DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

Q-BERNSTEIN POLYNOMIALS ON THE INTERVAL [a,b]

by

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July, 2014
İZMİR

Q-BERNSTEIN POLYNOMIALS ON THE INTERVAL [a,b]

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M.Sc THESIS EXAMINATION RESULT FORM

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Q-BERNSTEIN POLYNOMIALS ON THE INTERVAL [a,b]

ABSTRACT

We first define a new one parameter family of q-Bernstein polynomial on an arbitrary interval. It reduces to the classical Bernstein polynomial on any interval when $q = 1$. This polynomial also inherits some geometric properties of the classical Bernstein polynomial on any interval. However, the convergence of q-Bernstein polynomial on an arbitrary interval is very different from that of classical Bernstein polynomial on any interval. The uniform convergence of this polynomial for a given f in $C[a,b]$ depends on parameters a, b and q. We then consider the limit function of the generalized q-Bernstein polynomial on any interval and show that when q is fixed on the interval 0 and 1, the limit of this polynomial is $f(x)$ as n tends to infinity if and only if $f(x)$ is linear. Moreover we find the degree of approximation by modulus of continuity. We also show that this new q-Bernstein polynomial has symmetry property provided that f is symmetric on a closed symmetric interval.

Keywords: Generalized q-Bernstein polynomials on any closed interval, uniform convergence, modulus of continuity.

[a,b] ARALIĞINDA Q-BERNSTEIN POLİNOMLARI

ÖZ

İlk olarak herhangi bir kapalı aralıkta bir parametreli q-Bernstein polinom ailesini tanımladık. Parametreyi $q = 1$ olarak seçtiğimizde, bu polinom herhangi bir kapalı aralıkta tanımlı klasik Bernstein polinomuna dönüşür. Ayrıca bu polinom herhangi bir kapalı aralıkta tanımlı klasik Bernstein polinomunun bazı geometrik özelliklerine de sahiptir. Ancak, herhangi bir kapalı aralıkta tanımlı q-Bernstein polinomunun yakınsaklığı klasik Bernstein polinomununkinden oldukça farklıdır. a, b kapalı aralığında tanımlı sürekli fonksiyon için bu polinomun düzgün yakınsaklığı a, b ve q parametrelerine bağlıdır. Sonra n sonsuza giderken bu fonksiyonunun limitini inceledik ve q parametresini 0 ve 1 arasında sabitlediğimizde, bu fonksiyonun limiti $f(x)$ ancak ve ancak $f(x)$ doğrusal fonksiyon olduğunda sağladığını gösterdik. Ayrıca, simetrik kapalı bir aralıkta tanımlı f fonksiyonun simetrik olması koşulu altında genelleştirilmiş q-Bernstein polinomunun ilginç bir simetri özelliğine sahip olduğunu ispatladık .

Anahtar kelimeler: a, b kapalı aralığında genelleştirilmiş q-Bernstein polinomları, düzgün yakınsaklık, süreklilik modülü.

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CHAPTER ONE

INTRODUCTION

Among many proofs of the Weierstrass Approximation Theorem, probably the one given by S. N. Bersntein in 1912 is the most well-known. He introduced the following polynomials, so called *Bernstein polynomials*

$$B_n(f;x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}, \tag{1.1}$$

where $f(x)$ is a function on the interval $[0,1]$ and n is a positive integer. These polynomials have many remarkable properties. Since they are particularly useful for both approximation and curves and surfaces design, their analytic properties have been studied extensively for several decades. The books by Lorentz (1986) and Farin (2002) are the most comprehensive guide for these purposes.

We now briefly describe some of these properties. Firstly, it is easily verified that the operators $B_n f$ is a *linear monotone* operators on $[0,1]$ for $n = 1,2,\ldots$. Thus, we can apply Bohman-Korovkin theorem (see, Cheney (1984)) which states that for a *linear monotone* operator $\mathcal{L}_n$, the convergence of $\mathcal{L}_n f \to f$ for $f(x) = 1, x, x^2$ is sufficient for the operator $\mathcal{L}_n$ to have the uniform convergence property $\mathcal{L}_n f \to f$ for all $f \in C[a,b]$. This justifies the uniform convergence of $B_n f$ to f for all f in $C[0,1]$. Similarly, derivatives of the Bernstein polynomial $B_n^{(k)}(f;x)$ converges uniformly to $f^{(k)}(x)$ on $[0,1]$. Yet there is another property due to Voronovskaya, that is if f is bounded on $[0,1]$ and $f''(x)$ exists for some $x \in [0,1]$, then

$$\lim_{n\to\infty} n(B_n(f;x) - f(x)) = \frac{x(1-x)}{2} f''(x), \tag{1.2}$$

which gives asymptotic error estimates for the Bernstein polynomials. In other words, the convergence by Bernstein polynomial is very slow, like the sequence $1/n$.

These polynomials are also *variation diminishing*, which yields some shape preserving properties. Namely, the number of sign changes in $B_n(f;x)$ is bounded by that of $f(x)$. Furthermore, when a function f is monotone, Bernstein polynomials of f are monotone and it also yields a convex function whenever f is convex. Schoenberg

(1959) proved that if $f(x)$ is a convex function then the Bernstein polynomials are monotonic in the sense that

$$B_n(f;x) \geq B_{n+1}(f;x) \geq f(x), \quad x \in [0,1]. \tag{1.3}$$

The converse of this result showed by Kosmak (1960). That is, if $B_n(f;x) \geq B_{n+1}(f;x)$ for all $n \in N$, then $f(x)$ is convex.

Due to their benefit in geometric modelling, Bernstein-Bézier techniques are fundamental in Computer Aided Geometric Design (CAGD). Bézier curves independently discovered by P. de Casteljau at Citröen and by P. Bézier at Renault. A Bézier curve is defined by

$$P(t) = \sum_{j=0}^{n} p_j B_j^n(t), \quad t \in [0,1] \tag{1.4}$$

where $B_j^n(t) = \binom{n}{j} t^j (1-t)^{n-j}$ is Bernstein basis and $p_0, p_1, \ldots, p_n$ are control points in $\mathbb{R}^2$ or $\mathbb{R}^3$. As the curve lies in the convex hull of its control points, the points can be graphically displayed and used to manipulate the curve intuitively. Affine transformations such as translation and rotation can be applied on the curve by applying the respective transform on the control points of the curve.

The de Casteljau algorithm is recursive method to evaluate polynomials in Bernstein form or Bézier curves. The algorithm can also be used to split a single Bézier curves at an arbitrary parameter value. It has an elegant geometric interpretation. This algorithm is so fundamental that it is used both subdivision and in blossoming. Furthermore, in 1970's it is generalized to generate B-spline curves, so called *de Boor algorithm*.

Blossoms or polar forms simplify the construction of polynomial and piecewise polynomial curves and surfaces and lead to new surface representations and algorithms. The blossom bases on symmetric multiaffine mapping. A map $T : \mathbb{R} \to \mathbb{R}$ is *affine* if it preserves affine combinations, that is

$$T\left(\sum_{j=1}^{n} a_j t_j\right) = \sum_{j=1}^{n} a_j T(t_j), \quad \forall a_j, t_j \in \mathbb{R} \quad \text{with} \quad \sum_{j=1}^{n} a_j = 1. \tag{1.5}$$

A map $p : \mathbb{R}^n \to \mathbb{R}$ is *multiaffine*, *symmetric* and *diagonal* is called *polar form(blossom)*. Every polynomial $P : \mathbb{R} \to \mathbb{R}$ of degree $\leqslant n$ has a unique n-variate polar form or blossom

p. This polar form p is n-affine, symmetric and diagonal, see Goldman (2003). We will explain this technique in more detail in Chapter 2.2 .

In view of development of q-Calculus , Phillips (1997) proposed a generalization of the Bernstein polynomials, based on the q-integers. While for $q = 1$ these polynomials reduce to classical Bernstein polynomials. However, for $q \neq 1$ they exhibit interesting properties. The convergence of q-Bernstein polynomials on $[0, 1]$ was first studied by Phillips (1997). He showed that the convergence properties of these polynomials are different from classical Bernstein polynomials on $[0, 1]$. In particular, taking a sequence $q = q_n$ with $q_n \to 1$ such that $[n] \to \infty$ as $n \to \infty$, and using Bohman-Korovkin theorem he showed that $B_n(f; q, x)$ converges uniformly to f for all $f \in C[0, 1]$. Then, this convergence property is examined for fixed real q, $0 < q < 1$ and for $q \geq 1$. Oruç & Tuncer (2002) proved for a fixed q, $0 < q < 1$, the uniform convergence holds if and only if f is linear on $[0, 1]$. In addition, if $q \geq 1$, then $B_n f \to f$ as $n \to \infty$ for polynomial f. Furthermore, Il'inskii & Ostrovska (2002) studied the convergence of the limiting function $B_\infty(f; q, x)$.

1.1 The q-Calculus

Since our subsequent work invokes q-integers, q-binomial coefficients and q-series we begin here with a brief review of the q-calculus. For a more comprehensive discussion of the q-calculus, see Kac & Cheung (2002).

For any real fixed number q, the q-integer $[r]$ is defined as

$$[r] = \begin{cases} (1 - q^r)/(1 - q), & q \neq 1, \\ r, & q = 1. \end{cases} \tag{1.6}$$

Note that $[r]$ is a continuous function of q. We next define $[r]!$, where r is a nonnegative integer, as

$$[r]! = \begin{cases} [r][r-1]\ldots[1], & r \geq 1, \\ 1, & r = 0, \end{cases} \tag{1.7}$$

and call $[r]!$ a q-factorial. Later we shall also need the q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$ which is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[n-r]![r]!} = \frac{[n]\cdot[n-1]\cdots[n-r+1]}{[r]\cdot[r-1]\cdots[1]} \tag{1.8}$$

for $n \geq r \geq 1$, and has the value 1 when $r = 0$ and the value zero otherwise. Note that we use $\begin{bmatrix} n \\ r \end{bmatrix}_q$ to emphasize its dependence on q, whenever needed.

The q-binomial coefficients can be computed recursively by the q-Pascal identities

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^r \begin{bmatrix} n-1 \\ r \end{bmatrix} + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} \tag{1.9}$$

or

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r \end{bmatrix} + q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}. \tag{1.10}$$

Note that $\begin{bmatrix} n \\ r \end{bmatrix}$ is a polynomial in q of degree $r(n-r)$, whose coefficients may be regarded as the generating functions for restricted partitions of integers, see Andrews (1976).

1.2 The q-Bernstein Polynomials on the Interval [0,1]

Phillips (1997) proposed the following generalization of the Bernstein polynomials, based on the q-integers. For each positive integer n,

$$B_n(f;q,x) = \sum_{k=0}^{n} f_k \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{j=0}^{n-k-1} (1-q^j x), \ x \in [0,1], \tag{1.11}$$

where an empty product denotes 1, the parameter $q \in \mathbb{R}^+$ is fixed and $f_k = f([k]/[n])$. These polynomials possess the following properties:

i) When $q = 1$, it reduces to the classical Bernstein polynomial, $B_n(f;x)$.

ii) It satisfies the end point interpolation conditions

$$B_n(f;q,0) = f(0), \quad B_n(f;q,1) = f(1). \tag{1.12}$$

iii) $B_n(f;q,x)$ is a monotone linear operator if $0 < q \leqslant 1$ that is if $f(x) \geq g(x)$, then $B_n(f;q,x) \geq B_n(g;q,x)$.

iv) It reproduces linear functions

$$B_n(ax+b;q,x) = ax+b, \quad a,b \in \mathbb{R}. \tag{1.13}$$

v) The q-Bernstein polynomial may be expressed in terms of q-differences as

$$B_n(f;q,x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \Delta_q^k f_0 \, x^k, \tag{1.14}$$

where

$$\Delta_q^k f_j = \Delta_q^{k-1} f_{j+1} - q^{k-1}\Delta_q^{k-1} f_j, \quad k \geq 1, \tag{1.15}$$

with $\Delta_q^0 f_j = f_j = f([j]/[n])$. When $q = 1$ the q-differences become ordinary forward differences with equidistant step size $h = 1/n$. Furthermore, given a polynomial $f(x)$ of degree at most m, $\Delta_q^k f_0$ vanishes for all $k > m$. Therefore $B_n(f;q,x)$ is a polynomial of degree $\min(m,n)$.

We can deduce easily from q-difference form (1.14) that

$$B_n(1;q,x) = 1, \quad B_n(t;q,x) = x, \quad B_n(t^2;q,x) = x^2 + \frac{x(1-x)}{[n]}. \tag{1.16}$$

vi) It is shown in Phillips (1996) that (1.11) may be evaluated by the following de Casteljau type algorithm:

$$\begin{aligned} &\textbf{Given} \;:\; f_0^{[0]}, f_1^{[0]}, \ldots, f_n^{[0]} \\ &\textbf{Compute} \;:\; f_r^{[m]} = (q^r - q^{m-1}x) f_r^{[m-1]} + x f_{r+1}^{[m-1]} \begin{cases} m = 1, \ldots, n \\ r = 0, \ldots, n-m. \end{cases} \end{aligned} \tag{1.17}$$

We see that when $q = 1$, it reduces the classical de Casteljau type algorithm and has a nice geometric interpretation *subdivision*, see Goldman (2003).

vii) If the parameter is taken as $q = q_n \to 1$ from below as $n \to \infty$ then $B_n(f;q,x)$ converges uniformly to f, see Phillips (1997).

viii) The degree of approximation interms of modulus of continuity, see its definition in chapter (3.3), is given in Oruç (1998) as

$$\| B_n(f;q,x) - f(x) \| \leqslant \frac{3}{2}\omega(1/[n]^{1/2}),\ 0 < q \leqslant 1. \tag{1.18}$$

Here $\| . \|$ denotes the maximum norm on $[0,1]$.

The geometric properties are shown in Goodman, *et.al.* (1999), Oruç (1998) and Oruç & Phillips (2003):

ix) If $f(x)$ is convex on $[0,1]$ then $B_n(f;q,x)$ is also convex for $0 < q \leqslant 1$. Furthermore, if $0 < q_1 \leqslant q_2 \leqslant 1$, then $B_n(f;q_2,x) \leqslant B_n(f;q_1,x)$, see Goodman, *et.al.* (1999).

x) The approximation to a convex function f by q-Bernstein polynomials is one-sided, from above f. Namely,

$$B_{n-1}(f;q,x) \geq B_n(f;q,x) \geq f(x), \quad n = 2,3,\ldots, \text{ and } \quad 0 < q \leqslant 1. \tag{1.19}$$

The convergence of $B_n(f;q,x)$ very much depends on the parameter q. The first investigation of convergence with $q > 1$ is given in Tuncer (2001). Then the limiting $B_n(f;q,x)$ has been studied by Il'inskii & Ostrovska (2002).

xi) When $0 < q < 1$ is fixed, $\lim\limits_{n\to\infty} B_n(f;q,x) = f(x)$ if and only if $f(x)$ is linear.

CHAPTER TWO

Q-BERSNTEIN BASES ON [a,b]

After the introduction of q-Bernstein polynomials on $[0,1]$ by Phillips (1997), the paper by Simeonov, *et.al.* (2012) extended the q-Bernstein basis polynomials over arbitrary intervals. The main purpose of the latter work was to develop q-blossoming and subdivision techniques to generate q-Bézier curves.

In this chapter we will investigate q-Bernstein bases on arbitrary intervals, introduced by Simeonov, *et.al.* (2012) . Then, we verify that the basis function form partition of unity. Finally, we will give a brief summary of q-blossoming and subdivision techniques for q-Bézier curves.

2.1 q-Identities for q-Bernstein Bases over Arbitrary Intervals

We begin by explaining the notation to be used. Simeonov, *et.al.* (2012) defined the following q-Bernstein basis functions:

$$B_k^n(t;[a,b];q) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{\prod_{j=0}^{k-1}(t-aq^j)\prod_{j=0}^{n-k-1}(b-q^jt)}{\prod_{j=0}^{n-1}(b-aq^j)}, \quad k=0,\ldots,n, \quad t\in[a,b]. \tag{2.1}$$

In the above notation, values of q for which $b-aq^j=0$ for some $1\leqslant j\leqslant n-1$ are excluded. Note that when $a=0$ and $b=1$, the formula (2.1) reduces to

$$B_k^n(t;q) = \begin{bmatrix} n \\ k \end{bmatrix} t^k \prod_{j=0}^{n-k-1}(1-q^jt), \tag{2.2}$$

the q-Bernstein basis functions on the interval $[0,1]$. The limit of (2.1) as $q\to 1$ gives the classical Bernstein basis functions over the interval $[a,b]$,

$$B_k^n(t;[a,b];1) = \binom{n}{k}\frac{(t-a)^k(b-t)^{n-k}}{(b-a)^n}. \tag{2.3}$$

The q-Bernstein basis functions on $[a,b]$ satisfy the recurrence relations

$$B_k^n(t;[a,b];q) = \left(\frac{b-tq^{n-k-1}}{b-aq^{n-1}}\right) B_k^{n-1}(t;[a,b];q) + q^{n-k}\left(\frac{t-aq^{k-1}}{b-aq^{n-1}}\right) B_{k-1}^{n-1}(t;[a,b];q) \tag{2.4}$$

and

$$B_k^n(t;[a,b];q) = q^k\left(\frac{b-tq^{n-k-1}}{b-aq^{n-1}}\right) B_k^{n-1}(t;[a,b];q) + \left(\frac{t-aq^{k-1}}{b-aq^{n-1}}\right) B_{k-1}^{n-1}(t;[a,b];q) \tag{2.5}$$

These recurrence relation can be verified using q-Pascal identities (1.9), (1.10) respectively. Cubic bases functions $B_k^3(t;[a,b];q)$ for $k=0,1,2,3$ and various values of a, b and q are depicted below.

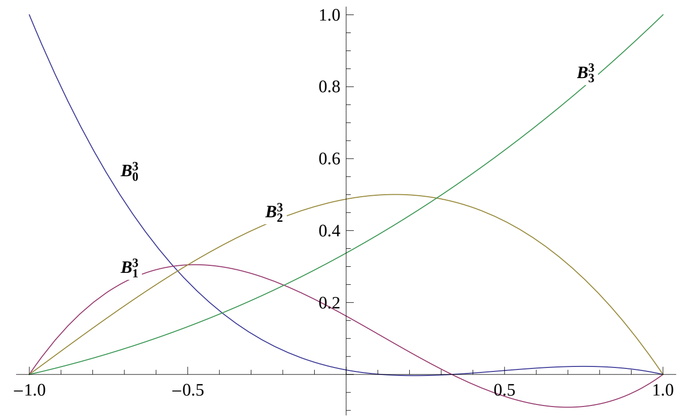

Figure 2.1 Cubic q-Bernstein bases on $[-1,1]$ for $q=3$

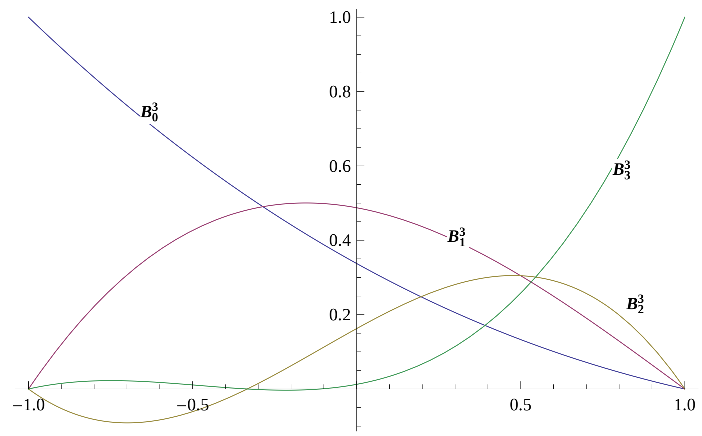

Figure 2.2 Cubic q-Bernstein bases on $[-1,1]$ for $q=1/3$

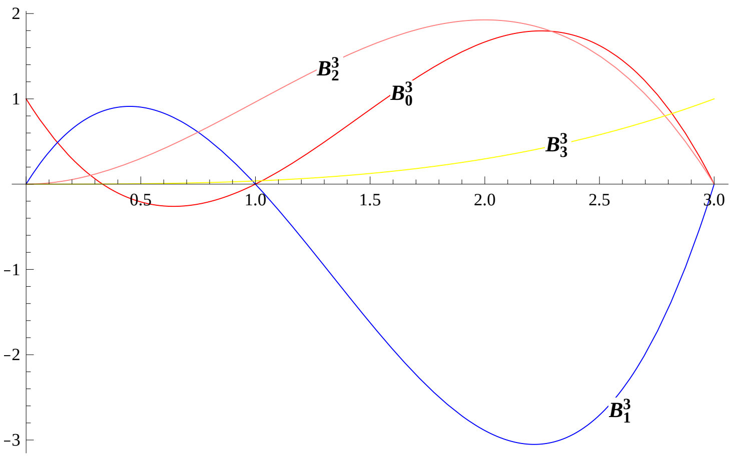

Figure 2.3 Cubic q-Bernstein bases on $[0,3]$ for $q = 3$

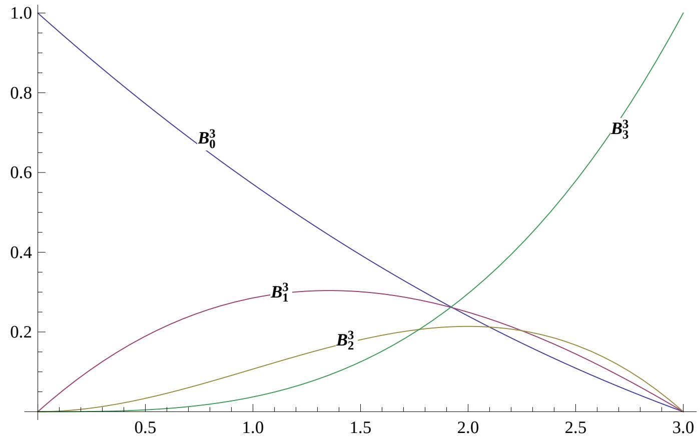

Figure 2.4 Cubic q-Bernstein bases on $[0,3]$ for $q = 1/3$

Notice that we lost an important feature of the classical Bernstein basis and the q-Bernstein basis, the nonnegativity.

Proposition 2.1.1. *The basis functions* $B_k^n(t;[a,b];q) \geq 0$ *for* $k = 0,1,2,\ldots,n$ *and* $t \in [a,b]$ *if* $0 \leqslant a < b$ *and* $0 < q \leqslant 1$.

Proof. This is straightforward from the definition (2.1).

Another property is that although the classical Bernstein basis function has just multiple zeros at the endpoints of the interval, this new basis B_k^n posseses n distinct real zeros on $\mathbb{R}$.

Although partition of unity property is used many times in Simeonov, *et.al.* (2012), its proof is not given. We now give its proof by induction.

It is clear that for $n = 0$, we have

$$B_0^0(t;[a,b];q) = 1.$$

Suppose that for $n \geq 1$,

$$\sum_{k=0}^{n} B_k^n(t;[a,b];q) = 1. \tag{2.6}$$

On using the recurrence relation (2.4), we get

$$\begin{aligned}\sum_{k=0}^{n+1} B_k^{n+1}(t;[a,b];q) &= \sum_{k=0}^{n+1}\left(\frac{b-tq^{n-k}}{b-aq^n}\right)B_k^n(t;[a,b];q)\\ &+ \sum_{k=0}^{n+1} q^{n-k+1}\left(\frac{t-aq^{k-1}}{b-aq^n}\right)B_{k-1}^n(t;[a,b];q).\end{aligned} \tag{2.7}$$

Note that $B_k^n(t;[a,b];q)$ and $B_{k-1}^n(t;[a,b];q)$ are not defined for $k = n+1, k = 0$, respectively. We may rewrite the last equation as

$$\begin{aligned}\sum_{k=0}^{n+1} B_k^{n+1}(t;[a,b];q) &= \sum_{k=0}^{n}\left(\frac{b-tq^{n-k}}{b-aq^n}\right)B_k^n(t;[a,b];q)\\ &+ \sum_{k=1}^{n+1} q^{n-k+1}\left(\frac{t-aq^{k-1}}{b-aq^n}\right)B_{k-1}^n(t;[a,b];q).\end{aligned} \tag{2.8}$$

Then shifting the index of the second summation, we obtain

$$\sum_{k=0}^{n} B_k^{n+1}(t;[a,b];q) = \sum_{k=0}^{n}\left[\left(\frac{b-tq^{n-k}}{b-aq^n}\right)+q^{n-k}\left(\frac{t-aq^k}{b-aq^n}\right)\right]B_k^n(t;[a,b];q)$$

$$= \sum_{k=0}^{n} B_k^n(t;[a,b];q) = 1. \tag{2.9}$$

2.2 Subdivision and q-Blossoming

Blossom is first introduced by Ramshaw (1989) . Blossoming is an effective technique for deriving change of basis algorithms and analyzing the properties of Bézier curves and Bernstein bases. Degree elevation, Subdivision, Conversion from monomial to Bézier form are easily derived from blossoming. It is extended to q-blossom by Simeonov, *et.al.* (2012). The q-blossom form of a polynomial $P(t)$ of degree n is the unique symmetric multiaffine function $p(u_1, \ldots, u_n; q)$ that reduces to $P(t)$ along the q-diagonal. That is, $p(u_1, \ldots, u_n; q)$ is the unique multivariate polynomial satisfying the following three axioms:

1. Symmetry:

$$p(u_1, \ldots, u_n; q) = p(u_{\sigma(1)}, \ldots, u_{\sigma(n)}; q) \tag{2.10}$$

   for every permutation σ of the set $\{1, \ldots, n\}$.

2. Multiaffine:

$$\begin{aligned} p(u_1, \ldots, (1-\alpha)u_k + \alpha v_k, \ldots, u_n; q) &= (1-\alpha)p(u_1, \ldots, u_k, \ldots, u_n; q) \\ &+ \alpha p(u_1, \ldots, v_k, \ldots, u_n; q) \end{aligned} \tag{2.11}$$

3. q-Diagonal:

$$p(t, qt, \ldots, q^{n-1}t; q) = P(t). \tag{2.12}$$

When $q = 1$, we have classical blossoms.

2.2.1 *q-Blossom of Cubic Polynomials*

Let us consider the monomials 1, t, t^2, and t^3 as cubic polynomials. The q-blossom of these monomials $p(u_1,u_2,u_3;q)$ are given below respectively:

$$\begin{aligned} p(u_1,u_2,u_3;q) &= 1, \\ p(u_1,u_2,u_3;q) &= \frac{u_1+u_2+u_3}{1+q+q^2}, \\ p(u_1,u_2,u_3;q) &= \frac{u_1u_2+u_2u_3+u_3u_1}{q(1+q+q^2)}, \\ p(u_1,u_2,u_3;q) &= \frac{u_1u_2u_3}{q^3}. \end{aligned} \tag{2.13}$$

Each of these functions above $p(u_1,u_2,u_3;q)$ satisfy q-blossoming axioms. Thus, we can obtain q-blossom of any cubic polynomial $P(t) = at^3+bt^2+ct+d$ as

$$p(u_1,u_2,u_3;q) = a\frac{u_1u_2u_3}{q^3} + b\frac{u_1u_2+u_2u_3+u_3u_1}{q(1+q+q^2)} + c\frac{u_1+u_2+u_3}{1+q+q^2} + d. \tag{2.14}$$

Moreover we can apply a similar technique to find q-blossom polynomials of any degree n by using q-blossoming of monomials t^r, for $r = 0,\dots,n$.

Splitting a Bézier curve into smaller pieces is useful as a divide and conquer strategy for intersection algorithms. The process of splitting a Bézier curve into two or more Bézier curves that represents the exactly same curve is called *subdivision*. The de Casteljau algorithm, $q = 1$ in (1.17), is a subdivision algorithm. This algorithm was extended to q-Bézier curves by Simeonov, *et.al.* (2012). The convergence of recursive subdivision for q-Bezier curve was investigated. The following result concerning the rate of approximation by subdivision of q-Bézier curve was obatined in their work.

Theorem 2.2.1. *Let $P(t)$ be a q-Bézier curve defined on $[a,b]$. Then the control polygons generated by q-Bézier subdivision converge to the q-Bézier curve $P(t)$ uniformly on the interval $[a,b]$ at the rate of 2^{-N}, where N is the number of iterations.*

CHAPTER THREE

CONVERGENCE OF Q-BERNSTEIN POLYNOMIAL ON [a,b]

In this chapter, we define a new one parameter family of q-Bernstein polynomials $B_n(f;[a,b];q,x)$, based on the bases functions $B_k^n(t;[a,b];q)$ given in Simeonov, *et.al.* (2012). We discuss convergence properties and find the degree of approximation by modulus of continuity for $B_n(f;[a,b];q,x)$. In addition, it is shown that if f is symmetric on the interval $[-a,a]$, the corresponding q-Bernstein polynomials satisfy the property

$$B_n(f;[-a,a];q,x) = B_n(f;[-a,a];1/q,-x).$$

3.1 The q-Bernstein Polynomials over an Arbitrary Interval

Definition 3.1.1. Given a function $f(x)$ with $x \in [a,b]$ and a fixed real number q, we define q-Bernstein polynomial by

$$B_n(f;[a,b];q,x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} \frac{\prod_{s=0}^{r-1}(x-aq^s)\prod_{s=0}^{n-r-1}(b-q^s x)}{\prod_{s=0}^{n-1}(b-aq^s)}, \tag{3.1}$$

for each $n \in \mathbb{Z}^+$, where f_r denotes the value of the function f at

$$x = (a+(b-a)\frac{[r]}{[n]}), \quad \text{for} \quad r = 0,1,2,\ldots,n.$$

Here an empty product in (3.1) denotes 1. When we put q=1 in (3.1), we obtain the classical Bernstein polynomials on the interval $[a,b]$,

$$B_n(f;[a,b];1,x) = \sum_{r=0}^{n} f\left(a+(b-a)\frac{r}{n}\right)\binom{n}{r}\frac{(x-a)^r(b-x)^{n-r}}{(b-a)^n}. \tag{3.2}$$

We can also see easily from (3.1) that $B_n(f;[a,b];q,x)$ interpolates endpoints:

$$B_n(f;[a,b];q,a) = f(a) \quad \text{and} \quad B_n(f;[a,b];q,b) = f(b) \tag{3.3}$$

as in for the classical Bernstein polynomials. It is clear that $B_n f$ is a linear operator

$$B_n(\lambda f + \mu g) = \lambda B_n f + \mu B_n g \tag{3.4}$$

for all functions f and g defined on $C[a,b]$, and all real λ and μ. We can also see that $B_n f$ is a monotone operator on $C[a,b]$ if $0 < q \leqslant 1$ and $0 \leqslant a < b$.
Note that cubic q-Bernstein bases are symmetric in the sense that

$$B_k^n(t;[-a,a];q) = B_{n-k}^n(-t;[-a,a];1/q) \tag{3.5}$$

as shown in the figures below.

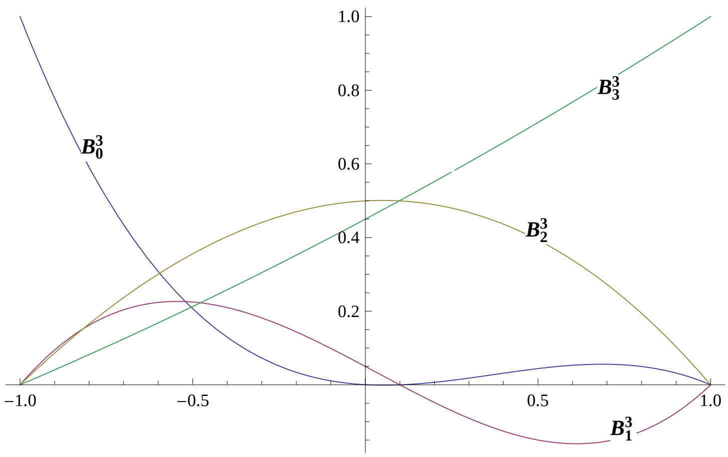

Figure 3.1 Cubic q-Bernstein bases on $[-1,1]$ for $q = 10$.

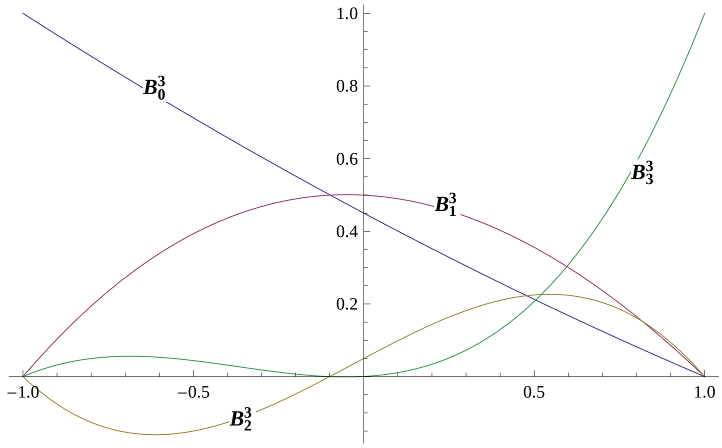

Figure 3.2 Cubic q-Bernstein bases on $[-1,1]$ for $q = 1/10$.

Then from this observation, we have the following proposition.

Proposition 3.1.2. *If $f(x)$ is symmetric on $[-a,a]$, then*

$$B_n(f;[-a,a];q,x) = B_n(f;[-a,a];1/q,-x). \tag{3.6}$$

Proof. Assume that $f(x)$ is symmetric on $[-a,a]$ such that $f(x) = f(-x)$. Firstly, let us manipulate the basis functions

$$\begin{aligned} B^n_{n-r}([-a,a];1/q,-x) &= \begin{bmatrix} n \\ n-r \end{bmatrix}_{1/q} \frac{\prod_{s=0}^{n-r-1}(-x+a/q^s)\prod_{s=0}^{r-1}(a+x/q^s)}{\prod_{s=0}^{n-1}(a+a/q^s)} \\ &= q^{r^2-nr}\begin{bmatrix} n \\ n-r \end{bmatrix}_{q} \frac{\prod_{s=0}^{n-r-1}(a-xq^s)\prod_{s=0}^{r-1}(x+aq^s)}{\prod_{s=0}^{n-1}(a+aq^s)} q^{nr-r^2} \\ &= B^n_r([-a,a];q,x). \end{aligned} \tag{3.7}$$

Then, using definition (3.1), we get

$$B_n(f;[-a,a];1/q,-x) = \sum_{r=0}^{n} f_{n-r} B^n_{n-r}([-a,a];1/q,-x), \tag{3.8}$$

where $f_{n-r} = f(-a+2a\frac{[n-r]_{1/q}}{[n]_{1/q}})$.

Since

$$\frac{[n-r]_{1/q}}{[n]_{1/q}} = q^r \frac{[n-r]_q}{[n]_q} = \frac{[n]_q-[r]_q}{[n]_q} \tag{3.9}$$

and by assumption, we obtain

$$\begin{aligned} B_n(f;[-a,a];1/q,-x) &= \sum_{r=0}^{n} f(-a+2a\frac{[r]_q}{[n]_q}) B^n_r([-a,a];q,x) \\ &= \sum_{r=0}^{n} f_r B^n_r([-a,a];q,x) \qquad (3.10) \\ &= B_n(f;[-a,a];q,x) \end{aligned}$$

which completes the proof.

The following figure shows behaviour of $B_n(f;[-1,1];q,x)$ while approximating $f(x) = x^2$.

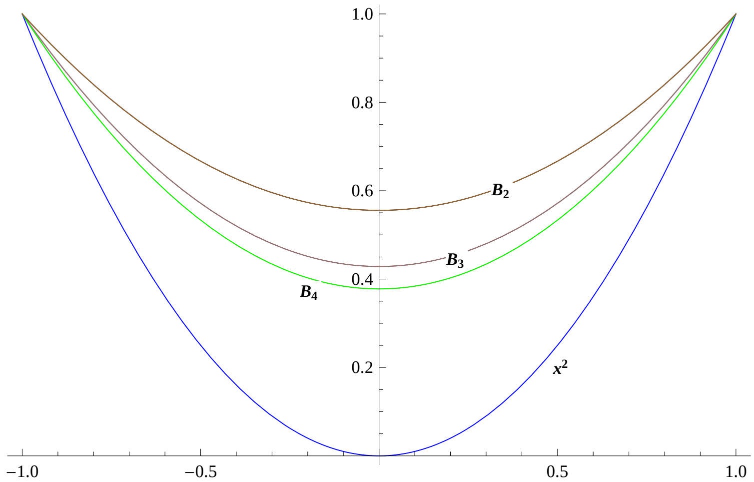

Figure 3.3 q-Bernstein polynomials of x^2 on $[-1,1]$ for $q=2$ and $n=2,3,4$.

3.2 q-de Casteljau type Algorithm and q-Differences

We now give two algorithms based on Phillips (1996) and Goldman (2003), for evaluating the q-Bernstein polynomials over arbitrary intervals. When $a=0$ and $b=1$ these algorithms reduce to de Casteljau type algorithm (1.17).

Algorithms 3.2.1 These algorithms start with the value of q and the values of f at the $n+1$ points $a+(b-a)\frac{[r]}{[n]}$, $0 \leqslant r \leqslant n$, and computes $B_n(f;[a,b];q,x) = f_0^{[n]}$, which is the final point generated by the algorithms. Notice that intermediate points are not affine combinations of the preceding points in the first algorithm, but this affinity holds in the second q-de Casteljau type algorithm. According to Simeonov, *et.al.* (2012), there are $n!$ number such algorithm whose final points are exactly the same $f_0^{[n]}$.

Algorithm 1:

$$\textbf{Given}: \quad q, f(a), f(a+\frac{(b-a)}{[n]}), \ldots, f(b)$$

$$\textbf{Compute}: \quad f_r^{[m]} = \frac{(bq^r - q^{m-1}x) f_r^{[m-1]} + (x-aq^r) f_{r+1}^{[m-1]}}{b-aq^{m-1}} \begin{cases} m = 1, \ldots, n \\ r = 0, \ldots, n-m. \end{cases} \tag{3.11}$$

Algorithm 2:

Given : $\quad q, f(a), f(a+\dfrac{(b-a)}{[n]}), \ldots, f(b)$

Compute : $\quad f_r^{[m]} = \left(\dfrac{b-q^{n-m-r}x}{b-aq^{n-m}}\right) f_r^{[m-1]} + \left(1-\dfrac{b-q^{n-m-r}x}{b-aq^{n-m}}\right) f_{r+1}^{[m-1]} \begin{cases} m=1,\ldots,n \\ r=0,\ldots,n-m. \end{cases}$

(3.12)

The following theorem justifies the Algorithm 1.

Theorem 3.2.1. *For $0 \leqslant m \leqslant n$ and $0 \leqslant r \leqslant n-m$, we have*

$$f_r^{[m]} = \sum_{s=0}^{m} f_{r+s} \begin{bmatrix} m \\ s \end{bmatrix} \frac{\prod_{j=0}^{s-1}(x-aq^{j+r}) \prod_{j=0}^{m-s-1}(bq^r - q^j x)}{\prod_{j=0}^{m-1}(b-aq^j)} \tag{3.13}$$

and , in particular

$$f_0^{[n]} = B_n(f;[a,b];q,x).$$

Proof. We use induction on m. The result holds for $m=0$ and $0 \leqslant r \leqslant n$, since $f_r^{[0]} = f(a+(b-a)\frac{[r]}{[n]})$. We assume (3.13) holds for some m such that $0 \leqslant m \leqslant n$, and for all r such that $0 \leqslant r \leqslant n-m$. Then for $0 \leqslant r \leqslant n-m-1$, it follows from the algorithm (3.11) that

$$f_r^{[m+1]} = \frac{(bq^r - q^m x) f_r^{[m]} + (x-aq^r) f_{r+1}^{[m]}}{b-aq^m}. \tag{3.14}$$

On using (3.13), we have

$$f_r^{[m+1]} = (bq^r - q^m x) \sum_{s=0}^{m} f_{r+s} \begin{bmatrix} m \\ s \end{bmatrix} \frac{\prod_{j=0}^{s-1}(x-aq^{j+r}) \prod_{j=0}^{m-s-1}(bq^r - q^j x)}{\prod_{j=0}^{m}(b-aq^j)}$$

$$+ (x-aq^r) \sum_{s=0}^{m} f_{r+s+1} \begin{bmatrix} m \\ s \end{bmatrix} \frac{\prod_{j=0}^{s-1}(x-aq^{j+r+1}) \prod_{j=0}^{m-s-1}(bq^{r+1} - q^j x)}{\prod_{j=0}^{m}(b-aq^j)}. \tag{3.15}$$

The coefficient of f_{r+m+1} on the right of the above equation is

$$\frac{(x-aq^r) \prod_{j=0}^{m-1}(x-aq^{j+r+1})}{\prod_{j=0}^{m}(b-aq^j)} = \frac{\prod_{j=0}^{m}(x-aq^{j+r})}{\prod_{j=0}^{m}(b-aq^j)}, \tag{3.16}$$

and the coefficient of f_r is

$$\frac{(bq^r - q^m x)\prod_{j=0}^{m-1}(bq^r - q^j x)}{\prod_{j=0}^{m}(b - aq^j)} = \frac{\prod_{j=0}^{m}(bq^r - q^j x)}{\prod_{j=0}^{m}(b - aq^j)}. \tag{3.17}$$

For $1 \leq s \leq m$, shifting the index in $f_r^{[m+1]}$ yields the coefficients of f_{r+s} as

$$(bq^r - q^m x)\sum_{s=1}^{m}\begin{bmatrix} m \\ s \end{bmatrix}\frac{\prod_{j=0}^{s-1}(x - aq^{j+r})\prod_{j=0}^{m-s-1}(bq^r - q^j x)}{\prod_{j=0}^{m}(b - aq^j)}$$

$$+(x - aq^r)\sum_{s=1}^{m}\begin{bmatrix} m \\ s-1 \end{bmatrix}\frac{\prod_{j=0}^{s-2}(x - aq^{j+r+1})\prod_{j=0}^{m-s}(bq^{r+1} - q^j x)}{\prod_{j=0}^{m}(b - aq^j)}. \tag{3.18}$$

We see that

$$\prod_{j=0}^{m-s}(bq^{r+1} - q^j x) = (bq^{r+1} - x)q^{m-s}\prod_{j=0}^{m-s-1}(bq^r - q^j x) \tag{3.19}$$

and

$$(x - aq^r)\prod_{j=0}^{s-2}(x - aq^{j+r+1}) = \prod_{j=0}^{s-1}(x - aq^{j+r}). \tag{3.20}$$

Substituting the last equations into (3.18) and using q-Pascal's identities (1.9), (1.10) simplifies the expression (3.18) to

$$\sum_{s=1}^{m}\frac{\begin{bmatrix} m+1 \\ s \end{bmatrix}\prod_{j=0}^{s-1}(x - aq^{j+r})\prod_{j=0}^{m-s}(bq^r - q^j x)}{\prod_{j=0}^{m}(b - aq^j)}. \tag{3.21}$$

Therefore, the coefficient of f_{r+s}, for $1 \leqslant s \leqslant m$, in (3.15) is

$$\frac{\begin{bmatrix} m+1 \\ s \end{bmatrix}\prod_{j=0}^{s-1}(x - aq^{j+r})\prod_{j=0}^{m-s}(bq^r - q^j x)}{\prod_{j=0}^{m}(b - aq^j)}, \tag{3.22}$$

and it also holds for $s = 0$ and $s = m+1$. Consequently, we obtain

$$f_r^{[m+1]} = \sum_{s=0}^{m+1} \frac{f_{r+s}\begin{bmatrix} m+1 \\ s \end{bmatrix} \prod_{j=0}^{s-1}(x-aq^{j+r}) \prod_{j=0}^{m-s}(bq^r - q^j x)}{\prod_{j=0}^{m}(b-aq^j)}, \tag{3.23}$$

which completes the proof by induction.

Similarly, in Algorithm (3.11), as we see in (3.13), each intermediate point $f_r^{[m]}$ has a form that resembles that of the final number $f_0^{[n]} = B_n(f;[a,b];q,x)$. We now show that each $f_r^{[m]}$ can also be expressed in terms of q-differences.

Theorem 3.2.2. *For $0 \leqslant m \leqslant n$ and $0 \leqslant r \leqslant n-m$, we have*

$$f_r^{[m]} = \sum_{t=0}^{m} q^{(m-t)r} \Delta_q^t f_r \begin{bmatrix} m \\ t \end{bmatrix} \prod_{s=0}^{t-1} \left(\frac{x-aq^{s+r}}{b-aq^s} \right). \tag{3.24}$$

Proof. We use induction on m. It is true for $m=0$ and $0 \leqslant r \leqslant n$, because $f_r^{[0]} = f(a+(b-a)\frac{[r]}{[n]})$. We assume (3.13) holds for some m such that $0 \leqslant m \leqslant n$, and for all r such that $0 \leqslant r \leqslant n-m$. Then for $0 \leqslant r \leqslant n-m-1$, it follows from the algorithm that

$$f_r^{[m+1]} = \frac{(bq^r - q^m x) f_r^{[m]} + (x-aq^r) f_{r+1}^{[m]}}{(b-aq^m)}. \tag{3.25}$$

On using (3.13), we obtain

$$\begin{aligned} f_r^{[m+1]} &= \frac{1}{(b-aq^m)} \left\{ (bq^r - q^m x) \sum_{t=0}^{m} q^{(m-t)r} \Delta_q^t f_r \begin{bmatrix} m \\ t \end{bmatrix} \prod_{s=0}^{t-1} \left(\frac{x-aq^{s+r}}{b-aq^s} \right) \right\} \\ &+ \frac{1}{(b-aq^m)} \left\{ (x-aq^r) \sum_{t=0}^{m} q^{(m-t)(r+1)} \begin{bmatrix} m \\ t \end{bmatrix} \Delta_q^t f_{r+1} \prod_{s=0}^{t-1} \left(\frac{x-aq^{s+r+1}}{b-aq^s} \right) \right\}. \end{aligned} \tag{3.26}$$

Then we use

$$\Delta_q^{t+1} f_r = \Delta_q^t f_{r+1} - q^t \Delta_q^t f_r, \tag{3.27}$$

in the last equation and rearrange the terms to get

$$f_r^{[m+1]} = \frac{1}{(b-aq^m)}\left\{\sum_{t=0}^{m}\left[bq^r - aq^{t+r+m})\right]q^{(m-t)r}\begin{bmatrix} m \\ t \end{bmatrix}\Delta_q^t f_r \prod_{s=0}^{t-1}\left(\frac{x-aq^{s+r}}{b-aq^s}\right)\right\}$$

$$+\frac{1}{(b-aq^m)}\left\{\sum_{t=0}^{m} q^{(m-t)r+(m-t)}\begin{bmatrix} m \\ t \end{bmatrix}\Delta_q^{t+1} f_r \frac{\prod_{s=0}^{t}(x-aq^{s+r})}{\prod_{s=0}^{t-1}(b-aq^s)}\right\}. \tag{3.28}$$

We may write this as

$$f_r^{[m+1]} = q^{(m+1)r}\Delta_q^0 f_r$$

$$+\frac{1}{(b-aq^m)}\left\{\sum_{t=1}^{m}(b-aq^{m+t})q^{(m-t+1)r}\begin{bmatrix} m \\ t \end{bmatrix}\Delta_q^t f_r \prod_{s=0}^{t-1}\left(\frac{x-aq^{s+r}}{b-aq^s}\right)\right\}$$

$$+\frac{1}{(b-aq^m)}\left\{\sum_{t=0}^{m-1} q^{(m-t)r+(m-t)}\begin{bmatrix} m \\ t \end{bmatrix}\Delta_q^{t+1} f_r \frac{\prod_{s=0}^{t}(x-aq^{s+r})}{\prod_{s=0}^{t-1}(b-aq^s)}\right\} \tag{3.29}$$

$$+\frac{1}{(b-aq^m)}\left\{\Delta_q^{m+1} f_r \frac{\prod_{s=0}^{m}(x-aq^{s+r})}{\prod_{s=0}^{m-1}(b-aq^s)}\right\}.$$

Shifting the index of the second summation and using q-Pascal identitities (1.9) and (1.10) we find the coefficient $\Delta_q^t f_r$ as

$$\begin{bmatrix} m+1 \\ t \end{bmatrix} q^{(m-t+1)r}\prod_{s=0}^{t-1}\left(\frac{x-aq^{s+r}}{b-aq^s}\right). \tag{3.30}$$

Therefore

$$f_r^{[m+1]} = \sum_{t=0}^{m+1} q^{(m-t+1)r}\begin{bmatrix} m+1 \\ t \end{bmatrix}\Delta_q^t f_r \prod_{s=0}^{t-1}\left(\frac{x-aq^{s+r}}{b-aq^s}\right) \tag{3.31}$$

and the induction is complete.

Consequently, for $r = 0$ in the above equation (3.24), we may express the generalized q-Bernstein polynomial defined by (3.1) in terms of q-differences.

Corollary 3.2.3.

$$B_n(f;[a,b];q,x) = \sum_{r=0}^{n} \Delta_q^r f_0 \begin{bmatrix} n \\ r \end{bmatrix} \prod_{s=0}^{r-1} \left(\frac{x - aq^s}{b - aq^s} \right). \tag{3.32}$$

Next we investigate the divided differences at $x_i = a + (b-a)\frac{[i]}{[n]}$ for $i = 0, 1, \ldots, n$, to express $B_n f$ interms of divided difference so as to find the eigenvalues of the operator $B_n f : C[a,b] \to C[a,b]$. First we have

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{\frac{(b-a)}{[n]} q^i} = \frac{\Delta_q f(x_i)}{\frac{(b-a)}{[n]} q^i}. \tag{3.33}$$

Then second-order divided difference becomes,

$$f[x_i, x_{i+1}, x_{i+2}] = \left(\frac{\Delta_q f(x_{i+1}) - q\Delta_q f(x_i)}{\left(\frac{b-a}{[n]}\right)^2 q^{2i+1}[2]} \right) \tag{3.34}$$

which is equivalent to

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{\Delta_q^2 f(x_i)}{\left(\frac{b-a}{[n]}\right)^2 q^{2i+1}[2]}. \tag{3.35}$$

We now state the following proposition similar to Phillips (2003), which shows the relation between q-differences and divided differences on $[a,b]$.

Proposition 3.2.4. *For all* $i, k \geq 0$, *we have*

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{\Delta_q^k f(x_i)}{\left(\frac{b-a}{[n]}\right)^k q^{k(2i+k-1)/2}[k]!}, \tag{3.36}$$

where $x_i = a + (b-a)\frac{[i]}{[n]}$.

Proof. The proof given for $x_i = [i]$ in Phillips (2003) may be easily adapted for the values $x_i = a + (b-a)\frac{[i]}{[n]}$.

What follows now using (3.2.4), we can rewrite the generalized q-Bersntein polynomials on an arbitrary interval in terms of divided difference.

Proposition 3.2.5.

$$B_n(f;[a,b];q,x) = \sum_{r=0}^{n}\left(1-\frac{1}{[n]}\right)\cdots\left(1-\frac{[r-1]}{[n]}\right)\frac{(b-a)^r}{\prod_{s=0}^{r-1}(b-aq^s)} \tag{3.37}$$

$$f[x_0,\ldots,x_r]\prod_{s=0}^{r-1}(x-aq^s).$$

Proof. We use (3.36) to rearrange. We get

$$\Delta_q^r f(x_0) = f[x_0,x_1,\ldots,x_r]\left(\frac{b-a}{[n]}\right)^r q^{r(r-1)/2}[r]!. \tag{3.38}$$

Substituiting this expression into (3.32), gives

$$B_n(f;[a,b];q,x) = \sum_{r=0}^{n}\begin{bmatrix} n \\ r \end{bmatrix}[r]!\left(\frac{b-a}{[n]}\right)^r q^{r(r-1)/2} \tag{3.39}$$

$$f[x_0,\ldots,x_r]\prod_{s=0}^{r-1}\left(\frac{x-aq^s}{b-aq^s}\right).$$

We have

$$\frac{[n]![r]!}{[n-r]![r]![n]^r} = \frac{[n][n-1]\ldots[n-r+1]}{[n]^r}$$

$$= \frac{1}{q}\left(1-\frac{1}{[n]}\right)\ldots\frac{1}{q^{r-1}}\left(1-\frac{[r-1]}{[n]}\right) \tag{3.40}$$

$$= \frac{1}{q^{r(r-1)/2}}\left\{\left(1-\frac{1}{[n]}\right)\ldots\left(1-\frac{[r-1]}{[n]}\right)\right\}, \quad r = 1,2,\ldots,n.$$

Substituiting the last expresion in (3.39) gives,

$$B_n(f;[a,b];q,x) = \sum_{r=0}^{n}\left(1-\frac{1}{[n]}\right)\cdots\left(1-\frac{[r-1]}{[n]}\right)\frac{(b-a)^r}{\prod_{s=0}^{r-1}(b-aq^s)} \tag{3.41}$$

$$f[x_0,x_1,\ldots,x_r]\prod_{s=0}^{r-1}(x-aq^s).$$

Because rth order divided differences annihilates all polynomials of degree less than r, it follows that $B_n(f;[a,b];q,x)$ is a polynomial of degree $\min(n,m)$ when $f(x) \in \mathbb{P}_m$. The values $\lambda_0 = 1,\ \lambda_1 = 1$ are eigenvalues corresponding to the eigenfunctions $f(x) = 1$, $f(x) = x$ respectively . Furthermore

$$\lambda_r = \left(1 - \frac{1}{[n]}\right)\cdots\left(1 - \frac{[r-1]}{[n]}\right)\frac{(b-a)^r}{\prod_{s=0}^{r-1}(b-aq^s)}, \quad r = 2,3,4,\ldots,n \tag{3.42}$$

are the eigenvalues of the operator B_nf . So far no explicit formula of eigenfunctions neither for q-Bernstein polynomials nor for classical ones is known.

3.3 Modulus of Continuity and Convergence

We will discuss the uniform convergence of generalized q-Bernstein operator on $[a,b]$. Since it is a *monotone linear* operator for $0 < q \leqslant 1$ and $0 \leqslant a < b$, we can employ the Bohman-Korovkin Theorem (see, Cheney (1984)) which states that for a *linear monotone* operator $\mathcal{L}_n$, the convergence of $\mathcal{L}_nf \to f$ for $f(x) = 1, x, x^2$ is sufficient for the operator $\mathcal{L}_n$ to have the uniform convergence property $\mathcal{L}_nf \to f$, for all $f \in C[a,b]$.

Firstly, we need to evaluate $B_n(f;[a,b];q,x)$ for $f = 1, t, t^2$. We can see from (3.1) and (2.9) that

$$\begin{aligned} B_n(1;[a,b];q,x) &= \sum_{r=0}^{n}\begin{bmatrix} n \\ r \end{bmatrix}\frac{\prod_{s=0}^{r-1}(x-aq^s)\prod_{s=0}^{n-r-1}(b-q^sx)}{\prod_{s=0}^{n-1}(b-aq^s)} \\ &= \sum_{r=0}^{n}B_r^n(x;[a,b];q) = 1. \end{aligned} \tag{3.43}$$

For $f(t) = t$, and $x \in [a,b]$, consider $\sum_{r=0}^{n} f_rB_r^n(x;[a,b];q)$ where $f_r = f(\frac{aq^r[n-r]+b[r]}{[n]})$, then

$$B_n(t;[a,b];q,x) = \sum_{r=0}^{n} \left(\frac{aq^r[n-r] + b[r]}{[n]} \right) B_r^n(x;[a,b];q)$$

$$= \sum_{r=0}^{n} aq^r \frac{[n-r]}{[n]} B_r^n(x;[a,b];q) + \sum_{r=0}^{n} b \frac{[r]}{[n]} B_r^n(x;[a,b];q). \tag{3.44}$$

From the identities

$$\frac{[n-r]}{[n]} \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r \end{bmatrix} \quad \text{and} \quad \frac{[r]}{[n]} \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}, \tag{3.45}$$

we may write (3.44) as

$$B_n(t;[a,b];q,x) = a \sum_{r=0}^{n-1} q^r \begin{bmatrix} n-1 \\ r \end{bmatrix} \frac{\prod_{j=0}^{r-1}(x-aq^j) \prod_{j=0}^{n-r-1}(b-q^j x)}{\prod_{j=0}^{n-1}(b-aq^j)}$$

$$+ b \sum_{r=1}^{n} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} \frac{\prod_{j=0}^{r-1}(x-aq^j) \prod_{j=0}^{n-r-1}(b-q^j x)}{\prod_{j=0}^{n-1}(b-aq^j)}. \tag{3.46}$$

Shifting the indices we get

$$B_n(t;[a,b];q,x) = a \sum_{r=0}^{n-1} \frac{q^r(b-xq^{n-r-1})}{(b-aq^{n-1})} B_r^{n-1}(x;[a,b];q)$$

$$+ b \sum_{r=0}^{n} (x-aq^r) \begin{bmatrix} n-1 \\ r \end{bmatrix} \frac{\prod_{j=0}^{r-1}(x-aq^j) \prod_{j=0}^{n-r-2}(b-q^j x)}{\prod_{j=0}^{n-1}(b-aq^j)}. \tag{3.47}$$

Rearranging the last expression gives

$$B_n(t;[a,b];q,x) = \sum_{r=0}^{n-1} \left(\frac{aq^r(b-xq^{n-r-1}) + b(x-aq^r)}{(b-aq^{n-1})} \right) B_r^{n-1}(x;[a,b];q)$$

$$= x \sum_{r=0}^{n-1} B_r^{n-1}(x;[a,b];q) = x. \tag{3.48}$$

A shorter way to establish the last identity is to use the difference form. Since $\Delta_q^0 f_0 = a$, $\Delta_q f_0 = \frac{b-a}{[n]}$ and $\Delta_q^r f_0 = 0$ for $r \geq 2$, we find that

$$B_n(t;[a,b];q,x) = x. \tag{3.49}$$

Finally, for $f(t) = t^2$, we compute

$$f_0 = a^2, \quad \text{and} \quad \Delta_q f_0 = \left(a + \frac{(b-a)}{[n]}\right)^2 - a^2. \tag{3.50}$$

Using (3.27), we have

$$\Delta_q^2 f_0 = \left(a + [2]\frac{(b-a)}{[n]}\right)^2 - [2]\left(a + \frac{(b-a)}{[n]}\right)^2 + a^2 q. \tag{3.51}$$

Thus, from (3.32)

$$\begin{aligned} B_n(t^2;[a,b];q,x) &= a^2 + \left(\frac{(b-a)}{[n]} + 2a\right)(x-a) + \left(\frac{[n]-1}{[n]}\right)\frac{(b-a)(x-a)(x-aq)}{(b-aq)} \\ &= \left(1 - \frac{1}{[n]}\right)\left(\frac{b-a}{b-aq}\right)x^2 + \frac{b+a}{b-aq}\left(\frac{(b-a)}{[n]} + a(1-q)\right)x \\ &\quad - \frac{ab}{b-aq}\left(\frac{(b-a)}{[n]} + a(1-q)\right). \end{aligned} \tag{3.52}$$

For a fixed value of q with $0 < q < 1$,

$$[n] \to \frac{1}{1-q} \quad as \quad n \to \infty. \tag{3.53}$$

Therefore $B_n(t^2;[a,b];q,x)$ does not converge to x^2, but

$$B_n(t^2;[a,b];q,x) \to \frac{(b-a)q}{(b-aq)}x^2 + \frac{b(b+a)(1-q)}{(b-aq)}x - \frac{ab^2(1-q)}{(b-aq)}, \text{ as } \quad n \to \infty. \tag{3.54}$$

Thus, this explains the limitations of Theorem (3.3.1). We now state a theorem, similar to given by Phillips (1997).

Theorem 3.3.1. *Let* (q_n) *be a sequence such that* $0 < q_n < 1$ *and* $q_n \to 1 \quad as \quad n \to \infty$. *Then,*

$$B_n(f;[a,b];q,x) \to f(x), \quad \forall f \in C[a,b] \quad and \quad 0 \leqslant a < b, \tag{3.55}$$

where $B_n(f;[a,b];q,x)$ *is defined by* (3.1) *with* $q = q_n$.

Proof. This becomes a consequence of the Bohman-Korovkin Theorem.
Next we will consider convergence of generalized q-Bernstein polynomials as $q \to 0$.
As $q \to 0$ we have

$$\begin{aligned}
f_0\begin{bmatrix} n \\ 0 \end{bmatrix}\frac{(b-x)\dots(b-xq^{n-1})}{(b-a)\dots(b-aq^{n-1})} &= f(a)\frac{(b-x)}{(b-a)}, \\
f_1\begin{bmatrix} n \\ 1 \end{bmatrix}\frac{(x-a)(b-x)\dots(b-xq^{n-2})}{(b-a)\dots(b-aq^{n-1})} &= f(b)\frac{(x-a)(b-x)b^{n-2}}{(b-a)b^{n-1}}, \\
&\vdots \\
f_{n-1}\begin{bmatrix} n \\ n-1 \end{bmatrix}\frac{(x-a)\dots(x-aq^{n-2})(b-x)}{(b-a)\dots(b-aq^{n-1})} &= f(b)\frac{(x-a)x^{n-2}(b-x)}{(b-a)b^{n-1}}, \\
f_n\begin{bmatrix} n \\ n \end{bmatrix}\frac{(x-a)\dots(x-aq^{n-1})}{(b-a)\dots(b-aq^{n-1})} &= f(b)\frac{(x-a)x^{n-1}}{(b-a)b^{n-1}}.
\end{aligned} \tag{3.56}$$

After some algebraic manipulations, we obtain that

$$\begin{aligned}
B_n(f;[a,b];0,x) &= f(a)\frac{(b-x)}{(b-a)} + f(b)\frac{(x-a)x^{n-1}}{(b-a)b^{n-1}} \\
&\quad + f(b)\left[\frac{(x-a)(b-x)}{(b-a)b^{n-1}}\left(\frac{(x+b)^n - b^n - x^n}{bx}\right)\right].
\end{aligned} \tag{3.57}$$

Phillips (1997) gave an upper bound for the error $f(x) - B_n(f;q,x)$ in terms of the modulus of continuity. We will prove a similar result for the generalized q-Bernstein polynomials $B_n(f;[a,b];q,x)$. Let us recall the modulus of continuity.

Definition 3.3.2. The modulus of continuity $\omega(\delta)$ of a function f on $[a,b]$ is defined by

$$\omega(\delta) = \sup_{\substack{|x-y|\leqslant\delta \\ x,y\in[a,b]}} |f(x) - f(y)|, \quad \delta \geq 0. \tag{3.58}$$

The modulus of continuity has the following properties:

(i) Monotonicity: if $0 < \delta_1 \leqslant \delta_2$, then $\omega(\delta_1) \leqslant \omega(\delta_2)$,

(ii) Subadditivity: $\omega(\delta_1 + \delta_2) \leqslant \omega(\delta_1) + \omega(\delta_2)$,

(iii) if $\lambda > 0$, then $\omega(\lambda\delta) \leqslant (1+\lambda)\omega(\delta)$,

(iv) f is uniformly continuous on $[a,b]$ if and only if $\lim_{\delta\to 0}\omega(\delta) = 0$.

The proofs can be found in Rivlin (1969).

Theorem 3.3.3. *If f is bounded on [a,b], then for* $0 < q \leqslant 1$ *and* $0 \leqslant a < b$,

$$\|f(x) - B_n(f;[a,b];q,x)\| \leqslant \left(1 + \frac{(b-a)}{2}\left\{\frac{b-a((1-q)[n]-1)}{(b-aq)}\right\}^{1/2}\right)\omega\left(1/[n]^{1/2}\right), \tag{3.59}$$

where $\|.\|$ *denotes the maximum norm.*

Proof. We adapt the result in Rivlin (1969) as follows. We have

$$\|f(t) - B_n(f;[a,b];q,t)\| = \|\sum_{r=0}^{n}(f(t) - f_r)B_r^n(t;[a,b],q)\|$$

$$\leqslant \sum_{r=0}^{n}\|f(t) - f_r\|B_r^n(t;[a,b],q) \tag{3.60}$$

$$\leqslant \sum_{r=0}^{n}\omega\left(|t-(a+(b-a)\frac{[r]}{[n]})|\right)B_r^n(t;[a,b],q)$$

From property of the modulus of continuity (iii) , we have

$$\omega\left(|t-(a+(b-a)\frac{[r]}{[n]})|\right)=\omega\left([n]^{1/2}|t-(a+(b-a)\frac{[r]}{[n]})|\frac{1}{[n]^{1/2}}\right)$$

$$\leqslant\left(1+[n]^{1/2}|t-(a+(b-a)\frac{[r]}{[n]})|\right)\omega\left(\frac{1}{[n]^{1/2}}\right), \tag{3.61}$$

so that

$$\|f(t)-B_n(f;[a,b];q,t)\|\leqslant\sum_{r=0}^{n}\left(1+[n]^{1/2}|t-(a+(b-a)\frac{[r]}{[n]})|\right)\omega\left(\frac{1}{[n]^{1/2}}\right)B_r^n(t;[a,b],q)$$

$$\leqslant\omega\left(\frac{1}{[n]^{1/2}}\right)\left[1+[n]^{1/2}\sum_{r=0}^{n}|t-(a+(b-a)\frac{[r]}{[n]})|B_r^n(t;[a,b],q)\right]. \tag{3.62}$$

Applying Cauchy-Schwartz inequality on the last summation gives

$$\sum_{r=0}^{n}\|t-(a+(b-a)\frac{[r]}{[n]})\|B_r^n(t;[a,b],q)\leqslant\left(\sum_{r=0}^{n}\left(t-(a+(b-a)\frac{[r]}{[n]})\right)^2B_r^n(t;[a,b],q)\right)^{1/2}$$

$$\leqslant\left(\sum_{r=0}^{n}\left(t^2-2t(a+(b-a)\frac{[r]}{[n]})+(a+(b-a)\frac{[r]}{[n]})^2\right)B_r^n(t;[a,b],q)\right)^{1/2}. \tag{3.63}$$

On using (3.48) and (3.52), we get

$$\left\{\left((1-\frac{1}{[n]})(\frac{b-a}{b-aq})-1\right)t^2+\frac{b+a}{b-aq}\left(\frac{(b-a)}{[n]}+a(1-q)\right)t-\frac{ab}{b-aq}\left(\frac{(b-a)}{[n]}+a(1-q)\right)\right\}^{1/2}$$

$$\leqslant\left\{\frac{(b-a)^2}{4(b-aq)}\left(\frac{b-a}{[n]}+a(1-q)\right)\right\}^{1/2} \tag{3.64}$$

and thus, from (3.62),

$$\|f(x)-B_n(f;[a,b];q,x)\|\leqslant\left(1+\frac{(b-a)}{2}\left\{\frac{b-a((1-q)[n]-1)}{(b-aq)}\right\}^{1/2}\right)\omega\left(1/[n]^{1/2}\right).$$

We now investigate the limits of $B_n(f;[a,b];q,x)$ as $n \to \infty$ for a fixed real q such that $0 < q < 1$. For each $r = 0,1,2,\ldots,$ we have

$$\lim_{n\to\infty}\left(a+(b-a)\frac{[r]}{[n]}\right) = a+(b-a)(1-q^r), \tag{3.65}$$

$$\begin{aligned}\lim_{n\to\infty} B_r^n([a,b];q,x) &= \frac{1}{(1-q)^r[r]!}\prod_{s=0}^{r-1}(x-aq^s)\prod_{s=0}^{\infty}\left(\frac{b-xq^s}{b-aq^s}\right)\\ &:= B_r^\infty([a,b];q,x).\end{aligned} \tag{3.66}$$

Notice that $B_r^\infty([a,b];q,x) \geq 0$, for all $x \in [a,b]$ and each $q \in (0,1)$, and $0 \leqslant a < b$. The function B_r^∞ is not a polynomial but a transcendantal function. Moreover it satisfies the partition of unity for $|\frac{a}{b}| < 1$, $|x| < b$ and $|q| < 1$. To show this property, we first give the following identity, which appears in the study of hypergeometric functions, is due to Cauchy, see Andrews (1976).

Theorem 3.3.4. *If* $|q| < 1$,$|t| < 1$*, then*

$$1+\sum_{n=1}^{\infty}\frac{(1-\alpha)(1-\alpha q)\ldots(1-\alpha q^{n-1})}{(1-q)\ldots(1-q^n)}t^n = \prod_{n=0}^{\infty}\left(\frac{1-\alpha t q^n}{1-tq^n}\right). \tag{3.67}$$

In q-calculus, it is also sensible to use the following *Pochhammer symbol*:

$$\begin{aligned}(\alpha;q)_n &= (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}),\\ (\alpha;q)_\infty &= \prod_{k=0}^{\infty}(1-\alpha q^k).\end{aligned} \tag{3.68}$$

We have

$$\begin{aligned}\sum_{r=0}^{\infty} B_r^\infty([a,b];q,x) &= \sum_{r=0}^{\infty}\frac{1}{(1-q^r)[r]!}\prod_{s=0}^{r-1}(x-aq^s)\prod_{s=0}^{\infty}\frac{(b-xq^s)}{(b-aq^s)}\\ &= \prod_{s=0}^{\infty}\frac{(b-xq^s)}{(b-aq^s)}\left[1+\sum_{r=1}^{\infty}\frac{(x-a)(x-aq)\ldots(x-aq^{r-1})}{(1-q^r)[r]!}\right].\end{aligned} \tag{3.69}$$

Since $|\frac{a}{b}| < 1$, $|x| < b$ and $|q| < 1$, using Cauchy identity (3.3.4) and *Pochhammer symbol* (3.68)

$$\sum_{r=0}^{\infty} B_r^{\infty}([a,b];q,x) = \prod_{s=0}^{\infty} \frac{(1-\frac{xq^s}{b})}{(1-\frac{aq^s}{b})}\left[1+\sum_{r=1}^{\infty}\frac{(x-a)(x-aq)\dots(x-aq^{r-1})}{(1-q)\dots(1-q^r)}\right]$$

$$= \frac{(\frac{x}{b};q)_\infty}{(\frac{a}{b};q)_\infty}\left[1+\sum_{r=1}^{\infty}\frac{(1-\frac{a}{x})(1-\frac{aq}{x})\dots(1-\frac{aq^{r-1}}{x})}{(1-q)\dots(1-q^r)}x^r\right]. \tag{3.70}$$

We consider the expression

$$1+\sum_{r=1}^{\infty}\frac{(1-\frac{a}{x})(1-\frac{aq}{x})\dots(1-\frac{aq^{r-1}}{x})}{(1-q)\dots(1-q^r)}x^r. \tag{3.71}$$

Replacing α by $\frac{a}{b}$, and t by $\frac{x}{b}$ in the Cauchy identity (3.67) gives

$$\begin{aligned} 1+\sum_{r=1}^{\infty}\frac{(1-\frac{a}{x})(1-\frac{aq}{x})\dots(1-\frac{aq^{r-1}}{x})(\frac{x}{b})^r}{(1-q)\dots(1-q^r)} &= \sum_{r=0}^{\infty}\frac{(\frac{a}{x};q)_r(\frac{x}{b})^r}{(q;q)_r} \\ &= \frac{(\frac{a}{b};q)_\infty}{(\frac{x}{b};q)_\infty}. \end{aligned} \tag{3.72}$$

Then, substituting the last expression in (3.70) yields

$$\sum_{r=0}^{\infty} B_r^{\infty}([a,b];q,x) = \frac{(\frac{x}{b};q)_\infty}{(\frac{a}{b};q)_\infty}\frac{(\frac{a}{b};q)_\infty}{(\frac{x}{b};q)_\infty} = 1. \tag{3.73}$$

In what follows, we define the limits of the q-Bernstein polynomial on $[a,b]$ as $n \to \infty$ by

$$B_\infty(f;[a,b];q,x) = \begin{cases} \sum_{r=0}^{\infty} f(a+(b-a)(1-q^r))B_r^{\infty}([a,b];q,x), & \text{if } x \in [a,b)\ , \\ \\ f(b), & \text{if } x = b. \end{cases} \tag{3.74}$$

The convergence of this new generalized q-Bernstein polynomial on $[a,b]$ is very different from classical Bernstein polynomial on $[a,b]$ but immitates that of q-Bernstein polynomial on $[0,1]$ under certain conditions.

Theorem 3.3.5. *Let* $0 < q < 1$ *be a fixed real number and* $0 \leqslant a < b$. *Then*

$$\lim_{n\to\infty} B_n(f;[a,b];q,x) = f(x), \quad \textit{for all } f \in C[a,b] \tag{3.75}$$

if and only if $f(x) = a_0 + a_1 x$, *where* a_0 *and* a_1 *are constants.*

Proof. We will adapt the proof given in Il'inskii & Ostrovska (2002).
Suppose that $f(t) = a_0 + a_1 t$. Since $B_n(f;[a,b];q,x)$ is linear, we have

$$B_n(a_0 + a_1 t;[a,b];q,x) = a_0 + a_1 t, \quad \forall q > 0 \quad \text{and} \quad n = 1,2,3,\dots. \tag{3.76}$$

and so

$$B_\infty(f;[a,b];q,x) = \lim_{n\to\infty} B_n(f;[a,b];q,x) = f(x). \tag{3.77}$$

Conversly assume that $B_\infty(f;[a,b];q,x) = f(x)$ for all $f \in C[a,b]$ and all x. Let us consider the function

$$g(x) = f(x) - (f(b) - f(a))\left(\frac{x-a}{b-a}\right). \tag{3.78}$$

It is easily seen that $g(a) = g(b)$ and $B_\infty(g;[a,b];q,x) = g(x)$. We will prove $g(a) = g(b) = g(x)$ for all $x \in [a,b]$. Let $M = \max_{x\in[a,b]} g(x)$ *and* $M > g(b)$. Then $M = g(z)$ for some $z \in (a,b)$ and $g(a + (1-q^r)(b-a)) < M$ for r sufficiently large. On using positivity of $B_r^\infty([a,b];q,x)$ and the fact that $\sum_{r=0}^\infty B_r^\infty([a,b];q,x) = 1$, we get

$$\begin{aligned} M = g(z) &= \sum_{r=0}^{\infty} g(a + (1-q^r)(b-a)) B_r^\infty([a,b];q,x) \\ &< \sum_{r=0}^{\infty} M B_r^\infty([a,b];q,x) = M. \end{aligned} \tag{3.79}$$

This contradiction implies that $g(x) \leqslant g(b), \forall x \in [a,b]$. Similarly, let $N = \min_{x \in [a,b]} g(x)$ and $N < g(b)$. Then $N = g(z)$ for some $z \in (a,b)$ and $g(a+(1-q^r)(b-a)) > N$ for r sufficiently large. So,

$$\begin{aligned} N = g(z) &= \sum_{r=0}^{\infty} g(a+(1-q^r)(b-a)) B_r^{\infty}([a,b];q,x) \\ &> \sum_{r=0}^{\infty} N B_r^{\infty}([a,b];q,x) = N. \end{aligned} \tag{3.80}$$

which shows $g(x) \geq g(b)$ for each $x \in [a,b]$. Hence $g(x) = c$ for some $c \in \mathbb{R}$. Consequently,

$$\begin{aligned} g(x) &= f(x) - (f(b)-f(a))\left(\frac{x-a}{b-a}\right), \\ f(x) &= g(x) + \frac{(f(b)-f(a))}{(b-a)}x - \frac{(f(b)-f(a))a}{(b-a)} \\ &= a_0 + a_1 x \end{aligned} \tag{3.81}$$

CHAPTER FOUR
CONCLUSION

After many studies of q-Bernstein polynomial on $[0,1]$ introduced by Phillips (1997), Simeonov, *et.al.* (2012) extended q-Bernstein basis polynomial over an arbitrary interval to generate q-Bézier curves using subdivision and blossoming techniques. Based on this recent work, we have defined generalized q-Bernstein polynomial $B_n(f;[a,b];q,x)$ on an arbitary interval. Setting $q = 1$ reduces to the classical Bernstein polynomial on $[a,b]$. The polynomial $B_n(f;[a,b];q,x)$ exhibits interesting properties. One of them is the convergence property. When (q_n) is a sequence such that $0 < q_n < 1$ and $q_n \to 1 \quad as \quad n \to \infty$, it is shown that $B_n(f;[a,b];q,x)$ uniformly converges to f for all $f \in C[a,b]$, with $q = q_n$. Then we investigate the convergence of the limit function $B_\infty(f;[a,b];q,x)$, that is

$$\lim_{n\to\infty} B_n(f;[a,b];q,x) = f(x), \quad \text{for all } f \in C[a,b] \tag{4.1}$$

if and only if $f(x)$ is linear function, where $0 < q < 1$ is a fixed real number and $0 \leqslant a < b$. We find the degree approximation by modulus of continuity for this polynomial. Moreover if f is symmetric on $[-a,a]$, then q-Bernstein polynomial satisfies

$$B_n(f;[-a,a];q,x) = B_n(f;[-a,a];1/q,-x).$$

Further convergence properties as well as geometric properties will be investigated in future work.

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