DOKUZ EYLÜL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

Q-EULERIAN POLYNOMIALS AND B-SPLINES

by

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İZMİR

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We have read the thesis entitled **"Q-EULERIAN POLYNOMIALS AND B-SPLINES"** completed by **ŞULE ULUTAŞ** under supervision of **ASSIST.PROF. DR. ÇETİN DİŞİBÜYÜK** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Q-EULERIAN POLYNOMIALS AND B-SPLINES

ABSTRACT

We investigate the relations between q-Eulerian polynomials and B-splines with knots both at q-integers and in geometric progression. We give q-analogue of exponential splines and use it to derive q-Euler-Frobenius polynomials. Using q-Euler-Frobenius polynomials the relation between q-Eulerian numbers and B-splines are derived for both knot sequences. It is shown that B-splines with knots at q-integers and B-splines with knots in geometric progression have same values on their knot points. We also construct q-analogues of Marsden's identity and these identities lead us to q-analogue of Worpitzky identity. Finally, we derive two identities for B-splines with knots in geometric progression which generate the symmetry property of B-splines with knots at integers.

Keywords: Eulerian numbers, Eulerian polynomials, Euler-Frobenius polynomials, B-splines, q-Eulerian numbers, q-Eulerian polynomials.

Q-EULERİAN POLİNOMLARI VE B-SPLİNE FONKSİYONLARI

ÖZ

Düğümleri q-tamsayılarda ve geometrik dizide olan B-spline fonksiyonları ile q-Eulerian polinomları arasındaki ilişkiyi inceledik. Üstel spline fonksiyonlarının q benzerini tanımladık ve bunu Euler-Frobenius polinomlarının q benzerini türetmek için kullandık. q-Eulerian sayıları ile B-spline fonksiyonları arasındaki bağıntıyı her iki düğüm dizisi için q-Euler-Frobenius polinomlarını kullanarak elde ettik. Düğümleri q-tamsayılarda olan B-spline fonksiyonu ve düğümleri geometrik dizide olan B-spline fonksiyonlarının düğüm noktalarında aynı değerleri aldığını gösterdik. Ayrıca Marsden özdeşliğinin q benzerlerini oluşturduk ve bu özdeşlikler Worpitzky özdeşliğinin q benzerini bulmamızı sağladı. Son olarak, düğümleri tamsayılarda olan B-spline fonksiyonlarının simetri özelliğini genelleyen düğümleri geometrik dizide olan B-spline fonksiyonları için iki tane özdeşlik türettik.

Anahtar sözcükler : Eulerian sayıları, Eulerian polinomları, Euler-Frobenius polinomları, B-spline fonksiyonları, q-Eulerian sayıları, q-Eulerian polinomları.

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INTRODUCTION

B-splines are first introduced in Schoenberg (1946) for equally spaced knots. In Cury & Schoenberg (1947) B-splines for arbitrary knots are defined by applying divided differences to truncated power functions. But calculating B-splines using truncated power functions is numerically unstable process. A stable and efficient way of evaluating B-splines described independently by de Boor (1972) and Cox (1972).

In He (2011) an relationship between Eulerian polynomials and B-splines is presented. It is also shown in He (2011) that there is a relation between Eulerian numbers and cardinal B-spline values at knot points.

There are q-analogues of Eulerian polynomials. These polynomials are constructed by using the joint distribution of MacMahon and Eulerian statistics. For more details see Shareshian & Wachs (2007).

Our aim is to find the relationship between q-Eulerian polynomials and B-splines. We proceed in the following fashion. In Chapter One we give a short review of Eulerian numbers, Eulerian polynomials, Euler-Frobenius polynomials and B-splines. Then we give relationship between Eulerian polynomials and cardinal B-splines. Using this relationship we also present the relation between Eulerian numbers and cardinal B-splines. Chapter Two deals with q-analogue of Eulerian numbers and Eulerian polynomials. We also derive a q-analogue of Worpitzky identity. In Chapter Three we examine two special types of B-splines, B-splines with knots at q-integers and B-splines with knots in geometric progression. The q-analogue of Worpitzky's identity is obtained using q-analogues of Marsden's identity in terms of both B-splines. We also introduce two different q-analogues of exponential spline. These q-exponential splines lead us to a q-analogue of Euler-Frobenius polynomials. Finally, using q-Euler-Frobenius polynomial we showed that the values B-splines with knots at q-integers and B-splines with knots in geometric progression are same on their knots and these values may be expressed in terms of q-Eulerian numbers.

1.1 Eulerian Numbers

Euler first introduced

$$\frac{1-x}{1-xe^{\lambda(1-x)}} = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} A(n,k)x^k \frac{\lambda^n}{n!} \tag{1.1}$$

in his famous book "Institutiones calculi differentialis" in 1755. The integers $A(n,k)$, $k = 1,2,...,n$, on the right hand side of the above equation are known as the Eulerian numbers. As Bernoulli numbers, Stirling numbers, Harmonic numbers and Binomial coefficients, Eulerian numbers used in some context of enumerative combinatorics. Another notation for Eulerian number is $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ which combinatorically gives the number of permutations of $\{1,2,...,n\}$ having k descents in (Graham, Knuth, & Patashrink, 1994, p. 267). A slightly different definition for Eulerian number $A(n,k)$ was given by (Comtet, 1974, p.241) who defines the number of permutations of length n with k rises is related to the number of descents such that k rises implying $k-1$ descents. The relation between them is $A(n,k) = \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle$. In the rest of context we will use $A(n,k)$.

Definition 1.1.1. $A(n,k)$ is the number of permutations $\pi_1\pi_2...\pi_n$ of $\{1,2,...,n\}$ that have $k-1$ descents, namely, $k-1$ places where $\pi_{j-1} > \pi_j$ for all j.

Definition 1.1.1 may be used to obtain a recurrence relation for Eulerian numbers. We give this recurrence relation in the same way given in (Graham, Knuth, & Patashrink, 1994, p. 268).

Inserting the new element n in each permutation $\tilde{\pi} = \pi_1...\pi_{n-1}$ of $\{1,...,n-1\}$ in all possible ways, we have n permutations of $\{1,...,n\}$. If we put n in position j, we obtain the permutation $\pi = \pi_1...\pi_{j-1}n\pi_j...\pi_{n-1}$. The number of descents in π is the same as the descent number in $\tilde{\pi}$ if $j = n$ or $\pi_{j-1} > \pi_j$; on the other hand the number of descents increase by 1 in π if $\pi_{j-1} < \pi_j$ or $j = 1$. Therefore π has $k-1$ descents

Table 1.1 First values of Eulerian Numbers

n	$A(n,1)$	$A(n,2)$	$A(n,3)$	$A(n,4)$	$A(n,5)$	$A(n,6)$	$A(n,7)$	$A(n,8)$
1	1	0						
2	1	1	0					
3	1	4	1	0				
4	1	11	11	1	0			
5	1	26	66	26	1	0		
6	1	57	302	302	57	1	0	
7	1	120	1191	2416	1191	120	1	0

in a total of $kA(n-1,k)$ ways from permutations $\tilde{\pi}$ that have $k-1$ descents, plus a total of $((n-2)-(k-2)+1)A(n-1,k-1)$ ways from permutations $\tilde{\pi}$ that have $k-2$ descents. Thus the recurrence is

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k) \qquad n > 0 \tag{1.2}$$

where $A(0,0) = 1$ and $A(0,k) = 0$ for $k \neq 0$. We will assume that $A(n,k) = 0$ if $k < 1$ and $A(n,k) = 0$ if $k \geq n+1$.

It is well known that Eulerian numbers satisfy the following symmetry property. See Charalambides (2002).

Proposition 1.1.2. *Given a positive integer n and* $1 \leq k \leq n$,

$$A(n,k) = A(n,n-k+1). \tag{1.3}$$

Proof. We will show (1.3) by using induction. Clearly (1.3) is true for $n = 1$. Now suppose that (1.3) is true for any $n \geq 1$ and $1 \leq k \leq n$. Replacing k by $n-k+2$ in (1.3), we obtain the relation

$$A(n,n-k+2) = A(n,k-1). \tag{1.4}$$

Using (1.2) and induction hypothesis with (1.3) and (1.4), we have

$$\begin{aligned} A(n+1,n-k+2) &= kA(n,n-k+1) + (n-k+2)A(n,n-k+2) \\ &= kA(n,k) + (n-k+2)A(n,k-1) \\ &= A(n+1,k) \end{aligned} \tag{1.5}$$

which shows that (1.3) is true for $n+1$.

Another elegant proof of (1.3) can be given by using the combinatorial definition of $A(n,k)$. That is, the permutation $\pi_1\pi_2...\pi_n$ has $n-k$ descents if and only if its reflection $\pi_n...\pi_2\pi_1$ has $k-1$ descents.

Proposition 1.1.3.

$$\sum_{k=1}^{n} A(n,k) = n! \tag{1.6}$$

Proof. We use induction on n. For $n=1$, it is trivial. Suppose that (1.6) is true for $n>1$. Then by recurrence relation

$$\sum_{k=1}^{n+1} A(n+1,k) = \sum_{k=1}^{n+1} \{(n-k+2)A(n,k-1)+kA(n,k)\}. \tag{1.7}$$

Since $A(n,0)=A(n,n+1)=0$, we have

$$\sum_{k=1}^{n+1} A(n+1,k) = \sum_{k=2}^{n+1}(n-k+2)A(n,k-1)+\sum_{k=1}^{n} kA(n,k). \tag{1.8}$$

Shifting the index of the first summation on the right gives

$$\begin{aligned}\sum_{k=1}^{n+1} A(n+1,k) &= (n+1)\sum_{k=1}^{n} A(n,k) \\ &= (n+1)!\end{aligned} \tag{1.9}$$

Similarly (1.6) can be proved using the combinatorial interpretation. Clearly, the summation of $A(n,k)$ for all $1 \leq k \leq n$ is the number of all the permutations of $\{1,2,...,n\}$.

In 1833 Worpitzky proved that the monomial x^n can be expressed in terms of Eulerian numbers.

Proposition 1.1.4.

$$x^n = \sum_{k=1}^{n} A(n,k)\binom{x+k-1}{n} \quad n \geq 1 \tag{1.10}$$

Proof. The proof is by induction on n. For $n = 1$, it is trivial. Now assume that for any $n > 1$, (1.10) is true and consider the expression $\sum_{k=1}^{n+1} A(n+1,k)\binom{x+k-1}{n+1}$. Since $A(n,0) = 0$ for $n \geq 1$, then by (1.2)

$$\begin{aligned}\sum_{k=1}^{n+1} A(n+1,k)\binom{x+k-1}{n+1} &= \sum_{k=2}^{n+1}(n-k+2)A(n,k-1)\binom{x+k-1}{n+1} \\ &\quad + \sum_{k=1}^{n+1} kA(n,k)\binom{x+k-1}{n+1}.\end{aligned} \tag{1.11}$$

Shifting the index of the first summation and rearraging the terms by using $A(n,k) = 0$ for $k \geq n+1$ gives

$$\begin{aligned}\sum_{k=1}^{n+1} A(n+1,k)\binom{x+k-1}{n+1} &= \sum_{k=1}^{n} A(n,k)\binom{x+k-1}{n} k\frac{x+k-n-1}{n+1} \\ &\quad + \sum_{k=1}^{n} A(n,k)\binom{x+k-1}{n}(n-k+1)\frac{x+k}{n+1}.\end{aligned} \tag{1.12}$$

Since

$$k\frac{x+k-n-1}{n+1} + (n-k+1)\frac{x+k}{n+1} = x \tag{1.13}$$

we have

$$\sum_{k=1}^{n+1} A(n+1,k)\binom{x+k-1}{n+1} = x\sum_{k=1}^{n} A(n,k)\binom{x+k-1}{n}. \tag{1.14}$$

Using inductive hypothesis gives

$$\sum_{k=1}^{n+1} A(n+1,k)\binom{x+k-1}{n+1} = x^{n+1}. \tag{1.15}$$

This completes the proof.

Euler showed that $A(n,k)$ can be calculated directly by the following explicit formula.

Proposition 1.1.5.

$$A(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n,\ 1 \leq k \leq n. \tag{1.16}$$

Proof. We will prove the identity (1.16) by using induction on $n \geq 1$. If $n = 1$ and $k = 1$, then (1.16) is satisfied since $A(1,1) = 1$. Now assume that (1.16) is true for

$n-1$ and substitute it into the recurrence relation (1.2)

$$\begin{aligned} A(n,k) &= k\sum_{j=0}^{k}(-1)^j\binom{n}{j}(k-j)^{n-1}+(n-k+1)\sum_{j=0}^{k-1}(-1)^j\binom{n}{j}(k-j-1)^{n-1} \\ &= k^n + k\sum_{j=1}^{k}(-1)^j\binom{n}{j}(k-j)^{n-1}+(n-k+1)\sum_{j=1}^{k}(-1)^{j-1}\binom{n}{j-1}(k-j)^{n-1} \\ &= k^n + \sum_{j=1}^{k}\left(k\binom{n}{j}-(n-k+1)\binom{n}{j-1}\right)(-1)^j(k-j)^{n-1} \\ &= k^n + \sum_{j=1}^{k}(-1)^j\binom{n+1}{j}(k-j)^n \\ &= \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n. \end{aligned} \tag{1.17}$$

Hence the formula (1.16) is true for n.

In He (2011), the generating function of Eulerian numbers, known as Eulerian polynomials $A_n(z)$, defined by

$$A_n(z)=\sum_{k=1}^{n}A(n,k)z^k, \qquad A_0(z)=1. \tag{1.18}$$

Definition 1.1.6. (He (2011)) Eulerian polynomial sequence $\{A_n(z)\}_{n\geq 0}$ is given by

$$\sum_{l\geq 0} l^n z^l = \frac{A_n(z)}{(1-z)^{n+1}}, \qquad |z|<1. \tag{1.19}$$

There is also a combinatorial interpretation of the Eulerian polynomials which is given by Garsia (1979)

$$A_n(z)=z\sum_{\pi\varepsilon\delta_n} z^{des(\pi)}, \qquad n>0 \tag{1.20}$$

where $des(\pi)$ is the number of descents in the permutation π and δ_n is the symmetric group on the set $\{1,2,...,n\}$ and $\pi=\pi_1\pi_2...\pi_n\in\delta_n$.

A formula for exponential generating function of Eulerian polynomials is given in (Stanley, 2011, p. 41) in the following way.

Proposition 1.1.7.

$$\sum_{n\geq 0} A_n(z)\frac{t^n}{n!}=\frac{(1-z)}{1-ze^{t(1-z)}}. \tag{1.21}$$

Proof. Multiply (1.19) by $t^n/n!$ and then take the summation we have

$$\begin{aligned}\sum_{n\geq 0}\frac{A_n(z)}{(1-z)^{n+1}}\frac{t^n}{n!} &= \sum_{n\geq 0}\sum_{l\geq 0} l^n z^l \frac{t^n}{n!}\\ &= \sum_{l\geq 0} z^l e^{lt}\\ &= \tfrac{1}{1-ze^t}.\end{aligned} \tag{1.22}$$

Multiplying the both sides of the last equation by $1-z$ and repalacing t by $t(1-z)$ we obtain (1.21).

Proposition 1.1.8. *(He (2011)) The Eulerian polynomials is computed by recurrence*

$$A_0(z) = 1, \qquad A_{n+1}(z) = z(1-z)A_n'(z) + z(n+1)A_n(z) \tag{1.23}$$

Proof. Multiplying (1.19) by $(1-z)^{n+1}$ we get

$$A_n(z) = (1-z)^{n+1}\sum_{l\geq 0} l^n z^l. \tag{1.24}$$

Taking the derivative of (1.24) with respect to z, we obtain

$$A_n'(z) = -(n+1)(1-z)^n\sum_{l\geq 0} l^n z^l + \frac{1}{z}(1-z)^{n+1}\sum_{l\geq 0} l^{n+1} z^l. \tag{1.25}$$

To compute the proof multiply both sides of the above identity by $z(1-z)$ and use (1.24).

Proposition 1.1.9.

$$A_0(z) = 1, \qquad A_n(z) = z\sum_{k=0}^{n}\binom{n}{k}(1-z)^{n-k}A_k(z). \tag{1.26}$$

Proof. Multiplying (1.21) by $1-ze^{t(1-z)}$ yields

$$\begin{aligned}1-z &= \sum_{n\geq 0} A_n(z)\frac{t^n}{n!} - z\sum_{k\geq 0}\frac{t^k(1-z)^k}{k!}\sum_{n\geq 0} A_n(z)\frac{t^n}{n!}\\ &= \sum_{n\geq 0} A_n(z)\frac{t^n}{n!} - \sum_{n\geq 0}\left(z\sum_{k=0}^{n}\binom{n}{k}(1-z)^{n-k}A_k(z)\right)\frac{t^n}{n!}\\ &= \sum_{n\geq 0}\left(A_n(z) - z\sum_{k=0}^{n}\binom{n}{k}(1-z)^{n-k}A_k(z)\right)\frac{t^n}{n!}.\end{aligned} \tag{1.27}$$

Now comparing the coefficients of $t^n/n!$ gives (1.26).

1.2 B-Splines

Definition 1.2.1. (Cury & Schoenberg (1947)) For any positive integer n, let

$$(t-x)_+^{n-1} = \begin{cases} (t-x)^{n-1}, & \text{if } -\infty < x \le t \\ 0, & \text{if } t > x. \end{cases} \tag{1.28}$$

Given any knot sequence $...x_{-2} < x_{-1} < x_0 < x_1 < x_2 < ....$, the B-spline sequence B_i^{n-1} of degree $n-1$ is defined by

$$B_i^{n-1}(x) := (x_{i+n} - x_i)[x_i, ..., x_{i+n}](t-x)_+^{n-1}. \tag{1.29}$$

The B-spline have the following properties. See (Phillips, 2003, chap.6, sec. 2) and (Schoenberg, 1993, chap.2, sec.1).

(1) B-spline is indeed a spline, i.e., a piecewise polynomial function of degree $n-1$ with the knots $x_i, x_{i+1}, ..., x_{i+n}$. In particular, when this knots are distinct, we have the explicit representation

$$B_i^{n-1}(x) = B_i^{n-1}(x; x_i, x_{i+1}, ..., x_{i+n}) = \sum_{j=i}^{i+n} \frac{(x_{i+n} - x_i)(x_j - x)_+^{n-1}}{\omega'(x_j)} \tag{1.30}$$

where $\omega = (x - x_i)...(x - x_{i+n})$. The above equation follows from the formula for divided difference. For more information about divided difference see (Phillips, 2003, chap.1, sec. 1). In view of the identity $x_+^{n-1} = x^{n-1} + (-1)^n(-x)_+^{n-1}$, we may write (1.30) as follows

$$B_i^{n-1}(x) = B_i^{n-1}(x; x_i, x_{i+1}, ..., x_{i+n}) = (-1)^n \sum_{j=i}^{i+n} \frac{(x_{i+n} - x_i)(x - x_j)_+^{n-1}}{\omega'(x_j)} \tag{1.31}$$

It is clear that $B_i^{n-1}(x) \in C^{n-2}$.

(2) B-splines have a finite support

$$suppB_i^{n-1}(x) = [x_i, x_{i+n}]. \tag{1.32}$$

Because, if $x \ge x_{i+n}$, then $(t-x)_+^{n-1} = 0$ at $t = x_i, ..., x_{i+n}$, hence $[x_i, ..., x_{i+n}](t-x)_+^{n-1} = 0$, and if $x \le x_i$, then $(t-x)_+^{n-1} = (t-x)^{n-1}$ at $t = x_i, ..., x_{i+n}$, hence $[x_i, ..., x_{i+n}](t-x)_+^{n-1} = 0$.

(3) If $f \in C^n(\mathbb{R})$, then taking the Taylor formula (see (Phillips, 2003, p. 147))

$$f(t) = p_{n-1}(t) + \frac{1}{(n-1)!}\int_{x_i}^{x_{i+n}} (t-x)_+^{n-1} f^{(n)}(x)dx \tag{1.33}$$

and applying the divided difference to both sides, we obtain

$$[x_i,...,x_{i+n}]f = \frac{1}{(n-1)!(x_{i+n}-x_i)}\int_{x_i}^{x_{i+n}} B_i^{n-1}(x;x_i,x_{i+1},...,x_{i+n})f^{(n)}(x)dx, \tag{1.34}$$

i.e., B-spline $B_i^{n-1}(x;x_i,x_{i+1},...,x_{i+n})$ is the Peano kernel in the integral representation of the functional $[x_i,...,x_{i+n}]$.

(4) The B-splines satisfy the following expression

$$\frac{1}{x_{i+n}-x_i}\int_{x_i}^{x_{i+n}} B_i^{n-1}(x;x_i,x_{i+1},...,x_{i+n})(x)dx = \frac{1}{n}. \tag{1.35}$$

The condition follows from (1.34) if we take $f(x) = x^n$.

(5) (**de Boor recurrence**) The following formula relates two B-splines of degree $n-2$ with the supports $[x_i,x_{i+n-1}]$ and $[x_{i+1},x_{i+n}]$ with that of degree $n-1$ with the support $[x_i,x_{i+n}]$.

$$B_i^{n-1}(x) = \left(\frac{x-x_i}{x_{i+n-1}-x_i}\right)B_i^{n-2}(x) + \left(\frac{x_{i+n}-x}{x_{i+n}-x_{i+1}}\right)B_{i+1}^{n-2}(x) \tag{1.36}$$

(6) B_i^0 is a step function, that is

$$B_i^0(x) := \begin{cases} 1, & x_i \le x < x_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.37}$$

Theorem 1.2.2. *The derivative of B-spline can be calculated for $n \ge 3$ as*

$$\frac{d}{dx}B_i^{n-1}(x) = \left(\frac{n-1}{x_{i+n-1}-x_i}\right)B_i^{n-2}(x) - \left(\frac{n-1}{x_{i+n}-x_{i+1}}\right)B_{i+1}^{n-2}(x) \tag{1.38}$$

for all real x. For $n = 2$, (1.38) holds for all x except at x_i, x_{i+1}, and x_{i+2} since the derivative of B_i^1 is not defined on these knots.

Proof. See (Phillips, 2003, p. 218).

Theorem 1.2.3. *(Marsden's identity) For any $n \ge 1$,*

$$(t-x)^{n-1} = \sum_{i=-\infty}^{\infty}(t-x_{i+1})...(t-x_{i+n-1})B_i^{n-1}(x). \tag{1.39}$$

If $n=1$, *then* $(t-x_{i+1})\dots(t-x_{i+n})$ *is taken to be* 1.

Proof. See (Phillips, 2003, p. 222).

Now, we consider special case of knot points, that is $x_i = i$. The corresponding B-splines are called cardinal B-splines. Since the knot sequence is equally spaced the above formulas turn into much simpler form. An important property of cardinal B-splines is thst they have translation property. That is

$$B_i^{n-1}(x) = B_0^{n-1}(x-i). \tag{1.40}$$

Here and in the sequel we will use the notation

$$B_n(x) := B_0^{n-1}(x). \tag{1.41}$$

B_n is completely determimined by $n+1$ knots $0,1,\dots,n$.

If we choose the knots as $0,1,\dots,n$, the recurrence relation (1.36) becomes

$$B_n(x) = \frac{x}{n-1}B_{n-1}(x) + \frac{n-x}{n-1}B_{n-1}(x-1), \tag{1.42}$$

and B-spline $B_1(x)$ of degree zero is given by

$$B_1(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{1.43}$$

The derivative formula for cardinal B-splines of degree $n-1$ follows from the derivative formula of genaral B-splines (1.38) and the translation property. For $n \ge 3$, we have

$$\frac{d}{dx}B_n(x) = B_{n-1}(x) - B_{n-1}(x-1) \tag{1.44}$$

for all real x. For $n=2$, (1.44) holds for all x, except at the knots $0,1$, and 2, since the derivative of B_2 is not defined on these knots.

We obtain the following proposition by using the expression in (Phillips, 2003, p. 229) and the translation property of B-splines.

Proposition 1.2.4. *Each cardinal B-spline B_n of degree $n-1$ is symetric about the centre of its interval of support in particular $[0,n]$, so that*

$$B_n(x) = B_n(n-x), \qquad -\infty < x < \infty. \tag{1.45}$$

Theorem 1.2.5. *$B_n(x)$ can be calculated by the following explicit formula*

$$B_n(x) = \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)_+^{n-1}. \tag{1.46}$$

Proof. When we take the knots as $0,1,...,n$, it is derived from (1.31).

$$\begin{aligned} B_n(x) &= (-1)^n \sum_{j=0}^{n} \frac{n(x-j)_+^{n-1}}{\omega'(j)} \\ &= (-1)^n \sum_{j=0}^{n} \frac{n(x-j)_+^{n-1}}{(j-0)(j-1)...(j-(j-1))(j-(j+1))...(j-n)} \\ &= (-1)^n \sum_{j=0}^{n} \frac{n(x-j)_+^{n-1}}{j!(-1)^{n-j}(n-j)!}. \end{aligned} \tag{1.47}$$

Then multiplying both numerator and denominator of the latter equation by $n!$ gives (1.46).

Corollary 1.2.6.

$$B_n(k) = \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (k-j)_+^{n-1} = \frac{1}{(n-1)!} \sum_{j=0}^{k} (-1)^j \binom{n}{j} (k-j)^{n-1} \tag{1.48}$$

for $1 \leq k \leq n-1$. Here $(k-j)_+^0 = 1$ when $k \geq j$ and 0 otherwise.

1.2.1 Euler-Frobenius Polynomials

Euler-Frobenius polynomials $\Pi_n(z)$ are introduced in 1749 in the paper "Remarques sur un beau rapport entre les series des puissances tant direct que reciproques" in the form

$$\sum_{l=0}^{n} (l+1)^n z^l = \frac{\Pi_n(z)}{(1-z)^{n+1}} \tag{1.49}$$

to calculate the Dirichlet $\eta-$ function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{1.50}$$

at $s=-1,-2,\ldots$. Altough the series (1.49) converges for $|z|<1$, Euler showed that

$$\eta(-n)=\Pi_n(-1)2^{-n-1} \tag{1.51}$$

for $n=0,1,2,\ldots$. The polynomial $\Pi_n(z)$ is the generating function of Eulerian numbers. After MacMahon (1915) had showed equidistribution of descent and excadence numbers, Riordan (1958) showed that

$$\Pi_n(z)=\sum_{\pi\in\delta_n} z^{des(\pi)}=\sum_{\pi\in\delta_n} z^{exc(\pi)} \tag{1.52}$$

Later the polynomials were studied by Frobenius to obtain their interrelationship with Bernoulli numbers. Euler-Frobenius polynomials are also related to Eulerian polynomials and their properties. In He (2011), a different approach is followed to construct Euler-Frobenius polynomials. He considered the space S_n of splines with knots at integers satisfying $f(x+1)=zf(x)$ for $z\neq 0,1$. The most general element of S_n satisfying this equation is stated in (Schoenberg, 1993, p. 17) which leads us to the concept of an exponential spline defined by

$$\phi_n(x;z)=\sum_{-\infty}^{\infty} z^j B_{n+1}(x-j), \qquad z\neq 0,1. \tag{1.53}$$

$\phi_n(x;z)$ is called the exponential spline of degree n to the base z.

If we differentiate (1.53) and use derivative formula of cardinal B-splines (1.44) we find that

$$\begin{aligned}\phi_n'(x;z) &= \sum_{-\infty}^{\infty} z^j B_{n+1}'(x-j)\\ &= \sum_{-\infty}^{\infty} z^j (B_n(x-j)-B_n(x-j-1))\\ &= \sum_{-\infty}^{\infty} z^j B_n(x-j)-\sum_{-\infty}^{\infty} z^{j-1}B_n(x-j)\end{aligned} \tag{1.54}$$

and thus

$$\phi_n'(x;z)=(1-z^{-1})\phi_{n-1}(x;z). \tag{1.55}$$

Repeating this process n times, we obtain

$$\phi_n^{(n)}(x;z)=(1-z^{-1})^n\phi_0(x;z)=(1-z^{-1})^n\sum_{-\infty}^{\infty} z^j B_1(x-j). \tag{1.56}$$

Since $B_1(x)=1$ for $0\leq x<1$, $B_1(x)=0$ otherwise, we find

$$\phi_n^{(n)}(x;z)=(1-z^{-1})^n, \qquad 0\leq x<1. \tag{1.57}$$

Its polynomial component in the interval $0 \le x < 1$ has the form

$$\phi(x;z) = \frac{1}{n!}(1-z^{-1})^n x^n + \textit{lower degree terms}. \tag{1.58}$$

Euler showed that the monic polynomial $n!(1-z^{-1})^{-n}\phi_n(x;z)$ is equivalent to exponential Euler polynomial $A_n(x;z)$ for $x \in [0,1]$. Hence

$$A_n(x,z) := n!(1-z^{-1})^{-n}\phi_n(x,z), \qquad 0 \le x \le 1, \ \ z \ne 0,1. \tag{1.59}$$

The generating function of $\{A_n(x,z)\}$ is (see (Schoenberg, 1993, p. 21))

$$\frac{z-1}{z-e^t}e^{xt} = \sum_{n\ge 0} A_n(x,z)\frac{t^n}{n!}. \tag{1.60}$$

Writing $x = 0$, (1.60) becomes

$$\frac{z-1}{z-e^t} = \sum_{n\ge 0} \beta_n(z)\frac{t^n}{n!} \tag{1.61}$$

where $\beta_n(z) = A_n(0,z)$. Substituting (1.61) into (1.60), we get

$$\begin{aligned} \sum_{n\ge 0} A_n(x;z)\frac{t^n}{n!} &= \tfrac{z-1}{z-e^t}e^{xt} \\ &= \sum_{j\ge 0} \beta_n(z)\frac{t^n}{n!}\sum_{j\ge 0}\frac{x^j t^j}{j!} \\ &= \sum_{n\ge 0}\left(\sum_{j=0}^{n}\binom{n}{j}\beta_j(z)x^{n-j}\right)\frac{t^n}{n!} \end{aligned} \tag{1.62}$$

and comparing the coefficients of $t^n/n!$ on both sides we obtain

$$A_n(x,z) = \sum_{j=0}^{n}\binom{n}{j}\beta_j(z)x^{n-j}. \tag{1.63}$$

Particularly,

$$A_n(1,z) = \sum_{j=0}^{n}\binom{n}{j}\beta_j(z). \tag{1.64}$$

Here $\beta_n(z)$ is called the Euler-Frobenius fraction. See He (2011).

Moreover, multiplying (1.61) by $z - e^t$ yields

$$\begin{aligned} z-1 &= (z-e^t)\sum_{n\ge 0}\beta_n(z)\frac{t^n}{n!} \\ &= z\sum_{n\ge 0}\beta_n(z)\frac{t^n}{n!} - e^t\sum_{n\ge 0}\beta_n(z)\frac{t^n}{n!} \\ &= z\sum_{n\ge 0}\beta_n(z)\frac{t^n}{n!} - \sum_{k\ge 0}\frac{t^k}{k!}\sum_{n\ge 0}\beta_n(z)\frac{t^n}{n!} \\ &= \sum_{n\ge 0}\left(z\beta_n(z) - z\sum_{k=0}^{n}\binom{n}{k}\beta_k(z)\right)\frac{t^n}{n!}. \end{aligned} \tag{1.65}$$

Now, comparing the coefficients of $t^n/n!$ gives

$$\beta_0(z) = 1, \qquad z\beta_n(z) = \sum_{k=0}^{n} \binom{n}{k} \beta_k(z). \tag{1.66}$$

Associating (1.64) with (1.66) yields

$$A_n(1,z) = zA_n(0,z). \tag{1.67}$$

Finally, $\Pi_n(z)$ is defined from a different viewpoint (see He (2011))

$$\Pi_n(z) := \beta_n(z)(z-1)^n \equiv A_n(0,z)(z-1)^n. \tag{1.68}$$

$\Pi_n(z)$ can be written by the following expression using the interval of support property of B-spline and Proposition 1.2.4

$$\Pi_n(z) = n! \sum_{-\infty}^{\infty} B_{n+1}(-j)z^{n+j} = n! \sum_{j=0}^{n-1} B_{n+1}(n-j)z^j = n! \sum_{j=0}^{n-1} B_{n+1}(j+1)z^j \tag{1.69}$$

since

$$A_n(0;z) = n!(1-z^{-1})^{-n}\phi(0;z) = n!(1-z^{-1})^{-n} \sum_{-\infty}^{\infty} z^j B_{n+1}(-j). \tag{1.70}$$

There is a relation between eulerian polynomials and euler-frobenius polynomials. The relation is given in He (2011) by the following theorem.

Theorem 1.2.7.

$$A_n(z) = \begin{cases} \Pi_n(z) = 1, & \textit{if } n = 0 \\ z\Pi_n(z), & \textit{if } n > 0. \end{cases} \tag{1.71}$$

Proof. See He (2011).

1.3 Relation Between Eulerian Numbers and B-Splines

The relation between eulerian numbers and the value of cardinal B-spline of degree n at $x = k$ is given in He (2011) by

$$A(n,k) = n!B_{n+1}(k) \qquad (n > 0). \tag{1.72}$$

As a corollary the identities of the Eulerian numbers which are previously proven such as (1.2), (1.3), and (1.10) can be proved by using the corresponding identities of the cardinal B-splines and the relation (1.72).

Recurrence for Eulerian numbers: From (1.36) we have

$$B_{n+1}(k) = \frac{k}{n}B_n(k) + \frac{n+1-k}{n}B_n(k-1) \tag{1.73}$$

multiplying the above eqation by $n!$ and using (1.72) we obtain

$$A(n,k) = kA(n-1,k) + (n+1-k)A(n-1,k-1). \tag{1.74}$$

Symetry to Eulerian numbers: From symmetry property of B-splines

$$B_{n+1}(k) = B_{n+1}(n+1-k). \tag{1.75}$$

The result immediately follows by multiplying the above equation by $n!$.

Worpitzky identity: (Wang, Xu, & Xu (2010)) From Marsden's identity

$$\begin{aligned}(t-x)^n &= \sum_{i=-\infty}^{\infty}(t-(i+1))...(t-(i+n))B_{n+1}(x-i)\\ &= \sum_{i=-\infty}^{\infty}(t-i-1))...(t-i-n))B_{n+1}(x-i)\\ &= \sum_{i=-\infty}^{\infty}B_{n+1}(x-i)\prod_{\gamma=1}^{n}(t-i-\gamma)\end{aligned} \tag{1.76}$$

replacing i by $-i$ yields

$$(t-x)^n = \sum_{i=-\infty}^{\infty}B_{n+1}(x+i)\prod_{\gamma=1}^{n}(t+i-\gamma). \tag{1.77}$$

Then write $x=0$

$$t^n = \sum_{i=1}^{n} n!B_{n+1}(i)\prod_{\gamma=1}^{n}\frac{(t+i-\gamma)}{n!}. \tag{1.78}$$

Using the relation (1.72),

$$t^n = \sum_{i=1}^{n}A(n,i)\frac{(t+i-1)(t+i-2)...(t+i-n)}{n!} \tag{1.79}$$

and hence

$$t^n = \sum_{i=1}^{n}A(n,i)\binom{t+i-1}{n}. \tag{1.80}$$

CHAPTER TWO
Q-EULERIAN NUMBERS AND Q-EULERIAN POLYNOMIALS

2.1 q-Calculus

In this section we will mention the basic concepts of the q-calculus briefly, which is used in our subsequent study. For more information on q-calculus see Kac & Cheung (2002).

For a given value of $q > 0$ and any number n we define the q-integer $[n]$ by

$$[n] = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \neq 1 \\ n, & \text{if } q = 1. \end{cases} \tag{2.1}$$

We next define the q-factorial $[n]!$ by

$$[n]! = \begin{cases} [n][n-1]...[1], & \text{if } n = 1,2,... \\ 1, & \text{if } n = 0. \end{cases} \tag{2.2}$$

We also need the concept of the q-binomial coefficients which is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \qquad 0 \leq k \leq n. \tag{2.3}$$

For $q = 1$, it reduces to the usual binomial coefficients.

The q-binomial coefficient satisfies the Pascal type identities,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{2.4}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{2.5}$$

The q-analogue of $(1+x)^n$ is the polynomial

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)...(1+q^{n-1}x), & \text{if } n = 1,2,... \\ 1, & \text{if } n = 0. \end{cases} \tag{2.6}$$

The q-derivative of the function $f(x)$ is

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x} \tag{2.7}$$

The q-derivative of the product $f(x)$ and $g(x)$ is

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x) \tag{2.8}$$

or

$$D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x) \tag{2.9}$$

2.2 q-Eulerian Numbers

q-Eulerian numbers $A_{n,k}$ are defined in Carlitz (1954) by

$$[x]^n = \sum_{k=1}^{n} A_{n,k} \begin{bmatrix} x+k-1 \\ n \end{bmatrix}. \tag{2.10}$$

In this paper, we examine an another construction of q-Eulerian numbers $A_q(n,k)$ defined in Carlitz (1975), which is different from $A_{n,k}$. However, there is a relation between them such that $A_q(n,k) = A_{n,n-k+1}$. The construction of recurrence relation for q-Eulerian numbers is stated in Carlitz (1975) as follows.

Let π denote a permutation of $\{1,2,...,n\}$ with k rises and therefore $n-k+1$ falls. We count a conventional rise on the extreme left and a conventional fall on the extreme right. We shall label both rises and falls by the positions of their left hand elements. Let the rises of π have the positions $i_0, i_1, ..., i_{k-1}$ and let the falls have the positions $j_1, j_2, ..., j_{n-k+1}$.
Put

$$i = i_0 + i_1 + ... + i_{k-1}, \qquad j = j_1 + j_2 + ... + j_{n-k+1}, \tag{2.11}$$

so that

$$i + j = \frac{1}{2}n(n+1) \tag{2.12}$$

For example, the permutation •3o1•2•4•5o6o has rises at positions $0,2,3,4$ and falls at $1,5,6$. Here •, o indicates rises and falls.

Let $\tilde{a}(n,k,i)$ denote the number of permutations π of $\{1,2,...,n\}$ with k rises and i as defined in (2.11). Consider the effect of inserting the element $n+1$ in π. If it is inserted in the rise of position t, the number of rises remains unchanged but i

becomes $i' = i+k-t-1$. If it is inserted in the fall of position t, then the number of rises becomes $k+1$ but the number of falls remains unchanged. Moreover, j becomes $j' = j+n-k-t+2 = \frac{1}{2}(n+1)(n+2)-i-k-t+1$, by (2.12). Hence i becomes

$$i' = \frac{1}{2}(n+1)(n+2) - j' = i+k+t-1 \tag{2.13}$$

It follows that

$$\begin{aligned} \tilde{a}(n+1,k,i) &= \sum_{t=0}^{k-1} \tilde{a}(n,k,i-k+t+1) + \sum_{t=1}^{n-k+2} \tilde{a}(n,k-1,i-k-t+2) \\ &= \sum_{t=0}^{k-1} \tilde{a}(n,k,i-t) + \sum_{t=0}^{n-k+1} \tilde{a}(n,k-1,i-n+t). \end{aligned} \tag{2.14}$$

Introduce

$$A_q(n,k) = \sum_i \tilde{a}(n,k,i)q^i. \tag{2.15}$$

It follows from (2.14) that

$$\begin{aligned} A_q(n+1,k) &= \sum_i q^i \sum_{t=0}^{k-1} \tilde{a}(n,k,i-t) + \sum_i q^i \sum_{t=0}^{n-k+1} \tilde{a}(n,k-1,i-n+t) \\ &= \sum_i \tilde{a}(n,k,i)q^i \sum_{t=0}^{k-1} q^t + \sum_i \tilde{a}(n,k-1,i)q^i \sum_{t=0}^{n-k+1} q^{n-t}. \end{aligned} \tag{2.16}$$

Replacing n by $n-1$ yields

$$A_q(n,k) = [k]A_q(n-1,k) + q^{k-1}[n-k+1]A_q(n-1,k-1) \tag{2.17}$$

where $A_q(0,0) = 1$ and $A_q(0,k) = 0$ for $k \neq 0$. We will asume that $A_q(n,k) = 0$ if $k < 1$ and $A_q(n,k) = 0$ if $k \geq n+1$. The following table gives few values of q-Eulerian numbers.

Table 2.1 First values of q-Eulerian numbers

$A_q(n,k)$	1	2	3
1	1		
2	1	q	
3	1	$2q+2q^2$	q^3
4	1	$3q^3+5q^2+3q$	$3q^5+5q^4+3q^3$
5	1	$4q^4+9q^3+9q^2+4q$	$6q^7+16q^6+22q^5+16q^4+6q^3$

Proposition 2.2.1.

$$\sum_{k=1}^{n} A_q(n,k) = [n]! \tag{2.18}$$

Proof. It can be proved by induction on n. For $n = 1$, it is trivial. We supoose that (2.18) is true for $n > 1$. Then by recurrence relation

$$\sum_{k=1}^{n+1} A_q(n+1,k) = \sum_{k=1}^{n+1} \{[k]A_q(n,k) + q^{k-1}[n-k+2]A_q(n,k-1)\} \tag{2.19}$$

since $A_q(n,0) = 0$ when $n \geq 1$ and $A_q(n,n+1) = 0$

$$\sum_{k=1}^{n+1} A_q(n+1,k) = \sum_{k=1}^{n} [k]A_q(n,k) + \sum_{k=2}^{n} q^{k-1}[n-k+2]A_q(n,k-1). \tag{2.20}$$

Shifting the index of the second summation and rearranging the terms we obtain

$$\begin{aligned} \sum_{k=1}^{n+1} A_q(n+1,k) &= \sum_{k=1}^{n} \Big([k] + q^k[n-k+1]\Big) A_q(n,k) \\ &= [n+1] \sum_{k=1}^{n} A_q(n,k). \end{aligned} \tag{2.21}$$

Using the inductive hypothesis we see that (2.18) is true for $n+1$, which completes the proof.

Proposition 2.2.2. *q-Eulerian numbers satisfy the following symetry relations*

$$A_q(n,k) = q^{-n(n-2k+1)/2} A_q(n,n-k+1) \tag{2.22}$$

and

$$A_q(n,k) = q^{n(k-1)} A_{1/q}(n,k) \tag{2.23}$$

Proof. We prove both of them by induction.

Proof of (2.22)**:** For $n = 1$ and $k = 0$, $A_q(1,0) = 0 = q^{-1}A_q(1,2)$ is true by definition of $A_q(n,k)$. For $n = 1$ and $k = 1$, clearly $q^{-1(1-2+1)/2}A_q(1,1-1+1) = q^0 A_q(1,1) = A_q(1,1)$. Assume that it is true for any $n > 1$. By (2.17)

$$\begin{aligned} A_q(n+1,k) &= [k]A_q(n,k) + q^{k-1}[n-k+2]A_q(n,k-1) \\ &= [k]q^{-n(n-2k+1)/2}A_q(n,n-k+1) \\ &\quad + q^{k-1}[n-k+2]q^{-n(n-2k+3)/2}A_q(n,n-k+2) \\ &= q^{-(n+1)(n-2k+2)/2}[k]q^{n-k+1}A_q(n,n-k+1) \\ &\quad + q^{-(n+1)(n-2k+2)/2}[n-k+2]A_q(n,n-k+2). \end{aligned} \tag{2.24}$$

Using (2.17) we get

$$A_q(n+1,k) = q^{-(n+1)(n-2k+2)/2} A_q(n+1,n-k+2). \tag{2.25}$$

Proof of (2.23)**:** For $n=1$ and $k=0$, $A_q(1,0)=0=q^{-1}A_{1/q}(1,0)$. For $n=1$ and $k=1$, $A_q(1,1)=1=A_{1/q}(1,1)$. Assume that (2.23) is true for any $n>1$. Then

$$\begin{aligned} A_q(n+1,k) &= [k]A_q(n,k)+q^{k-1}[n-k+2]A_q(n,k-1) \\ &= [k]q^{n(k-1)}A_{1/q}(n,k)+q^{k-1}[n-k+2]q^{n(k-2)}A_{1/q}(n,k-1) \\ &= q^{(n+1)(k-1)}\Big\{[k]q^{1-k}A_{1/q}(n,k)+q^{-n}[n-k+2]A_{1/q}(n,k-1)\Big\}. \end{aligned} \tag{2.26}$$

Since

$$A_{1/q}(n,k) = [k]q^{1-k}A_{1/q}(n-1,k)+[n-k+1]q^{1-n}A_{1/q}(n-1,k-1) \tag{2.27}$$

we have

$$A_q(n+1,k) = q^{(n+1)(k-1)}A_{1/q}(n+1,k). \tag{2.28}$$

$A_q(n,k)$ can be calculated directly by the following explicit formula.

Corollary 2.2.3.

$$A_q(n,k) = \sum_{j=0}^{k}(-1)^j q^{j(j-1)/2}\begin{bmatrix} n+1 \\ j \end{bmatrix}[k-j]^n\,, \quad 1\le k\le n \tag{2.29}$$

Proof. It can be shown by induction on $n\ge 1$. For $n=1$, it is trivial by Table (2.1). We assume that (2.29) is true for $n-1$. Substituting (2.29) into the recurrence relation (2.17) we obtain

$$\begin{aligned} A_q(n,k) &= [k]\sum_{j=0}^{k}(-1)^j q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix}[k-j]^{n-1} \\ &\quad +q^{k-1}[n-k+1]\sum_{j=0}^{k-1}(-1)^j q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix}[k-1-j]^{n-1}. \end{aligned} \tag{2.30}$$

Shifting the index of the second summation and rearranging

$$\begin{aligned}
A_q(n,k) &= [k]^n + [k]\sum_{j=1}^{k}(-1)^j q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix}[k-j]^{n-1} \\
&\quad + q^{k-1}[n-k+1]\sum_{j=1}^{k}(-1)^{j-1}q^{(j-1)(j-2)/2}\begin{bmatrix} n \\ j-1 \end{bmatrix}[k-j]^{n-1} \\
&= [k]^n \\
&\quad + \sum_{j=1}^{k}\left\{[k]\begin{bmatrix} n \\ j \end{bmatrix} - q^{k-j}[n-k+1]\begin{bmatrix} n \\ j-1 \end{bmatrix}\right\}(-1)^j q^{j(j-1)/2}[k-j]^{n-1}.
\end{aligned} \tag{2.31}$$

Using the identities $q^{k-j}[n-k+1] = [n-j+1]-[k-j]$ and $[k] = q^j[k-j]+[j]$ we obtain

$$\begin{aligned}
A_q(n,k) &= [k]^n + \sum_{j=1}^{k}\left\{q^j[k-j]+[j]\right\}\begin{bmatrix} n \\ j \end{bmatrix}(-1)^j q^{j(j-1)/2}[k-j]^{n-1} \\
&\quad - \sum_{j=1}^{k}\left\{[n-j+1]-[k-j]\right\}\begin{bmatrix} n \\ j-1 \end{bmatrix}(-1)^j q^{j(j-1)/2}[k-j]^{n-1} \\
&= [k]^n \\
&\quad + \sum_{j=1}^{k}\left\{\left\{\begin{bmatrix} n \\ j-1 \end{bmatrix} + q^j\begin{bmatrix} n \\ j \end{bmatrix}\right\}[k-j] + [j]\begin{bmatrix} n \\ j \end{bmatrix}\right\}(-1)^j q^{j(j-1)/2}[k-j]^{n-1} \\
&\quad - \sum_{j=1}^{k}[n-j+1]\begin{bmatrix} n \\ j-1 \end{bmatrix}(-1)^j q^{j(j-1)/2}[k-j]^{n-1}.
\end{aligned} \tag{2.32}$$

Finally, using the Pascal-type relation and rearranging the terms, we get

$$\begin{aligned}
A_q(n,k) &= [k]^n + \sum_{j=1}^{k}(-1)^j q^{j(j-1)/2}\begin{bmatrix} n+1 \\ j \end{bmatrix}[k-j]^n \\
&= \sum_{j=0}^{k}(-1)^j q^{j(j-1)/2}\begin{bmatrix} n+1 \\ j \end{bmatrix}[k-j]^n
\end{aligned} \tag{2.33}$$

Next we derive a q-analogue of Worpitzky identity.

Theorem 2.2.4.

$$\sum_{k=1}^{n} A_q(n,k)q^{(-2kn+n(n+1))/2}\begin{bmatrix} x+k-1 \\ n \end{bmatrix} = [x]^n. \tag{2.34}$$

Proof. We prove (2.34) by induction on n. For $n=1$, it is trivial. Now assuming the truth of (2.34) for any $n>1$, set $A = \sum_{k=1}^{n+1} A_q(n+1,k)q^{(n+1)(n-2k+2)/2}\begin{bmatrix} x+k-1 \\ n+1 \end{bmatrix}$. Using the recurrence relation (2.17) and using $A_q(n,0)=0$, we get

$$\begin{aligned} A &= \sum_{k=1}^{n+1}[k]A_q(n,k)q^{(n+1)(n-2k+2)/2}\begin{bmatrix} x+k-1 \\ n+1 \end{bmatrix} \\ &\quad + \sum_{k=2}^{n+1}q^{k-1}[n-k+2]A_q(n,k-1)q^{(n+1)(n-2k+2)/2}\begin{bmatrix} x+k-1 \\ n+1 \end{bmatrix}. \end{aligned} \tag{2.35}$$

Shifting the index of the second summation and rearranging

$$\begin{aligned} A &= \sum_{k=1}^{n}A_q(n,k)\begin{bmatrix} x+k-1 \\ n \end{bmatrix}q^{(n+1)(n-2k+2)/2}[k]\frac{[x+k-n-1]}{[n+1]} \\ &\quad + \sum_{k=1}^{n}A_q(n,k)\begin{bmatrix} x+k-1 \\ n \end{bmatrix}q^{(-2kn+n(n+1))/2}[n-k+1]\frac{[x+k]}{[n+1]} \\ &= \sum_{k=1}^{n}A_q(n,k)\begin{bmatrix} x+k-1 \\ n \end{bmatrix}q^{(-2kn+n(n+1))/2}q^{n-k+1}\frac{[x+k-n-1][k]}{[n+1]} \\ &\quad + \sum_{k=1}^{n}A_q(n,k)\begin{bmatrix} x+k-1 \\ n \end{bmatrix}q^{(-2kn+n(n+1))/2}\frac{[n-k+1][x+k]}{[n+1]}. \end{aligned} \tag{2.36}$$

Since

$$q^{n-k+1}\frac{[x+k-n-1][k]}{[n+1]}+\frac{[n-k+1][x+k]}{[n+1]}=[x] \tag{2.37}$$

we have

$$A=[x]\sum_{k=1}^{n}A_q(n,k)q^{(-2kn+n(n+1))/2}\begin{bmatrix} x+k-1 \\ n \end{bmatrix}. \tag{2.38}$$

Using the inductive hypothesis we obtain (2.34) for $n+1$, which completes the proof.

2.3 q-Eulerian Polynomials

Since $\lim_{q\to 1}(1-z)_q^{n+1}=(1-z)^{n+1}$ and $\lim_{q\to 1}[l]=l$, Definition 1.1.6 suggests that a new sequence of polynomials exist.

Definition 2.3.1. q-Eulerian polynomial sequence $\{A_n(z,q)\}_{n\geq 0}$ is given by setting

$$\frac{A_{n,q}(z)}{\prod_{i=0}^{n}(1-zq^i)}=\sum_{l\geq 0}[l]^n z^l \tag{2.39}$$

Each q-Eulerian polynomials can be presented as a generating function of q-Eulerian numbers.

$$A_{n,q}(z)=\sum_{k=1}^{n}A_q(n,k)z^k, \qquad A_{0,q}(z)=1. \tag{2.40}$$

There is also a combinatorial interpretation of the q-Eulerian polynomials. It is defined in Foata (2010) by means of fundamental permutations statistics; descent numbers and inversion numbers.

$$A_{n,q}(z) = z \sum_{\pi \varepsilon \delta_n} z^{des(\pi)} q^{inv(\pi)}, \qquad n \geq 1 \tag{2.41}$$

where $inv(\pi) := \#\{(\pi_i, \pi_j) : 1 \leq i < j \leq n, \pi_i > \pi_j\}$ and δ_n is the symetric group on the set $\{1, 2, ..., n\}$ and $\pi = \pi_1 \pi_2 ... \pi_n \in \delta_n$.

The corresponding exponential generating function is stated in Foata (2010) as

$$\sum_{n \geq 0} A_{n,q}(z) \frac{t^n}{[n]!} = \frac{1-z}{1 - z e_q^{t(1-z)}} \tag{2.42}$$

where $e_q^z = \sum_{j=0}^{\infty} \frac{z^j}{[j]!}$.

Using q-calculus, we find the following q-analogue of the equation (1.23).

Theorem 2.3.2. *The q-Eulerian polynomials can be computed by recurrence*

$$A_{n+1,q}(z) = z[n+1] A_{n,q}(z) + z(1-z) D_q A_{n,q}(z) \tag{2.43}$$

Proof. Multiplying (2.3.1) by $\prod_{i=0}^{n} (1 - zq^i)$ we get

$$A_{n,q}(z) = \prod_{i=0}^{n} (1 - zq^i) \sum_{l \geq 0} [l]^n z^l. \tag{2.44}$$

Taking the q-derivative of (2.44) with respect to z,

$$D_q A_{n,q}(z) = \prod_{i=0}^{n} (1 - zq^{i+1}).D_q \left\{ \sum_{l \geq 0} [l]^n z^l \right\} + \sum_{l \geq 0} [l]^n z^l.D_q \left\{ \prod_{i=0}^{n} (1 - zq^i) \right\}. \tag{2.45}$$

From the definition of the q-derivative, we obtain

$$D_q A_{n,q}(z) = \frac{1}{z} \prod_{i=1}^{n+1} (1 - zq^i) \sum_{l \geq 0} [l]^{n+1} z^l - \frac{[n+1]}{(1-z)} \prod_{i=0}^{n} (1 - zq^i) \sum_{l \geq 0} [l]^n z^l. \tag{2.46}$$

Multiply both sides of the above equation by $z(1-z)$ gives

$$z(1-z) D_q A_{n,q}(z) = \prod_{i=0}^{n+1} (1 - zq^i) \sum_{l \geq 0} [l]^{n+1} z^l - z[n+1] \prod_{i=0}^{n} (1 - zq^i) \sum_{l \geq 0} [l]^n z^l. \tag{2.47}$$

Using (2.44), we obtain (2.43).

Theorem 2.3.3.

$$A_{0,q}(z) = 1, \qquad A_{n,q}(z) = z\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (1-z)^{n-k} A_{k,q}(z) \tag{2.48}$$

Proof. Multiplying (2.42) by $1 - ze_q^{t(1-z)}$ yields

$$\begin{aligned} 1-z &= (1-ze_q^{t(1-z)}) \sum_{n\geq 0} A_n(z,q) \frac{t^n}{[n]!} \\ &= \sum_{n\geq 0} A_n(z,q) \frac{t^n}{[n]!} - ze_q^{t(1-z)} \sum_{n\geq 0} A_n(z,q) \frac{t^n}{[n]!} \\ &= \sum_{n\geq 0} A_n(z,q) \frac{t^n}{[n]!} - z\sum_{k\geq 0} \frac{t^k(1-z)^k}{[k]!} \sum_{n\geq 0} A_n(z,q) \frac{t^n}{[n]!} \\ &= \sum_{n\geq 0} \left(A_{n,q}(z) - z\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (1-z)^{n-k} A_{k,q}(z) \right) \frac{t^n}{[n]!}. \end{aligned} \tag{2.49}$$

Now, comparing the coefficients of $t^n/[n]!$ gives (2.48).

CHAPTER THREE

B-SPLINES

In this chapter we examine the B-splines with both at q-integers and in geometric progression.

3.1 B-splines with Knots at q-integers

The B-splines of degree $n-1$ with knots at q-integers are not translates of one another as in cardinal splines but there is a relation between them. This relation is

$$B_i^{n-1}(x) = B_n\left(\frac{x-[i]}{q^i}\right) \tag{3.1}$$

B_n is completely determined by the $n+1$ knots $[0],[1],...,[n]$.

With knots at $[0],...,[n]$, the recurrence relation (1.36) becomes

$$B_n(x) = \frac{x}{[n-1]}B_{n-1}(x) + \frac{[n]-x}{q[n-1]}B_{n-1}\left(\frac{x-1}{q}\right), \tag{3.2}$$

and the B-spline B_1 is given by

$$B_1(x) = \begin{cases} 1, & [0] \leq x < [1], \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Substituting $x=[k]$ in the equation (3.2) and using $[n]-[k]=q^k[n-k]$, we see that the B-splines with knots at q-integers satisfy

$$B_n([k]) = \frac{[k]}{[n-1]}B_{n-1}([k]) + q^{k-1}\frac{[n-k]}{[n-1]}B_{n-1}([k-1]). \tag{3.4}$$

Using equation (1.38) together with (3.1), for $n \geq 3$ we have

$$\frac{d}{dx}B_n(x) = \frac{n-1}{[n-1]}B_{n-1}(x) - \frac{n-1}{q[n-1]}B_{n-1}\left(\frac{x-1}{q}\right) \tag{3.5}$$

for all real x. Similar to cardinal B-splines, when $n=2$, (3.5) holds for all x except at the three knots $[0],[1]$, and $[2]$, since the derivative of B_2 is not defined on these knots.

The following theorem gives a q-analogue of Marsden's identity in terms of B-splines with knots at q-integers.

Theorem 3.1.1. *(Marsden's identity) For $n \geq 0$,*

$$([t]-x)^n = \sum_{i=-\infty}^{\infty} ([t]-[i+1])...([t]-[i+n])B_{n+1}\Big(\frac{x-[i]}{q^i}\Big). \tag{3.6}$$

When $n=0$ we take $([t]-[i+1])...([t]-[i+n]) = 1$.

Proof. We use induction on n. It follows from (1.37) that

$$\sum_{i=-\infty}^{\infty} B_i^0(x) = 1. \tag{3.7}$$

Then using (3.1) we obtain

$$\sum_{i=-\infty}^{\infty} B_1\Big(\frac{x-[i]}{q^i}\Big) = 1, \tag{3.8}$$

which shows that (3.6) is true for $n=0$. Assume that (3.6) is true for any $n \geq 0$. We need the following identity to complete the rest of the proof

$$\frac{[t]-[i+n+1]}{[i]-[i+n+1]}([i]-x) + \frac{[t]-[i]}{[i+n+1]-[i]}([i+n+1]-x) = [t]-x. \tag{3.9}$$

This is the linear interpolating function of $[t]-x$ which interpolates at $t=i$ and $t=i+n+1$. Mutiplying (3.6) by $([t]-x)$ yields

$$\begin{aligned} ([t]-x)^{n+1} \quad &= \sum_{-\infty}^{\infty}([t]-[i+1])...([t]-[i+n+1])\frac{[i]-x}{[i]-[i+n+1]}B_{n+1}\Big(\frac{x-[i]}{q^i}\Big) \\ &\quad + \sum_{-\infty}^{\infty}([t]-[i])...([t]-[i+n])\frac{[i+n+1]-x}{[i+n+1]-[i]}B_{n+1}\Big(\frac{x-[i]}{q^i}\Big). \end{aligned} \tag{3.10}$$

Shifting the index of the second summation

$$\begin{aligned} ([t]-x)^{n+1} = &\sum_{-\infty}^{\infty}([t]-[i+1])...([t]-[i+n+1])\frac{[i]-x}{[i]-[i+n+1]}B_{n+1}\Big(\frac{x-[i]}{q^i}\Big) \\ &+ \sum_{-\infty}^{\infty}([t]-[i+1])...([t]-[i+n+1])\frac{[i+n+2]-x}{[i+n+2]-[i+1]}B_{n+1}\Big(\frac{x-[i+1]}{q^{i+1}}\Big). \end{aligned} \tag{3.11}$$

Since $[i]-[i+n+1] = -q^i[n+1]$ and $[i+n+2]-[i+1] = q^{i+1}[n+1]$

$$\begin{aligned} ([t]-x)^{n+1} \quad &= \sum_{-\infty}^{\infty}([t]-[i+1])...([t]-[i+n+1])\frac{x-[i]}{q^i[n+1]}B_{n+1}\Big(\frac{x-[i]}{q^i}\Big) \\ &\quad + \sum_{-\infty}^{\infty}([t]-[i+1])...([t]-[i+n+1])\frac{[i+n+2]-x}{q^{i+1}[n+1]}B_{n+1}\Big(\frac{x-[i+1]}{q^{i+1}}\Big) \end{aligned} \tag{3.12}$$

using (3.2) we obtain

$$([t]-x)^{n+1}=\sum_{i=-\infty}^{\infty}([t]-[i+1])...([t]-[i+n+1])B_{n+2}\Big(\frac{x-[i]}{q^i}\Big) \tag{3.13}$$

which completes the proof.

B-splines with knots at integers have the symmetry property about the midpoint of interval of support but this property is not valid for B-splines with knots at q-integers. In (Phillips, 2003, p. 241) the following two generalizations of (1.45) are given.

Theorem 3.1.2. *The B-splines of degree $n-1$ with knots at q-integers satisfy the relation*

$$B_n(x)=q^{-n(n-1)/2}(1-(1-q)x)^{n-1}B_n\left(\frac{[n]-x}{1-(1-q)x}\right) \tag{3.14}$$

for all integers $n>1$, all real $q>0$ and x, for $n=1$ and all x except for $x=[0]$ and $[1]$.

Proof. See Kocak & Phillips (1994).

Theorem 3.1.3. *The B-splines of degree $n-1$ with knots at q-integers satisfy the relation*

$$B_n(x;q)=B_n(q^{-n+1}([n]-x);1/q), \tag{3.15}$$

for all inregers $n>1$, all real $q>0$ and x, for $n=1$ and all x except from $x=[0]$ and $[1]$.

Proof. See Kocak & Phillips (1994).

Theorem 3.1.4. *(Phillips, 2003, p. 241) $B_n(x)$ can be calculated by the following explicit formula*

$$B_n(x)=\frac{1}{[n-1]!}\sum_{j=0}^{n}(-1)^j q^{j(j-2n+1)/2}\begin{bmatrix}n\\ j\end{bmatrix}(x-[j])_+^{n-1}. \tag{3.16}$$

Proof. It follows from (1.31) with knots at $[0],[1],...,[n]$ that

$$\begin{aligned} B_n(x) &= (-1)^n \sum_{j=0}^{n} \frac{[n](x-[j])_+^{n-1}}{\omega'([j])} \\ &= (-1)^n \sum_{j=0}^{n} \frac{[n](x-[j])_+^{n-1}}{([j]-[0])([j]-[1])...([j]-[j-1])([j]-[j+1])...([j]-[n])}. \end{aligned} \tag{3.17}$$

Using $[j]-[k] = q^k[j-k]$ we obtain

$$B_n(x) = (-1)^n \sum_{j=0}^{n} \frac{[n](x-[j])_+^{n-1}}{q^{j(j-1)/2}[j]!(-1)^{n-j}q^{j(n-j)}[n-j]!}. \tag{3.18}$$

Multiplying both numerator and denominator of the latter equation by $[n]!$ gives (3.16).

Corollary 3.1.5.

$$\begin{aligned} B_n([k]) &= \tfrac{1}{[n-1]!} \sum_{j=0}^{n} (-1)^j q^{j(j-2n+1)/2} \begin{bmatrix} n \\ j \end{bmatrix} ([k]-[j])_+^{n-1} \\ &= \tfrac{1}{[n-1]!} \sum_{j=0}^{k} (-1)^j q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix} [k-j]^{n-1} \end{aligned} \tag{3.19}$$

for $1 \le k \le n-1$, *where* $([k]-[j])_+^0 = 1$ *if* $k \ge j$ *and* 0 *otherwise, and* $([k]-[j])_+^{n-1} = ([k]-[j])^{n-1} = q^{j(n-1)}[k-j]^{n-1}$ *if* $k \ge j$ *and* 0 *otherwise.*

3.1.1 q-Euler-Frobenius Polynomials

Euler-Frobenius polynomials which are related to Eulerian polynomials were constructed previously. Now, to construct the q-analogue of Euler-Frobenius polynomials we consider the space S_n of splines with knots at q-integers satisfying $f(qx+1) = zq^n f(x)$ for $z \neq 0,1$ and each $f \in S_n$. This consideration requires that we should define the q-analogue of exponential spline by

$$\phi_{n,q}(x;z) := \sum_{-\infty}^{\infty} (q^n z)^j B_{n+1}\left(\frac{x-[j]}{q^j}\right), \qquad z \neq 0,1. \tag{3.20}$$

We call $\phi_{n,q}(x;z)$ as the q-analogue of exponential spline of degree n to the base z.

If we differentiate (3.20) and use the derivative formula of B-spline functions with knots at q-integers (3.5) we find that

$$\begin{aligned}
\phi_{n,q}'(x;z) &= \sum_{-\infty}^{\infty}(q^n z)^j \left(B_{n+1}\left(\frac{x-[j]}{q^j}\right)\right)' \\
&= \sum_{-\infty}^{\infty}(q^{n-1}z)^j B_{n+1}'\left(\frac{x-[j]}{q^j}\right) \\
&= \sum_{-\infty}^{\infty}(q^{n-1}z)^j \frac{n}{[n]}\left\{B_n\left(\frac{x-[j]}{q^j}\right) - \frac{1}{q}B_n\left(\frac{x-[j+1]}{q^{j+1}}\right)\right\} \\
&= \tfrac{n}{[n]}\left\{\sum_{-\infty}^{\infty}(q^{n-1}z)^j B_n\left(\frac{x-[j]}{q^j}\right) - (q^n z)^{-1}\sum_{-\infty}^{\infty}(q^{n-1}z)^j B_n\left(\frac{x-[j]}{q^j}\right)\right\}.
\end{aligned} \tag{3.21}$$

Hence

$$\phi_{n,q}'(x;z) = \frac{n}{[n]}(1-(q^n z)^{-1})\phi_{n-1,q}(x;z). \tag{3.22}$$

Repeating the operation n times we have

$$\begin{aligned}
\phi_{n,q}^{(n)}(x;z) &= \tfrac{n!}{[n]!}(1-(q^n z)^{-1})(1-(q^{n-1}z)^{-1})...(1-(qz)^{-1})\phi_{0,q}(x;z) \\
&= \tfrac{n!}{[n]!}(1-(q^n z)^{-1})(1-(q^{n-1}z)^{-1})...(1-(qz)^{-1})\sum_{-\infty}^{\infty} z^j B_1\left(\frac{x-[j]}{q^j}\right).
\end{aligned} \tag{3.23}$$

Since $B_1(x) = 1$ in $[0] < x < [1]$, $B_1(x) = 0$ elsewhere, we find that

$$\phi_{n,q}^{(n)}(x;z) = \frac{n!}{[n]!}\frac{(z-q^{-n})(z-q^{-n+1})...(z-q^{-1})}{z^n}, \quad [0] < x < [1] \tag{3.24}$$

and its polynomial component in the interval $[0] < x < [1]$ has the form

$$\phi_{n,q}(x;z) = \frac{1}{[n]!}\frac{(z-q^{-n})_q^n}{z^n}x^n + \textit{lower degree terms}. \tag{3.25}$$

From (3.25) we also know that the coefficient of x^n is $(z-q^{-n})_q^n/([n]!z^n)$. We generate the following q-analogue of exponential Euler polynomial.

Definition 3.1.6. We define the monic polynomial $A_{n,q}(x;z) = x^n + (\textit{lower degree terms})$ by

$$A_{n,q}(x;z) = \frac{[n]!z^n}{(z-q^{-n})_q^n}\phi_{n,q}(x;z) \tag{3.26}$$

in $[0] \le x \le [1]$, $z \ne 0$, $z \ne 1$.

Substituting $x = 0$ in (3.26), we have

$$\begin{aligned}
A_{n,q}(0;z) &= \tfrac{[n]!z^n}{(z-q^{-n})_q^n}\phi_{n,q}(0;z) = \tfrac{[n]!z^n}{(z-q^{-n})_q^n}\sum_{-\infty}^{\infty}(q^n z)^j B_{n+1}([-j]) \\
&= \tfrac{[n]!}{(z-q^{-n})_q^n}\sum_{-\infty}^{\infty} q^{n(j-n)}z^j B_{n+1}([n-j]).
\end{aligned} \tag{3.27}$$

From Theorem 3.1.2 we have $B_{n+1}[n-j] = q^{n(n-1-2j)/2}B_{n+1}[j+1]$. Thus

$$A_{n,q}(0;z) = \frac{[n]!}{(z-q^{-n})_q^n} q^{-n(n+1)/2} \sum_{j=0}^{n-1} z^j B_{n+1}([j+1]). \tag{3.28}$$

Set

$$\Pi_{n,q}(z) := q^{n(n+1)/2} A_{n,q}(0;z)(z-q^{-n})_q^n. \tag{3.29}$$

The q-analogue of Euler-Frobenius polynomials and then (3.28) becomes

$$\Pi_{n,q}(z) = [n]! \sum_{j=0}^{n-1} z^j B_{n+1}([j+1]) \tag{3.30}$$

The following theorem gives the relation between $\Pi_{n,q}(z)$ and $A_{n,q}(z)$.

Theorem 3.1.7.

$$A_{n,q}(z) = \begin{cases} \Pi_{n,q}(z) = 1, & \text{if } n = 0 \\ z\Pi_{n,q}(z), & \text{if } n > 0. \end{cases} \tag{3.31}$$

Proof. We prove the above relation by using explicit forms of $A_q(n,k)$ and B-spline with knots at q-integers. Shifting the index of (3.30), we have

$$\begin{aligned} \Pi_{n,q}(z) &= [n]!z^{-1} \sum_{j=1}^{n} z^j B_{n+1}([j]) \\ &= [n]!z^{-1} \sum_{j=1}^{n} z^j \left(\frac{1}{[n]!} \sum_{i=0}^{j} (-1)^i q^{i(i-1)/2} \begin{bmatrix} n+1 \\ i \end{bmatrix} [j-i]^n \right) \\ &= z^{-1} \sum_{j=1}^{n} z^j A_q(n,j) \\ &= z^{-1} A_{n,q}(z). \end{aligned} \tag{3.32}$$

In case $n = 0$, it follows from (3.29) and (2.40).

Using (3.31) in the comparision between (2.40) and (3.30), we obtain the following corollary.

Corollary 3.1.8. *The relation between q-eulerian numbers and the value of B-spline of degree n at $x = [k]$ with knots at q-integers is*

$$A_q(n,k) = [n]!B_{n+1}([k]), \qquad 1 \le k \le n. \tag{3.33}$$

Proof. We use induction on n. For $n = 1$, $A_q(1,1) = 1 = B_2([1])$. Assume that (3.33) is true for any $n > 1$. First we write the recurrence relation of q-Eulerian numbers

$$A_q(n+1,k) = [k]A_q(n,k) + q^{k-1}[n-k+2]A_q(n,k-1) \tag{3.34}$$

and then by inductive hypothesis we obtain

$$A_q(n+1,k) = [n+1]!\left\{\frac{[k]}{[n+1]}B_{n+1}([k]) + q^{k-1}\frac{[n-k+1]}{[n+1]}B_{n+1}([k-1])\right\}. \tag{3.35}$$

From (3.4) we see that (3.33) is also true for $n+1$, which completes the proof.

Furthermore, previously proven identities such as (2.17), (2.22), (2.23) and (2.34) can be proved by using the corresponding identities of the B-splines with knots at q-integers and (3.33).

Proof of (2.17)**:** From (1.36) we have

$$B_{n+1}([k]) = \frac{[k]}{[n]}B_n([k]) + q^{k-1}\frac{[n-k+1]}{[n]}B_n([k-1]). \tag{3.36}$$

Then multiply both sides of the identity with $[n]!$

$$A_q(n,k) = [k]A_q(n-1,k) + q^{k-1}[n-k+1]A_q(n-1,k-1) \tag{3.37}$$

Proof of (2.22)**:** From the relation (3.14) we get

$$B_{n+1}([k]) = q^{-n(n+1)/2}(1-(1-q)[k])^n B_{n+1}\left(\frac{[n+1]-[k]}{1-(1-q)[k]}\right). \tag{3.38}$$

Since $1-(1-q)[k] = q^k$

$$B_{n+1}([k]) = q^{-n(n+1-2k)/2}B_{n+1}([n+1-k]). \tag{3.39}$$

Equation (2.22) immediately follows from multiplying the above equation by $[n]!$.

Proof of (2.23)**:** From the relation (3.15) we have

$$\begin{aligned} B_{n+1}([k],q) &= B_{n+1}\Big(q^{-n}([n+1]-[k]);1/q\Big) \\ &= B_{n+1}\Big(q^{k-n}[n+1-k];1/q\Big). \end{aligned} \tag{3.40}$$

Multiplying both sides of the above equation by $[n]_q!$ and using the identity $[n]_{1/q}! = q^{-n(n-1)/2}[n]_q!$ we obtain

$$[n]_q!B_{n+1}([k];q) = q^{n(n-1)/2}[n]_{1/q}!B_{n+1}\Big([n+1-k]_{1/q};1/q\Big). \tag{3.41}$$

Using the relation (3.33)

$$A_q(n,k) = q^{n(n-1)/2} A_{1/q}(n, n-k+1). \tag{3.42}$$

From (2.22), we obtain

$$A_q(n,k) = q^{n(k-1)} A_{1/q}(n,k). \tag{3.43}$$

Proof of (2.34)**:** From (3.6) we have

$$\begin{aligned}
([t]-x)^n &= \sum_{i=-\infty}^{\infty} ([t]-[i+1])...([t]-[i+n]) B_{n+1}\Big(\frac{x-[i]}{q^i}\Big) \\
&= \sum_{i=-\infty}^{\infty} B_{n+1}\Big(\frac{x-[i]}{q^i}\Big) \prod_{\gamma=1}^{n} ([t]-[i+\gamma]) \\
&= \sum_{i=-\infty}^{\infty} B_{n+1}\Big(\frac{x-[i]}{q^i}\Big) \prod_{\gamma=1}^{n} q^{i+\gamma}[t-i-\gamma].
\end{aligned} \tag{3.44}$$

Replacing i by $-i$ yields

$$\begin{aligned}
([t]-x)^n &= \sum_{i=-\infty}^{\infty} B_{n+1}\Big(\frac{x-[-i]}{q^{-i}}\Big) \prod_{\gamma=1}^{n} q^{-i+\gamma}[t+i-\gamma] \\
&= \sum_{i=-\infty}^{\infty} B_{n+1}(xq^i+[i]) \prod_{\gamma=1}^{n} q^{-i+\gamma}[t+i-\gamma].
\end{aligned} \tag{3.45}$$

Substituting $x=0$ gives

$$[t]^n = \sum_{i=-\infty}^{\infty} B_{n+1}([i]) \prod_{\gamma=1}^{n} q^{-i+\gamma}[t+i-\gamma]. \tag{3.46}$$

Now using the relation (3.33) and $B_{n+1}([i]) > 0$ for $1 \le i \le n$ we obtain

$$\begin{aligned}
[t]^n &= \sum_{i=1}^{n} A_q(n,i) \frac{\prod_{\gamma=1}^{n} q^{-i+\gamma}[t+i-\gamma]}{[n]!} \\
&= \sum_{i=1}^{n} A_q(n,i) q^{(-2in+n(n+1))/2} \frac{[t+i-1][t+i-2]...[t+i-n]}{[n]!} \\
&= \sum_{i=1}^{n} A_q(n,i) q^{(-2in+n(n+1))/2} \begin{bmatrix} t+i-1 \\ n \end{bmatrix}.
\end{aligned} \tag{3.47}$$

The following corollary obtained from (3.30) and (3.33).

Corollary 3.1.9.

$$\Pi_{n,q}(z) = \sum_{j=0}^{n-1} A_q(n, j+1) z^j \tag{3.48}$$

We see that $\Pi_{n,q}(z)$ is identical with $A_n(z,q)$ defined in Foata (2010) as q-Eulerian polynomials. For this reason we quote from Foata (2010) that q-analogue of Euler-Frobenius polynomials can be expressed by the following proposition.

Proposition 3.1.10. *For $n \geq 0$ we have*

$$\frac{\Pi_{n,q}(z)}{\prod_{i=0}^{n}(1-zq^i)} = \sum_{l\geq 0} z^l([l+1])^n \tag{3.49}$$

The corresponding exponential genearting function is defined as follows.

Theorem 3.1.11. *(Stanley (1976))*

$$\sum_{n\geq 0} \Pi_{n,q}(z)\frac{t^n}{[n]!} = \frac{1-z}{E_q(t(z-1))-z} \tag{3.50}$$

where $E_q(z) = \sum_{n\geq 0} q^{n(n-1)/2}\frac{z^n}{[n]!}$.

The recurrence relation for q-Euler-Frobenius polynomials given by

Corollary 3.1.12. *(Foata (2010))*

$$\Pi_{n+1,q}(z) = (1+qz[n])\Pi_{n,q}(z) + qz(1-z)D_q\Pi_{n,q}(z) \tag{3.51}$$

Proof. Multiplying (3.49) by $\prod_{i=0}^{n}(1-zq^i)$ gives

$$\Pi_{n,q}(z) = \prod_{i=0}^{n}(1-zq^i)\sum_{l\geq 0}[l+1]^n z^l. \tag{3.52}$$

Taking the q-derivative of both sides of the above identity we obtain

$$\begin{aligned}
D_q\Pi_{n,q}(z) &= \prod_{i=0}^{n}(1-zq^{i+1}).D_q\Big\{\sum_{l\geq 0}[l+1]^n z^l\Big\} + \sum_{l\geq 0}[l+1]^n z^l.D_q\Big\{\prod_{i=0}^{n}(1-zq^i)\Big\} \\
&= \prod_{i=0}^{n}(1-zq^{i+1})\sum_{l\geq 0}[l+1]^n[l]z^{l-1} - \frac{[n+1]}{(1-z)}\prod_{i=0}^{n}(1-zq^i)\sum_{l\geq 0}[l+1]^n z^l \\
&= \tfrac{1}{qz}\prod_{i=0}^{n}(1-zq^{i+1})\sum_{l\geq 0}([l+1]^{n+1}-[l+1]^n)z^l \\
&\quad -\tfrac{[n+1]}{(1-z)}\prod_{i=0}^{n}(1-zq^i)\sum_{l\geq 0}[l+1]^n z^l.
\end{aligned} \tag{3.53}$$

Multiplying both sides of the last equation by $qz(1-z)$ we get

$$\begin{aligned}
qz(1-z)D_q\Pi_{n,q}(z) &= \prod_{i=0}^{n+1}(1-zq^i)\sum_{l\geq 0}[l+1]^{n+1}z^l - \prod_{i=0}^{n+1}(1-zq^i)\sum_{l\geq 0}[l+1]^n z^l \\
&\qquad -qz(1-z)\prod_{i=0}^{n}(1-zq^i)\sum_{l\geq 0}[l+1]^n z^l \\
&= \Pi_{n+1,q}(z) - (1-zq^{n+1}+qz[n+1])\Pi_{n,q}(z).
\end{aligned} \tag{3.54}$$

Finally, rearraging inside of the paranthesis gives (3.51).

3.2 B-splines with Knots in Geometric Progression

Similar to the B-splines with knots at q-integers, the B-splines with knots in geometric progression don't have translation property, nevertheless, they are closely associated with one another. Namely, with the indexing $x_i = q^i$, for all i, we get

$$B_i^{n-1}(x) = B_n\left(\frac{x}{q^i}\right). \tag{3.55}$$

B_n is completely determined by the $n+1$ knots $q^0, q^1, ..., q^n$.

With knots at $q^0, ..., q^n$, the recurrence relation (1.36) becomes

$$B_n(x) = \frac{x-1}{q^{n-1}-1}B_{n-1}(x) + \frac{q^n - x}{q(q^{n-1}-1)}B_{n-1}(xq^{-1}), \tag{3.56}$$

and $B_1(x)$ is given by

$$B_1(x) = \begin{cases} 1, & q^0 \leq x < q^1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.57}$$

The derivative formula of B-splines with knots in geometric progression follows from the derivative formula of general B-splines (1.38) and (3.55). For $n \geq 3$, we have

$$\frac{d}{dx}B_n(x) = \frac{1}{q-1}\left\{\frac{n-1}{[n-1]}B_{n-1}(x) - \frac{n-1}{q[n-1]}B_{n-1}\left(\frac{x}{q}\right)\right\} \tag{3.58}$$

for all real x. For $n = 2$, (3.58) holds for all x except at the three knots q^0, q^1, and q^2, where the derivative of B_2 is not defined.

The following theorem gives a q-analogue of Marsden's identity for B-splines with knots in geometric progression.

Theorem 3.2.1. *For any* $n \geq 0$,

$$(q^t - x)^n = \sum_{i=-\infty}^{\infty}(q^t - q^{i+1})...(q^t - q^{i+n})B_{n+1}\left(\frac{x}{q^i}\right). \tag{3.59}$$

When $n = 0$, *we take* $(q^t - q^{i+1})...(q^t - q^{i+n}) = 1$.

Proof. We use induction on n. From (3.55) we obtain

$$\sum_{i=-\infty}^{\infty} B_1\left(\frac{x}{q^i}\right) = 1, \tag{3.60}$$

which shows that (3.59) is true for $n = 0$. Assume that (3.59) is true for any $n \geq 0$. We need the following identity to complete the rest of the proof

$$\frac{q^t - q^{i+n+1}}{q^i - q^{i+n+1}}(q^i - x) + \frac{q^t - q^i}{q^{i+n+1} - q^i}(q^{i+n+1} - x) = q^t - x. \tag{3.61}$$

This is the linear interpolating function of $q^t - x$ which interpolates at $t = i$ and $t = i+n+1$. Mutiplying (3.59) by $(q^t - x)$ yields

$$\begin{aligned}(q^t - x)^{n+1} &= \sum_{-\infty}^{\infty}(q^t - q^{i+1})\dots(q^t - q^{i+n+1})\frac{q^i - x}{q^i - q^{i+n+1}}B_{n+1}\left(\frac{x}{q^i}\right) \\ &\quad + \sum_{-\infty}^{\infty}(q^t - q^i)\dots(q^t - q^{i+n})\frac{q^{i+n+1} - x}{q^{i+n+1} - q^i}B_{n+1}\left(\frac{x}{q^i}\right).\end{aligned} \tag{3.62}$$

Shifting the index of the second summation

$$\begin{aligned}(q^t - x)^{n+1} &= \sum_{-\infty}^{\infty}(q^t - q^{i+1})\dots(q^t - q^{i+n+1})\frac{q^i - x}{q^i - q^{i+n+1}}B_{n+1}\left(\frac{x}{q^i}\right) \\ &\quad + \sum_{-\infty}^{\infty}(q^t - q^{i+1})\dots(q^t - q^{i+n+1})\frac{q^{i+n+2} - x}{q^{i+n+2} - q^{i+1}}B_{n+1}\left(\frac{x}{q^{i+1}}\right).\end{aligned} \tag{3.63}$$

Using (3.56) we obtain

$$(q^t - x)^{n+1} = \sum_{i=-\infty}^{\infty}(q^t - q^{i+1})\dots(q^t - q^{i+n+1})B_{n+2}\left(\frac{x}{q^i}\right) \tag{3.64}$$

which completes the proof.

Similarly, the B-splines with knots in geometric progression are not symmetric about the midpoint of interval of support.

Theorem 3.2.2. *The B-splines with knots in geometric progression satisfy the relation*

$$B_n(x) = q^{-n(n-1)/2}x^{n-1}B_n\left(\frac{q^n}{x}\right) \tag{3.65}$$

for all integers $n > 1$, all real $q > 1$ and x, when $n = 1$ (3.65) satisfied for all $q > 1$ and all x except for $x = q^0$ and q^1.

Proof. We use induction. For $n = 1$, we have $B_1(x) = B_1\left(\frac{q^1}{x}\right)$. This identity is not valid at the end points of the interval of the support of $B_1(x)$ but is valid for $x \in (q^0, q^1)$

since for $x = q^0$ we have $\frac{q}{x} = q$, but $B_1(q) = 0$ while $B_1(q^0) = 1$. Now we must show that it is true for $n = 2$. From (3.56) we have

$$B_2(x) = \frac{x-1}{q-1}B_1(x) + \frac{q^2-x}{q(q-1)}B_1(xq^{-1}). \tag{3.66}$$

Replacing x by $\frac{q^2}{x}$

$$B_2\left(\frac{q^2}{x}\right) = \frac{q^2-x}{x(q-1)}B_1\left(\frac{q^2}{x}\right) + \frac{xq^2-q^2}{xq(q-1)}B_1\left(\frac{q}{x}\right) \tag{3.67}$$

and multiplying the both sides of the last equation by $\frac{x}{q}$

$$\frac{x}{q}B_2\left(\frac{q^2}{x}\right) = \frac{q^2-x}{q(q-1)}B_1\left(\frac{q^2}{x}\right) + \frac{x-1}{q-1}B_1\left(\frac{q}{x}\right). \tag{3.68}$$

Using the truth of (3.65) for $n = 1$ we obtain $\frac{x}{q}B_2\left(\frac{q^2}{x}\right) = B_2(x)$ which shows that (3.65) is true for $n = 2$. For $x = q^0$, $B_2(q^0) = 0 = \frac{1}{q}B_2(q^2)$ and for $x = q^2$, $B_2(q^2) = 0 = qB_2(q^0)$ since the interval of support of $B_2(x)$ is $x \in [q^0, q^2]$. Suppose that (3.65) holds for $n-1$. Then we see from the recurrence relation

$$B_n(x;q) = \frac{x-1}{q^{n-1}-1}B_{n-1}(x) + \frac{q^n-x}{q(q^{n-1}-1)}B_{n-1}(xq^{-1}), \tag{3.69}$$

using inductive hypothesis yields

$$\begin{aligned} B_n(x) \quad &= \tfrac{x-1}{q^{n-1}-1}q^{-(n-1)(n-2)/2}x^{n-2}B_{n-1}\left(\tfrac{q^{n-1}}{x}\right) \\ &\quad + \tfrac{q^n-x}{q(q^{n-1}-1)}q^{-(n-1)(n-2)/2}(xq^{-1})^{n-2}B_{n-1}\left(\tfrac{q^{n-1}}{xq^{-1}}\right). \end{aligned} \tag{3.70}$$

After rearranging the terms we have

$$\begin{aligned} B_n(x) \quad &= q^{-n(n-1)/2}x^{n-1}\left\{\tfrac{q^{n-1}(x-1)}{x(q^{n-1}-q)}B_{n-1}\left(\tfrac{q^{n-1}}{x}\right) + \tfrac{q(q^n-x)}{x(q^n-q)}B_{n-1}\left(\tfrac{q^n}{x}\right)\right\} \\ &= q^{-n(n-1)/2}x^{n-1}B_n\left(\tfrac{q^n}{x}\right) \end{aligned} \tag{3.71}$$

which completes the proof.

Theorem 3.2.3. *The B-splines of degree $n-1$ with knots in geometric progression satisfy the relation*

$$B_n(x;q) = B_n(q^{-n}x;1/q) \tag{3.72}$$

for all integers $n > 1$ and i, all real $q > 1$ and x, and for $n = 1$ and all x except for $x = q^0$ and q^1.

Proof. We prove by induction. For $n=1$, (3.72) reduces to $B_1(x;q) = B_1(q^{-1}x;1/q)$. This identity is not provided for $x=q^0$ and $x=q^1$ since while $B_1(x;q)=1$ for $x \in [q^0,q^1)$ and 0 otherwise, $B_1(q^{-1}x;1/q)=1$ for $q^{-1}x \in [1/q,1/q^0)$ and 0 otherwise. Now, we show the truth of (3.72) for $n=2$. From (3.56) we have

$$B_2(x;q) = \frac{x-1}{q-1}B_1(x;q) + \frac{q^2-x}{q(q-1)}B_1(xq^{-1};q). \tag{3.73}$$

Replacing q by $1/q$

$$B_2(x;1/q) = \frac{q(x-1)}{1-q}B_1(x;1/q) + \frac{1-xq^2}{1-q}B_1(xq;1/q) \tag{3.74}$$

and substituting x/q^2 for x

$$B_2(x/q^2;1/q) = \frac{x-q^2}{q(q-1)}B_1(x/q^2;1/q) + \frac{1-x}{1-q}B_1(x/q;1/q). \tag{3.75}$$

By using the truth of (3.72) for $n=1$ we have $B_2(x/q^2;1/q) = B_2(x;q)$. For $x=q^0$, $B_2(q^0;q) = 0 = B_2(q^{-2};1/q)$ and $x=q^2$, $B_2(q^2;q) = 0 = B_2(q^0;1/q)$ since the interval of support of $B_2(x;q)$ is $[q^0,q^2]$ while the interval of support of $B_2(x;1/q)$ is $[1/q^0,1/q^2]$. Suppose that (3.72) holds for $n-1$. Then we write the recurrence relation

$$B_n(x;q) = \frac{x-1}{q^{n-1}-1}B_{n-1}(x) + \frac{q^n-x}{q(q^{n-1}-1)}B_{n-1}(xq^{-1}), \tag{3.76}$$

using the inductive hypothesis we obtain

$$\begin{aligned} B_n(x;q) &= \tfrac{x-1}{q^{n-1}-1}B_{n-1}(q^{-n+1}x;1/q) + \tfrac{q^n-x}{q^n-q}B_{n-1}(q^{-n}x;1/q) \\ &= B_n(q^{-n}x;1/q). \end{aligned} \tag{3.77}$$

Theorem 3.2.4. *The B-spline $B_n(x)$ of degree $n-1$ can be calculated by the following explicit formula*

$$B_n(x) = \frac{1}{(q-1)^{n-1}[n-1]!}\sum_{j=0}^{n}(-1)^j q^{j(j-2n+1)/2}\begin{bmatrix} n \\ j \end{bmatrix}(x-q^j)_+^{n-1}. \tag{3.78}$$

Proof. Substituting $x_i = q^i$, $\forall i$ in (1.31) gives

$$\begin{aligned} B_n(x) &= (-1)^n \sum_{j=0}^{n} \frac{(q^n-1)(x-q^j)_+^{n-1}}{\omega'(q^j)} \\ &= (-1)^n \sum_{j=0}^{n} \frac{(q^n-1)(x-q^j)_+^{n-1}}{(q^j-q^0)(q^j-q^1)...(q^j-q^{j-1})(q^j-q^{j+1})...(q^j-q^n)} \\ &= (-1)^n \sum_{j=0}^{n} \frac{(q^n-1)(x-q^j)_+^{n-1}}{(q^j-1)q(q^{j-1}-1)...q^{j-1}(q-1)q^j(1-q)...q^j(1-q^{n-j})}. \end{aligned} \tag{3.79}$$

Multiply and divide the denominator by $(q-1)^n$

$$B_n(x) = (-1)^n \sum_{j=0}^{n} \frac{(q^n-1)(x-q^j)_+^{n-1}}{q^{j(j-1)/2}[j]!(-1)^{n-j}(q-1)^n q^{j(n-j)}[n-j]!} \tag{3.80}$$

after multiplying both numerator and denominator of the above equation by $[n]!$ and rearranging the terms we obtain (3.78).

Corollary 3.2.5.

$$B_n(q^k) = \frac{1}{[n-1]!} \sum_{j=0}^{k} (-1)^j q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix} [k-j]^{n-1} \tag{3.81}$$

for $q > 1$ and $1 \le k \le n-1$, where $(q^k - q^j)_+^0 = 1$ if $k \ge j$ and 0 otherwise, and $(q^k - q^j)_+^{n-1} = (q^k - q^j)^{n-1} = q^{j(n-1)}(q^{k-j}-1)^{n-1}$ if $k \ge j$ and 0 otherwise.

3.2.1 q-Euler-Frobenius Polynomials

We constructed the q-analogue of Euler-Frobenius polynomials with knots at q-integers. In this section we show that q-Euler-Frobenius polynomials can be derived from the B-splines with knots in geometric pregression. First we define the elements of S_n with knots in geometric progression satisfying $f(qx) = zq^n f(x)$ by

$$\tilde{\phi}_{n,q}(x;z) := \sum_{-\infty}^{\infty} (q^n z)^j B_{n+1}\left(\frac{x}{q^j}\right), \qquad z \ne 0, 1. \tag{3.82}$$

We call $\tilde{\phi}_{n,q}(x;z)$ as the q-analogue of exponential spline of degree n with knots in geometric progression to the base z.

If we differentiate (3.82) and use the derivative formula of B-spline functions with knots in geometric progression (3.58) we find that

$$\begin{aligned}
\tilde{\phi}'_{n,q}(x;z) &= \sum_{-\infty}^{\infty} (q^{n-1}z)^j B'_{n+1}\left(\frac{x}{q^j}\right) \\
&= \sum_{-\infty}^{\infty} (q^{n-1}z)^j \frac{n}{[n]} \frac{1}{q-1} \left(B_n\left(\frac{x}{q^j}\right) - \frac{1}{q} B_n\left(\frac{x}{q^{j+1}}\right) \right) \\
&= \tfrac{n}{[n](q-1)} \left\{ \sum_{-\infty}^{\infty} (q^{n-1}z)^j B_n\left(\frac{x}{q^j}\right) - (q^n z)^{-1} \sum_{-\infty}^{\infty} (q^{n-1})^j B_n\left(\frac{x}{q^j}\right) \right\}.
\end{aligned} \tag{3.83}$$

Hence

$$\tilde{\phi}'_{n,q}(x;z) = \frac{n(1-(q^n z)^{-1})}{[n](q-1)} \tilde{\phi}_{n-1,q}(x;z). \tag{3.84}$$

Repeating the operation n times we have

$$\begin{aligned}\tilde{\phi}^{(n)}_{n,q}(x;z) &= \tfrac{n!(1-(q^nz)^{-1})(1-(q^{n-1}z)^{-1})\ldots(1-(qz)^{-1})}{[n]!(q-1)^n}\tilde{\phi}_{0,q}(x;z) \\ &= \tfrac{n!(1-(q^nz)^{-1})(1-(q^{n-1}z)^{-1})\ldots(1-(qz)^{-1})}{[n]!(q-1)^n}\sum_{-\infty}^{\infty} z^j B_1\left(\frac{x}{q^j}\right).\end{aligned} \tag{3.85}$$

Since $B_1(x) = 1$ in $q^0 < x < q^1$, $B_1(x) = 0$ elsewhere, we find that

$$\tilde{\phi}^{(n)}_{n,q}(x;z) = \frac{n!(z-q^{-n})(z-q^{-n+1})\ldots(z-q^{-1})}{[n]!(q-1)^n z^n}, \quad q^0 < x < q^1 \tag{3.86}$$

and its polynomial component in the interval $q^0 < x < q^1$ has the form

$$\tilde{\phi}_{n,q}(x;z) = \frac{(z-q^{-n})^n_q}{[n]!(q-1)^n z^n} x^n + lower\ degree\ terms. \tag{3.87}$$

Thus we generate a monic polynomial from (3.87).

Definition 3.2.6. We define the monic polynomial $\tilde{A}_{n,q}(x;z) = x^n + (lower\ degree\ terms)$ by

$$\tilde{A}_{n,q}(x;z) = \frac{(q-1)^n [n]! z^n}{(z-q^{-n})^n_q}\tilde{\phi}_{n,q}(x;z) \tag{3.88}$$

for $q^0 \le x \le q^1$, $z \ne 0,1$ and call it the exponential Euler polynomial with knots in geometric progression.

Substituting $x = 1$ in (3.88), we have

$$\begin{aligned}\tilde{A}_{n,q}(1;z) &= \tfrac{(q-1)^n[n]!z^n}{(z-q^{-n})^n_q}\tilde{\phi}_{n,q}(1;z) \\ &= \tfrac{(q-1)^n[n]!z^n}{(z-q^{-n})^n_q}\sum_{\infty}^{\infty}(q^nz)^j B_{n+1}(q^{-j}) \\ &= \tfrac{(q-1)^n[n]!}{(z-q^{-n})^n_q}\sum_{-\infty}^{\infty} z^j q^{n(j-n)} B_{n+1}(q^{n-j}).\end{aligned} \tag{3.89}$$

Using $B_{n+1}(q^{n-j}) = q^{n(n-1-2j)/2} B_{n+1}(q^{j+1})$

$$\tilde{A}_{n,q}(1;z) = \frac{(q-1)^n[n]!}{(z-q^{-n})^n_q} q^{-n(n+1)/2} \sum_{j=0}^{n-1} z^j B_{n+1}(q^{j+1}). \tag{3.90}$$

Define

$$\tilde{\Pi}_{n,q}(z) := \frac{q^{n(n+1)/2}\tilde{A}_{n,q}(1;z)(z-q^{-n})^n_q}{(q-1)^n}. \tag{3.91}$$

Then (3.90) becomes

$$\tilde{\Pi}_{n,q}(z) = [n]! \sum_{j=0}^{n-1} z^j B_{n+1}(q^{j+1}) \tag{3.92}$$

Theorem 3.2.7. *Let* $\tilde{\Pi}_{n,q}(z)$ *and* $A_{n,q}(z)$ *be the polynomials defined by* (3.90) *and* (2.39)*, respectively. Then we can construct the relation between them as follows*

$$A_{n,q}(z) = \begin{cases} \tilde{\Pi}_{n,q}(z) = 1, & \text{if } n = 0 \\ z\tilde{\Pi}_{n,q}(z), & \text{if } n > 0 \end{cases} \tag{3.93}$$

Proof. We will prove the above relation by using the explicit forms of Eulerian numbers and B-spline with knots in geometric progression. Shifting the index of (3.92) yields

$$\begin{aligned} \tilde{\Pi}_{n,q}(z) &= [n]!z^{-1}\sum_{j=1}^{n} z^j B_{n+1}(q^j) \\ &= [n]!z^{-1}\sum_{j=1}^{n} z^j \left(\frac{1}{[n]!}\sum_{i=0}^{j}(-1)^i q^{i(i-1)/2}\begin{bmatrix} n+1 \\ i \end{bmatrix}[j-i]^n\right) \\ &= z^{-1}\sum_{j=1}^{n} z^j A_q(n,j) \\ &= z^{-1}A_{n,q}(z). \end{aligned} \tag{3.94}$$

The relation is also true for $n = 0$ since $\tilde{\Pi}_{0,q}(z) = 1$ and $A_{0,q}(z) = 1$ which follows from (3.90) and (2.40), respectively.

We conclude from Theorem (3.1.7) and Theorem (3.2.7) that the relation between $\tilde{\Pi}_{n,q}(z)$ and $\Pi_{n,q}(z)$ can be expressed in the following corollary.

Corollary 3.2.8. *For* $n \geq 0$

$$\tilde{\Pi}_{n,q}(z) = \Pi_{n,q}(z). \tag{3.95}$$

Using (3.93) in the comparision between (2.40) and (3.92), we obtain a relation between q-Eulerian numbers and the value of B-splines with knots in geometric progression that evaluated at $x = q^k$.

Proposition 3.2.9.

$$A_q(n,k) = [n]!B_{n+1}(q^k), \qquad 1 \leq k \leq n. \tag{3.96}$$

Proof. We prove the above identity by induction on $n \geq 1$. If $n = 1$ and $k = 1$, then $A_q(1,1) = 1 = B_2(q^1)$. We assume that it is true for $n-1$. We use (3.56) and the inductive hypothesis we get

$$\begin{aligned} A_q(n,k) &= [k][n-1]!B_n(q^k) + q^{k-1}[n-k+1]B_n(q^{k-1}) \\ &= [n]!\left(\tfrac{[k]}{[n]}B_n(q^k) + q^{k-1}\tfrac{n-k+1}{[n]}B_n(q^{k-1}) \right) \\ &= [n]!B_{n+1}(q^k). \end{aligned} \tag{3.97}$$

Similarly (2.17), (2.22), (2.23) and (2.34) can be proven by using corresponding identities of B-splines with knots in geometric progression and (3.96).

Proof of (2.17)**:** From (1.36) we have

$$B_{n+1}(x) = \frac{[k]}{[n]}B_n(q^k) + q^{k-1}\frac{[n-k+1]}{[n]}B_n(q^{k-1}). \tag{3.98}$$

Multiplying the above equation by $[n]!$ and using (3.96) we obtain

$$A_q(n,k) = [k]A_q(n-1,k) + q^{k-1}[n-k+1]A_q(n-1,k-1). \tag{3.99}$$

Proof of (2.22)**:** From (3.65) we have

$$B_{n+1}(q^k) = q^{-n(n-2k+1)/2}B_{n+1}(q^{n+1-k}). \tag{3.100}$$

The result follows from multiplying the above equation by $[n]!$.

Proof of (2.34)**:** From (3.59) we get

$$\begin{aligned} (q^t - x)^n &= \sum_{i=-\infty}^{\infty} (q^t - q^{i+1})...(q^t - q^{i+n})B_{n+1}\left(\frac{x}{q^i}\right) \\ &= \sum_{i=-\infty}^{\infty} B_{n+1}\left(\frac{x}{q^i}\right)\prod_{\gamma=1}^{n}(q^t - q^{i+\gamma}). \end{aligned} \tag{3.101}$$

Replacing i by $-i$ gives

$$(q^t - x)^n = \sum_{i=-\infty}^{\infty} B_{n+1}\left(\frac{x}{q^{-i}}\right)\prod_{\gamma=1}^{n} q^{-i+\gamma}(q^{t+i-\gamma} - 1). \tag{3.102}$$

Writing $x = 1$ and using $(q^t - 1)^n = (q-1)^n[t]^n$

$$(q-1)^n[t]^n = \sum_{i=-\infty}^{\infty} B_{n+1}(q^i)\prod_{\gamma=1}^{n} q^{-i+\gamma}(q^{t+i-\gamma} - 1). \tag{3.103}$$

Dividing both sides of the above equation by $(q-1)^n$ and using the interval of support of $B_{n+1}(q^i)$ yields

$$[t]^n = \sum_{i=1}^{n} B_{n+1}(q^i)q^{(-2in+n(n+1))/2}[t+i-1][t+i-2]...[t+i-n]. \tag{3.104}$$

From (3.96)

$$\begin{aligned}
[t]^n &= \sum_{i=1}^{n} A_q(n,i) q^{(-2in+n(n+1))/2} \frac{[t+i-1][t+i-2]...[t+i-n]}{[n]!} \\
&= \sum_{i=1}^{n} A_q(n,i) q^{(-2in+n(n+1))/2} \begin{bmatrix} t+i-1 \\ n \end{bmatrix}.
\end{aligned} \tag{3.105}$$

CHAPTER FOUR
CONCLUSIONS

In this study, we give the relation between q-Eulerian polynomials and B-splines with knots at q-integers and in geometric progression. We find two different q-analogues of Euler-Frobenius polynomials by constructing q-analogues of exponential splines. We state the relation between q-Eulerian numbers and B-spline values with knots at q-integers and in geometric progression via q-analogues of Euler-Frobenius polynomial as the following

$$A_q(n,k) = [n]!B_{n+1}([k]), \qquad 1 \leq k \leq n \tag{4.1}$$

and

$$A_q(n,k) = [n]!B_{n+1}(q^k), \qquad 1 \leq k \leq n. \tag{4.2}$$

We also find these q-analogues of Euler-Frobenius polynomials can be written in terms of B-splines with knots at q-integers and in geometric progression. We show that both B-spline has same value on their knots. For the B-splines with knots in geometric progression, we construct the following identities for all integers $n > 1$ and i, all real $q > 1$ and x, and for $n = 1$ and all x except for $x = q^0$ and q^1

$$B_n(x) = q^{-n(n-1)/2}x^{n-1}B_n\left(\frac{q^n}{x}\right) \tag{4.3}$$

and

$$B_n(x;q) = B_n(q^{-n}x;1/q), \tag{4.4}$$

which generalize the symmetry property of cardinal B-splines. We establish q-analogues of Marsden's identity in terms of B-splines with knots at q-integers and in geometric progression. These q-analogues of Marsden's identity is used in the construction of q-analogue of Worpitzky's identity.

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