DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

RATIONAL BÉZIER CURVES

by
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İZMİR

RATIONAL BÉZIER CURVES

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"RATIONAL BÉZIER CURVES"** completed by **ÇETİN DİŞİBÜYÜK** under supervision of **Assistant Prof. HALİL ORUÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Çetin Dişibüyük

RATIONAL BÉZIER CURVES

ABSTRACT

In this thesis, we introduce a generalization of rational Bézier curves using q-Bernstein Bézier polynomilas. We generate these curves by a de Casteljau algorithm, which is a generalization of that relating to classical case. The explicit formula of intermediate points of de Casteljau algorithm is obtained. These points of de Casteljau algorithm are expressed in terms of q-differences. In the process of subdivision, the change of basis matrix between Bernstein Bézier basis and q-Bernstein Bézier basis is used. We study degree elevation of rational q-Bernstein Bézier curves. Finally, it is shown that rational q-Bernstein Bézier curves can be represented in matrix form.

Keywords: q-Bernstein polynomials, Rational q-Bernstein Bézier curves, de Casteljau algorithm, subdivision, degree elevation.

RASYONEL BÉZİER EĞRİLERİ

ÖZ

q-Bernstein Bézier polinomları kullanılarak rasyonel Bézier eğrileri genelleştirildi. Bu eğriler genelleştirilmiş de Casteljau algoritması kullanılarak elde edildi. de Casteljau algoritmasının ara noktaları q-farklar ile ifade edildi. Bernstein Bézier tabanı ve q-Bernstein Bézier tabanı arasında dönüşüm matrisi kullanarak subdivision yapıldı. q-Bernstein Bézier eğrilerinin derecesi yükseltildi. Son olarak, q-Bernstein Bézier eğrileri matris formunda gösterildi.

Anahtar sözcükler: q-Bernstein polinomları, Rasyonel q-Bernstein Bézier eğrileri, de Casteljau algoritması, subdivision, derece yükseltme.

CONTENTS

CHAPTER ONE
INTRODUCTION

We first give some basics of Bernstein Bézier polynomials which may be found in (Farin, 2002). We investigate certain properties of Bézier curves and rational Bézier curves.

1.1 BERNSTEIN BÉZIER POLYNOMIALS

In Computer Aided Design (CAD) and Computer Aided Geometric Design (CAGD) systems, it is important to have information about the shape of the curve we use. In general, it is not possible to talk about the shape of the curve represented in the form $\sum b_i t^i$ by investigating the coefficients b_i's. However, it is possible for the curve which has the form

$$\sum_{i=0}^{n} b_i \binom{n}{i} t^i (1-t)^{n-i}, \qquad t \in [0,1], \quad b_i \in \mathbb{E}^2 \text{ or } \mathbb{E}^3.$$

This representation is known as Bézier representation. The points b_i's give information about the shape of the curve. We first give some of the basics of this representation. The basis functions

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \qquad i = 0, 1, \ldots, n, \tag{1.1.1}$$

is called Bernstein Bézier polynomials of degree n. It can easily be verified that Bernstein Bézier polynomials satisfy the recurrence relation

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t). \tag{1.1.2}$$

The Bernstein Bézier polynomials have partition of unity property which follows from the Binomial Theorem

$$1 = ((1-t)+t)^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i},$$

the endpoint condition

$$B_i^n(0) = \delta_{i,0}, \qquad B_i^n(1) = \delta_{i,n}$$

and symmetry property

$$B_i^n(t) = B_{n-i}^n(1-t). \tag{1.1.3}$$

These properties are significant for design purpose. Figure 1.1 shows the graphs of cubic Bernstein Bézier polynomials for $t \in [0,1]$.

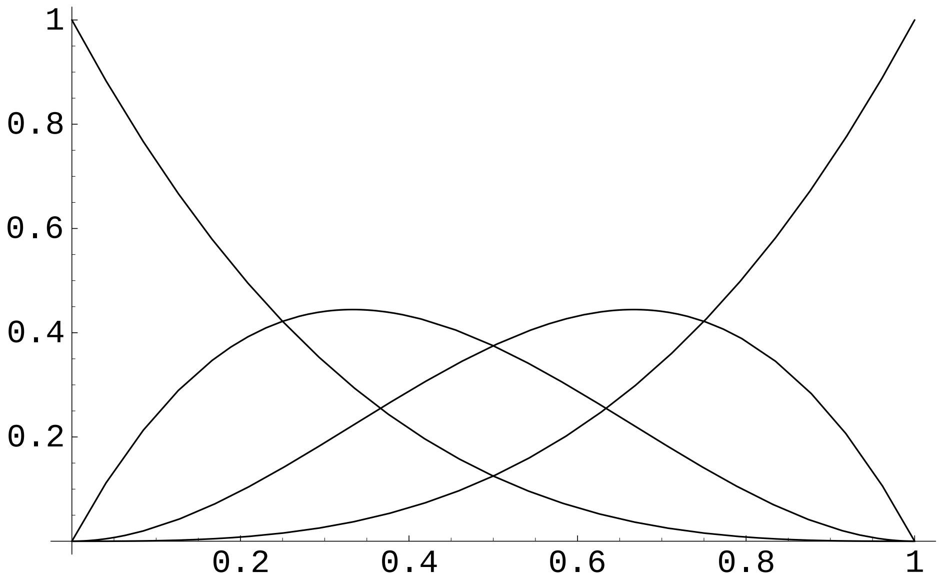

Figure 1.1: Cubic Bernstein Bézier polynomials.

1.2 BÉZIER CURVES

A parametric Bézier curve is defined by

$$P(t) = \sum_{i=0}^{n} \mathsf{b}_i B_i^n(t). \tag{1.2.1}$$

The points $\mathsf{b}_i \in \mathbb{E}^2$ or $\mathbb{E}^3$ are called Bézier points, or control points and the polygon formed by connecting b_i with b_{i+1} for all i is called Bézier polygon, or control polygon. Figure 1.2 illustrates a Bézier polygon and its quantic Bézier curve $P(t)$.

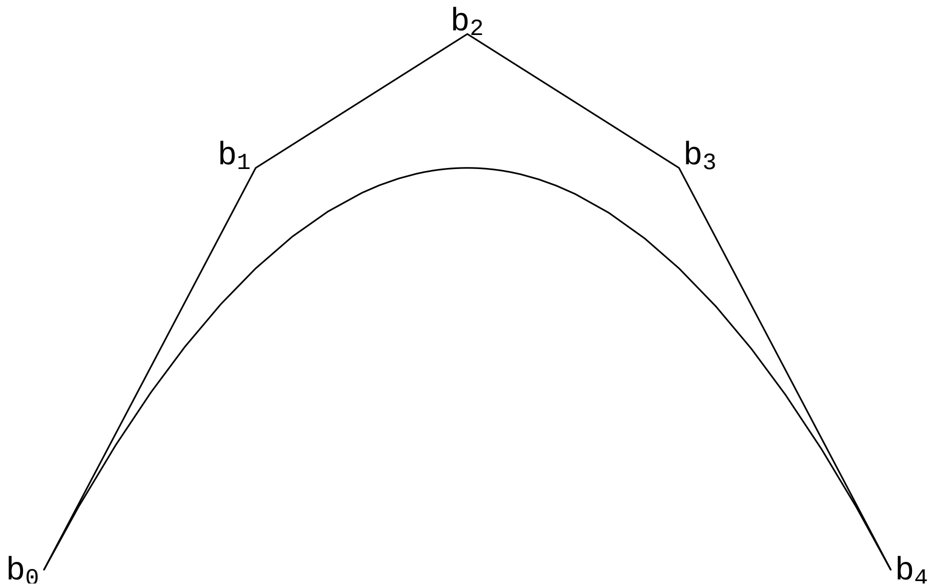

Figure 1.2: A Bézier polygon and its quantic Bézier curve.

The properties of Bézier curves:

1. Convex hull property: Convex hull of a point set is the convex region formed by all convex combinations of points. Since Bernstein Bézier polynomials have partition of unity and nonnegative for all $t \in [0, 1]$, the Bézier curve lies in the convex hull of Bézier polygon.

2. Affine invariance property: Bézier curves are invariant under affine maps. This means that the following procedures give the same result:

i) Compute $P(t)$ and then apply an affine map to it.

ii) Apply the map to the control points then evaluate Bézier curve.

These two curves are the same.

3. Endpoint interpolation property: The curve passes through the points b_0 and b_n. That is,

$$P(0) = \mathsf{b}_0, \qquad P(1) = \mathsf{b}_n$$

4. Variation diminishing property: The number of times any straight line intersect the curve is bounded by the number of times the line intersect the control polygon. Namely, the curve does not oscillate about any straight line more often than the control polygon.

5. Symmetry property: Let $P_1(t)$ be a Bézier curve with the control points $\mathsf{b}_0, \ldots, \mathsf{b}_n$ and $P_2(t)$ be a Bézier curve with the control points $\mathsf{c}_i = \mathsf{b}_{n-i}$, $i = 0, \ldots, n$. These two curves that correspond to the two different ordering of polygons look the same. They differ only in the direction in which they are traversed,

$$\sum_{i=0}^{n} \mathsf{b}_i B_i^n(t) = \sum_{i=0}^{n} \mathsf{b}_{n-i} B_{n-i}^n(1-t).$$

As a result of these properties, the shape of the curve mimics the shape of the control polygon.

One of the shortcomings of this simple but powerful Bézier curve technique is that making a change in a control point globally changes the curve. If a Bézier point b_i is moved to a new position $\tilde{\mathsf{b}}_i$, then all points on the Bézier curve move towards $\tilde{\mathsf{b}}_i$ in a direction parallel to $\tilde{\mathsf{b}}_i - \mathsf{b}_i$. Figure 1.3 shows the effect of moving one point of the Bézier points.

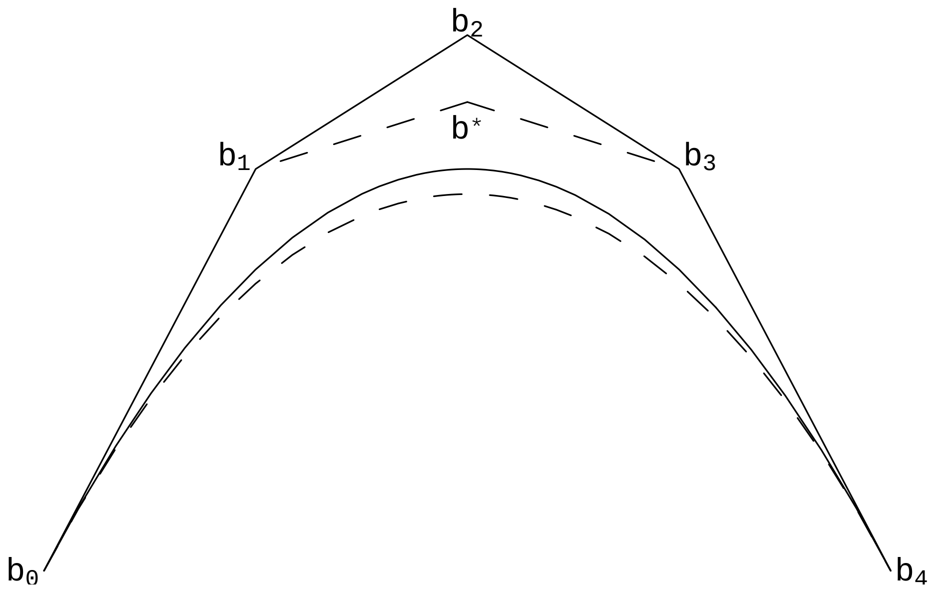

Figure 1.3: Effect of moving the point b_2 to the point b^*.

1.2.1 The de Casteljau Algorithm

Given the points $\mathsf{b}_0, \mathsf{b}_1, \dots, \mathsf{b}_n$ and $t \in \mathbb{R}$, set

$$\mathsf{b}_i^r(t) = (1-t)\mathsf{b}_i^{r-1}(t) + t\mathsf{b}_{i+1}^{r-1}(t), \qquad \begin{cases} r = 1, \dots, n \\ i = 0, \dots, n-r \end{cases} \tag{1.2.2}$$

and $\mathsf{b}_i^0(t) = \mathsf{b}_i$ for all i. Then it can be shown by induction on n that $\mathsf{b}_0^n(t)$ is the point with the parameter value t on the Bézier curve $P(t)$. Hence by continuity $\mathsf{b}_0^n(t) = P(t)$. The intermediate points $\mathsf{b}_i^r(t)$ can be put in a triangular array of points.

$$\begin{array}{ccccc}
\mathsf{b}_0 & & & & \\
\mathsf{b}_1 & \mathsf{b}_0^1 & & & \\
. & . & . & & \\
. & . & . & . & \\
\mathsf{b}_n & \mathsf{b}_{n-1}^1 & . & . & \mathsf{b}_0^n
\end{array}$$

The intermediate points of the de Casteljau algorithm obtained explicitly as

$$\mathsf{b}_i^r(t) = \sum_{j=0}^{r} \mathsf{b}_{i+j} B_j^r(t) \tag{1.2.3}$$

which are also Bézier curves.

1.2.2 Subdivision

Using the de Casteljau algorithm we can subdivide a Bézier curve into two Bézier curve segments which join together at a point $t_0 \in (0,1)$. The part of the curve that corresponds to the interval $[0, t_0]$ can be defined by a control polygon whose vertices $\mathsf{b}_i^{l(l)}$ are $\mathsf{b}_i^{(l)} = \mathsf{b}_0^i(t_0)$. It follows from the symmetry property that the control points $\mathsf{b}_i^{(r)}$ for the part corresponding to $[t_0, 1]$ are given by $\mathsf{b}_i^{(r)} = \mathsf{b}_i^{n-i}(t_0)$.

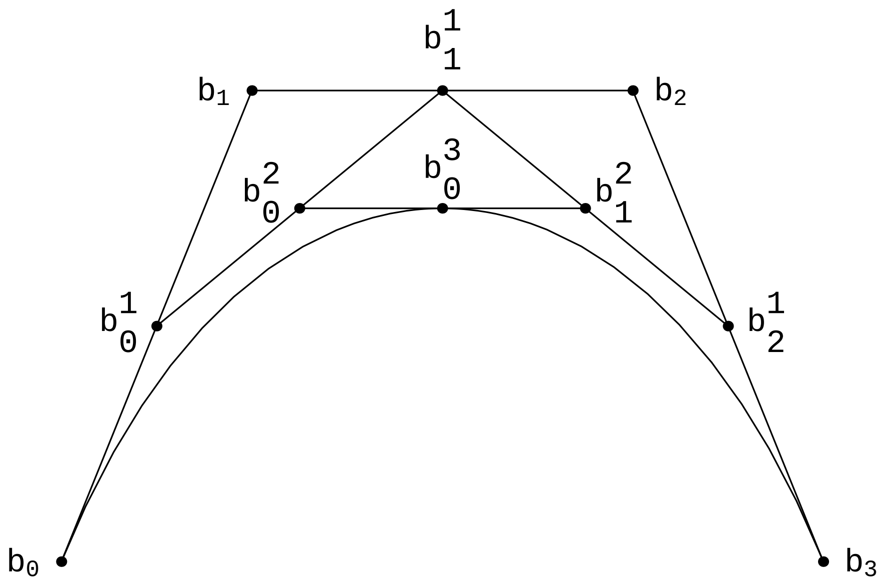

Figure 1.4: Subdivision of cubic Bézier curve in the de Casteljau algorithm.

Thus the curve segments are

$$P_{[0,t_0]}(t) = \sum_{i=0}^{n} \mathsf{b}_i^{(l)} B_i^n(t), \qquad P_{[t_0,1]}(t) = \sum_{i=0}^{n} \mathsf{b}_i^{(r)} B_i^n(t)$$

and

$$P_{[0,1]}(t) = P_{[0,t_0]}(t) \cup P_{[t_0,1]}(t) = \sum_{i=0}^{n} \mathsf{b}_i B_i^n(t).$$

For further investigation of Bézier curves and surfaces (See Farin, 2002)

1.2.3 Degree elevation

One of the methods to make Bézier curve more flexible is to represent the same curve using a higher degree Bernstein Bézier polynomials associated with a different set of control points. This process is called degree elevation.
For this purpose we write $P(t) = (1-t)P(t) + tP(t)$. Since
$(1-t)B_i^n(t) = \frac{n+1-i}{n+1} B_i^{n+1}(t)$ and $tB_i^n(t) = \frac{i+1}{n+1} B_{i+1}^{n+1}(t)$ we have

$$P(t) = \sum_{i=0}^{n} \frac{n+1-i}{n+1} \mathsf{b}_i B_i^{n+1}(t) + \sum_{i=0}^{n} \frac{i+1}{n+1} \mathsf{b}_i B_{i+1}^{n+1}(t).$$

Extending the upper limit of the first sum to $n+1$, shifting the index of the second sum to the limits 1 to $n+1$ and then extending the lower limit to 0 we obtain

$$P(t) = \sum_{i=0}^{n+1} \frac{n+1-i}{n+1} \mathsf{b}_i B_i^{n+1}(t) + \sum_{i=0}^{n+1} \frac{i}{n+1} \mathsf{b}_{i-1} B_i^{n+1}(t).$$

Then

$$P(t) = \sum_{i=0}^{n+1} \left(\frac{n+1-i}{n+1} \mathsf{b}_i + \frac{i}{n+1} \mathsf{b}_{i-1} \right) B_i^{n+1}(t). \qquad (1.2.4)$$

Thus, the new control points denoted by b_i^1 are

$$\mathsf{b}_i^1 = \frac{i}{n+1} \mathsf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right) \mathsf{b}_i, \qquad i = 0, \ldots, n+1. \qquad (1.2.5)$$

Notice that control points $\mathsf{b}_0^1, \ldots, \mathsf{b}_{n+1}^1$ and $\mathsf{b}_0, \ldots, \mathsf{b}_n$ describe the same Bézier curve with the basis $B_i^{n+1}(t)$ and $B_i^n(t)$ respectively and degree elevation interpolates the end points, that is $\mathsf{b}_0^1 = \mathsf{b}_0$ and $\mathsf{b}_{n+1}^1 = \mathsf{b}_n$.

1.3 RATIONAL BÉZIER CURVES

Bézier curves can be used to represent a wide variety of curves. But the conic sections cannot be represented in Bézier form. In order to be able to include conic sections in the set of representable curves in Bézier form, we turn to Rational Bézier curves.

The motivating idea is to take an nth degree Rational Bézier curve in $\mathbb{E}^3$ as the projection of an nth degree Bézier curve in $\mathbb{E}^4$ into the hyperplane $w = 1$. Rational Bézier curve $R(t)$ is defined by

$$R(t) = \frac{\sum_{i=0}^{n} w_i \mathsf{b}_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)} \qquad R(t), \mathsf{b}_i \in \mathbb{E}^3 \tag{1.3.1}$$

The positive real values w_i are called weights and the points b_i are the control points which is the projection of the $4D$ control points $[w_i \mathsf{b}_i \quad w_i]^T$. If the weights are set to $w_i = 1$ for all i, then we obtain polynomial Bézier curves.

Rational Bézier curves have the following properties

1. Convex hull property holds when all $w_i > 0$ and the Bézier polygon approximately describe the shape of the curve.

2. Endpoint interpolation: The first and the last points of the curve coincide with the control points $\mathsf{b}_0, \mathsf{b}_n$ respectively.

3. if all $w_i > 0$ then variation diminishing property holds.

4. Rational Bézier curves are affinely and projectively invariant. Projectively invariant means that the following procedure gives the same result:

i) Compute $P(t)$ and then project it to the hyperplane $w = 1$.

ii) Project the control polygon points $[w_i\mathsf{b}_i \quad w_i]^T$ to the hyperplane and then evaluate Bézier curve.

If we increase the weight w_i then all points on the curve move towards the control point b_i, if we decrease the weight w_i then all point of the curve move away from b_i.

Figure 1.5 shows rational Bézier curves with different weights and the same Bézier polygon

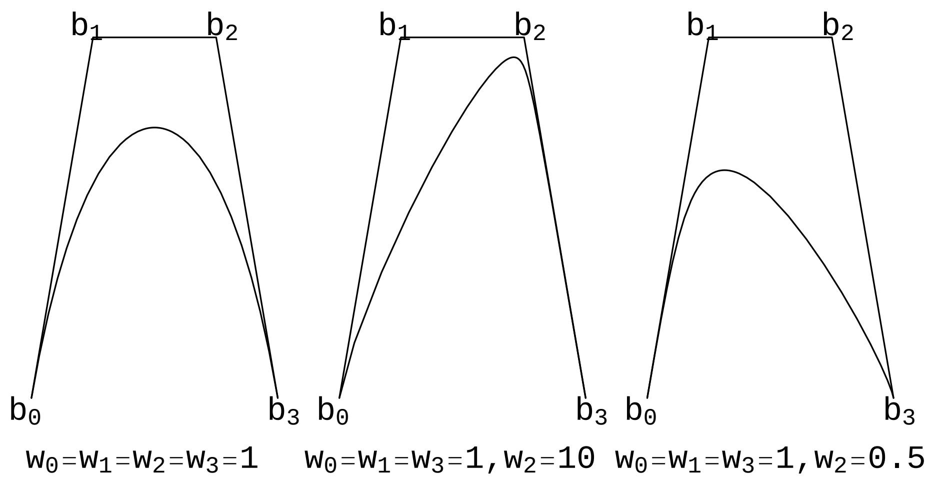

Figure 1.5: Rational Bézier curves with different weights and the same Bézier polygon.

1.3.1 The de Casteljau Algorithm

We now show that the de Casteljau algorithm can be extended to compute rational Bézier curves. By applying it to the homogeneous coordinates $[w_i\mathsf{b}_i \quad w_i]^T$. Namely, compute the de Casteljau algorithm for the weights and the weighted control points.

Let $\mathsf{c}_i = [w_i \mathsf{b}_i \quad w_i]^T$ be the control points of a $4D$ curve. Then by (1.2.2)

$$\mathsf{c}_i^r(t) = (1-t)\mathsf{c}_i^{r-1}(t) + t\mathsf{c}_{i+1}^{r-1}(t). \tag{1.3.2}$$

with $\mathsf{c}_i^0 = \mathsf{c}_i$ in $\mathbb{E}^4$ which we can write

$$w_i^r(t)\mathsf{b}_i^r(t) = (1-t)w_i^{r-1}(t)\mathsf{b}_i^{r-1}(t) + tw_{i+1}^{r-1}(t)\mathsf{b}_{i+1}^{r-1}(t) \tag{1.3.3}$$

where

$$w_i^r(t) = (1-t)w_i^{r-1}(t) + tw_{i+1}^{r-1}(t) \tag{1.3.4}$$

with $\mathsf{b}_i^0 = \mathsf{b}_i, \quad w_i^0 = w_i$ for all i. Then we see by induction on n that $\mathsf{c}_0^n(t) = P(t)$ and $\mathsf{b}_0^n(t) = R(t)$.

The explicit form of the intermediate point $\mathsf{b}_i^r(t)$ is given by

$$\mathsf{b}_i^r(t) = \frac{\sum_{j=0}^r w_{i+j}\mathsf{b}_{i+j}B_j^r(t)}{\sum_{j=0}^r w_{i+j}B_j^r(t)} \tag{1.3.5}$$

which are also rational Bézier curves.

As in the standard Bézier curves, de Casteljau algorithm maybe used to subdivide a rational Bézier curve into two curve segments. The control points and the weights corresponding to the interval $[0, t_0]$ are respectively given by

$$\mathsf{b}_i^{(l)} = \mathsf{b}_0^i(t_0), \qquad w_i^{(l)} = w_0^i(t_0) \qquad i = 0, 1, \ldots, n.$$

The control points and the weights corresponding to the interval $[t_0, 1]$ are

$$\mathsf{b}_i^{(r)} = \mathsf{b}_i^{n-i}(t_0), \qquad w_i^{(r)} = w_i^{n-i}(t_0) \qquad i = 0, 1, \ldots, n.$$

We may also represent a rational Bézier curve of degree n by a rational Bézier curve of degree $n+1$ to increase the flexibility of the curve. One can do this by degree elevating the $4D$ control polygon with vertices $[w_i \mathsf{b}_i \quad w_i]^T$ and then projecting it into hyperplane $w = 1$. Using a similar technique as in the Bézier curves, the control points b_i^1 of degree elevated curve are

$$\mathsf{b}_i^1 = \frac{\frac{i}{n+1}w_{i-1}\mathsf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right) w_i \mathsf{b}_i}{\frac{i}{n+1}w_{i-1} + \left(1 - \frac{i}{n+1}\right) w_i}, \qquad i = 0, 1, \ldots, n+1. \tag{1.3.6}$$

The weights w_i^1 of the new control points are

$$w_i^1 = \frac{i}{n+1}w_{i-1} + \left(1 - \frac{i}{n+1}\right) w_i, \qquad i = 0, 1, \ldots, n+1. \tag{1.3.7}$$

CHAPTER TWO
GENERALIZATION of BÉZIER CURVES

In this chapter, following the papers (Goodman, Oruç & Phillips (1999); Oruç & Phillips (1999); Phillips (1996)) we outline geometric properties of q-Bernstein Bézier curves which is a generalization of Bernstein Bézier curves.

2.1 q-BERNSTEIN BÉZIER POLYNOMIALS

q-Bernstein Bézier polynomial are first introduced in (See Phillips, 1997) as a generalization of Bernstein polynomial and studied in (Goodman, Oruç & Phillips (1999); Oruç & Phillips (1999); Phillips (1996)) in view of geometric modelling. One parameter family (q, the parameter) of Bernstein Bézier polynomials (called q-Bernstein Bézier polynomials) are defined by

$$B_i^{n,q}(t) = \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{s=0}^{n-i-1} (1 - q^s t), \qquad t \in [0,1], \quad 0 \leqslant i \leqslant n, \tag{2.1.1}$$

where an empty product denotes 1 and the parameter q is a positive real number. For the sake of simplicity, we denote $B_i^{n,q}(t)$ by $B_i^n(t)$ unless the parameter q is emphasized. The q-binomial coefficient $\begin{bmatrix} n \\ i \end{bmatrix}$, which is also called a Gaussian polynomial see (See Andrews, 1998), is defined as

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n][n-1]\cdots[n-i+1]}{[i][i-1]\cdots[1]} \tag{2.1.2}$$

for $0 \leqslant i \leqslant n$, and has the value 1 when $i = 0$ and the value 0 otherwise. Here $[i]$ denotes a q-integer, defined by

$$[i] = \begin{cases} (1-q^i)/(1-q), & q \neq 1, \\ i, & q = 1. \end{cases} \tag{2.1.3}$$

When q=1 the q-binomial coefficients reduces to the usual binomial coefficients. The q-binomial coefficient satisfies the following recurrence relations

$$\begin{bmatrix} n \\ i \end{bmatrix} = q^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ i \end{bmatrix} \tag{2.1.4}$$

and

$$\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + q^{i} \begin{bmatrix} n-1 \\ i \end{bmatrix} \tag{2.1.5}$$

Using (2.1.4) it is easily shown by induction on n that

$$(1-t)(1-qt)\cdots(1-q^{n-1}t) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} t^i \tag{2.1.6}$$

It follows by using (2.1.4) in (2.1.1) that $B_i^n(t)$ can be computed recursively by

$$B_i^n(t) = q^{n-i} t B_{i-1}^{n-1}(t) + (1 - q^{n-i-1}t) B_i^{n-1}(t). \tag{2.1.7}$$

Similarly on using (2.1.5) in (2.1.1) we have

$$B_i^n(t) = t B_{i-1}^{n-1}(t) + (q^i - q^{n-1}t) B_i^{n-1}(t), \tag{2.1.8}$$

(See Oruç & Phillips, 2003).

q-Bernstein Bézier polynomials have partition of unity property and endpoint condition but do not hold the symmetry property (1.1.3).

The following new result will be needed to prove subdivision formula for q-Bernstein Bézier curves.

Lemma 2.1.1. *Let $B_i^n(t)$ be q-Bernstein Bézier polynomial and let $c \in (0,1)$ be a fixed real. Then*

$$B_i^n(ct) = \sum_{j=0}^{n} B_i^j(c) B_j^n(t) \tag{2.1.9}$$

Proof. Let M be a $(n+1) \times (n+1)$ matrix and its elements be $M_{i,j} = B_j^i(ct)$ $i = 0, \ldots, n$ and $j = 0, \ldots, n$.

It is clear that the matrix M is lower triangular matrix with nonzero diagonal elements. It follows that each diagonal element of M is an eigenvalue of M. Since the eigenvalues are distinct the matrix M can be written as $M = PDP^{-1}$ where D is diagonal matrix whose elements $D_{i,i}$ are the eigenvalues of M, $D_{i,i} = c^i t^i$. It is computed from the product that the elements $P_{i,j}$ of the matrix P are $P_{i,j} = \begin{bmatrix} i \\ j \end{bmatrix}$ and the elements of the matrix P^{-1} are $(P^{-1})_{i,j} = (-1)^{i-j} q^{(i-j-1)(i-j)/2} \begin{bmatrix} i \\ j \end{bmatrix}$. These matrices are obtained in the factorization of the Vandermonde matrix at the q-integer nodes (See Oruç & Akmaz, 2004). Now we can write $M = PDP^{-1} = PD_1D_2P^{-1}$, where D_1 and D_2 are diagonal matrices with elements $(D_1)_{i,i} = t^i$ and $(D_2)_{i,i} = c^i$, $i = 0, 1, \ldots, n$. Then it follow from

$$M = PD_1P^{-1}PD_2P^{-1} = TC$$

that the matrices T and C have the elements $T_{i,j} = B^i_j(t)$ and $C_{i,j} = B^i_j(c)$ respectively. Thus by multiplication rule of two matrices we obtain

$$M_{n,i} = B^n_i(ct) = \sum_{j=0}^{n} T_{n,j} C_{j,i} = \sum_{j=0}^{n} B^n_j(t) B^j_i(c),$$

which completes the proof. □

2.2 ONE PARAMETER FAMILY of BÉZIER CURVES

One parameter family of Bézier curves (called q-Bernstein Bézier curves) of degree n is defined by

$$P(t) = \sum_{i=0}^{n} \mathsf{b}_i \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{j=0}^{n-i-1} (1 - q^j t). \tag{2.2.1}$$

Note that if we set the parameter q to the value 1, we obtain ordinary Bézier curve.

The properties of q-Bernstein Bézier curve:

1. Convex hull property holds when $0 < q \leqslant 1$ and the Bézier polygon approximately describe the shape of the curve.

2. Affine invariance property holds.

3. The curve passes through the endpoints b_0 and b_n.

$$P(0) = \mathsf{b}_0, \qquad P(1) = \mathsf{b}_n$$

4. If $q \in (0, 1]$ then the variation diminishing property holds.

Figure 2.6 depicts two q-Bernstein Bézier curves with the same control polygon but different values of q.

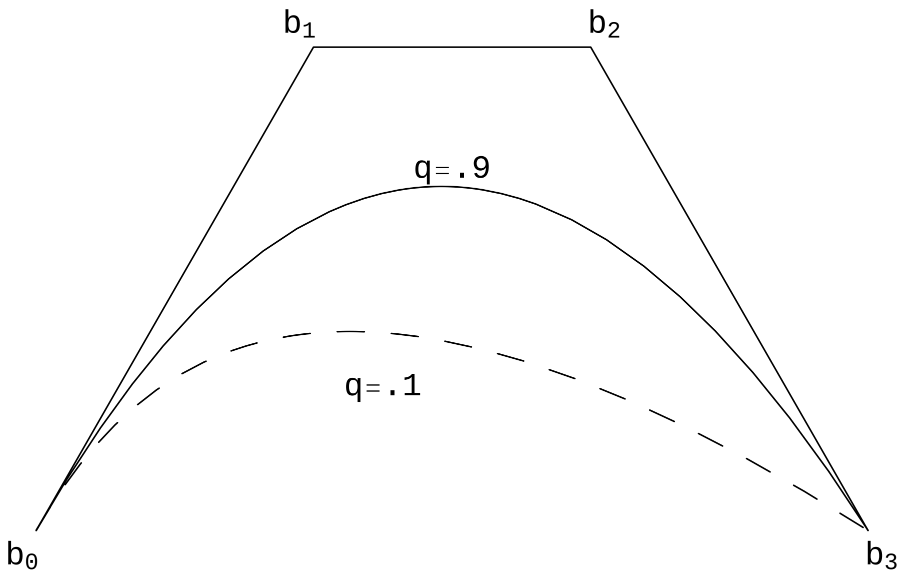

Figure 2.6: Two q-Bernstein Bézier curves with different values of q.

The q-Bernstein Bézier curve can be expressed in terms of q-differences. We define q-differences by

$$\Delta^0 \mathsf{b}_i = \mathsf{b}_i$$

for $i = 0, 1, \ldots, n$ and, recursively

$$\Delta^{k+1}\mathsf{b}_i = \Delta^k \mathsf{b}_{i+1} - q^k \Delta^k \mathsf{b}_i. \tag{2.2.2}$$

for $k = 0, 1, \ldots, n-i-1$. When $q = 1$, these q-differences reduces to ordinary forward differences. It is easily established by induction that

$$\Delta^k \mathsf{b}_i = \sum_{r=0}^{k} (-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} \mathsf{b}_{i+k-r}. \tag{2.2.3}$$

Then one may obtain the q-difference form

$$P(t) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r \mathsf{b}_0 \, t^r. \tag{2.2.4}$$

(See Phillips, 1997)

2.2.1 The de Casteljau Algorithm

A generalization of the de Casteljau algorithm to compute q-Bernstein Bézier curves, is given in (Phillips, 1996)

$$\mathsf{b}_i^r(t) = (q^i - q^{r-1}t)\mathsf{b}_i^{r-1} + t\mathsf{b}_{i+1}^{r-1}, \qquad \begin{cases} r = 0, 1, \ldots, n \\ i = 0, 1, \ldots, n-r \end{cases} \tag{2.2.5}$$

The intermediate points of the de Casteljau type algorithm obtained explicitly as

$$\mathsf{b}_i^r(t) = \sum_{j=0}^{r} \mathsf{b}_{i+j} \begin{bmatrix} r \\ j \end{bmatrix} t^j \prod_{s=0}^{r-j-1} (q^i - q^s t). \tag{2.2.6}$$

These points may be written in terms of q-differences as

$$\mathsf{b}_i^r(t) = \sum_{j=0}^{r} q^{(r-j)i} \begin{bmatrix} r \\ j \end{bmatrix} t^j \Delta^j \mathsf{b}_i \tag{2.2.7}$$

(See Phillips, 1996). Note that the parameter value $q = 1$ reduces to the usual de Casteljau algorithm.

2.2.2 Degree Elevation

This process for the q-Bernstein Bézier curve is studied in (Oruç & Phillips, 2003).The formulas

$$(1 - q^{n-i}t)B_i^n(t) = \frac{[n+1-i]}{[n+1]}B_i^{n+1} \tag{2.2.8}$$

and

$$(q^{n-i}t)B_i^n(t) = \left(1 - \frac{[n-i]}{[n+1]}\right) B_{i+1}^{n+1}(t) \tag{2.2.9}$$

which follow from (2.1.1) will be useful.

First write the curve $P(t)$ in the form

$$P(t) = (1 - q^{n-i}t)P(t) + q^{n-i}tP(t)$$

and then use the identities (2.2.8) and (2.2.9) to obtain

$$P(t) = \sum_{i=0}^{n} \frac{[n+1-i]}{[n+1]}\mathsf{b}_i B_i^{n+1}(t) + \sum_{i=0}^{n} \left(1 - \frac{[n-i]}{[n+1]}\right) \mathsf{b}_i B_{i+1}^{n+1}(t)$$

Now, we may write these two summation by shifting their limits

$$P(t) = \sum_{i=0}^{n+1} \frac{[n+1-i]}{[n+1]}\mathsf{b}_i B_i^{n+1}(t) + \sum_{i=0}^{n+1} \left(1 - \frac{[n+1-i]}{[n+1]}\right) \mathsf{b}_{i-1} B_i^{n+1}(t)$$

where b_{-1} is set to zero vector. Comparing coefficients of both sides of the last equation, the new control points b_i^1 are obtained as follows:

$$\mathsf{b}_i^1 = \left(1 - \frac{[n+1-i]}{[n+1]}\right) \mathsf{b}_{i-1} + \frac{[n+1-i]}{[n+1]}\mathsf{b}_i, \qquad i = 0, 1 \ldots, n+1. \tag{2.2.10}$$

CHAPTER THREE
GENERALIZATION of RATIONAL BÉZIER CURVES

In this chapter, we generalize rational Bézier curves. A de Casteljau algorithm is obtained to compute q-Bernstein Bézier curves. We give explicit formulas for the intermediate points of the de Casteljau algorithm. Subdivision and degree elevation of rational q-Bernstein Bézier curves are also studied. Finally, we represent this curve in matrix form by using a change of basis matrix.

3.1 ONE PARAMETER FAMILY of RATIONAL BÉZIER CURVES

We introduce a generalization of q-Bernstein Bézier curves via rational approach. An analogues technique is used as in the section (2.2). One may consider a one parameter family of rational q-Bernstein curve of degree n in $\mathbb{E}^3$ as the projection of nth degree q-Bernstein Bézier curve in $\mathbb{E}^4$ into hyperplane $w = 1$.

Let $R(t) \in \mathbb{E}^3$ be a point of nth degree rational q-Bernstein Bézier curve. We may identify $R(t) \in \mathbb{E}^3$ with $[R(t), \quad 1]^T \in \mathbb{E}^4$. This point for $t \in [0,1]$ is the projection of a point $[w(t)R(t) \quad w(t)]^T$ which lies on the curve of degree n in $4D$. The fourth component $w(t)$ of this point must be nth degree polynomial in t, and may be expressed in terms of q-Bernstein Bézier polynomials by

$$w(t) = \sum_{i=0}^{n} w_i B_i^n(t), \text{ where } w_i \in \mathbb{R}.$$

We now may write

$$w(t) \begin{bmatrix} R(t) \\ 1 \end{bmatrix} = \begin{bmatrix} R(t) \sum_{i=0}^{n} w_i B_i^n(t) \\ \sum_{i=0}^{n} w_i B_i^n(t) \end{bmatrix}.$$

The left hand side of the equation is an nth degree rational curve, and we have

$$\sum_{i=0}^{n}\begin{bmatrix}\mathsf{p}_i\\ w_i\end{bmatrix}B_i^n(t)=\begin{bmatrix}R(t)\sum_{i=0}^n w_iB_i^n(t)\\ \sum_{i=0}^n w_iB_i^n(t)\end{bmatrix}$$

with the some points $\mathsf{p}_i\in\mathbb{E}^3$. Thus,

$$\sum_{i=0}^{n}\mathsf{p}_iB_i^n(t)=R(t)\sum_{i=0}^{n}w_iB_i^n(t)$$

and hence

$$R(t)=\frac{\sum_{i=0}^n \mathsf{p}_iB_i^n(t)}{\sum_{i=0}^n w_iB_i^n(t)}.$$

Setting $\mathsf{p}_i=w_i\mathsf{b}_i$ gives

$$R(t)=\frac{\sum_{i=0}^n w_i\mathsf{b}_iB_i^n(t)}{\sum_{i=0}^n w_iB_i^n(t)}. \tag{3.1.1}$$

The points b_i form the control points of the rational curve $R(t)$. The numbers w_i are called weights associated with b_i. Here $w_i>0$ for all values of i.

3.1.1 The de Casteljau Algorithm

The usual de Casteljau algorithm may be adopted to rational q-Bernsetein Bézier curves by projecting each intermediate de Casteljau point $[w_i^r\mathsf{b}_i^r \quad w_i^r]^T$ into $\mathbb{E}^3$.

The de Casteljau type algorithm for a $4D$ polynomial Bézier curve is

$$\mathsf{c}_i^r(t)=(q^i-q^{r-1}t)\mathsf{c}_i^{r-1}(t)+t\mathsf{c}_{i+1}^{r-1}(t),\qquad \begin{cases} r=1,2,\ldots,n\\ i=0,1,\ldots,n-r\end{cases}$$

where c_i are the points of control polygon vertices of $4D$ curve in homogeneous form

$$\mathsf{c}_i=\begin{bmatrix}\mathsf{p}_i\\ w_i\end{bmatrix}=\begin{bmatrix}w_i\mathsf{b}_i\\ w_i\end{bmatrix}.$$

Hence,

$$\mathsf{c}_i^r(t) = \begin{bmatrix} w_i^r \mathsf{b}_i^r \\ w_i^r \end{bmatrix} = (q^i - q^{r-1}t) \begin{bmatrix} w_i^{r-1} \mathsf{b}_i^{r-1} \\ w_i^{r-1} \end{bmatrix} + t \begin{bmatrix} w_{i+1}^{r-1} \mathsf{b}_{i+1}^{r-1} \\ w_{i+1}^{r-1} \end{bmatrix}.$$

and

$$w_i^r = (q^i - q^{r-1}t) w_i^{r-1} + t w_{i+1}^{r-1}.$$

Since the projection of the points gives intermediate de Casteljau points for rational q-Bernstein Bézier curve we obtain

$$\mathsf{b}_i^r(t) = \frac{(q^i - q^{r-1}t) w_i^{r-1} \mathsf{b}_i^{r-1} + t w_{i+1}^{r-1} \mathsf{b}_{i+1}^{r-1}}{w_i^r}, \quad \begin{cases} r = 1, 2, \ldots, n \\ i = 0, 1, \ldots, n-r \end{cases} \tag{3.1.2}$$

where,

$$w_i^r = (q^i - q^{r-1}t) w_i^{r-1} + t w_{i+1}^{r-1}, \quad \begin{cases} r = 1, 2, \ldots, n \\ i = 0, 1, \ldots, n-r \end{cases}$$

Theorem 3.1.1. *Each intermediate point $\mathsf{b}_i^r(t)$ of the de Casteljau algorithm (3.1.2) can be expressed as*

$$\mathsf{b}_i^r(t) = \frac{\sum_{j=0}^{r} w_{i+j} \mathsf{b}_{i+j} \begin{bmatrix} r \\ j \end{bmatrix} t^j \prod_{s=0}^{r-j-1} (q^i - q^s t)}{\sum_{j=0}^{r} w_{i+j} \begin{bmatrix} r \\ j \end{bmatrix} t^j \prod_{s=0}^{r-j-1} (q^i - q^s t)}, \quad \begin{cases} r = 1, 2, \ldots, n \\ i = 0, 1, \ldots, n-r \end{cases} \tag{3.1.3}$$

Proof. We use induction on r. First denote the expression $\begin{bmatrix} r \\ j \end{bmatrix} t^j \prod_{s=0}^{r-j-1} (q^i - q^s t)$ by $C_{i,j}^r$. It is easily seen from (3.1.3) that

$$\mathsf{b}_i^r(t) = \frac{\sum_{j=0}^{r} w_{i+j} \mathsf{b}_{i+j} C_{i,j}^r}{\sum_{j=0}^{r} w_{i+j} C_{i,j}^r}$$

For $r = 0$ it is clear that $\mathsf{b}_i^0(t) = \mathsf{b}_i^0$, the control points.
Let us assume that (3.1.3) holds for a given r, $0 \leqslant r < n$ and for $0 \leqslant i \leqslant n - r$. The proof is complete if we show (3.1.3) is true for $r + 1$, $0 \leqslant r + 1 < n$ and for $0 \leqslant i \leqslant n - r - 1$. By (3.1.2) we have

$$\mathsf{b}_i^{r+1}(t) = \frac{(q^i - q^r t) w_i^r \mathsf{b}_i^r + t w_{i+1}^r \mathsf{b}_{i+1}^r}{w_i^{r+1}}, \qquad i = 0, 1, \ldots, n - r - 1$$

It follows from the fact $w_i^r(t) = \sum_{j=0}^r w_{i+j} C_{i,j}^r$ (See Phillips, 1996) that the last equation yields

$$\mathsf{b}_i^{r+1}(t) = \frac{(q^i - q^r t)\sum_{j=0}^r w_{i+j}\mathsf{b}_{i+j}C_{i,j}^r + t\sum_{j=0}^r w_{i+j+1}\mathsf{b}_{i+j+1}C_{i+1,j}^r}{\sum_{j=0}^{r+1} w_{i+j}C_{i,j}^{r+1}}$$

Shifting the index of the second summation of the numerator we have

$$\mathsf{b}_i^{r+1} = \frac{\sum_{j=0}^r (q^i - q^r t) w_{i+j}\mathsf{b}_{i+j}C_{i,j}^r + \sum_{j=1}^{r+1} t w_{i+j}\mathsf{b}_{i+j}C_{i+1,j-1}^r}{\sum_{j=0}^{r+1} w_{i+j}C_{i,j}^{r+1}} \tag{3.1.4}$$

Note that

$$\prod_{s=0}^{r-j}(q^{i+1} - q^s t) = (q^{i+1} - t)\prod_{s=1}^{r-j}(q^{i+1} - q^s t) = (q^{i+1} - t)q^{r-j}\prod_{s=1}^{r-j}(q^i - q^{s-1}t)$$

$$= (q^{i+1} - t)q^{r-j}\prod_{s=0}^{r-j-1}(q^i - q^s t).$$

Then using the last equation in (3.1.4) we have

$$\mathsf{b}_i^{r+1}(t) = \frac{w_i\mathsf{b}_i(q^i - q^r)C_{i,0}^{r+1} + \sum_{j=1}^r w_{i+j}\mathsf{b}_{i+j}t^j\prod_{s=0}^{r-j-1}(q^i - q^s t)A_{r,j} + w_{i+r+1}\mathsf{b}_{i+r+1}t^{r+1}}{\sum_{j=0}^{r+1} w_{i+j}C_{i,j}^{r+1}}$$

where

$$A_{r,j} = \left(\begin{bmatrix} r \\ j \end{bmatrix}(q^i - q^r t) + \begin{bmatrix} r \\ j-1 \end{bmatrix}(q^{i+1} - t)q^{r-j}\right).$$

Rearranging $A_{r,j}$ and using the identities (2.1.4) and (2.1.5) we obtain

$$A_{r,j} = q^i\left(\begin{bmatrix} r \\ j \end{bmatrix} + q^{r+1-j}\begin{bmatrix} r \\ j-1 \end{bmatrix}\right) - q^{r-j}t\left(q^j\begin{bmatrix} r \\ j \end{bmatrix} + \begin{bmatrix} r \\ j-1 \end{bmatrix}\right) = (q^i - q^{r-j}t)\begin{bmatrix} r+1 \\ j \end{bmatrix}.$$

Thus,

$$\mathsf{b}_i^{r+1}(t) = \frac{w_i\mathsf{b}_i C_{i,0}^{r+1} + \sum_{j=1}^r w_{i+j}\mathsf{b}_{i+j}C_{i,j}^{r+1} + w_{i+r+1}\mathsf{b}_{i+r+1}t^{r+1}}{\sum_{j=0}^{r+1} w_{i+j}C_{i,j}^{r+1}}$$

and

$$\mathsf{b}_i^{r+1}(t) = \frac{\sum_{j=0}^{r+1} w_{i+j}\mathsf{b}_{i+j}\begin{bmatrix} r+1 \\ j \end{bmatrix} t^j \prod_{s=0}^{r-j}(q^i - q^s t)}{\sum_{j=0}^{r+1} w_{i+j}\begin{bmatrix} r+1 \\ j \end{bmatrix} t^j \prod_{s=0}^{r-j}(q^i - q^s t)},$$

which completes the induction. □

Corollary 3.1.1. *The intermediate point $\mathsf{b}_0^n(t)$ of the de Casteljau algorithm, with a value t is on the rational q-Bernstein Bézier curve $R(t)$. Hence by continuity $\mathsf{b}_0^n(t) = R(t)$.*

Another way to deduce the above formula is to find the intermediate points of de Casteljau type algorithm of nth degree curve in $4D$ and project them into the hyperplane $w = 1$.

We can also show that $\mathsf{b}_i^r(t)$ can be expressed in terms of q-differences.

Theorem 3.1.2. *The intermediate points of the de Casteljau type algorithm can be expressed as*

$$\mathsf{b}_i^r(t) = \frac{\sum_{j=0}^{r} q^{(r-j)i}\begin{bmatrix} r \\ j \end{bmatrix} \triangle^j (w_i\mathsf{b}_i)t^j}{\sum_{j=0}^{r} q^{(r-j)i}\begin{bmatrix} r \\ j \end{bmatrix} \triangle^j w_i t^j}$$

where $\triangle^j (w_i\mathsf{b}_i) = \triangle^{j-1} (w_{i+1}\mathsf{b}_{i+1}) - q^{j-1} \triangle^{j-1} (w_i\mathsf{b}_i)$.

Proof. It is proved in (Phillips, 1996) that

$$w_i^r(t) = \sum_{j=0}^{r} w_{i+j}C_{i,j}^r = \sum_{j=0}^{r} q^{(r-j)i}\begin{bmatrix} r \\ j \end{bmatrix} \triangle^j w_i t^j, \qquad \begin{cases} r = 1, 2, \ldots, n \\ i = 0, 1, \ldots, n-r \end{cases} \tag{3.1.5}$$

From the equation (3.1.3) we have

$$\mathsf{b}_i^r(t) = \frac{\sum_{j=0}^{r} w_{i+j}\mathsf{b}_{i+j}C_{i,j}^r}{\sum_{j=0}^{r} w_{i+j}C_{i,j}^r}.$$

Setting $\mathsf{c}_{i+j} = w_{i+j}\mathsf{b}_{i+j}$ and writing (3.1.5) in the numerator and the dominator of the last equation gives

$$\mathsf{b}_i^r(t) = \frac{\sum_{j=0}^{r} \mathsf{c}_{i+j}C_{i,j}^r}{\sum_{j=0}^{r} w_{i+j}C_{i,j}^r} = \frac{\sum_{j=0}^{r} q^{(r-j)i}\begin{bmatrix} r \\ j \end{bmatrix} \triangle^j \mathsf{c}_i\, t^j}{\sum_{j=0}^{r} q^{(r-j)i}\begin{bmatrix} r \\ j \end{bmatrix} \triangle^j w_i\, t^j} = \frac{\sum_{j=0}^{r} q^{(r-j)i}\begin{bmatrix} r \\ j \end{bmatrix} \triangle^j (w_i\mathsf{b}_i)\, t^j}{\sum_{j=0}^{r} q^{(r-j)i}\begin{bmatrix} r \\ j \end{bmatrix} \triangle^j w_i\, t^j},$$

and this completes the proof. □

As a consequence of the above theorem we have the following result.

Corollary 3.1.2. *The rational q-Bernstein Bézier curve can be expressed in q-differences*

$$\mathsf{b}_0^n(t) = \frac{\sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \triangle^j \, (w_0 \mathsf{b}_0) \, t^j}{\sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \triangle^j \, w_0 \, t^j}.$$

3.1.2 Subdivision

In standard rational Bézier curves, in the first step of subdivision we use the intermediate points of the de Casteljau algorithm to have two different new curves which join together at the some point $t_0 \in (0,1)$ to form the original curve. The curve with the control points $\mathsf{b}_0^i(t_0)$ is the left part of the rational curve with the weights $w_0^i(t)$. The curve with the control points $\mathsf{b}_i^{n-i}(t_0)$ gives the right part of the rational curve with the weights $w_i^{n-i}(t)$. In general the simplest choice is to take $t_0 = 1/2$.

In the second step we use the same procedure to both curves and have four control polygons. If this procedure is repeated k times we have 2^k curve segments and the corresponding control polygons. As $k \to \infty$ all points of the control polygons lies on the original curve, (Micchelli, 1995).

Theorem 3.1.3. *Let $R(t)$ be a rational q-Bernstein Bézier curve of degree n with control points $\mathsf{b}_i, i = 0, 1, \ldots, n$. Then the part of the curve that correspond to the interval $[0, c], c \in (0, 1)$ denoted by $R_{[0,c]}(t)$ is*

$$R_{[0,c]}(t) = \frac{\sum_{j=0}^n w_0^j(c) \mathsf{b}_0^j(c) B_j^n(t)}{\sum_{j=0}^n w_0^j(c) B_j^n(t)}, \qquad t \in [0,1] \tag{3.1.6}$$

where $\mathsf{b}_0^j(c)$ and $w_0^j(c)$ are evaluated from the de Casteljau type algorithm (3.1.2).

Proof. From (3.1.1) we have

$$R_{[0,1]}(t) = \frac{\sum_{i=0}^{n} w_i \mathsf{b}_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}, \qquad t \in [0,1].$$

Note that if $t \in [0,1]$ then $ct \in [0,c]$ where $c \in (0,1)$. So we can find the part of the curve that correspond to the part $[0,c]$ as

$$R_{[0,c]}(t) = \frac{\sum_{i=0}^{n} w_i \mathsf{b}_i B_i^n(ct)}{\sum_{i=0}^{n} w_i B_i^n(ct)}, \qquad t \in [0,1].$$

By using (2.1.9) we obtain

$$R_{[0,c]}(t) = \frac{\sum_{i=0}^{n} w_i \mathsf{b}_i \sum_{j=0}^{n} B_i^j(c) B_j^n(t)}{\sum_{i=0}^{n} w_i \sum_{j=0}^{n} B_i^j(c) B_j^n(t)},$$

and equivalently

$$R_{[0,c]}(t) = \frac{\sum_{j=0}^{n} \sum_{i=0}^{n} w_i \mathsf{b}_i B_i^j(c) B_j^n(t)}{\sum_{j=0}^{n} \sum_{i=0}^{n} w_i B_i^j(c) B_j^n(t)}.$$

Since, $B_i^j(c) = 0$ for $i > j$ we have

$$R_{[0,c]}(t) = \frac{\sum_{j=0}^{n} \sum_{i=0}^{j} w_i \mathsf{b}_i B_i^j(c) B_j^n(t)}{\sum_{j=0}^{n} \sum_{i=0}^{j} w_i B_i^j(c) B_j^n(t)}.$$

It follows from (3.1.3) and (2.2.6) that

$$R_{[0,c]}(t) = \frac{\sum_{j=0}^{n} w_0^j(c) \mathsf{b}_0^j(c) B_j^n(t)}{\sum_{j=0}^{n} w_0^j(c) B_j^n(t)}, \qquad t \in [0,1].$$

□

However, the q-analogue of the de Casteljau type algorithm (3.1.2) does not lead to clear subdivision formula for the curve that correspond to the part $[c,1]$ because of the lack of the symmetry property (1.1.3). It may be too complicated to compute a q-Bernstein Bézier curve except the interval $[0,t_0]$. We cope with this difficulty as follows.

Let $\phi = \{\phi_0, \phi_1, \ldots, \phi_n\}$ and $\psi = \{\psi_0, \psi_1, \ldots, \psi_n\}$ with elements $\phi_i = \binom{n}{i} t^i (1-t)^{n-i}$, $\psi_i = \begin{bmatrix} n \\ i \end{bmatrix} t^i \prod_{s=0}^{n-i-1} (1-q^s t), i = 0, 1, \ldots, n$ represent the bases for polynomials of degree at most n.

The change of basis matrix M satisfying $\psi^T = M\phi^T$ is obtained in (Oruç & Phillips, 2003) and the elements $M_{i,j}$ of M are

$$M_{i,j} = \frac{\begin{bmatrix} n \\ i \end{bmatrix}}{\binom{n}{j}} (1-q)^{j-i} S(n-i-1, j-i),$$

where $S(n,k) = 0$ for $k < 0$ and $k > n$, $S(n,0) = 1$ for $n \geqslant -1$ and $S(n,k)$ satisfies the recurrence relation

$$S(n,k) = S(n-1,k) + [n]S(n-1,k-1).$$

So, for any rational q-Bernstein Bézier curve with $0 < q < 1$ to find new control polygon and new weights we transform the control polygon of projected non-rational Bézier curve by multiplying by the matrix M. That is

$$(\mathsf{c}_0, \mathsf{c}_1, \ldots, \mathsf{c}_n)M = (\tilde{\mathsf{c}}_0, \tilde{\mathsf{c}}_1, \ldots, \tilde{\mathsf{c}}_n)$$

where $\mathsf{c}_i = \begin{bmatrix} w_i \mathsf{b}_i \\ w_i \end{bmatrix}, \tilde{\mathsf{c}}_i = \begin{bmatrix} \tilde{w}_i \tilde{\mathsf{b}}_i \\ \tilde{w}_i \end{bmatrix}$. Here $\tilde{\mathsf{b}}_i$ are new control points and $\tilde{w}_i$ are corresponding new weights. Now we can find subdivision formulas for the curves that correspond to the part $[0, c]$ and $[c, 1]$. After finding curve segments and associated control polygons we transform these polygons by multiplying the matrix M^{-1} to represent the curve segments as q-Bernstein Bézier curves.

Example 3.1.1. Let us have rational q-Bernstein Bézier curve with $q = 1/2$, Bézier points

$$\mathsf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathsf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathsf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathsf{b}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

with the weights $w_0 = 1, w_1 = 2, w_2 = 2, w_3 = 1$ respectively.

Let us subdivide this rational q-Bernstein Bézier curve into two q-rational

Bézier curve segments which join together at the point corresponding to $t = 1/2$.

After transformation of rational q-Bernstein Bézier curve to the rational Bézier curve we get new control points and corresponding weights as:

$$\tilde{\mathsf{b}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tilde{\mathsf{b}}_1 = \begin{bmatrix} 14/19 \\ 14/19 \end{bmatrix}, \tilde{\mathsf{b}}_2 = \begin{bmatrix} 14/9 \\ 14/45 \end{bmatrix}, \tilde{\mathsf{b}}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\tilde{w}_0 = 1, \tilde{w}_1 = 19/12, \tilde{w}_2 = 15/8, \tilde{w}_3 = 1$$

now we can use standard subdivision procedure to get new control polygons these are

$$\tilde{\mathsf{b}}_0^l = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tilde{\mathsf{b}}_1^l = \begin{bmatrix} 14/31 \\ 14/31 \end{bmatrix}, \tilde{\mathsf{b}}_2^l = \begin{bmatrix} 126/145 \\ 14/29 \end{bmatrix}, \tilde{\mathsf{b}}_3^l = \begin{bmatrix} 122/99 \\ 50/99 \end{bmatrix}$$

with the weights

$$\tilde{w}_0^l = 1, \tilde{w}_1^l = 31/24, \tilde{w}_2^l = 145/96, \tilde{w}_3^l = 99/64$$

and

$$\tilde{\mathsf{b}}_0^r = \begin{bmatrix} 122/99 \\ 50/99 \end{bmatrix}, \tilde{\mathsf{b}}_1^r = \begin{bmatrix} 30/19 \\ 10/19 \end{bmatrix}, \tilde{\mathsf{b}}_2^r = \begin{bmatrix} 142/69 \\ 38/69 \end{bmatrix}, \tilde{\mathsf{b}}_3^r = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

with the weights

$$\tilde{w}_0^r = 99/64, \tilde{w}_1^r = 19/12, \tilde{w}_2^r = 23/16, \tilde{w}_3^r = 1$$

respectively.

Here $\tilde{\mathsf{b}}_i^l$'s denote the control point of left rational Bézier curve and $\tilde{\mathsf{b}}_i^r$'s denote the control points of the right rational Bézier curve. It is remain to transform these rational Bézier curves to the rational q-Bernstein Bézier curves. After transformation we get the control points for the left q-rational Bézier curve as:

$$\mathsf{b}_0^l = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathsf{b}_1^l = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}, \mathsf{b}_2^l = \begin{bmatrix} 14/13 \\ 6/13 \end{bmatrix}, \mathsf{b}_3^l = \begin{bmatrix} 122/99 \\ 50/99 \end{bmatrix}$$

with the weights

$$w_0^l = 1, w_1^l = 3/2, w_2^l = 13/8, w_3^l = 99/64$$

respectively, and the control points for the right rational q-Bernstein Bézier curve are

$$\mathsf{b}_0^r = \begin{bmatrix} 122/99 \\ 50/99 \end{bmatrix}, \mathsf{b}_1^r = \begin{bmatrix} 1310/721 \\ 390/721 \end{bmatrix}, \mathsf{b}_2^r = \begin{bmatrix} 1434/595 \\ 338/595 \end{bmatrix}, \mathsf{b}_3^r = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

with the weights

$$w_0^r = 99/64, w_1^r = 103/64, w_2^r = 85/64, w_3^r = 1$$

The graph of these two curves are shown in the following figure

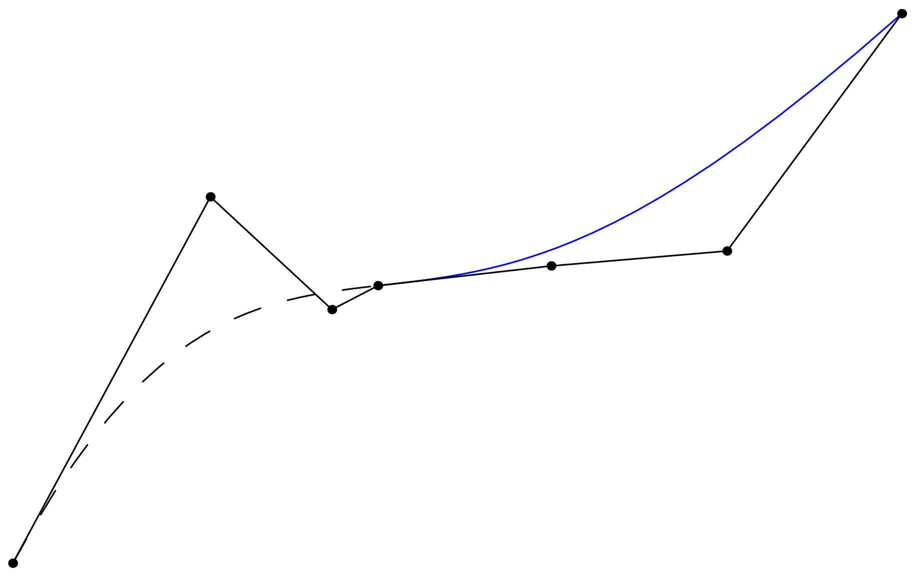

Figure 3.7: The dashed curve is curve segment corresponding to the interval $[0, 1/2]$ and the other curve segment is corresponding to the interval $[1/2, 1]$.

3.1.3 Degree Elevation

Degree elevation can be extended to rational q-Bernstein Bézier curves.

Theorem 3.1.4. *An nth degree rational q-Bernstein Bézier curve $R(t)$ can be represented as a rational q-Bernstein Bézier curve of degree $n+1$.*

$$R(t) = \frac{\sum_{i=0}^{n} w_i \mathsf{b}_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)} = \frac{\sum_{i=0}^{n+1} w_i^1 \mathsf{b}_i^1 B_i^{n+1}(t)}{\sum_{i=0}^{n+1} w_i^1 B_i^{n+1}(t)}$$

where

$$w_i^1 \mathsf{b}_i^1 = \left(1 - \frac{[n+1-i]}{[n+1]}\right) w_{i-1}\mathsf{b}_{i-1} + \frac{[n+1-i]}{[n+1]} w_i \mathsf{b}_i$$

$$w_i^1 = \left(1 - \frac{[n+1-i]}{[n+1]}\right) w_{i-1} + \frac{[n+1-i]}{[n+1]} w_i$$

for $i = 0, 1, \ldots, n+1$

Proof. For the q-Bernstein Bézier curve it is shown in section (2.2) that

$$\sum_{i=0}^{n} \mathsf{b}_i B_i^n(t) = \sum_{i=0}^{n+1} \mathsf{b}_i^1 B_i^{n+1}(t)$$

where

$$\mathsf{b}_i^1 = \left(1 - \frac{[n+1-i]}{[n+1]}\right) \mathsf{b}_{i-1} + \frac{[n+1-i]}{[n+1]} \mathsf{b}_i, \qquad i = 0, 1, \ldots, n+1$$

Since both the numerator and the denominator express q-Bernstein Bézier curve, we degree elevate them separetely giving

$$\frac{\sum_{i=0}^{n} w_i \mathsf{b}_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)} = \frac{\sum_{i=0}^{n+1} w_i^1 \mathsf{b}_i^1 B_i^{n+1}(t)}{\sum_{i=0}^{n} w_i^1 B_i^{n+1}(t)},$$

where

$$w_i^1 \mathsf{b}_i^1 = \left(1 - \frac{[n+1-i]}{[n+1]}\right) w_{i-1}\mathsf{b}_{i-1} + \frac{[n+1-i]}{[n+1]} w_i \mathsf{b}_i.$$

$\square$

3.1.4 Matrix Representation of Rational q-Bernstein Bézier Curves

We follow the work (Oruç & Phillips, 2003). Let $\Psi = (B_0^n(t), \ldots, B_n^n(t))$ and $\Phi = (1, t, \ldots, t^n)$. Both Ψ and Φ form a basis for the space of the polynomials of degree at most n. Thus we can find a transformation matrix M such that $\Psi^T = M\Phi^T$. Since

$$B_j^n(t) = \begin{bmatrix} n \\ j \end{bmatrix} t^j \prod_{s=0}^{n-j-1} (1 - q^s t),$$

from (2.1.6), we obtain

$$B_j^n(t) = \sum_{k=0}^{n-j} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix} t^{j+k}.$$

Shifting the limits of the above sum and writing

$$\begin{bmatrix} n-j \\ k-j \end{bmatrix} = \frac{\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}},$$

we deduce that

$$B_j^n(t) = \sum_{k=j}^{n} (-1)^{k-j} q^{(k-j)(k-j-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} t^k.$$

Since $\begin{bmatrix} k \\ j \end{bmatrix} = 0$ for $k < j$ we can write

$$B_j^n(t) = \sum_{k=0}^{n} (-1)^{k-j} q^{(k-j)(k-j-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} t^k.$$

Thus the elements $M_{i,j}$ of M are

$$M_{j,k} = (-1)^{k-j} q^{(k-j)(k-j-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} t^k.$$

The rational q-Bernstein Bézier curve of the form

$$R(t) = \frac{\sum_{i=0}^{n} w_i \mathsf{b}_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}$$

can be interpreted as

$$R(t) = \frac{[w_0\mathsf{b}_0, \ldots, w_n\mathsf{b}_n]\Psi^T}{[w_0, \ldots, w_n]\Psi^T}.$$

Thus we deduce that

$$R(t) = \frac{[w_0\mathsf{b}_0, \ldots, w_n\mathsf{b}_n]M\Phi^T}{[w_0, \ldots, w_n]M\Phi^T}. \tag{3.1.7}$$

If we set $q = 1$ then this representation reduce to the matrix representation of rational Bézier curves.

Another approach to find matrix representation of rational q-Bernstein Bézier curve $R(t)$ is to use homogeneous coordinates $\mathsf{c}_i = [w_i\mathsf{b}_i \quad w_i]^T$ which are control points of projected q-Bernstein Bézier curve $P(t)$ of degree n in $4D$. The matrix representation of this curve is $P(t) = [\mathsf{c}_0, \ldots, \mathsf{c}_n]\Psi^T = [\mathsf{c}_0, \ldots, \mathsf{c}_n]M\Phi^T$, if $P(t)$ is projected into the hyperplane $w = 1$ we obtain (3.1.7).

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