-K-cononsingular if, for every $N \trianglelefteq_d M$ and $N' \trianglelefteq N$ so that $\varphi(N') \neq 0, \forall \varphi \in End(N)$, we get that $N' \leq_e N$.*

Proposition 3.55 *Let M be an R-module.*

(i) *M is FI-K-nonsingular if and only if, for all $I \trianglelefteq S$, $r_M(I) \leq_e eM$ for $e^2 = e \in S = End(M)$, implies $I \cap Se = 0$.*

(ii) *M is FI-K-cononsingular if and only if, for all $N \trianglelefteq M$, $r_M(l_S(N)) \leq_d M$ implies $N \leq_e r_M(l_S(N))$.*

Proof. The proof follows on the same line as that of Proposition 3.21.□

The above definitions indeed generalize the notions of K-nonsingularity and K-cononsingularity, respectively.

Corollary 3.56 *We have the following implications:*

(i) *If M is K-nonsingular, then M is FI-K-nonsingular.*

(ii) *If M is K-cononsingular, then M is FI-K-cononsingular.*

Proof. The proof follows from Proposition 3.21 and Proposition 3.55.□

Any semiprime ring R which is not right nonsingular has the property that R_R is FI-K-nonsingular but not K-nonsingular. Any module which is Baer, FI-extending but not extending has the property that it is FI-K-cononsingular but not K-cononsingular.

Lemma 3.57 *Let M be FI-extending. Then M is FI-K-cononsingular.*

Proof. Let $N \trianglelefteq_d M$. Then by Proposition 3.46, N is FI-extending. Take $N' \trianglelefteq N$ such that $\varphi(N') \neq 0$, $\forall \varphi \in End(N)$. By the FI-extending property $N' \leq_e \overline{N'} \leq_d N$. Assume $\overline{N'} \oplus N_2 = N$ for some $N_2 \leq_d N$ where $N_2 \neq 0$. Then let π_2 be the canonical projection of N onto N_2 has the property that $\pi_2(\overline{N'}) = 0$, contradiction. Hence $N_2 = 0$, hence $N' \leq_e N$.□

Lemma 3.58 *Let M be FI-K-nonsingular FI-extending module. Then M is quasi-Baer.*

Proof. Let $I \trianglelefteq S = End(M)$. We want to show that $r_M(I) \leq_d M$. We have that $r_M(I) \trianglelefteq M$, and by FI-extending property we get $r_M(I) \leq_e eM, e^2 = e \in S = End(M)$. By FI-K-nonsingularity we get that $r_M(I) = eM$.□

Lemma 3.59 *Let M be quasi-Baer. Then M is FI-K-nonsingular.*

Proof. Let $I \trianglelefteq S = End(M)$, with $r_M(I) \leq_e eM$, where $e^2 = e \in S$. Then by quasi-Baer property, $r_M(I) \leq_d M$. As $r_M(I) \subseteq eM$ it follows that $r_M(I) \leq_d eM$. Since it is also essential, $r_M(I) = eM$.□

Lemma 3.60 *Let M be FI-K-cononsingular quasi-Baer module. Then M is FI-extending.*

Proof. Let $N \trianglelefteq M$, and $l_S(N) = Se$ (by quasi-Baer property). Hence, $N \subseteq (1-e)M$. Moreover, since $N \trianglelefteq M$, $Se \trianglelefteq S$ hence $(1-e)M \trianglelefteq_d M$. Now let $\varphi \in End((1-e)M)$, thus $\varphi = (1-e)\varphi(1-e) \in S = End(M)$. Suppose $\varphi(N) = 0 \Rightarrow \varphi = (1-e)\varphi(1-e) \in l_S(N) = Se$. But then $(1-e)\varphi(1-e) \in [(1-e)S](1-e) \cap Se = 0$. So, by the FI-K-cononsingularity of M we get that $N \leq_e (1-e)M$, hence M is FI-extending.□

Theorem 3.61 *A module M is FI-extending and FI-K-nonsingular if and only if M is quasi-Baer and FI-K-cononsingular.*

Proof.

(⇒:) By Lemma 3.57 and Lemma 3.58.

(⇐:) By Lemma 3.59 and Lemma 3.60.

□

Remark 3.62 *In the proof of Lemma 3.60 we also get that* $(1-e)M \trianglelefteq M$ *(*$N \trianglelefteq M \Rightarrow Se = l_S(N) \trianglelefteq S \Rightarrow (1-e)M = r_M(l_S(N)) \trianglelefteq M$*), and so we obtain that M is, in fact, strongly FI-extending.*

Corollary 3.63 *A ring R is right FI-extending and right FI-K-nonsingular if and only if R is quasi-Baer and right FI-K-cononsingular.*

Theorem 3.64 *Let M be a quasi-Baer module. Then for any* $N \leq_d M$*, N is also a quasi-Baer module.*

Proof. Since $N \leq_d M$, there exists $e^2 = e \in S = End(M)$ so that $N = eM$, and let $F \trianglelefteq N$. Using Lemma 3.12, there exists $G \trianglelefteq (1-e)M$ so that $F \oplus G \trianglelefteq M$. Since M is quasi-Baer module, $I = l_S(F \oplus G) \trianglelefteq_d S$. The endomorphism ring of $N = eM$ is eSe, and since $I \trianglelefteq S$, $eIe = eSe \cap I$ (one inclusion is obvious, while the other one results from the following argument : $i \in I \cap eSe \Rightarrow i = ese = e^2se^2 = eie \in eIe$). At the same time, $I = Sf$ where $f^2 = f \in S = End(M)$, and so $eIe = eSfe$. But, since $Sf \trianglelefteq S$, $fe \in Sf \Rightarrow fe = fef$; we can write hence $eIe = eSfe = eSfef = eSfefe = (eSfe)(efe)$. Notice that $(efe)^2 = efeefe = efefe = efee = efe$; we have $(eSfe)(efe) \subseteq (eSe)(efe)$, but also the reverse: let $(ese)(efe) \in (eSe)(efe)$; $eseefe = esefe = esefef = esefefe = e((se)f)efe = e((se)f)eefe = (e((se)f)e)(efe) \in (eSef)(efe)$. Hence we have that $eIe \leq_d eSe$ (in fact, it is a fully invariant direct summand because efe is a semi-central idempotent in eSe: $(efe)(ese) = efese = efesef = efesefe = (efe)(ese)(efe)$).

We only have to show that $eIe = l_{eSe}(F)$. It is clear that $(eIe)(F) = 0$: $eie(F) = ei(F) = e(0) = 0$. Assume there exists $0 \neq eje \in eSe$, $eje \notin eIe$ so that $eje(F) = 0$. But $ejeG \subseteq eje(1-e)M = 0$, and so $eje \in l_S(F \oplus G) = I$. But then $eje = eejee = e(eje)e \in eIe$, a contradiction. Hence $l_{eSe}(F) = eIe \leq_d eSe$; F was arbitrarily chosen, hence N is quasi-Baer.□

Theorem 3.65 *Let M_1 and M_2 be quasi-Baer modules. If we have the property $\psi(x) = 0$ for all $\psi \in Hom(M_i, M_j)$ implies $x = 0$ $(i \neq j; i, j = 1, 2)$ then $M_1 \oplus M_2$ is quasi-Baer.*

Proof. Let $S = End(M_1 \oplus M_2)$, and let $I \trianglelefteq S$. Then $r_{M_1 \oplus M_2}(I) \trianglelefteq M_1 \oplus M_2$, hence, using Lemma 3.11, $r_{M_1 \oplus M_2}(I) = N_1 \oplus N_2$, where $N_i \trianglelefteq M_i$, $i = 1, 2$. As mentioned,

$$S = \begin{pmatrix} S_1 & Hom(M_2, M_1) \\ Hom(M_1, M_2) & S_2 \end{pmatrix}$$

Since $I \trianglelefteq S$ we have the following properties;

$$I_{11} = \{\varphi \in S_1 \mid \varphi = \xi_{11}; (\xi_{ij})_{i,j=1,2} \in I\} \trianglelefteq S_1$$

$$I_{22} = \{\varphi \in S_2 \mid \varphi = \xi_{22}; (\xi_{ij})_{i,j=1,2} \in I\} \trianglelefteq S_2$$

We also define $I_{12} = \{\psi \in Hom(M_2, M_1) \mid \psi = \xi_{12}; (\xi_{ij})_{i,j=1,2} \in I\}$ and $I_{21} = \{\psi \in Hom(M_1, M_2) \mid \psi = \xi_{21}; (\xi_{ij})_{i,j=1,2} \in I\}$.

Let $N_1' = r_{M_1}(I_1)$. We have that $N_1 = N_1' \cap (\bigcap_{\psi \in I_{12}} Ker\psi)$. Since M_1 is quasi-Baer, we know that $r_{M_1}(I_1) \leq_d M_1$. We also have that $\psi(N_1')$ satisfies $\chi(\psi(N_1')) = 0$ for all $\chi \in Hom(M_2, M_1)$, since $\chi(\psi) \in I_{11}$ for $\psi \in I_{12}$. Since we have the property that $\chi(x) = 0$ for all $\chi \in Hom(M_2, M_1)$ implies $x = 0$ then we get that $\psi(N_1') = 0$ for all $\psi \in I_{12}$, and so $N_1 = N_1' \leq_d M_1$.

□

Proposition 3.66 *$M = \bigoplus_{i\in F} M_i$ is quasi-Baer if M_i is quasi-Baer and subisomorphic to (i.e., isomorphic to a submodule of) M_j, for all $i \neq j$; $i, j \in F$, where F is an index set.*

Proof. Let S_i be the endomorphism ring of M_i, for all $i \in F$. The endomorphism ring of M, S, is a ring of matrices, with elements of S_i in the ii-position, and maps $M_j \to M_i$ in ij-position, for all $i, j \in F$, $i \neq j$. We need to show, for all $I \trianglelefteq S$, $r_M(I) \leq_d M$. But since $r_M(I) \trianglelefteq M$, $r_M(I) = \bigoplus_{i\in F} r_M(I) \cap M_i$. We only have to analyze, hence, the column morphisms (i.e., matrices) taking M_i into M for an $i \in F$. Similar to our previous theorem, we have that the ith column of $I \trianglelefteq S$ has elements from an ideal $I_i \trianglelefteq S_i$ in the ith position, and certain elements from $Hom(M_i, M_j)$ in the remaining places (call the union of all these sets A). $r_M(I) \cap M_i = r_{M_i}(I_i) \cap (\bigcap_{\varphi\in A}(Ker(\varphi)))$. But $M_i' = r_{M_i}(I_i) \leq_d M_i$, since M_i is a quasi-Baer module. If we take a $\varphi \in A$, for example $\varphi : M_i \to M_j$, $i, j \in F$, $i \neq j$, then $\psi_{ji}\varphi \in I_i$, where $\psi_{ji} : M_j \to M_i$ is the monomorphism taking M_j into M_i; we obtain this by noting that if we multiply a morphism in I, having φ in the ji-position, with the morphism $(\chi_{kl})_{k,l\in F}$, $\chi_{kl} = 0$ for where $(k,l) \neq (i, j)$ and $\chi_{ij} = \psi_{ji}$, then we get a morphism in I with $\psi_{ji}\varphi : M_i \to M_i$ in the ii-position. This means that $\psi_{ji}\varphi(M_i') = 0$; as ψ_{ji} is a monomorphism, hence $\varphi(M_i') = 0$, thus $M_i' \subseteq Ker(\varphi)$. Since $\varphi \in A$ was arbitrarily chosen, $r_M(I) \cap M_i = r_{M_i}(I_i) \cap (\bigcap_{\varphi\in A}(Ker(\varphi))) = M_i' \leq_d M_i$. Using this argument for all $i \in F$ we obtain that $r_M(I) = \bigoplus_{i\in F} M_i' \leq_d \bigoplus_{i\in F} M_i = M$.

□

Corollary 3.67 *A free module over a quasi-Baer ring is a quasi-Baer module.*

Proof. Follows directly from the above result.

□

3.4 Endomorphism Rings

Theorem 3.68 *Let M be a Baer (respectively, quasi-Baer) module. Then $S = End(M)$ is Baer (respectively, quasi-Baer) ring.*

Proof. Let $I \leq S$ be a left (respectively, two-sided) ideal. Since M is Baer (respectively, quasi-Baer), $r_M(I) \leq_d M$, thus there exists $e^2 = e \in S$ such that $r_M(I) = eM$. We claim that $r_S(I) = eS$ also holds. For any $e\psi \in eS$, we observe that $Ie\psi = 0$, as for all $x \in M$, $Ie\psi(x) \subseteq IeM = 0$. Therefore $IeS = 0$, and $eS \subseteq r_S(I)$. Next, let $\varphi \in r_S(I)$ be any element; then we can write $\varphi = e\varphi + (1-e)\varphi$. Since $I\varphi = 0$, $I\varphi(M) = 0 \Rightarrow I(\varphi(M)) = 0$. Hence $\varphi(M) \subseteq r_M(I) = eM$. Let $m \in M$ be arbitrary; then $\varphi(m) = em'(m' \in M) \Rightarrow e\varphi(m) = em' = \varphi(m) \Rightarrow e\varphi = \varphi$. Hence $\varphi \in eS$ which yields $eS = r_S(I)$.□

Corollary 3.69 [4, *Proposition*4.8] *Let M be a K-nonsingular, FI-extending module. Then S=End(M) is a quasi-Baer ring.*

Proof. We know that if a module is FI-extending and FI-K-nonsingular, then it is quasi-Baer. Since K-nonsingularity implies FI-K-nonsingularity, we get that M is a quasi-Baer module. By Theorem 3.68, M is quasi-Baer module implies S=End(M) is quasi-Baer ring.□

Next example shows that the converse of Theorem 3.68 does not hold in general.

Example 3.70 *Let $M = \mathbb{Z}_{p^\infty}$, considered as a $\mathbb{Z}$-module. Then it is well-known that $End_{\mathbb{Z}}(M)$ is the ring of p-adic integers [19, Example 3, page 43]. Since the ring of p-adic integers is a commutative domain, it is a (quasi-) Baer ring. However $M = \mathbb{Z}_{p^\infty}$ is not a (quasi-) Baer module.*

Definition 3.71 *A module M is called retractable if $Hom(M, N) \neq 0$, $\forall\ 0 \neq N \leq M$ (or, equivalently, $\exists\ 0 \neq \varphi \in S = End(M)$ with $Im(\varphi) \subseteq N$).*

We recall a result of Khuri that already proved as Theorem 2.19.

Theorem 3.72 *Let M_R be nonsingular and retractable. Then $S = End(M)$ is right extending ring if and only if M is extending module.*

Proposition 3.73 *Let M be retractable. Then M is Baer if and only if S=End(M) is Baer ring.*

Proof. The direct implication has already been shown as Theorem 3.68. We now prove the reverse implication. Let $I \leq {}_S S$; since S=End(M) is Baer, $r_S(I) = eS$ for $e^2 = e \in S$. Hence, $r_M(I) \supseteq eM$. Assume there exists $m \in M \setminus eM$ so that $Im = 0$ without loss of generality we can assume $0 \neq m \in (1-e)M$. By retractability, there exists $0 \neq \varphi \in S, Im(\varphi) \subseteq mR$. But in this case, $I\varphi M \subseteq ImR = 0$, hence $\varphi \in r_S(I)$. But $\varphi = (1-e)\varphi \in eS \cap (1-e)S = 0$ which is a contradiction. Hence $r_M(I) = eM$, implying that M is a Baer module.□

Proposition 3.74 *Let M be retractable. Then M is quasi-Baer if and only if S=End(M) is a quasi-Baer ring.*

Proof. The proof is similar to the Baer case, discussed above.□

Theorem 3.75 $[29, Theorem 2, Theorem 3]$ *If R is a Baer ring with only countably many idempotents, then R has no infinite sets of orthogonal idempotents. If, in addition, R is a regular ring, then R is a semisimple Artinian ring.*

Proposition 3.76 *If M is a Baer module, with only countably many direct summands, then M contains no infinte direct sums of disjoint summands.*

Proof. Since M is Baer, S is Baer by Theorem 3.68. Since M has countably many direct summands, then S has only countably many idempotents. By Theorem 3.75 S has no infinite sets of orthogonal idempotents, hence there exists no infinite sets of mutually disjoint direct summands in M.□

Proposition 3.77 *Let M be an extending module such that its endomorphism ring S=End(M) is regular ring. Then M is a Baer module, and subsequently S is a Baer ring.*

Proof. In view of Theorem 3.29 we only have to show that M is K-nonsingular. Take $\varphi \in S = End(M)$ so that $r_M(S\varphi) = Ker(\varphi) \leq_e M$. Since S is regular, there exists $\psi \in S$ so that $\varphi = \varphi\psi\varphi$, hence $\psi\varphi = (\psi\varphi)(\psi\varphi)$ is an idempotent with the property that $S\varphi = S\psi\varphi$; but then $r_M(S\varphi) = r_M(\psi\varphi) = (1-\psi\varphi)M \leq_d M$. Hence $Ker(\varphi) = r_M(S\varphi) = M \Rightarrow \varphi = 0$.□

CHAPTER 4

ON K-NONSINGULAR MODULES AND APPLICATIONS

4.1 K-nonsingular Modules

In this chapter [40] was taken as a reference basically.

Definition 4.1 [43, $Proposition 11.1$] *A module M is called non-M-singular (polyform) if for all* $K \subseteq M$ *and* $0 \neq f : K \to M$, $Ker f$ *is not essential in K.*

Proposition 4.2 *Every non-M-singular (polyform) module M is K-nonsingular.*

Proof. In particular, all nonzero endomorphisms of M have kernels which are not essential in M, providing our assertion.□

The following examples show that K-nonsingularity of modules is a proper generalization of the concepts of non-M-singularity (or polyform property) and nonsingularity, i.e., the converse of Proposition 4.2 and Proposition 3.23 do not hold true.

Example 4.3 *In* $\mathbb{Z}$*-mod let* $M = Q \oplus \mathbb{Z}$*. Then* $Q \trianglelefteq M$ *and* $\mathbb{Z}_2 \trianglelefteq M$ $(Hom_{\mathbb{Z}}(Q, \mathbb{Z}_2) = 0$, $Hom_{\mathbb{Z}}(\mathbb{Z}_2, Q) = 0)$*.From Theorem 4.14 (later in this section) we can see that M is a K-nonsingular* $\mathbb{Z}$*-module since Q is K-nonsingular (in fact it is nonsingular) and* $\mathbb{Z}_2$ *is K-nonsingular. However, if we take* $\mathbb{Z} \leq Q$ *and* $\varphi : \mathbb{Z} \to Q \oplus \mathbb{Z}_2, \varphi(z) = (0, \hat{z})$*, then kernel of* φ *is* $2\mathbb{Z} \leq_e \mathbb{Z}$*, hence M cannot be non-M-singular (polyform) or nonsingular.*

However, when the module $M = R$, the base ring, the tree concepts coincide.

Proposition 4.4 *A ring R is K-nonsingular if and only if R is nonsingular if and only if R is non-R-singular (polyform).*

Proof. The proof easily follows as $End(R) = R$ consists of left multiplication by elements of R.□

Definition 4.5 *For a module M we define the K-singular submodule of M by* $Z^K(M) = \sum_{\varphi \in S, Ker\varphi \leq_e M} Im\varphi$ *(where the summation goes over all* $\varphi \in S = End(M)$ *with* $Ker\varphi \leq_e M$*).*

Proposition 4.6 *A module M is K-nonsingular if and only if* $Z^K(M) = 0$*.*

Proof. If $Z^K(M) = 0 \Leftrightarrow \varphi = 0, \forall \varphi \in S = End(M)$ with $Ker\varphi \leq_e M$. The result follows.□

Proposition 4.7 *Let M be a module. Then* $Z^K(M) \trianglelefteq M$*. Moreover,* $Z^K(M) \subseteq Z(M)$*.*

Proof. Let $\varphi \in S = End(M)$ so that $Ker\varphi \leq_e M$; for any $\psi \in S$, $Ker\psi\varphi \supseteq Ker\varphi$, and so $Ker\psi\varphi \leq_e M$.

For $x \in Z^K(M)$, let $x = x_1 + x_2 + ... + x_n$, where $x_i \in Im\varphi_i, \varphi_i \in S, i \in \{1, ..., n\}$, for some $n \in \mathbb{N}$. For each $x_i, \exists m_i \in M$ and $I_i \leq_e R_R$ so that $0 \neq x_i = \varphi_i m_i$, however $m_i I_i \subseteq Ker\varphi_i \Rightarrow \varphi_i(m_i I_i) = x_i I_i = 0$ $(\forall 1 \leq i \leq n)$. Taking $I = \bigcap_{1 \leq i \leq n} I_i$ we get that $xI = 0$ and $I \leq_e R_R$. Hence $x \in Z(M)$, and the result follows.□

Note that, if M is nonsingular, $Z(M) = 0$, hence by Proposition 4.7, $Z^K(M) = 0$. In view of Proposition 4.6, this provides another proof of the fact that nonsingularity implies K-nonsingularity.

Proposition 4.8 *Let* $M = \bigoplus_{i \in F} M_i$*. Then* $Z^K(M) \supseteq \bigoplus_{i \in F} Z^K(M_i)$*.*

Proof. Since $Z^K(M) \trianglelefteq M \Rightarrow Z^K(M) = \bigoplus_{i \in F}(Z^K(M) \cap M_i)$. We only need to show that $Z^K(M_i) \subseteq Z^K(M) \cap M_i$.

For a fixed $i \in F$, $x \in Z^K(M_i) \Rightarrow x = \varphi_1(x_1) + \varphi_2(x_2) + ... + \varphi_n(x_n)$, for some n, where $\varphi_i \in End(M_i)$ and $x_i \in M_i$, $Ker\varphi_i \leq_e M_i$, $\forall 1 \leq i \leq n$. Extending each φ_i to $\overline{\varphi_i} : M \to M$, by $\overline{\varphi_i}|_{M_j} = 0$ for $\forall i \neq j$ where $i, j \in F$ gets us $Ker\overline{\varphi_i} \leq_e M$ (as it is easily checked), and so $x \in Z^K(M_i) \Rightarrow x \in Z^K(M) \cap M_i$, $i \in F$.□

It is well known that nonsingularity is inherited by submodules of nonsingular modules, and in particular by direct summands of nonsingular modules. Also, a direct sum of nonsingular modules is nonsingular. On the contrary, for the case of K-nonsingularity, neither submodules always inherit K-nonsingularity, nor direct sums of K-nonsingular modules are K-nonsingular. The next example exhibits this.

Example 4.9 *It is easy to see that $\mathbb{Z} \oplus \mathbb{Z}_2 \leq Q \oplus \mathbb{Z}_2$ is not a K-nonsingular $\mathbb{Z}$-module (the map $(z, \hat{n}) \to (0, \hat{z})$ has essential kernel), even though $Q \oplus \mathbb{Z}_2$ is a K-nonsingular $\mathbb{Z}$-module. Also, $\mathbb{Z}$ and $\mathbb{Z}_2$ are both K-nonsingular, while $\mathbb{Z} \oplus \mathbb{Z}_2$ is not.*

The property of K-nonsingularity, however, is inherited by direct summands.

Proposition 4.10 *Alternative proof of Proposition 3.43 can be done by using Proposition 4.8 and 4.6.*

Proof. Let $M = N \oplus N'$. By Proposition 4.8 we obtain that $Z^K(M) \supseteq Z^K(N) \oplus Z^K(N')$; but $Z^K(M) = 0$ by Proposition 4.6, since M is K-nonsingular. Thus, $Z^K(N) = 0$, and applying Proposition 4.6 again, we obtain that N is K-nonsingular.□

Definition 4.11 *Let M and N be R-modules. We say that M is K-nonsingular relative to N if, $\forall \varphi \in Hom(M, N), Ker\varphi \leq_e M$ implies $\varphi = 0$*

Definition 4.12 *The R-modules M and N are relatively Rickart if for all $\varphi \in Hom_R(M, N)$, $Ker(\varphi) \leq_d M$ and for all $\psi \in Hom_R(N, M)$, $Ker(\psi) \leq_d N$ (this condition was termed "relatively Baer" in previous chapter).*

Remark 4.13 *M is K-nonsingular relative to M if and only if M is K-nonsingular. If M and N are relatively Rickart modules, then they are mutually relatively K-nonsingular.*

In the next result a necessary sufficient condition is provided for arbitrary direct sums of K-nonsingular modules to be K-nonsingular.

Theorem 4.14 *Let $(M_i)_{i\in F}$ be a family of modules. Then $M = \bigoplus_{i\in F} M_i$ is K-nonsingular if and only if M_i is K-nonsingular relative to M_j, $\forall i, j \in F$.*

Proof. Let $\psi \in End(M)$. Assume that $Ker\psi \leq_e M$. Thus, $Ker\psi \cap M_i \leq_e M_i, \forall i \in F$. Fix $i \in F$. We can split $\psi|_{M_i} : M_i \to \bigoplus M_i$ into $\bigoplus_{j \in I} \pi_j \psi|_{M_i}$, where π_j is canonical projection of M onto M_j, $j \in F$. $Ker\psi|_{M_i} = Ker\psi \cap M_i \leq_e M_i$, but $Ker\psi|_{M_i} = \bigcap_{j \in F} Ker\pi_j \psi|_{M_i} \Rightarrow Ker\pi_j \psi|_{M_i} \leq_e M_i$, $j \in F$. But, by relative K-nonsingularity in our hypothesis, $\pi_j \psi|_{M_i} = 0$, $\forall j \in F$. Hence $\psi|_{M_i} = 0$.

Since $i \in F$ is arbitrary, $\psi|_{M_i} = 0, \forall i \in F$, thus $\psi = 0$. This implies that M is K-nonsingular.

Converse holds by Proposition 4.10 and the fact that nonzero homomorphisms between any pair of summands of M can be extended to M in the obvious fashion, in which case each kernel must be non-essential.□

Proposition 4.15 *Let M be a module such that $E(M)$ is a K-nonsingular module. Then M is K-nonsingular.*

Proof. Any endomorphism $\varphi \in End(M)$ can be extended to an endomorphism $\overline{\varphi} \in End(E(M))$. Assume $Ker\varphi \leq_e M \leq_e E(M)$; since $Ker\varphi \subseteq Ker\overline{\varphi} \Rightarrow Ker\overline{\varphi} \leq_e E(M)$. Hence, since $E(M)$ is K-nonsingular, we obtain that $\overline{\varphi} = 0 \Rightarrow \varphi = 0$. In conclusion, M is also K-nonsingular.□

In the next example it is shown that the converse of Proposition 4.15 does not hold in general.

Example 4.16 *Let $M = \mathbb{Z}_p$. Even though M is K-nonsingular $\mathbb{Z}$-module (M is simple module in fact), essential extensions of M, in particular its injective hull $E(M) = \mathbb{Z}_{p^\infty}$, are not necessarily so. This, since the endomorphism of $E(M)$ obtained by multiplying elements by p has a nonzero essential kernel.*

Theorem 4.17 *Let R be a ring. The following are equivalent:*

- **(i)** *Every injective (right) R-module is Baer;*
- **(ii)** *Every (right) R-module is Baer;*
- **(iii)** *R is semisimple artinian.*

Proof. ($iii \Rightarrow ii \Rightarrow i$) are obvious.

To prove ($i \Rightarrow iii$), consider the module: $B = E(M) \oplus E(E(M)/M)$, where M is an arbitrary right R-module. B is injective (being the direct sum of two injective modules) and hence Baer by hypothesis. Let $\varphi : E(M) \to E(E(M)/M)$ be defined by $\varphi(x) = x + M$, $\forall x \in E(M)$. Then $Ker\varphi = M$ is a direct summand of $E(M)$. Since $M \leq_e E(M)$ we get $M = E(M)$. Since M was arbitrarily chosen, we get that all right R-modules are injective, hence R must be semisimple artinian.□

As a consequence, it is shown that the class of semisimple artinian rings is precisely also the class for which every R-module is K-nonsingular.

Corollary 4.18 *Let R be a ring. The following assertions are equivalent:*

(i) *Every (right) R-module is K-nonsingular;*

(ii) *Every injective (right) R-module is K-nonsingular;*

(iii) *R is semisimple artinian.*

Proof. ($i \Rightarrow ii$) is obvious. To prove ($ii \Rightarrow iii$) we observe that if a module M is injective and K-nonsingular, it is, in particular, extending and K-nonsingular. By Theorem 3.29 we obtain that M is Baer. Then by Theorem 4.17 R is semisimple artinian.

($iii \Rightarrow i$) is clear since the kernel of any nonzero endomorphism of any R-module must be a proper direct summand, hence cannot be essential.□

4.2 K-Nonsingularity and The Endomorphism Ring

Definition 4.19 *Recall that A right R-module M is called continuous (quasi-continuous) if M is CS-module satisfying the condition* (C_2) $((C_3))$.

Let R be a ring. Let M and N be R-modules with identity if every homomorphism from a submodule X of N to M extend from N to M then M is said to be N-injective. For every R-module N if M is N-injective then M is called injective module. If M is M-injective then M is called quasi-injective module. M and N are relatively injective if M is N-injective and N is M-injective. Also if M is R_R *injective then M is injective.*

Lemma 4.20 *Let A be a submodule of an arbitrary module M. If A is closed in a summand of M, then A is closed in M.*

Proof. Let $M = M_1 \oplus M_2$ with $A \leq_c M_1$. $\pi_1 : M_1 \oplus M_2 \to M_1$ be projection map. Assume that $A \leq_e B \leq M$. Then it is easy to see that $A = \pi(A) \leq_e \pi(B) \leq M_1$. Since $A \leq_c M_1$ then $A = \pi(A) = \pi(B) \leq B$. $(1-\pi)B \leq B \Rightarrow (1-\pi)B \cap A = 0$, $(1-\pi)B = 0$ $\Rightarrow B = \pi B \leq M_1$ so $A = B$□

Theorem 4.21 *The followings are equivalent for a module M.*

(i) *M is quasi-continuous.*

(ii) $M = X \oplus Y$ *for any two submodules X and Y which are complement of each other.*

(iii) $fM \leq M$ *for every idempotent* $f \in End(E(M))$.

(iv) $E(M) = \bigoplus_{i\in I} E_i$ *implies* $M = \bigoplus_{i\in I} M \cap E_i$.

Proof.

$(i \Rightarrow ii)$ $X, Y \leq_c M$ then $X, Y \leq_d M$. By condition (C_3) $X \oplus Y \leq_d M$. Since $X \oplus Y \leq_e M$ then $X \oplus Y = M$.

$(ii \Rightarrow iii)$ Let $A_1 = M \cap f(E(M))$ $(f : E(M) \to E(M))$ and $A_2 = M \cap (1-f)(E(M))$. Let B_1 be a complement of A_2 that contains A_1. Let B_2 be a complement of A_1 that contains A_2. Then $M = B_1 \oplus B_2$. Let π be the projection $B_1 \oplus B_2 \to B_1$. We claim that $M \cap (f-\pi)M = 0$. Let $x, y \in M$ such that $y = (f-\pi)(x) \Rightarrow y = f(x) - \pi(x) \Rightarrow$ $f(x) = y + \pi(x) \in M$ and hence $f(x) \in A_1$. Thus $(1-f)(x) \in M$ and so $(1-f)(x) \in A_2$. Therefore $\pi(x) = f(x)$, and consequently $y = 0$. $(x - f(x) \in A_2 \Rightarrow \pi(x - f(x)) = 0 \Rightarrow$ $\pi(x) - \pi(f(x)) = 0 \Rightarrow \pi(x) = f(x))$. $M \leq_e E(M) \Rightarrow (f-\pi)M = 0 \Rightarrow f(M) = \pi(M) \leq M$.

$(iii \Rightarrow iv)$ It is clear that $\bigoplus_{i\in I} M \cap E_i \leq M$. Let m be an arbitrary element in M. Then $m \in \bigoplus_{i\in F} E_i$ for a finite subset $F \subseteq I$. Write $E(M) = \bigoplus_{i\in F} E_i \oplus E^*$. Then there exists orthogonal idempotents $f_i \in End(E(M))$ $i \in F$ such that $E_i = f_i(E(M))$ since $f_iM \leq M$ by assumption, $m = (\sum_{i\in F} f_i)(m) = \sum_{i\in F} f_i(m) \in \bigoplus_{i\in F} M \cap E_i$ then $M \leq \bigoplus_{i\in F} M \cap E_i$ so $M = \bigoplus_{i\in I} M \cap E_i$.

($iv \Rightarrow i$) Let $A \leq M$. Then $E(M) = E(A) \oplus E^*$. Then $M = (M \cap E(A)) \oplus (M \cap E^*)$ with $A \leq_e M \cap E(A)$ hence M has (C_1). Let $M_1, M_2 \leq_d M$ such that $M_1 \cap M_2 = 0$. $E(M) = E(M_1) \oplus E(M_2) \oplus E'$ then $M = (M \cap E(M_1)) \oplus (M \cap E(M_2)) \oplus (M \cap E') = M_1 \oplus M_2 \oplus (M \cap E')$ so $M_1 \oplus M_2 \leq_d M$ ($M_i \leq_d M$, $M_i \leq_e E(M_i) \Rightarrow M_i \leq_e M \cap E(M_i) \leq M \Rightarrow M_i \leq_d M \cap E(M_i) \leq M \Rightarrow M_i = M \cap E(M_i)$).

□

Proposition 4.22 *If $M_1 \oplus M_2$ is quasi-continuous then M_1 and M_2 are relatively injective.*

Proof. Let $M = M_1 \oplus M_2$. We will show that M_2 is M_1-injective. Let $X \leq M_1$, $\varphi : X \to M_2$ and define $B = \{x - \varphi(x) : x \in X\} \leq M$. Assume there exists an element $y \in B \cap M_2$ then $y = x - \varphi(x)$; $x \in M_1$ so $x = y + \varphi(x) \in M_1 \cap M_2 = 0$ hence $y = 0$ then $B \cap M_2 = 0$. Now let M_1^* be a complement of M_2 that contains B. Since $M = M_1 \oplus M_2$ is quasi-continuous then we can define $M = M_1^* \oplus M_2$ where $M_1^* \cong M_1$. Let $\pi : M_1^* + M_2 \to M_2$ where $Ker(\pi) = M_1^* \supseteq B$. Let $x \in X$ so can define $x = m_1^* + m_2$ since $\pi(x - \varphi(x)) = 0$ then $\pi(x) = \pi(\varphi(x)) = \varphi(x)$. So we can extend $\varphi(x)$ to $\pi|_{M_1} : M_1 \to M_2; \pi(m_1) = m_2$.□

Corollary 4.23 *$\bigoplus_{i=1}^{n} M_i$ is quasi-continuous if and only if each M_i is quasi-continuous and M_j-injective for all $i \neq j$.*

Proposition 4.24 *Let M be a quasi-continuous module, $S = End(M)$, $\triangle = \{\alpha \in S : Ker(\alpha) \leq_e M\}$ and J the Jacobson radical of S. Then M is continuous if and only if $\triangle = J$ and $S/\triangle$ is regular.*

Proof. Suppose M is continuous. Let $\alpha \in S$ and let L be a complement of $K = Ker(\alpha)$ by (C_1) $L \leq_d M$. Since $\alpha|_L$ is a monomorphism; $\alpha|_L : L \to M$, $\alpha(L) \cong L$ then $\alpha(L) \leq_d M$ by (C_2). Hence there exists $\beta \in S$ such that $\beta\alpha = 1_L$. Then $(\alpha - \alpha\beta\alpha)(K \oplus L) = (\alpha - \alpha\beta\alpha)(L) = \alpha(L) - \alpha(L) = 0$ so $K \oplus L \subseteq Ker(\alpha - \alpha\beta\alpha) \leq_e M$ that implies $(\alpha - \alpha\beta\alpha) \in \triangle$ thus $S/\triangle$ is regular ring. This also proves that $J \leq \triangle$ ($J(S/\triangle) = 0 \Rightarrow$

$J(S) \subseteq \triangle$). Let $a \in \triangle$. Since $Ker(a) \cap Ker(1-a) = 0$ and $Ker(a) \leq_e M$ this implies that $Ker(1-a) = 0$ then $(1-a)(M) \leq_d M$ by (C_2). Since $Ker(a) \leq (1-a)M$ then $(1-a)M \leq_e M$. Hence, $(1-a)M = M$ then $(1-a)$ is unit in S. It follows that $a \in J$ then $\triangle \subseteq J$ so $\triangle = J$.

Conversely, assume that $\triangle = J$ and $S/\triangle$ is regular. Let $\varphi \in S$ be a monomorphism with essential image. There exists $\psi \in S$ such that $\varphi - \varphi\psi\varphi \in \triangle$. Then $(1-\varphi\psi)\varphi K = 0$ for some $K \leq_e M$. Since φ is a monomorphism, $\varphi K \leq_e \varphi M$ thus $\varphi K \leq_e M$ as $\varphi M \leq_e M$. Therefore, $1 - \varphi\psi \in \triangle = J$, and hence $\varphi\psi$ is a unit in S. Thus φ is onto and so φ is an isomorphism. Then M is continuous. □

Proposition 4.25 *If M is a K-nonsingular and continuous module, then S is regular and S is right continuous.*

Proof. M is continuous hence $J(S) = \{\varphi \mid Ker\varphi \leq_e M\}$, and $S/J(S)$ is von Neumann regular and right continuous. By K-nonsingularity, $J(S) = 0$, and the result follows.□

Proposition 4.26 *If M is a module such that S=End(M) is regular, then M is K-nonsingular.*

Proof. Let $\varphi \in S$ so that $Ker\varphi \leq_e M$. By regularity, $\exists\psi \in S$ so that $\varphi\psi\varphi = \varphi$. But that implies $\psi\varphi$ is an idempotent, and hence $Ker\psi\varphi \leq_d M$. But $Ker\varphi \subseteq Ker\psi\varphi \Rightarrow Ker\psi\varphi \leq_e M \Rightarrow Ker\psi\varphi = M \Rightarrow \psi\varphi = 0 \Rightarrow \varphi = \varphi\psi\varphi = 0$.□

Corollary 4.27 *Let M be an extending module such that its endomorphism ring S is a regular ring. Then M is Baer module, and subsequently S is Baer ring.*

Proof. By Theorem 3.29 and Proposition 4.26, M is Baer module. By Theorem 3.68, S=End(M) is a Baer ring.□

Proposition 4.28 *Let M be a module with semisimple artinian endomorphism ring S. Then M is Baer module.*

Proof. Since a semisimple artinian ring is Baer, S is Baer. Since every left ideal $I \leq {}_S S$ is a summand in ${}_S S$ (semisimple artinian ring is, in particular, semisimple left module over itself), $I = Se$ with $e^2 = e \in S, r_M(I) = (1 - e)M \leq_d M$, hence M is Baer.□

Proposition 4.29 *Let M be retractable. If M is K-nonsingular, then S=End(M) is right nonsingular.*

Proof. Let M be a K-nonsingular module. Let $\varphi \in S$, so that $r_S(\varphi) \leq_e S_S$. Assume $r_M(\varphi) = Ker(\varphi)$ is not essential in M; hence, there exists a nonzero complement $N \leq M$, $N \cap Ker(\varphi) = 0$. By retractability, $\exists 0 \neq \psi \in S, Im\psi \subseteq N$. But $\varphi\psi \neq 0$(as the image of ψ has zero intersection with the kernel of φ), thus $\psi S \cap r_S(\varphi) = 0$, since the image of any $\psi\psi'$ with $\psi' \in S$ is also a subset of N. This contradicts essentiality of $r_S(\varphi)$, hence $r_M(\varphi) \leq_e M \Rightarrow \varphi = 0$, by K-nonsingularity of M.□

Theorem 4.30 *Let M be a Baer module with only countably many direct summands. Then M is semisimple artinian if any of the following conditions hold:*

(i) *M is retractable and S=End(M) is a regular ring;*

(ii) *Every cyclic submodule of M is a direct summand of M; or*

(iii) $\forall m \in M, \exists f \in Hom(M, R_R)$ *such that* $m = mfm$.

Proof. Suppose (i) holds. By Theorem 3.68 and in view of Proposition 3.76, S is a regular ring with only countably many idempotents. Then S is a semisimple artinian ring, by [29, Theorem 3].

Since S is semisimple artinian ring, it can be decomposed into a finite, ring direct sum of simple artinian rings, $S = \bigoplus_{1 \leq i \leq n} Se_i$, where $n \in \mathbb{N}$ and all e_i are central idempotents. Hence we obtain the following module direct decomposition $M = \bigoplus_{1 \leq i \leq n} e_i M$, where $e_i M \trianglelefteq_d M$. We show that this implies, $\forall 1 \leq i \leq n$, $e_i M$ is retractable. Take $N \leq e_{i_0} M \leq M$, $1 \leq i_0 \leq n$; there exists a nonzero endomorphism $\varphi \in S$ so that $\varphi(M) \leq N \Rightarrow \pi_{i_o}\varphi = \varphi$ (where π_{i_o} is the canonical projection onto $e_{i_0}M$). Assume that $0 \neq x \in M$, so that $\varphi(x) \neq 0$; we have $x = e_1x + e_2x + ... + e_nx$ and by $e_iM \trianglelefteq M \Rightarrow \varphi(e_ix) = \pi_{i_o}\varphi(e_ix) \in e_iM \cap e_{i_0}M = 0$ unless $i = i_0$. This implies that $\varphi(x) = \varphi(e_{i_0}x) \neq 0$, and so the restricting of φ to $e_{i_0}M$ produces a nonzero endomorphism of $e_{i_0}M$ ($e_{i_0}M \trianglelefteq M \Rightarrow \varphi(e_{i_0}M) \in e_{i_0}M$), whose image is a subset of N. Since i_0 was arbitrary chosen, this implies that all of the above summands of M are retractable. If we can show that, for any $1 \leq i \leq n$, e_iM is semisimple artinian, we're done. To simplify notation, and without losing generality, we can assume S is simple artinian.

We know that a simple artinian ring is finite, $m \times m$ matrix ring over a field K, where $m \in \mathbb{N}$. Let ψ_i,$1 \leq i \leq m$ be the idempotent elements of S=End(M) having 1 in the ii-position, and 0 everywhere else. Thus M can be decomposed as $M = \psi_1M+\psi_2M+...+\psi_mM$. We want to show that each ψ_jM is a simple module, for $1 \leq j \leq m$. For a fixed j_0, take $P \leq \psi_{j_0}M$, where $1 \leq j_0 \leq m$. By retractability of M, we have that there exists $\chi \in S$ with $\chi(M) \leq P$. Take $y \in M$ so that $\chi y \neq 0$; we have $x = \psi_1x + \psi_2x + ... + \psi_mx$ and hence $0 \neq \chi\psi_1x + \chi\psi_2x + ... + \chi\psi_mx \Rightarrow \exists 1 \leq k_0 \leq m$ so that $\chi\psi_{k_0}x \neq 0$. If $k_0 = j_0$ we can restrict χ to $\psi_{j_0}M$, and so obtain an endomorphism of $\psi_{j_0}M$, whose image is a subset of N. But $S\psi_{j_0}$ is isomorphic, as a ring, with field K, which implies that all nonzero endomorphisms must be isomorphisms. This means that N, containing the image of such an endomorphism, must equal $\psi_{j_0}M$.

Assume now that $k_0 \neq j_0$. Since there exists an isomorphism $f : S\psi_{j_0} \rightarrow S\psi_{k_0}$, $0 \neq (\chi|_{S\psi_{k_0}}) \circ f : S\psi_{j_0} \rightarrow S\psi_{j_0}$, $Im(\chi|_{S\psi_{k_0}}) \circ f \subseteq N$. Now use the above argument to show that $N = \psi_{j_0}M$.

Since N was arbitrarily chosen, it implies that any nonzero submodule of $\psi_{j_0}M$ must be $\psi_{j_0}M$, hence $\psi_{j_0}M$ is simple. j_0 was arbitrarily chosen, hence M is semisimple artinian.□

Proposition 4.31 *If M is an indecomposable Baer module then $S = End(M)$ is a domain. If M is retractable and S is a domain, then M is an indecomposable Baer module.*

Proof. Let M be indecomposable Baer module. From Theorem 3.40 we get that all endomorphisms are monomorphisms, hence S is a domain.

Let M is retractable since S is a domain then S is Baer ring so M is a Baer module. Also, since S is a domain, it does not have any proper idempotents, thus M is indecomposable.□

REFERENCES

[1] Berberian, S. K. *Baer Rings*, Springer Verlag, Berlin - Heidelberg - New York (1972).

[2] Birkenmeier, G. F., Heatherly, H. E., Kim, J. Y., Park, J. K. *Triangular matrix representations*, J. Algebra 230, (2002a), 558-595.

[3] Birkenmeier, G. F., Kim, J. Y., Park, J. K. *Principally quasi- Baer rings*, Comm. Algebra 29, (2001b), 639-660.

[4] Birkenmeier, G. F., Kim, J. Y., Park, J. K. *A sheaf representation of quasi- Baer rings*, J. Pure Appl. Algebra 146, (2000b), 209-223.

[5] Birkenmeier, G. F., Kim, J. Y., Park, J. K. *Quasi- Baer ring extensions and Biregular rings*, Bull. Australian Math. Soc. 61, (2000c), 39-52

[6] Birkenmeier, G. F., Kim, J. Y., Park, J. K. *On quasi- Baer rings*, In: Huynh, D. V., Jain, S. K., Lopez- Permouth, S. R., eds. Algebra and Its Applications,. Contemp. Math., 259, Providence: Amer. Math. Soc., (2002d), 67-92.

[7] Birkenmeier, G. F., Kim, J. Y., Park, J. K.*Semicentral reduced algebras*, In: International Symposium on Ring Theory, Birkenmeier, G. F., Park, J. K., Park, Y. S., eds.; Trends in Math, Boston: Birkhäuser, (2001e), 67-84.

[8] Birkenmeier, G. F., Kim, J. Y., Park, J. K. *Rings with countably many direct summands*, Comm. Algebra, 28(2), (2000a), 757-769.

[9] Birkenmeier, G. F., Kim, J. Y., Park, J. K. *A counterexample for CS- rings*, Glasgow. Math. J. 42, (2000), 263-269.

[10] Birkenmeier, G. F., Müller, B. J., Rizvi, S. T. *Modules in which every fully invariant submodule is essential in a direct summand* , Comm. Algebra 30, (2002a), 1395-1415.

[11] Birkenmeier, G. F., Park, J. K. *Triangular matrix representations of ring extensions*, J. Algebra 265, (2003), 457-477.

[12] Birkenmeier, G. F., Park, J. K., Rizvi, S. T. *Modules with fully invariant submodules essential in fully invariant summands* , Comm. Algebra 30, (2001), 1833-1852.

[13] Birkenmeier, G. F., Park, J. K., Rizvi, S. T. *Generalized triangular matrix rings and fully invariant extending property*, Rocky Mountain J. Math. 32, (2002), 1299-1319.

[14] Birkenmeier, G. F., Park, J. K., Rizvi, S. T. *Ring hulls with applications to C*-Algebras*, Preprint.

[15] Clark, W. E. *Twisted matrix units semigroup algebras*, Duke Math. J. 34, (1967), 417-424.

[16] Chatters, A. W., Khuri, S. M. *Endomorphism rings of modules over nonsingular CS Rings*, J. London Math. Soc. 21(2), (1980), 434-444.

[17] Dung, Nguyen Viet, Huynh, Dinh Van, Smith Patrick, F., Wisbauer, Robert *Extending Modules*, Pitman Research Notes in Mathematics Series 314, Harlow: London Scientific and Technical (1994).

[18] Faith, C. *Lectures on injective modules and quotient rings*,Lecture notes in mathematics, Springer Verlag, Berlin - Heidelberg - New York 246, (1972).

[19] Fuchs, L. *Infinite Abelian Groups*, Pure and Applied Mathematics Series, Volume 36, New York- London: Academic Press (1970).

[20] Goodearl, K. R. *Von Neumann Regular Rings*, Robert E. Krieger Publishing Co., Inc. Malabar (1991).

[21] Goodearl, K. R., Boyle, A. K. *Dimension theory for nonsingular injective modules*, Memoirs of the American Mathematical Society 7(177), (1976).

[22] Hamdouni, A., Harmanci, A., Ozcan, A. Ç. *Modules having SSP and modules having SIP*, J. Algebra , Number Theorey and Its Applications (2005), 469-490.

[23] Kaplansky, I. *Ring of Operators,* Mathematics Lecture Note Series. New York: W. A. Benjamin (1968).

[24] Khuri, S. M. *Baer endomorphism rings and closure operators*, Canad. J. Math. 30, (1978), 1070-1078.

[25] Khuri, S. M. *Correspondence theorems for modules and their endomorphism rings*, J. Algebra 122, (1989), 380-396.

[26] Khuri, S. M. *Nonsingular retractable modules and their endomorphism rings,* Bull. Austral. Math. Soc. 43, (1991), 63-71.

[27] Khuri, S. M. *Endomorphism rings of nonsingular modules,* Ann. Sc. Math. Québec IV(2), (1980), 145-152.

[28] Khuri, S. M. *Endomorphism Rings of Modules,* Manuscript.

[29] Kim, J. Y., Park, J. K. *When is a regular ring semisimple artinian ring?*, Math. Japonica 45(2), (1997), 311-313.

[30] Lam, T. Y. *Lectures notes on Modules and Rings*, GTM 189, Springer Verlag, Berlin- Heidelberg- New York (1999).

[31] Lawrence, J. *A singular primitive ring*, Proc. Am. Math. Soc. 45, (1974), 59-62.

[32] Mewborn, A. C. *Regular rings and Baer rings*, Math. Z. 121, (1971), 211-219.

[33] Mohamed, S. H., Müller, B. J. *Continuous and Discrete Modules*, Cambridge University Press, Cambridge (1990).

[34] Müller, B. J., Rizvi, S. T. *On injective and quasi-continuous modules*, J. Pure Appl. Algebra 28(2), (1983), 197-210.

[35] Pollingher, A., Zaks, A. *On Baer and quasi-Baer rings*, Duke Math. J. 37, (1970), 127-138.

[36] Rizvi, S. T. *Commutative rings for which every continuous module is quasi-injective*, Arch. Math. 50(5), (1988), 435-442.

[37] Roman, C. S. *Baer and Quasi-Baer Modules*, Doctoral Dissertation, The Ohio State Univeristy (2004).

[38] Rizvi, S. T., Roman, C. S. *Baer and quasi-Baer modules*, Comm. Alg. 32(1), (2004), 103-123.

[39] Rizvi, S. T., Roman, C. S. *Baer property of modules and applications*, In: Chen, J. L., Ding, N. Q., Marubayashi, H., eds. Advances in Ring Theory, World Scientific, (1967), 225-241.

[40] Rizvi, S. T., Roman, C. S. *On K-Nonsingular Modules and Applications*, Comm. Alg. 35, (2007), 2960-2982.

[41] Stenström, B. *Rings of Quotients*, GTM 217, Springer Verlag, Berlin-Heidelberg- New York (1975).

[42] Utumi, Y. *On rings of which are one-sided quotient ring is two-sided*, Proc. Amer. Math. Soc. 14, (1963), 141-147.

[43] Wilson, G. V. *Modules with the summand intersection property*, Comm. Algebra 14, (1986), 21-38.

[44] Wisbauer, R. *Modules and Algebras: Bimodule Structure and Group Actions on Algebras*, Pitman Monographs and Survey in Pure and Applied Mathematics, Harlow, Longman, 81, (1996).

[45] Wolfson, K. G. *Baer rings of endomorphisms*, Math. Annalen 143, (1961), 19-28.

[46] Zelmanowitz, J. *Regular modules*, Trans. Am. Math. Soc. 163, (1972), 341-355.