DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON THE EIGENVALUES OF A SCHRÖDINGER OPERATOR

by
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İZMİR

ON THE EIGENVALUES OF A SCHRÖDINGER OPERATOR

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M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"ON THE EIGENVALUES OF A SCHRÖDINGER OPERATOR"** completed by **DİDEM COŞKAN** under supervision of **YRD. DOÇ. DR. SEDEF KARAKILIÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ON THE EIGENVALUES OF A SCHRÖDINGER OPERATOR

ABSTRACT

In this thesis, we study on the eigenvalues of the self-adjoint Schrödinger operator, with mixed boundary condition defined on a d-dimensional parallelepiped F.

Keywords: Eigenvalues, mixed boundary condition, Schrödinger operator.

SCHRÖDINGER OPERATÖRÜNÜN ÖZDEĞERLERİ ÜZERİNE

ÖZ

Bu tezde, d boyutlu bir prizma F üzerinde karışık sınır şartı ile tanımlanan kendine eş Schrödinger operatörünün özdeğerleri üzerine çalışılmıştır.

Anahtar sözcükler: Karışık sınır şartı, özdeğerler, Schrödinger operatörü.

CONTENTS

CHAPTER ONE
INTRODUCTION

1.1 Introduction

The time independent Schrödinger operator

$$L(u) = -\Delta u + q(x)u$$

is one of the fundamental operators in quantum mechanics. Due to its physical importance, it has been studied for a long time.

For one dimensional case the perturbation theory can be applied and asymptotic formulas for sufficiently large eigenvalues can be easily obtained

$$\lambda_n = n^2 + O(\frac{1}{n}),$$

where λ_n is the eigenvalue of the perturbed operator andn^2 is the eigenvalue of the unperturbed operator.

However, in multy dimensional cases the eigenvalues influence each other strongly and the regular perturbation theory does not work.

For the first time asymptotic formulas for the eigenvalues of the periodic (with respect to an arbitrary lattice) Schrödinger operator with quasiperiodic boundary conditions are obtained by Veliev(1987). By some other methods, the asymptotic formulas for quasiperiodic boundary conditions in two and three dimensional cases are obtained in Feldman, Knoerrer, & Trubowitz (1990), Feldman, Knoerrer, & Trubowitz (1991), Karpeshina (1992), Karpeshina (1996). The asymptotic formulas for the eigenvalues of the Schrödinger operator with periodic boundary conditions are obtained in Friedlanger (1990). When this operator is considered with Dirichlet boundary conditions in two dimensional rectangle, the asymptotic formulas for the eigenvalues are obtained in Hald, & McLaughlin (1996). Atılgan, Karakılıç, & Veliev (2002) obtained the asymptotic formulas for the non-resonance eigenvalues of the Schrödinger operator with Dirichlet and Neumann boundary conditions in an arbitrary dimension. Also, the asymptotic formulas for the resonance eigenvalues of the Schrödinger operator with Dirichlet and Neumann boundary conditions are obtained in Karakılıç, Atılgan, & Veliev (2005) and Karakılıç, Veliev, & Atılgan (2005).

In this thesis, we consider the d-dimensional Schrödinger operator defined by the differential expression

$$Lu = -\Delta u + q(x)u \tag{1.1.1}$$

in F with the mixed boundary condition

$$(\alpha u(x) + \frac{\partial u}{\partial n}) \mid_{\partial F} = 0, \tag{1.1.2}$$

where $x = (x_1, x_2, \ldots, x_d) \in F, F = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_d], a_1, a_2, \ldots, a_d \in R, d \geq 2$, ∂F is the boundary of F, $q(x)$ is a real-valued function in $L_2(F)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator in R^d, $\alpha > 0$, $\frac{\partial}{\partial n}$ is the differentiation along the outward normal.

We denote the operator defined by (1.1.1) and (1.1.2) in $L_2(F)$ by $L_M(q(x))$ and the eigenvalues and the corresponding eigenfunctions of the operator$L_M(q(x))$ by Λ_N and Ψ_N, respectively.

The aim of this thesis is to obtain an asymptotic formula for the eigenvalues of the operator $L_M(q(x))$. For this, we use the method which is introduced by Veliev (1987). He studied the periodic Schrödinger operator with quasiperiodic boundary conditions. In this method, the eigenvalues of the unperturbed operator are divided into two groups: Resonance and Non-Resonance. In this thesis, we obtain the asymptotic formula for the non-resonance eigenvalues.

1.2 The Eigenvalues and the Eigenfunctions of the Operator$L_M(0)$

We first consider the unperturbed operator $L_M(0)$ which is defined by the differential expression (1.1.1) when $q(x) = 0$ and the mixed boundary condition (1.1.2).

We find the eigenvalues of the operator $L_M(0)$, that is, we solve the following eigenvalue problem:

$$-\Delta u = \lambda u, \tag{1.2.1}$$
$$(\alpha u + \frac{\partial u}{\partial n}) \mid_{\partial F} = 0. \tag{1.2.2}$$

For this we use the method of separation of variables: We seek a non-zero solution of (1.2.1)-(1.2.2) in the following form

$$u(x) = u_1(x_1)u_2(x_2)\ldots u_d(x_d). \tag{1.2.3}$$

Substituting (1.2.3) into (1.2.1) we obtain

$$-u_1''(x_1)\cdots u_d(x_d)-\cdots-u_1(x_1)\cdots u_d''(x_d)=\lambda u_1(x_1)u_2(x_2)\cdots u_d(x_d).$$

Since $u(x)$ is assumed to be an eigenfunction it is nonzero. Dividing both sides of the last equation by $u(x)$, we get

$$-\frac{u_1''(x_1)}{u_1(x_1)}-\frac{u_2''(x_2)}{u_2(x_2)}-\cdots-\frac{u_d''(x_d)}{u_d(x_d)}=\lambda,$$

$$-\frac{u_k''(x_k)}{u_k(x_k)}=\lambda_k \quad k=1,2,\ldots,d,$$

where λ_k is a scalar for $k=1,2,\ldots,d$. So we have

$$\lambda=\lambda_1+\lambda_2+\cdots+\lambda_d, \tag{1.2.4}$$

and

$$u_k''(x_k)+\lambda_k u_k(x_k)=0 \quad k=1,2,\ldots,d. \tag{1.2.5}$$

On the other hand, the boundary of the domain F is formed by the hyperplanes

$$\Pi_k=\{x\in R^d:(x,e_k)=0,e_k=(0,\ldots,0,1,0,\ldots,0)\},$$

and its shifts

$$a_ke_k+\Pi_k=\{x\in R^d:(x,a_ke_k)=0,e_k=(0,\ldots,0,1,0,\ldots,0)\},$$

where the outer normal to Π_k is $-e_k=(0,\ldots,0,-1,0,\ldots,0)$ and the outer normal to $a_ke_k+\Pi_k$ is $e_k=(0,\ldots,0,1,0,\ldots,0)$ for every $k=1,2,\ldots,d$. Using this we write the boundary condition (1.2.2) explicitly as

$$(\alpha u(x)+\frac{\partial u}{\partial n})\mid_{\Pi_k}=0 \quad k=1,2,\ldots,d, \tag{1.2.6}$$

and

$$(\alpha u(x)+\frac{\partial u}{\partial n})\mid_{a_ke_k+\Pi_k}=0 \quad k=1,2,\ldots,d. \tag{1.2.7}$$

(1.2.3) and (1.2.6) imply

$$\alpha u_1(x_1)\ldots u_k(x_k)\ldots u_d(x_d)-u_1(x_1)\ldots u_k'(x_k)\ldots u_d(x_d)|_{x_k=0}=0,$$

$$\alpha u_1(x_1)\dots u_k(0)\dots u_d(x_d) - u_1(x_1)\dots u_k'(0)\dots u_d(x_d) = 0,$$

$$\alpha u_k(0) - u_k'(0) = 0 \quad k = 1,2,\dots,d.$$

(1.2.3) and (1.2.7) imply

$$\alpha u_1(x_1)\dots u_k(x_k)\dots u_d(x_d) + u_1(x_1)\dots u_k'(x_k)\dots u_d(x_d)|_{x_k=a_k} = 0,$$

$$\alpha u_1(x_1)\dots u_k(a_k)\dots u_d(x_d) + u_1(x_1)\dots u_k'(a_k)\dots u_d(x_d) = 0,$$

$$\alpha u_k(a_k) + u_k'(a_k) = 0 \quad k = 1,2,\dots,d.$$

That is,

$$\alpha u_k(0) - u_k'(0) = 0 \quad k = 1,2,\dots,d, \tag{1.2.8}$$

$$\alpha u_k(a_k) + u_k'(a_k) = 0 \quad k = 1,2,\dots,d. \tag{1.2.9}$$

From (1.2.5), (1.2.8) and (1.2.9) we obtain the following Sturm-Liouville problems for every $k = 1,2,\dots,d$

$$\begin{array}{rcll} u_k''(x_k) + \lambda_k u_k(x_k) & = & 0 & 0 < x_k < a_k \\ \alpha u_k(0) - u_k'(0) & = & 0 & \\ \alpha u_k(a_k) + u_k'(a_k) & = & 0 & \alpha > 0. \end{array} \tag{1.2.10}$$

The eigenvalues of (1.2.10) are

$$\lambda_{n_k} \quad n_k = 1,2,3,\dots, \quad k = 1,2,\dots,d, \tag{1.2.11}$$

where $\lambda_{n_k} = (\frac{\mu_{n_k}}{a_k})^2$, μ_{n_k} are the positive roots of $\cot\mu = \frac{\mu}{\alpha a_k} - \frac{\alpha a_k}{\mu}$ and satisfy $n_k\pi < \mu_{n_k} < (n_k+1)\pi$, so that $\frac{n_k\pi}{a_k} < \sqrt{\lambda_{n_k}} < \frac{(n_k+1)\pi}{a_k}$, $n_k \in Z^+ \bigcup \{0\}$, $k = 1,2,\dots,d.$

The eigenfunctions of (1.2.10) corresponding to the eigenvalues λ_{n_k} are

$$u_{n_k}(x_k) = e^{i\sqrt{\lambda_{n_k}}x_k} - \frac{\alpha - i\sqrt{\lambda_{n_k}}}{\alpha + i\sqrt{\lambda_{n_k}}} \quad e^{-i\sqrt{\lambda_{n_k}}x_k} \quad k = 1,2,\dots,d. \tag{1.2.12}$$

By (1.2.4) and (1.2.11) the eigenvalues of $L_M(0)$ are

$$\lambda = \lambda_{n_1} + \lambda_{n_2} + \dots + \lambda_{n_d}. \tag{1.2.13}$$

Since we assumed $u(x) = u_1(x_1)u_2(x_2)\dots u_d(x_d)$, by (1.2.5) and (1.2.12) the eigenfunctions of $L_M(0)$ corresponding to the eigenvalues λ are

$$u_\lambda(x) = \prod_{k=1}^{d} \quad [e^{i\sqrt{\lambda_{n_k}}x_k} - \frac{\alpha - i\sqrt{\lambda_{n_k}}}{\alpha + i\sqrt{\lambda_{n_k}}} \quad e^{-i\sqrt{\lambda_{n_k}}x_k}]. \tag{1.2.14}$$

We define a lattice Ω in R^d by

$$\Omega = \{\sum_{k=1}^{d} m_k\omega_k : m_k \in Z, k = 1, 2, \dots, d\}$$

with the basis

$$\omega_1 = (a_1, 0, \dots, 0), \omega_2 = (0, a_2, 0, \dots, 0), \dots, \omega_d = (0, \dots, 0, a_d)$$

and the dual lattice Γ of Ω by

$$\Gamma = \{\sum_{k=1}^{d} n_k\beta^k : n_k \in Z, k = 1, 2, \dots, d\}$$

with the basis

$$\beta^1 = (\frac{2\pi}{a_1}, 0, \dots, 0), \beta^2 = (0, \frac{2\pi}{a_2}, 0, \dots, 0), \dots, \beta^d = (0, \dots, 0, \frac{2\pi}{a_d}).$$

Notice that ω_i and β^j are bi-orthogonal vectors, that is,

$$(\omega_i, \beta^j) = 2\pi\delta_{ij},$$

where $(\cdot, \cdot)$ is the standard inner product in R^d, δ_{ij} is the Kronecker delta.

According to dual lattice Γ, we introduce the following notations:

$$\frac{\Gamma}{2} = \{\gamma = (\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \dots \frac{n_d\pi}{a_d}) : \quad n_i \in Z \quad \forall i = 1, 2, \dots, d\},$$

$$\frac{\Gamma^{+0}}{2} = \{\gamma = (\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \dots \frac{n_d\pi}{a_d}) : \quad n_i \in Z^+ \bigcup\{0\} \quad \forall i = 1, 2, \dots, d\}.$$

Since the solutions λ_{n_i} of the Sturm-Liouville's problem (1.2.10) for $k = i$ satisfy $\frac{n_i\pi}{a_i} < \sqrt{\lambda_{n_i}} < \frac{(n_i+1)\pi}{a_i}$, $n_i \in Z^+ \bigcup\{0\}$, $i = 1, 2, \dots, d$, the eigenvalues λ of $L_M(0)$ are indeed $|\lambda_\gamma|^2$

where $\lambda_\gamma = (\sqrt{\lambda_{n_1}}, \ldots, \sqrt{\lambda_{n_d}})$, $\gamma = (\frac{n_1\pi}{a_1}, \ldots, \frac{n_d\pi}{a_d})$. We denote by $S_{\frac{\Gamma+0}{2}}$ the set of all λ_γ, $\gamma \in \frac{\Gamma+0}{2}$ where $|\lambda_\gamma|^2$ are the eigenvalues of $L_M(0)$. Also we denote the eigenfunction $u_\lambda(x)$ in (1.2.14) by $u_\gamma(x)$ which corresponds to the eigenvalue $|\lambda_\gamma|^2$.

For the sake of simplicity in calculations, we write the eigenfunctions$u_\gamma(x)$ of the unperturbed operator $L_M(0)$ corresponding to the eigenvalues $|\lambda_\gamma|^2$ for any $\gamma \in \frac{\Gamma+0}{2}$ as

$$u_\gamma(x) = \sum_{\substack{\beta \in A_{\lambda_\gamma} \\ \theta_\beta \in B_{\lambda_\gamma}}} e^{i\theta_\beta} e^{i(\beta,x)},$$

where

$$A_{\lambda_\gamma} = \{\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in S_{\frac{\Gamma}{2}} : \quad | \beta_k |=| \sqrt{\lambda_{n_k}} |, k = 1, 2, \ldots, d\},$$

and

$$\begin{gathered} B_{\lambda_\gamma} = \{\theta_\beta = \textstyle\sum_{n_k:\beta_k=-\sqrt{\lambda_{n_k}}} \theta_{n_k} : \quad \theta_{n_k} = \arg[-\frac{\alpha - i\sqrt{\lambda_{n_k}}}{\alpha + i\sqrt{\lambda_{n_k}}}], \\ k = 1, 2, \ldots, d, \beta = (\beta_1, \beta_2, \ldots, \beta_d) \in A_{\lambda_\gamma}\}. \end{gathered}$$

(Note that $\theta_\beta = 0$ if $\beta = \lambda_\gamma$)

Let $\gamma \in \frac{\Gamma+0}{2}$, then there correspond an eigenvalue $|\lambda_\gamma|^2$ and an eigenfunction $u_\gamma(x)$ of $L_M(0)$. Suppose that any j-th component of γ is changed by its negative, that is, $\sqrt{\lambda_{n_j}}$ is changed by $-\sqrt{\lambda_{n_j}}$. Then

$$\begin{aligned}
u_{n_j}(x_j) \mid_{\sqrt{\lambda_{n_j}} \to -\sqrt{\lambda_{n_j}}} &= \{e^{i\sqrt{\lambda_{n_j}}x_j} + e^{i\theta_{n_j}} e^{-i\sqrt{\lambda_{n_j}}x_j}\} \mid_{\sqrt{\lambda_{n_j}} \to -\sqrt{\lambda_{n_j}}} \\
&= \{e^{i\sqrt{\lambda_{n_j}}x_j} - \frac{\alpha - i\sqrt{\lambda_{n_j}}}{\alpha + i\sqrt{\lambda_{n_j}}} e^{-i\sqrt{\lambda_{n_j}}x_j}\} \mid_{\sqrt{\lambda_{n_j}} \to -\sqrt{\lambda_{n_j}}} \\
&= e^{i(-\sqrt{\lambda_{n_j}}x_j)} - \frac{\alpha - i(-\sqrt{\lambda_{n_j}})}{\alpha + i(-\sqrt{\lambda_{n_j}})} e^{-i(-\sqrt{\lambda_{n_j}}x_j)} \\
&= e^{-i\sqrt{\lambda_{n_j}}x_j} - \frac{\alpha + i\sqrt{\lambda_{n_j}}}{\alpha - i\sqrt{\lambda_{n_j}}} e^{i\sqrt{\lambda_{n_j}}x_j}\} \\
&= \{-\frac{\alpha + i\sqrt{\lambda_{n_j}}}{\alpha - i\sqrt{\lambda_{n_j}}}\}\{e^{i\sqrt{\lambda_{n_j}}x_j} - \frac{\alpha - i\sqrt{\lambda_{n_j}}}{\alpha + i\sqrt{\lambda_{n_j}}} e^{-i\sqrt{\lambda_{n_j}}x_j}\} \\
&= \frac{1}{e^{i\theta_{n_j}}}[e^{i\sqrt{\lambda_{n_j}}x_j} + e^{i\theta_{n_j}} e^{-i\sqrt{\lambda_{n_j}}x_j}] \\
&= e^{-i\theta_{n_j}} u_{n_j}(x_j).
\end{aligned}$$

Denote the function $u_{n_j}(x_j) \mid_{\sqrt{\lambda_{n_j}} \to -\sqrt{\lambda_{n_j}}}$ by $u_{-n_j}(x_j)$. Thus if we change $\sqrt{\lambda_{n_j}}$ by $-\sqrt{\lambda_{n_j}}$ then

$$u_{-n_j}(x_j) = e^{-i\theta_{n_j}} u_{n_j}(x_j).$$

Any component of γ can be changed by its negative. Taking into consideration all these changes, θ_γ is defined as $\theta_\gamma = \sum_{n_j:\beta_j=-\sqrt{\lambda_{n_j}}} \theta_{n_j}$. So we have

$$u_{\gamma'}(x) = e^{-i\theta_\gamma} u_\gamma(x) \quad \forall \lambda_{\gamma'} \in A_{\lambda_\gamma}, \tag{1.2.15}$$

where $\theta_\gamma \in B_{\lambda_\gamma}$.

By direct substitution we show that $|\lambda_{\gamma'}|^2$ is an eigenvalue of $L_M(0)$ and $u_{\gamma'}(x) = e^{-i\theta_\gamma} u_\gamma(x)$ is an eigenfunction corresponding to the eigenvalue $|\lambda_{\gamma'}|^2$.

Using (1.2.15), we get

$$\begin{aligned}
-\Delta u_{\gamma'}(x) &= -\Delta[e^{-i\theta_\gamma} u_\gamma(x)] \\
&= (e^{-i\theta_\gamma})(-\Delta u_\gamma(x)) \\
&= (e^{-i\theta_\gamma})(|\lambda_\gamma|^2 u_\gamma(x)) \\
&= (e^{-i\theta_\gamma})(|\lambda_{\gamma'}|^2 u_\gamma(x)) \\
&= |\lambda_{\gamma'}|^2 (e^{-i\theta_\gamma} u_\gamma(x)) \\
&= |\lambda_{\gamma'}|^2 u_{\gamma'}(x).
\end{aligned}$$

So (1.1.1) is satisfied and again by (1.2.15)

$$\begin{aligned}(\alpha u_{\gamma'}(x)+\frac{\partial u_{\gamma'}(x)}{\partial n})\mid_{\partial F} &= (\alpha e^{-i\theta_\gamma}u_\gamma(x)+\frac{\partial e^{-i\theta_\gamma}u_\gamma(x)}{\partial n})\mid_{\partial F}\\ &= e^{-i\theta_\gamma}[(\alpha u_\gamma(x)+\frac{\partial u_\gamma(x)}{\partial n})\mid_{\partial F}]\\ &= e^{-i\theta_\gamma}0\\ &= 0.\end{aligned}$$

So (1.1.2) is satisfied.

It is clear that the system of eigenfunctions $u_\gamma(x)$ of $L_M(0)$, that is,

$$\{u_\gamma(x)\}_{\gamma\in\frac{\Gamma+0}{2}}=\{\sum_{\substack{\beta\in A_{\lambda_\gamma}\\ \theta_\beta\in B_{\lambda_\gamma}}} e^{i\theta_\beta}e^{i(\beta,x)}\}_{\gamma\in\frac{\Gamma+0}{2}}$$

forms an orthogonal basis in $L_2(F)$. Hence any $Q(x)\in L_2(F)$ is equal to its Fourier series

$$Q(x)=\sum_{\gamma\in\frac{\Gamma+0}{2}} Q_\gamma u_\gamma(x),$$

where $Q_\gamma=\frac{<Q(x),u_\gamma(x)>}{\|u_\gamma(x)\|^2}$ are the Fourier coefficients of $Q(x)$ with respect to the basis $\{u_\gamma(x)\}_{\gamma\in\frac{\Gamma+0}{2}}$, $<\cdot,\cdot>$ denotes the inner product in $L_2(F)$.

By this result and (1.2.15) any function $Q(x)$ in $L_2(F)$ has the following Fourier series expansion

$$Q(x)=\sum_{\gamma'\in\frac{\Gamma}{2}} Q_{\gamma'}u_{\gamma'}(x), \tag{1.2.16}$$

where $Q_{\gamma'}=\frac{1}{|A_{\lambda_{\gamma'}}|}\frac{<Q(x),u_{\gamma'}(x)>}{\|u_{\gamma'}(x)\|^2}$ are the Fourier coefficients of $Q(x)$ with respect to the basis $\{u_{\gamma'}(x)\}_{\gamma'\in\frac{\Gamma}{2}}$, and $\mid A_{\lambda_{\gamma'}}\mid$ is the number of vectors in $A_{\lambda_{\gamma'}}$.

Indeed, let $\gamma\in\frac{\Gamma+0}{2}$. Consider $\beta_1,\beta_2,\ldots,\beta_r\in A_{\lambda_\gamma}$ where $r=\mid A_{\lambda_\gamma}\mid$, $\mid A_{\lambda_\gamma}\mid$ is the number of vectors in A_{λ_γ}. By (1.2.15)

$$\begin{aligned}
< Q(x), u_{\beta_1}(x) > u_{\beta_1}(x) &= < Q(x), e^{-i\theta_\gamma} u_\gamma(x) > e^{-i\theta_\gamma} u_\gamma(x) \\
&= < Q(x), u_\gamma(x) > u_\gamma(x) \\
< Q(x), u_{\beta_2}(x) > u_{\beta_2}(x) &= < Q(x), e^{-i\theta_\gamma} u_\gamma(x) > e^{-i\theta_\gamma} u_\gamma(x) \\
&= < Q(x), u_\gamma(x) > u_\gamma(x) \\
\vdots &= \vdots \\
< Q(x), u_{\beta_r}(x) > u_{\beta_r}(x) &= < Q(x), e^{-i\theta_\gamma} u_\gamma(x) > e^{-i\theta_\gamma} u_\gamma(x) \\
&= < Q(x), u_\gamma(x) > u_\gamma(x)
\end{aligned}$$

Summing both sides we have

$$| A_{\lambda_\gamma} | < Q(x), u_\gamma(x) > u_\gamma(x) = \sum_{\beta \in A_{\lambda_\gamma}} < Q(x), u_\beta(x) > u_\beta(x),$$

or

$$< Q(x), u_\gamma(x) > u_\gamma(x) = \frac{1}{| A_{\lambda_\gamma} |} \sum_{\beta \in A_{\lambda_\gamma}} < Q(x), u_\beta(x) > u_\beta(x). \qquad (1.2.17)$$

On the other hand, by (1.2.15) for any $\lambda_{\beta_i} \in A_{\lambda_\gamma}$ we have

$$\begin{aligned}
\| u_{\beta_i}(x) \|^2 &= < u_{\beta_i}(x), u_{\beta_i}(x) > \\
&= < e^{-i\theta_\gamma} u_\gamma(x), e^{-i\theta_\gamma} u_\gamma(x) > \\
&= e^{-i\theta_\gamma} e^{i\theta_\gamma} < u_\gamma(x), u_\gamma(x) > \\
&= \| u_\gamma(x) \|^2 . \qquad (1.2.18)
\end{aligned}$$

Substituting (1.2.17) and (1.2.18) into the Fourier expansion of $Q(x)$ with respect to the basis $\{u_\gamma(x)\}_{\gamma \in \frac{\Gamma+0}{2}}$, we obtain

$$\begin{aligned}
Q(x) &= \sum_{\gamma \in \frac{\Gamma+0}{2}} \frac{< Q(x), u_\gamma(x) >}{\| u_\gamma(x) \|^2} u_\gamma(x) \\
&= \sum_{\gamma \in \frac{\Gamma+0}{2}} \frac{1}{| A_{\lambda_\gamma} |} \sum_{\beta \in A_{\lambda_\gamma}} \frac{< Q(x), u_\beta(x) >}{\| u_\beta(x) \|^2} u_\beta(x).
\end{aligned}$$

Using the facts $S_{\frac{\Gamma}{2}} = \bigcup_{\gamma \in \frac{\Gamma+0}{2}} A_{\lambda_\gamma}$ and $| A_{\lambda_\gamma} | = | A_{\lambda_{\gamma'}} | = 2^d$ for all $\gamma, \gamma' \in \frac{\Gamma+0}{2}$ the last expression reduces to

$$Q(x) = \sum_{\gamma' \in \frac{\Gamma}{2}} \frac{< Q(x), u_{\gamma'}(x) >}{| A_{\lambda_{\gamma'}} ||| u_{\gamma'}(x) \|^2} u_{\gamma'}(x). \tag{1.2.19}$$

Letting $Q_{\gamma'} = \frac{<Q(x), u_{\gamma'}(x)>}{|A_{\lambda_{\gamma'}}|\|u_{\gamma'}(x)\|^2}$ the result (1.2.16) follows.

CHAPTER TWO
ASYMPTOTIC FORMULA

2.1 Resonance and Non-Resonance Domains

As in papers Veliev (1987) and Veliev (1988) we divide the eigenvalues of the unperturbed operator $L_M(0)$ into two groups.

Consider the eigenvalues $|\lambda_\gamma|^2$ of $L_M(0)$ for all $\gamma \in \frac{\Gamma}{2}$ such that $|\gamma| \sim \rho$. $|\gamma| \sim \rho$ means that $|\gamma|$ and ρ are asymptotically equal, that is, $c_1\rho \leq |\gamma| \leq c_2\rho$ where c_i, $i = 1, 2$ are positive real constants which do not depend on ρ.

Let $\alpha < \frac{1}{d+20}$, $\alpha_1 = 3\alpha$ and define the following sets

$$V_b(\rho^{\alpha_1}) \equiv \{x \in R^d : \quad || x |^2 - | x + b |^2| < \rho^{\alpha_1}\}$$

$$E_1(\rho^{\alpha_1}, p) \equiv \bigcup_{b \in \Gamma(p\rho^{\alpha_1})} V_b(\rho^{\alpha_1})$$

$$U(\rho^{\alpha_1}, p) \equiv R^d \setminus E_1(\rho^{\alpha_1}, p),$$

where $\Gamma(p\rho^{\alpha_1}) \equiv \{b \in \frac{\Gamma}{2} : 0 < | b | < p\rho^{\alpha_1}\}$. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain, and the eigenvalue $|\lambda_\gamma|^2$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$, for all $b \in \Gamma(p\rho^{\alpha})$ are called resonance domains and the eigenvalue $|\lambda_\gamma|^2$ is a resonance eigenvalue if $\gamma \in V_b(\rho^{\alpha_1})$.

Remark 2.1.1. Note that, the elements of the single resonance domain

$$V_b(\rho^{\alpha_1}) = \{x \in R^d : \quad || x |^2 - | x + b |^2| < \rho^{\alpha_1}\}$$

are contained between the two hyperplanes

$$\Pi_1 = \{x : \quad || x |^2 - | x + b |^2| = -\rho^{\alpha_1}\}$$

and

$$\Pi_2 = \{x : \quad || x |^2 - | x + b |^2| = \rho^{\alpha_1}\}.$$

Since

$$\begin{aligned} |x|^2 - |x+b|^2 = (x,x) - (x+b, x+b) &= -2(x,b) - |b|^2 = \mp\rho^{\alpha_1}, \\ (x,b) + \frac{|b|^2}{2} \mp \frac{\rho^{\alpha_1}}{2} &= 0, \end{aligned}$$

we have

$$\begin{aligned} \Pi_1 = \{x : (x + \frac{b}{2} + \frac{\rho^{\alpha_1} b}{2|b|^2}, b) = 0\} &= \Pi_b + (\frac{b}{2} + \frac{\rho^{\alpha_1} b}{2|b|^2}) \\ \Pi_2 = \{x : (x + \frac{b}{2} - \frac{\rho^{\alpha_1} b}{2|b|^2}, b) = 0\} &= \Pi_b + (\frac{b}{2} - \frac{\rho^{\alpha_1} b}{2|b|^2}), \end{aligned}$$

where $\Pi_b = \{x : (x,b) = 0\}$ is the hyperplane passing through the origin. It is clear that the distance between the two hyperplanes Π_1 and Π_2 is $\frac{\rho^{\alpha_1}}{|b|}$.

Lemma 2.1.2. *The non-resonance domain $U(\rho^{\alpha_1}, p)$ has asymptotically full measure on R^d, that is,*

$$\frac{\mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho))}{\mu(B(\rho))} \to 1 \quad as \quad \rho \to \infty,$$

where $B(\rho) = \{x \in R^d : \quad |x| \leq \rho\}$.

Proof. It is clear that $V_b(\rho^{\alpha_1}) \cap B(\rho)$ is the part of $B(\rho)$ which is contained between the two hyperplanes Π_1 and Π_2. Since the distance between these hyperplanes is $\frac{\rho^{\alpha_1}}{|b|}$, we have

$$\mu(V_b(\rho^{\alpha_1}) \bigcap B(\rho)) = O(\rho^{d-1+\alpha_1}),$$

where O is an order relation and we say that a function f is in the O relation with g for $\xi \to \infty$ and we write it $f(\xi) = O(g(\xi))$ for $\xi \to \infty$ if there is a constant c such that $|f(\xi)| < c |g(\xi)|$ at some neighborhood of ∞.

The number of vectors γ in $\Gamma(p\rho^{\alpha})$ is $O(\rho^{d\alpha})$ and $\mu(B(\rho)) \sim \rho^d$. Thus,

$$\begin{aligned} \mu(\bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1}) \bigcap B(\rho)) &= O(\rho^{d-1+\alpha_1+d\alpha}) \\ &= \mu(B(\rho)) O(\rho^{d\alpha+\alpha_1-1}). \end{aligned} \tag{2.1.1}$$

Using that, $R^d = U(\rho^{\alpha_1}, p) \cup E_1$, and

$$R^d \bigcap B(\rho) = (U(\rho^{\alpha_1}, p) \bigcap B(\rho)) \bigcup (E_1 \bigcap B(\rho))$$

we have,

$$\mu(B(\rho)) = \mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho)) + \mu(E_1 \bigcap B(\rho)),$$

which together with (2.1.1) implies

$$\mu(U(\rho^{\alpha_1}, p) \bigcap B(\rho)) = \mu(B(\rho))(1 - O(\rho^{\alpha_1 + d\alpha - 1})).$$

Thus from the last equation the result follows, since $\alpha_1 + d\alpha < 1$. That is, the domain $U(\rho^{\alpha_1}, p)$ has asymptotically full measure on R^d. $\square$

Note that this lemma implies that the number of non-resonance eigenvalues is greater than the number of resonance eigenvalues.

2.2 On the Potential of $L_M(q(x))$

If we consider the functions $q(x)u_\gamma(x)$ in $L_2(F)$ for any $\gamma \in \frac{\Gamma}{2}$, by a rearrangement in indexing, the Fourier series of $q(x)u_\gamma(x)$ can be written as

$$Q(x) = q(x)u_\gamma(x) = \sum_{\gamma' \in \frac{\Gamma}{2}} Q_{\gamma+\gamma'} u_{\gamma+\gamma'}(x), \tag{2.2.1}$$

where

$$Q_{\gamma+\gamma'} = \frac{< q(x)u_\gamma(x), u_{\gamma+\gamma'}(x) >}{| A_{\lambda_{\gamma+\gamma'}} | \| u_{\gamma+\gamma'}(x) \|^2}. \tag{2.2.2}$$

Suppose that for all $\gamma \in \frac{\Gamma}{2}$ such that $| \gamma | \sim \rho$ the Fourier coefficients (2.2.2) satisfy

$$\sum_{\gamma' \in \frac{\Gamma}{2}} | Q_{\gamma+\gamma'} |^2 (1 + | \gamma' |^{2l}) < \infty, \tag{2.2.3}$$

where $l > \frac{(d+20)(d-1)}{2} + d + 3$.

Therefore,

$$Q(x) = \sum_{\gamma' \in \Gamma(\rho^\alpha)} Q_{\gamma+\gamma'} u_{\gamma+\gamma'}(x) + O(\rho^{-p\alpha}), \tag{2.2.4}$$

where $\Gamma(\rho^\alpha) = \{\gamma' \in \frac{\Gamma}{2} : 0 <| \gamma' |< \rho^\alpha\}$, $p = l - d$, $\alpha < \frac{1}{d+20}$ and ρ is a large parameter, $O(\rho^{-p\alpha})$ is a function in $L_2(F)$ with norm of order $\rho^{-p\alpha}$.

Indeed, for $\gamma' \notin \Gamma(\rho^\alpha)$

$$\sum_{|\gamma'|>\rho^\alpha} |\, Q_{\gamma+\gamma'} \,|^2 = \sum_{|\gamma'|>\rho^\alpha} \frac{|\, Q_{\gamma+\gamma'} \,|^2 |\, \gamma' \,|^{2l}}{|\, \gamma' \,|^{2l}} \leq [\sum_{|\gamma'|>\rho^\alpha} \frac{|\, Q_{\gamma+\gamma'} \,||\, \gamma' \,|^l}{|\, \gamma' \,|^l}]^2$$
$$\leq [(\sum_{|\gamma'|>\rho^\alpha} |\, Q_{\gamma+\gamma'} \,|^2 |\, \gamma' \,|^{2l})^{\frac{1}{2}} (\sum_{|\gamma'|>\rho^\alpha} \frac{1}{|\, \gamma' \,|^{2l}})^{\frac{1}{2}}]^2$$
$$= (\sum_{|\gamma'|>\rho^\alpha} |\, Q_{\gamma+\gamma'} \,|^2 |\, \gamma' \,|^{2l})(\sum_{|\gamma'|>\rho^\alpha} \frac{1}{|\, \gamma' \,|^{2l}}) = O(\rho^{-p\alpha}).$$

Because the first series on the right hand side of the inequality is convergent by (2.2.3) and the norm of the second series is in the big O relation with $\rho^{-p\alpha}$ which we show by using the integral test.

Let $f(x) = \frac{1}{x^{2l}}$. Clearly, $f(x)$ is a continuous positive decreasing function on $[1, \infty)$. On the other hand, $\int_1^\infty \frac{dx}{x^{2l}}$ is convergent. Because $2l > 1$, and

$$\int_{\rho^\alpha}^\infty \frac{dx}{x^{2l}} < \int_1^\infty \frac{dx}{x^{2l}} < \infty.$$

$$\begin{aligned}\int_{\rho^\alpha}^\infty \frac{dx}{x^{2l}} &= \lim_{t\to\infty} \int_{\rho^\alpha}^t \frac{dx}{x^{2l}} = \lim_{t\to\infty} \frac{x^{-2l+1}}{-2l+1} \Big|_{x=\rho^\alpha}^t = \lim_{t\to\infty} \frac{t^{-2l+1} - \rho^{\alpha(-2l+1)}}{-2l+1} \\ &= \frac{\rho^{\alpha(-2l+1)}}{2l-1} \leq \frac{\rho^{-p\alpha}}{2l-1}\end{aligned}$$

since $p = l - d$. Thus

$$\int_{\rho^\alpha}^\infty \frac{dx}{x^{2l}} = O(\rho^{-p\alpha}).$$

Letting $a_{\gamma'} = f(|\, \gamma' \,|) = \frac{1}{|\gamma'|^{2l}}$, the series $\sum_{|\gamma'|>\rho^\alpha} a_{\gamma'} = \sum_{|\gamma'|>\rho^\alpha} \frac{1}{|\gamma'|^{2l}}$ is convergent by the integral test and

$$\sum_{|\gamma'|>\rho^\alpha} \frac{1}{|\, \gamma' \,|^{2l}} = O(\rho^{-p\alpha}).$$

On the other hand,

$$\sum_{\gamma' \in \frac{\Gamma}{2}} |\, Q_{\gamma+\gamma'} \,| = \sum_{\gamma' \in \frac{\Gamma}{2}} \frac{|\, Q_{\gamma+\gamma'} \,||\, \gamma' \,|^l}{|\, \gamma' \,|^l} \leq (\sum_{\gamma' \in \frac{\Gamma}{2}} |\, Q_{\gamma+\gamma'} \,|^2 |\, \gamma' \,|^{2l})^{\frac{1}{2}} (\sum_{\gamma' \in \frac{\Gamma}{2}} \frac{1}{|\, \gamma' \,|^{2l}})^{\frac{1}{2}} < \infty.$$

Because the first series on the right hand side of the inequality is convergent by (2.2.3) and the second series is easily found to be convergent by the integral test.

So say

$$M(\gamma) = \sum_{\gamma' \in \frac{\Gamma}{2}} \mid Q_{\gamma+\gamma'} \mid . \tag{2.2.5}$$

2.3 Asymptotic Formula

To obtain an asymptotic formula for the eigenvalues Λ_N of the operator $L_M(q(x))$ in a non-resonance domain we use the binding formula

$$(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) > = < \Psi_N(x), q(x) u_\gamma(x) > . \tag{2.3.1}$$

To obtain the binding formula, we multiply both sides of the equation

$$-\Delta \Psi_N(x) + q(x)\Psi_N(x) = \Lambda_N \Psi_N(x)$$

by $u_\gamma(x)$. That is,

$$< -\Delta \Psi_N(x) + q(x)\Psi_N(x), u_\gamma(x) > = < \Lambda_N \Psi_N(x), u_\gamma(x) > .$$

Using the properties of inner product, we obtain

$$< -\Delta \Psi_N(x), u_\gamma(x) > + < q(x)\Psi_N(x), u_\gamma(x) > = \Lambda_N < \Psi_N(x), u_\gamma(x) > .$$

Since $L_M(0) = -\Delta$ is self adjoint and $q(x)$ is real valued

$$< \Psi_N(x), -\Delta u_\gamma(x) > + < q(x)\Psi_N(x), u_\gamma(x) > = \Lambda_N < \Psi_N(x), u_\gamma(x) >,$$

and $u_\gamma(x)$ is an eigenfunction of $L_M(0)$

$$< \Psi_N(x), \mid \lambda_\gamma \mid^2 u_\gamma(x) > + < q(x)\Psi_N(x), u_\gamma(x) > = \Lambda_N < \Psi_N(x), u_\gamma(x) > .$$

Consequently, we obtain from the last equation

$$\mid \lambda_\gamma \mid^2 < \Psi_N(x), u_\gamma(x) > + < q(x)\Psi_N(x), u_\gamma(x) > = \Lambda_N < \Psi_N(x), u_\gamma(x) >,$$

$$< q(x)\Psi_N(x), u_\gamma(x) >= \Lambda_N < \Psi_N(x), u_\gamma(x) > - \mid \lambda_\gamma \mid^2 < \Psi_N(x), u_\gamma(x) >,$$

$$< q(x)\Psi_N(x), u_\gamma(x) >= (\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) > .$$

Lemma 2.3.1. *Let $\gamma \in U(\rho^{\alpha_1}, p)$, that is, $\mid \lambda_\gamma \mid^2$ be a non-resonance eigenvalue of $L_M(0)$ and $b \in \Gamma(p\rho^\alpha)$. Then*

$$\mid\mid \lambda_\gamma \mid^2 - \mid \lambda_{\gamma+b} \mid^2 \mid > \rho^{\alpha_1}.$$

Proof. If $\gamma \in U(\rho^{\alpha_1}, p)$ then for all $b \in \Gamma(p\rho^\alpha)$ we have

$$\mid\mid \gamma \mid^2 - \mid \gamma + b \mid^2 \mid \geq \rho^{\alpha_1}. \tag{2.3.2}$$

Let us denote $\gamma \in U(\rho^{\alpha_1}, p)$ by $\gamma = (\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2}, \dots, \frac{n_d\pi}{a_d})$. $\mid \lambda_\gamma \mid^2$ is an eigenvalue of $L_M(0)$ for $\lambda_\gamma \in S_{\frac{\Gamma}{2}}$. So we have

$$\frac{n_k\pi}{a_k} < \sqrt{\lambda_{n_k}} < \frac{(n_k+1)\pi}{a_k} \quad k = 1, 2, \dots, d.$$

We obtain from this relation,

$$\mid \gamma \mid^2 < \mid \lambda_\gamma \mid^2 < \mid \gamma + e \mid^2, \tag{2.3.3}$$

where $e = (\frac{\pi}{a_1}, \frac{\pi}{a_2}, \dots, \frac{\pi}{a_d})$.

Similarly,

$$\mid \gamma + b \mid^2 < \mid \lambda_{\gamma+b} \mid^2 < \mid \gamma + b + e \mid^2 . \tag{2.3.4}$$

Then using (2.3.3), (2.3.4) and (2.3.2), we get

$$\mid\mid \lambda_\gamma \mid^2 - \mid \lambda_{\gamma+b} \mid^2 \mid > \mid\mid \gamma \mid^2 - \mid \gamma + b + e \mid^2 \mid > \rho^{\alpha_1}.$$

□

Lemma 2.3.2. *Let $\gamma \in U(\rho^{\alpha_1}, p)$, that is, $\mid \lambda_\gamma \mid^2$ be a non-resonance eigenvalue of $L_M(0)$ and Λ_N be the eigenvalue of $L_M(q(x))$ satisfying the inequality*

$$\mid \Lambda_N - \mid \lambda_\gamma \mid^2 \mid < \frac{1}{2}\rho^{\alpha_1}.$$

Then

$$\mid \Lambda_N - \mid \lambda_{\gamma+b} \mid^2 \mid > \frac{1}{2}\rho^{\alpha_1}$$

for all $b \in \Gamma(p\rho^{\alpha})$.

Proof. If $\gamma \in U(\rho^{\alpha_1}, p)$ then for all $b \in \Gamma(p\rho^{\alpha})$ we have from Lemma(2.3.1)

$$|| \lambda_\gamma |^2 - | \lambda_{\gamma+b} |^2| > \rho^{\alpha_1}$$

which together with $| \Lambda_N - | \lambda_\gamma |^2| < \frac{1}{2}\rho^{\alpha_1}$ gives

$$| \Lambda_N - | \lambda_{\gamma+b} |^2| = | \Lambda_N - | \lambda_\gamma |^2 + | \lambda_\gamma |^2 - | \lambda_{\gamma+b} |^2| \geq$$
$$|| \Lambda_N - | \lambda_\gamma |^2| - || \lambda_{\gamma+b} |^2 - | \lambda_\gamma |^2|| > | \rho^{\alpha_1} - \frac{1}{2}\rho^{\alpha_1} | = \frac{1}{2}\rho^{\alpha_1}.$$

□

Lemma 2.3.3. *Let* $| \lambda_\gamma |^2$ *be an eigenvalue of the operator* $L_M(0)$ *where* $| \gamma | \sim \rho$. *Then there is an integer* N *such that* $| \Lambda_N - | \lambda_\gamma |^2| < 2M$ *and*

$$|< \Psi_N(x), u_\gamma(x) >| > c_3 \rho^{\frac{-(d-1)}{2}}, \tag{2.3.5}$$

where $M = \| q(x) \|$.

Proof. We use a result from the general perturbation theory, the N-th eigenvalue of the operator $L_M(q(x))$ lies in M-neighborhood of the N-th eigenvalue of the operator $L_M(0)$.

Let the N-th eigenvalues of $L_M(q(x))$ and $L_M(0)$ be Λ_N and $| \lambda_\gamma |^2$, respectively. It is clear that the eigenfunctions $\Psi_N(x)$ of $L_M(q(x))$ form an orthonormal basis for $L_2(F)$. So

$$u_\gamma(x) = \sum_{N=1}^{\infty} < \Psi_N(x), u_\gamma(x) > \Psi_N(x).$$

Without loss of generality, assume that $\| u_\gamma(x) \| = 1$. By Parseval's relation

$$1 = \| u_\gamma(x) \| = \sum_{N=1}^{\infty} |< \Psi_N(x), u_\gamma(x) >|^2$$

$$= \sum_{N:|\Lambda_N - |\lambda_\gamma|^2| > 2M} |< \Psi_N(x), u_\gamma(x) >|^2 + \sum_{N:|\Lambda_N - |\lambda_\gamma|^2| \leq 2M} |< \Psi_N(x), u_\gamma(x) >|^2 .$$

Using the binding formula (2.3.1), Bessel's inequality and $M = \| q(x) \|$ we have

$$\sum_{N:|\Lambda_N - |\lambda_\gamma|^2| > 2M} |< \Psi_N(x), u_\gamma(x) >|^2 = \sum_{N:|\Lambda_N - |\lambda_\gamma|^2| > 2M} \frac{|< \Psi_N(x), q(x)u_\gamma(x) >|^2}{|\Lambda_N - |\lambda_\gamma|^2|^2}$$

$$< \frac{1}{4M^2} \sum_{N:|\Lambda_N - |\lambda_\gamma|^2| > 2M} |< \Psi_N(x), q(x)u_\gamma(x) >|^2 < \frac{1}{4M^2} \| q(x) \|^2 \| u_\gamma(x) \|^2 < \frac{1}{4}.$$

Therefore, by Parseval's relation

$$\sum_{N:|\Lambda_N - |\lambda_\gamma|^2| \le 2M} |< \Psi_N(x), u_\gamma(x) >|^2 \ge \frac{3}{4}.$$

On the other hand, if $a \sim \rho$ then the number of $\gamma \in \frac{\Gamma}{2}$ satisfying $|| \gamma |^2 - a^2 | < 1$ is less than $c_4 \rho^{d-1}$. Therefore, the number of eigenvalues of $L_M(0)$ lying in $(a^2 - 1, a^2 + 1)$ is less than $c_5 \rho^{d-1}$. By this result and a result of perturbation theory the number of eigenvalues Λ_N of $L_M(q(x))$ in the interval $I = [| \lambda_\gamma |^2 - 2M, | \lambda_\gamma |^2 + 2M]$ is less than $c_6 \rho^{d-1}$. Thus there is $N \in I$ such that

$$\frac{3}{4} < \sum_{N:|\Lambda_N - |\lambda_\gamma|^2| \le 2M} |< \Psi_N(x), u_\gamma(x) >|^2 < c_3 \rho^{d-1} |< \Psi_N(x), u_\gamma(x) >|^2 .$$

That is,

$$|< \Psi_N(x), u_\gamma(x) >| > c_3 \rho^{\frac{-(d-1)}{2}}.$$

□

Theorem 2.3.4. *For every non-resonance eigenvalue* $| \lambda_\gamma |^2$, $| \gamma | \sim \rho$, *of the operator* $L_M(0)$ *there exists an eigenvalue* Λ_N *of the operator* $L_M(q(x))$ *satisfying*

$$\Lambda_N = | \lambda_\gamma |^2 + O(\rho^{-\alpha_1}). \tag{2.3.6}$$

Proof. We do an iteration by using the binding formula.

In order to start iteration we substitute the decomposition (2.2.4) of $q(x)u_\gamma(x)$ into the binding formula (2.3.1)

$$(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) >$$

$$\begin{aligned} &= < \Psi_N(x), \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} Q_{\gamma+\gamma_1} u_{\gamma+\gamma_1}(x) > + O(\rho^{-p\alpha}) \\ &= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} Q_{\gamma+\gamma_1} < \Psi_N(x), u_{\gamma+\gamma_1}(x) > + O(\rho^{-p\alpha}). \end{aligned} \tag{2.3.7}$$

Since $\gamma \in U(\rho^{\alpha_1}, p)$, $\gamma_1 \in \Gamma(\rho^\alpha)$ by Lemma(2.3.2)

$$\mid \Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2 \mid > \frac{1}{2}\rho^{\alpha_1}.$$

So the binding formula

$$(\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2) < \Psi_N(x), u_{\gamma+\gamma_1}(x) > = < \Psi_N(x), q(x) u_{\gamma+\gamma_1}(x) >$$

implies

$$< \Psi_N(x), u_{\gamma+\gamma_1}(x) > = \frac{< \Psi_N(x), q(x) u_{\gamma+\gamma_1}(x) >}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2}. \tag{2.3.8}$$

Substituting (2.3.8) into (2.3.7)

$$(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) >$$

$$= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2} < \Psi_N(x), q(x) u_{\gamma+\gamma_1}(x) > + O(\rho^{-p\alpha}). \tag{2.3.9}$$

At the second step of the iteration we substitute the decomposition (2.2.4) of$q(x)u_{\gamma+\gamma_1}(x)$ into (2.3.9)

$$(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) >$$

$$\begin{aligned} &= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2} < \Psi_N(x), \sum_{\gamma_2 \in \Gamma(\rho^\alpha)} Q_{\gamma+\gamma_1+\gamma_2} u_{\gamma+\gamma_1+\gamma_2}(x) > + O(\rho^{-p\alpha}) \\ &= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2} \sum_{\gamma_2 \in \Gamma(\rho^\alpha)} Q_{\gamma+\gamma_1+\gamma_2} < \Psi_N(x), u_{\gamma+\gamma_1+\gamma_2}(x) > + O(\rho^{-p\alpha}). \end{aligned}$$

Isolating the terms with the coefficient$< \Psi_N(x), u_\gamma(x) >$ in the last expression, we obtain

$$
\begin{aligned}
&(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) > \\
&= \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha) \\ \gamma_1+\gamma_2=0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2} Q_{\gamma+\gamma_1+\gamma_2} < \Psi_N(x), u_\gamma(x) > \\
&+ \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha) \\ \gamma_1+\gamma_2\neq 0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2} Q_{\gamma+\gamma_1+\gamma_2} < \Psi_N(x), u_{\gamma+\gamma_1+\gamma_2}(x) > \\
&+ O(\rho^{-p\alpha}). \qquad (2.3.10)
\end{aligned}
$$

Since $\gamma \in U(\rho^{\alpha_1}, p)$, $\gamma_1 + \gamma_2 \in \Gamma(2\rho^\alpha)$ by Lemma(2.3.2)

$$
\mid \Lambda_N - \mid \lambda_{\gamma+\gamma_1+\gamma_2} \mid^2 \mid > \frac{1}{2}\rho^{\alpha_1}.
$$

So the binding formula

$$
(\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\gamma_2} \mid^2) < \Psi_N(x), u_{\gamma+\gamma_1+\gamma_2}(x) > = < \Psi_N(x), q(x) u_{\gamma+\gamma_1+\gamma_2}(x) >
$$

implies

$$
< \Psi_N(x), u_{\gamma+\gamma_1+\gamma_2}(x) > = \frac{< \Psi_N(x), q(x) u_{\gamma+\gamma_1+\gamma_2}(x) >}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\gamma_2} \mid^2}. \qquad (2.3.11)
$$

Substituting (2.3.11) into (2.3.10)

$$
\begin{aligned}
&(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) > \\
&= \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha) \\ \gamma_1+\gamma_2=0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2} Q_{\gamma+\gamma_1+\gamma_2} < \Psi_N(x), u_\gamma(x) > \\
&+ \sum_{\substack{\gamma_1,\gamma_2 \in \Gamma(\rho^\alpha) \\ \gamma_1+\gamma_2\neq 0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2} \frac{Q_{\gamma+\gamma_1+\gamma_2}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\gamma_2} \mid^2} < \Psi_N(x), q(x) u_{\gamma+\gamma_1+\gamma_2}(x) > \\
&+ O(\rho^{-p\alpha}). \qquad (2.3.12)
\end{aligned}
$$

At the third step of the iteration we substitute the decomposition (2.2.4) of$q(x)u_{\gamma+\gamma_1+\gamma_2}(x)$ into (2.3.12)

$$(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) >$$

$$\begin{aligned}
&= \sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1}\mid^2} Q_{\gamma+\gamma_1+\gamma_2} < \Psi_N(x), u_\gamma(x) > \\
&+ \sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2\neq 0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1}\mid^2} \frac{Q_{\gamma+\gamma_1+\gamma_2}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\gamma_2}\mid^2} \\
&\quad < \Psi_N(x), \sum_{\gamma_3\in\Gamma(\rho^\alpha)} Q_{\gamma+\gamma_1+\gamma_2+\gamma_3} u_{\gamma+\gamma_1+\gamma_2+\gamma_3}(x) > \\
&+ O(\rho^{-p\alpha}) \\
&= \sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1}\mid^2} Q_{\gamma+\gamma_1+\gamma_2} < \Psi_N(x), u_\gamma(x) > \\
&+ \sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2\neq 0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1}\mid^2} \frac{Q_{\gamma+\gamma_1+\gamma_2}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\gamma_2}\mid^2} \\
&\quad \sum_{\gamma_3\in\Gamma(\rho^\alpha)} Q_{\gamma+\gamma_1+\gamma_2+\gamma_3} < \Psi_N(x), u_{\gamma+\gamma_1+\gamma_2+\gamma_3}(x) > \\
&+ O(\rho^{-p\alpha}).
\end{aligned}$$

Isolating the terms with the coefficient $< \Psi_N(x), u_\gamma(x) >$ in the last expression, we obtain

$$(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) >$$

$$\begin{aligned}
&= \sum_{\substack{\gamma_1,\gamma_2\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2=0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1}\mid^2} Q_{\gamma+\gamma_1+\gamma_2} \\
&\quad < \Psi_N(x), u_\gamma(x) > \\
&+ \sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\gamma_3=0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1}\mid^2} \frac{Q_{\gamma+\gamma_1+\gamma_2}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\gamma_2}\mid^2} Q_{\gamma+\gamma_1+\gamma_2+\gamma_3} \\
&\quad < \Psi_N(x), u_\gamma(x) > \\
&+ \sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\gamma_3\neq 0}} \frac{Q_{\gamma+\gamma_1}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1}\mid^2} \frac{Q_{\gamma+\gamma_1+\gamma_2}}{\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\gamma_2}\mid^2} Q_{\gamma+\gamma_1+\gamma_2+\gamma_3} \\
&\quad < \Psi_N(x), u_{\gamma+\gamma_1+\gamma_2+\gamma_3}(x) > \\
&+ O(\rho^{-p\alpha}).
\end{aligned}$$

By the same method, repeating the iteration p times and isolating the terms with multiplicand

$< \Psi_N(x), u_\gamma(x) >$, we get

$$(\Lambda_N - \mid \lambda_\gamma \mid^2) < \Psi_N(x), u_\gamma(x) >= \{\sum_{k=1}^{p} S_k\} < \Psi_N(x), u_\gamma(x) > +R_p + O(\rho^{-p\alpha}), \tag{2.3.13}$$

where

$$S_k = \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{k+1}\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\cdots+\gamma_{k+1}=0}} \frac{Q_{\gamma+\gamma_1}\cdots Q_{\gamma+\gamma_1+\cdots+\gamma_k+\gamma_{k+1}}}{(\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2)\cdots(\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\cdots+\gamma_k} \mid^2)},$$

$$\begin{aligned} R_p \;=\; & \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{p+1}\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\cdots+\gamma_{p+1}\neq 0}} \frac{Q_{\gamma+\gamma_1}\cdots Q_{\gamma+\gamma_1+\cdots+\gamma_p+\gamma_{p+1}}}{(\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2)\cdots(\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\cdots+\gamma_p} \mid^2)} \\ & < \Psi_N(x), u_{\gamma+\gamma_1+\cdots+\gamma_{p+1}}(x) >, \end{aligned}$$

$\gamma_k \in \Gamma(\rho^\alpha)$ and $\mid \gamma_1 + \gamma_2 + \cdots + \gamma_k \mid < p\rho^\alpha$ for all $k = 1, 2, \ldots, p$. Therefore, using Lemma(2.3.2) and (2.2.5)

$\mid S_k \mid=$

$$\begin{aligned} &= \mid \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{k+1}\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\cdots+\gamma_{k+1}=0}} \frac{Q_{\gamma+\gamma_1}\cdots Q_{\gamma+\gamma_1+\cdots+\gamma_k+\gamma_{k+1}}}{(\Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2)\cdots(\Lambda_N - \mid \lambda_{\gamma+\gamma_1+\cdots+\gamma_k} \mid^2)} \mid \\ &\le \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{k+1}\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\cdots+\gamma_{k+1}=0}} \frac{\mid Q_{\gamma+\gamma_1} \mid \cdots \mid Q_{\gamma+\gamma_1+\cdots+\gamma_k+\gamma_{k+1}} \mid}{\mid \Lambda_N - \mid \lambda_{\gamma+\gamma_1} \mid^2 \mid \cdots \mid \Lambda_N - \mid \lambda_{\gamma+\gamma_1+\cdots+\gamma_k} \mid^2 \mid} \\ &\le (\frac{1}{2}\rho^{\alpha_1})^{-k} \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{k+1}\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\cdots+\gamma_{k+1}=0}} \mid Q_{\gamma+\gamma_1} \mid \cdots \mid Q_{\gamma+\gamma_1+\cdots+\gamma_k+\gamma_{k+1}} \mid \\ &= (\frac{1}{2}\rho^{\alpha_1})^{-k} M(\gamma) \sum_{\substack{\gamma_2,\ldots,\gamma_{k+1}\in\Gamma(\rho^\alpha)\\ \gamma_1+\gamma_2+\cdots+\gamma_{k+1}=0}} \mid Q_{\gamma+\gamma_1+\gamma_2} \mid \cdots \mid Q_{\gamma+\gamma_1+\cdots+\gamma_k+\gamma_{k+1}} \mid \\ &\vdots \\ &= (\frac{1}{2}\rho^{\alpha_1})^{-k} M(\gamma)M(\gamma+\gamma_1)\cdots M(\gamma+\gamma_1+\cdots+\gamma_k), \end{aligned}$$

that is,

$$\mid S_k \mid \le (\frac{1}{2}\rho^{\alpha_1})^{-k} M(\gamma)M(\gamma+\gamma_1)\cdots M(\gamma+\gamma_1+\cdots+\gamma_k),$$

or

$$S_k = O(\rho^{-k\alpha_1}) \tag{2.3.14}$$

for every $k = 1, 2, \ldots, p$ which implies

$$\sum_{k=1}^{p} S_k = O(\rho^{-\alpha_1}). \tag{2.3.15}$$

Using Lemma(2.3.2), (2.2.5) and $|< \Psi_N(x), u_{\gamma+\gamma_1+\ldots+\gamma_{p+1}}(x) >| \leq K$ for some constant K

$| R_p |$

$$\begin{aligned}
&= \Big| \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{p+1}\in\Gamma(\rho^{\alpha}) \\ \gamma_1+\gamma_2+\cdots+\gamma_{p+1}\neq 0}} \frac{Q_{\gamma+\gamma_1}\cdots Q_{\gamma+\gamma_1+\cdots+\gamma_p+\gamma_{p+1}}}{(\Lambda_N - |\lambda_{\gamma+\gamma_1}|^2)\cdots(\Lambda_N - |\lambda_{\gamma+\gamma_1+\cdots+\gamma_p}|^2)} \Big| \\
&\quad |< \Psi_N(x), u_{\gamma+\gamma_1+\ldots+\gamma_{p+1}}(x) >| \\
&\leq \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{p+1}\in\Gamma(\rho^{\alpha}) \\ \gamma_1+\gamma_2+\cdots+\gamma_{p+1}\neq 0}} \frac{|Q_{\gamma+\gamma_1}|\cdots|Q_{\gamma+\gamma_1+\cdots+\gamma_p+\gamma_{p+1}}|}{|\Lambda_N - |\lambda_{\gamma+\gamma_1}|^2|\cdots|\Lambda_N - |\lambda_{\gamma+\gamma_1+\cdots+\gamma_p}|^2|} \\
&\quad |< \Psi_N(x), u_{\gamma+\gamma_1+\ldots+\gamma_{p+1}}(x) >| \\
&\leq K(\frac{1}{2}\rho^{\alpha_1})^{-p} \sum_{\substack{\gamma_1,\gamma_2,\ldots,\gamma_{p+1}\in\Gamma(\rho^{\alpha}) \\ \gamma_1+\gamma_2+\cdots+\gamma_{p+1}\neq 0}} |Q_{\gamma+\gamma_1}|\cdots|Q_{\gamma+\gamma_1+\cdots+\gamma_p+\gamma_{p+1}}| \\
&= K(\frac{1}{2}\rho^{\alpha_1})^{-p} M(\gamma) \sum_{\substack{\gamma_2,\ldots,\gamma_{p+1}\in\Gamma(\rho^{\alpha}) \\ \gamma_1+\gamma_2+\cdots+\gamma_{p+1}\neq 0}} |Q_{\gamma+\gamma_1+\gamma_2}|\cdots|Q_{\gamma+\gamma_1+\cdots+\gamma_p+\gamma_{p+1}}| \\
&\vdots \\
&= K(\frac{1}{2}\rho^{\alpha_1})^{-p} M(\gamma)M(\gamma+\gamma_1)\cdots M(\gamma+\gamma_1+\cdots+\gamma_p),
\end{aligned}$$

that is,

$$| R_p | \leq K(\frac{1}{2}\rho^{\alpha_1})^{-p} M(\gamma)M(\gamma+\gamma_1)\cdots M(\gamma+\gamma_1+\cdots+\gamma_p),$$

or

$$R_p = O(\rho^{-p\alpha_1}). \tag{2.3.16}$$

Substituting (2.3.15) and (2.3.16) into (2.3.13) we obtain

$$(\Lambda_N - |\lambda_\gamma|^2) < \Psi_N(x), u_\gamma(x) > =$$

$$O(\rho^{-\alpha_1}) < \Psi_N(x), u_\gamma(x) > + O(\rho^{-p\alpha_1}) + O(\rho^{-p\alpha}).$$

Dividing both sides of the last equation by $< \Psi_N(x), u_\gamma(x) >$

$$\Lambda_N - \mid \lambda_\gamma \mid^2 = O(\rho^{-\alpha_1}) + \frac{O(\rho^{-p\alpha_1})}{< \Psi_N(x), u_\gamma(x) >} + \frac{O(\rho^{-p\alpha})}{< \Psi_N(x), u_\gamma(x) >},$$

using Lemma(2.3.3)

$$\Lambda_N - \mid \lambda_\gamma \mid^2 = O(\rho^{-\alpha_1}) + \frac{O(\rho^{-p\alpha_1})}{O(\rho^{\frac{-(d-1)}{2}})} + \frac{O(\rho^{-p\alpha})}{O(\rho^{\frac{-(d-1)}{2}})},$$

using $\alpha_1 = 3\alpha > \alpha$

$$\Lambda_N - \mid \lambda_\gamma \mid^2 = O(\rho^{-\alpha_1}) + \frac{O(\rho^{-p\alpha})}{O(\rho^{\frac{-(d-1)}{2}})},$$

choosing p such that $p > \frac{d-1}{2\alpha} + 1$

$$\Lambda_N = \mid \lambda_\gamma \mid^2 + O(\rho^{-\alpha_1}).$$

This completes the proof. $\square$

CONCLUSIONS

In this thesis, we obtained an asymptotic formula for the non-resonance eigenvalues of the self-adjoint Schrödinger operator, with mixed boundary condition defined on a d-dimensional parallelepiped F.

For every non-resonance eigenvalue $\mid \lambda_\gamma \mid^2$, $\mid \gamma \mid \sim \rho$, of the operator $L_M(0)$ there exists an eigenvalue Λ_N of the operator $L_M(q(x))$ satisfying

$$\Lambda_N = \mid \lambda_\gamma \mid^2 + O(\rho^{-\alpha_1}).$$

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